Testing Membership in Commutative Transformation Semigroups

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1 Introduction

The majority of results in this paper are taken from [1].

Throughout this paper we will use the following notations:

- We will represent the associated monoid of a semigroup S by S^1.
- The action of elements of a semigroup on a set of states will be represented on the right.
- The identity function on a set X will be represented by id or id_X.
- The symmetric group on a set X will be denoted S_X.
- The order of an element g in a group will be denoted $o(g)$.
- The natural numbers, denoted $\mathbb{N}$, will be defined as the set of positive integers.

2 Commutative Transformation Semigroups

Lemma 2.1 *Let S be a commutative transformation semigroup, generated by a set A, acting on a finite set X. Let*

$$U = \bigcup_{a \in A} \mathrm{im}(a).$$

Then S is a monoid if and only if A contains an element t such that $t \restriction_U$ is a permutation of U.

In such a case, the identity of S is t^n, where n is the order of $t \restriction_U$ in S_U.

Proof ($\Rightarrow$): Suppose S is a monoid. Then $\exists e \in S$ such that $eg = ge$, $\forall g \in S$. Let $x \in U$. We have that $\exists a \in A,\ y \in X$ such that $x = ya$. Suppose, for contradiction, that $xe \neq x$.

$$\implies yae \neq ya \implies ae \neq a,$$

a contradiction. Hence e fixes all elements of U. We have that $\exists a_1, a_2, \ldots, a_k \in A$, for some $k \in \mathbb{N}$ such that

$$a_1 a_2 \cdots a_k = e \implies a_1 a_2 \cdots a_k \restriction_U = \mathrm{id}_U .$$

Since $|\mathrm{im}(\mathrm{id}_U)| = |U|$, we have that all the generators that product to e have image of at least the same size, when restricted to U. Since we only have $|U|$ states to map from, the sizes of the images the generators are at most $|U|$, and hence equal to U. Hence they are surjective maps from U to

U, and therefore permutations of U. In particular, we have at least one permutation of U in A, when restricting to U.

($\Leftarrow$): Suppose A contains an element, t such that $t \restriction_U$ is a permutation of U. Since $t \restriction_U \in S_U$, the symmetric group on U, we have that $(t \restriction_U)^n = \mathrm{id}_U$ for some $n \in \mathbb{N}$, which is the order of $t \restriction_U$. Define a transformation $g = t^n$ (acting on X). Let $a \in A,\ x \in X$. If $x \in U$, then

$$xg = xt^n = xt^n \restriction_U = x,$$

$$\implies xga = xa,$$

If $x \notin U$, then

$$xga = xag = xat^n.$$

Since $xa \in U$, we have that t^n fixes it. Hence

$$xga = xa.$$

Hence $xga = xa,\ \forall x \in X,\ a \in A$, and hence g acts as the identity on the generators of S. Let $h \in S$. Then $\exists a_1, a_2, \ldots, a_k \in A$, for some $k \in \mathbb{N}$, possibly with repeats, such that $h = a_1 a_2 \cdots a_k$. Note $a_k g = g$. Then

$$\begin{aligned} hg &= a_1 a_2 \cdots a_k g \\ &= a_1 a_2 \cdots a_k \\ &= h \end{aligned}$$

Hence g is the identity of S, and S is a monoid.

Definition 2.2 Let S be a finite semigroup, $g \in S$. Let t be the smallest non-negative integer such that $\exists k \in \mathbb{N}$ such that

$$g^t = g^{t+k}.$$

The *threshold* of g is defined to be t, and the *period* to be the smallest $p \in \mathbb{N}$ such that

$$g^t = g^{t+p}.$$

The *threshold* and *period* of S are defined to be the maximum of the thresholds and the lowest common multiple of the periods of the elements of S. If the period of S is 1, S is called *aperiodic.*

Lemma 2.3 *Let A be a set of transformations of a finite set X. If the elements of A commute with one another, then $\langle A \rangle$ is commutative.*

Proof Let $g, h \in \langle A \rangle$. We have that $g = a_1 \cdots a_k,\ h = b_1 \cdots b_l$, for some $k, l \in \mathbb{N},\ a_i, b_j \in A, \forall i \leq k,\ j \leq l$. Then

$$gh = a_1 \cdots a_k b_1 \cdots b_l = b_1 \cdots b_l a_1 \cdots a_k = hg.$$

Lemma 2.4 *Let S be a transformation semigroup, generated by a set A. Let an element $a_i \in A$ have threshold t_i and period p_i. Then for any $g \in \langle A \rangle$, with threshold t and period p, we have*

$$t \leq \max_i(t_i), \qquad p \mid \mathrm{lcm}_i(p_i).$$

Proof Let $g, h \in S$ and let T be the maximum of the thresholds of g and h. Let p_g, p_h denote the periods of g, h respectively. Then

$$(gh)^{T+p_g p_h} = g^{T+p_g p_h} h^{T+p_g p_h} = g^T h^T = (gh)^T.$$

Hence the threshold of gh is less than or equal to T. Therefore, it follows that any element of S has threshold less than the maximum of the thresholds of the generators, by induction on the number of generators in the factorisation.

Let P be the lowest common multiple of the periods of the elements of A. Let $g \in S$ have threshold t and period p. Then $g = a_1 \cdots a_k$, for some $a_1, \ldots, a_k \in A$.

$$g^{t+P} = a_1^{t+P} \cdots a_k^{t+P} = a_1^t \cdots a_k^t = g^t.$$

Hence $p \leq P$. Note $P - p \in \mathbb{N}_0$. We have

$$g^{t+P-p} = g^{t+P} = g^t.$$

Let $q = \left\lfloor \frac{P}{p} \right\rfloor$. Then

$$g^{t+P-qp} = g^t.$$

as $P - qp \geq 0$. Suppose $P - qp \neq 0$. Then $P - p \left\lfloor \frac{P}{p} \right\rfloor < p$, a contradiction to the minimality of p. Hence $P = qp$ and $p|P$.

Definition 2.5 Let S be a commutative semigroup acting on a finite set X. The *centraliser* of S, denoted C_S is the set of all transformations of X that commute with all elements of S.

Note that elements of the centraliser are not necessarily in S, and do not have to commute with all elements of the centraliser.

Definition 2.6 Let X be a finite set. The *full transformation monoid* on X, denoted T_X, is the set of all transformations from X to X, together with the binary operation of composition of functions.

Lemma 2.7 *Let S be a commutative transformation semigroup, generated by a set A acting on a finite set X. Let $f \in T_X$. Then*

$$f \in C_S \iff af = fa, \quad \forall a \in A.$$

Proof ($\Rightarrow$): Suppose $f \in C_S$. Then f commutes with all elements of S, and hence all elements of A, since $A \subseteq S$.

($\Leftarrow$): Suppose $af = fa$, $\forall a \in A$. Let $g \in S$. We have that $\exists a_1, a_2, \ldots a_k \in A$, for some $k \in \mathbb{N}$ such that

$$g = a_1 a_2 \cdots a_k.$$

Hence

$$fg = fa_1 a_2 \cdots a_k = a_1 f a_2 \cdots a_k = \cdots = a_1 a_2 \cdots a_k f = gf.$$

Hence f commutes with g and it follows that $f \in C_S$.

3 The Induced Aperiodic Semigroup

Definition 3.1 Let S be a transformation semigroup acting on a finite set X. Given a state $x \in X$, define the *strongly connected component (SCC)* of x, denoted $\bar{x}$, by

$$\bar{x} = \{y \in X : \exists g, h \in S^1 \text{ such that } x = yg, y = xh\}.$$

The set of all SCCs of S will be denoted

$$\bar{X} = \{\bar{x} : x \in X\}.$$

Lemma 3.2 *Let S be a transformation semigroup acting on a finite set X. The SCCs of the elements of X partition X.*

Proof Define the relation $\sim$ on X, for $x, y \in X$ by,

$$x \sim y \text{ if } y \in \bar{x}.$$

Note for any $x \in X$, we have $x = x * e = e * x$, where e is the identity of S^1, so $x \in \bar{x}$ and hence $x \sim x$ and $\sim$ is reflexive.

Let $x, y \in X$ such that $x \sim y$. Hence $\exists g, h \in S^1$ such that

$$x = yg, \quad y = xh.$$

Hence $x \in \bar{y}$ and we have that $y \sim x$, so $\sim$ is symmetric.

Let $x, y, z \in X$, such that $x \sim y, y \sim z$. Then $\exists g_1, g_2, h \in S^1$, such that

$$x = yg_1, \quad y = xg_2, \quad y = zh.$$

$$\implies x = zhg_2, \quad z = xg_2h.$$

Since S^1 is closed, we have that $hg_2, g_2h \in S^1$. Hence $z \in \bar{x}$ and we have that $x \sim z$ and $\sim$ is transitive.

Hence $\sim$ is an equivalence relation, and therefore it partitions X into equivalence classes. The equivalence class of a given $x \in X$ is

$$\{y \in X : y \in \bar{x}\} = \bar{x},$$

so the equivalence classes are the SCCs.

Proposition 3.3 *Let S be a commutative semigroup acting on a finite set X. We have that*

$$\forall x \in X, g \in C_S, \qquad \bar{x}g \subseteq \overline{xg}.$$

Let $g \in C_S$. We have that there exists a transformation $\bar{g}$ of $\bar{X}$, such that $\forall x \in X$, $\bar{x}\bar{g} = \overline{xg}$. In addition, there is a homomorphism that maps every $g \in C_S$ to the corresponding $\bar{g}$.

Proof Let $x \in X, g \in C_S$. We have

$$y \in \bar{x}g \implies \exists z \in \bar{x} \text{ such that } y = zg.$$

Since $z \in \bar{x}$, $\exists f, h \in S$ such that $z = xf, x = zh$.

$$\implies y = zg = xfg = xgf, \qquad xg = zhg = zgh = yh.$$

So $y \in \overline{xg}$ and it follows that $\bar{x}g \subseteq \overline{xg}$.

Define $\bar{g}$ as a transformation of $\bar{X}$ as follows. For any $x \in X$, $\bar{x}\bar{g} = \overline{xg}$. We will show this function is well-defined. Let $x, y \in X$ such that $\bar{x} = \bar{y}$. Since $\bar{X}$ is a partition of X, and $\bar{x}\bar{g}, \bar{y}\bar{g}$ are SCCs, we have that they are equal if they are not disjoint.

$$xg \in \bar{x}g \subseteq \overline{xg} = \bar{x}\bar{g}$$

$$xg \in \bar{y}g \subseteq \overline{yg} = \bar{y}\bar{g}$$

Hence $xg \in \bar{y}\bar{g} \cap \bar{x}\bar{g}$, and it follows that $\bar{x}\bar{g} = \bar{y}\bar{g}$.

Let $g, h \in C_S, \bar{x} \in \bar{X}$. We have

$$\bar{x}\bar{g}\bar{h} = \overline{xg}\bar{h} = \overline{xgh} = \bar{x}\overline{gh}.$$

So the map that sends an element $g \in C_S$ to $\bar{g}$ is a homomorphism.

Lemma 3.4 *A commutative transformation semigroup acting on a finite set is aperiodic if and only if all of its SCCs are singletons.*

Proof Let S be a commutative transformation semigroup acting on a finite set X.

($\Rightarrow$): Suppose S is aperiodic. Let $x, y \in X$, $x \neq y$. Suppose, for contradiction, that $x, y \in \bar{x}$. Then $\exists g, h \in S^1$ such that $x = yg$, $y = xh$. Let T be the maximum of the thresholds of g, h. We have

$$ygh = y \implies y = y(gh)^T = yg^T h^T \implies xg^T h^T g = x \implies yg^{T+1}h^T = x \neq y.$$

We therefore have that $g^T \neq g^{T+1}$, so the threshold of g is greater than T, a contradiction.

($\Leftarrow$): Suppose all the SCCs of S are singletons. Let $g \in S$, $x \in X$. Let $n = |X|$. Let t be the threshold of g. Then $xg^t = xg^j$, for some $j \leq n, t < j$, and let j be minimal. We have

$$xg^{t+1} = xg^t g,$$

and

$$xg^t = xg^j = xg^{t+1}g^{j-t-1},$$

which implies $xg^t, xg^{t+1} \in \overline{xg^t}$. Since this is a singleton, we have that $xg^t = xg^{t+1}$ and hence the period of g is 1, so S is aperiodic.

Definition 3.5 Let S be a commutative transformation semigroup, acting on a finite set X. The *induced aperiodic semigroup* of S, denoted $\bar{S}$, is defined by

$$\bar{S} = \{\bar{g} : g \in S\},$$

together with the binary operation composition of functions.

Corollary 3.6 *Let S be a commutative transformation semigroup, generated by a set A acting on a finite set X. The induced aperiodic semigroup of S is a commutative aperiodic semigroup, generated by the set $\bar{A} = \{\bar{g} : g \in A\}$.*

Proof Since mapping any $g \in S$ to $\bar{g}$ is a homomorphism, and $\bar{S}$ is the homomorphic image of this homomorphism, we have that $\bar{S}$ is a semigroup.

Let $\alpha, \beta \in \bar{S}$. Then $\exists g, h \in S$ such that $\alpha = \bar{g},\ \beta = \bar{h}$. We have

$$\alpha\beta = \bar{g}\bar{h} = \overline{gh} = \overline{hg} = \bar{h}\bar{g} = \beta\alpha,$$

and therefore $\bar{S}$ is commutative.

To show $\bar{S}$ is aperiodic, it suffices to show that the SCCs of $\bar{S}$ are singletons, by Lemma 3.4. Let $x \in X$. Suppose that $\exists y \in X,\ g, h \in S$ such that

$$\bar{y} = \bar{x}\bar{g}, \quad \bar{x} = \bar{y}\bar{h}.$$

$$\implies xg \in \bar{y}.$$

Hence $\exists f_1 \in S$ such that $y = xgf_1$. We also have

$$yh \in \bar{x}.$$

It therefore follows that $\exists f_2 \in S$ such that $x = yhf_2$. Together with $y = xgf_1$, we conclude that $x, y \in \bar{x}$ and hence $\bar{x} = \bar{y}$, recalling that $\bar{X}$ is a partition of X by Lemma 3.2. Hence the SCC that $\bar{x}$ is in, with respect to $\bar{S}$ is a singleton.

Let $g \in S$. Since $S = \langle A \rangle$, $\exists a_1, \dots a_n \in A$ such that $g = a_1 \cdots a_n$. Since mapping any $h \in S$ to $\bar{h}$ is a homomorphism, we have

$$\bar{g} = \overline{a_1 \cdots a_n} = \bar{a_1} \cdots \bar{a_n}.$$

So $\bar{g}$ can be written as a product of elements of $\bar{A}$, and hence

$$\bar{S} = \langle \bar{A} \rangle.$$

4 Stabilisers of Strongly Connected Components

Definition 4.1 Let S be a transformation semigroup, acting on a finite set X. Let $\bar{x} \in \bar{S}$. A transformation $g \in S$ *stabilises* $\bar{x}$, if $\bar{x}\bar{g} = \bar{x}$. The *stabiliser* of $\bar{x}$, denoted $\mathrm{Stab}(\bar{x})$, is the set of all elements of S that stabilise $\bar{x}$.

Definition 4.2 Let G be a permutation group acting on a finite set X. The group G is *transitive* if $\forall x, y \in X,\ \exists g \in G$ such that $y = xg$. If, in addition, this g is always unique, then G has *regular action.*

Lemma 4.3 *Every transitive abelian group has regular action.*

Proof Let G be a transitive abelian group, acting on a set X. Let $x, y \in X$. By transitivity of G, $\exists g \in G$ such that $y = xg$. Suppose $\exists h \in G$ such that $y = xh$. Then $xh = xg$. Let $z \in X$. By transitivity $\exists f \in G$ such that $x = zf$. Therefore,

$$xg = xh \implies zfg = zfh \implies zgf = zhf \implies zg = zh.$$

Hence $zh = zf$, $\forall z \in X$, so $g = h$.

Proposition 4.4 *Let S be a commutative transformation semigroup, generated by a set A, acting on a finite set X. Let $x \in X$. If* $\mathrm{Stab}(\bar{x})$ *is non-empty, then the restriction of* $\mathrm{Stab}(\bar{x})$ *to the SCC $\bar{x}$ is an abelian permutation group with regular action, generated by* $\mathrm{Stab}(\bar{x}) \cap A$.

Proof Let $x \in X$, $g, h \in \mathrm{Stab}(\bar{x})$. We therefore have that $\bar{x}\bar{g} = \bar{x}$, $\bar{x}\bar{h} = \bar{x}$ so $\bar{x}\bar{g}\bar{h} = \bar{x}\bar{h} = \bar{x}$ and hence $gh \in \mathrm{Stab}(\bar{x})$.

Let $y, z \in \bar{x}$. We have that $\exists g \in S^1$ such that $y = zg$. Let $w \in \bar{x}$. Hence $\exists h \in S^1$ such that $w = yh$. By Proposition 3.3, we have

$$\begin{aligned} \bar{w}g &\subseteq \overline{wg} \\ \implies \bar{x}g &\subseteq \overline{wg}. \end{aligned}$$

Note that $y \in \bar{x}$ and $y = zg$ so $y \in \bar{x}g \subseteq \overline{wg}$. Hence $y \in \bar{x} \cap \overline{wg}$. Since these are both SCCs, and SCCs are disjoint or equal, because $\bar{X}$ partitions X, it follows that $\overline{wg} = \bar{x}$. In particular $wg \in \bar{x}$ and hence $g \in \mathrm{Stab}(\bar{x})$. We therefore have that the maps that send elements of an SCC to each other, as mentioned in the definition of an SCC, are in the stabiliser of that SCC.

Let $g_1 \in \mathrm{Stab}(\bar{x})$. Let $y = xg_1$. We have that $\exists g_2 \in \mathrm{Stab}(\bar{x})$ such that $x = yg_2$. Let $z \in \bar{x}$. We have that $\exists h \in \mathrm{Stab}(\bar{x})$ such that $z = xh$. Then

$$zg_2g_1 = zg_1g_2 = xhg_1g_2 = xg_1g_2h = yg_2h = xh = z,$$

and hence g_2g_1 acts as the identity on $\bar{x}$ and hence is the identity of $\mathrm{Stab}(\bar{x})$, when restricted to $\bar{x}$. We also have that g_2 is the inverse of g_1, on $\bar{x}$. Since it has an inverse, g_1 must be injective, when restricted to $\bar{x}$, and since it is a transformation of a finite set, we have that g_1 is a permutation of $\bar{x}$.

Therefore $\mathrm{Stab}(\bar{x})$ is a permutation group, when restricted to $\bar{x}$. Since the transformation that maps any element of the $\bar{x}$ to any other is always in $\mathrm{Stab}(\bar{x})$, it follows that $\mathrm{Stab}(\bar{x})$ is transitive.

We now have that $\mathrm{Stab}(\bar{x})$ is a transitive abelian group, when acting on $\bar{x}$. By Lemma 4.3, it has regular action.

To prove that $\mathrm{Stab}(\bar{x})$ is generated by $\mathrm{Stab}(\bar{x}) \cap A$, we will proceed by contradiction. Let $g \in \mathrm{Stab}(\bar{x})$ be expressed as a product of elements of A by

$$g = a_1^{\gamma_1} a_2^{\delta_2} \cdots a_k^{\gamma_k} b_1^{\delta_1} b_2^{\delta_2} \cdots b_l^{\delta_l},$$

where $k, l \in \mathbb{N}$, and the generators denoted a_i for some $i \in \mathbb{N}$ are elements of $\mathrm{Stab}(\bar{x})$, and the generators denoted b_i for some $i \in \mathbb{N}$ are not. Let $h = b_1^{\delta_1 - 1} b_2^{\delta_2} \cdots b_l^{\delta_l}$. We have

$$\bar{x} = \bar{x}\bar{g} = \bar{x}\bar{b_1}\bar{h}.$$

Since $b_1 \notin \mathrm{Stab}(\bar{x})$, $\bar{x}\bar{b_1} \neq \bar{x}$ but $\bar{x}\bar{b_1}\bar{h_1} = \bar{x}$, which means that $\bar{x}$ and $\bar{x}\bar{b_1}$ are non-equal, but in the same SCC of $\bar{S}$. This contradicts $\bar{S}$ being aperiodic, and therefore only having singleton SCCs, by Corollary 3.6.

Corollary 4.5 *Let S be a commutative transformation semigroup acting on a finite set X. Let $x \in X, g \in S$. If g stabilises $\bar{x}$, then g acts as a permutation on $\bar{x}$.*

Proof Let g stabilise $\bar{x}$. By Proposition 4.4, we have that since $\mathrm{Stab}(\bar{x})$ is a permutation group, when restricted to $\bar{x}$, we have that $g \restriction_{\bar{x}}$ is a permutation.

Corollary 4.6 *Let S be a commutative transformation semigroup acting on a finite set X. Let $g \in S$. The thresholds of g and $\bar{g}$ are equal.*

Proof Let t be the threshold of g and p be the period. Let u be the threshold of $\bar{g}$. Let $x \in X$. We have

$$\bar{x}\bar{g}^t = \bar{x}\overline{g^t} = \overline{xg^t},$$

since the map that sends every $f \in C_S$ to $\bar{f}$ is a homomorphism. It therefore follows that

$$\bar{x}\bar{g}^{t+p} = \overline{xg^{t+p}} = \overline{xg^t} = \bar{x}\bar{g}^t,$$

and hence $u \leq t$.

Note also that

$$\overline{xg^u} = \bar{x}\bar{g}^u = \bar{x}\bar{g}^u\bar{g} = \overline{xg^u}\bar{g},$$

so g stabilises $\overline{xg^u}$. By Corollary 4.5, g permutes $\overline{xg^u}$. We have

$$xg^u \in \overline{xg^u} \implies xg^u g^{o\left(g\restriction_{\overline{xg^u}}\right)} = xg^u,$$

and hence $t \leq u$.

Definition 4.7 Let S be a commutative transformation semigroup acting on a finite set X. A state $x \in X$ is a *source* if $\forall y \in X \backslash \{x\}, g \in S,\ yg \neq x$. A *source-SCC* is defined to be a source of $\bar{S}$. The set of all source-SCCs of S will be denoted $\bar{Y}$, the union of all source-SCCs will be denoted Y.

Lemma 4.8 *Let S be a commutative transformation semigroup acting on a finite set. Let $f, g \in C_S$. If*

$$zg = zf, \quad \forall z \in Y,$$

then $f = g$.

Proof Let X be the set that S acts on. We already have that f, g coincide on all states in source-SCCs. Let $\bar{x} \in \bar{X} \backslash \bar{Y}$. Then there $\exists \bar{z} \in \bar{X},\ \bar{g} \in \bar{S}$ such that $\bar{x} = \bar{z}\bar{g}$. If $\bar{z}$ is a source-SCC, we have a sequence of maps in $\bar{S}$ that take a source-SCC to $\bar{x}$. Otherwise apply this process to $\bar{z}$, and continue until a source-SCCs is reached.

This must happen after a finite number of steps, since $\bar{X}$ is finite, and no repeats can appear in the sequence. This is because $\bar{S}$ is aperiodic, and hence all of its SCCs are singletons, by Lemma 3.4. If we can get from one SCC to another by an element of $\bar{S}$, and back again, then these two element are in the same SCC of $\bar{S}$, a contradiction. We also have that there is a sequence of maps in S that when the homomorphism $\psi : \ h \mapsto \bar{h}$ is applied to them, we obtain the sequence of maps in $\bar{S}$ that take a source-SCC to $\bar{x}$. Let these maps be

$$h_1, h_2, \ldots, h_k,$$

where $k \in \mathbb{N}$. Let $w \in X$, such that $\bar{w}$ is a source SCC such that

$$\bar{x} = \bar{w}\bar{h_1}\bar{h_2} \cdots \bar{h_k}.$$

Let

$$v = wh_1h_2 \cdots h_k.$$

Note $v \in \bar{x}$. Let $z \in \bar{x}$. Then $\exists l \in S$ such that $vl = z$. Hence

$$z = wh_1h_2 \cdots h_kl.$$

$$zf = wh_1h_2 \cdots h_klf = wfh_1h_2 \cdots h_kl = wgh_1h_2 \cdots h_kl = wh_1h_2 \cdots h_klg = zg,$$

and we have that f and g coincide on all states that are not in source-SCCs, and hence must be equal.

5 The Source Action Group

Definition 5.1 Let S be a commutative transformation semigroup acting on a finite set X, $f \in S$. For a transformation $g \in \bigcap_{\bar{x} \in \bar{Y}\bar{f}} \mathrm{Stab}(\bar{x})$, define the *source action transformation* of g with respect to f, denoted $\hat{g}_f$, by

$$\begin{aligned} \hat{g}_f : X &\to X \\ x &\mapsto \left\{ \begin{array}{cc} xg & \bar{x} \in \bar{Y}\bar{f} \\ x & \text{otherwise} \end{array} \right. . \end{aligned}$$

Define the *source action group* of f, denoted $\hat{S}_f$ by

$$\hat{S}_f = \left\{ \hat{g}_f : g \in \bigcap_{\bar{x} \in \bar{Y}\bar{f}} \mathrm{Stab}(\bar{x}) \right\} \cup \{\mathrm{id}\}.$$

Define

$$\hat{A}_f = \left\{ \hat{g}_f : g \in A \cap \left(\bigcap_{\bar{x} \in \bar{Y}\bar{f}} \mathrm{Stab}(\bar{x}) \right) \right\} \cup \{\mathrm{id}\}.$$

Lemma 5.2 *Let S be a commutative transformation semigroup acting on a finite set X, $f \in S$. The source action group of f is an abelian group, generated by $\hat{A}_f$, and isomorphic to*

$$\bigcap_{\bar{x} \in \bar{Y}\bar{f}} \mathrm{Stab}(\bar{x}) \cup \{\mathrm{id}\},$$

restricted to Yf. It has regular action when restricted to any $\bar{x} \in \bar{Y}\bar{f}$.

Proof Let

$$G = \left\{ g \restriction_{Yf} : g \in \bigcap_{\bar{x} \in \bar{Y}\bar{f}} \mathrm{Stab}(\bar{x}) \cup \{\mathrm{id}\} \right\}.$$

Define the mapping

$$\begin{aligned} \phi : G &\to \hat{S}_f \\ g &\mapsto \hat{g}_f \end{aligned}$$

As an intersection of abelian groups, we have that G is an abelian group, using Proposition 4.4, when restricted to Yf. We have, by definition of $\hat{S}_f$, that ϕ is surjective. Let $g, h \in S$ such that $g \restriction_{Yf}, h \restriction_{Yf} \in G$ and $\phi(g) = \phi(h)$. Let $x \in Yf$. We have

$$\hat{g}_f = \hat{h}_f \implies xg = xh \implies g \restriction_{Yf} = h \restriction_{Yf}.$$

Hence ϕ is injective. Let $g, h \in S$ such that $g \restriction_{Yf}, h \restriction_{Yf} \in G$. Let $x \in Yf$. We have

$$\begin{aligned} (x)\phi(g \restriction_{Yf} h \restriction_{Yf}) &= x\widehat{gh}_f \\ &= xgh \\ &= x\hat{g}_f\hat{h}_f \\ &= (x)\phi(g \restriction_{Yf})\phi(h \restriction_{Yf}) \end{aligned}$$

Hence ϕ is a homomorphsim. We conclude that ϕ is an isomorphism from G to $\hat{S}_f$, and hence $\hat{S}_f$ is a group. Since G is abelian, we have that $\hat{S}_f$ is an abelian group. By Proposition 4.4, we have that $G \restriction_{\bar{x}}$ has regular action, for any $\bar{x} \in \bar{Y}\bar{f}$. Let $x, y \in \bar{x}$. By regularity of the action of $G \restriction_{\bar{x}}$, there is a unique $g \in G$ such that $xg = y$. Note that, since $x \in Yf$, we have

$$xg = y \implies x\hat{g}_f = y,$$

so $\hat{S}_f \restriction_{\bar{x}}$ is transitive. As a transitive abelian group, we have that $\hat{S}_f \restriction_{\bar{x}}$ has regular action, by Lemma 4.3.

Let $g \in G$, such that $g = hk$ for some $h, k \in S \restriction_{\bar{Y}\bar{f}}$. Let $\bar{x} \in \bar{Y}\bar{f}$. Suppose, for contradiction, that $\bar{x}\bar{h} = \bar{y} \neq \bar{x}$. Then, since $\bar{x}\bar{g} = \bar{x}$, as $g \in \mathrm{Stab}(\bar{x})$, we have that $\bar{y}\bar{k} = \bar{x}$. Hence $\bar{x}, \bar{y}$ are strongly connected, a contradiction to the fact that they are SCCs. Therefore, the list of generators in A that product to g all stabilise all $\bar{x} \in \bar{Y}\bar{f}$. Hence G is generated by

$$\left\{ g \restriction_{Yf} : \ g \in A \cap \left(\bigcap_{\bar{x} \in \bar{Y}\bar{f}} \mathrm{Stab}(\bar{x}) \right) \cup \{\mathrm{id}\} \right\}.$$

Applying ϕ to this gives

$$\left\{ \hat{g}_f : \ g \in A \cap \left(\bigcap_{\bar{x} \in \bar{Y}\bar{f}} \mathrm{Stab}(\bar{x}) \right) \cup \{\mathrm{id}\} \right\} = \hat{A}_f.$$

6 Membership Testing in Semilattices

Definition 6.1 Let S be a semigroup. An element $g \in S$ is an *idempotent* if $g^2 = g$. A *semilattice* is a commutative semigroup, such that every element is an idempotent.

Lemma 6.2 *Let S be a transformation semilattice acting on a finite set X, and generated by a set A. Let $f \in C_S$. Let*

$$B = \{a \in A : \ af = f\}.$$

Then $\exists C \subseteq A$, such that

$$f = \prod_{c \in C} c,$$

and

$$f \neq \prod_{c \in C \cup \{d\}} c,$$

for any $d \in A \backslash C$. In addition, $C = B$.

Proof Define C to be the union of all subsets $D \subseteq A$ such that

$$f = \prod_{d \in D} d.$$

Let D_1 be one of these subsets. We have that $C = D_1 \cup C \backslash D_1$ and that

$$f = \prod_{d \in D_1} d.$$

Let $g \in C \backslash D_1$. Then $\exists D_2 \subseteq A$ which products to f and contains g.

$$g \prod_{d \in D_1} d = gf = g \prod_{d \in D_2} d = \prod_{d \in D_2} d = f.$$

By repeating this, it follows that C products to f, due to the finiteness of A. Let $h \in A \notin C$. As h is not in any factorisation of f, we have that

$$f \neq \prod_{c \in C \cup \{h\}} c.$$

Let $b \in B$. We have that $bf = f$ and hence

$$f = \prod_{c \in C \cup \{b\}} c,$$

which implies $b \in C$. Hence $B \subseteq C$.

Let $c \in C$. Then

$$cf = c \prod_{c \in C} c = \prod_{c \in C} c = f,$$

so $c \in B$, and we have that $C \subseteq B$. Hence $B = C$.

Theorem 6.3 *Let S be a transformation semilattice acting on a finite set X, and generated by a set A. Let $f \in C_S$. Let*

$$B = \{a \in A : \; af = f\}.$$

Then $f \in S$ if and only if

$$f = \prod_{b \in B} b.$$

Proof $(\Rightarrow)$: Suppose $f \in S$. By Lemma 6.2, we have that $B = C$, which is a factorisation of f over A.
$(\Leftarrow)$: We have that $f \in S$, since f is a product of elements of S.

Algorithm 6.4 Let S be a transformation semilattice acting on a finite set X, and generated by a set A. Let $f \in C_S$. The following algorithm shows how to test if $f \in S$. The time complexity of this algorithm is $O(mn)$, where $m = |X|, \; n = |A|$.

1. Calculate B, as defined in Theorem 6.3.
2. Check if
$$f = \prod_{b \in B} b.$$
If so $f \in S$. Otherwise $f \notin S$.

Note that B is a factorisation for f into the generators of S.

7 Membership Testing for Transformations of Threshold 1

Lemma 7.1 *Let S be an commutative aperiodic transformation semigroup, generated by a set $A = \{a_1, a_2, \ldots, a_n\}$, acting on a finite set X. Let $t_1, t_2, \ldots, t_n$ be the thresholds of $a_1, a_2, \ldots, a_n$, respectively. Let $f \in C_S$ be an idempotent. Then $f \in S$ if and only if*

$$f \in \langle a_1^{t_1}, a_2^{t_2}, \ldots, a_n^{t_n} \rangle.$$

Proof $(\Rightarrow)$: Suppose $f \in S$. Then $f = a_1^{i_1} \cdots a_k^{i_k}$, for some $a_1, \ldots a_k \in A, \; i_1, \ldots i_k \in \mathbb{N}$. Let $b \in \{a_1, \ldots a_k\}$. Suppose $bf \neq f$. WLOG let $b = a_1$. We have that $\exists x \in X$ such that $xfa_1 \neq xf$. Let $y = xf, \; z = xfa_1$. We have

$$\begin{aligned} y &= xf \\ &= xf^2 \\ &= xfa_1^{i_1}a_2^{i_2} \cdots a_k^{i_k} \\ &= xfa_1a_1^{i_1-1}a_2^{i_2} \cdots a_k^{i_k} \\ &= za_1^{i_1-1}a_2^{i_2} \cdots a_k^{i_k} \end{aligned}$$

By definition of $y, \; z$, we have that $ya_1 = z$. Together with the above, we have that y and z are in the same SCC of S, a contradiction to Lemma 3.4.

Therefore, we have that $bf = f$.

$$\implies a_1^{t_1 - i_1} f = f$$

$$\implies a_1^{t_1-i_1} f \cdot a_k^{t_k-i_k} f = f$$

$$\implies f = a_1^{t_1-i_1} f \cdot a_k^{t_k-i_k} a_1^{i_1} \cdots a_k^{i_k} = a_1^{t_1} \cdots a_k^{t_k} \in \langle a_1^{t_1}, \ldots, a_n^{t_n} \rangle.$$

($\Leftarrow$): Suppose $f \in \langle a_1^{t_1}, \ldots, a_n^{t_n} \rangle$. Since this is a subset of S, we have that $f \in S$.

Lemma 7.2 *Let $f \in T_X$ for some finite set X. Then f has threshold less than or equal to 1 if and only if f permutes* $\operatorname{im} f$*.*

Proof ($\Rightarrow$): Suppose f has threshold less than or equal to 1. Let p be the period of f, $x \in \operatorname{im} f$. Hence $x = yf$ for some $y \in X$ and $f^{p+1} = f$. We have

$$f = f^{p+1} \implies yf = yf^{p+1} \implies x = xf^p.$$

Hence $f^p \restriction_{\operatorname{im} f} = \mathrm{id}_{\operatorname{im} f}$, and therefore $f \restriction_{\operatorname{im} f}$ is a permutation, since

$$|\operatorname{im} f \restriction_{\operatorname{im} f}| = \cdots = |\operatorname{im} f^p \restriction_{\operatorname{im} f}|, \qquad \operatorname{im} f^p \subseteq \cdots \subseteq \operatorname{im} f.$$

($\Leftarrow$): Suppose $f \restriction_{\operatorname{im} f}$ is a permutation. Then $(f \restriction_{\operatorname{im} f})^{o(f \restriction_{\operatorname{im} f})} = \mathrm{id} \restriction_{\operatorname{im} f}$. Let $x \in X$. We have that $xf \in \operatorname{im} f$. Hence

$$xf \cdot f^{o(f \restriction_{\operatorname{im} f})} = xf.$$

Therefore

$$f = f \cdot f^{o(f \restriction_{\operatorname{im} f})} = f^{o(f \restriction_{\operatorname{im} f})+1}.$$

Hence the threshold of g is less than or equal to 1. Since we know that $g \neq \mathrm{id}$, we have that g has threshold less than or equal to 1.

Theorem 7.3 *Let S be a commutative transformation semigroup, generated by a set A, acting on a finite set X. Let $f \in C_S$ have threshold less than or equal to 1. Suppose $\exists \lambda_f \in S^1$ such that $\bar{\lambda_f} = \bar{f}$. Then $f \in S^1$ if and only if $\hat{\mu_f} \in \hat{S}$ such that*

$$yf = y\lambda_f \hat{\mu_f}, \qquad \forall y \in Y.$$

Proof ($\Rightarrow$): Suppose $f \in S^1$. By Corollary 4.6, we have that the threshold of $\bar{f}$ is less than or equal to 1. Since $\bar{f}$ is aperiodic, it follows that $\bar{f}$ is an idempotent. Let $\bar{Y}$ denote the set of source-SCCs of S. We have $\forall \bar{x} \in \bar{Y}$ that

$$\bar{x}\bar{f}^2 = \bar{x}\bar{f} \implies \bar{x}f \subseteq \bar{x}f^2.$$

Since f is a permutation on its image, by Lemma 7.2, we have that $\bar{x}f = \bar{x}$, $\forall \bar{x} \in \bar{Y}\bar{f}$, and hence $f \in \mathrm{Stab}(\bar{x})$, $\forall \bar{x} \in \bar{Y}\bar{f}$. Note that since $\bar{\lambda_f} = \bar{f}$, we have that the threshold of λ_f is also less than or equal to 1, and hence $\lambda_f \in \mathrm{Stab}(\bar{x})$, $\forall \bar{x} \in \bar{Y}\bar{f}$. Try $\hat{\mu} = \hat{f}_f(\hat{\lambda_f}_f)^{-1}$. Let $y \in Y$. We have

$$\begin{aligned} y\lambda_f \hat{\mu} &= y\lambda_f \hat{f}_f(\hat{\lambda_f}_f)^{-1} \\ &= y\lambda_f f(\hat{\lambda_f}_f)^{-1} \\ &= yf\lambda_f(\hat{\lambda_f}_f)^{-1} \\ &= yf\hat{\lambda_f}_f(\hat{\lambda_f}_f)^{-1} \\ &= yf \end{aligned}$$

($\Leftarrow$): Suppose $\exists \lambda_f \in S^1,\ \hat{\mu_f} \in \hat{S}^1$ such that

$$\bar{f} = \bar{\lambda_f}, \quad yf = y\lambda_f\hat{\mu_f}, \qquad \forall y \in Y.$$

Let $g \in \bigcap_{\bar{x} \in \bar{Y}\bar{f}} \mathrm{Stab}(\bar{x}) \cup \{\mathrm{id}\}$ be such that $\hat{g}_f = \hat{\mu_f}$. Let $y \in Y$. We have

$$\bar{y}\bar{f} = \bar{y}\bar{\lambda_f}, \quad \bar{y}\bar{f}\bar{g} = \bar{y}\bar{f},$$

since g is a transformation which stabilises every SCC in $\bar{Y}\bar{f}$. We also have that

$$y\lambda_f \in \bar{y}\bar{\lambda_f} = \bar{y}\bar{f} \in \bar{Y}\bar{f}.$$

Hence

$$yf = y\lambda_f\hat{\mu_f} = y\lambda_f\hat{g}_f = y\lambda_f g.$$

By Lemma 4.8, we have $f = \lambda_f g$. Therefore $f \in S^1$.

Algorithm 7.4 Let S be a commutative transformation semigroup, generated by a set A, acting on a finite set X. Let $f \in T_X$ have threshold less than or equal to 1. The following algorithm shows how to test if $f \in S$ in polynomial time.

1. Check if f acts as the identity on S. If so, check if A contains an element that permutes the set $\bigcup_{a \in A} \mathrm{im}\, a$. If so and f is equal to the identity of S, then $f \in S$. Otherwise $f \notin S$.
2. Check if $af = fa, \quad \forall a \in A$. If not $f \notin S$.
3. Build $\bar{X}, Y, \bar{A}, \hat{A}_f$.
4. Check if $\bar{f} \in \bar{S}$. If not $f \notin S$. If so find a $\lambda_f \in S^1$, written as a product of elements of A, such that $\bar{f} = \bar{\lambda_f}$.
5. Construct a spanning tree T, with root xf for some $x \in X$, of the digraph with vertex set Yf, and an edge from any $y \in Yf$ to any $z \in Yf$, if $\exists a \in \hat{A}$ such that $ya = z$.
6. For each element of $\bar{y}$ of $\bar{Y}$ and any $x \in \bar{y}$, find the path in T that takes $x\lambda_f$ to the root, xf, and define $\hat{\mu_f}$, on this subset of Yf to be the product of the sequence of generators in this path.
7. Check if $yf = y\lambda_f\hat{\mu_f},\ \forall y \in Y$. If not $f \notin S$.
8. Check if $\hat{\mu_f} \in \hat{S}_f$. If so $f \in S$. Otherwise $f \notin S$.

If f acts as the identity on S, then Step 1 checks if S is a monoid, and if so, checks if f is the identity of S. This allows subsequent steps to assume S is a monoid. This uses Lemma 2.1.

Step 2 checks if $f \in C_S$, using Lemma 2.7.

Step 3 constructs the generating sets of the induced aperiodic semigroup and source action group of S, with respect to f. This uses Corollary 3.6 and Lemma 4.8.

Step 4 requires membership testing and factorisation in a semilattice. We have that $\bar{f}$ is aperiodic and has threshold less than or equal to 1, by Corollary 4.6. Therefore $\bar{f}$ is an idempotent. By Lemma 7.1, it suffices to check if $\bar{f}$ is in the semigroup generated by the generators of $\bar{S}$, raised to the power of their thresholds, which is a semilattice. Since the membership testing algorithm for semilattices in Algorithm 6.4 yields a factorisation, we can obtain a factorisation for $\bar{f}$ into powers of the generators of $\bar{S}$. Since the mapping $g \mapsto \bar{g}$ is a homomorphism, we can obtain a λ_f by producting and powering the generators in A, that are sent to the generators in $\bar{A}$, in the factorisation of $\bar{f}$.

Steps 5 and 6 are used to construct a transformation $\hat{\mu_f}$. It suffices to check one element of each SCC, because of the regular action of $\hat{S}_f$ on each SCC in $\bar{Y}\bar{f}$. Together with λ_f, we potentially have the conditions mentioned in Theorem 7.3, provided $yf = y\lambda_f\hat{\mu_f},\ \forall y \in Y$.

Step 7 checks is $f = \lambda_f\hat{\mu_f},\ \forall y \in Y$. Step 8 checks if $\hat{\mu_f} \in \hat{S}_f$. This can be done in polynomial time, using the Schreier-Sims algorithm.

8 Testing Membership in Semigroups of Threshold 1

Definition 8.1 A semigroup S is *regular* if $\forall g \in S,\ \exists h \in S$ such that $ghg = g$.

Note that this is not the same as having regular action.

Lemma 8.2 *Let S be a regular commutative transformation semigroup acting on a finite set. Then S has threshold less than or equal to 1.*

Proof Let $g \in S$. Then $\exists h \in S$ such that $ghg = g$. Hence $g^2h = g$. Let $x \in \operatorname{im} g$. We have $(xg)h = x,\ (x)g = (xg)$, and hence x, xg are in the same SCC. Therefore $\bar{g} = \bar{g}^2$, so $\bar{g}$ has threshold less than or equal to 1. By Corollary 4.6 we have that the threshold of g is less than or equal to 1.

Definition 8.3 A semigroup S is *completely regular* if every element of S lies in a subgroup of S.

Lemma 8.4 *Every completely regular semigroup is regular.*

Proof Let S be a completely regular semigroup. Let $g \in S$ be in a subgroup $G \leq S$. Let $h \in S$ be the element that acts as the inverse of g in G. Then

$$g = ghg.$$

Definition 8.5 A regular semigroup S is a *Clifford semigroup* if its idempotents are central.

Algorithm 8.6 Let S be a commutative transformation semigroup, acting on a finite set X, of threshold less than or equal to 1. Let $f \in T_X$. The following algorithm tests if $f \in S$ in polynomial time.

1. Check if f permutes its image. If not $f \notin S$.

2. Run Algorithm 7.4 to test if $f \in S$.

Step 1 uses Lemma 7.2 to test if f has threshold less than or equal to 1. Step 2 can therefore assume that f has threshold 1, and use Algorithm 7.4.

9 Counter-Example to Algorithm in [1]

Throughout this section, a transformation f of the set $\{1, \ldots, n\}$ will be represented as the list $(1f\ \ldots\ nf)$. The notation used throughout the paper will be mostly the same as the notation in [1].

Algorithm 9.1 The following algorithm is given in [1] as a general algorithm for membership testing in a commutative transformation semigroup. We have adapted the notation to be consistent with the notation defined throughout this paper. Here the semigroup in question will be S, generated by a set A, and acting on a finite set X. The test transformation will be $f \in T_X$.

1. Test whether $af = fa$ for every $a \in A$.
2. Build $\bar{X},\ \bar{Y},\ \bar{A},\ \hat{A}_f$, the latter two generate the semigroups $\bar{S},\ \hat{S}_f$.
3. Test whether $\bar{f} \in \bar{S}$. If so, find a $\lambda_f \in S$ such that $\bar{\lambda_f} = \bar{f}$.
4. Build a transformation $\hat{\mu_f}$ such that for every $x \in Y,\ x\lambda_f\hat{\mu_f} = xf$.
5. Test whether $\hat{\mu_f} \in \hat{S}_f$.

Example 9.2 Here we will give a counter-example to Algorithm 9.1.

Let $X = \{1, \ldots, 7\}$. Let $A = \{a, b, c\}$, where

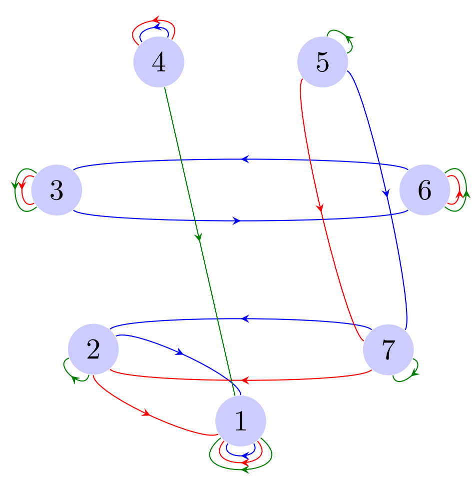
$$a = (1\ 1\ 6\ 4\ 7\ 3\ 2),\ b = (1\ 1\ 3\ 4\ 7\ 6\ 2),\ c = (1\ 2\ 3\ 1\ 5\ 6\ 7).$$

Let $S = \langle A \rangle$. Note that

$$ab = (1\ 1\ 6\ 4\ 2\ 3\ 1) = ba,\ ac = (1\ 1\ 6\ 1\ 7\ 3\ 2) = ca,\ bc = (1\ 1\ 3\ 1\ 7\ 6\ 2) = cb.$$

Hence S is commutative. Let $f = (1\ 1\ 3\ 1\ 7\ 6\ 2)$. We have

$$af = (1\ 1\ 6\ 1\ 2\ 3\ 1) = fa,\ bf = (1\ 1\ 3\ 1\ 2\ 6\ 1) = fb,\ cf = (1\ 1\ 3\ 1\ 7\ 6\ 2) = fc,$$



We therefore have that the set of SCCs is

$$\bar{X} = \{\{1\},\ \{3,6\},\ \{4\},\ \{5\},\ \{7\}\},$$

and the set of source-SCCs is

$$\bar{Y} = \{\{3,6\},\ \{4\},\ \{5\}\}.$$

Note that

$$\bar{f} = \begin{pmatrix} \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} & \bar{7} \\ \bar{1} & \bar{1} & \bar{3} & \bar{1} & \bar{7} & \bar{2} \end{pmatrix},$$

$$\bar{a} = \begin{pmatrix} \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} & \bar{7} \\ \bar{1} & \bar{1} & \bar{3} & \bar{4} & \bar{7} & \bar{2} \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} & \bar{7} \\ \bar{1} & \bar{1} & \bar{3} & \bar{4} & \bar{7} & \bar{2} \end{pmatrix}, \quad \bar{c} = \begin{pmatrix} \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} & \bar{7} \\ \bar{1} & \bar{2} & \bar{3} & \bar{1} & \bar{5} & \bar{7} \end{pmatrix}.$$

Hence $\bar{Y}\bar{f} = \{\bar{3},\ \bar{1},\ \bar{7}\}$ and $Yf = \{1,\ 3,\ 6,\ 7\}$. The only $a \in A$ which fixes all SCCs in $\bar{Y}\bar{f}$ is $c = (1\ 2\ 3\ 1\ 5\ 6\ 7)$. Note that $\hat{c}_f = id$. Hence $\hat{S}_f = \langle \mathrm{id} \rangle = \{\mathrm{id}\}$.

The elements of S are

$$(1\ 1\ 6\ 4\ 7\ 3\ 2),\ (1\ 1\ 3\ 4\ 7\ 6\ 2),\ (1\ 2\ 3\ 1\ 5\ 6\ 7),\ (1\ 1\ 3\ 4\ 2\ 6\ 1),\ (1\ 1\ 6\ 4\ 2\ 3\ 1),$$

$$(1\ 1\ 6\ 1\ 7\ 3\ 2),\ (1\ 1\ 3\ 1\ 7\ 6\ 2),\ (1\ 1\ 6\ 4\ 1\ 3\ 1),\ (1\ 1\ 3\ 4\ 1\ 6\ 1),$$

$$(1\ 1\ 3\ 1\ 2\ 6\ 1),\ (1\ 1\ 6\ 1\ 2\ 3\ 1),\ (1\ 1\ 6\ 1\ 1\ 3\ 1),\ (1\ 1\ 3\ 1\ 1\ 6\ 1).$$

Therefore, the only potential candidates for λ_f, as mentioned in [1], are

$$(1\ 1\ 6\ 1\ 7\ 3\ 2),\ (1\ 1\ 3\ 1\ 7\ 6\ 2),$$

the latter one being f.

Try taking $\lambda_f = (1\ 1\ 6\ 1\ 7\ 3\ 2)$. The algorithm in [1], suggests that $\exists \hat{\mu_f} \in \hat{S}_f$ such that $\forall y \in Y,\ yf = y\lambda_f\hat{\mu_f}$. However, since the only potential candidate for $\hat{\mu_f}$ is id, we have that

$$3\lambda_f\hat{\mu_f} = 6 \neq 3f, \quad \forall \hat{\mu_f} \in \hat{S}_f.$$

Therefore by the algorithm in [1], it should follow that $f \notin S$, which is not true.

References

[1] Martin Beaudry. Membership testing in commutative transformation semigroups. *Information and Computing*, 79:84–93, October 1988.

[2] J. D. Mitchell et al. *Semigroups - GAP package, Version 3.0.3*, Jun 2017.