JOURNAL OF ALGEBRA 134, 1–27 (1990)

Operators Related to E-Disjunctive and Fundamental Completely Regular Semigroups

MARIO PETRICH AND NORMAN R. REILLY

Department of Mathematics and Statistics, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6

Communicated by T. E. Hall

Received April 19, 1988

1. INTRODUCTION AND SUMMARY

We may regard completely regular semigroups S as algebras with the (binary) operation of multiplication and the (unary) operation of inversion. The latter operation associates with each element a of S its inverse in the maximal subgroup of S which contains a. The variety $\mathscr{CR}$ of these algebras is determined by the identities

$$a(bc) = (ab)c, \qquad (a^{-1})^{-1} = a, \qquad a = aa^{-1}a, \qquad aa^{-1} = a^{-1}a. \quad (1)$$

The lattice $\mathscr{L}(\mathscr{CR})$ of all subvarieties of $\mathscr{CR}$ has been the subject of considerable investigations in recent years. Various approaches to these investigations showed that $\mathscr{L}(\mathscr{CR})$, even though complex, can be successfully studied both locally and globally. In the study of the latter , the main tools turn out to be various partitions of the lattice $\mathscr{L}(\mathscr{CR})$, more precisely, complete congruences on $\mathscr{L}(\mathscr{CR})$.

One source of such complete congruences is the lattice $\mathscr{C}$ of fully invariant congruences on a free completely regular semigroup $F\mathscr{CR}$ on a countably infinite set. Since $F\mathscr{CR}$ is a regular semigroup, the relation T on $\mathscr{C}$ which identifies any two congruences having the same trace is a complete congruence [1, 13]. In addition, it was proved in [4] that the relation K on $\mathscr{C}$ which identifies any two congruences having the same kernel is also a complete congruence on $\mathscr{C}$. By the standard anti-isomorphism π of $\mathscr{C}$ onto $\mathscr{L}(\mathscr{CR})$, both K and T induce complete congruences on $\mathscr{L}(\mathscr{CR})$. As shown in [4, 6], both of these congruences enjoy a number of properties which can be used profitably to study $\mathscr{L}(\mathscr{CR})$ in a global way.

In Section 3, we let $\mathscr{D}$ be the class of all E-disjunctive completely regular semigroups (with $\tau = \varepsilon$, that is, without proper idempotent pure congruences) and consider a certain lattice of its subclasses which we denote by

0021-8693/90 $3.00

$\mathscr{L}_\tau(\mathscr{CR})$. We discuss the mapping $\theta_{\mathscr{D}}: \mathscr{V} \to \mathscr{V} \cap \mathscr{D}$, which maps $\mathscr{L}(\mathscr{CR})$ onto $\mathscr{L}_\tau(\mathscr{CR})$ and has some interesting properties. In particular, we show that the classes of the equivalence relation $\bar{\theta}_{\mathscr{D}}$ induced by $\theta_{\mathscr{D}}$ are intervals and describe their upper and lower bounds. In Section 4 we investigate the operators on $\mathscr{L}(\mathscr{CR})$ mapping $\mathscr{V}$ to the upper or lower bounds of $\mathscr{V}\bar{\theta}_{\mathscr{D}}$. In Section 5 we consider the associated relation K.

In a dual way, in Section 6 we let $\mathscr{F}$ be the class of all fundamental completely regular semigroups (with $\mu = \varepsilon$, that is, without proper idempotent separating congruences) and perform a similar analysis with the mapping $\theta_{\mathscr{F}}: \mathscr{V} \to \mathscr{V} \cap \mathscr{F}$ and relate it to the congruence T on $\mathscr{C}$. Again the classes of the equivalence relation $\bar{\theta}_{\mathscr{F}}$ induced by $\theta_{\mathscr{F}}$ are intervals and in Section 7 we investigate operators on $\mathscr{L}(\mathscr{CR})$ associated with the bounds of these intervals. In Section 8, a similar analysis is made with left fundamental completely regular semigroups.

2. Preliminaries

Besides the standard terminology and notation, we will need the following.

If θ is a mapping of a set A, then $\bar{\theta}$ denotes the equivalence relation on A induced by θ. If also $B \subseteq A$, then $\theta|_B$ denotes the restriction of θ to B. The equality relation on any set is denoted by ε.

Let S be a completely regular semigroup. Then $E(S)$ denotes the set of idempotents of S, $C(S)$ denotes the *core* of S, that is, the subsemigroup of S generated by $E(S)$, and $\mathscr{C}(S)$ denotes the lattice of fully invariant congruences on S. Let ρ be a congruence on S. Then

$$\ker \rho = \{a \in S \mid a \,\rho\, e \text{ for some } e \in E(S)\}$$

is the *kernel* of ρ and $\operatorname{tr} \rho = \rho|_{E(S)}$ is the *trace* of S. Any congruence on a completely regular semigroup is uniquely determined by its kernel and trace. The greatest congruence on S for which $\ker \rho = E(S)$ is denoted by τ and the greatest congruence such that $\operatorname{tr} \rho = \varepsilon$ is denoted by μ. If $\tau = \varepsilon$, then S is *E-disjunctive*, and if $\mu = \varepsilon$, then S is *fundamental*. If $a \in S$, then a^0 denotes the element $a^{-1}a$.

For any class $\mathscr{A}$ of completely regular semigroups $\mathbf{H}\mathscr{A}$ and $\mathbf{S}\mathscr{A}$ denote, respectively, the homomorphic closure and the (completely regular) subsemigroup closure of $\mathscr{A}$.

We denote by $\mathscr{CR}$ the variety of all completely regular semigroups, given by the identities (1). The lattice of subvarieties of a variety $\mathscr{V}$ is denoted by $\mathscr{L}(\mathscr{V})$. For any $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, $\mathrm{F}\mathscr{V}$ denotes the (relatively) free object in

$\mathcal{V}$ on $\aleph_0$ generators. The familiar anti-isomorphism of $\mathcal{C}(F\mathcal{CR})$ onto $\mathcal{L}(\mathcal{CR})$ is denoted by π; in addition we write

$$\pi: \rho \to [\rho], \qquad \pi^{-1}: \mathcal{V} \to \rho_{\mathcal{V}}.$$

The following subvarieties of $\mathcal{CR}$ will figure prominently in our discussions.

$\mathcal{T}$ = trivial semigroups
$\mathcal{LZ}$ = left zero semigroups
$\mathcal{RZ}$ = right zero semigroups
$\mathcal{RB}$ = rectangular bands
$\mathcal{R}e\mathcal{G}$ = rectangular groups
$\mathcal{S}$ = semilattices
$\mathcal{B}$ = bands
$\mathcal{NB}$ = normal bands (those satisfying $axya = ayxa$)
$\mathcal{G}$ = groups
$\mathcal{AG}$ = abelian groups
$\mathcal{A}_n$ = abelian groups of exponent n
$\mathcal{LG}$ = left groups
$\mathcal{RG}$ = right groups
$\mathcal{SG}$ = semilattices of groups
$\mathcal{CS}$ = completely simple semigroups
$\mathcal{OG}$ = orthogroups (orthodox completely regular semigroups)
$\mathcal{CG}$ = cryptogroups (completely regular semigroups for which $\mathcal{H}$ is a congruence)
$\mathcal{NCG}$ = normal cryptogroups (completely regular semigroups for which $\mathcal{H}$ is congruence and $S/\mathcal{H}$ is a normal band)

In addition we shall use the following notation.

$c(u)$ = the set of variables appearing in u
θ^0 = the largest congruence on a semigroup S for which the equivalence relation θ on S is a union of classes
A^0 = the largest congruence on S for which the subset A of S S is a union of classes ($=\theta^0$, where $\theta = (A \times A) \cup (S \setminus A) \times (S \setminus A)$)
$\langle A \rangle$ = the variety generated by the class (or object) A
$[u = v]$ = the variety of completely regular semigroups determined by the identity $u = v$

We recall that θ^0 can be described by

$$a\,\theta^0\,b \Leftrightarrow xay\,\theta\,xby \qquad \text{for all} \quad x, y \in S^1.$$

Important special cases of θ^0 and A^0 are $\mu = \mathcal{H}^0$ and $\tau = E(S)^0$.

Let $\mathscr{A}$ and $\mathscr{B}$ be any classes of completely regular semigroups. If ρ is a congruence on a semigroup S such that all idempotent ρ-classes are in $\mathscr{A}$, then we say that ρ is *over* $\mathscr{A}$. The *Malcev product* of $\mathscr{A}$ and $\mathscr{B}$ is defined by

$$\mathscr{A} \circ \mathscr{B} = \{S \in \mathscr{CR} \mid \text{there exists a congruence } \rho \text{ on } S \text{ over } \mathscr{A} \text{ with } S/\rho \in \mathscr{B}\}.$$

For certain choices of varieties $\mathscr{A}$ and $\mathscr{B}$, $\mathscr{A} \circ \mathscr{B}$ is also a variety, but not always (see [3]).

LEMMA 2.1 [3, Theorem 5.1]. *For all $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$ and $\mathscr{U} \in \{\mathscr{G}, \mathscr{LG}, \mathscr{RG}, \mathscr{ReG}\}$, $U \circ \mathscr{V} \in \mathscr{L}(\mathscr{CR})$ and, for all $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$ with $\mathscr{S} \subseteq \mathscr{V}$, $\mathscr{B} \circ \mathscr{V} \in \mathscr{L}(\mathscr{CR})$.*

By a *radical congruence system* $\mathbf{K}$ we mean a family of congruences $\{\kappa_S \mid S \in \mathscr{CR}\}$, where each κ_S is a congruence on S, such that

$\mathbf{K}$(i) for any $S = \prod_{\alpha \in A} S_\alpha$, $S_\alpha \in \mathscr{CR}$, $\kappa_S = \prod_{\alpha \in A} \kappa_{S_\alpha}$, that is, if $a = (a_\alpha)$, $b = (b_\alpha) \in S$, then

$$a \,\kappa_S\, b \Leftrightarrow a_\alpha \,\kappa_{S_\alpha}\, b_\alpha. \qquad \text{for all} \quad \alpha \in I.$$

$\mathbf{K}$(ii) for any $S \in \mathscr{CR}$, $\kappa_{(S/\kappa_S)} = \varepsilon$.

We have in mind here such "standard" congruences on a completely regular semigroup as μ, τ, etc. For any radical congruence system $\mathbf{K}$, we define the *radical class of* $\mathbf{K}$ to be

$$\mathscr{K} = \{S \in \mathscr{CR} \mid \kappa_S = \varepsilon\}.$$

A subclass $\mathscr{C}$ of $\mathscr{K}$ is a *κ-variety* if it satisfies

(i) $\mathscr{C}$ is closed under isomorphisms and the formation of direct products,

(ii) if $S \in \mathbf{HS}\mathscr{C}$, then $S/\kappa_S \in \mathscr{C}$.

Note that an ε-variety is a variety in the usual sense.

Let $\mathscr{L}_\kappa(\mathscr{CR})$ denote the class of all κ-varieties ordered by inclusion.

PROPOSITION 2.2. *Let $\mathbf{K} = \{\kappa_S \mid S \in \mathscr{CR}\}$ be a radical congruence system and $\mathscr{K} = \{S \mid \kappa_S = \varepsilon\}$.*

(i) *The mapping*

$$\theta_\kappa : \mathscr{V} \to \mathscr{V} \cap \mathscr{K} \qquad (\mathscr{V} \in \mathscr{L}(\mathscr{CR}))$$

is a surjection of $\mathscr{L}(\mathscr{CR})$ onto $\mathscr{L}_\kappa(\mathscr{CR})$ respecting arbitrary intersections.

(ii) *$\mathscr{L}_\kappa(\mathscr{CR})$ is a complete lattice with maximum element $\mathscr{K}$.*

Proof. (i) Let $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$. Clearly $\mathscr{V} \cap \mathscr{K}$ is closed under isomorphisms, since both of $\mathscr{V}$ and $\mathscr{K}$ are. For each $\alpha \in A$, let $S_\alpha \in \mathscr{V} \cap \mathscr{K}$ and $S = \prod_{\alpha \in A} S_\alpha$. Since $\mathscr{V}$ is a variety, $S \in \mathscr{V}$. By **K**(i), $\kappa_S = \varepsilon$ so that $S \in \mathscr{V} \cap \mathscr{K}$ and $\mathscr{V} \cap \mathscr{K}$ is closed under direct products.

Now let $S \in \mathbf{HS}(\mathscr{V} \cap \mathscr{K})$. Then $S \in \mathscr{V}$, since $\mathscr{V}$ is a variety, so that by **K**(ii), $S/\kappa_S \in \mathscr{V} \cap \mathscr{K}$. Consequently, $\mathscr{V} \cap \mathscr{K} \in \mathscr{L}_\kappa(\mathscr{CR})$ and θ_κ maps $\mathscr{L}(\mathscr{CR})$ into $\mathscr{L}_\kappa(\mathscr{CR})$. Clearly θ_κ respects arbitrary intersections.

Let $\mathscr{A} \in \mathscr{L}_\kappa(\mathscr{CR})$ and set $\mathscr{V} = \langle \mathscr{A} \rangle$. Since $\mathscr{A}$ is closed under direct products, we have $\mathscr{V} = \mathbf{HS}\mathscr{A}$. Hence, if $S \in \mathscr{V} \cap \mathscr{K}$ then $S \in \mathbf{HS}\mathscr{A} \cap \mathscr{K}$ and thus $S = S/\kappa_S \in \mathscr{A} \cap \mathscr{K} = \mathscr{A}$. Therefore $\mathscr{V} \cap \mathscr{K} \subseteq \mathscr{A}$ and equality prevails. Consequently $\mathscr{V}\theta_{\mathscr{K}} = \mathscr{V} \cap \mathscr{K} = \mathscr{A}$ and $\theta_{\mathscr{K}}$ maps $\mathscr{L}(\mathscr{CR})$ onto $\mathscr{L}_\kappa(\mathscr{CR})$.

(ii) By (i), $\mathscr{L}_\kappa(\mathscr{CR})$ is a set. Also $\mathscr{K} = \mathscr{CR} \cap \mathscr{K} \in \mathscr{L}_\kappa(\mathscr{CR})$, so that $\mathscr{K}$ must clearly be the greatest element of $\mathscr{L}_\kappa(\mathscr{CR})$. It therefore suffices to show that $\mathscr{L}_\kappa(\mathscr{CR})$ is closed under arbitrary intersections.

Let $\mathscr{U}_\alpha \in \mathscr{L}_\kappa(\mathscr{CR})$, $\alpha \in A$, and let $\mathscr{U} = \bigcap_{\alpha \in A} \mathscr{U}_\alpha$. Since each $\mathscr{U}_\alpha$ is closed under direct products, so is $\mathscr{U}$. Let $S \in \mathbf{HS}\mathscr{U}$. Then $S \in \mathbf{HS}\mathscr{U}_\alpha$, for each $\alpha \in A$, whence $S/\kappa_S \in \mathscr{U}_\alpha$ and $S/\kappa_S \in \mathscr{U}$. Therefore $\mathscr{U} \in \mathscr{L}_\kappa(\mathscr{CR})$ and it follows that $\mathscr{L}_\kappa(\mathscr{CR})$ is complete lattice.

One interesting consequence of Proposition 2.2(i) is that the κ-varieties constitute a set of cardinality at most $2^{\aleph_0}$.

We will require the following two results of a lattice theoretical nature.

LEMMA 2.3. *Let ρ be an equivalence relation on a complete lattice L such that, for each $a \in L$, $a\rho$ contains a maximum element a^*. Let $L^* = \{a^* \mid a \in L\}$ and define*

$$\rho^* : a \to a^* \qquad (a \in L).$$

The following statements are equivalent.

(i) *ρ^* is order preserving.*

(ii) *ρ respects arbitrary joins.*

(iii) *L^* is a complete $\wedge$-subsemilattice of L.*

Proof. Let $\{a_\alpha \mid \alpha \in A\}$ be an arbitrary subset of L.

(i) *implies* (ii). Since $a_\alpha \leqslant a_\alpha^*$ we have $\bigvee_{\alpha \in A} a_\alpha \leqslant \bigvee_{\alpha \in A} a_\alpha^*$ and by (i),

$$\left(\bigvee_{\alpha \in A} a_\alpha\right)^* \leqslant \left(\bigvee_{\alpha \in A} a_\alpha^*\right)^*. \tag{2}$$

Also, the fact that $a_\alpha \leqslant \bigvee_{\alpha\in A} a_\alpha$ implies, again by (i), that $a_\alpha^* \leqslant (\bigvee_{\alpha\in A} a_\alpha)^*$ whence

$$\bigvee_{\alpha\in A} a_\alpha^* \leqslant \Big(\bigvee_{\alpha\in A} a_\alpha\Big)^*. \tag{3}$$

Hence

$$\begin{aligned}\Big(\bigvee_{\alpha\in A} a_\alpha^*\Big)^* &\leqslant \Big(\Big(\bigvee_{\alpha\in A} a_\alpha\Big)^*\Big)^* && \text{by (3) and (i)}\\ &= \Big(\bigvee_{\alpha\in A} a_\alpha\Big)^* \\ &\leqslant \Big(\bigvee_{\alpha\in A} a_\alpha^*\Big)^* && \text{by (2).}\end{aligned}$$

Consequently $(\bigvee_{\alpha\in A} a_\alpha)^* = (\bigvee_{\alpha\in A} a_\alpha^*)^*$ so that $\bigvee_{\alpha\in A} a_\alpha \,\rho\, \bigvee_{\alpha\in A} a_\alpha^*$, from which (ii) follows easily.

(ii) *implies* (i). Let $a \leqslant b$. Then $a^* \vee b \,\rho\, a \vee b = b$ so that $(a^* \vee b)^* = b^*$ and

$$a^* \leqslant a^* \vee b \leqslant (a^* \vee b)^* = b^*.$$

(i) *implies* (iii). Let $c = \bigwedge_{\alpha\in A} a_\alpha^*$. By (i), $c^* \leqslant (a_\alpha^*)^* = a_\alpha^*$ for all $\alpha \in A$, so that $c^* \leqslant \bigwedge_{\alpha\in A} a_\alpha^* = c \leqslant c^*$. Hence $c = c^* \in L^*$.

(iii) *implies* (i). Let $a \leqslant b$. Then $a = a \wedge b \leqslant a^* \wedge b^* \leqslant a^*$ so that $a \,\rho\, a^* \wedge b^*$. But, by (iii), $a^* \wedge b^* \in L^*$ and therefore we must have $a^* \wedge b^* = a^*$ so that $a^* \leqslant b^*$.

LEMMA 2.4 [5, Lemma 4.10]. *Let ρ be a complete congruence on a complete lattice L. For each $x \in L$, let x^* be the least element of $x\rho$. Then for any $A \subseteq L$, we have $\bigvee_{x\in A} x^* = (\bigvee_{x\in A} x)^*$.*

3. E-DISJUNCTIVE COMPLETELY REGULAR SEMIGROUPS

For any $S \in \mathscr{C}\mathscr{R}$, let $\tau = \tau_S$ denote the largest idempotent pure congruence on S, that is, the largest congruence on S such that $E(S)$ is a union of τ-classes. Recall that S is said to be *E-disjunctive* if $\tau_S = \varepsilon$. Let $\mathscr{D}$ denote the class of E-disjunctive completely regular semigroups.

LEMMA 3.1. *Let S and S_α, $\alpha \in A$, be completely regular semigroups.*

(i) *For $a, b \in S$, $a\,\tau\,b$ if and only if*

$$xay \in E(S) \Leftrightarrow xby \in E(S) \qquad (x, y \in S^1).$$

(ii) *S/τ is E-disjunctive.*

(iii) *For $T = \prod_{\alpha \in A} S_\alpha$, $\tau_T = \prod_{\alpha \in A} \tau_{T_\alpha}$.*

(iv) *$T = \{\tau_S : S \in \mathscr{CR}\}$ is a radical congruence system.*

(v) *For any congruence ρ on S there is a maximum congruence ρ^K and a minimum congruence ρ_K on S with the same kernel as ρ. If ρ is fully invariant, then so are ρ^K and ρ_K.*

Proof. Part (i) is well known.

(ii) Let ρ be a congruence on S/τ such that $\ker \rho = E(S/\tau)$. Let

$$\lambda = \{(a, b) \in S \times S \mid (a\tau)\,\rho\,(b\tau)\}.$$

Then λ is a congruence on S and clearly $\tau \subseteq \lambda$. Let $a \in S$, $e \in E(S)$, and $a\,\lambda\,e$. Then $(a\tau)\,\rho\,(e\tau)$. Since $\ker \rho = E(S/\tau)$, $a\tau \in E(S/\tau)$ so that $a \in \ker \tau = E(S)$ and $\ker \lambda \subseteq E(S)$. Therefore $\ker \lambda = E(S)$ and consequently $\lambda \subseteq \tau$. Hence $\lambda = \tau$ and so $\rho = \varepsilon$. Thus S/τ is E-disjunctive.

(iii) Clearly $E(S) = \prod_{\alpha \in A} E(S_\alpha)$, and the claim then follows easily from part (i).

(iv) This is an immediate consequence of (ii) and (iii).

(v) Let $L = \ker \rho$. It is evident that

$$\rho^K = ((L \times L) \cup (S \backslash L) \times (S \backslash L))^0$$

is the maximum congruence and that

$$\rho_K = \bigcap \{\lambda \mid \ker \lambda = \ker \rho\}$$

is the minimum congruence with the same kernel as ρ. From these descriptions, it is straightforward to verify that ρ^K and ρ_K are fully invariant if ρ is.

In the light of Lemma 3.1(iv) and in keeping with the notation of Section 2, we will denote by $\mathscr{L}_\tau(\mathscr{CR})$ the complete lattice of τ-varieties.

We define the relation K on $\mathscr{C}(F\mathscr{CR})$ by

$$\lambda\,K\,\rho \Leftrightarrow \ker \lambda = \ker \rho$$

and use the same notation for the relation that it induces on $\mathscr{L}(\mathscr{CR})$ via the anti-isomorphism π. The classes of K on $\mathscr{L}(\mathscr{CR})$ will then also be intervals. For $\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R})$ we define $\mathscr{V}^K$ and $\mathscr{V}_K$ by $\mathscr{V}K = [\mathscr{V}_K, \mathscr{V}^K]$. Let $\mathscr{V} = [\rho]$. Then it is clear that $\mathscr{V}^K = [\rho_K]$ and $\mathscr{V}_K = [\rho^K]$.

We shall have occasion to use the equivalence relation Θ on $\mathscr{L}(\mathscr{C}\mathscr{R})$ with classes

$$\mathscr{L}(\mathscr{B}),\ \mathscr{L}(\mathscr{C}\mathscr{S})\backslash\mathscr{L}(\mathscr{R}\mathscr{B}),\ \mathscr{L}(\mathscr{C}\mathscr{R})\backslash(\mathscr{L}(\mathscr{B}) \cup \mathscr{L}(\mathscr{C}\mathscr{S})).$$

Note that Θ is not a congruence on $\mathscr{L}(\mathscr{C}\mathscr{R})$ since $\mathscr{A}_2\,\Theta\,\mathscr{A}_3$, $\mathscr{A}_2$ is not Θ-related to $\mathscr{T}$, but $\mathscr{A}_2 \cap \mathscr{A}_3 = \mathscr{T}$.

The main purpose of this section is to prove the following result.

THEOREM 3.2. *The mapping*

$$\theta_{\mathscr{D}}: \mathscr{V} \to \mathscr{V} \cap \mathscr{D} \qquad (\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R}))$$

has the following properties.

(i) *It is a complete $\cap$-homomorphism of $\mathscr{L}(\mathscr{C}\mathscr{R})$ onto $\mathscr{L}_\tau(\mathscr{C}\mathscr{R})$.*

(ii) *It is not a $\vee$-homomorphism.*

(iii) *For any $\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R})$, we have $\mathscr{V}\bar{\theta}_{\mathscr{D}} = [\langle \mathscr{V} \cap \mathscr{D} \rangle, \mathscr{V}D]$, where*

$$\mathscr{V}D = \begin{cases} \mathscr{R}\mathscr{B} \circ \mathscr{V} & \textit{if} \quad \mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{S})\backslash(\mathscr{R}\mathscr{B}) \\ \mathscr{B} \circ \mathscr{V} & \textit{otherwise.} \end{cases}$$

(iv) *$\bar{\theta}_{\mathscr{D}} = \Theta \cap K$ and is a complete congruence.*

(v) *The restriction of $\theta_{\mathscr{D}}$ to $\mathscr{L}(\mathscr{C}\mathscr{S})$ is a complete endomorphism of $\mathscr{L}(\mathscr{C}\mathscr{S})$.*

Proof. Part (i). This follows immediately from Proposition 2.2 and Lemma 3.1(iv).

Part (ii). The following example will establish this part.

EXAMPLE 3.3. Indeed,

$$(\mathscr{A}_2 \cap \mathscr{D}) \vee (\mathscr{S} \cap \mathscr{D}) = \mathscr{A}_2 \vee \mathscr{T} = \mathscr{A}_2$$

whereas $(Z/(2))^0 \in (\mathscr{A}_2 \vee \mathscr{S}) \cap \mathscr{D}$.

Part (iii). Let $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R})$ be such that $\mathscr{U} \cap \mathscr{D} = \mathscr{V} \cap \mathscr{D}$. Then $\langle \mathscr{V} \cap \mathscr{D} \rangle = \langle \mathscr{U} \cap \mathscr{D} \rangle \subseteq \mathscr{U}$. Also, for $S \in \mathscr{U}$, we have $S/\tau \in \mathscr{U} \cap \mathscr{D} = \mathscr{V} \cap \mathscr{D} \subseteq \mathscr{V}$ so that $S \in \mathscr{B} \circ \mathscr{V}$. Consequently $\mathscr{U} \subseteq \mathscr{B} \circ \mathscr{V}$, which proves that

$$\mathscr{U} \cap \mathscr{D} = \mathscr{V} \cap \mathscr{D} \Rightarrow \langle \mathscr{V} \cap \mathscr{D} \rangle \subseteq \mathscr{U} \subseteq \mathscr{B} \circ \mathscr{V}. \tag{4}$$

We now claim that

$$\mathscr{U} \notin \mathscr{L}(\mathscr{C}\mathscr{S}) \cup \mathscr{L}(\mathscr{B}) \Rightarrow \mathscr{U} \cap \mathscr{D} \notin \mathscr{L}_\tau(\mathscr{C}\mathscr{S}). \tag{5}$$

Indeed, assume that $\mathcal{U} \notin \mathcal{L}(\mathcal{CS}) \cup \mathcal{L}(\mathcal{B})$. Then $\mathcal{U}$ contains the variety of semilattices and in particular the two element semilattice $Y_2 = \{0, 1\}$ belongs to $\mathcal{U}$. Also $\mathcal{U}$ contains a nontrivial group G. Since $G^0 \simeq (G \times Y_2)/(G \times \{0\})$, we conclude that $G^0 \in \mathcal{U}$. Letting e be the identity of G, we see $(e, 0) \notin \tau$, whence $G^0 = G^0/\tau \in \mathcal{U} \cap \mathcal{D}$ but $G^0/\tau \notin \mathcal{CS}$. This proves (5).

Suppose again that $\mathcal{U} \cap \mathcal{D} = \mathcal{V} \cap \mathcal{D}$. We consider two cases.

Case 1. $\mathcal{V} \in \mathcal{L}(\mathcal{CS}) \setminus \mathcal{L}(\mathcal{RB})$. Then $\mathcal{V} \cap \mathcal{G}$ is nontrivial and $\mathcal{V} \cap \mathcal{G} \subseteq \mathcal{V} \cap \mathcal{D}$. Thus $\mathcal{V} \cap \mathcal{D} \subseteq \mathcal{CS}$ and $\mathcal{V} \cap \mathcal{D}$ is nontrivial, which by (5) yields that $\mathcal{U} \in \mathcal{L}(\mathcal{CS})$. Now (4) gives that $\mathcal{U} \subseteq \mathcal{RB} \circ \mathcal{V}$. In the light of Lemma 2.1, we have that $\mathcal{RB} \circ \mathcal{V} \in \mathcal{L}(\mathcal{CR})$, which together with (4) yields

$$\mathcal{V}\bar{\theta}_{\mathcal{D}} \subseteq [\langle \mathcal{V} \cap \mathcal{D} \rangle, \mathcal{RB} \circ \mathcal{V}]. \tag{6}$$

Case 2. $\mathcal{V} \in (\mathcal{L}(\mathcal{CR}) \setminus \mathcal{L}(\mathcal{CS})) \cup \mathcal{L}(\mathcal{B})$. If $\mathcal{V} \notin \mathcal{L}(\mathcal{CS})$, then by Lemma 2.1, we have that $\mathcal{B} \circ \mathcal{V} \in \mathcal{L}(\mathcal{CR})$. Suppose that $\mathcal{V} \in \mathcal{L}(\mathcal{B})$. For $S \in \mathcal{B} \circ \mathcal{V}$ and $a \in S$, we have $a \,\tau\, a^0$ and hence $a = a^0$. It follows that $\mathcal{B} \circ \mathcal{V} = \mathcal{B}$. Thus, in all cases, we get $\mathcal{B} \circ \mathcal{V} \in \mathcal{L}(\mathcal{CR})$. By (4), we obtain

$$\mathcal{V}\bar{\theta}_{\mathcal{D}} \subseteq [\langle \mathcal{V} \cap \mathcal{D} \rangle, \mathcal{B} \circ \mathcal{V}]. \tag{7}$$

In any case, $\mathcal{V} \cap \mathcal{D} \subseteq \mathcal{V}$ so that $\langle \mathcal{V} \cap \mathcal{D} \rangle \subseteq \mathcal{V}$ whence $\langle \mathcal{V} \cap \mathcal{D} \rangle \cap \mathcal{D} \subseteq \mathcal{V} \cap \mathcal{D}$, and the opposite inclusion is trivial. Therefore $\langle \mathcal{V} \cap \mathcal{D} \rangle \in \mathcal{V}\bar{\theta}_{\mathcal{D}}$. In Case 1, if $S \in (\mathcal{RB} \circ \mathcal{V}) \cap \mathcal{D}$, then $S = S/\tau \in \mathcal{V}$ and thus $(\mathcal{RB} \circ \mathcal{V}) \cap \mathcal{D} \subseteq \mathcal{V} \cap \mathcal{D}$ and the reverse inclusion is trivial; therefore $\mathcal{RB} \circ \mathcal{V} \in \mathcal{V}\bar{\theta}_{\mathcal{D}}$. The same kind of argument gives $\mathcal{B} \circ \mathcal{V} \in \mathcal{V}\bar{\theta}_{\mathcal{D}}$ in Case 2. By convexity of $\bar{\theta}_{\mathcal{D}}$-classes, we deduce that both in (6) and (7) equality prevails.

The content of the next lemma is a detailed reformulation of the first part of (iv).

LEMMA 3.4. *Let* $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$. *Then*

$$\mathcal{U} \cap \mathcal{D} = \mathcal{V} \cap \mathcal{D} \Leftrightarrow \textit{either } 1.\ \mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{B}),$$
$$\textit{that is, } \ker \rho_{\mathcal{U}} = \ker \rho_{\mathcal{V}} = F\mathcal{CR},$$
$$\textit{or } 2.\ \mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CS})$$
$$\textit{and } \ker \rho_{\mathcal{U}} = \ker \rho_{\mathcal{V}} \neq F\mathcal{CR},$$
$$\textit{or } 3.\ \mathcal{S} \subseteq \mathcal{U} \cap \mathcal{V}$$
$$\textit{and } \ker \rho_{\mathcal{U}} = \ker \rho_{\mathcal{V}} \neq F\mathcal{CR}.$$

Proof. First suppose that $\mathcal{U} \cap \mathcal{D} = \mathcal{V} \cap \mathcal{D}$. Then

$$F\mathcal{U}/\tau \in \mathcal{U} \cap \mathcal{D} = \mathcal{V} \cap \mathcal{D} \subseteq \mathcal{V}.$$

Let κ denote the congruence on $F\mathscr{C}\mathscr{R}$ which contains $\rho_{\mathscr{U}}$ and induces τ on $F\mathscr{U}$. Since τ is idempotent pure, $\ker \kappa = \ker \rho_{\mathscr{U}}$ and, since $F\mathscr{U}/\tau \in \mathscr{V}$, we must have $\rho_{\mathscr{V}} \subseteq \kappa$. Hence $\ker \rho_{\mathscr{V}} \subseteq \ker \kappa = \ker \rho_{\mathscr{U}}$. Similarly, $\ker \rho_{\mathscr{U}} \subseteq \ker \rho_{\mathscr{V}}$ and equality follows.

Thus in all cases,

$$\mathscr{U} \cap \mathscr{D} = \mathscr{V} \cap \mathscr{D} \Rightarrow \ker \rho_{\mathscr{U}} = \ker \rho_{\mathscr{V}}.$$

Now, if $\mathscr{U} \in \mathscr{L}(\mathscr{B})$, then $F\mathscr{C}\mathscr{R} = \ker \rho_{\mathscr{U}} = \ker \rho_{\mathscr{V}}$ so that $\mathscr{V} \in \mathscr{L}(\mathscr{B})$. If $\mathscr{S} \subseteq \mathscr{U}$ and $\ker \rho_{\mathscr{U}} \neq F\mathscr{C}\mathscr{R}$, then there exists a nontrivial group $G \in \mathscr{U}$ so that $G^0 \in \mathscr{U} \cap \mathscr{D} = \mathscr{V} \cap \mathscr{D} \subseteq \mathscr{V}$. But then $\mathscr{V}$ contains $G^0/\mathscr{H}$, a two element semilattice so that $\mathscr{S} \subseteq \mathscr{V}$. Thus, when $\ker \rho_{\mathscr{U}} = \ker \rho_{\mathscr{V}} \neq F\mathscr{C}\mathbf{R}$, $\mathscr{S} \subseteq \mathscr{U}$ if and only if $\mathscr{S} \subseteq \mathscr{V}$.

This completes the proof in one direction. For the converse, we consider the three cases separately. Let $\ker \rho_{\mathscr{U}} = \ker \rho_{\mathscr{V}} = L$.

Case 1. $\mathscr{U}, \mathscr{V} \subseteq \mathscr{B}$ implies that $\mathscr{U} \cap \mathscr{D}$ and $\mathscr{V} \cap \mathscr{D}$ both consist of just the one element semigroups.

Case 2. Let $\mathscr{W} = \mathscr{R}\mathscr{B} \circ \mathscr{U}$. By Lemma 2.1, $\mathscr{W} \in \mathscr{L}(\mathscr{C}\mathscr{S})$. Clearly $\ker \rho_{\mathscr{W}} = \ker \rho_{\mathscr{U}} = L$. Let $\rho = (\rho_{\mathscr{U}})_K$.

Let $\mathscr{X} = [\rho] \cap \mathscr{C}\mathscr{S}$. Then $\rho_{\mathscr{X}} = \rho \vee \rho_{\mathscr{C}\mathscr{S}} \subseteq \rho_{\mathscr{U}}$ so that

$$L \subseteq \ker \rho_{\mathscr{X}} \subseteq \ker \rho_{\mathscr{U}} = L.$$

Thus $\ker \rho_{\mathscr{X}} = L$ and, since $\rho_{\mathscr{X}} = \rho \vee \rho_{\mathscr{C}\mathscr{S}}$, $\mathscr{X}$ must be the largest variety in $\mathscr{L}(\mathscr{C}\mathscr{S})$ with kernel L. Hence $F\mathscr{U}$ must be an idempotent pure homomorphic image of $F\mathscr{X}$. Therefore $F\mathscr{X} \in \mathscr{R}\mathscr{B} \circ \mathscr{U}$ so that $\mathscr{X} \subseteq \mathscr{R}\mathscr{B} \circ \mathscr{U} = \mathscr{W}$ and thus $\mathscr{X} = \mathscr{R}\mathscr{B} \circ \mathscr{U}$ by the maximality of $\mathscr{X}$. Similarly $\mathscr{X} = \mathscr{R}\mathscr{B} \circ \mathscr{V}$ and we have $\mathscr{R}\mathscr{B} \circ \mathscr{U} = \mathscr{R}\mathscr{B} \circ \mathscr{V}$. Therefore

$$\mathscr{U} \cap \mathscr{D} = (\mathscr{R}\mathscr{B} \circ \mathscr{U}) \cap \mathscr{D} = (\mathscr{R}\mathscr{B} \circ \mathscr{V}) \cap \mathscr{D} = \mathscr{V} \cap \mathscr{D},$$

as required.

Case 3. Since $\mathscr{S} \subseteq \mathscr{U} \cap \mathscr{V}$, it follows from Lemma 2.1 that $\mathscr{B} \circ \mathscr{U}$ and $\mathscr{B} \circ \mathscr{V}$ are varieties. Let $\mathscr{W} = \mathscr{B} \circ \mathscr{U}$. Clearly $\mathscr{W}$ is the largest variety with $\ker \rho_{\mathscr{W}} = \ker \rho_{\mathscr{U}}$. Therefore we must also have $\mathscr{W} = \mathscr{B} \circ \mathscr{V}$. Thus $\mathscr{B} \circ \mathscr{U} = \mathscr{B} \circ \mathscr{V}$ and

$$\mathscr{U} \cap \mathscr{D} = (\mathscr{B} \circ \mathscr{U}) \cap \mathscr{D} = (\mathscr{B} \circ \mathscr{V}) \cap \mathscr{D} = \mathscr{V} \cap \mathscr{D},$$

as required.

To see that $\Theta \cap K$ is a complete congruence it suffices to observe that the mappings

$$\mathscr{V} \to \langle \mathscr{V} \cap \mathscr{D} \rangle, \qquad \mathscr{V} \to \mathscr{V} D$$

associated with the lower and upper bounds of the interval $\mathscr{V}\bar{\theta}_{\mathscr{D}}$ are clearly order preserving and to invoke Lemma 2.3 and its dual.

In order to prove part (v) we first establish an auxiliary result of some independent interest.

LEMMA 3.5. *In a completely simple semigroup S, we have*

$$a\,\tau\,b \Leftrightarrow xay = xby \qquad \textit{for all} \quad x, y \in S.$$

Proof. Let $a\,\tau\,b$ and $x, y \in S$. Then $u = xay\,\mathscr{H}\,xby$ and thus $u^0 = (u^{-1}x)\,ay\,\mathscr{H}\,(u^{-1}x)\,by$. Since $a\,\tau\,b$, it follows that $(u^{-1}x)\,by \in E(S)$ and hence $u^{-1}xay = u^0 = u^{-1}xby$. But then $u^0xay = u^0xby$ so that $xay = xby$.

Conversely, suppose that $xay = xby$ for all $x, y \in S$. Let $x, y \in S$. If $xay \in E(S)$, then $xby = xay \in E(S)$. Next assume $xa \in E(S)$. Then $xb(xa) = xa(xa) \in E(S)$ and thus $xb \in E(S)$ since $R_{xa} = R_{xb}$ is a right group. Similarly $ay \in E(S)$ implies $by \in E(S)$. Therefore $a\,\tau\,b$, as required.

Part (v). By part (i) it suffices to prove that $\theta_{\mathscr{D}}|_{\mathscr{L}(\mathscr{CS})}$ is a complete $\vee$-homomorphism.

Let A be a nonempty index set and for each $\alpha \in A$, let $\mathscr{V}_\alpha \in \mathscr{L}(\mathscr{CS})$ and let $S \in (\bigvee_{\alpha \in A} \mathscr{V}_\alpha) \cap \mathscr{D}$. Then there exist $V_\alpha \in \mathscr{V}_\alpha$, a subdirect product T of V_α, $\alpha \in A$, and a homomorphism φ of T onto S. Now let (a_α), $(b_\alpha) \in T$ be such that $(a_\alpha)\,\tau_T\,(b_\alpha)$. Since T is completely simple, Lemma 3.5 yields

$$(x_\alpha)(a_\alpha)(y_\alpha) = (x_\alpha)(b_\alpha)(y_\alpha) \qquad ((x_\alpha), (y_\alpha) \in T). \tag{8}$$

Applying φ to this equation and noting that φ maps T onto S we deduce, again by Lemma 3.5, that $(a_\alpha)\varphi\,\tau_S\,(b_\alpha)\varphi$. Since $S \in \mathscr{D}$, we get $(a_\alpha)\varphi = (b_\alpha)\varphi$. This shows that

$$\psi : (a_\alpha)\tau \to (a_\alpha)\varphi \qquad ((a_\alpha) \in T) \tag{9}$$

is a homomorphism of T/τ_T onto S.

Continuing with the same hypothesis, also let (x_α), $(y_\alpha) \in \prod_{\alpha \in A} V_\alpha$. Fix $\zeta \in A$. Since T is a subdirect product of V_α, $\alpha \in A$, there is a choice of elements x'_α for $\alpha \neq \zeta$ and $x'_\zeta = x_\zeta$ such that $(x'_\alpha) \in T$. Make an analogous choice of and element $(y'_\alpha) \in T$. By (8), we have $(x'_\alpha)(a_\alpha)(y'_\alpha) = (x'_\alpha)(b_\alpha)(y'_\alpha)$, which in the ζ-component gives $x_\zeta a_\zeta y_\zeta = x_\zeta b_\zeta y_\zeta$. Since this holds for any $\zeta \in A$, we conclude that $(x_\alpha)(a_\alpha)(y_\alpha) = (x_\alpha)(b_\alpha)(y_\alpha)$ for all (x_α), $(y_\alpha) \in \prod_{\alpha \in A} V_\alpha$. By Lemma 3.5, we then have $(a_\alpha)\,\tau_{\prod V_\alpha}\,(b_\alpha)$. It follows that $\tau_T = \tau_{\prod V_\alpha}|_T$ again by Lemma 3.5. The same reference also gives that for any (a_α), $(b_\alpha) \in \prod_{\alpha \in A} V_\alpha$,

$$(a_\alpha)\,\tau_{\prod V_\alpha}\,(b_\alpha) \Leftrightarrow a_\alpha\,\tau_{V_\alpha}\,b_\alpha \qquad \text{for all} \quad \alpha \in A.$$

It now follows that T/τ_T is a subdirect product of V_α/τ_{V_α}, $\alpha \in A$, which together with the homomorphism ψ in (9) gives that $S \in \bigvee_{\alpha\in A}(\mathcal{V}_\alpha \cap \mathcal{D})$ by the definition of a τ-variety since $S = S/\tau$.

We have proved that

$$\left(\bigvee_{\alpha\in A} \mathcal{V}_\alpha\right) \cap \mathcal{D} \subseteq \bigvee_{\alpha\in A} (\mathcal{V}_\alpha \cap \mathcal{D}).$$

Since the opposite inclusion is trivial, we have that $\theta_{\mathcal{D}}|_{\mathcal{L}(\mathcal{CS})}$ is a complete $\vee$-homomorphism.

Corollary 3.6. *For any $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$, we have*

$$\mathcal{U}\Theta \cap K\mathcal{V} \Leftrightarrow \langle \mathcal{U} \cap \mathcal{D}\rangle = \langle \mathcal{V} \cap \mathcal{D}\rangle \Leftrightarrow \mathcal{U} \cap \mathcal{D} = \mathcal{V} \cap \mathcal{D}$$
$$\Leftrightarrow \begin{cases} \mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CS})\backslash\mathcal{L}(\mathcal{RB}) \text{ and } \mathcal{B}\circ\mathcal{U} = \mathcal{B}\circ\mathcal{V} \text{ or} \\ \mathcal{U}, \mathcal{V} \notin \mathcal{L}(\mathcal{CS})\backslash\mathcal{L}(\mathcal{RB}) \text{ and } \mathcal{B}\circ\mathcal{U} = \mathcal{B}\circ\mathcal{V}. \end{cases}$$

4. Operators Related to E-Disjunctive Completely Regular Semigroups

From Theorem 3.2(iii) we know that the classes of $\bar{\theta}_{\mathcal{D}}$ are intervals. This leads naturally to questions, explored in this section, regarding the lower and upper ends of these intervals and of the intervals induced by the associated relation K.

Lemma 4.1. *Neither of the mappings*

$$\mathcal{V} \to \langle \mathcal{V} \cap \mathcal{D}\rangle, \qquad \mathcal{V} \to \mathcal{V}D \qquad (\mathcal{V} \in \mathcal{L}(\mathcal{CR}))$$

is an $\cap$- or $\vee$-homomorphism.

Proof. To see that the first mapping is not an $\cap$-homomorphism, let $\mathcal{U} = \mathcal{A}_2 \vee \mathcal{S}$ and $\mathcal{V} = \mathcal{A}_3 \vee \mathcal{S}$. Then $\mathcal{U} \cap \mathcal{V} = \mathcal{S}$ so that

$$\langle \mathcal{U} \cap \mathcal{V} \cap \mathcal{D}\rangle = \langle \mathcal{S} \cap \mathcal{D}\rangle = \mathcal{T} \neq \mathcal{S} = \mathcal{U} \cap \mathcal{V} = \langle \mathcal{U} \cap \mathcal{D}\rangle \cap \langle \mathcal{V} \cap \mathcal{D}\rangle.$$

Nor is the first mapping a $\vee$-homomorphism. Consider

$$\langle \mathcal{G} \cap \mathcal{D}\rangle \vee \langle \mathcal{S} \cap \mathcal{D}\rangle = \mathcal{G} \vee \mathcal{T} = \mathcal{G}$$

while

$$\langle (\mathcal{G} \vee \mathcal{S}) \cap D\rangle = \langle \mathcal{S}\mathcal{G} \cap \mathcal{D}\rangle \neq \mathcal{G},$$

since any 2 component semilattice of groups with a noninjective homomorphism belongs to $\mathscr{S}\mathscr{G} \cap \mathscr{D}$.

Further,

$$\mathscr{B} \circ (\mathscr{S} \cap \mathscr{C}\mathscr{S}) = \mathscr{B} \circ \mathscr{T} = \mathscr{B}.$$

$$\mathscr{B} \circ \mathscr{S} \cap \mathscr{R}\mathscr{B} \circ \mathscr{C}\mathscr{S} = \mathscr{B} \cap \mathscr{C}\mathscr{S} = \mathscr{R}\mathscr{B},$$

so that the second mapping is not an $\cap$-homomorphism; finally using [12, Corollary 7.4(ii)], we obtain

$$\mathscr{B} \circ (\mathscr{S} \vee \mathscr{C}\mathscr{S}) = \mathscr{B} \circ \mathscr{N}\mathscr{C}\mathscr{G} \supseteq \mathscr{B} \circ \mathscr{S}\mathscr{G} = \mathscr{O}\mathscr{G},$$

$$\mathscr{B} \circ \mathscr{S} \vee \mathscr{R}\mathscr{B} \circ \mathscr{C}\mathscr{S} = \mathscr{B} \vee \mathscr{C}\mathscr{S} \subseteq \mathscr{C}\mathscr{G},$$

where $\mathscr{C}\mathscr{G}$ does not contain $\mathscr{O}\mathscr{G}$, so the second mapping is not a $\vee$-homomorphism.

Despite the negative conclusions of Lemma 4.1, we are able to provide some additional information regarding the ends of the intervals associated with $\theta_{\mathscr{S}}$. We first construct a basis for the identities for $\mathscr{V}D$ in terms of a basis for $\mathscr{V}$.

LEMMA 4.2. *Let* $\mathscr{V} = [u_\alpha = v_\alpha]_{\alpha \in A} \in \mathscr{L}(\mathscr{C}\mathscr{R})$ *with* $\mathscr{V} = [a^2 = a, u = v]$ *if* $\mathscr{V} \in \mathscr{L}(\mathscr{B})$. *Then*

$$\mathscr{V}D = [(xu_\alpha y)(xv_\alpha y)^{-1} \in E]_{\alpha \in A},$$

where $w \in E$ *means* $w^2 = w$ *and* $x, y \notin c(u_\alpha) \cup c(v_\alpha)$ *for all* $\alpha \in A$.

Proof. If $\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{S}) \setminus \mathscr{L}(\mathscr{R}\mathscr{B})$, then

$$\begin{aligned}\mathscr{R}\mathscr{B} \circ \mathscr{V} &= \{S \in \mathscr{C}\mathscr{S} \mid S/\tau \in \mathscr{V}\} = [xu_\alpha y = xv_\alpha y]_{\alpha \in A} \\ &= [(xu_\alpha y)(xv_\alpha y)^{-1} \in E]_{\alpha \in A},\end{aligned}$$

the second equality by Lemma 3.5 and the third by complete simplicity. Let $\mathscr{V} \in \mathscr{L}(\mathscr{B})$ and let

$$S \in [(xa^2y)(xay)^{-1} \in E, (xuy)(xvy)^{-1} \in E];$$

for $x = y = a^0$, we get $a = a^2a^{-1} \in E(S)$ and S is a band. Hence

$$[(xa^2y)(xay)^{-1} \in E, (xuy)(xvy)^{-1} \in E] \subseteq \mathscr{B}$$

and the opposite inclusion is trivial. Thus equality prevails and also $\mathscr{B} \circ \mathscr{V} = \mathscr{B}$, which yields the desired conclusion.

The remaining case where $\mathscr{V} \in \mathscr{L}(\mathscr{CR}) \backslash (\mathscr{L}(\mathscr{CS}) \cup \mathscr{L}(\mathscr{B}))$ follows from [3, Corollary 6.5].

For the case where $\mathscr{V} \in \mathscr{L}(\mathscr{CS})$, we have additional information as the next result indicates.

LEMMA 4.3. *If* $\mathscr{V} = [u_\alpha = v_\alpha]_{\alpha \in A} \in \mathscr{L}(\mathscr{CS})$, *then*

$$\mathscr{RB} \circ \mathscr{V} = \mathscr{RB} \vee \mathscr{V} = [xu_\alpha y = xv_\alpha y]_{\alpha \in A},$$

where $x, y \notin c(u_\alpha) \cup c(v_\alpha)$ *for all* $\alpha \in A$.

Proof. Let $S \in \mathscr{RB} \circ \mathscr{V}$. There exists a congruence ρ on S over $\mathscr{RB}$ such that $S/\rho \in \mathscr{V}$. Hence ρ is idempotent pure whence $\mathscr{H} \cap \rho = \varepsilon$ since $\mathscr{H}$ is an idempotent separating congruence. Thus S is a subdirect product of the rectangular band $S/\mathscr{H}$ and the semigroup S/ρ and therefore $S \in \mathscr{RB} \vee \mathscr{V}$. Clearly $\mathscr{RB} \vee \mathscr{V} \subseteq [xu_\alpha y = xv_\alpha y]_{\alpha \in A}$. Let $S \in [xu_\alpha y = xv_\alpha y]_{\alpha \in A}$. In view of Lemma 3.5, we see that S/τ satisfies $u_\alpha = v_\alpha$ for all $\alpha \in A$ and hence $S/\tau \in \mathscr{V}$ so that $S \in \mathscr{RB} \circ \mathscr{V}$.

Remark. The first part of the above proof shows that for any $\mathscr{V} \in [\mathscr{S}, \mathscr{CG}]$, every member of $\mathscr{B} \circ \mathscr{V}$ is a subdirect product of a band and a semigroup in $\mathscr{V}$; in particular $(\mathscr{B} \circ \mathscr{V}) \cap \mathscr{CG} = \mathscr{B} \vee \mathscr{V}$. This may be contrasted with: if $\mathscr{V} \in \mathscr{L}(\mathscr{OG})$, then $(\mathscr{G} \circ \mathscr{V}) \cap \mathscr{OG} = \mathscr{G} \vee \mathscr{V}$; see Lemma 7.9.

5. THE KERNEL RELATION

From Theorem 3.2(iv) we see that the relation $\bar{\theta}_{\mathscr{Q}}$ is very closely related to K. By Lemma 3.1(iii), the classes of K are intervals and so, for any $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, we define $\mathscr{V}_K$ and $\mathscr{V}^K$ by

$$\mathscr{V}K = [\mathscr{V}_K, \mathscr{V}^K].$$

In order to complete the picture we devote this section to the relation K. We will provide new information about the lower bounds $\mathscr{V}_K$ and alternative derivations of other facts.

THEOREM 5.1. *The mapping*

$$\mathscr{V} \to \mathscr{V}^K \qquad (\mathscr{V} \in \mathscr{L}(\mathscr{CR}))$$

is a complete endomorphism of $\mathscr{L}(\mathscr{CR})$. *Moreover, for any* $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$,

$$\mathscr{V}^K = \mathscr{B} \circ (\mathscr{V} \vee \mathscr{S}).$$

Proof. The first assertion follows from [10, Theorem 1(3)] via [4, Theorem 14]; the second statement is [3, Proposition 7.2(ii)].

In order to describe $\mathscr{V}_K$, we require some preparation. We first wish to characterize those varieties $\mathscr{V}$ for which $\mathscr{V}_K \in \mathscr{L}(\mathscr{C}\mathscr{S})$.

Define an operator L by

$$L: \mathscr{V} \to \{S \in \mathscr{C}\mathscr{R} \mid eSe \in \mathscr{V} \text{ for all } e \in E(S)\} \qquad (\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R})).$$

For all $\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R})$, *$\mathscr{V}L$ is a variety* (see [2, Proposition 4.1]).

The lattice $\mathscr{L}(\mathscr{N}\mathscr{C}\mathscr{G})$ is nicely described in the following result.

LEMMA 5.2 [8, Theorem 4.7]. *The mapping*

$$\mathscr{V} \to (\mathscr{V} \cap \mathscr{S}, \mathscr{V} \cap \mathscr{C}\mathscr{S}) \qquad (\mathscr{V} \in \mathscr{L}(\mathscr{N}\mathscr{C}\mathscr{G}))$$

is an isomorphism of $\mathscr{L}(\mathscr{N}\mathscr{C}\mathscr{G})$ onto $\mathscr{L}(\mathscr{S}) \times \mathscr{L}(\mathscr{C}\mathscr{S})$ with inverse

$$(\mathscr{P}, \mathscr{Q}) \to \mathscr{P} \vee \mathscr{Q} \qquad (\mathscr{P} \in \mathscr{L}(\mathscr{S}), \mathscr{Q} \in \mathscr{L}(\mathscr{C}\mathscr{S})).$$

LEMMA 5.3. $\mathscr{O}\mathscr{G}L = (\mathscr{C}\mathscr{S})^K$.

Proof. We have

$$\begin{aligned} \mathscr{O}\mathscr{G}L &= (\mathscr{G}^K)L && \text{by [12, Corollary 7.4(ii)]} \\ &= (\mathscr{G}L)^K && \text{by [9, Lemma 6.3]} \\ &= (\mathscr{C}\mathscr{S})^K. \end{aligned}$$

LEMMA 5.4. *Let $\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R})$. Then $\mathscr{V}_K \in \mathscr{L}(\mathscr{C}\mathscr{S})$ if and only if $\mathscr{V} \in \mathscr{L}(\mathscr{O}\mathscr{G}L)$.*

Proof. If $\mathscr{V}_K = \mathscr{L}(\mathscr{C}\mathscr{S})$, then

$$\begin{aligned} \mathscr{V} \subseteq \mathscr{V}^K = (\mathscr{V}_K)^K & \\ = \mathscr{B} \circ (\mathscr{V}_K \vee \mathscr{S}) &\qquad \text{by Proposition 5.1} \\ \subseteq \mathscr{B} \circ (\mathscr{C}\mathscr{S} \vee \mathscr{S}) & \\ = \mathscr{C}\mathscr{S}^K &\qquad \text{by Proposition 5.1} \\ = \mathscr{O}\mathscr{G}L &\qquad \text{by Lemma 5.3.} \end{aligned}$$

Conversely, let $\mathscr{V} \in \mathscr{L}(\mathscr{O}\mathscr{G}L) = \mathscr{L}(\mathscr{C}\mathscr{S}^K)$. Let $F = \mathrm{F}\mathscr{V}$. Then $F \in \mathscr{V} \subseteq \mathscr{C}\mathscr{S}^K = \mathscr{B} \circ (\mathscr{C}\mathscr{S} \vee \mathscr{S})$ so that $F/\tau \in \mathscr{C}\mathscr{S} \vee \mathscr{S}$. Clearly $\mathscr{V}_K = \langle F/\tau \rangle$. Consequently $\mathscr{V}_K \subseteq \mathscr{C}\mathscr{S} \vee \mathscr{S}$ and, by Lemma 5.2, $\mathscr{V}_K = \mathscr{P} \vee \mathscr{Q}$, where

$\mathscr{P}=\mathscr{V}_K\cap\mathscr{S}$ and $\mathscr{Q}=\mathscr{V}_K\cap\mathscr{CS}$. By Proposition 5.1, we have $\mathscr{V}_K\, K\,\mathscr{Q}$. Since $\mathscr{Q}\subseteq\mathscr{V}_K$, we must therefore have $\mathscr{V}_K=\mathscr{Q}\in\mathscr{L}(\mathscr{CS})$, as required.

LEMMA 5.5. *For any* $\mathscr{V}\in\mathscr{L}(\mathscr{CR})$,

$$\mathscr{V}_K=\begin{cases}\langle\mathscr{V}\cap\mathscr{CS}\cap\mathscr{D}\rangle & \text{if } \mathscr{V}\in\mathscr{L}(\mathscr{OG}L)\\ \langle\mathscr{V}\cap\mathscr{D}\rangle & \text{otherwise.}\end{cases}$$

Proof. Let $\mathscr{V}\in\mathscr{L}(\mathscr{OG}L)\backslash\mathscr{L}(\mathscr{B})$. By Lemma 5.4, we have $\mathscr{V}_K\in\mathscr{L}(\mathscr{CS})$. Then $\mathscr{V}_K\subseteq\mathscr{V}\cap\mathscr{CS}\subseteq\mathscr{V}$ so that $\mathscr{V}_K=(\mathscr{V}\cap\mathscr{CS})_K$. Since $\mathscr{V}\notin\mathscr{L}(\mathscr{B})$, we have $\mathscr{V}\cap\mathscr{CS}\notin\mathscr{L}(\mathscr{RB})$. Now $\mathscr{U}\in\mathscr{L}(\mathscr{CS})\backslash\mathscr{L}(\mathscr{RB})$ implies that $\mathscr{U}_K\in\mathscr{L}(\mathscr{CS})\backslash\mathscr{L}(\mathscr{RB})$. Hence

$$K|_{\mathscr{L}(\mathscr{CS})\backslash\mathscr{L}(\mathscr{RB})}=(\Theta\cap K)|_{\mathscr{L}(\mathscr{CS})\backslash\mathscr{L}(\mathscr{RB})}$$

so that $(\mathscr{V}\cap\mathscr{CS})_K$ is the smallest element in the $(\Theta\cap K)$-class containing $\mathscr{V}\cap\mathscr{CS}$. By Theorem 3.2(iii), $(\mathscr{V}\cap\mathscr{CS})_K=\langle\mathscr{V}\cap\mathscr{CS}\cap\mathscr{D}\rangle$. On the other hand, if $\mathscr{V}\in\mathscr{L}(\mathscr{RB})$, then

$$\mathscr{V}_K=\mathscr{T}=\mathscr{V}\cap\mathscr{D}=\mathscr{V}\cap\mathscr{CS}\cap\mathscr{D}=\langle\mathscr{V}\cap\mathscr{CS}\cap\mathscr{D}\rangle.$$

If $\mathscr{V}\in\mathscr{L}(\mathscr{B})$, then $\mathscr{V}_K=\mathscr{T}=\langle\mathscr{V}\cap\mathscr{D}\rangle$ trivially.

Now let $\mathscr{V}\in\mathscr{L}(\mathscr{CR})\backslash\mathscr{L}(\mathscr{OG}L)$. By Lemma 5.4, we have $\mathscr{V}_K\notin\mathscr{L}(\mathscr{CS})$. Hence $\mathscr{V}K=\mathscr{V}\bar{\theta}_{\mathscr{D}}$ so that $\mathscr{V}_K$ is the minimum element in the $\bar{\theta}_{\mathscr{D}}$-class containing $\mathscr{V}$. Therefore, by Theorem 3.2(iii), we get $\mathscr{V}_K=\langle\mathscr{V}\cap\mathscr{D}\rangle$.

The description of $\mathscr{V}_K$ in Lemma 5.5 for the case $\mathscr{V}\in\mathscr{L}(\mathscr{OG}L)$ can be broken down and simplified. In order to do so we require some additional preparation.

LEMMA 5.6 [7, Lemma 1]. *Let S be a completely regular semigroup with components S_α, $\alpha\in Y$. Then $S\in\mathscr{OG}$ if and only if $S_\alpha\in\mathscr{OG}$ for all $\alpha\in Y$.*

LEMMA 5.7. *For any* $\mathscr{V}\in\mathscr{L}(\mathscr{CS})\backslash\mathscr{L}(\mathscr{R}e\mathscr{G})$, $\mathscr{V}=\mathscr{V}_K$.

Proof. This follows from [10, Lemma 8] via [4, Theorem 14].

THEOREM 5.8 (Cf. [10, Theorem 2]). *For any* $\mathscr{V}\in\mathscr{L}(\mathscr{CR})$,

$$\mathscr{V}_K=\begin{cases}\mathscr{V}\cap\mathscr{G} & \text{if } \mathscr{V}\in\mathscr{L}(\mathscr{OG})\\ \mathscr{V}\cap\mathscr{CS} & \text{if } \mathscr{V}\in\mathscr{L}(\mathscr{OG}L)\backslash\mathscr{L}(\mathscr{OG})\\ \langle\mathscr{V}\cap\mathscr{D}\rangle & \text{otherwise.}\end{cases}$$

Proof. If $\mathcal{V} \in \mathcal{L}(\mathcal{OG})$, then

$$\begin{aligned} \mathcal{V}_K &= \langle \mathcal{V} \cap \mathcal{CS} \cap \mathcal{D} \rangle && \text{by Lemma 5.5} \\ &= \langle \mathcal{V} \cap \mathcal{R}e\mathcal{G} \cap \mathcal{D} \rangle && \text{since } \mathcal{V} \in \mathcal{L}(\mathcal{OG}) \\ &= \langle \mathcal{V} \cap \mathcal{G} \rangle \\ &= \mathcal{V} \cap \mathcal{G}. \end{aligned}$$

Now let $\mathcal{V} \in \mathcal{L}(\mathcal{OG}L)\backslash\mathcal{L}(\mathcal{OG})$ and $\mathcal{U} = \mathcal{V} \cap \mathcal{CS}$. By Lemma 5.6, $\mathcal{U} \notin \mathcal{L}(\mathcal{R}e\mathcal{G})$. Hence

$$\begin{aligned} \mathcal{V}_K &= \langle \mathcal{V} \cap \mathcal{CS} \cap \mathcal{D} \rangle && \text{by Lemma 5.5} \\ &= \langle \mathcal{U} \cap \mathcal{CS} \cap \mathcal{D} \rangle = \mathcal{U}_K = \mathcal{U} && \text{by Lemma 5.7} \\ &= \mathcal{V} \cap \mathcal{CS}. \end{aligned}$$

The remaining case is just a restatement of part of Lemma 5.5.

Now that two of the three cases in Theorem 5.8 are expressed directly in terms of varieties it is natural to wonder whether $\mathcal{V} \cap \mathcal{D}$ might always be a variety. However, it is easily verified that $\mathcal{SG} = \langle \mathcal{SG} \cap \mathcal{D} \rangle \neq \mathcal{SG} \cap \mathcal{D}$.

We conclude this section with an observation about the mapping $\mathcal{V} \to \mathcal{V}_K$.

COROLLARY 5.9. *The mapping*

$$\varphi : \mathcal{V} \to \mathcal{V}_K \qquad (\mathcal{V} \in \mathcal{L}(\mathcal{CR}))$$

is a complete $\vee$*-endomorphism but is not an* $\cap$*-homomorphism.*

Proof. Since the relation K is a complete congruence, by Theorem 5.1, Lemma 2.4 implies that φ is a complete $\vee$-endomorphism.

Let $\mathcal{CSA}_n$ denote the variety of completely simple semigroups all of whose subgroups are abelian of exponent n. Clearly $\mathcal{CSA}_2 \cap \mathcal{CSA}_3 = \mathcal{RB}$. However, for any $n > 1$, $\mathcal{CSA}_n \notin \mathcal{L}(\mathcal{OG})$ and consequently, by Theorem 5.8, $(\mathcal{CSA}_n)_K = \mathcal{CSA}_n \cap \mathcal{CS} = \mathcal{CSA}_n$. Thus

$$\begin{aligned} (\mathcal{CSA}_2)_K \cap (\mathcal{CSA}_3)_K &= \mathcal{CSA}_2 \cap \mathcal{CSA}_3 = \mathcal{RB} \\ &\neq \mathcal{T} = (\mathcal{RB})_K = (\mathcal{CSA}_2 \cap \mathcal{CSA}_3)_K. \end{aligned}$$

6. Fundamental Completely Regular Semigroups

Denote by $\mathcal{F}$ the class of all fundamental completely regular semigroups. Recall that these are the semigroups in which $\mu = \varepsilon$, that is, the equality relation is the only idempotent separating congruence.

LEMMA 6.1. *$M = \{\mu_S : S \in \mathscr{CR}\}$ is a radical congruence system.*

Proof. Let $S_\alpha \in \mathscr{CR}$, $\alpha \in A$, and $a = (a_\alpha)$, $b = (b_\alpha) \in \prod_{\alpha \in A} S_\alpha$. Then

$$a \mathscr{H} b \Leftrightarrow a_\alpha \mathscr{H} b_\alpha \qquad \text{for all} \quad \alpha \in A.$$

From this it follows easily that

$$a \mathscr{H}^0 b \Leftrightarrow a_\alpha \mathscr{H}^0 b_\alpha \qquad \text{for all} \quad \alpha \in A. \tag{10}$$

But $\mu = \mathscr{H}^0$, and so **K**(i) holds. Since μ is the largest congruence contained in $\mathscr{H}$, it is clear that we also have $\mu_{(S/\mu)} = \varepsilon$ and the result holds.

Let $\mathscr{L}_\mu(\mathscr{CR})$ denote the complete lattice of μ-varieties ordered by inclusion, as defined in Section 2.

The relation T defined on $\mathscr{C}(\mathrm{F}\mathscr{CR})$ by

$$\lambda \; T \; \rho \Leftrightarrow \operatorname{tr} \lambda = \operatorname{tr} \rho$$

is a complete congruence; see [5, Lemma 2.5 and Theorem 4.20]. We will use the same notation for the relation T induces on $\mathscr{L}(\mathscr{CR})$ via the anti-isomorphism π.

The main purpose of this section is to establish the following result.

THEOREM 6.2. *The mapping*

$$\theta_{\mathscr{F}} : \mathscr{V} \to \mathscr{V} \cap \mathscr{F} \qquad (\mathscr{V} \in \mathscr{L}(\mathscr{CR}))$$

is a complete homomorphism of $\mathscr{L}(\mathscr{CR})$ onto $\mathscr{L}_\mu(\mathscr{CR})$ which induces T. Moreover, for any $\mathscr{V} \in \mathscr{L}(\mathscr{CR})$, we have

$$\mathscr{V}\bar{\theta}_{\mathscr{F}} = [\langle \mathscr{V} \cap \mathscr{F} \rangle, \mathscr{G} \circ \mathscr{V}]. \tag{11}$$

Proof. 1. *$\theta_{\mathscr{F}}$ is a complete homomorphism.* By Proposition 2.2 and Lemma 6.1, $\theta_{\mathscr{F}}$ is a complete intersection homomorphism of $\mathscr{L}(\mathscr{CR})$ onto $\mathscr{L}_\mu(\mathscr{CR})$. Let $\mathscr{V}_\alpha \in \mathscr{L}(\mathscr{CR})$, for each $\alpha \in A$, and let $S \in (\bigvee_{\alpha \in A} \mathscr{V}_\alpha) \cap \mathscr{F}$. Then there exist $V_\alpha \in \mathscr{V}_\alpha$, a completely regular subsemigroup T of $\prod_{\alpha \in A} V_\alpha$, and a homomorphism φ of T onto S. By (10) we have

$$\prod_{\alpha \in A} (V_\alpha/\mu) \cong \left(\prod_{\alpha \in A} V_\alpha\right)\Big/\mu, \tag{12}$$

where the latter semigroup has $T/(\mu|_T)$ as a subsemigroup and $\mu|_T \subseteq \mathscr{H}_T$. Let $a, b \in T$ be such that $a \, \mu \, b$. Then $a \, \mu_T \, b$ and thus $(a\varphi) \, \mu_S \, (b\varphi)$ since $\mu = \mathscr{H}^0$. But then $a\varphi = b\varphi$ since $S \in \mathscr{F}$. It follows that the mapping $a\mu \to a\varphi$ $(a \in T)$ is a homomorphism of $T/(\mu|_T)$ onto S. In view of (12), $T/(\mu|_T)$ is isomorphic to a subsemigroup of $\prod_{\alpha \in A} (V_\alpha/\mu) \in \bigvee_{\alpha \in A} (\mathscr{V}_\alpha \cap \mathscr{F})$. Therefore, since $S = S/\mu$, we get $S \in \bigvee_{\alpha \in A} (\mathscr{V}_\alpha \cap \mathscr{F})$, which proves that $\theta_{\mathscr{F}}$ is also a complete $\vee$-homomorphism.

2. *Relation* (11) *holds*. Let $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{CR})$ be such that $\mathscr{U} \cap \mathscr{F} = \mathscr{V} \cap \mathscr{F}$. Then $\langle \mathscr{V} \cap \mathscr{F} \rangle = \langle \mathscr{U} \cap \mathscr{F} \rangle \subseteq \mathscr{U}$. Also, for $S \in \mathscr{U}$, we have $S/\mu \in \mathscr{U} \cap \mathscr{F} = \mathscr{V} \cap \mathscr{F}$ so that $S \in \mathscr{G} \circ \mathscr{V}$. Consequently $\mathscr{U} \subseteq \mathscr{G} \circ \mathscr{V}$, which then proves that

$$\mathscr{V}\bar{\theta}_{\mathscr{F}} \subseteq [\langle \mathscr{V} \cap \mathscr{F} \rangle, \mathscr{G} \circ \mathscr{V}]. \tag{13}$$

We next verify

$$\langle \mathscr{V} \cap \mathscr{F} \rangle \cap \mathscr{F} = \mathscr{V} \cap \mathscr{F} = (\mathscr{G} \circ \mathscr{V}) \cap \mathscr{F}. \tag{14}$$

Indeed, $\mathscr{V} \cap \mathscr{F} \subseteq \mathscr{V}$ implies that $\langle \mathscr{V} \cap \mathscr{F} \rangle \subseteq \mathscr{V}$ and thus $\langle \mathscr{V} \cap \mathscr{F} \rangle \cap \mathscr{F} \subseteq \mathscr{V} \cap \mathscr{F}$. Clearly $\mathscr{V} \cap \mathscr{F} \subseteq \langle \mathscr{V} \cap \mathscr{F} \rangle \cap \mathscr{F}$. Since $\mathscr{V} \subseteq \mathscr{G} \circ \mathscr{V}$, we get $\mathscr{V} \cap \mathscr{F} \subseteq (\mathscr{G} \circ \mathscr{V}) \cap \mathscr{F}$. Finally, let $S \in (\mathscr{G} \circ \mathscr{V}) \cap \mathscr{F}$. Then $S/\mu \in \mathscr{V}$ and $\mu = \varepsilon$ since $S \in \mathscr{F}$ and hence $S \in \mathscr{V}$. It follows that $(\mathscr{G} \circ \mathscr{V}) \cap \mathscr{F} \subseteq \mathscr{V} \cap \mathscr{F}$. This establishes (14), which implies that $\langle \mathscr{V} \cap \mathscr{F} \rangle$, $\mathscr{G} \circ \mathscr{V} \in \mathscr{V}\bar{\theta}_{\mathscr{F}}$, which by convexity gives the reverse inclusion in (13). Note that $\mathscr{G} \circ \mathscr{V} \in \mathscr{L}(\mathscr{CR})$ in view of Lemma 2.1.

3. $\theta_{\mathscr{F}}$ *induces* T. By [3, Proposition 7.2], for any $\rho \in \mathscr{C}(\mathrm{F}\mathscr{CR})$, we have $[\rho_T] = \mathscr{G} \circ [\rho]$, where ρ_T is the least (fully invariant) congruence on $\mathrm{F}\mathscr{CR}$ with the same trace as ρ. Hence, for two such congruences, we obtain, using (11),

$$\begin{aligned} \lambda\, T\, \rho \Leftrightarrow \quad \lambda_T = \rho_T \quad &\Leftrightarrow [\lambda_T] = [\rho_T] \\ &\Leftrightarrow \mathscr{G} \circ [\lambda] = \mathscr{G} \circ [\rho] \Leftrightarrow [\lambda]\, \theta_{\mathscr{F}} = [\rho]\, \theta_{\mathscr{F}}, \end{aligned}$$

as required.

COROLLARY 6.3. *For any $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{CR})$, we have*

$$\mathscr{U}\, T\, \mathscr{V} \Leftrightarrow \mathscr{U} \cap \mathscr{F} = \mathscr{V} \cap \mathscr{F} \Leftrightarrow \langle \mathscr{U} \cap \mathscr{F} \rangle = \langle \mathscr{V} \cap \mathscr{F} \rangle \Leftrightarrow \mathscr{G} \circ \mathscr{U} = \mathscr{G} \circ \mathscr{V}.$$

Let $\rho \in \mathscr{C}(\mathrm{F}\mathscr{CR})$. It follows from the above theorem that there exist ρ_T and ρ^T, the least and the greatest members of $\mathscr{C}(\mathrm{F}\mathscr{CR})$ with the same trace as ρ, and that they satisfy

$$[\rho_T] = \mathscr{G} \circ [\rho], \qquad [\rho^T] = \langle [\rho] \cap \mathscr{F} \rangle.$$

The first of these was used in the proof of the theorem and was proved in [3, Proposition 7.2].

Denoting by $\mathscr{E}(X)$ the lattice of equivalence relations on a set X, we also have the diagram

$$\begin{array}{ccc} \mathscr{L}(\mathscr{C}\mathscr{R}) & \xrightarrow{\theta_{\mathscr{F}}} & \mathscr{L}_{\mu}(\mathscr{C}\mathscr{R}) \\ {\scriptstyle\pi}\uparrow & & \downarrow \\ \mathscr{C}(\mathrm{F}\mathscr{C}\mathscr{R}) & \xrightarrow{\mathrm{tr}} & \mathscr{C}(E(\mathrm{F}\mathscr{C}\mathscr{R})) \end{array}$$

where the mapping $\mathrm{tr}: \rho \to \mathrm{tr}\, \rho$ was proved in [5, Theorem 4.20] to be a complete homomorphism. By the theorem, both $\pi\theta_{\mathscr{F}}$ and tr induce the same congruence. Thus there is a natural injection of $\mathscr{L}_{\mu}(\mathscr{C}\mathscr{R})$ into $\mathscr{E}(E(\mathrm{F}\mathscr{C}\mathscr{R}))$.

7. Operators Related to Fundamental Completely Regular Semigroups

In this section we consider properties associated with the lower and upper bounds of the intervals identified in Theorem 6.2. In order to treat the case of the lower bound, we need some preparation.

LEMMA 7.1. *For any congruence ρ on a completely regular semigroup S, $\rho \vee \mathscr{H} = \mathscr{H}\rho\mathscr{H}$ in the lattice of equivalence relations on S.*

Proof. Let $a\, \rho\mathscr{H}\rho\, b$. Then $a\, \rho\, x\, \mathscr{H}\, y\, \rho\, b$ for some $x, y \in S$, which implies that $a^0\, \rho\, x^0 = y^0\, \rho\, b^0$. But then $a\, \mathscr{H}\, a^0\, \rho\, b^0\, \mathscr{H}\, b$ so that $a\, \mathscr{H}\rho\mathscr{H}\, b$. This implies that $\rho\mathscr{H}\rho \subseteq \mathscr{H}\rho\mathscr{H}$, which evidently gives that $\rho \vee \mathscr{H} = \mathscr{H}\rho\mathscr{H}$.

COROLLARY 7.2. *With the hypotheses of Lemma* 7.1,

$$a\, \rho \vee \mathscr{H}\, b \Leftrightarrow a^0\, \rho\, b^0 \qquad (a, b \in S).$$

Proof. Straightforward.

LEMMA 7.3. *Let $S \in \mathscr{C}\mathscr{R}$. Then the mapping $\rho \to \rho \vee \mathscr{H}$ is a complete homomorphism of the lattice of congruences on S into the lattice of equivalence relations on S.*

Proof. Clearly the mapping preserves arbitrary joins. So let ρ_α be a congruence on S for each $\alpha \in A$. Then $(\bigcap_{\alpha \in A} \rho_\alpha) \vee \mathscr{H} \subseteq \bigcap_{\alpha \in A} (\rho_\alpha \vee \mathscr{H})$, and it remains to establish the reverse inclusion. Let $(a, b) \in \bigcap_{\alpha \in A} (\rho_\alpha \vee \mathscr{H})$.

By Corollary 7.2, we have $a^0 \rho_\alpha b^0$ for all $\alpha \in A$. Thus $(a^0, b^0) \in \bigcap_{\alpha \in A} \rho_\alpha$ so that

$$(a, b) \in \mathscr{H}\left(\bigcap_{\alpha \in A} \rho_\alpha\right)\mathscr{H} \subseteq \left(\bigcap_{\alpha \in A} \rho_\alpha\right) \vee \mathscr{H}.$$

Hence $(\bigcap_{\alpha \in A} \rho_\alpha) \vee \mathscr{H} = \bigcap_{\alpha \in A} (\rho_\alpha \vee \mathscr{H})$ as required.

COROLLARY 7.4. *The mapping*

$$\mathscr{V} \to \mathscr{V} \cap \mathscr{B} \qquad (\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{G}))$$

is a complete endomorphism of $\mathscr{L}(\mathscr{C}\mathscr{G})$.

Proof. As usual, there is a anti-isomorphism π of $\mathscr{L}(\mathscr{C}\mathscr{G})$ onto the lattice of fully invariant congruences on $F\mathscr{C}\mathscr{G}$. In this mapping, the variety $\mathscr{B}$ of bands corresponds to the Green relation $\mathscr{H}$ on $F\mathscr{C}\mathscr{G}$. The desired result now follows from Lemma 7.3 in view of the fact that π is an anti-isomorphism.

LEMMA 7.5. *The following conditions on* $\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R})$ *are equivalent.*

(i) $\mathscr{V} \subseteq \mathscr{C}\mathscr{G}$. (ii) $\langle \mathscr{V} \cap \mathscr{F} \rangle \subseteq \mathscr{B}$. (iii) $\mathscr{V} \cap \mathscr{F} = \mathscr{V} \cap \mathscr{B}$.

Proof. (i) *implies* (ii). If $S \in \mathscr{V} \cap \mathscr{F}$, then $S \in \mathscr{C}\mathscr{G} \cap \mathscr{F} = \mathscr{B}$. Hence $\mathscr{V} \cap \mathscr{F} \subseteq \mathscr{B}$ and thus $\langle \mathscr{V} \cap \mathscr{F} \rangle \subseteq \mathscr{B}$.

(ii) *implies* (iii). The hypothesis implies that $\mathscr{V} \cap \mathscr{F} \subseteq \mathscr{V} \cap \mathscr{B}$. But $\mathscr{B} \subseteq \mathscr{F}$ and hence $\mathscr{V} \cap \mathscr{B} \subseteq \mathscr{V} \cap \mathscr{F}$ always holds. Therefore $\mathscr{V} \cap \mathscr{F} = \mathscr{V} \cap \mathscr{B}$.

(iii) *implies* (i). If $S \in \mathscr{V}$, then $S/\mu \in \mathscr{V} \cap \mathscr{F} = \mathscr{V} \cap \mathscr{B}$, which evidently implies that $\mu = \mathscr{H}$ and $S \in \mathscr{C}\mathscr{G}$.

PROPOSITION 7.6. *The mapping*

$$\mathscr{V} \to \langle \mathscr{V} \cap \mathscr{F} \rangle \qquad (\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R}))$$

is a complete $\vee$*-endomorphism of* $\mathscr{L}(\mathscr{C}\mathscr{R})$ *but is not an* $\cap$*-homomorphism Its restriction to* $\mathscr{L}(\mathscr{C}\mathscr{G})$ *is a complete endomorphism of* $\mathscr{L}(\mathscr{C}\mathscr{G})$.

Proof. The first assertion follows directly from Theorem 6.2 and Lemma 2.4.

Now let $\mathscr{U}$ (respectively $\mathscr{V}$) be the variety of all completely regular semigroups which are semilattices of left (respectively right) groups. Let U (respectively V) be the semigroup of all transformations on a set of two elements written on the left (respectively right). One verifies easily that

$U \in \mathscr{U} \cap \mathscr{F}$ and $V \in \mathscr{V} \cap \mathscr{F}$. Taking the nonuniversal Rees congruence on U and V, we see both U and V admit a 2 element group with zero H as a homomorphic image. Therefore $H \in \langle \mathscr{U} \cap \mathscr{V} \rangle \cap \langle \mathscr{V} \cap \mathscr{F} \rangle$. But

$$\langle (\mathscr{U} \cap \mathscr{V}) \cap \mathscr{F} \rangle = \langle \mathscr{S}\mathscr{G} \cap \mathscr{F} \rangle = \langle \mathscr{S} \rangle = \mathscr{S},$$

which shows that the above mapping is not an $\cap$-homomorphism.

The final assertion follows directly from Corollary 7.4 and Lemma 7.5.

In order to perform a similar analysis with the mapping $\mathscr{V} \rightarrow \mathscr{G} \circ \mathscr{V}$ associated with the upper bound of the interval figuring in Theorem 6.2, we again need some preparation.

LEMMA 7.7. *Let $S \in \mathscr{O}\mathscr{G}$. Then $S/\tau \in \mathscr{S}\mathscr{G}$.*

Proof. By [12, Corollary 7.4(ii)], we have $\mathscr{O}\mathscr{G} = \mathscr{B} \circ \mathscr{S}\mathscr{G}$ whence $S/\tau \in \mathscr{S}\mathscr{G}$.

LEMMA 7.8 [3, Section 6]. *For any $\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R})$,*

$$\mathscr{G} \circ \mathscr{V} = \{S \in \mathscr{C}\mathscr{R} \mid S/\mu \in \mathscr{V}\}.$$

LEMMA 7.9. *For any $\mathscr{V} \in \mathscr{L}(\mathscr{O}\mathscr{G})$ we have*

$$\mathscr{G} \circ \mathscr{V} \cap \mathscr{O}\mathscr{G} = \{S \in \mathscr{O}\mathscr{G} \mid S/\mu \in \mathscr{V}\} = \mathscr{V} \vee \mathscr{G}.$$

Proof. The first equality is an immediate consequence of Lemma 7.8.

Next let $S \in \mathscr{O}\mathscr{G}$ be such that $S/\mu \in \mathscr{V}$. Since $\mu \cap \tau = \varepsilon$, we have that S is a subdirect product of S/μ, which is in $\mathscr{V}$, and of S/τ. If $\mathscr{V} \subseteq \mathscr{C}\mathscr{S}$, then, by Lemma 7.7, $S/\tau \in \mathscr{C}\mathscr{S} \cap \mathscr{S}\mathscr{G} = \mathscr{G}$ so that $S \in \mathscr{V} \vee \mathscr{G}$. Otherwise, $\mathscr{S} \subseteq \mathscr{V}$ while again by Lemma 7.7, $S/\tau \in \mathscr{S}\mathscr{G} = \mathscr{S} \vee \mathscr{G}$ so that $S \in \mathscr{V} \vee \mathscr{S} \vee \mathscr{G} = \mathscr{V} \vee \mathscr{G}$.

Finally, since $\mathscr{G} \circ \mathscr{V}$ is a variety, so also is $\mathscr{G} \circ \mathscr{V} \cap \mathscr{O}\mathscr{G}$. It evidently contains both $\mathscr{V}$ and $\mathscr{G}$ and hence also $\mathscr{V} \vee \mathscr{G}$. This completes the circle of containments.

PROPOSITION 7.10. *The mapping*

$$\mathscr{V} \rightarrow \mathscr{G} \circ \mathscr{V} \qquad (\mathscr{V} \in \mathscr{L}(\mathscr{C}\mathscr{R}))$$

is a complete $\cap$-endomorphism of $\mathscr{L}(\mathscr{C}\mathscr{R})$ but is not a $\vee$-homomorphism. Moreover, the mapping

$$\mathscr{V} \rightarrow \mathscr{G} \circ \mathscr{V} \cap \mathscr{O}\mathscr{G} \qquad (\mathscr{V} \in \mathscr{L}(\mathscr{O}\mathscr{G}))$$

is a complete endomorphism of $\mathscr{L}(\mathscr{O}\mathscr{G})$.

Proof. Let $\mathscr{V}_\alpha \in \mathscr{L}(\mathscr{CR})$ for $\alpha \in A$, $\mathscr{P} = \bigcap_{\alpha \in A} (\mathscr{G} \circ \mathscr{V}_\alpha)$, and $\mathscr{Q} = \mathscr{G} \circ (\bigcap_{\alpha \in A} \mathscr{V}_\alpha)$ so that it suffices to prove that $\mathscr{P} \subseteq \mathscr{Q}$. Let $S \in \mathscr{P}$. Then for every $\alpha \in A$, by Lemma 7.8, $S/\mu \in \mathscr{V}_\alpha$. Hence $S/\mu \in \bigcap_{\alpha \in A} \mathscr{V}_\alpha$ and so, again by Lemma 7.8, $S \in Q$. Thus the first mapping is a complete $\cap$-endomorphism. That the second mapping is a complete $\vee$-endomorphism follows from the second equality in Lemma 7.9.

For the second assertion of the proposition, consider

$$\mathscr{G} \circ \mathscr{L}\mathscr{Z} \vee \mathscr{G} \circ \mathscr{R}\mathscr{Z} = \mathscr{L}\mathscr{G} \vee \mathscr{R}\mathscr{G} = \mathscr{R}e\mathscr{G},$$

$$\mathscr{G} \circ (\mathscr{L}\mathscr{Z} \vee \mathscr{R}\mathscr{Z}) = \mathscr{G} \circ \mathscr{R}\mathscr{B} = \mathscr{C}\mathscr{S}.$$

A similar counterexample also occurs above semilattices if we consider the varieties of left normal bands and right normal bands.

The next lemma provides two bases for the identitities of $\mathscr{G} \circ \mathscr{V}$.

LEMMA 7.11. *If* $\mathscr{V} = [u_\alpha = v_\alpha]_{\alpha \in A} \in \mathscr{L}(\mathscr{CR})$, *then*

$$\begin{aligned}\mathscr{G} \circ \mathscr{V} &= [u_\alpha^0 = v_\alpha^0,\ (xu_\alpha y)^0 = (xv_\alpha y)^0]_{\alpha \in A} \\ &= [u_\alpha^{-1}(u_\alpha x u_\alpha)^0\, u_\alpha = v_\alpha^{-1}(v_\alpha x v_\alpha)^0 v_\alpha]_{\alpha \in A},\end{aligned}$$

where $x, y \notin c(u_\alpha) \cup c(v_\alpha)$ *for all* $\alpha \in A$.

Proof. The first characterization of $\mathscr{G} \circ \mathscr{V}$ can be found in [12, Theorem 3.9]. Since the proof there is not quite complete, we argue as follows. Recall Lemma 7.5. Since $\mu = \mathscr{H}^0$, it suffices to express the requirement that $S/\mathscr{H}^0$ satisfies $u_\alpha = v_\alpha$ for all $\alpha \in A$. But

$$\begin{aligned} u_\alpha \mathscr{H}^0 v_\alpha &\Leftrightarrow xu_\alpha y \mathscr{H} xv_\alpha y && \text{for all} \quad x, y \in S^1 \\ &\Leftrightarrow (xy_\alpha y)^0 = (xv_\alpha y)^0 && \text{for all} \quad x, y \in S^1. \end{aligned} \tag{15}$$

If (15) holds, then for $x = y = 1$ and for $x \neq 1 \neq y$, we obtain the expressions in the statement of the lemma. If we have these special cases, then

$$(xu_\alpha)^0 = (xu_\alpha u_\alpha^0)^0 = (xv_\alpha u_\alpha^0)^0 = (xv_\alpha v_\alpha^0)^0 = (xv_\alpha)^0$$

and the case $u_\alpha x$ is symmetric. This establishes the first characterization. The proof of the second follows similarly to that of [3, Corollary 6.4].

8. LEFT FUNDAMENTAL COMPLETELY REGULAR SEMIGROUPS

Denote by $\mathscr{L}\mathscr{F}$ the class of all left fundamental completely regular semigroups. Recall that these are the semigroups in which $\mathscr{L}^0 = \varepsilon$, that is, the equality is the only congruence contained in the Green relation $\mathscr{L}$.

LEMMA 8.1. *$L=\{\mathcal{L}^0_S : S\in\mathcal{CR}\}$ is a radical congruence system.*

Proof. The proof is entirely analogous to that of Lemma 6.1.

Let $\mathcal{L}_{\mathcal{L}^0}(\mathcal{CR})$ denote the complete lattice of $\mathcal{L}^0$-varieties ordered by inclusion, as defined in Section 2.

The relation T_l defined on $\mathcal{C}(\mathrm{F}\mathcal{CR})$ by

$$\lambda\, T_l\, \rho \Leftrightarrow l\,\mathrm{tr}\,\lambda = l\,\mathrm{tr}\,\rho,$$

where $l\,\mathrm{tr}\,\rho=\mathrm{tr}(\rho\vee\mathcal{L})^0$ is the *left trace* of ρ, is a complete congruence; see [5, Theorem 4.7]. We will use the same notation for the relation that T_l induces on $\mathcal{L}(\mathcal{CR})$ via the anti-isomorphism π.

The main purpose of this section is to establish the following result.

THEOREM 8.2. *The mapping*

$$\theta_{\mathcal{LF}} : \mathcal{V}\to\mathcal{V}\cap\mathcal{LF}\qquad (\mathcal{V}\in\mathcal{L}(\mathcal{CR}))$$

is a complete homomorphism of $\mathcal{L}(\mathcal{CR})$ onto $\mathcal{L}_{\mathcal{L}^0}(\mathcal{CR})$ which induces T_l. Moreover, for $\mathcal{V}\in\mathcal{L}(\mathcal{CR})$, we have

$$\mathcal{V}\bar{\theta}_{\mathcal{LF}}=[\langle\mathcal{V}\cap\mathcal{LF}\rangle, \mathcal{LG}\circ\mathcal{V}]. \tag{16}$$

Proof. 1. *$\theta_{\mathcal{LF}}$ is a complete homomorphism.* It follows from Proposition 2.2 and Lemma 8.1 that $\theta_{\mathcal{LF}}$ is a complete intersection homomorphism of $\mathcal{L}(\mathcal{CR})$ onto $\mathcal{L}_{\mathcal{L}^0}(\mathcal{CR})$. Let $\mathcal{V}_\alpha\in\mathcal{L}(\mathcal{CR})$, for each $\alpha\in A$, and let $S\in(\bigvee_{\alpha\in A}\mathcal{V}_\alpha)\cap\mathcal{LF}$. Then there exist $V_\alpha\in\mathcal{V}_\alpha$, a completely regular subsemigroup T of $\prod_{\alpha\in A}V_\alpha$, and a homomorphism φ of T onto S. By **K**(i), we have

$$\prod_{\alpha\in A}(V_\alpha/\mathcal{L}^0)\cong\Big(\prod_{\alpha\in A}V_\alpha\Big)\Big/\mathcal{L}^0, \tag{17}$$

where the latter semigroup has $T/(\mathcal{L}^0|_T)$ as a subsemigroup and $\mathcal{L}^0|_T\subseteq\mathcal{L}_T$. Let $a,b\in T$ be such that $a\,\mathcal{L}^0\,b$. Then $a\,\mathcal{L}^0_T\,b$ and thus $a\varphi\,\mathcal{L}^0_S\,b\varphi$. But then $a\varphi=b\varphi$ since $S\in\mathcal{LF}$. It follows that the mapping $a\mathcal{L}^0\to a\varphi$ $(a\in T)$ is a homomorphism of $T/(\mathcal{L}^0|_T)$ onto S. In view of (17), $T/(\mathcal{L}^0|_T)$ is isomorphic to a subsemigroup of $\prod_{\alpha\in A}(V_\alpha/\mathcal{L}^0)\in\prod_{\alpha\in A}(\mathcal{V}_\alpha\cap\mathcal{LF})$, which proves that $\theta_{\mathcal{LF}}$ is also a complete $\vee$-homomorphism.

2. *Relation* (16) *holds.* Let $\mathcal{U},\mathcal{V}\in\mathcal{L}(\mathcal{CR})$ be such that $\mathcal{U}\cap\mathcal{LF}=\mathcal{V}\cap\mathcal{LF}$. Then $\langle\mathcal{V}\cap\mathcal{LF}\rangle=\langle\mathcal{U}\cap\mathcal{LF}\rangle\subseteq\mathcal{U}$. Also, for

$S \in \mathscr{U}$, we have $S/\mathscr{L}^0 \in \mathscr{U} \cap \mathscr{LF}$ so that $S \in \mathscr{LG} \circ \mathscr{V}$. Consequently $\mathscr{U} \subseteq \mathscr{LG} \circ \mathscr{V}$, which then proves that

$$\mathscr{V}\bar{\theta}_{\mathscr{LF}} \subseteq [\langle \mathscr{V} \cap \mathscr{LF} \rangle, \mathscr{LG} \circ \mathscr{V}]. \tag{18}$$

We next verify

$$\langle \mathscr{V} \cap \mathscr{LF} \rangle \cap \mathscr{LF} = \mathscr{V} \cap \mathscr{LF} = (\mathscr{LG} \circ \mathscr{V}) \cap \mathscr{LF}. \tag{19}$$

Now, $\mathscr{V} \cap \mathscr{LF} \subseteq \mathscr{V}$ implies $\langle \mathscr{V} \cap \mathscr{LF} \rangle \subseteq \mathscr{V}$ so that $\langle \mathscr{V} \cap \mathscr{LF} \rangle \cap \mathscr{LF} \subseteq \mathscr{V} \cap \mathscr{LF}$. Clearly $\mathscr{V} \cap \mathscr{LF} \subseteq \langle \mathscr{V} \cap \mathscr{LF} \rangle \cap \mathscr{F}$ and the first equality holds. Since $\mathscr{V} \subseteq \mathscr{LG} \circ \mathscr{V}$, we get $\mathscr{V} \cap \mathscr{LF} \subseteq (\mathscr{LG} \circ \mathscr{V}) \cap \mathscr{LF}$. Finally, let $S \in (\mathscr{LF} \circ \mathscr{V}) \cap \mathscr{LF}$. Then $S/\mathscr{L}^0 \in \mathscr{V}$ and $\mathscr{L}^0 = \varepsilon$ since $S \in \mathscr{LF}$ and hence $S \in \mathscr{V}$. It follows that $(\mathscr{LG} \circ \mathscr{V}) \cap \mathscr{LF} \subseteq \mathscr{V} \cap \mathscr{LF}$. This establishes (19), which implies that $\langle \mathscr{V} \cap \mathscr{LF} \rangle$, $\mathscr{LG} \circ \mathscr{V} \in \mathscr{V}\theta_{\mathscr{LF}}$, which by convexity gives the reverse inclusion in (18). Note that $\mathscr{LG} \circ \mathscr{V} \in \mathscr{L}(\mathscr{CR})$ in view of Lemma 2.1.

3. *$\theta_{\mathscr{LF}}$ induces T_l.* By the left hand analogue of [4, Lemma 3] for any $\rho \in \mathscr{C}(\mathrm{F}\mathscr{CR})$, we have $[\rho_{T_l}] = \mathscr{LG} \circ [\rho]$, where ρ_{T_l} is the least (fully invariant) congruence on $\mathrm{F}\mathscr{CR}$ with the same left trace as ρ. Hence, for such congruences, we obtain, using (19), the desired equivalence as follows:

$$\begin{aligned} \lambda\, T_l\, \rho \Leftrightarrow \quad \lambda_{T_l} = \rho_{T_l} \quad &\Leftrightarrow [\lambda_{T_l}] = [\rho_{T_l}] \\ \Leftrightarrow \mathscr{LG} \circ [\lambda] = \mathscr{LG} \circ [\rho] &\Leftrightarrow [\lambda]\,\theta_{\mathscr{LF}} = [\rho]\,\theta_{\mathscr{LF}}. \end{aligned}$$

COROLLARY 8.3. *For any $\mathscr{U}, \mathscr{V} \in \mathscr{L}(\mathscr{CR})$, we have*

$$\begin{aligned} \mathscr{U}\, T_l\, \mathscr{V} &\Leftrightarrow \mathscr{U} \cap \mathscr{LF} = \mathscr{V} \cap \mathscr{LF} \\ &\Leftrightarrow \langle \mathscr{U} \cap \mathscr{LF} \rangle = \langle \mathscr{V} \cap \mathscr{LF} \rangle \Leftrightarrow \mathscr{LG} \circ \mathscr{U} = \mathscr{LG} \circ \mathscr{V}. \end{aligned}$$

Let $\rho \in \mathscr{C}(\mathrm{F}\mathscr{CR})$. It follows from the above theorem that there exist ρ_{T_l} and ρ^{T_l}, the least and the greatest members of $\mathscr{C}(\mathrm{F}\mathscr{CR})$ with the same left trace as ρ, and that they satisfy

$$[\rho^{T_l}] = \langle [\rho] \cap \mathscr{LF} \rangle, \qquad [\rho_{T_l}] = \mathscr{LG} \circ [\rho]. \tag{20}$$

PROPOSITION 8.4. *The mapping*

$$\mathscr{V} \to \langle \mathscr{V} \cap \mathscr{LF} \rangle \qquad (\mathscr{V} \in \mathscr{L}(\mathscr{CR}))$$

is a complete endomorphism of $\mathscr{L}(\mathscr{CR})$.

Proof. This follows directly from the first part of relation (20) and [4, Theorem 8] via the anti-isomorphism π.

Remark. Note that the endomorphism in the above proposition induces the complete congruence T_l on $\mathcal{L}(\mathcal{CR})$. It follows from the dual of Lemma 2.4 that the mapping

$$\mathcal{V} \to \mathcal{LG} \circ \mathcal{V} \qquad (\mathcal{V} \in \mathcal{L}(\mathcal{CR}))$$

is a complete $\cap$-endomorphism since T_l is a complete congruence. It remains an open questions whether or not this mapping is a (complete) $\vee$-homomorphism.

In conclusion, we give two bases for the identities of $\mathcal{LF} \circ \mathcal{V}$. For an alternative basis, see [11, Theorem 1.7(2)].

LEMMA 8.5. *If* $\mathcal{V} = [u_\alpha = v_\alpha]_{\alpha \in A} \in \mathcal{L}(\mathcal{CR})$, *then*

$$\begin{aligned}\mathcal{LG} \circ \mathcal{V} &= \{S \in \mathcal{CR} \mid S/\mathcal{L}^0 \in \mathcal{V}\} \\ &= [(xu_\alpha)^0 = (xu_\alpha xv_\alpha)^0, (xv_\alpha)^0 = (xv_\alpha xu_\alpha)^0] \\ &= [xu_\alpha = xu_\alpha(xv_\alpha)^0, xv_\alpha = xv_\alpha(xu_\alpha)^0],\end{aligned}$$

where $x \notin c(u_\alpha) \cup c(v_\alpha)$ *for all* $\alpha \in A$.

Proof. Let $S \in \mathcal{LG} \circ \mathcal{V}$. Then there exists a congruence ρ on S over $\mathcal{LG}$ such that $S/\rho \in \mathcal{V}$. Since a subsemigroup of S which is a left group must be contained in an $\mathcal{L}$-class of S, we conclude that $\rho \subseteq \mathcal{L}$. But then $\rho \subseteq \mathcal{L}^0$ and $S/\mathcal{L}^0$ is a homomorphic image of S/ρ and thus $S/\mathcal{L}^0 \in \mathcal{V}$. Conversely, if $S/\mathcal{L}^0 \in \mathcal{V}$, then $S \in \mathcal{LG} \circ \mathcal{V}$ since $\mathcal{L}^0$ is evidently over left groups. This establishes the first equality.

In any completely regular semigroup S, we have

$$a \,\mathcal{L}\, b \Leftrightarrow a \,\mathcal{H}\, ab,\ b \,\mathcal{H}\, ba$$

as is easily verified and hence

$$a \,\mathcal{L}\, b \Leftrightarrow a^0 = (ab)^0,\ b^0 = (ba)^0.$$

Furthermore

$$a \,\mathcal{L}^0\, b \Leftrightarrow xa \,\mathcal{L}\, xb \qquad \text{for all} \quad x \in S^1$$

since $\mathcal{L}$ is a right congruence. The second equality in the statement of the lemma is a simple consequence of these remarks.

The last expression follows similarly from

$$a \,\mathcal{L}\, b \Leftrightarrow a = ab^0,\ b = ba^0.$$

From [10, Theorem 1.5(5) and Theorem 1.2(1)–(6); 4, Lemma 7] one can derive a basis for the identities of $\mathcal{V}_{T_r}$ in terms of a basis for the identities of $\mathcal{V}$.

REFERENCES

1. T. E. HALL, On the lattice of congruences on a regular semigroup, *Bull. Austral. Math. Soc.* **1** (1969), 231–235.
2. T. E. HALL AND P. R. JONES, On the lattice of varieties of bands of groups, *Pacific J. Math.* **91** (1980), 327–337.
3. P. R. JONES, Mal'cev product of varieties of completely regular semigroups, *J. Austral. Math. Soc.* **42** (1987), 227–246.
4. F. PASTIJN, The lattice of completely regular semigroup varieties, preprint.
5. F. PASTIJN AND M. PETRICH, Congruences on regular semigroups, *Trans. Amer. Math. Soc.* **295** (1986), 607–633.
6. F. J. PASTIJN AND P. G. TROTTER, Lattices of completely regular semigroup varieties, *Pacific J. Math.* **119** (1985), 191–214.
7. M. PETRICH, Semigroups certain of whose subsemigroups have identities, *Czech. Math. J.* **16** (1966), 186–198.
8. M. PETRICH, Certain varieties and quasivarieties of completely regular semigroups, *Canad. J. Math.* **29** (1977), 1171–1197.
9. M. PETRICH AND N. R. REILLY, Semigroups generated by certain operators on varieties of completely regular semigroups, *Pacific J. Math.* **132** (1988), 151–175.
10. L. POLÁK, On varieties of completely regular semigroups, I, *Semigroup Forum* **32** (1985), 97–123.
11. L. POLÁK, On varieties of completely regular semigroups, II, *Semigroup Forum* **36** (1987), 253–284.
12. N. R. REILLY, Varieties of completely regular semigroups, *J. Austral. Math. Soc.* **38** (1985), 372–393.
13. H. E. SCHEIBLICH, Certain congruence and quotient lattices related to completely 0-simple and primitive regular semigroups, *Glasgow Math. J.* **10** (1969), 21–24.