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Anneaux de Grothendieck quantiques, algèbres amassées et catégorie $\mathcal{O}$ affine quantique

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Résumé/Abstract

Résumé

L'objectif de cette thèse est de construire et d'étudier une structure d'anneau de Grothendieck quantique pour une catégorie $\mathcal{O}$ de représentations de la sous-algèbre de Borel $\mathcal{U}_q(\hat{\mathfrak{b}})$ d'une algèbre affine quantique $\mathcal{U}_q(\hat{\mathfrak{g}})$.

On s'intéresse dans un premier lieu à la construction de modules standards asymptotiques pour la catégorie $\mathcal{O}$, qui sont des analogues des modules standards existant dans la catégorie des représentations de dimension finie de $\mathcal{U}_q(\hat{\mathfrak{g}})$. Une construction complète de ces modules est proposée dans le cas où l'algèbre de Lie simple sous-jacente $\mathfrak{g}$ est $\mathfrak{sl}_2$.

Ensuite, nous définissons un tore quantique qui étend le tore quantique contenant l'anneau de Grothendieck quantique de la catégorie des représentations de dimension finie. Nous utilisons pour cela des notions liées aux algèbres amassées quantiques. Dans le même esprit, nous proposons une construction d'une structure d'algèbre amassée quantique sur l'anneau de Grothendieck quantique $K_t(\mathscr{C}_{\mathbb{Z}}^-)$ d'une sous-catégorie monoïdale $\mathscr{C}_{\mathbb{Z}}^-$ de la catégorie des représentations de dimension finie.

Puis, nous définissons un anneau de Grothendieck quantique $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ d'une sous-catégorie $\mathcal{O}_{\mathbb{Z}}^+$ de la catégorie $\mathcal{O}$, comme une algèbre amassée quantique. Nous établissons ensuite que cet anneau de Grothendieck quantique contient celui de la catégorie des représentations de dimension finie. Ce résultat est montré directement en type A, puis en tout type simplement lacé en utilisant la structure d'algèbre amassée quantique de $K_t(\mathscr{C}_{\mathbb{Z}}^-)$.

Enfin, nous définissons des (q,t)-caractères pour des représentations simples de dimension infinie remarquables de la catégorie $\mathcal{O}$. Ceci nous permet d'écrire des versions t-déformées de relations importantes dans l'anneau de Grothendieck classique de la catégorie $\mathcal{O}_{\mathbb{Z}}^+$, qui ont des liens avec les systèmes intégrables quantiques associés.

Mots-clefs

Groupes quantiques, théorie des représentations, algèbres affines quantiques, catégorie $\mathcal{O}$, anneaux de Grothendieck quantiques, algèbres amassées quantiques.

Quantum Grothendieck rings, cluster algebras and quantum affine category $\mathcal{O}$

Abstract

The aim of this thesis is to construct and study some quantum Grothendieck ring structure for the category $\mathcal{O}$ of representations of the Borel subalgebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ of a quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$.

First of all, we focus on the construction of asymptotical standard modules, analogs in the context of the category $\mathcal{O}$ of the standard modules in the category of finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules. A construction of these modules is given in the case where the underlying simple Lie algebra $\mathfrak{g}$ is $\mathfrak{sl}_2$.

Next, we define a new quantum torus, which extends the quantum torus containing the quantum Grothendieck ring of the category of finite-dimensional modules. In order to do this, we use notions linked to quantum cluster algebras. In the same spirit, we build a quantum cluster algebra structure on the quantum Grothendieck ring of a monoidal subcategory $\mathscr{C}_{\mathbb{Z}}^-$ of the category of finite-dimensional representations.

With this quantum torus, we define the quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ of a subcategory $\mathcal{O}_{\mathbb{Z}}^+$ of the category $\mathcal{O}$ as a quantum cluster algebra. Then, we prove that this quantum Grothendieck ring contains that of the category of finite-dimensional representation. This result is first shown directly in type A, and then in all simply-laced types using the quantum cluster algebra structure of $K_t(\mathscr{C}_{\mathbb{Z}}^-)$.

Finally, we define (q,t)-characters for some remarkable infinite-dimensional simple representations in the category $\mathcal{O}_{\mathbb{Z}}^+$. This enables us to write t-deformed analogs of important relations in the classical Grothendieck ring of the category $\mathcal{O}$, which are related to the corresponding quantum integrable systems.

Keywords

Quantum groups, representation theory, quantum affine algebras, category $\mathcal{O}$, quantum Grothendieck rings, quantum cluster algebra.

Contents

Introduction

I.1 Contexte

I.1.1 Spectres des systèmes intégrables quantiques

Les motivations originelles des questions étudiées dans ce travail de thèse se situent en mécanique statistique et dans l'étude des systèmes intégrables quantiques.

Le modèle qui nous intéresse particulièrement est le modèle à 6 sommets, aussi appelé modèle de la glace. Il a été introduit par Pauling en 1935 pour décrire certaines structures cristallines, en particulier celle des cristaux de glace. Ce système est un important objet d'étude, pour les physiciens autant que pour les mathématiciens (voir par exemple [Bax82, Chapitre 8] et [Res10]).

Le système est modélisé par un réseau à deux dimensions pour lequel chaque sommet est relié à exactement quatre autres sommets. Un état du système consiste en un choix d'orientation de chacune des arêtes du réseau. De plus, on impose une condition appelée *condition de la glace* : les arêtes sont orientées de façon à ce que chaque sommet ait exactement deux flèches entrantes et deux flèches sortantes. Ainsi, chaque sommet a six configurations possibles, d'où le nom de ce modèle.

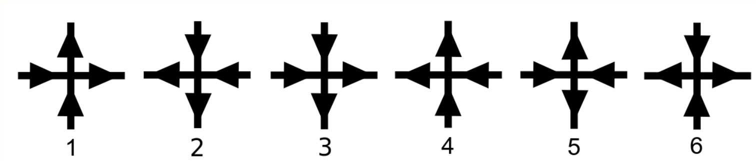


FIGURE 1 – Configurations possibles

L'état physique du système, c'est-à-dire le choix d'orientation de chaque arête, est encodé par une fonction $\mathcal{Z}$ appelée *fonction de partition*. Les systèmes considérés comportent généralement un nombre gigantesque de particules. Ainsi, une manière d'accéder à cette fonction de partition est d'étudier le comportement moyen, ou macroscopique, du système. La fonction de partition peut être écrite comme la trace d'un opérateur $\mathcal{T}$ appelé *matrice de transfert* :

$$\mathcal{Z} = \mathrm{Tr}_W(\mathcal{T}),$$

où W est l'espace des états. Donc, dans le but d'étudier $\mathcal{Z}$, on cherche à obtenir les valeurs propres de l'opérateur $\mathcal{T}$:

$$\mathrm{Tr}_W(\mathcal{T}) = \sum_j \lambda_j.$$

En 1967, à l'aide d'une technique appelée *Ansatz de Bethe*, Lieb [Lie67] a résolu le modèle à six sommets en construisant explicitement les vecteurs propres. Puis, en 1972, Baxter [Bax72] a résolu algébriquement le modèle à huit sommets, qui est une généralisation du modèle à six sommets, affranchi de la condition de la glace. Il a montré que les valeurs propres de la matrice de transfert avaient une forme particulière :

$$\lambda_j = A(z)\frac{Q_j(zq^2)}{Q_j(z)} + D(z)\frac{Q_j(zq^{-2})}{Q_j(z)}, \tag{I.1.1.1}$$

où z et q sont des paramètres du système, les fonctions $A(z)$ et $D(z)$ sont universelles, et le seul terme dépendant de la valeur propre est $Q_j(z)$, qui est une fonction polynomiale. La relation (I.1.1.1) est appelée *relation de Baxter*. Elle sera importante dans la suite.

L'approche de Baxter pour comprendre ce modèle est basée sur une famille commutative de matrices de transfert. La condition de commutation de deux de ces matrices pouvant être traduite par une relation, appelée par Baxter *"star-triangle relation"*.

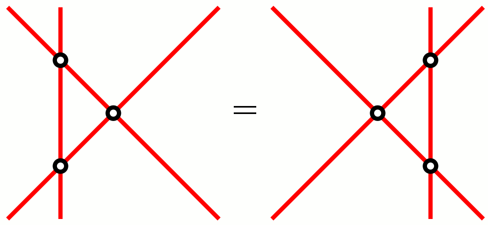


FIGURE 2 – Relation de Yang-Baxter

Cette relation est plus connue sous le nom de *Relation de Yang-Baxter*. Les travaux de Baxter ont mené à d'importantes généralisations des techniques de Bethe, et ont permis de mettre en évidence la richesse de la structure algébrique de ce modèle. Quand Sklyanin [Skl79], simultanément avec Kulish et Reshetikhin [KRS81], ont mis au jour le lien entre la relation de Yang-Baxter dans le modèle à six sommets et la quantification de systèmes intégrables classiques, cela a ouvert la voie pour de fructueuses recherches dans le domaine des systèmes intégrables quantiques dans les décennies qui ont suivi (voir [JM95] ou [KBI93] pour des vues d'ensemble). C'est dans la continuité de ces questions que les groupes quantiques émergeront.

I.1.2 Algèbres affines quantiques

Dans les années 1980, Drinfeld [Dri87] et Jimbo [Jim85] ont indépendamment défini, pour toute algèbre de Lie simple de dimension finie $\mathfrak{g}$, une famille d'algèbres de Hopf $\mathcal{U}_q(\mathfrak{g})$, dépendant d'un paramètre $q \in \mathbb{C}^\times$, appelées *algèbres enveloppantes universelles quantifiées*. Il s'agit de q-déformations de l'algèbre enveloppante universelle $\mathcal{U}(\mathfrak{g})$ de l'algèbre de Lie $\mathfrak{g}$.

Une des caractéristiques principales de ces types de groupes quantiques réside dans le fait qu'à toute représentation de dimension finie V de $\mathcal{U}_q(\mathfrak{g})$, l'on puisse associer un opérateur $\mathcal{R} \in \mathrm{End}(V \otimes V)$ qui satisfait l'équation de Yang-Baxter :

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}, \tag{I.1.2.1}$$

où $\mathcal{R}_{12} = \mathcal{R} \otimes \mathrm{Id} \in \mathrm{End}(V \otimes V \otimes V)$, et $\mathcal{R}_{13}$ et $\mathcal{R}_{23}$ sont définis de manière similaire. La relation (I.1.2.1) peut aussi être représentée par le diagramme de la Figure 2.

Intéressons-nous maintenant à l'algèbre de Kac-Moody affine (non-twistée) $\hat{\mathfrak{g}}$ correspondant à l'algèbre de Lie simple $\mathfrak{g}$, il s'agit d'une extension centrale $\hat{g} = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c$ de l'algèbre des lacets

$$\mathcal{L}(\mathfrak{g}) := \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}].$$

Kac [Kac68] et Moody [Moo69] ont montré indépendamment que l'algèbre $\hat{\mathfrak{g}}$ avait une présentation analogue à la Serre des algèbres simples complexes de dimension finie. Ceci qui permet d'appliquer la construction générique des groupes quantiques de Drinfeld et Jimbo à $\hat{\mathfrak{g}}$. L'algèbre enveloppant q-déformée obtenue s'appelle *l'algèbre affine quantique* $\mathcal{U}_q(\hat{\mathfrak{g}})$.

Dans [Dri88], Drinfeld donna une nouvelle réalisation de l'algèbre affine quantique par un procédé d'affinisation quantique de l'algèbre enveloppante universelle quantifiée de l'algèbre de Lie simple $\mathfrak{g}$ correspondante à $\hat{\mathfrak{g}}$. À l'origine de cette nouvelle présentation se trouvent des théories de Lusztig sur l'action des groupes de tresses sur l'algèbre enveloppante quantifiée. Cette présentation implique une infinité de générateurs, semblables aux générateurs de l'algèbre des lacets ; si $I = \{1, \ldots, n\}$ avec n le rang de l'algèbre de Lie simple $\mathfrak{g}$, ces derniers s'écrivent :

$$x_{i,r}^{\pm} \quad (i \in I, r \in \mathbb{Z}), \quad \phi_{i,\pm m}^{\pm} \quad (i \in I, m \geqslant 0), \quad k_i^{\pm 1} \quad (i \in I), \tag{I.1.2.2}$$

où $x_{i,r}^{\pm}$ est l'équivalent quantique de l'élément $x_i^{\pm 1} \otimes t^r$ de l'algèbre des lacets $\mathcal{L}(\mathfrak{g})$.

Drinfeld énonça l'existence d'un isomorphisme entre cette nouvelle présentation et la présentation classique, et Beck [Bec94] donna un isomorphisme explicite entre les deux présentations (précisé plus récemment par Damiani dans [Dam15]). Ainsi, les deux procédés, d'affinisation puis de quantification, ou de quantification, puis d'affinisation quantique commutent. Ceci se résume par le diagramme commutatif suivant :

$$\begin{tikzcd}
 & \mathfrak{g} \arrow[dl, "\text{quantification}"'] \arrow[dr, "\text{affinisation}"] & \\
\mathcal{U}_q(\mathfrak{g}) \arrow[dr, "\text{affinisation quantique}"'] & & \hat{\mathfrak{g}} \arrow[dl, "\text{quantification}"] \\
 & \mathcal{U}_q(\hat{\mathfrak{g}}) &
\end{tikzcd}.$$

De plus, Beck [Bec94] explicita une *décomposition triangulaire* de $\mathcal{U}_q(\hat{\mathfrak{g}})$, grâce à la nouvelle présentation de Drinfeld :

$$\mathcal{U}_q(\hat{\mathfrak{g}}) \simeq \mathcal{U}_q(\hat{\mathfrak{g}})^- \otimes \mathcal{U}_q(\hat{\mathfrak{g}})^0 \otimes \mathcal{U}_q(\hat{\mathfrak{g}})^+, \tag{I.1.2.3}$$

où $\mathcal{U}_q(\hat{\mathfrak{g}})^{\pm}$ (resp. $\mathcal{U}_q(\hat{\mathfrak{g}})^0$) est la sous-algèbre de $\mathcal{U}_q(\hat{\mathfrak{g}})$ engendrée par les générateurs $x_{i,r}^{\pm}$ (resp. $\phi_{i,\pm m}^{\pm}, k_i^{\pm 1}$) donnés en (I.1.2.2).

Drinfeld montra [Dri85] que l'algèbre affine quantique possède une R-matrice universelle $\mathcal{R}$. Il s'agit d'une solution de l'équation de Yang-Baxter (I.1.2.1), appartenant à une complétion du produit tensoriel de deux copies de $\mathcal{U}_q(\hat{\mathfrak{g}})$:

$$\mathcal{R}(z) \in \mathcal{U}_q(\hat{\mathfrak{g}}) \hat{\otimes}\, \mathcal{U}_q(\hat{\mathfrak{g}})[[z]]. \tag{I.1.2.4}$$

Une formule explicite pour la R-matrice universelle a été obtenue par d'autres auteurs, d'une part par Khoroshkin et Tolstoy [KT94] et par une autre méthode par Levendorsky-Soibelman-Stukopin [LSS93] pour $\mathfrak{g} = \mathfrak{sl}_2$, puis Damiani dans le cas général [Dam98].

La théorie des représentations des algèbres affines quantiques est très riche et a été intensivement étudiées ces 30 dernières années, avec des liens vers de très nombreux domaines des mathématiques (géométrie, topologie, combinatoire, etc) et de la physique (théorie conforme des champs, systèmes intégrables, etc).

I.1.3 Représentations de dimension finie des algèbres affines quantiques

Rappelons ici quelques résultats importants de l'étude des représentations de dimension finie des algèbres affines quantiques, qui est un domaine de recherche très fertile, comportant encore de nombreuses questions ouvertes.

Soit $\mathscr{C}$ la catégorie des représentations de dimension de l'algèbre affine quantique $\mathcal{U}_q(\hat{\mathfrak{g}})$. Comme $\mathcal{U}_q(\hat{\mathfrak{g}})$ est une algèbre de Hopf, le produit tensoriel de deux de représentations de $\mathcal{U}_q(\hat{\mathfrak{g}})$ possède aussi une structure de $\mathcal{U}_q(\hat{\mathfrak{g}})$-module, et $\mathscr{C}$ est une *catégorie tensorielle.*

Une question importante du domaine est la description et la classification des représentations simples de la catégorie $\mathscr{C}$, c'est-à-dire des représentations irréductibles de dimension finie de $\mathcal{U}_q(\hat{\mathfrak{g}})$. Quand $\mathfrak{g} = \mathfrak{sl}_2$, cette question a été complètement résolue par Chari et Pressley dans [CP91], qui ont donné une description explicite des représentations irréductibles. Pour les autres types, cette question est beaucoup plus difficile, et reste encore majoritairement ouverte, bien qu'elle ait été considérablement étudiée. Parmi les nombreuses références sur le sujet, on peut en particulier citer [CP95a], [CP91], [CP95b], [CP96], [GV93], [Vas98], [KS95], [AK97], [FR96], et [Nak04], mais cette liste n'est bien évidemment pas exhaustive.

En comparaison, dans la limite classique $q \to 1$, la classification des représentations simples de dimension finie de $\hat{\mathfrak{g}}$ est beaucoup plus accessible. Pour tout $a \in \mathbb{C}^\times$, considérons le *morphisme d'évaluation* $ev_a : \hat{\mathfrak{g}} \to \mathfrak{g}$, défini en évaluant les polynômes de Laurent de l'algèbre des lacets $\mathcal{L}(\mathfrak{g})$ en a. Le tiré en arrière par ev_a de n'importe quel $\mathfrak{g}$-module irréductible $V(\lambda)$ de plus haut poids λ est un $\hat{\mathfrak{g}}$-module irréductible noté $V_a(\lambda)$. Alors, toutes les représentations irréductibles (à isomorphisme près) de $\hat{\mathfrak{g}}$ sont de la forme

$$V_{a_1}(\lambda_1) \otimes V_{a_2}(\lambda_2) \otimes \cdots \otimes V_{a_k}(\lambda_k),$$

avec $a_i \neq a_j$ pour $i \neq j$ [Cha86]. Dans le cas quantique, le morphisme d'évaluation ev_a ne s'étend pas en un morphisme $\mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\hat{\mathfrak{g}})$ si $\mathfrak{g}$ n'est pas de type A.

Néanmoins, les représentations de dimension finie de $\mathcal{U}_q(\hat{\mathfrak{g}})$ ont tout de même une classification qui peut être vue comme un analogue quantique de la classification de Cartan des représentations simples de dimension finie de $\mathfrak{g}$ par leur plus haut poids.

La partie $\mathcal{U}_q(\hat{\mathfrak{g}})^0$ de la décomposition triangulaire (I.1.2.3) joue le rôle de la sous-algèbre de Cartan. Cette algèbre, appelée *sous-algèbre de Drinfeld-Cartan*, agit comme une sous-algèbre commutative de $\mathcal{U}_q(\hat{\mathfrak{g}})$ sur les représentations de dimension finie. Un vecteur v d'un $\mathcal{U}_q(\hat{\mathfrak{g}})$-module est dit de *$\ell$-poids* $\mathbf{\Psi}$ s'il appartient à un sous-espace de Jordan pour l'action simultanée des générateurs $\{\phi^\pm_{i,\pm m}, k_i^{\pm 1}\}_{i\in I, m\geqslant 0}$ de la sous-algèbre de Drinfeld-Cartan. Un *ℓ-poids* est donc une famille $(\Psi^\pm_{i,\pm m})_{m\geqslant 0}$, que l'on écrit par sa fonction génératrice :

$$\mathbf{\Psi} = (\Psi_i(z))_{i\in I}, \quad \text{où } \Psi_i(z) = \sum_{m\geqslant 0} \Psi_{i,\pm m} z^{\pm m}.$$

Un vecteur v d'un $\mathcal{U}_q(\hat{\mathfrak{g}})$-module est dit *de plus haut ℓ-poids* s'il est de ℓ-poids et s'il est annulé par les générateurs $x^+_{i,r}$ de la partie positive de la décomposition triangulaire (I.1.2.3).

Chari et Pressley [CP95b] ont montré le résultat fondamental suivant : toute représentation irréductible de $\mathcal{U}_q(\hat{\mathfrak{g}})$ est engendrée par un vecteur de plus haut ℓ-poids, de plus, une représentation irréductible engendrée par un vecteur de plus haut ℓ-poids $\boldsymbol{\Psi}$ est de dimension finie si et seulement si celui-ci est de la forme :

$$\boldsymbol{\Psi} = (\Psi_i(z))_{i\in I}, \quad \Psi_i(z) = q_i^{\deg(P_i)} \frac{P_i(zq_i^{-1})}{P_i(zq_i)}, \tag{I.1.3.1}$$

où les P_i sont des polynômes de coefficients constants égaux à 1, appelés *polynômes de Drinfeld*, et $q_i := q^{d_i}$, où $D = \mathrm{diag}(d_1, \dots, d_n)$ est l'unique matrice diagonale telle que $B = DC$ est symétrique et les d_i sont des entiers positifs premiers entre eux.

En particulier, pour tout $i \in I$, et tout $a \in \mathbb{C}^\times$, si on considère le n-uplet de polynômes :

$$(1, \dots, 1, \underbrace{1 - az}_{i}, 1, \dots, 1), \tag{I.1.3.2}$$

la représentation irréductible de dimension finie associée, dite *représentation fondamentale*, est de plus haut poids ω_i comme $\mathcal{U}_q(\mathfrak{g})$-module, on la note $V_i(a)$. En type A, il s'agit d'une représentation d'évaluation. Pour les autres types, comme mentionné précédemment il n'existe pas de morphisme d'évaluation, et en particulier $V_i(a)$ est en général plus grand que $V(\omega_i)$ comme $\mathcal{U}_q(\mathfrak{g})$-module. Il s'agit là d'une différence fondamentale entre le cas classique et le cas quantique, et un des points de difficulté, mais aussi d'intérêt, de l'étude des représentations des algèbres affines quantiques.

La description des représentations simples de $\mathcal{U}_q(\hat{\mathfrak{g}})$ donnée par Chari et Pressley nous donne une paramétrisation efficace de ces modules, malheureusement elle ne donne que peu d'informations vis-à-vis de leur structure. Par exemple, leur dimension n'est pas connue en général.

I.1.4 Anneaux de Grothendieck et q-caractères pour les représentations de dimension finie

Notons que la catégorie $\mathscr{C}$ n'est pas semi-simple. Elle admet un anneau de Grothendieck $K_0(\mathscr{C})$, qui est défini de la manière suivante (voir [EGNO15] pour plus de détails). Pour tout objet simple (représentation irréductible) X_i de la catégorie $\mathscr{C}$ et tout objet Y, on note $[Y : X_i]$ la multiplicité de X_i dans la suite de composition de Y (la catégorie $\mathscr{C}$ est localement finie).

On définit le *groupe de Grothendieck* comme le groupe abélien libre engendré par les classes d'isomorphismes $[X_i]$ des objets simples (représentations irréductibles) de $\mathscr{C}$. A chaque objet X de $\mathscr{C}$, on associe sa classe $[X]$ dans le groupe de Grothendieck, définie par

$$[X] = \sum_i [X : X_i][X_i].$$

Comme la catégorie $\mathscr{C}$ est abélienne, pour chaque suite exacte courte $0 \to X \to Y \to Z \to 0$, on a la relation $[Y] = [X] + [Z]$ dans le groupe de Grothendieck. Comme de plus la catégorie $\mathscr{C}$ est linéaire et monoïdale, et que le foncteur produit tensoriel est biexact, alors ce produit tensoriel induit une loi multiplicative associative sur le groupe de Grothendieck de la catégorie $\mathscr{C}$. celle-ci est définie par, pour tous objets simples X_i et X_j de $\mathscr{C}$:

$$[X_i] \cdot [X_j] = \sum_k [X_i \otimes X_j : X_k][X_k]$$

L'anneau de Grothendieck $K_0(\mathscr{C})$ est l'anneau formé par le groupe de Grothendieck muni de cette multiplication.

Dans [FR96] et [FR99], Frenkel et Reshetikhin ont défini des $\mathcal{W}$-algèbres *déformées*. Dans le cas classique ($q = 1$), les $\mathcal{W}$-algèbres permettent de décrire le centre de l'algèbre enveloppante $\mathcal{U}(\hat{\mathfrak{g}})$ comme une intersection de noyaux d'opérateurs dits "d'écrantage". Pour le cas quantique, l'anneau de Grothendieck $K_0(\mathscr{C})$ de la catégorie $\mathscr{C}$ peut être vu comme une $\mathcal{W}$-algèbre q-déformée.

Dans ce contexte, ils ont pu définir un morphisme d'anneaux injectif, appelé morphisme de *q-caractère* sur l'anneau de Grothendieck de la catégorie $\mathscr{C}$ vers un anneau de polynômes de Laurent en un nombre infini de variables :

$$\chi_q : K_0(\mathscr{C}) \to \mathcal{Y} = \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^\times}. \tag{I.1.4.1}$$

La construction de ce morphisme en elle-même est basée sur l'existence de la R-matrice universelle $\mathcal{R}$ évoquée en (I.1.2.4). Pour toute représentation de dimension finie (V, ρ_V) de $\mathcal{U}_q(\hat{\mathfrak{g}})$, on peut construire une matrice de transfert associée à V de la façon suivante :

$$t_V(z) = \mathrm{Tr}_V\left(\rho_{V(z)} \otimes \mathrm{Id})(\mathcal{R}\right) \quad \in \mathcal{U}_q(\hat{\mathfrak{g}})[[z]], \tag{I.1.4.2}$$

où $V(z)$ est une version twistée de V, et z est un paramètre formel.

Remarquons que ces matrices de transfert sont celles évoquées en Section I.1.1 dans la construction de Baxter. En particulier, elles commutent, pour tous V, V', z, z',

$$\left[t_V(z), t_{V'}(z')\right] = 0. \tag{I.1.4.3}$$

Si $\mathfrak{g} = \mathfrak{sl}_2$ et V est une représentation irréductible de dimension 2, et W est un produit tensoriel de N copies de V, alors t_V agissant sur W devient précisément l'opérateur de Baxter.

Le morphisme de q-caractère est construit à partir des opérateurs t_V ainsi qu'un analogue du morphisme d'Harish-Chandra.

De façon cruciale, le morphisme obtenu rend compte de la décomposition des représentations en espaces de ℓ-poids. Précisons cela. Frenkel et Reshetikhin ont montré [FR99] que les ℓ-poids des représentations de dimension finie de $\mathcal{U}_q(\hat{\mathfrak{g}})$ étaient de la forme

$$\boldsymbol{\Psi} = (\Psi_i(z))_{i\in I}, \quad \Psi_i(z) = q_i^{\deg(Q_i)-\deg(R_i)} \frac{Q_i(zq_i^{-1})R_i(zq_i)}{Q_i(zq_i)R_i(zq_i^{-1})}, \tag{I.1.4.4}$$

où les Q_i et R_i sont des polynômes de coefficients constants égaux à 1, généralisant (I.1.3.1). À chaque paire de n-uplet de polynômes $((Q_i)_{i\in I}, (R_i)_{i\in I})$, décomposés en

$$Q_i(z) = \prod_{r=1}^{k_i}(1 - a_{i_r}z), \quad R_i(z) = \prod_{s=1}^{l_i}(1 - b_{i_s}z),$$

on associe le monôme de Laurent

$$\prod_{i\in I}\left(\prod_{r=1}^{k_i} Y_{i,a_{i_r}} \prod_{s=1}^{l_i} Y_{i,b_{i_s}}^{-1}\right).$$

En résumé la variable $Y_{i,a}$ correspond au ℓ-poids

$$Y_{i,a} \rightsquigarrow \left(1, \ldots, 1, \underbrace{q_i \frac{1 - aq_i^{-1}z}{1 - aq_i z}}_{i}, 1, \ldots, 1\right). \tag{I.1.4.5}$$

Ainsi, la classification des modules simples dans la catégorie $\mathscr{C}$ de Chari-Pressley peut se traduire de la façon suivante : les $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules simples de dimension finie sont indexés par les monômes en les variables $(Y_{i,a})_{i\in I, a\in\mathbb{C}^\times}$. Pour tout monôme $m = \prod_{k=1}^{r} Y_{i_k,a_k}$, on note $L(m)$ le module simple de plus haut ℓ-poids m par la correspondance (I.1.4.5).

Ensuite, il a été montré ([FR99], [FM01]) que le q-caractère d'un $\mathcal{U}_q(\hat{\mathfrak{g}})$-module donnait les dimensions de ces espaces de ℓ-poids :

$$\chi_q(V) = \sum_{\boldsymbol{\Psi}\ \ell\text{-poids}} \dim(V_{\boldsymbol{\Psi}})[\boldsymbol{\Psi}] \quad \in \mathcal{Y},$$

où $[\boldsymbol{\Psi}]$ désigne le monôme de Laurent associé au ℓ-poids $\boldsymbol{\Psi}$.

Par exemple, pour $\mathfrak{g} = \mathfrak{sl}_2$, la représentation fondamentale $V_1(a)$ est de dimension 2 et son q-caractère est :

$$\chi_q(V_1(a)) = Y_{1,a} + Y_{1,aq^2}^{-1}. \tag{I.1.4.6}$$

Malheureusement, il n'existe pas de formule explicite, équivalente à la formule de Weyl, donnant les q-caractères en règle générale. À la place, Frenkel et Mukhin [FM01] proposent un algorithme permettant de les calculer, communément appelé *algorithme de Frenkel-Mukhin*. Ils ont prouvé que cet algorithme permettait de calculer les q-caractères des représentations fondamentales, en revanche il a été démontré [NN11] qu'il ne permet pas de calculer les q-caractères de toutes les représentations irréductibles.

I.1.5 Anneaux de Grothendieck quantiques

Dans cette section, sauf mention explicite du contraire, les algèbres de Lie sont considérées *simplement lacées* : de type A, D ou E.

Nakajima [Nak01b] proposa un nouvel algorithme, celui-ci permettant de calculer les q-caractères de tous les modules simples. Cet algorithme se base sur la connaissance des q-caractères des représentations fondamentales, par l'algorithme de Frenkel-Mukhin, et sur la contribution de la géométrie, plus particulièrement les *variétés de carquois*.

Les variétés de carquois de Nakajima sont des variétés algébriques complexes associées à des carquois, elles forment en elles-mêmes une direction importante de recherche (voir [Nak96], [Sch08], [Gin12], [MO12] et le livre [Kir16]). Le premier lien entre représentations de carquois et algèbres de Kac-Moody date des années 90 avec les travaux de Ringel [Rin90]. Celui-ci réalisa la partie positive $\mathcal{U}_q(\mathfrak{n})$ de l'algèbre enveloppante quantifiée $\mathcal{U}_q(\mathfrak{g})$ d'une algèbre de Lie $\mathfrak{g}$ de dimension finie comme une algèbre de Hall associée à un carquois qui n'est autre qu'une version orientée du diagramme de Dynkin de $\mathfrak{g}$. Peu de temps après, Lusztig [Lus91] combina les travaux de Ringel avec la théorie des faisceaux pervers pour construire une *base canonique* de $\mathcal{U}_q(\mathfrak{n})$.

Dans l'approche de Nakajima, des représentations de l'algèbre enveloppante complète sont construites géométriquement, ce qui généralise les constructions en termes de l'algèbre de convolution en K-théorie équivariante des variétés de drapeaux partiels en type A [GV93], [Vas98]. Ces variétés sont en effet des cas particuliers de variétés de carquois associées à n'importe quelle algèbre de Kac-Moody symétrique [Nak94], [Nak98]. Ceci mena à une construction géométrique de représentations de l'algèbre enveloppante quantifiée via l'algèbre de convolution en K-théorie équivariante des variétés de carquois [VV99], [Nak01a].

A l'aide de cette approche géométrique, Nakajima [Nak01a] définit une nouvelle base de l'anneau de Grothendieck $K_0(\mathscr{C})$ de la catégorie $\mathscr{C}$ des représentations de dimension finie de $\mathcal{U}_q(\hat{\mathfrak{g}})$, formées par les *modules standards*. Comme la base des modules simples, celle-ci est indexée par les monômes en les variables $(Y_{i,a})_{i\in I, a\in\mathbb{C}^\times}$. Les modules standards, notés $M(m)$, sont les équivalents pour $\mathcal{U}_q(\hat{\mathfrak{g}})$ des modules de Weyl. On peut aussi les voir comme des produits tensoriels de modules fondamentaux [VV02a], si m est un monôme $m = \prod_{k=1}^r Y_{i_k,a_k}$, alors

$$M(m) = \bigotimes_{k=1}^{r} L(Y_{i_k,a_k}). \tag{I.1.5.1}$$

Ensuite, Nakajima montra que la matrice de passage entre la base des modules simples et celle des modules standards était triangulaire supérieure, pour un certain ordre partiel sur les ℓ-poids. Pour tout monôme m, dans l'anneau de Grothendieck de $\mathscr{C}$,

$$[M(m)] = [L(m)] + \sum_{m'<m} P_{m',m}[L(m')]. \tag{I.1.5.2}$$

En fait, Nakajima montre une version t-déformée de la décomposition (I.1.5.2) :

$$[M(m)]_t = t^{\alpha}\left([L(m)]_t + \sum_{m'<m} P_{m',m}(t)[L(m')]_t\right), \tag{I.1.5.3}$$

où $\alpha \in \mathbb{Z}$ et les $P_{m',m}(t) \in \mathbb{Z}[t]$ sont des polynômes sans coefficients constants.

Cette relation a lieu dans une version t-déformée de l'anneau de Grothendieck de la catégorie $\mathscr{C}$, introduite dans [VV03] comme une déformation plate de l'anneau de Grothendieck, en termes de faisceaux pervers. Cet *anneau de Grothendieck quantique*, que l'on notera $K_t(\mathscr{C})$, est contenu dans un *tore quantique* $\mathcal{Y}_t$,

$$K_t(\mathscr{C}) \subset \mathcal{Y}_t. \tag{I.1.5.4}$$

Le tore quantique $\mathcal{Y}_t$ est une t-déformation non-commutative de l'anneau des ℓ-poids $\mathcal{Y}$ de (I.1.4.1), il s'agit d'une $\mathbb{Z}[t^{\pm 1}]$-algèbre engendrée par les variables $Y_{i,a}^{\pm 1}$, et des relations de t-commutation de la forme suivante :

$$Y_{i,a} * Y_{j,b} = t^{\mathcal{N}_{i,j}(a,b)} Y_{j,b} * Y_{i,a}. \tag{I.1.5.5}$$

Ce tore quantique se projette par l'évaluation en $t = 1$ sur l'anneau de polynômes de Laurent qui contient l'image du morphisme de q-caractère par (I.1.4.1) :

$$ev_{t=1} : \mathcal{Y}_t \to \mathcal{Y}.$$

On peut noter que l'anneau de Grothendieck quantique est noté $K_t(\mathscr{C})$ alors qu'il faut l'évaluer en $t = 1$ pour retrouver l'anneau de Grothendieck classique, noté $K_0(\mathscr{C})$.

Géométriquement, les t-déformations des classes des modules standards $[M(m)]_t$ dans la relation (I.1.5.3) sont interprétées comme des faisceaux constants et les classes des modules simples $[L(m)]_t$ comme des complexes de cohomologie d'intersection (*IC sheaves*), d'où une t-graduation qui arrive naturellement. Cette interprétation géométrique a permis à Nakajima de montrer que les coefficients des polynômes $P_{m',m}(t)$ étaient positifs et que l'évaluation à $t = 1$ donnait la multiplicité du module dans le module standard :

$$P_{m',m}(1) = [M(m), L(m')].$$

C'est la relation (I.1.5.3) qui donne un algorithme pour obtenir les q-caractères de tous les modules simples de la catégorie $\mathscr{C}$. En effet, les classes $[L(m)]_t$ et $[M(m)]_t$ dans l'anneau de Grothendieck quantique peuvent être vues comme des t-déformations des q-caractères, c'est pour cela qu'elles sont souvent appelées (q,t)-*caractères*. Tout d'abord, ce sont les (q,t)-caractères des représentations fondamentales qui sont définis, grâce à une version t-déformée de l'algorithme de Frenkel-Mukhin, puis par multiplication et par (I.1.5.1), ceux des modules standards. Par construction, ils vérifient

$$[M(m)]_t \xrightarrow{t=1} \chi_q(M(m)).$$

Ensuite, les (q,t)-caractères des modules simples sont définis par un algorithme à la Kazhdan-Lusztig : il s'agit de l'unique famille d'éléments de l'anneau de Grothendieck quantique qui vérifient la propriété de décomposition (I.1.5.3), ainsi qu'une certaine propriété d'invariance. De façon remarquable, les éléments obtenus sont aussi des t-déformations des q-caractères [Nak01b] :

$$[L(m)]_t \xrightarrow{t=1} \chi_q(L(m)).$$

Ainsi, cette méthode fournit un algorithme permettant de calculer les (q,t)-caractères, et donc les q-caractères, de toutes les représentations simples de dimension finie. En revanche, cet algorithme ne donne toujours pas de formule explicite, et peut être très complexe à mettre en œuvre. On remarque que la première étape de cet algorithme commence par le calcul des (q,t)-caractères des représentations fondamentales. En type E_8, par exemple, le calcul du (q,t)-caractère de la cinquième représentation fondamentale prend 350h sur un super-ordinateur et l'écriture du résultat nécessite 180Go [Nak10].

Par la suite, un certain nombre de ces résultats ont pu être étendus au cas non-simplement lacé (quand $\mathfrak{g}$ est de type B, C, F ou G). Hernandez [Her04] a défini algébriquement des (q,t)-caractères pour les représentations fondamentales en tout type. Sa méthode est aussi basée sur une t-déformation de l'algorithme de Frenkel-Mukhin, dont il faut prouver dans ce cas qu'il n'échoue pas. Dans le cas simplement lacé, on retrouve par cette méthode les (q,t)-caractères de Nakajima. En revanche, sans l'apport de la géométrie, la positivité des coefficients dans la décomposition (I.1.5.3) n'est pas garantie et reste pour l'instant au stade de conjecture [Her04], sauf en type B où elle a récemment été prouvée partiellement par Hernandez-Oya [HO19].

I.1.6 Catégorie $\mathcal{O}$

Le contexte de la catégorie $\mathcal{O}$ de représentations définie par Hernandez et Jimbo [HJ12] est une direction naturelle de généralisation de ces résultats.

En regardant le résultat de classification des représentations de $\mathcal{U}_q(\hat{\mathfrak{g}})$ irréductibles de dimension finie (cf (I.1.3.1)), il est en effet naturel de se demander quelles sont les représentations de plus haut ℓ-poids dont le plus haut ℓ-poids est formé plus généralement de fractions rationnelles.

Pour répondre à cette question il faut s'intéresser aux représentations de la sous-algèbre de Borel $\mathcal{U}_q(\hat{\mathfrak{b}})$ de $\mathcal{U}_q(\hat{\mathfrak{g}})$, pour la présentation de Drinfeld-Jimbo, il s'agit aussi d'une algèbre de Hopf. La décomposition triangulaire de (I.1.2.3) passe à cette sous-algèbre de Borel et la notion de ℓ-poids aussi.

La catégorie $\mathcal{O}$, appelé ici *affine quantique*, est définie comme la catégorie des $\mathcal{U}_q(\hat{\mathfrak{b}})$-modules qui sont sommes de leurs espaces de poids (comme $\mathcal{U}_q(\mathfrak{g})$-modules), dont ces espaces de poids sont de dimension finie, et dont les poids sont dans une union finie de cônes, pour l'ordre partiel de Chevalley sur les poids.

Une représentation irréductible de plus haut ℓ-poids de $\mathcal{U}_q(\hat{\mathfrak{b}})$ est alors dans la catégorie $\mathcal{O}$ si et seulement si son plus haut ℓ-poids $\boldsymbol{\Psi} = (\Psi_i(z))_{i\in I}$ est formé de fractions rationnelles ne s'annulant pas en 0, et n'ayant pas de pôle en 0 [HJ12]. En général, ces représentations irréductibles ne sont pas de dimension finie, mais en tant que $\mathcal{U}_q(\mathfrak{g})$-modules, ont des espaces de poids de dimension finie [HJ12].

Cette catégorie possède deux familles de représentations irréductibles particulières : les *représentations préfondamentales*, positives et négatives. Pour tout $i \in I$ et tout $a \in \mathbb{C}^\times$, soit $L^\pm_{i,a}$ la représentation simple de plus haut ℓ-poids

$$\mathbf{\Psi}^{\pm 1}_{i,a} := (1, \ldots, 1, \underbrace{(1 - az)^{\pm 1}}_{i}, 1, \ldots, 1).$$

Contrairement à l'écriture (I.1.3.2), il s'agit du ℓ-poids et non des polynômes de Drinfeld. On remarque le lien suivant entre les $Y_{i,a}$ et les $\mathbf{\Psi}^{\pm 1}_{i,a}$:

$$Y_{i,a} = q_i \mathbf{\Psi}_{i,aq_i^{-1}} \mathbf{\Psi}^{-1}_{i,aq_i}, \quad \forall i \in I, a \in \mathbb{C}^\times. \tag{I.1.6.1}$$

Néanmoins, là n'est pas la première occurrence des représentations préfondamentales. Elles sont apparues pour $\mathfrak{g} = \mathfrak{sl}_2$ sous le nom de *représentations de q-oscillation* dans les travaux de Bazhanov-Lukyanov-Zamolodchikov [BLZ99] en théorie conforme des champs, qui les ont construites explicitement. Puis, elles ont été définies pour $\mathfrak{g} = \mathfrak{sl}_3$ par Bazhanov-Hibberd-Khoroshkin [BHK02] et pour $\mathfrak{g} = \mathfrak{sl}_n$ et $i = 1$ par Kojima [Koj08].

La catégorie $\mathcal{O}$ est une catégorie monoïdale. Les objets de la catégorie $\mathcal{O}$ ne sont pas nécessairement de longueur finie, mais sa sous-catégorie formée des objets de longueur finie n'est pas stable par produit tensoriel (voir [BJM+09, Lemme C.1]). Néanmoins, comme pour la catégorie $\mathcal{O}$ d'une algèbre de Kac-Moody, la multiplicité d'un module simple dans un module donné est bien définie (voir [Kac90, Section 9.6]). On s'intéresse à son anneau de Grothendieck, qui est formé de sommes formelles

$$\chi = \sum_{\mathbf{\Psi}} \lambda_{\mathbf{\Psi}} [L(\mathbf{\Psi})] \quad \in K_0(\mathcal{O}),$$

qui vérifient les hypothèses de construction de la catégorie $\mathcal{O}$ (espaces de poids de dimension finie et poids dans une union finie de cônes).

Hernandez et Jimbo [HJ12] ont montré que le morphisme de q-caractère de Frenkel-Reshetikhin, donné par la décomposition en espaces de ℓ-poids, s'étendait aux représentations de la catégorie $\mathcal{O}$:

$$\chi_q(V) = \sum_{\mathbf{\Psi}} \dim(V_{\mathbf{\Psi}})[\Psi] \quad \in \mathcal{E}_\ell,$$

où $\mathcal{E}_\ell$ est un anneau commutatif étendant $\mathcal{Y} = \mathbb{Z}[Y^{\pm 1}_{i,a}]_{i \in I, a \in \mathbb{C}^\times}$.

Par exemple, le q-caractère de la représentation préfondamentale positive $L^+_{i,a}$ est :

$$\chi_q(L^+_{i,a}) = [\mathbf{\Psi}_{i,a}]\chi_i, \tag{I.1.6.2}$$

où χ_i est le caractère de la représentation $L^+_{i,a}$ (pour l'action de $\mathcal{U}_q(\mathfrak{g})$), et ne dépend pas de $a \in \mathbb{C}^\times$ (voir [FH15]).

De plus, l'application ainsi définie est un morphisme d'anneaux injectif sur l'anneau de Grothendieck de la catégorie $\mathcal{O}$:

$$\chi_q : K_0(\mathcal{O}) \to \mathcal{E}_\ell.$$

Cette catégorie de représentations permet de donner des interprétations catégoriques à certaines relations liées aux systèmes intégrables quantiques. Dans [BLZ99], la matrice de transfert de la représentation de "q-oscillation" mentionnée plus haut est identifiée à l'opérateur étudié par Baxter dans [Bax72] (voir Section I.1.1). Cela permet d'interpréter la relation de Baxter (I.1.1.1) comme une relation dans l'anneau de Grothendieck de la catégorie $\mathcal{O}$.

En particulier, ce point de vue apporté par la catégorie $\mathcal{O}$ a permis à Hernandez-Frenkel [FH15] de démontrer une conjecture de Frenkel-Reshetikhin [FR99] sur les valeurs propres des matrices de transfert des représentations de dimension finies de $\mathcal{U}_q(\hat{\mathfrak{g}})$. Pour toute représentation de dimension finie V de $\mathcal{U}_q(\hat{\mathfrak{g}})$, si l'on remplace chaque variable $Y_{i,a}$ dans son q-caractère par le quotient des classes des représentations préfondamentales $[L^+_{i,aq^{-1}}]/[L^+_{i,aq}]$, multiplié par la classe $[\omega_1]$ de la représentation de dimension 1 de $\mathcal{U}_q(\hat{\mathfrak{b}})$, sur laquelle les générateurs $k_j^{\pm 1}$ de l'algèbre de Cartan agissent par le poids fondamental ω_i, alors le résultat obtenu est égal à la classe de la représentation V dans l'anneau de Grothendieck $K_0(\mathcal{O})$. Par exemple, pour $\mathfrak{g} = \mathfrak{sl}_2$, la cas de la relation de Baxter est le suivant : soit $V_1(a) = L(Y_{1,a})$, une représentation fondamentale de $\mathcal{U}_q(\hat{\mathfrak{g}})$, de dimension 2, dont le q-caractère a été rappelé en (I.1.4.6). La relation suivante est satisfaite dans l'anneau de Grothendieck $K_0(\mathcal{O})$:

$$[L(Y_{1,a})] \cdot [L^+_{1,aq}] = [\omega_1][L^+_{1,aq^{-1}}] + [-\omega_1][L^+_{1,aq^3}]. \tag{I.1.6.3}$$

Il s'agit d'une version *catégorifiée* de la relation de Baxter (I.1.1.1).

Cette dernière relation (I.1.6.3) a aussi une toute autre interprétation : comme *relation d'échange* dans une algèbre amassée.

I.1.7 Algèbres amassées

Les algèbres amassées ont été introduites par Fomin et Zelevinsky au début des années 2000 dans une série d'articles [FZ02], [FZ03a], [BFZ05], [FZ07], en lien avec les bases canoniques duales des groupes quantiques. Il s'agit par définition d'objets fortement combinatoires, mais des connections avec des domaines très variés des mathématiques ont ensuite été trouvées.

Par exemple dans [FZ03c], Fomin et Zelevinsky les utilisent pour démontrer une conjecture de Zamolodchikov sur certains systèmes dynamiques discrets appelé Y-systèmes, provenant d'une version thermodynamique de l'Ansatz de Bethe. Ces Y-systèmes sont fortement liés [KNS94] à un autre système de relations fonctionnelles liées aux représentations des algèbres affines quantiques, appelés T-systèmes. En lien plus direct avec les questions qui nous intéressent, Hernandez et Leclerc ont introduit la notion de catégorification monoïdale d'une algèbre amassée dans [HL10]. Dans ce travail, ils mettent en valeur une structure d'algèbre amassée pour l'anneau de Grothendieck d'une sous-catégorie monoïdale de $\mathscr{C}$.

Par la suite, cette approche a été généralisée, par les mêmes auteurs d'une part ([HL13], [HL16a], [HL16b]), mais aussi par d'autres, notoirement par Nakajima [Nak11], Qin [Qin16], Kang-Kashiwara-Kim-Oh [KKKO18].

La structure d'algèbre amassée qui est utilisée plus particulièrement dans ce travail de thèse est la suivante. Considérons la sous-catégorie $\mathcal{O}^+$ (resp. $\mathcal{O}^-$) [HL16b] de la catégorie $\mathcal{O}$ formées des représentations dont les constituants simples ont des plus hauts ℓ-poids qui sont des monômes en les variables $Y_{i,a}$ et $\mathbf{\Psi}_{i,a}$ (resp. $\mathbf{\Psi}_{i,a}^{-1}$). Il s'agit de sous-catégories monoïdales de $\mathcal{O}$ [HL16b]. Soient $\mathcal{O}^{\pm}_{\mathbb{Z}}$ les sous-catégories monoïdales de $\mathcal{O}^{\pm}$, où l'on a restreint les variables $Y_{i,a}$ et $\mathbf{\Psi}_{i,a}^{\pm 1}$ des plus hauts ℓ-poids à une certaine famille dénombrable. Ce type de restriction avait déjà été considéré par les mêmes auteurs dans [HL10] au sujet de la catégorie $\mathscr{C}$, elle contient en fait toute l'information importante de la catégorie complète.

Hernandez et Leclerc ont alors montré que, pour toute algèbre de Lie simple de dimension finie $\mathfrak{g}$, l'anneau de Grothendieck de la catégorie $\mathcal{O}_{\mathbb{Z}}^+$ de représentations de $\mathcal{U}_q(\hat{\mathfrak{g}})$ avait une structure d'algèbre amassée :

$$\mathcal{A}(\Gamma)\hat{\otimes}\,\mathcal{E} \xrightarrow{\sim} K_0(\mathcal{O}_{\mathbb{Z}}^+), \tag{I.1.7.1}$$

où Γ est un carquois infini basé sur le diagramme de Dynkin de $\mathfrak{g}$, $\mathcal{E}$ correspond à l'image de $K_0(\mathcal{O})$ par le morphisme de caractère, et le produit tensoriel est complété pour admettre des sommes formelles (dont les espaces de poids sont de dimension finie). De plus, l'isomorphisme (I.1.7.1) est obtenu en prenant comme graine initiale les représentations préfondamentales positives. Un énoncé symétrique existe pour la catégorie $\mathcal{O}_{\mathbb{Z}}^-$.

Notons que la relation de Baxter catégorifiée (I.1.6.3) correspond à la relation d'échange d'une mutation au sommet (i,a) du carquois Γ.

Nous n'entrerons pas dans les détails techniques concernant les algèbres amassées dans cette introduction, mais toutes les notions que nous utilisons dans ce domaine sont rappelées en Annexe B.

I.2 Résultats de la thèse

L'objectif de cette thèse est de construire et d'étudier une structure d'anneau de Grothendieck quantique pour la catégorie $\mathcal{O}$ de représentations de l'algèbre de Borel $\mathcal{U}_q(\hat{\mathfrak{b}})$.

Nous présentons en premier lieu dans le Chapitre 1 des résultats directement liés à cette question principale, concernant la construction d'équivalents des modules standards dans le contexte de la catégorie $\mathcal{O}$. Ensuite, dans le Chapitre 2, nous entamons la construction de l'anneau de Grothendieck quantique en commençant par définir un tore quantique $\mathscr{T}_t$ étendant le tore quantique $\mathcal{Y}_t$. Nous utilisons pour cela des notions liées aux algèbres amassées quantiques. Dans cet esprit, nous construisons dans le Chapitre 3 une structure d'algèbre amassée quantique sur l'anneau de Grothendieck quantique de la catégorie $\mathscr{C}_{\mathbb{Z}}^-$, une sous-catégorie monoïdale de $\mathscr{C}$. Puis, nous définissons dans le Chapitre 2 l'anneau de Grothendieck quantique $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ de la catégorie $\mathcal{O}_{\mathbb{Z}}^+$ comme une algèbre amassée quantique. Pour que ces résultats soient cohérents, il nous faut montrer que cet anneau de Grothendieck quantique contient celui de la catégorie $\mathscr{C}_{\mathbb{Z}}^-$. Ce résultat est montré directement en type A dans le Chapitre 2, et en tout type simplement lacé dans le Chapitre 3, en utilisant la structure d'algèbre amassée quantique de $K_t(\mathscr{C}_{\mathbb{Z}}^-)$. Subséquemment, dans le Chapitre 2, sont définis des (q,t)-caractères pour les représentations fondamentales positives, ce qui permet en particulier d'écrire une version t-déformée de la relation de Baxter (I.1.6.3). Enfin, dans le Chapitre 4, nous présentons certains résultats supplémentaires obtenus dans le cas où l'algèbre de Lie simple de dimension finie sous-jacente $\mathfrak{g}$ est $\mathfrak{sl}_2$, ainsi que divers prolongements possibles, notamment pour généraliser nos résultats en tout type non nécessairement simplement lacé.

I.2.1 Modules standards asymptotiques

Une première question qui vient naturellement lorsque l'on cherche à généraliser l'approche de Nakajima pour calculer les (q,t)-caractères des modules simples par un algorithme à la Kazhdan-Lusztig est celle de la définition d'analogues des modules standards dans le contexte de la catégorie $\mathcal{O}$. En effet, pour pouvoir mettre en place un algorithme similaire pour calculer les (q,t)-caractères des représentations simples de la catégorie $\mathcal{O}$, il faudrait expliciter une famille de modules qui engendreraient l'anneau de Grothendieck de la catégorie $\mathcal{O}$, et dont les (q,t)-caractères seraient faciles à calculer. La définition de modules standards par produit tensoriel de modules fondamentaux (I.1.5.1) est particulièrement propice pour ce dernier point. L'approche de Nakajima et de Varagnolo-Vasserot pour définir ces modules standards pour la catégorie $\mathscr{C}$ est basée sur une construction géométrique, qui n'existe malheureusement pas dans le contexte de la catégorie $\mathcal{O}$, il nous faut donc proposer une approche plus ad hoc.

Lorsque l'on s'intéresse aux premières motivations derrière l'introduction de la catégorie $\mathcal{O}$ et des représentations préfondamentales, on trouve le résultat suivant de Nakajima [Nak03a], démontré dans les cas non-simplement lacés par Hernandez dans [Her06].

Pour tout $k \in \mathbb{N}^\times$, $a \in \mathbb{C}^\times$ et $i \in I$, considérons le module de Kirillov-Reshetikhin (ou module KR) :

$$W_{k,a}^{(i)} = L(Y_{i,a}Y_{i,aq_i^2}\cdots Y_{i,aq_i^{2k-2}}). \tag{I.2.1.1}$$

Alors le q-caractère normalisé du module de Kirillov-Reshetikhin est un polynôme en des variables $A_{j,b}^{-1}$, et la suite des q-caractères normalisés des modules $W_{N,aq_i^{-2N+2}}^{(i)}$ a une limite quand $k \to +\infty$ comme série formelle en ces variables. De plus, dans [HJ12], Hernandez et Jimbo montrent que cette limite est égale au q-caractère normalisé de la représentation préfondamentale négative $\mathbf{\Psi}_{i,aq_i}^{-1}$. Ce résultat vient du fait que, par (I.1.6.1), le ℓ-poids $\Psi_{i,a}^{-1}$ peut être vu comme un produit infini de $Y_{j,b}$ normalisés :

$$\Psi_{i,a}^{-1} = \lim_{k\to+\infty} \tilde{Y}_{i,aq_i^{-1}}\tilde{Y}_{i,aq_i^{-3}}\cdots\tilde{Y}_{i,aq_i^{-2k+1}},$$

où $\tilde{Y}_{i,a} = \frac{Y_{i,a}}{q_i}$.

En se basant sur la même idée, nous cherchons donc à construire un module standard $M(\mathbf{\Psi}_{i,a}^{-1})$ comme produit tensoriel infini de modules fondamentaux :

$$L(\tilde{Y}_{i,aq_i^{-1}})\otimes L(\tilde{Y}_{i,aq_i^{-3}})\otimes\cdots\otimes L(\tilde{Y}_{i,aq_i^{-2k+1}})\otimes\cdots \rightsquigarrow M(\mathbf{\Psi}_{i,a}^{-1}).$$

La première étape est de vérifier que cette définition peut faire sens du point de vue des q-caractères, car le but de la définition de ces modules standard asymptotiques est de calculer des q-caractères.

On répond à cette question par la positive comme premier résultat du Chapitre 1.

Théorème 1. *(Théorème 1.2.2) La suite des q-caractères normalisés d'un produit tensoriel d'un nombre croissant de modules fondamentaux converge comme série formelle en les variables $A_{j,b}^{-1}$:*

$$\tilde{\chi}_q\left(L(\tilde{Y}_{i,aq_i^{-1}})\otimes L(\tilde{Y}_{i,aq_i^{-3}})\otimes\cdots\otimes L(\tilde{Y}_{i,aq_i^{-2N+1}})\right) \xrightarrow{N\to+\infty} \chi_{i,a}^{\infty} \in \mathbb{Z}[[A_{j,b}^{-1}]].$$

La deuxième étape est de donner un sens du côté de la théorie des représentations à cette limite, donc de construire un $\mathcal{U}_q(\hat{\mathfrak{b}})$-module $M(\Psi_{i,a}^{-1})$ dont le q-caractère normalisé serait $\chi_{i,a}^{\infty}$.

C'est ce qui est proposé dans la suite du Chapitre 1. Pour des raisons techniques, le reste de ce chapitre est concentré sur le cas où $\mathfrak{g} = \mathfrak{sl}_2$. Tout d'abord, nous donnons un sens au produit tensoriel infini :

$$T = L(\tilde{Y}_{q^{-1}})\otimes L(\tilde{Y}_{q^{-3}})\otimes L(\tilde{Y}_{q^{-5}})\otimes L(\tilde{Y}_{q^{-7}})\otimes\cdots.$$

Celui-ci est construit de façon "localement finie", de manière à ce qu'il soit de façon naturelle une limite d'une suite croissante de modules standards.

On construit ensuite explicitement une action de l'algèbre $\mathcal{U}_q(\hat{\mathfrak{b}})^{\geqslant 0}$, qui correspond au produit tensoriel de la partie positive de $\mathcal{U}_q(\hat{\mathfrak{b}})$ et de la partie Drinfeld-Cartan, dans la décomposition triangulaire (I.1.2.3), sur l'espace vectoriel T (Proposition 1.3.3). On montre ensuite qu'il s'agit d'un bon candidat pour notre module standard asymptotique, avec le résultat suivant.

Proposition 2. *(Proposition 1.3.7) Le $\mathcal{U}_q(\hat{\mathfrak{b}})^{\geqslant 0}$-module T est engendré par ses vecteurs de ℓ-poids, ses espaces de ℓ-poids sont de dimension finie, et son q-caractère vérifie*

$$\chi_q(T) = [\mathbf{\Psi}_1^{-1}]\chi_{1,1}^{\infty}.$$

Malheureusement, cette action ne s'étend pas naturellement à l'algèbre $\mathcal{U}_q(\hat{\mathfrak{b}})$ entière, et pour obtenir un $\mathcal{U}_q(\hat{\mathfrak{b}})$-module ayant le bon q-caractère, il est nécessaire de passer par un procédé d'*induction* :

$$T^c := T \otimes_{\mathcal{U}_q(\hat{\mathfrak{b}})^{\geqslant 0}} \mathcal{U}_q(\hat{\mathfrak{b}})^{-}.$$

L'argument clé à cette étape est de montrer que cette induction ne produit pas de nouveaux vecteurs de ℓ-poids, qui changeraient le q-caractère. C'est effectivement ce que nous montrons en Proposition 1.3.18. Ce qui amène naturellement au

Théorème 3. *(Théorème 1.3.19) L'espace vectoriel T^c est un $\mathcal{U}_q(\hat{\mathfrak{b}})$-module dont les espaces de ℓ-poids sont de dimension finies et son q-caractère vérifie :*

$$\chi_q(T^c) = [\mathbf{\Psi}_1^{-1}]\chi_{1,1}^{\infty}.$$

Ce résultat est bien sûr étendu aux ℓ-poids négatifs plus généraux que $\mathbf{\Psi}_1^{-1}$, mais la construction centrale est celle de $M(\mathbf{\Psi}_1^{-1})$.

Dans la dernière section du Chapitre 1, nous nous proposons d'étudier les propriétés de décomposition et de multiplicativité du q-caractère sur ces nouveaux modules standards asymptotiques. En particulier, nous démontrons que $\chi_q(T^c(\mathbf{\Psi}_1^{-1}))$ a une décomposition en somme de modules simples de la forme de (I.1.5.2) :

Théorème 4. *(Théorème 1.4.3)*

$$\chi_q(T^c(\mathbf{\Psi}_1^{-1})) = \chi_q(L_1^-) + \sum_{\mathbf{\Psi} < \mathbf{\Psi}_1^{-1}} P_{m',m}\chi_q(L(\mathbf{\Psi})),$$

pour un ordre partiel $<$ qui étend celui considéré par Nakajima.

I.2.2 Tore quantique étendu

Comme expliqué précédemment, l'autre pan de ce travail est la construction concrète d'un anneau de Grothendieck quantique pour la catégorie $\mathcal{O}_{\mathbb{Z}}^+$, et l'étude de sa structure.

La première étape d'un travail dans cette direction consiste à la mise en place d'un *tore quantique étendu*, contenant le tore quantique $\mathcal{Y}_t$ de (I.1.5.4), et pouvant contenir les (q,t)-caractères des représentations préfondamentales.

Dans la suite, l'algèbre de Lie $\mathfrak{g}$ simple de dimension finie à l'origine de l'algèbre de Borel $\mathcal{U}_q(\hat{\mathfrak{b}})$ dont nous étudions les représentations, est supposée *simplement lacée*.

Comme mentionné précédemment, le fait de s'intéresser à la sous-catégorie monoïdale $\mathcal{O}_{\mathbb{Z}}^+$ de $\mathcal{O}^+$ n'est que peu restrictif car celle-ci est assez riche pour contenir toute l'information importante de la catégorie $\mathcal{O}^+$. Dorénavant, les ℓ-poids considérés seront les

$$\mathbf{\Psi}_{i,r}^{\pm 1} = \mathbf{\Psi}_{i,q^r}^{\pm 1},$$
$$Y_{j,s}^{\pm 1} = Y_{j,q^s}^{\pm 1},$$

où $(i,r) \in \hat{I}$ et $(j,s) \in \hat{J}$, et $\hat{I}$ et $\hat{J}$ sont des sous-ensembles de $I \times \mathbb{Z}$ dénombrables.

Pour pouvoir construire des (q,t)-caractères pour la catégorie $\mathcal{O}_{\mathbb{Z}}$ dans ce nouveau tore quantique, celui-ci doit contenir les ℓ-poids préfondamentaux $\mathbf{\Psi}_{i,r}^{\pm 1}$, ainsi que des sommes (potentiellement infinies) et des produits de ceux-ci.

On commence donc par former le tore quantique $\mathcal{T}_t$ comme une $\mathbb{Z}[t^{\pm 1}]$-algèbre engendrée par les $\mathbf{\Psi}_{i,r}^{\pm 1}$ et par des relations de t-commutations de la forme :

$$\mathbf{\Psi}_{i,r} * \mathbf{\Psi}_{j,s} = t^{\mathcal{F}_{ij}(s-r)} \mathbf{\Psi}_{j,s} * \mathbf{\Psi}_{i,r}, \quad \forall ((i,r),(j,s) \in \hat{I}). \tag{I.2.2.1}$$

Bien évidemment, la loi de t-commutation $\mathcal{F}$ doit être choisie de façon cohérente avec les relations de t-commutation sur $\mathcal{Y}_t$ (I.1.5.5), et avec le lien entre les variables $Y_{i,r}$ et les $\mathbf{\Psi}_{i,r}$ (I.1.6.1).

La relation suivante est donc imposée sur la fonction $\mathcal{F}$:

$$\mathcal{N}_{ij}(r,s) = 2\mathcal{F}_{ij}(s-r) - \mathcal{F}_{ij}(s-r+2) - \mathcal{F}_{ij}(s-r-2), \quad \forall ((i,r),(j,s) \in \hat{I}). \tag{I.2.2.2}$$

Il semblerait donc qu'il existe un certain degré de liberté dans le choix de fonction de t-commutation $\mathcal{F}$. Cependant, nous voulons que certaines relations particulières soient vérifiées dans notre anneau de Grothendieck quantique, comme par exemple une version quantifiée de la relation de Baxter (I.1.6.3).

Comme expliqué en Section I.1.7, la relation de Baxter est en fait une relation d'échange correspondant à une mutation dans une algèbre amassée. Les algèbres amassées possèdent des t-déformations non-commutatives naturelles en *algèbres amassées quantiques*, introduites par Berenstein et Zelevinsky dans [BZ05]. Pour pouvoir définir une telle algèbre amassée quantique, l'on doit disposer d'une part d'un carquois Γ, qui correspond à la donnée d'une algèbre amassée classique, et d'autre part d'une matrice anti-symétrique $\Lambda = (\Lambda_{ij})$ encodant les relatons de t-commutation entre les variables initiales :

$$X_i * X_j = t^{\Lambda_{ij}} X_j * X_i.$$

De plus, ces deux objets doivent vérifier une condition de compatibilité : (Γ, Λ) doit former ce qu'on appelle une *paire compatible* (voir Chapitre 2, Section B.2).

La structure d'algèbre amassée de l'anneau de Grothendieck de la catégorie $\mathcal{O}_{\mathbb{Z}}^+$ de (I.1.7.1) est construite sur un carquois infini Γ, en prenant comme graine initiale les classes des représentations préfondamentales positives $[L_{i,r}^+]$. De plus, par (I.1.6.2) nous savons que les q-caractères de ces représentations sont égaux, à un inversible près, à leur plus haut ℓ-poids $[\mathbf{\Psi}_{i,r}]$. Ainsi, pour pouvoir obtenir une structure d'algèbre amassée quantique qui t-déforme la structure d'algèbre amassée de $K_0(\mathcal{O}_{\mathbb{Z}}^+)$, il faut nécessairement vérifier la compatibilité entre le carquois Γ et la fonction de t-commutation entre les $\mathbf{\Psi}_{i,r}$.

Nous fixons donc une famille de fonctions $\mathcal{F} = (\mathcal{F}_{ij})_{i,j \in I}$ telle que la relation (I.2.2.2) soit vérifiée, et que la "matrice $\hat{I} \times \hat{I}$" donnée par

$$\Lambda((i,r),(j,s)) = \mathcal{F}_{ij}(s-r),$$

forme une paire compatible avec le carquois Γ de (I.1.7.1), utilisé pour définir une structure d'algèbre amassée sur l'anneau de Grothendieck $K_0(\mathcal{O}_{\mathbb{Z}}^+)$.

Proposition 5. *(Proposition 2.4.10) Soit $\tilde{B}$ est la matrice d'adjacence du carquois Γ, alors $(\tilde{B}^T \Lambda)$ forme une* paire compatible*, plus précisément,*

$$(\tilde{B}^T \Lambda)((i,r),(j,s)) = \begin{cases} -2 & si\ (i,r) = (j,s), \\ 0 & sinon \end{cases}, \tag{I.2.2.3}$$

Notons que le carquois Γ est un carquois infini, ce qui ne correspond pas à la définition d'algèbre amassée quantique telle qu'elle apparait dans [BZ05]. Cependant, nous montrons que nous pouvons en fait considérer une suite croissante de carquois finis Γ_N, qui forment des paires compatibles (Γ_N, Λ_N), où Λ_N est la restriction de Λ à la partie finie du carquois. Comme les variables d'amas sont toujours obtenues par un nombre fini de mutations, ce procédé permet effectivement d'obtenir l'algèbre amassée quantique de "rang infini" comme limite d'algèbres amassées quantiques de rangs finis.

On conclue la construction du tore quantique étendu en posant :

$$\mathscr{T}_t := \mathcal{T}_t \hat{\otimes}_{\mathbb{Z}[t^{\pm 1/2}]} \mathcal{E}, \tag{I.2.2.4}$$

où le produit tensoriel est complété pour admettre certaines sommes infinies, comme dans (I.1.7.1).

Par construction, on a donc naturellement une inclusion de tores quantiques :

Proposition 6. *(Proposition 2.3.3) Il existe un morphisme injectif de $\mathbb{Z}[t^{\pm 1/2}]$-algèbres*

$$\mathcal{J} : \mathcal{Y}_t \to \mathscr{T}_t. \tag{I.2.2.5}$$

I.2.3 L'anneau de Grothendieck quantique $K_t(\mathscr{C}_{\mathbb{Z}}^-)$ comme algèbre amassée quantique

Cette construction de tore quantique étendu a introduit une nouvelle donnée au problème : si les anneaux de Grothendieck des catégories de représentations d'algèbres affines quantique peuvent avoir des structures d'algèbres amassées, alors leurs anneaux de Grothendieck quantiques peuvent avoir des structures d'algèbres amassées quantiques !

Revenons au cas des représentations de dimension finies. Dans ce cas, l'anneau de Grothendieck est déjà connu, ainsi que la structure d'algèbre amassée, nous pouvons donc vérifier que celle-ci se déforme bien en une algèbre amassée quantique, et que nous retrouvons par ce moyen la structure d'anneau de Grothendieck quantique. Il s'agit là du travail présenté dans la première moitié du Chapitre 3.

Tout d'abord, intéressons-nous à la catégorie $\mathscr{C}$ des représentations de dimension finie de l'algèbre affine quantique $\mathcal{U}_q(\hat{\mathfrak{g}})$. Nous pouvons comme précédemment se restreindre à l'étude d'une sous-catégorie tensorielle $\mathscr{C}_{\mathbb{Z}}$, engendrée par les modules simples dont les plus hauts ℓ-poids sont des monômes en les $(Y_{i,r})_{(i,r)\in\hat{J}}$. Hernandez et Leclerc ont considéré dans [HL16a] la catégorie $\mathscr{C}_{\mathbb{Z}}^-$, sous-catégorie monoïdale $\mathscr{C}_{\mathbb{Z}}^-$ de $\mathscr{C}_{\mathbb{Z}}$, où nous avons restreint les plus hauts ℓ-poids des modules simples aux monômes en les $Y_{i,r}$, avec $r \leqslant 0$. Ils ont montré que l'anneau de Grothendieck de cette catégorie avait une structure d'algèbre amassée, basée sur un carquois "semi-infini" G^-, qui correspond à la moitié du carquois infini Γ. Ils montrent de plus que les q-caractères de certains modules de Kirillov-Reshetikhin (et donc en particulier de certaines représentations préfondamentales) s'obtiennent comme variables d'amas à partir de la graine initiale

$$u_{i,r} = \prod_{r+2k\leqslant 0} Y_{i,r+2k}, \tag{I.2.3.1}$$

et de suites finies de mutations. L'idée est de voir ces variables d'amas comme des *q-caractères tronqués*, c'est-à-dire où l'on n'a gardé que les variables $Y_{i,r}^{\pm 1}$, avec $r \leqslant 0$. Les variables d'amas initiales sont effectivement des q-caractères tronqués, où il ne reste du q-caractère que le terme de plus haut ℓ-poids. Ensuite, une suite particulière de mutations est explicitée, telle qu'à chaque étape, la relation d'échange correspondant à la mutation est une relation de T-systèmes, comme mentionnés en Section I.1.7. Ces relations de T-systèmes sont des relations vérifiées dans l'anneau de Grothendieck, elles ont été démontrées par Nakajima [Nak04] en type ADE, et par Hernandez [Her06] en tout type. Ceci permet de montrer que toutes les variables d'amas apparaissant suite à ces mutations successives sont des q-caractères tronqués, et qu'en appliquant ce procédé un certain nombre de fois, les q-caractères tronqués deviennent complets. Notons que les résultats de [HL16a] ont été énoncés ici dans le cas simplement lacé, mais sont en fait vérifiés dans les cas non-simplement lacés.

Dans notre travail, nous nous proposons de démontrer que, dans le cas où $\mathfrak{g}$ est simplement lacé, l'anneau de Grothendieck quantique de la catégorie $\mathscr{C}_{\mathbb{Z}}^-$ a une structure d'algèbre amassée quantique, en t-déformant cette structure d'algèbre amassée.

On commence par montrer que si nous considérons le tore quantique $T \subset \mathcal{Y}_t$, engendré par les variables $u_{i,r}^{\pm 1}$ données en (I.2.3.1), alors si L est la matrice donnant les relations de t-commutation entre ces variables

$$u_{i,r} * u_{j,s} = t^{L((i,r),(j,s))} u_{j,s} * u_{i,r}, \tag{I.2.3.2}$$

(qui est entièrement déterminée par la famille de fonctions $\mathcal{N}$ de (I.1.5.5)), celle-ci forme une paire compatible, au sens des algèbres amassées quantiques, avec le carquois G^-.

Proposition 7. *(Proposition 3.4.1) Si B_- est la matrice d'adjacence du carquois G^-, alors celle-ci forme une paire compatible avec la fonction L :*

$$(B_-^T L)((i,r),(j,s)) = \begin{cases} -2 & \text{si } (i,r) = (j,s), \\ 0 & \text{sinon} \end{cases}.$$

Ainsi, nous pouvons définir à partir de la donnée initiale (G^-, L), une algèbre amassée quantique :

$$\mathcal{A}_t := \mathcal{A}_t(G^-, L).$$

Il s'agit d'une sous $\mathbb{Z}[t^{\pm 1/2}]$-algèbre du tore quantique T.

Il reste ensuite à montrer que cette algèbre amassée quantique est isomorphe à l'anneau de Grothendieck $K_t(\mathscr{C}_{\mathbb{Z}}^-)$ de la catégorie $\mathscr{C}_{\mathbb{Z}}^-$. Pour cela, l'idée est d'adapter en version quantique les arguments présentés dans [HL16a], en gérant les difficultés apportées par la non-commutativité.

Premièrement, les relations de T-systèmes ont des analogues dans l'anneau de Grothendieck quantique, appelés *T-systèmes quantiques*, qui ont été démontrés dans les cas simplement lacés par Nakajima dans [Nak03a]. Nous remarquons en Section 3.3.6 que ces relations s'étendent aux (q,t)-caractères tronqués (la troncature est faite de manière identique à la troncature des q-caractères). Nous constatons aussi qu'en prenant comme graine initiale les variables $u_{i,r}$, celles-ci sont aussi égales aux (q,t)-caractères tronqués des modules de Kirillov-Reshetikhin correspondants. Ainsi, en considérant la même suite de mutations *quantiques* dans le carquois G^- que Hernandez et Leclerc, nous obtenons le résultat suivant :

Proposition 8. *(Proposition 3.5.2) Les variables d'amas qui apparaissent dans cette suite de mutations sont des (q,t)-caractères tronqués de modules de Kirillov-Reshetikhin. De plus, en appliquant cette suite de mutations un certain nombre de fois, nous obtenons des (q,t)-caractères complets.*

Un élément important qui diffère de la preuve de [HL16a] est le fait que pour vérifier que les relations d'échange de la suite de mutations quantiques sont effectivement des T-systèmes quantiques, il faut montrer que les puissances en t qui apparaissent sont les mêmes. Pour cela, nous utilisons le fait que les variables d'amas quantiques et les (q,t)-caractères des modules simples sont des éléments de leurs tores quantiques respectifs invariants par des *involutions barres*, et que ces involutions correspondent à la même application, à l'identification des tores quantiques près. Il est intéressant de noter ici que ce résultat peut aussi être obtenu en utilisant la positivité des variables d'amas quantiques démontrée par Davison dans [Dav18], mais qu'il s'agit d'un résultat beaucoup plus fort, et qui n'est finalement pas nécessaire dans ce cas.

Ensuite, il ne reste qu'à montrer le résultat suivant :

Théorème 9. *(Théorème 3.4.6) L'identification entre les variables d'amas initiales et les (q,t)-caracyères tronqués des modules de Kirillov-Reshetikhin donne un isomorphisme*

$$A_t \xrightarrow{\sim} K_t(\mathscr{C}_{\mathbb{Z}}^-). \tag{I.2.3.3}$$

Là encore, il s'agit en partie d'une généralisation de la preuve de [HL16a, Théorème 5.1]. En revanche, comme les variables vérifient des relations de t-commutation, les identifications des variables doivent se faire avec précaution, nous démontrons donc dans ce but le Lemme 3.5.4.

L'inclusion $K_t(\mathscr{C}_{\mathbb{Z}}^-) \subset \mathcal{A}_t$ a essentiellement déjà été démontrée ci-dessus. En effet, l'anneau de Grothendieck quantique $K_t(\mathscr{C}_{\mathbb{Z}}^-)$ est engendré par les (q,t)-caractères tronqués des représentations préfondamentales, et nous avons déjà vu que ceux-ci s'obtenaient comme variables d'amas quantiques dans l'algèbre amassée quantique $\mathcal{A}_t$.

Pour l'inclusion inverse, il s'agit de montrer que toutes les variables d'amas quantiques sont des éléments de l'anneau de Grothendieck quantique. Nous procédons par récurrence sur la longueur des chemins dans le graphe d'échange de l'algèbre amassée quantique. Les variables d'amas initiales sont effectivement dans l'anneau de Grothendieck quantique, comme démontré en Proposition 3.5.2. Pour l'induction, nous utilisons la caractérisation des éléments de l'anneau de Grothendieck quantique comme zéros d'une famille d'opérateurs appelés *opérateurs d'écrantage*. Ce point de la preuve est particulièrement délicat à généraliser par rapport à ce qui a été fait dans [HL16a]. En fait, Hernandez a défini dans [Her03] des t-déformations de ces opérateurs d'écrantage. Il s'agit d'une famille d'opérateurs

$$S_{i,t} : \mathcal{Y}_t \to \mathcal{Y}_{i,t}, \tag{I.2.3.4}$$

où $\mathcal{Y}_{i,t}$ est un certain $\mathcal{Y}_t$-module, qui sont des dérivations, et tels que

$$K_t(\mathscr{C}_{\mathbb{Z}}) = \bigcap_{i\in I} \ker(S_{i,t}). \tag{I.2.3.5}$$

Le fait que ces opérateurs soient des dérivations permet de transmettre naturellement la propriété d'appartenance à l'anneau de Grothendieck quantique aux nouvelles variables d'amas crées par mutations. En effet, si la nouvelle relation d'échange est de la forme

$$Z * Z_1 = t^{\alpha} M_1 + t^{\beta} M_2,$$

où nous supposons par récurrence que la variable d'amas Z, ainsi que les monômes d'amas M_1 et M_2 sont dans l'anneau de Grothendieck quantique, alors l'application des opérateurs d'écrantage $S_{i,t}$ permet d'obtenir

$$0 = Z * S_{i,t}(Z_1).$$

Pour en déduire que la nouvelle variable d'amas Z_1 est aussi dans le noyau de l'opérateur nous utilisons le fait que le module $\mathcal{Y}_{i,t}$ est *libre*. Dans le cas non t-déformé, la liberté de ce module est donnée dans la construction [FR99] ; en revanche dans notre situation elle n'est pas évidente car ce module est défini comme un quotient. Néanmoins, nous le montrons dans le Lemme 3.3.4 de la Section 3.3.3. On peut noter que la caractérisation (I.2.3.5) de l'anneau de Grothendieck quantique comme intersection des noyaux des opérateurs d'écrantage est valable pour l'anneau de Grothendieck quantique $K_t(\mathscr{C}_\mathbb{Z})$ et non pour celui qui nous intéresse $K_t(\mathscr{C}_\mathbb{Z}^-)$. Pour outrepasser cette difficulté, il nous faut au préalable se ramener au cas de $K_t(\mathscr{C}_\mathbb{Z})$ en appliquant à notre algèbre amassée des suites infinies de mutations, qui sont en fait localement des suites finies.

En conclusion, nous avons montré que cette structure d'algèbre amassée quantique, construite à partir de la structure d'algèbre amassée de l'anneau de Grothendieck $K_0(\mathscr{C}_\mathbb{Z}^-)$, et des relations de t-commutations du tore quantique $\mathcal{Y}_t$, est égale à l'anneau de Grothendieck quantique $K_t(\mathscr{C}_\mathbb{Z}^-)$.

L'avantage de cette approche est qu'elle fournit un algorithme permettant de calculer les (q,t)-caractères des modules de Kirillov-Reshetikhin, et donc en particulier des modules fondamentaux. Dans le Chapitre 3, nous nous sommes intéressés aux nombre précis d'étapes qui étaient nécessaires pour obtenir ces (q,t)-caractères complets et il semble clair que le nombre d'étapes de calcul est largement inférieur à celui de l'algorithme de Frenkel-Mukhin (voir Remarque 3.6.10). De plus, cet algorithme ne nécessite pas de garder en mémoire toutes les étapes de calcul précédentes, mais uniquement la graine courante, ce qui pourrait permettre de palier aux problèmes de mémoire rencontrés lorsque l'on essaie de calculer explicitement ces (q,t)-caractères en dimension élevée (voir [Nak10]).

Notons que Qin a obtenu dans [Qin16] des résultats parallèles à ceux présentés ici. Il a démontré que pour certaines sous-catégories de $\mathscr{C}$ engendrées par un nombre fini de modules fondamentaux, leur anneau de Grothendieck quantique avait une structure d'algèbre amassée quantique. Dans notre travail, nous présentons une preuve plus directe de ce résultat, en explicitant une suite de mutation permettant d'obtenir les (q,t)-caractères des modules fondamentaux.

I.2.4 Anneau de Grothendieck quantique pour la catégorie $\mathcal{O}^+$ et calcul des (q,t)-caractères

On a donc montré que pour les représentations de dimension finie, l'anneau de Grothendieck quantique avait une structure d'algèbre amassée quantique. De plus, nous avons déjà utilisé en Section I.2.2 l'apport des algèbres amassées quantiques pour fixer les fonctions de t-commutation entres les variables $\mathbf{\Psi}_{i,r}^{\pm 1}$. Tout cela nous encourage à construire l'anneau de Grothendieck quantique $K_t(\mathcal{O}_\mathbb{Z}^+)$ *comme* une algèbre amassée.

D'autant plus que, dans le cas de la catégorie $\mathcal{O}$, les approches usuelles ne fonctionnent pas. En effet, l'approche géométrique est inenvisageable, car il n'existe pas de construction géométrique des représentations de la catégorie $\mathcal{O}$. Une autre approche usuelle est de voir l'anneau de Grothendieck quantique comme un espace invariant par des symétries de Weyl. Par exemple, la caractérisation (I.2.3.5) pour la catégorie $\mathscr{C}_\mathbb{Z}$ est ce qui a permis à Hernandez de définir un anneau de Grothendieck quantique et des (q,t)-caractères pour les cas non-simplement lacés dans [Her04]. Malheureusement, il n'existe pas de caractérisation semblable pour la catégorie $\mathcal{O}$, et ainsi l'approche par les algèbres amassées quantiques semble être celle qui fonctionne.

Nous avons déjà présenté le fait que le carquois infini Γ et la fonction de t-commutation, encodée dans la matrice $\hat{I} \times \hat{I}$ anti-symétrique $\tilde{B}$ formaient une paire compatible. Nous posons donc en Définition 2.4.15,

$$K_t(\mathcal{O}_{\mathbb{Z}}^+) := \mathcal{A}_t(\Gamma, \tilde{B})\hat{\otimes}_{\mathbb{Z}[t^{\pm 1/2}]}\mathcal{E} \quad \subset \mathscr{T}_t, \tag{I.2.4.1}$$

où le produit tensoriel est complété comme en (I.1.7.1) et en (I.2.2.4).

Bien sûr, il est important de vérifier que cette définition d'anneau de Grothendieck quantique pour la catégorie $\mathcal{O}_{\mathbb{Z}}^+$ est cohérente avec les structures préexistantes. En particulier, nous savons que la catégorie $\mathcal{O}_{\mathbb{Z}}^+$ contient la catégorie $\mathscr{C}_{\mathbb{Z}}$, il faut donc déjà voir que $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ contient l'anneau de Grothendieck quantique $K_t(\mathscr{C}_{\mathbb{Z}})$.

Pour cela, nous utilisons la structure d'algèbre amassée quantique de $K_t(\mathscr{C}_{\mathbb{Z}}^-)$ que nous venons d'étudier. Nous montrons qu'en appliquant la même suite de mutations qui permettent dans $K_t(\mathscr{C}_{\mathbb{Z}}^-)$ d'obtenir les (q,t)-caractères (non tronqués) des représentations fondamentales dans l'algèbre amassée $K_t(\mathcal{O}_{\mathbb{Z}}^+)$, nous obtenons encore les (q,t)-caractères des représentations fondamentales. Ainsi, ces (q,t)-caractères sont des variables d'amas dans l'algèbre affine quantique $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ et nous en déduisons le résultat principal du Chapitre 3 :

Théorème 10. *(Théorème 3.6.5) L'anneau de Grothendienck quantique $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ contient celui de la catégorie $\mathscr{C}_{\mathbb{Z}}^-$:*

$$K_t(\mathscr{C}_{\mathbb{Z}}) \subset K_t(\mathcal{O}_{\mathbb{Z}}^+). \tag{I.2.4.2}$$

Pour pouvoir obtenir ce résultat, il s'agit de voir l'algèbre amassée quantique $\mathcal{A}_t(G^-, L)$ comme une sous-structure de $\mathcal{A}_t(\Gamma, \Lambda)$. Sur ce point, les algèbres amassées quantiques sont plus délicates à gérer que leurs homologues classiques. La bonne définition de sous-structure pour les algèbres amassées quantiques a été donnée par Grabowski et Gratz dans [GG18] et elle nécessite que les carquois soient *seulement connectés par coefficients*, ce qui signifie que les sommets du plus petit carquois qui sont reliés à des sommets qui ne sont que dans le plus grand carquois correspondent à des variables gelées. Nous constatons que les carquois Γ et G^- ne sont pas uniquement connectés par coefficients. Nous introduisons donc un carquois intermédiaire Γ^- en ajoutant des variables gelées à G^-. Les carquois Γ et Γ^- sont alors uniquement connectés par coefficients, et $\mathcal{A}_t(\Gamma^-, \Lambda)$ est une sous-algèbre amassée quantique de $\mathcal{A}_t(\Gamma, \Lambda)$.

De plus, l'algèbre amassée quantique $\mathcal{A}_t(\Gamma^-, \Lambda)$ peut être obtenue comme une version twistée de $\mathcal{A}_t(G^-, L)$, dans les sens des twists apparaissant dans les travaux de Grabowski et Launois [GL14]. Nous en déduisons que les variables d'amas de $\mathcal{A}_t(\Gamma^-, \Lambda)$ sont des variables d'amas de $\mathcal{A}_t(G^-, L)$, à multiplication par les variables gelées qui ont été ajoutées près. La combinaison de ces deux approches permet de faire un lien entre $\mathcal{A}_t(G^-, L)$ et $\mathcal{A}_t(\Gamma, \Lambda)$ et de voir les (q,t)-caractères des représentations fondamentales comme des variables d'amas de $\mathcal{A}_t(\Gamma, \Lambda)$ et de montrer l'inclusion (I.2.4.2).

Dans le Chapitre 2, cette inclusion avait été présentée sous forme de conjecture, mais démontrée lorsque l'algèbre de Lie simple de dimension finie $\mathfrak{g}$ est de type A. Dans ce cas, les arguments utilisés ne sont pas exactement les mêmes. En type A, il est connu que les espaces de ℓ-poids des modules fondamentaux sont de dimension 1 (voir [FR96, Section 11] et les références qui y sont mentionnées). De plus, nous savons par [HL16b] que dans l'algèbre amassée (classique) $\mathcal{A}(\Gamma)$, il existe des suites de mutations permettant d'obtenir les classes des représentations fondamentales comme variables d'amas. Dans la Proposition 2.5.9 du Chapitre 2, nous constatons que les variables d'amas quantiques obtenues en appliquant la même suite de mutations dans l'algèbre amassée quantique $\mathcal{A}_t(\Gamma, \Lambda)$ sont des éléments invariants par l'involution barre, qui redonnent les q-caractères des représentations fondamentales en les évaluant en $t = 1$, et dont la décomposition sur la base des monômes commutatifs est à coefficients positifs. En type A, cela suffit à montrer que ces variables d'amas quantique s'identifient aux (q,t)-caractères des représentations fondamentales :

Théorème 11. *(Théorème 2.6.1) Les variables d'amas obtenues en appliquant cette suite de mutations sont les (q,t)-caractères des représentations fondamentales.*

On remarque que dans ce cas, nous utilisons de façon cruciale le résultat de positivité des variables d'amas quantiques de Davison [Dav18]. De plus, cette preuve ne fonctionne pas en dehors du type A, où des multiplicité apparaissent et où les coefficients des q-caractères des modules fondamentaux ont plusieurs t-déformations possibles.

En particulier, en Section 3.7, nous calculons explicitement le (q,t)-caractère de la représentation fondamentale $L(Y_{2,-6})$, quand $\mathfrak{g}$ est de type D_4. Il s'agit de l'exemple le plus simple où une multiplicité apparait (voir Remarque 3.7.1).

Enfin, à partir de cette donnée, nous pouvons construire des (q,t)-caractères pour les représentations préfondamentales positives dans la catégorie $\mathcal{O}^+_{\mathbb{Z}}$. En effet, si nous suivons la même logique, les (q,t)-caractères doivent être obtenus dans l'algèbre amassée quantique $\mathcal{A}_t(\Gamma,\Lambda)$ par les mêmes mutations utilisées pour obtenir les classes des représentations préfondamentales dans l'algèbre amassée $\mathcal{A}(\Gamma)$. Or, dans l'isomorphisme (I.1.7.1), les classes des représentations préfondamentales sont prises comme variables d'amas initiales, à un poids près :

$$z_{i,r} \equiv [L^+_{i,q^r}] \otimes \left[\frac{-r}{2}\omega_i\right].$$

Pour obtenir leurs q-caractères, nous multiplions ces variables par un poids inversible :

$$\chi_q(L^+_{i,q^r}) = [\mathbf{\Psi}_{i,r}]\chi_i \quad \in \mathcal{E}_\ell,$$

en utilisant la notation $[\mathbf{\Psi}_{i,r}] = z_{i,r}\left[\frac{r}{2}\omega_i\right]$.

Le même procédé dans l'anneau de Grothendieck quantique $K_t(\mathcal{O}^+_{\mathbb{Z}})$ donne donc comme (q,t)-caractères (voir Définition 2.5.3.1) :

$$[L_{i,q^r}]_t := [\mathbf{\Psi}_{i,r}] \otimes \chi_i \quad \in \mathscr{T}_t. \tag{I.2.4.3}$$

I.2.5 Relations remarquables dans l'anneau de Grothendieck quantique

Après avoir défini l'anneau de Grothendieck quantique $K_t(\mathcal{O}^+_{\mathbb{Z}})$, ainsi que les (q,t)-caractères des représentations préfondamentales positives $[L_{i,q^r}]$, il est possible de travailler avec ces objets. Par exemple, nous pouvons écrire des relations qui sont vérifiées dans cet anneau. De plus, la $\mathbb{Z}[t^{\pm 1/2}]$-algèbre $K_t(\mathcal{O}^+_{\mathbb{Z}})$ a été définie comme une algèbre amassée quantique, une structure qui est riche en relations.

On peut commencer par écrire une relation d'échange provenant d'une mutation quelconque à n'importe quel sommet du carquois Γ. Sur la base des monômes commutatifs, pour tout $(i,r) \in \hat{I}$, il existe $\alpha, \beta \in \frac{1}{2}\mathbb{Z}$ tels que :

$$z^{(1)}_{i,r} * z_{i,r} = t^\alpha z_{i,r+2}\prod_{j\sim i} z_{j,r-1} + t^\beta z_{i,r-2}\prod_{j\sim i} z_{j,r+1}. \tag{I.2.5.1}$$

Lorsque $\mathfrak{g} = \mathfrak{sl}_2$, cette relation a une traduction particulièrement intéressante en termes de (q,t)-caractères. Dans ce cas, (I.2.5.1) devient :

$$z^{(1)}_{2r} * z_{2r} = t^{1/2} z_{2r+2} + t^{-1/2} z_{2r-2}. \tag{I.2.5.2}$$

Or, dans ce cas, c'est cette relation d'échange qui permet d'obtenir le (q,t)-caractère de la représentation fondamentale, par le raisonnement expliqué précédemment,

$$z_{2r}^{(1)} = \mathcal{J}([L(Y_{2r-1})]_t).$$

Donc (I.2.5.2) se traduit par le résultat suivant :

Proposition 12. *(Proposition 2.6.3) Pour tout $r \in \mathbb{Z}$, on a, dans l'anneau de Grothendieck quantique $K_t(\mathcal{O}_{\mathbb{Z}}^+)$,*

$$[L(Y_{2r-1})]_t * [L_{2r}]_t = t^{-1/2}[\omega_1][L_{2r-2}]_t + t^{1/2}[-\omega_1][L_{2r+2}]_t. \tag{I.2.5.3}$$

Il s'agit d'une version quantique de la relation de Baxter (I.1.6.3), qui était une version catégorifiée de la relation originelle de Baxter (I.1.1.1).

Quand $\mathfrak{g} = \mathfrak{sl}_2$, nous remarquons une structure particulièrement intéressante dans une sous-algèbre de $K_t(\mathcal{O}_{\mathbb{Z}}^+)$. Ce sont les résultats détaillés dans la Section 2.6.2. Si nous nous intéressons à une sous-algèbre amassée quantique $\mathcal{A}_t(\Gamma_1,\Lambda_1)$ de $\mathcal{A}_t(\Gamma,\Lambda)$ de type fini, alors nous pouvons fixer des variables d'amas particulières

$$E, F, K, K' \quad \in \mathcal{A}_t(\Gamma_1,\Lambda_1),$$

qui engendrent l'algèbre amassée $\mathcal{A}_t(\Gamma_1,\Lambda_1)$ et qui permettent de définir un isomorphsime entre $\mathcal{A}_t(\Gamma_1,\Lambda_1)$ et un quotient du double de Drinfeld de paramètre $-t^{1/2}$.

Le double de Drinfeld $\mathfrak{D}_2$ que nous considérons est le double de Drinfeld, au sens de [Dri87], de la sous-algèbre de Borel du groupe quantique (non-affinisé) $\mathcal{U}_q(\mathfrak{sl}_2)$. Il s'agit de la $\mathbb{C}(q)$-algèbre engendrée par E, F, K, K' et par les relations

$$\begin{array}{ll} KE = q^2EK, & K'E = q^{-2}EK' \\ KF = q^{-2}FK, & K'F = q^2FK' \\ KK' = K'K, & \text{ et } [E,F] = (q-q^{-1})(K-K'). \end{array}$$

L'élément de Casimir est alors :

$$C_q := EF + qK + q^{-1}K'.$$

Proposition 13. *(Proposition 2.6.6) L'algèbre amassée quantique $\mathcal{A}(\Lambda_1,\Gamma_1)$ est isomorphsime au quotient du double de Drinfeld par l'élément de Casimir :*

$$\mathcal{A}(\Lambda_1,\Gamma_1) \xrightarrow{\sim} \mathfrak{D}_2/C_{-t^{1/2}}. \tag{I.2.5.4}$$

Ce résultat est à rapprocher du travail de Schrader et Shapiro [SS19], où des liens sont explicités entre des variétés d'amas et le groupe quantique $\mathcal{U}_q(\mathfrak{sl}_{n+1})$.

Il est intéressant de noter que l'isomorphisme (I.2.5.4) fait intervenir le groupe quantique $\mathcal{U}_q(\mathfrak{sl}_2)$ entier. Il existe de nombreux résultats de ce type, comme par exemple dans [GLS13], ou [HL15], mais qui ne réalisent que la partie positive du groupe quantique.

Dans le Chapitre 4, nous présentons quelques pistes d'ouvertures des travaux présentés dans cette thèse. En particulier, la Section 4.1 présente des résultats plus avancés dans le cas où $\mathfrak{g} = \mathfrak{sl}_2$ que nous espérons généraliser par la suite. Nous trouvons dans cette section d'autres relations importantes qui peuvent être écrites en type A_1.

Pour ce type, nous définissons un (q,t)-caractère pour les représentations préfondamentales négatives de la façon suivante (voir (4.2.1.2)) :

$$[L_{q^{2r}}^-]_t := [\boldsymbol{\Psi}_{q^{2r}}^{-1}]\left(1 + \sum_{m=1}^{+\infty}\prod_{l=0}^{m} A_{q^{2r-2l}}^{-1}\right) \quad \in \mathscr{T}_t.$$

Ce qui nous permet d'écrire une nouvelle version de la relation de Baxter quantifiée (I.2.5.3), en termes de représentations préfondamentales négatives.

Proposition 14. *(Proposition 4.1.4) Pour tout $r \in \mathbb{Z}$:*

$$[L(Y_{q^{2r+1}})]_t * [L_{q^{2r}}]_t = t^{1/2}[\omega_1][L^-_{q^{2r+2}}]_t + t^{-1/2}[-\omega_1][L^-_{q^{2r-2}}]_t.$$

On peut aussi former une relation entre les (q,t)-caractères des représentations préfondamentales positives et négatives. Cette relation, appelée *Relation Wronskien quantique*, a des généralisation à d'autres types sous le nom de *systèmes* $Q\tilde{Q}$. Ce type de relation est fortement liée aux systèmes intégrables quantiques dont il a été question en Section I.1.1 car elles permettent d'encoder en une relation le système d'équations de l'Ansatz de Bethe.

La version t-déformée de la relation de Wronskien quantique est la suivante :

Proposition 15. *(Proposition 4.1.11) Pour tout $r \in \mathbb{Z}$,*

$$[L^+_{q^{2r}}]_t * [L^-_{q^{2r}}]_t - t^{-1}[-\alpha_1][L^+_{q^{2r+2}}]_t * [L^+_{q^{2r-2}}]_t = \chi_1.$$

I.2.6 Pistes d'ouverture

Dans le Chapitre 4, en plus de résultats supplémentaires dans le cas $\mathfrak{g} = \mathfrak{sl}_2$, sont présentées quelques pistes de généralisations de certains de ces résultats.

En particulier, nous proposons en Section 4.2.1 une définition des (q,t)-caractères des représentations préfondamentales négatives, comme des éléments du tore quantique $\mathscr{T}_t$. Ces (q,t)-caractères sont vus comme des limites de (q,t)-caractères de modules de Kirillov-Reshetikhin (voir (4.2.1.2)).

De plus, nous présentons en Section 4.2.2 une méthode pour définir un anneau de Grothendieck quantique $K_t(\mathcal{O}^-_{\mathbb{Z}})$ pour la catégorie $\mathcal{O}^-_{\mathbb{Z}}$, basée elle-aussi sur une structure d'algèbre amassée quantique, ainsi que des arguments en faveur de cette méthode.

Enfin, en Section 4.2.3, nous évoquons la généralisation de nos résultats aux algèbres de Lie $\mathfrak{g}$ non-simplement lacées.

Chapter 1

Asymptotics of standard modules of quantum affine algebras

This chapter is an adapted version of [Bit18], published in *Algebras and Representation Theory*.

Abstract. We introduce a sequence of q-characters of standard modules of a quantum affine algebra and we prove it has a limit as a formal power series. For $\mathfrak{g} = \hat{\mathfrak{sl}}_2$, we establish an explicit formula for the limit which enables us to construct corresponding asymptotical standard modules associated to each simple module in the category $\mathcal{O}$ of a Borel subalgebra of the quantum affine algebra. Finally, we prove a decomposition formula for the limit formula into q-characters of simple modules in this category $\mathcal{O}$.

Introduction

The quantized universal enveloping algebra of a finite-dimensional simple Lie algebra $\mathfrak{g}$ was introduced independently by Drinfeld [Dri87] and Jimbo [Jim85] in 1985. It is a q-deformation of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, for q a generic complex number, and it has a Hopf algebra structure. If $\mathfrak{g}$ is an untwisted affine Kac-Moody algebra, then its universal enveloping algebra also admits a q-deformation, called the *quantum affine* *algebra* $\mathcal{U}_q(\mathfrak{g})$. The quantum affine algebra can also be obtained as a quantum affinization of the quantized universal enveloping algebra of the corresponding simple Lie algebra. Both processes (affinization then quantization and quantization then quantum affinization) commute ([Dri88, Bec94]). The quantum affine algebra has a presentation with the so-called "Drinfeld-Jimbo generators", and from this presentation one can define its Borel subalgebra $\mathcal{U}_q(\mathfrak{b}) \subset \mathcal{U}_q(\mathfrak{g})$.

Both the quantum affine algebra and the Borel algebra are Hopf algebras, thus their categories of finite-dimensional representations are monoidal categories. In [FR99], Frenkel and Reshetikhin defined the q-character χ_q, which is an injective ring morphism on the Grothendieck ring of the category of finite-dimensional representations of $\mathcal{U}_q(\mathfrak{g})$, mimicking the usual character morphism. For every finite-dimensional representation V of $\mathcal{U}_q(\mathfrak{g})$, one can consider its q-character $\chi_q(V)$. It describes the decomposition of V into ℓ-weight spaces the action of the ℓ-Cartan subalgebra of $\mathcal{U}_q(\mathfrak{g})$. These ℓ-weight are more suited than the weights spaces to the study of finite-dimensional $\mathcal{U}_q(\mathfrak{g})$-modules (for detailed results, see [FM01] for example).

Hernandez and Jimbo introduced in [HJ12] a category $\mathcal{O}$ of representations of the Borel algebra $\mathcal{U}_q(\mathfrak{b})$. Objects in this category $\mathcal{O}$ are sums of their ℓ-weight spaces and have finite-dimensional weight spaces, with weights in a finite union of certain cones. The category $\mathcal{O}$ contains the finite-dimensional representations, as well as some other remarkable representations, called *prefundamental representations*. For $\mathfrak{g} = \hat{\mathfrak{sl}}_2$, the latter appeared naturally, under the name *q-oscillator representations* in Conformal Field Theory (for example in [BLZ99]). In general, the prefundamental representations are constructed as limits of particular sequences of simple finite-dimensional modules [HJ12]. Note that the prefundamental representations were used in [FH15] to prove the Frenkel-Reshetikhin's conjecture on the spectra of quantum integrable systems [FR99] (generalizing certain results of Baxter [Bax72] on the spectra of the 6 and 8-vertex models).

The q-character morphism extends to a well-defined injective ring morphism on the Grothendieck ring of the category $\mathcal{O}$ ([HJ12]).

In the category $\mathcal{O}$, objects do not necessarily have finite lengths (see [BJM$^+$09, Lemma C.1], the tensor product of some prefundamental representations presents an infinite filtration of submodules), thus the category $\mathcal{O}$ is not a Krull-Schmidt category. However, as in the case of the classical category $\mathcal{O}$ of representations of a Kac-Moody Lie algebra, the multiplicity of a simple module in a module is well-defined. As such, all the elements of the Grothendieck ring of $\mathcal{O}$ can be written as (infinite) sums of classes of simple modules.

There is a second basis of the Grothendieck ring of finite-dimensional $\mathcal{U}_q(\mathfrak{b})$-modules formed by the *standard modules*, introduced in [Nak01a]. Both simple and standard modules are indexed by the same set: the monomials m in the ring $\mathbb{Z}[Y_{i,a}]_{i\in I, a\in\mathbb{C}^\times}$. In [Nak01a], Nakajima showed that for the simply laced types, the multiplicities of the simple modules in the standard modules can be realized as dimensions of certain varieties. Nakajima showed that the transition matrix between theses two bases is upper triangular:

$$[M(m)] = [L(m)] + \sum_{m'<m} P_{m',m}[L(m')], \tag{1.0.0.1}$$

for some partial order $\leqslant$ on the monomials in $\mathbb{Z}[Y_{i,a}]_{i\in I, a\in\mathbb{C}^\times}$, the exact definition of the order is recalled in (1.2.2.3), in the proof of Theorem 1.2.2.

Moreover, the coefficients $P_{m',m} \in \mathbb{Z}$ are non-negative. Moreover, using t-deformations of the q-characters, Nakajima showed ([Nak04]) that the coefficients $P_{m',m}$ can be computed as the evaluation at $t = 1$ of some polynomials (analogs of the Kazhdan-Lusztig polynomials for Weyl groups). With this type of formula, one can hope to deduce from the q-characters of the standard modules the q-characters of the simple modules, which are not known in the general case.

One would want to have the same type of decomposition in the category $\mathcal{O}$, in which the simple modules also form a basis. For that, the first step is to build analogs of the standard modules corresponding to each simple module in the category $\mathcal{O}$.

In order to do that, let us recall that the q-characters of the prefundamental representations can be obtained as limits (as formal power series) of sequences of q-characters of simple modules.

In the present Chapter, we establish that a limit of a particular sequence of q-characters of standard modules exists. We conjecture that this limit is the q-character of a certain asymptotical standard module. In the case $\mathfrak{g} = \hat{\mathfrak{sl}}_2$ we give an explicit formula for the limit and we prove this conjecture. In this case, the simple modules are known for certain natural monoidal subcategories (see [HL16b, Theorem 7.9]), thus we can work the other way and deduce from the simple modules information about the standard modules. Furthermore, will we show at the end of the chapter that the q-characters of these asymptotical standard modules satisfy a decomposition formula of the type (1.0.0.1).

This chapter is organized as follows. The first section presents the background for our work. It gathers the definitions and already known properties of the objects we study. We start with the definitions of the quantum affine algebra and the Borel algebra. Section 1.1.2 presents some results on the representations of the Borel algebra, specifically the category $\mathcal{O}$ of Hernandez-Jimbo. Finally, we present the q-character theory for the category $\mathcal{O}$.

In Section 1.2, we recall the definitions of standard modules for finite-dimensional representations, and study more precisely their q-characters, which will be the base for the construction of the asymptotical standard modules. We prove the convergence, as a formal power series, of some sequences of normalized q-character of standard modules (Theorem 1.2.2). Then we conjecture (Conjecture 1.2.7) that these limits are q-characters of certain asymptotical standard modules.

From Section 1.3 on, we focus on the case where $\mathfrak{g} = \hat{\mathfrak{sl}}_2$. Section 1.3 tackles the technical part of the construction of the standard modules, with the two main theorems. First, we build a $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$-module T, with finite-dimensional ℓ-weight spaces, such that its normalized q-character is the limit obtained from the result of Theorem 1.2.2 (Theorem 1.3.12). Then we induce from T a $\mathcal{U}_q(\mathfrak{b})$-module with finite-dimensional ℓ-weight spaces and the same q-character as T (Theorem 1.3.19).

Finally, in Section 1.4 we present some results for the decomposition of the q-characters of our asymptotical standard modules. The last theorem is a result of the type of (1.0.0.1): the limit obtained in Theorem 1.2.2, which from Theorem 1.3.19 can be realized as the q-character of a $\mathcal{U}_q(\mathfrak{b})$-module, admits a decomposition into a sum of q-characters of simple modules in the category $\mathcal{O}$. Moreover, the coefficients are non-negative integers (Theorem 1.4.3).

1.1 Background

1.1.1 The quantum affine algebra and its Borel subalgebra

Let us start by recalling the definitions of the two main algebras we study: the quantum affine algebra $\mathcal{U}_q(\mathfrak{g})$ and its Borel subalgebra for the Drinfeld-Jimbo generators $\mathcal{U}_q(\mathfrak{b})$.

Quantum affine algebra

Let $\mathfrak{g}$ be an untwisted affine Kac-Moody algebra, with $C = (C_{i,j})_{0\leqslant i,j\leqslant n}$ its Cartan matrix. Let $\dot{\mathfrak{g}}$ be the associated finite-dimensional simple Lie algebra, let $I = \{1, \dots, n\}$ be the vertices of its Dynkin diagram and $\dot{C} = (C_{i,j})_{i,j\in I}$ its Cartan matrix.

Let $(\alpha_i)_{i\in I}$, $(\alpha_i^\vee)_{i\in I}$ and $(\omega_i)_{i\in I}$ be the simple roots, the simple coroots and the fundamental weights of $\dot{\mathfrak{g}}$, respectively. We use the usual lattices $Q = \bigoplus_{i\in I}\mathbb{Z}\alpha_i$, $Q^+ = \bigoplus_{i\in I}\mathbb{N}\alpha_i$ and $P = \bigoplus_{i\in I}\mathbb{Z}\omega_i$. Let $P_{\mathbb{Q}} = P\otimes\mathbb{Q}$, endowed with the partial ordering: $\omega \leqslant \omega'$ if and only if $\omega' - \omega \in Q^+$. Let $D = \mathrm{diag}(d_0, d_1, \dots, d_n)$ be the unique diagonal matrix such that $B = DC$ is symmetric and the $(d_i)_{0\leqslant i\leqslant n}$ are relatively prime positive integers.

Fix an non-zero complex number q, which is not a root of unity, and $h \in \mathbb{C}$ such that $q = e^h$. Then for all $r \in \mathbb{Q}, q^r := e^{rh}$. Since q is not a root of unity, for $r, s \in \mathbb{Q}$, we have $q^r = q^s$ if and only if $r = s$. We set $q_i := q^{d_i}$, for $0 \leqslant i \leqslant n$.

We use the following notations, for $m, r, s \in \mathbb{N}, r \geqslant s$ and $z \in \mathbb{C}^\times$,

$$[m]_z = \tfrac{z^m - z^{-m}}{z - z^{-1}}, \quad [m]_z! = \textstyle\prod_{j=1}^m [j]_z, \quad \begin{bmatrix} r \\ s \end{bmatrix}_z = \tfrac{[r]_z!}{[s]_z![r-s]_z!}\ .$$

Definition 1.1.1. One defines the *quantum affine algebra* $\mathcal{U}_q(\mathfrak{g})$ as the $\mathbb{C}$-algebra generated by $e_i, f_i, k_i^{\pm 1}, 0 \leqslant i \leqslant n$, together with the following relations, for $0 \leqslant i, j \leqslant n$,

$$k_i k_j = k_j k_i, \quad [e_i, f_j] = \delta_{i,j}\frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \quad k_i k_i^{-1} = k_i^{-1}k_i = 1,$$
$$k_i e_j k_i^{-1} = q_i^{C_{i,j}} e_j, \quad k_i f_j k_i^{-1} = q_i^{-C_{i,j}} e_j,$$
$$\sum_{r=0}^{1-C_{i,j}} (-1)^r e_i^{(1-C_{i,j}-r)} e_j e_i^{(r)} = 0, \ (i \neq j),$$
$$\sum_{r=0}^{1-C_{i,j}} (-1)^r f_i^{(1-C_{i,j}-r)} f_j f_i^{(r)} = 0, \ (i \neq j)$$

where $x_i^{(r)} = x_i^r/[r]_{q_i}!, (x_i = e_i, f_i)$.

The algebra $\mathcal{U}_q(\mathfrak{g})$ has another presentation, with the *Drinfeld generators* ([Dri88], [Bec94])

$$x_{i,r}^{\pm}(i \in I, r \in \mathbb{Z}), \quad \phi_{i,\pm m}^{\pm}(i \in I, m \geqslant 0), \quad k_i^{\pm 1}(i \in I),$$

and some relations we will not recall here, but which are also q-deformations of the Weyl and Serre relations.

Example 1.1.2. For $\mathfrak{g} = \hat{\mathfrak{sl}}_2$, one has the following correspondence

$$e_1 = x_{1,0}^+, \quad f_1 = x_{1,0}^-,$$
$$e_0 = k_1^{-1} x_{1,1}^-, \quad f_0 = x_{1,-1}^+ k_1.$$

Let us introduce the generating series, for $i \in I$

$$\phi_i^{\pm}(z) = \sum_{m\geqslant 0} \phi_{i,\pm m}^{\pm} z^{\pm m} = k_i^{\pm 1}\exp\left(\pm(q_i - q_i^{-1})\sum_{r>0} h_{i,\pm r} z^{\pm r}\right) \quad \in \mathcal{U}_q(\mathfrak{h})[z^{\pm 1}].$$

Thus, the $(\phi_{i,\pm m}^{\pm})_{i\in I, m\geqslant 0}$ and the $(k_i^{\pm 1}, h_{i,\pm r})_{i\in I, r>0}$ generate the same subalgebra of $\mathcal{U}_q(\mathfrak{g})$: the ℓ-Cartan subalgebra $\mathcal{U}_q(\mathfrak{g})^0$.

The $(h_{i,\pm r})_{i\in I, r>0}$ can be useful because their relations with the Drinfeld generators are simpler to write. For example, for all $i,j\in I$, $r\in\mathbb{Z}\backslash\{0\}$ and $m\in\mathbb{Z}$,

$$[h_{i,r}, x^{\pm}_{j,m}] = \pm\frac{[r\dot{C}_{i,j}]_q}{r}x^{\pm}_{j,r+m}. \tag{1.1.1.1}$$

The quantum affine algebra has a triangular decomposition, associated to the Drinfeld generators: let $\mathcal{U}_q(\mathfrak{g})^{\pm}$ be the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by the $(x^{\pm}_{i,r})_{i\in I, r\in\mathbb{Z}}$. Then ([Bec94]),

$$\mathcal{U}_q(\mathfrak{g}) \simeq \mathcal{U}_q(\mathfrak{g})^- \otimes \mathcal{U}_q(\mathfrak{g})^0 \otimes \mathcal{U}_q(\mathfrak{g})^+. \tag{1.1.1.2}$$

The algebra $\mathcal{U}_q(\mathfrak{g})$ has a natural Q-grading, with, for all $i\in I, m\in\mathbb{Z}, r>0$,

$$\deg(x^{\pm}_{i,m}) = \pm\alpha_i, \quad \deg(h_{i,r}) = \deg(k_i^{\pm 1}) = 0. \tag{1.1.1.3}$$

The Borel subalgebra

Definition 1.1.3. The *Borel algebra* $\mathcal{U}_q(\mathfrak{b})$ is the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by the $e_i, k_i^{\pm 1}$, for $0\leqslant i\leqslant n$.

Let $\mathcal{U}_q(\mathfrak{b})^{\pm} = \mathcal{U}_q(\mathfrak{g})^{\pm}\cap\mathcal{U}_q(\mathfrak{b})$ and $\mathcal{U}_q(\mathfrak{b})^0 = \mathcal{U}_q(\mathfrak{g})^0\cap\mathcal{U}_q(\mathfrak{b})$, then [BCP99]

$$\mathcal{U}_q(\mathfrak{b})^+ = \langle x^+_{i,m}\rangle_{i\in I, m\geqslant 0}, \quad \mathcal{U}_q(\mathfrak{b})^0 = \langle \phi^+_{i,r}, k_i^{\pm}\rangle_{i\in I, r>0}.$$

Remark 1.1.4. In general, such a nice description does not exist for $\mathcal{U}_q(\mathfrak{b})^-$, except when $\mathfrak{g} = \hat{\mathfrak{sl}}_2$. In that case, $\mathcal{U}_q(\mathfrak{b})^-$ is isomorphic to the algebra defined by the generators $(x^-_{1,m})_{m\geqslant 1}$, together with the relations ([Jan96, Section 4.21]), for all $m,l\geqslant 1$, $i,j\in I$

$$x^{\pm}_{i,m+1}x^{\pm}_{j,l} - q^{\pm\dot{C}_{i,j}}x^{\pm}_{j,l}x^{\pm}_{i,m+1} = q^{\pm\dot{C}_{i,j}}x^{\pm}_{i,m}x^{\pm}_{j,l+1} - x^{\pm}_{j,l+1}x^{\pm}_{i,m}. \tag{1.1.1.4}$$

We also use the subalgebra $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$:

$$\mathcal{U}_q^{\geqslant 0}(\mathfrak{b}) := \langle x^+_{i,m}, \phi^+_{i,r}, k_i^{\pm}\rangle_{i\in I, m\geqslant 0, r>0}. \tag{1.1.1.5}$$

The triangular decomposition of (1.1.1.2) carries over ([Bec94]):

$$\mathcal{U}_q(\mathfrak{b}) \simeq \mathcal{U}_q(\mathfrak{b})^- \otimes \mathcal{U}_q(\mathfrak{b})^0 \otimes \mathcal{U}_q(\mathfrak{b})^+.$$

From now on, we are going to consider representations of the Borel algebra $\mathcal{U}_q(\mathfrak{b})$.

Example 1.1.5. The algebra $\mathcal{U}_q(\mathfrak{g})$ has only one 1-dimensional representation (of type 1), but $\mathcal{U}_q(\mathfrak{b})$ has an infinite family of one-dimensional representations, indexed by $P_{\mathbb{Q}}$: for each $\omega\in P_{\mathbb{Q}}$, $[\omega]$ denote the one-dimensional representation on which the $(e_i)_{i\in I}$ act trivially and k_i act by multiplication by $q_i^{\omega(\alpha_i)}$.

It may seem that by studying representations of $\mathcal{U}_q(\mathfrak{b})$ we consider many more representations, but we will see that for finite-dimensional representations, the simple modules are essentially the same.

Hopf algebra structure

The algebra $\mathcal{U}_q(\mathfrak{g})$ has a Hopf algebra structure, where the coproduct and the antipode are given by, for $i \in \{0, \dots, n\}$,

$$\begin{aligned} \Delta(e_i) &= e_i \otimes 1 + k_i \otimes e_i, \\ \Delta(f_i) &= f_i \otimes k_i^{-1} + 1 \otimes f_i, \\ \Delta(k_i) &= k_i \otimes k_i, \quad S(k_i) = k_i^{-1} \\ S(e_i) &= -k_i^{-1} e_i, \quad S(f_i) = -f_i k_i. \end{aligned} \tag{1.1.1.6}$$

With these coproducts and antipodes, the Borel algebra $\mathcal{U}_q(\mathfrak{b})$ is a Hopf subalgebra of $\mathcal{U}_q(\mathfrak{g})$.

We have the following result for the coproducts in $\mathcal{U}_q(\mathfrak{b})$, where the Q-grading follows from the Q-grading on $\mathcal{U}_q(\mathfrak{g})$ defined in (1.1.1.3).

Proposition 1.1.6. *[Dam98, Proposition 7.1] For $r > 0$ and $m \in \mathbb{Z}$,*

$$\Delta(h_{i,r}) \in h_{i,r} \otimes 1 + 1 \otimes h_{i,r} + \tilde{\mathcal{U}}_q^-(\mathfrak{b}) \otimes \tilde{\mathcal{U}}_q^+(\mathfrak{b}),$$

$$\Delta(x_{i,m}^+) \in x_{i,m}^+ \otimes 1 + \mathcal{U}_q(\mathfrak{b}) \otimes (\mathcal{U}_q(\mathfrak{b}) X^+),$$

where $\tilde{\mathcal{U}}_q^+(\mathfrak{b})$ (resp. $\tilde{\mathcal{U}}_q^-(\mathfrak{b})$) is the subalgebra of $\mathcal{U}_q(\mathfrak{b})$ consisting of elements of positive (resp. negative) Q-degree, and $X^+ = \sum_{i \in I, m \in \mathbb{Z}} \mathbb{C} x_{i,m}^+ \subset \mathcal{U}_q^+(\mathfrak{b})$.

These relations of "approximate coproducts" are going to be crucial in the definition of the asymptotical standard modules in Section 1.4.3.

1.1.2 Representations of the Borel algebra

In this Section, we recall some results on the representations of $\mathcal{U}_q(\mathfrak{b})$. First of all, we recall the notion of ℓ-weights and highest ℓ-weight modules. These notions are at the center of the study of $\mathcal{U}_q(\mathfrak{b})$-modules, as are weights and highest weight modules in the study of representations of semi-simple Lie algebras. Then, we cite some results on the finite-dimensional representations of the Borel algebra. And finally, we recall Hernandez-Jimbo's category $\mathcal{O}$ for the representations of $\mathcal{U}_q(\mathfrak{b})$.

Highest ℓ-weight modules

Let V be a $\mathcal{U}_q(\mathfrak{b})$-module and $\omega \in P_{\mathbb{Q}}$ a weight. One defines the weight space of V of weight ω by

$$V_\omega := \{v \in V \mid k_i v = q_i^{\omega(\alpha_i^\vee)} v, 0 \leqslant i \leqslant n\}.$$

The vector space V is said to be *Cartan diagonalizable* if $V = \bigoplus_{\omega \in P_{\mathbb{Q}}} V_\omega$.

Definition 1.1.7. A series $\mathbf{\Psi} = (\psi_{i,m})_{i \in I, m \geqslant 0}$ of complex numbers, such that $\psi_{i,0} \in q_i^{\mathbb{Q}}$ for all $i \in I$ is called an *ℓ-weight*. The set of ℓ-weights is denoted by P_ℓ. One identifies the ℓ-weight $\mathbf{\Psi}$ to its generating series:

$$\mathbf{\Psi} = (\psi_i(z))_{i \in I}, \quad \psi_i(z) = \sum_{m \geqslant 0} \psi_{i,m} z^m.$$

The sets $P_{\mathbb{Q}}$ and P_ℓ have group structures (the elements of P_ℓ are invertible formal series) and one has a surjective group morphism $\varpi : P_\ell \to P_{\mathbb{Q}}$ which satisfies $\psi_i(0) = q_i^{\varpi(\mathbf{\Psi})(\alpha_i^\vee)}$. Let V be a $\mathcal{U}_q(\mathfrak{b})$-module and $\mathbf{\Psi} \in P_\ell$ an ℓ-weight. One defines the ℓ-weight space of V of ℓ-weight $\mathbf{\Psi}$ by

$$V_{\mathbf{\Psi}} := \{v \in V \mid \exists p \geqslant 0, \forall i \in I, \forall m \geqslant 0, (\phi_{i,m}^+ - \psi_{i,m})^p v = 0\}.$$

A non-zero vector $v \in V$ which belongs to an ℓ-weight space is called an *ℓ-weight vector.*

Remark 1.1.8. As $\phi_{i,0}^+ = k_i$, one has $V_{\mathbf{\Psi}} \subset V_{\varpi(\mathbf{\Psi})}$.

Definition 1.1.9. Let V be a $\mathcal{U}_q(\mathfrak{b})$-module. It is said to be *of highest ℓ-weight* $\mathbf{\Psi} \in P_\ell$ if there is $v \in V$ such that $V = \mathcal{U}_q(\mathfrak{b})v$,

$$e_i v = 0, \forall i \in I \quad \text{and} \quad \phi_{i,m}^+ v = \psi_{i,m} v, \ \forall i \in I, m \geqslant 0.$$

In that case, the ℓ-weight $\mathbf{\Psi}$ is entirely determined by V, it is called the ℓ-weight of V, and v is the highest ℓ-weight vector of V.

Proposition 1.1.10. *[HJ12] For all $\mathbf{\Psi} \in P_\ell$ there is, up to isomorphism, a unique simple highest ℓ-weight module of ℓ-weight $\mathbf{\Psi}$, denote it by $L(\mathbf{\Psi})$.*

Example 1.1.11. For $\omega \in P_{\mathbb{Q}}$, the one-dimensional representation defined in Example 1.1.5 is the simple representation $[\omega] = L(\mathbf{\Psi}_\omega)$, with $(\mathbf{\Psi}_\omega)_i(z) = q_i^{\omega(\alpha_i^\wedge)}$, for all $i \in I$.

Let us define some particular simple modules.

Definition 1.1.12. For all $i \in I$ and $a \in \mathbb{C}^\times$, define the *fundamental representation* $V_{i,a}$ as the simple module $L(Y_{i,a})$, where $Y_{i,a}(z)_j = \begin{cases} q_i \frac{1 - a q_i^{-1} z}{1 - a q_i z} & \text{if } j = i \\ 1 & \text{if } j \neq i \end{cases}$.

If we generalize this definition:

Definition 1.1.13. For $i \in I, a \in \mathbb{C}^\times$ and $k \geqslant 0$, the *Kirillov-Reshetikhin module* (or KR-module) $W_{k,a}^{(i)}$ is the simple $\mathcal{U}_q(\mathfrak{b})$-module,

$$W_{k,a}^{(i)} = L(Y_{i,a} Y_{i,aq_i^2} \cdots Y_{i,aq_i^{2(k-1)}}).$$

We are going to see in the next section that these are finite-dimensional representations.

Let us define another family of ℓ-weights. For $i \in I$ and $a \in \mathbb{C}^\times$, let $\mathbf{\Psi}_{i,a}^{\pm 1}$ be the ℓ-weight satisfying

$$(\psi_{i,a}^{\pm 1})_j(z) = \begin{cases} (1 - az)^{\pm 1} & \text{if } i = j, \\ 1 & \text{otherwise.} \end{cases}$$

Then,

Definition 1.1.14. For $i \in I$ and $a \in \mathbb{C}^\times$, define

$$L_{i,a}^{\pm} := L(\mathbf{\Psi}_{i,a}^{\pm 1}).$$

The modules $L_{i,a}^+$ (resp. $L_{i,a}^-$) are the *positive prefundamental representations* (resp. *negative prefundamental representations*).

The construction of the $(L_{i,a}^{\pm})_{i \in I, a \in \mathbb{C}^\times}$ is detailed in [HJ12]. It is an asymptotical construction, in particular, they are infinite dimensional representations.

For all ℓ-weight $\mathbf{\Psi}$, one can consider the *normalized* ℓ-weight

$$\tilde{\mathbf{\Psi}} = (\mathbf{\Psi}(0))^{-1} \mathbf{\Psi}, \tag{1.1.2.1}$$

which is an ℓ-weight of weight 0. For example, for $i \in I, a \in \mathbb{C}^\times$, $\tilde{Y}_{i,a} = \mathbf{\Psi}_{i,aq_i^{-1}} (\mathbf{\Psi}_{i,aq_i})^{-1}$.

Finite-dimensional representations

Let $\mathscr{C}$ be the category of finite-dimensional $\mathcal{U}_q(\mathfrak{b})$-modules. The $([\omega])_{\omega\in P_{\mathbb{Q}}}$ and the Kirillov-Reshetikhin modules are examples of finite dimensional simple $\mathcal{U}_q(\mathfrak{b})$-modules.

As stated before, these are not so different from the $\mathcal{U}_q(\mathfrak{g})$-modules. In particular, one has

Proposition 1.1.15. *[HJ12, References for Proposition 3.5] Let V be a simple finite-dimensional $\mathcal{U}_q(\mathfrak{g})$-module. Then V is simple as a $\mathcal{U}_q(\mathfrak{b})$-module.*

Using this result and the classification of finite-dimensional simple module of quantum affine algebras in [CP95a], as well as [FH15, remark 3.11], one has

Proposition 1.1.16. *Let $\mathbf{\Psi} \in P_\ell$. Then the simple $\mathcal{U}_q(\mathfrak{b})$-module $L(\mathbf{\Psi})$ is finite dimensional if and only if, there exists $\omega \in P_{\mathbb{Q}}$ such that $\mathbf{\Psi}' = \mathbf{\Psi}/\mathbf{\Psi}_\omega$ satisfies: for all $i \in I$, $\Psi_i'(z)$ is of the form*

$$\Psi_i'(z) = q_i^{\deg(P_i)} \frac{P_i(zq_i^{-1})}{P_i(zq_i)},$$

where P_i are polynomials.

Moreover, in that case, the action of $\mathcal{U}_q(\mathfrak{b})$ can be uniquely extended to an action of $\mathcal{U}_q(\mathfrak{g})$.

Remark 1.1.17. Equivalently, $L(\mathbf{\Psi})$ is finite-dimensional if and only if $\mathbf{\Psi}$ is a monomial in the $(Y_{i,a})_{i\in I, a\in\mathbb{C}^\times}$. In particular, fundamental representations, and more generally KR-modules, are examples of finite-dimensional $\mathcal{U}_q(\mathfrak{b})$-modules.

Example 1.1.18. For $\mathfrak{g} = \hat{\mathfrak{sl}}_2$, for all $a \in \mathbb{C}^\times$,

$$V_a = \mathbb{C}v_a^+ \oplus \mathbb{C}v_a^-,$$

with v_a^+ of ℓ-weight Y_a and v_a^- of ℓ-weight $Y_{aq^2}^{-1}$.

This example will be used later.

Category $\mathcal{O}$

The category $\mathcal{O}$ of representations of the Borel algebra was first defined in [HJ12], mimicking the usual definition of the category $\mathcal{O}$ BGG for Kac-Moody algebras. Here, we use the definition in [HL16b], which is slightly different.

For all $\lambda \in P_{\mathbb{Q}}$, define $D(\lambda) := \{\omega \in P_{\mathbb{Q}} \mid \omega \leqslant \lambda\}$.

Definition 1.1.19. A $\mathcal{U}_q(\mathfrak{b})$-module V is in the category $\mathcal{O}$ if

1. V is Cartan diagonalizable,
2. For all $\omega \in P_{\mathbb{Q}}$, one has $\dim(V_\omega) < \infty$,
3. There is a finite number of $\lambda_1, \dots, \lambda_s \in P_{\mathbb{Q}}$ such that all the weights that appear in V are in the cone $\bigcup_{j=1}^s D(\lambda_j)$.

The category $\mathcal{O}$ is a monoidal category [HJ12].

Example 1.1.20. All finite dimensional $\mathcal{U}_q(\mathfrak{b})$-modules are in the category $\mathcal{O}$. Moreover the positive and negative prefundamental representations are also in the category $\mathcal{O}$.

Let $P_\ell^{\mathfrak{r}}$ be the set of ℓ-weights Ψ such that, for all $i \in I$, $\Psi_i(z)$ is rational. We need the following result.

Theorem 1.1.21. *[HJ12] Let $\Psi \in P_\ell$. Simple objects in the category $\mathcal{O}$ are highest ℓ-weight modules. The simple module $L(\Psi)$ is in the category $\mathcal{O}$ if and only if $\Psi \in P_\ell^{\mathfrak{r}}$. Moreover, if V is in the category $\mathcal{O}$ and $V_\Psi \neq 0$, then $\Psi \in P_\ell^{\mathfrak{r}}$.*

1.1.3 A q-character theory

The q-characters of finite-dimensional representations of quantum affine algebras were introduced in [FR99] using transfer-matrices. Here, we consider representations in the category $\mathcal{O}$, which are not necessarily finite-dimensional. Hence we use the q-character morphism on the Grothendieck ring of the category $\mathcal{O}$ defined in [HJ12]. More precisely, it is the version of [HL16b] we use here (since we also use the definition of the category $\mathcal{O}$ from [HL16b]).

After recalling the definition of the q-characters, we use it to define some interesting subcategories of the category $\mathcal{O}$: the categories $\mathcal{O}^+, \mathcal{O}^-$ and $\mathcal{O}^+_{\mathbb{Z}}, \mathcal{O}^-_{\mathbb{Z}}$, as in [HL16b].

q-characters for category $\mathcal{O}$

Let $\mathcal{E}_\ell$ be the additive group of all maps $c : P_\ell^{\mathfrak{r}} \to \mathbb{Z}$ whose support, $\operatorname{supp}(c) = \{\Psi \in P_\ell^{\mathfrak{r}} \mid c(\Psi) \neq 0\}$ satisfies: $\varpi(\operatorname{supp}(c))$ is contained in a finite union of sets of the form $D(\mu)$, and, for all $\omega \in P_{\mathbb{Q}}$, the set $\operatorname{supp}(c) \cap \varpi^{-1}(\omega)$ is finite. Similarly, $\mathcal{E}$ is the additive group of maps $c : P_{\mathbb{Q}} \to \mathbb{Z}$ whose support is contained in a finite union of sets of the form $D(\mu)$. The map ϖ naturally extends to a surjective morphism $\varpi : \mathcal{E}_\ell \to \mathcal{E}$.

For all $\mathbf{\Psi} \in P_\ell^{\mathfrak{r}}$, let $[\mathbf{\Psi}] = \delta_{\mathbf{\Psi}} \in \mathcal{E}_\ell$ (resp. $[\omega] = \delta_\omega \in \mathcal{E}$, for all $\omega \in P_{\mathbb{Q}}$).

Remark 1.1.22. One notices that this notation is coherent with the ones from Example 1.1.11. Indeed, for all $\omega \in P_{\mathbb{Q}}$, the simple one-dimensional representation $[\omega] = L(\mathbf{\Psi}_\omega)$ is identified with the map $\delta_\omega \in \mathcal{E}$.

Definition 1.1.23. Let V be a module in the category $\mathcal{O}$. The *q-character of V* is the following element of $\mathcal{E}_\ell$:

$$\chi_q(V) = \sum_{\mathbf{\Psi} \in P_\ell^{\mathfrak{r}}} \dim(V_{\mathbf{\Psi}})[\mathbf{\Psi}]. \tag{1.1.3.1}$$

The *character of V* is the following element of $\mathcal{E}$:

$$\chi(V) = \varpi(\chi_q(V)) = \sum_{\omega \in P_{\mathbb{Q}}} \dim(V_\omega)[\omega].$$

Remark 1.1.24. [HL16b, Section 3.2] In the category $\mathcal{O}$, every object does not necessarily have finite length. But, as for the category $\mathcal{O}$ of a classical Kac-Moody algebra (see [Kac90]), the multiplicity of a simple module is well-defined. Hence we have its Grothendieck ring $K_0(\mathcal{O})$. Its elements are formal sums, for each $M \in \mathcal{O}$,

$$[M] = \sum_{\mathbf{\Psi} \in P_\ell^{\mathfrak{r}}} \lambda_{\mathbf{\Psi},M}[L(\mathbf{\Psi})], \tag{1.1.3.2}$$

where $\lambda_{\mathbf{\Psi},M}$ is the multiplicity of $L(\mathbf{\Psi})$ in M. These coefficients satisfy

$$\sum_{\mathbf{\Psi} \in P_\ell^{\mathfrak{r}}, \omega \in P_{\mathbb{Q}}} |\lambda_{\mathbf{\Psi},M}| \dim((L(\mathbf{\Psi}))_\omega)[\omega] \quad \in \mathcal{E}.$$

$K_0(\mathcal{O})$ has indeed a ring structure, because of 3, in Definition 1.1.19.

The *q-character morphism* is the group morphism,

$$\chi_q : K_0(\mathcal{O}) \mapsto \mathcal{E}_\ell$$

which sends a class $[V]$ of a representation V to $\chi_q(V)$. It is well defined, as χ_q is compatible with exact sequences.

Remark 1.1.25. The definition of a q-character makes sense for more general representations than that in the category $\mathcal{O}$. For every representation V with finite-dimensional ℓ-weight spaces, (1.1.3.1) has a sense. However the resulting q-character is not necessarily in the ring $\mathcal{E}_\ell$. Moreover, the module is not necessarily the sum of its ℓ-weight spaces (for example, the Verma modules associated to the ℓ-weights, as in [HJ12, Section 3.1]).

For V a module in the category $\mathcal{O}$ having a unique ℓ-weight $\mathbf{\Psi}$ whose weight is maximal, one can consider its normalized q-character $\tilde{\chi}_q(V)$:

$$\tilde{\chi}_q(V) := [\mathbf{\Psi}^{-1}] \cdot \chi_q(V).$$

For $i \in I$ and $a \in \mathbb{C}^\times$, define $A_{i,a}$ as

$$Y_{i,aq_i^{-1}} Y_{i,aq_i} \left(\prod_{\{j \in I | C_{j,i} = -1\}} Y_{j,a} \prod_{\{j \in I | C_{j,i} = -2\}} Y_{j,aq^{-1}} Y_{j,aq} \prod_{\{j \in I | C_{j,i} = -3\}} Y_{j,aq^{-2}} Y_{j,aq} Y_{j,aq^2} \right)^{-1}.$$

For all $i \in I, a \in \mathbb{C}^\times$, $\varpi(A_{i,a}) = \alpha_i$.

Theorem 1.1.26. *[FR99, FM01] For V a simple finite-dimensional $\mathcal{U}_q(\mathfrak{g})$-module, one has*

$$\tilde{\chi}_q(V) \in \mathbb{Z}[A_{i,a}^{-1}]_{i \in I, a \in \mathbb{C}^\times}.$$

One has more precise results when V is a fundamental representation, we will use some later on.

Categories $\mathcal{O}^+$ and $\mathcal{O}^-$

Let us now recall the definitions of some subcategories of the category $\mathcal{O}$, introduced in [HL16b]. These categories are interesting to study because the ℓ-weights of the simple modules have some unique decomposition.

Definition 1.1.27. An ℓ-weight of $P_\ell^{\mathfrak{r}}$ is said to be *positive* (resp. *negative*) if it is a monomial in the following ℓ-weights:

- the $Y_{i,a} = q_i \mathbf{\Psi}_{i,aq_i}^{-1} \mathbf{\Psi}_{i,aq_i^{-1}}$, where $i \in I$ and $a \in \mathbb{C}^\times$,
- the $\mathbf{\Psi}_{i,a}$ (resp. $\mathbf{\Psi}_{i,a}^{-1}$), where $i \in I$ and $a \in \mathbb{C}^\times$,
- the $[\omega]$, where $\omega \in P_\mathbb{Q}$.

Let us denote by P_ℓ^+ (resp. P_ℓ^-) the ring of positive (resp. negative) ℓ-weights.

Definition 1.1.28. The category $\mathcal{O}^+$ (resp. $\mathcal{O}^-$) is the category of representations in $\mathcal{O}$ whose simple constituents have a positive (resp. negative) highest ℓ-weight, in the sense of (1.1.3.2), that is: for M in $\mathcal{O}^\pm$, one can write,

$$\chi_q(M) = \sum_{\mathbf{\Psi} \in P_\ell^\pm} \lambda_{\mathbf{\Psi},M} [L(\mathbf{\Psi})].$$

Remark 1.1.29. (i) The category $\mathcal{O}^+$ (resp. $\mathcal{O}^-$) contains $\mathscr{C}$, the category of finite-dimensional representations, as well as the positive (resp. negative) prefundamental representations $L_{i,a}^+$ (resp. $L_{i,a}^-$), for all $i \in I, a \in \mathbb{C}^\times$.

(ii) The generalized Baxter's relations in [FH15] are satisfied in the Grothendieck rings $K_0(\mathcal{O}^\pm)$.

(iii) Positive ℓ-weights have a unique factorization into a product of $Y_{i,a}$ and $\mathbf{\Psi}_{i,a}$. In particular, for $\mathfrak{g} = \hat{\mathfrak{sl}}_2$, this implies a unique factorization of simple modules into products of prime simple representations in $\mathcal{O}^+$ (see [HL16b, Theorem 7.9]).

Theorem 1.1.30. *[HL16b] The categories $\mathcal{O}^+$ and $\mathcal{O}^-$ are monoidal categories.*

The categories $\mathcal{O}^+_{\mathbb{Z}}$ and $\mathcal{O}^-_{\mathbb{Z}}$

First, let us recall the infinite quiver defined in [HL16a, Section 2.1.2]. Let $\tilde{V} = I \times \mathbb{Z}$ and $\tilde{\Gamma}$ be the quiver with vertex set $\tilde{V}$ whose arrows are given by

$$((i,r) \to (j,s)) \iff (C_{i,j} \neq 0 \text{ and } s = r + d_i C_{i,j}).$$

Select one of the two connected components of $\tilde{\Gamma}$ (see [HL16a, Lemma 2.2]) and call it Γ. The vertex set of Γ is denoted by V.

Example 1.1.31. For $\mathfrak{g} = \mathfrak{sl}_2$, the infinite quiver is

$$\tilde{\Gamma} : \quad \begin{array}{l} \cdots \to \overset{(1,-2)}{\bullet} \longrightarrow \overset{(1,0)}{\bullet} \longrightarrow \overset{(1,2)}{\bullet} \longrightarrow \overset{(1,4)}{\bullet} \to \cdots \\[1em] \qquad \cdots \to \underset{(1,-1)}{\bullet} \longrightarrow \underset{(1,1)}{\bullet} \longrightarrow \underset{(1,3)}{\bullet} \longrightarrow \underset{(1,5)}{\bullet} \to \cdots \end{array}$$

In that case, the choice of a connected component is the choice of a parity.

Definition 1.1.32. [HL16b] Define the category $\mathcal{O}^+_{\mathbb{Z}}$ (resp. $\mathcal{O}^-_{\mathbb{Z}}$) as the subcategory of representations of $\mathcal{O}^+$ (resp. $\mathcal{O}^-$) whose simple components have a highest ℓ-weight $\boldsymbol{\Psi}$ such that the roots and poles of $\Psi_i(z)$ are of the form q_i^r, with $(i,r) \in V$.

Remark 1.1.33. (i) [HL16b, 4.3] One does not lose any information by only studying the subcategories $\mathcal{O}^\pm_{\mathbb{Z}}$, instead of $\mathcal{O}^\pm$. Indeed, as in the case of finite dimensional representations, each simple object in $\mathcal{O}^\pm$ has a decomposition into a tensor product of simple objects which are essentially in $\mathcal{O}^\pm_{\mathbb{Z}}$.

(ii) Moreover, these categories are interesting to study by themselves, because they are categorification of some cluster algebras (see [HL16b, Theorem 4.2]).

1.2 Limits of q-characters of standard modules

In this section, we recall the definition of standard modules for finite-dimensional representations. Then we show that the q-characters of a specific sequence of standard modules converge to some limit. We conjecture that this limit is the q-character of some $\mathcal{U}_q(\mathfrak{b})$-module which respects some of the structure of the finite-dimensional standard modules.

1.2.1 Standard modules for finite dimensional representations

For finite-dimensional representations of a quantum affine algebra, one can define the *standard module* associated to a given highest ℓ-weight. Let m be a monomial $m = Y_{i_1,a_1} Y_{i_2,a_2} \cdots Y_{i_N,a_N}$, then the associated standard module is the following tensor product of fundamental representations

$$M(m) := V_{i_1,a_1} \otimes V_{i_2,a_2} \otimes \cdots \otimes V_{i_N,a_N}, \tag{1.2.1.1}$$

where the tensor product is written so that:

$$\text{if } k < l, \text{ then } a_l/a_k \notin q^{\mathbb{N}}. \tag{1.2.1.2}$$

Recall the following result, which is a weaker form of [Cha02, Theorem 5.1 and Corollary 5.3]:

Lemma 1.2.1. *For $(a,b) \in (\mathbb{C}^\times)^2$ such that $a/b \notin q^{\mathbb{Z}}$, and all $i,j \in I$, then*

$$V_{i,a} \otimes V_{j,b} \simeq V_{j,b} \otimes V_{i,a}, \qquad \textit{and this module is irreducible.}$$

The definition of the standard module $M(m)$ by the expression (1.2.1.1) is unambiguous, it does not depend on the order of the factors, as long as (1.2.1.2) is satisfied. Indeed, with Lemma 1.2.1, as long as the condition (1.2.1.2) is satisfied, two such tensor products are isomorphic.

Moreover, each simple finite-dimensional representation has a highest ℓ-weight m that can be uniquely written as a monomial in the $Y_{i,a}$'s.

One would want to define analogs of the standard modules for a larger category of representations, namely the category $\mathcal{O}$.

1.2.2 Limits of q-character

It is known ([Nak03a, Theorem 1.1] for simply laced types and [Her06, Theorem 3.4] in general) that the normalized q-characters of KR-modules, seen as polynomials in $A_{i,a}^{-1}$, have limits as formal power series. In [HJ12, Section 6.1], the normalized q-character of a negative prefundamental representation is explicitly obtained, as a formal power series in $\mathbb{Z}[[A_{i,a}^{-1}]]$, as a limit of a sequence of normalized q-characters of KR-module (hence finite-dimensional representations).

The idea here is to consider the limit of a specific sequence of normalized q-characters of finite-dimensional standard modules. The formal power series we obtain is a conjectural normalized q-character for a potential infinite-dimensional standard module.

For $i \in I$ and $a \in \mathbb{C}^\times$, consider the negative ℓ-weight $\mathbf{\Psi} = \mathbf{\Psi}_{i,a}^{-1}$.

One has $(\mathbf{\Psi})_j = 0$ for $j \neq i$ and $(\mathbf{\Psi})_i(z) = \frac{1}{1-az}$. Heuristically, one wants to write

$$\begin{aligned}\frac{1}{1-az} &\approx \frac{1-aq_i^{-2}z}{1-az} \times \frac{1-aq_i^{-4}z}{1-aq_i^{-2}z} \times \frac{1-aq_i^{-6}z}{1-aq_i^{-4}z} \times \frac{1-aq^{-8}z}{1-aq^{-6}z} \times \cdots \\ &\approx \left(\tilde{Y}_{i,aq_i^{-1}}\tilde{Y}_{i,aq_i^{-3}}\tilde{Y}_{i,aq_i^{-5}}\tilde{Y}_{i,aq_i^{-7}}\cdots\right)_i(z),\end{aligned}$$

with the normalized ℓ-weights defined in (1.1.2.1). Thus consider, for $N \geqslant 1$,

$$m_N := \prod_{k=0}^{N-1} \tilde{Y}_{i,aq_i^{-2k-1}}.$$

As stated before, from [HJ12, Theorem 6.1], one knows that, as formal power series, the normalized q-characters satisfy

$$\tilde{\chi}_q(L(m_N)) \xrightarrow[N\to+\infty]{} \tilde{\chi}_q(L_{i,a}^-).$$

For $N \geqslant 1$, one can look at the normalized q-character of the standard module associated to the same ℓ-weight m_N. Consider the standard module

$$S_N := V_{i,aq_i^{-1}} \otimes V_{i,aq_i^{-3}} \otimes V_{i,aq_i^{-5}} \otimes \cdots \otimes V_{i,aq_i^{-2N+1}}. \tag{1.2.2.1}$$

And its normalized q-character

$$\tilde{\chi}_N := \tilde{\chi}_q(S_N) = \prod_{k=0}^{N-1} \tilde{\chi}_q\left(V_{i,aq_i^{-2k-1}}\right) = \prod_{k=0}^{N-1} \tilde{\chi}_q\left(L(\tilde{Y}_{i,aq_i^{-2k-1}})\right).$$

Theorem 1.2.2. *For all $N \geqslant 1$, $\tilde{\chi}_N \in \mathbb{Z}[A^{-1}_{j,aq^l}]_{j\in I, l\in\mathbb{Z}}$.*

As a formal power series in $(A^{-1}_{j,aq^l})_{j\in I,l\in\mathbb{Z}}$, $\tilde{\chi}_N$ has a limit as $N \to +\infty$,

$$\tilde{\chi}_N \xrightarrow[N\to+\infty]{} \chi^{\infty}_{i,a} \in \mathbb{Z}[[A^{-1}_{j,aq^l}]]_{j\in I,l\in\mathbb{Z}}. \tag{1.2.2.2}$$

Remark 1.2.3. Here we consider formal sums of monomials in the $(A^{-1}_{j,aq^l})_{j\in I,l\in\mathbb{Z}}$:

$$\mathbb{Z}[[A^{-1}_{j,aq^l}]]_{j\in I,l\in\mathbb{Z}} \ni \sum_{\alpha\in\mathbb{N}^{(I\times\mathbb{Z})}} c_\alpha A_\alpha^{-1},$$

where each α is a finitely supported sequence of non-negative integers on $I \times \mathbb{Z}$, $A_\alpha = \prod_{(j,l)\in I\times\mathbb{Z}} A^{\alpha(j,l)}_{j,aq^l}$. This definition is necessary as the result we have to deal with has homogeneous parts (for the standard degree $\deg(A_\alpha^{-1}) = \sum_{(j,l)\in I\times\mathbb{Z}} \alpha(j,l)$) with infinitely many terms, which was not the case in the limits considered in [HJ12]. For example, here we want to be able to consider elements such as $\sum_{l\in\mathbb{Z}} A^{-1}_{j,aq^l}$, for all $j \in I$. We get a well-defined ring of formal power series.

The topology on $\mathbb{Z}[[A^{-1}_{j,aq^l}]]_{j\in I,l\in\mathbb{Z}}$ is the one of the pointwise convergence: a sequence of its elements converges only if for each monomial A_α^{-1} the corresponding coefficient converges, or more precisely is eventually constant, as these coefficients are integers.

Proof. From Theorem 1.1.26, for all $N \geqslant 1$, $\tilde{\chi}_N \in \mathbb{Z}[A^{-1}_{j,b}]_{j\in I,b\in\mathbb{C}^\times}$. Moreover, by [FM01, Lemma 6.1], each $\tilde{\chi}_q(V_{i,aq_i^{-2k-1}})$ is in $\mathbb{Z}[A^{-1}_{j,aq^l}]_{j\in I,l\in\mathbb{Z}}$, thus $\tilde{\chi}_N$ too.

By [FM01, proof of Lemma 6.5], for all $j \in I, b \in \mathbb{C}^\times$,

$$\tilde{\chi}_q(V_{j,b}) = 1 + \sum m, \text{ with } m \leqslant A^{-1}_{j,bq_j},$$

for Nakajima's partial order on the monomials in $(Y^{\pm}_{j,b})_{j\in I,b\in\mathbb{C}^\times}$ [Nak04]:

$$m \leqslant m' \Leftrightarrow m(m')^{-1} = \prod_{\text{finite}} A^{-1}_{j,b}. \tag{1.2.2.3}$$

Let us fix a monomial $m \in \mathbb{Z}[A^{-1}_{j,aq^l}]_{j\in I,l\in\mathbb{Z}}$. If the factor $\tilde{\chi}_q(V_{i,aq_i^{-2k-1}})$ contributes to the multiplicity of m in $\tilde{\chi}_N$, then $A^{-1}_{i,aq_i^{-2k}}$ is a factor of m. Thus, only a finite number of factors $\tilde{\chi}_q(V_{i,aq_i^{-2k-1}})$ can contribute to this multiplicity. In particular, for k large enough, the factor $\tilde{\chi}_q(V_{i,aq_i^{-2k-1}})$ does not contribute to the multiplicity of m in $\tilde{\chi}_N$, for all $N > k$. Thus the multiplicity of m in $\tilde{\chi}_N$ is stationary, as $N \to +\infty$.

As this is true for all monomials in $\mathbb{Z}[A^{-1}_{j,aq^l}]_{j\in I,l\in\mathbb{Z}}$, the limit of $\tilde{\chi}_N$ as $N \to +\infty$ is well defined as a formal power series (see Remark 1.2.3). ☐

Example 1.2.4. For $\mathfrak{g} = \hat{\mathfrak{sl}}_2$, one can compute this formula explicitly. Consider the ℓ-weight $\mathbf{\Psi} = \mathbf{\Psi}^{-1}_{1,1}$. In this case, the normalized q-character of S_N is known,

$$\tilde{\chi}_N = \prod_{k=0}^{N-1}\left(1 + A^{-1}_{1,q^{-2k}}\right).$$

One can write

$$\tilde{\chi}_N = \sum_{m=0}^{N-1} \sum_{0\leqslant k_1<k_2<\cdots<k_m\leqslant N-1} \prod_{i=1}^{m} A^{-1}_{1,q^{-2k_i}}.$$

This formal power series has a limit

$$\tilde{\chi}_N \xrightarrow[N\to+\infty]{} \sum_{m=0}^{+\infty} \sum_{0\leqslant k_1<k_2<\cdots<k_m} \prod_{i=1}^{m} A^{-1}_{1,q^{-2k_i}} \quad \in \mathbb{Z}[[A^{-1}_{1,q^{-2k}}]]_{k\in\mathbb{N}}. \tag{1.2.2.4}$$

Remark 1.2.5. (i) Except for the weight 0, all weight spaces are infinite-dimensional. Thus this formula is not the (normalized) q-character of a representation in the category $\mathcal{O}$. Nor is the result a formal power series is the "classical" sense. Nonetheless, it can still be a q-character, as stated in Remark 1.1.25.

(ii) We will see later that this q-character also has a decomposition into a sum of q-characters of simple representations, as in (1.1.3.2).

Let us generalize the statement of Theorem 1.2.2. Let $\mathbf{\Psi}$ be a negative ℓ-weight in P_ℓ^-. It can be written as a finite product of $(Y_{i,a})_{i\in I,a\in\mathbb{C}^\times}$, $(\mathbf{\Psi}^{-1}_{i,a})_{i\in I,a\in\mathbb{C}^\times}$ and $[\omega]$, with $\omega\in P_Q$. Let us write

$$\mathbf{\Psi} = [\omega] \times \prod_{k=1}^{r} Y_{i_k,a_k} \times \prod_{l=1}^{s} \mathbf{\Psi}^{-1}_{j_l,b_l}. \tag{1.2.2.5}$$

Each factor $\mathbf{\Psi}^{-1}_{i,a}$ can be seen as a limit of $\prod \tilde{Y}_{i,aq_i^{-2k-1}}$. For $N\geqslant 1$, consider the finite-dimensional standard module S_N, which is the tensor product of the V_{i_k,a_k}, for $1\leqslant k\leqslant r$, and the $\bigotimes_{k=0}^{\overrightarrow{N-1}} V_{j_l,b_lq_{j_l}^{-2k-1}}$, for $1\leqslant l\leqslant s$, ordered so as to satisfy the condition (1.2.1.2) of Section 1.2.1 (this is a direct generalization of the S_N defined in (1.2.2.1)). Then,

Corollary 1.2.6. *The sequence of normalized q-characters of S_N converges as $N\to+\infty$, as a formal power series. The limit $\chi^\infty_{\mathbf{\Psi}}$ can be written*

$$\chi^\infty_{\mathbf{\Psi}} = \prod_{k=1}^{r} (\tilde{\chi}_q(V_{i_k,a_k})) \cdot \prod_{l=1}^{s} \chi^\infty_{j_l,b_l} \quad \in \mathbb{Z}[[A^{-1}_{i,a}]]_{i\in I,a\in\mathbb{C}^\times}.$$

1.2.3 A conjecture

As explained in the previous section, the formal power series

$$\chi^\infty_{\mathbf{\Psi}} = \lim_{N\to+\infty} \tilde{\chi}_q(S_N)$$

is a good candidate for the normalized q-character of the asymptotical standard module associated to the negative ℓ-weight $\mathbf{\Psi}$.

We would like to build, for all negative ℓ-weight $\mathbf{\Psi}$, a $\mathcal{U}_q(\mathfrak{b})$-module $M(\mathbf{\Psi})$ with finite-dimensional ℓ-weight spaces, whose q-character is $[\mathbf{\Psi}]\cdot\chi^\infty_{\mathbf{\Psi}}$. The following conjecture claims that a module with the right q-character and which retains some structure of the finite-dimensional standard modules exists.

Conjecture 1.2.7. *For all negative ℓ-weight $\mathbf{\Psi}$, there exists a $\mathcal{U}_q(\mathfrak{b})$-module $M(\mathbf{\Psi})$ with finite-dimensional ℓ-weight spaces, such that $\chi_q(M(\mathbf{\Psi})) = [\mathbf{\Psi}]\cdot\chi^\infty_{\mathbf{\Psi}}$ and the sum of these ℓ-weights spaces is a $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$-module containing a sub-$\mathcal{U}_q(\mathfrak{b})^+$-module isomorphic to S_N for any $N\geqslant 0$.*

Remark 1.2.8. Let us note here that the resulting module is not necessarily equal to the sum of its ℓ-weight spaces. We will see later that even for $\mathfrak{g}=\hat{\mathfrak{sl}}_2$, for certain ℓ-weights $\mathbf{\Psi}$, there is no $\mathcal{U}_q(\mathfrak{b})$-module $M(\mathbf{\Psi})$ satisfying the properties of the Conjecture, which is the sum of its ℓ-weight spaces.

Section 1.3 will prove this conjecture when $\mathfrak{g}=\hat{\mathfrak{sl}}_2$.

1.3 Asymptotical standard modules

From now on, let us assume that $\mathfrak{g} = \hat{\mathfrak{sl}_2}$. We hope in other work to extend these results to the type A and then to other types.

For simplicity, we omit the notation for the node from now on: $Y_{1,a} = Y_a$ and $\Psi_{1,a} = \Psi_a$.

The aim of this section is to prove Conjecture 1.2.7 for $\mathfrak{g} = \hat{\mathfrak{sl}_2}$. We build, for every negative ℓ-weight $\mathbf{\Psi}$ such that $L(\mathbf{\Psi})$ is in $\mathcal{O}_{\mathbb{Z}}^{-}$, an asymptotical standard $\mathcal{U}_q(\mathfrak{b})$-module $M(\mathbf{\Psi})$. This module has finite-dimensional ℓ-weight spaces and its q-character is the formal power series $\chi_{\mathbf{\Psi}}^{\infty}$.

The construction will take place in two parts

1. Recall the definition of $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$ in (1.1.1.5). First, we build a $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$-module T, with finite-dimensional ℓ-weight spaces and the correct q-character.
2. Then, we build by induction from T a $\mathcal{U}_q(\mathfrak{b})$-module T^c. We show that T^c still has finite-dimensional ℓ-weight spaces and the same q-character.

1.3.1 Construction of the $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$-module.

Fundamental example: the case where $\mathbf{\Psi} = \mathbf{\Psi}_1^{-1}$

We build a $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$-module T with finite-dimensional ℓ-weight spaces whose q-character is exactly the limit in (1.2.2.4).

For $N \geqslant 1$, consider the standard module S_N as defined in (1.2.2.1), and the normalized standard module $\tilde{S}_N$, obtained by normalizing the ℓ-weights in each factor of S_N.

$$\tilde{S_N} := L(\tilde{Y}_{q^{-1}}) \otimes L(\tilde{Y}_{q^{-3}}) \otimes L(\tilde{Y}_{q^{-5}}) \otimes \cdots \otimes L(\tilde{Y}_{q^{-2N+1}}). \tag{1.3.1.1}$$

As in example 1.1.18, each one of the factors of $\tilde{S}_N$ is a two dimensional representation

$$L(\tilde{Y}_{q^{-2r-1}}) = \mathbb{C}v_r^+ \oplus \mathbb{C}v_r^-,$$

where v_r^+ is of ℓ-weight $\tilde{Y}_{q^{-2r-1}}$ and v_r^- is of ℓ-weight $[-2\omega_1](\tilde{Y}_{q^{-2r+1}})^{-1}$.

Let $\mathcal{P}_f(\mathbb{N})$ be the set of finite subsets of $\mathbb{N}$. Let

$$T := \bigoplus_{J \in \mathcal{P}_f(\mathbb{N})} \mathbb{C}v_J. \tag{1.3.1.2}$$

For $J \in \mathcal{P}_f(\mathbb{N})$, the element v_J represents the infinite simple tensor:

$$v_J = v_{-1}^{\epsilon_0(J)} \otimes v_{-3}^{\epsilon_1(J)} \otimes v_{-5}^{\epsilon_2(J)} \otimes v_{-7}^{\epsilon_3(J)} \otimes \cdots,$$

where $\epsilon_r(J) = \begin{cases} - & \text{if } r \in J, \\ + & \text{if } r \notin J. \end{cases}$

Example 1.3.1. For example, $\begin{cases} v_{\varnothing} = v_{-1}^+ \otimes v_{-3}^+ \otimes v_{-5}^+ \otimes v_{-7}^+ \otimes \cdots \\ v_{\{2\}} = v_{-1}^+ \otimes v_{-3}^+ \otimes v_{-5}^- \otimes v_{-7}^+ \otimes \cdots \end{cases}$.

Now, we endow the vector space T with a $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$-module structure. The key ingredients are the approximate coproducts formulas of Proposition 1.1.6. Let us recall their expressions in this context. For $r > 0$ and $m \in \mathbb{Z}$,

$$\Delta(h_r) \in h_r \otimes 1 + 1 \otimes h_r + \tilde{\mathcal{U}_q^-}(\mathfrak{b}) \otimes \tilde{\mathcal{U}_q^+}(\mathfrak{b}), \tag{1.3.1.3}$$

$$\Delta(x_m^+) \in x_m^+ \otimes 1 + \mathcal{U}_q(\mathfrak{b}) \otimes (\mathcal{U}_q(\mathfrak{b})X^+), \tag{1.3.1.4}$$

where $\tilde{\mathcal{U}}_q^+(\mathfrak{b})$ (resp. $\tilde{\mathcal{U}}_q^-(\mathfrak{b})$) is the subalgebra of $\mathcal{U}_q(\mathfrak{b})$ consisting of elements of positive (resp. negative) degree, and $X^+ = \sum_{m\in\mathbb{Z}} \mathbb{C}x_m^+ \subset \mathcal{U}_q^+(\mathfrak{b})$.

Let $J \in \mathcal{P}_f(\mathbb{N})$ and $N > i_0 := \max(J)$. Consider the truncated simple tensor $v_J^{(N)}$, which is an element of the $\mathcal{U}_q(\mathfrak{b})$-module $\tilde{S}_N$ defined in (1.3.1.1),

$$v_J^{(N)} := v_{-1}^{\epsilon_0(J)} \otimes v_{-3}^{\epsilon_1(J)} \otimes v_{-5}^{\epsilon_2(J)} \otimes \cdots \otimes v_{-2N+1}^{\epsilon_N(J)} \quad \in \tilde{S}_N,$$

where $\epsilon_r(J)$ is defined as above.

Action of $\mathcal{U}_q(\mathfrak{b})^+$: The algebra $\mathcal{U}_q(\mathfrak{b})^+$ is generated by the $(x_m^+)_{m\geqslant 0}$.

For $N > i_0$, write

$$v_J^{(N)} = v_J^{(i_0)} \otimes v_{-2i_0-3}^+ \otimes \cdots \otimes v_{-2N+1}^+ = v_J^{(i_0)} \otimes u^{(N)}, \tag{1.3.1.5}$$

Then, using (1.3.1.4), one has, for $m \geqslant 0$,

$$x_m^+ \cdot v_J^{(N)} = \left(x_m^+ \cdot v_J^{(i_0)}\right) \otimes u^{(N)} + 0,$$

as $\mathcal{U}_q(\mathfrak{b})X^+$ acts by 0 on $u^{(N)}$, which is a highest ℓ-weight vector and $v_J^{(i_0)} \in \tilde{S}_{i_0}$, which is a $\mathcal{U}_q(\mathfrak{b})$-module. That way the action of $(x_m^+)_{m\geqslant 0}$ on v_J is defined as

$$x_m^+ \cdot v_J := \left(x_m^+ \cdot v_J^{(i_0)}\right) \otimes u, \tag{1.3.1.6}$$

where $u = v_{-2i_0-3}^+ \otimes \cdots \otimes v_{-2N+1}^+ \otimes \cdots$. As $x_m^+ \cdot v_J^{(i_0)} \in \tilde{S}_{i_0}$, it is a linear combination of $v_K^{(i_0)}$, with $\max(K) \leqslant i_0$. Hence, $x_m^+ \cdot v_J$ is the same linear combination, but with the v_K instead of the $v_K^{(i_0)}$.

Action of $\mathcal{U}_q(\mathfrak{b})^0$: The algebra $\mathcal{U}_q(\mathfrak{b})^0$ is generated by the $(h_r, k_1^{\pm 1})_{r\geqslant 1}$.

As we normalized the action of k_1, for all $N > i_0$, $k_1 \cdot v_{\mathcal{I}}^{(N)} = q^{-2|J|} v_J^{(N)}$. Hence, naturally

$$k_1 \cdot v_J := q^{-2|J|} v_J. \tag{1.3.1.7}$$

For the action of the h_r's, let us write, for $N > i_0$, $v_J^{(N)}$ as in (1.3.1.5). Then, using (1.3.1.3), one has, for $r \geqslant 1$,

$$h_r \cdot v_J^{(N)} = \left(h_r \cdot v_J^{(i_0)}\right) \otimes u^{(N)} + v_J^{(i_0)} \otimes \left(h_r \cdot u^{(N)}\right) + 0,$$

as $\tilde{\mathcal{U}}_q^+(\mathfrak{b})$ sends $u^{(N)}$ to a higher weight space, which is $\{0\}$.

The vector $u^{(N)}$ is a highest ℓ-weight vector of $\tilde{S}_n$, whose ℓ-weight is the product $\tilde{Y}_{q^{-2i_0-3}} \tilde{Y}_{q^{-2i_0-5}} \cdots \tilde{Y}_{q^{-2N+1}}$. Hence,

$$h_r \cdot u^{(N)} = \frac{q^{-2i_0-2} - q^{-2N}}{q - q^{-1}} u^{(N)}.$$

Thus it is natural to define, for $r \geqslant 1$, and $N > i_0$

$$h_r \cdot v_J := \left(\left(h_r + \frac{q^{-2i_0-2}}{q - q^{-1}} \operatorname{Id}\right) \cdot v_J^{(i_0)}\right) \otimes u. \tag{1.3.1.8}$$

As before, $\left(h_r + \frac{q^{-2i_0-2}}{q-q^{-1}}\,\mathrm{Id}\right)\cdot v_J^{(i_0)} \in \tilde{S}_{i_0}$ is a linear combination of $v_K^{(i_0)}$, and $h_r \cdot v_J$ is the same linear combination of v_K.

Example 1.3.2. As $\Delta(h_1) = h_1 \otimes 1 + 1 \otimes h_1 - (q^2 - q^{-2})x_1^- \otimes x_0^+$, then $h_1 \cdot \left(v_{-1}^+ \otimes v_{-3}^-\right) = -q^{-1}(q^2 - q^{-2})v_{-1}^- \otimes v_{-3}^+$. Hence,

$$h_1 \cdot v_{\{1\}} = \frac{q^{-4}}{q-q^{-1}} v_{\{1\}} - q^{-1}(q^2 - q^{-2})v_{\{0\}}.$$

Moreover, $h_1 \cdot v_{\{0\}} = (-q + \frac{q^{-2}}{q-q^{-1}})v_{\{0\}}$, and $h_1(v_{\{1\}} - v_{\{0\}}) = \frac{q^{-4}}{q-q^{-1}}(v_{\{1\}} - v_{\{0\}})$. We will see that $v_{\{0\}}$ is of ℓ-weight $(\boldsymbol{\Psi}_1)^{-1}A_1^{-1}$ and $v_{\{1\}} - v_{\{0\}}$ is of ℓ-weight $(\boldsymbol{\Psi}_1)^{-1}A_{q^{-2}}^{-1}$.

The combination of the last two paragraphs gives us the following result. As the actions of the generators of $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$ on T defined in (1.3.1.6), (1.3.1.7) and (1.3.1.8) are based on actions on finite-dimensional $\mathcal{U}_q(\mathfrak{b})$-modules, they naturally satisfy the relations in $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$.

Proposition 1.3.3. *The vector space T has a $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$-module structure.*

Moreover, the action of $\mathcal{U}_q(\mathfrak{b})^0$ on the v_J's is upper-triangular, which allows us to explicit ℓ-weight vectors.

Partial order on $\mathcal{P}_f(\mathbb{N})$: We define some order on $\mathcal{P}_f(\mathbb{N})$. It has nothing to do with Nakajima's partial order on the ℓ-weights recalled in (1.2.2.3).

Remark 1.3.4. For all $J \in \mathcal{P}_f(\mathbb{N})$, v_J is a weight vector of weight $[|J|\omega_1]$. Moreover, from (1.3.1.8), for all $r \geqslant 1$,

$$h_r \cdot v_J \in \bigoplus_{|K|=|J|, K \subset \mathrm{Conv}(J)} \mathbb{C} v_K,$$

where $\mathrm{Conv}(J)$ is the convex hull of J, $\mathrm{Conv}(J) = \{j \in \mathbb{N} \mid \min(J) \leqslant j \leqslant \max(J)\}$.

In particular, sets in $\mathcal{P}_f(\mathbb{N})$ correspond to ℓ-weights which are generically incomparable for Nakajima's partial order.

Definition 1.3.5. For all $N \in \mathbb{N}$, let $\mathcal{P}_N(\mathbb{N}) := \{J \in \mathcal{P}_f(\mathbb{N}) \mid |J| = N\}$. The set $\mathcal{P}_N(\mathbb{N})$ is equipped with the lexicographic order on N-tuples, noted $\preceq$.

Lemma 1.3.6. *For $J \in \mathcal{P}_f(\mathbb{N})$ and $r \geqslant 1$,*

$$h_r \cdot v_J \in h_{r,J} v_J + \sum_{K \subset \mathrm{Conv}(J), K \prec J} \mathbb{C} v_K,$$

where the $h_{r,J}$ are the coefficients arising from the action of h_r on each component of v_J.

Proof. Using (1.3.1.3) recursively, one has, for $N \geqslant 1$,

$$\begin{aligned}\Delta^N(h_r) \in \sum_{k=1}^{N} 1 \otimes \cdots \otimes \underbrace{h_r}_{k} \otimes 1 \otimes \cdots \otimes 1 + \tilde{\mathcal{U}}_q^-(\mathfrak{b}) \otimes \mathcal{U}_q(\mathfrak{b}) \otimes \cdots \otimes \mathcal{U}_q(\mathfrak{b}) \\ + 1 \otimes \tilde{\mathcal{U}}_q^-(\mathfrak{b}) \otimes \mathcal{U}_q(\mathfrak{b}) \otimes \cdots \otimes \mathcal{U}_q(\mathfrak{b}) + \cdots + 1 \otimes \cdots \otimes 1 \otimes \tilde{\mathcal{U}}_q^-(\mathfrak{b}) \otimes \tilde{\mathcal{U}}_q^+(\mathfrak{b}).\end{aligned}$$

Hence, using the previous notations, for $N > i_0$,

$$h_r \cdot v_J = \left(\left(h_r + \frac{q^{-2i_0-2}}{q-q^{-1}}\,\mathrm{Id}\right)\cdot v_J^{(i_0)}\right) \otimes u \quad \in h_{r,J} v_J + \sum_{K \subset \mathrm{Conv}(J), K \prec J} \mathbb{C} v_K.$$

$\square$

Proposition 1.3.7. *The vector space T has a basis of ℓ-weights vectors. More precisely,*

$$T = \bigoplus_{J \in \mathcal{P}_f(\mathbb{N})} \mathbb{C} w_J, \quad \textit{with } w_J \textit{ of } \ell\textit{-weight } (\mathbf{\Psi}_1)^{-1} \prod_{j \in J} A^{-1}_{q^{-2j}}.$$

As the ℓ-weight spaces are finite dimensional, one can define a q-character for T,

$$\chi_q(T) = \sum_{J \in \mathcal{P}_f(\mathbb{N})} [\mathbf{\Psi}_1^{-1}] \prod_{j \in J} A^{-1}_{q^{-2j}}. \tag{1.3.1.9}$$

Remark 1.3.8. The normalized q-character of T is exactly the limit $\chi^{\infty}_{1,1}$ of the normalized q-characters of the sequence of standard modules $(S_N)_{N \geqslant 1}$ obtained in (1.2.2.4).

Proof. By Lemma 1.3.6, for all $r \in \mathbb{N}^*$, $J \in \mathcal{P}_f(\mathbb{N})$,

$$h_r \cdot v_J \in \left(\sum_{m=0}^{+\infty} h_r^{J,m} \right) v_J + \sum_{K \subset \mathrm{Conv}(J), K < J} \mathbb{C} v_K,$$

where as before $h_r^{J,m}$ is the coefficient coming from the action of h_r on the m-th component of v_J.

Hence the action of the $(h_r)_{r \in \mathbb{N}^*}$ is simultaneously diagonalizable. Let us look at the diagonal terms. For $r \geqslant 1, J \in \mathcal{P}_f(\mathbb{N})$,

$$h_r^{J,m} = \begin{cases} q^{-(2m+1)r} \frac{[r]_q}{r} & \text{if } m \notin J \\ -q^{-(2m-1)r} \frac{[r]_q}{r} & \text{if } m \in J \end{cases}.$$

Hence,

$$\sum_{r=1}^{+\infty} \left(\sum_{m=0}^{+\infty} h_r^{J,m} \right) z^r = \frac{1}{q - q^{-1}} \log \left(\frac{1}{1-z} \prod_{m \in J} \frac{1 - q^{-2m+2} z}{1 - q^{-2m-2} z} \right).$$

Thus the vector space T contains a basis of ℓ-weight vectors $(w_J)_{J \in \mathcal{P}_f(\mathbb{N})}$, where for each $J \in \mathcal{P}_f(\mathbb{N})$, w_J is a ℓ-weight $(\mathbf{\Psi}_1)^{-1} \prod_{j \in J} A^{-1}_{q^{-2j}}$. Moreover,

$$w_J \in v_J + \mathrm{Vect}(v_K, K \preceq J, K \neq J).$$

□

Remark 1.3.9. However, the action cannot be extended to the full Borel algebra $\mathcal{U}_q(\mathfrak{b})$ the same way. For example, for $N \in \mathbb{N}$ if we consider the truncated pure tensor vector,

$$v_{\varnothing}^{(N)} = v_{-1}^+ \otimes v_{-3}^+ \otimes v_{-5}^+ \otimes \cdots v_{-2N+1}^+ \otimes v_{-2N-1}^+,$$

then x_1^- acts on $v_{\varnothing}^{(N)} \in L(\tilde{Y}_{q^{-1}}) \otimes L(\tilde{Y}_{q^{-3}}) \otimes \cdots \otimes L(\tilde{Y}_{q^{-2N-1}})$, as

$$x_1^- \cdot v_{\varnothing}^{(N)} = \sum_{k=0}^{N} q^{-2k-1} v_{\{k\}}.$$

Which does not have a limit in T as $N \to +\infty$.

Remark 1.3.10. Hence, if we consider the Conjecture 1.2.7, then necessarily, the sum of the ℓ-weight spaces of any potential asymptotical standard module $M(\mathbf{\Psi}_1^{-1})$ is the infinite tensor product T (it is the only $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$-module containing a sub-$\mathcal{U}_q(\mathfrak{b})^+$-module isomorphic to S_N for any $N \geqslant 0$). As the action on T cannot be extended, the module $M(\mathbf{\Psi}_1^{-1})$ must be defined differently.

Generalization of this construction

Let $\mathbf{\Psi} \in P_\ell^{\mathfrak{r}}$ be a negative ℓ-weight such that $L(\mathbf{\Psi})$ is in $\mathcal{O}_{\mathbb{Z}}^-$. In this section, we generalize Proposition 1.3.7 to this context.

As seen in example 1.1.31, the roots and poles of $\Psi_1(z)$ all have the same parity. Let us write $\mathbf{\Psi}$ as a finite product

$$\mathbf{\Psi} = [\omega] \times m \times \left(\prod_{r=R_1}^{R_2} \mathbf{\Psi}_{q^{2r}}^{-b_r} \right),$$

where $\omega \in P_{\mathbb{Q}}$, m is a monomial in the $(Y_{q^{2l+1}})_{l\in\mathbb{Z}}$ and $b_r \in \mathbb{N}$.

In Section 1.2.2, we have seen that each $\mathbf{\Psi}_{q^{2r}}^{-1}$ can be written as an infinite product of $(\tilde{Y}_{q^{2k-1}})_{k\leqslant r}$. Hence, the ℓ-weight $\mathbf{\Psi}$ can be seen as a limit of a series of monomials,

$$\mathbf{\Psi} \approx \lim_{N\to+\infty} [\omega'] \prod_{r=-N}^{R} \tilde{Y}_{q^{2r-1}}^{a_r} \quad \left(= \lim_{N\to+\infty} m_N \right),$$

where the sequence $(a_r)_{-\infty<r\leqslant R}$ is ultimately stationary.

Remark 1.3.11. If $\mathbf{\Psi}$ is only a product of $[\omega]$ and $(Y_{q^{2l+1}})_{l\in\mathbb{Z}}$ (if $L(\mathbf{\Psi})$ is finite-dimensional, with Proposition 1.1.16), this sequence is stationary. In that case, the vector space T is finite-dimensional.

We also know that the sequence of normalized q-characters of the standard modules associated to m_N converges as $N \to +\infty$. With notations from Section 1.2.2,

$$\chi_{\mathbf{\Psi}}^{\infty} = \lim_{N\to+\infty} (\tilde{\chi}_q(S_N)) = \lim_{N\to+\infty} \left(\prod_{r=-N}^{R} \left(1 + A_{q^{2r}}^{-1}\right)^{a_r} \right) \in \mathbb{Z}[[A_{q^{2r}}^{-1}]]_{r\leqslant R}. \tag{1.3.1.10}$$

Theorem 1.3.12. *There exists a $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$-module $T_{\mathbf{\Psi}}$, with finite dimensional ℓ-weight spaces, whose q-character is*

$$\chi_q(T_{\mathbf{\Psi}}) = [\mathbf{\Psi}] \cdot \chi_{\mathbf{\Psi}}^{\infty}.$$

Let us introduce a few notations. Define the set

$$\mathcal{J} = \{(r,k) \mid r \leqslant R, 1 \leqslant k \leqslant a_r\}.$$

We consider the following order on $\mathcal{P}_f(\mathcal{J})$ (generalized from the order defined in 1.3.5).

Definition 1.3.13. Let $N \in \mathbb{N}$ and $J, K \in \mathcal{P}_f(\mathcal{J})$ such that $|K| = |J| = N$.

We say that $J \preceq K$ if and only if they are ordered that way for the lexicographical order on N-tuples, while elements of $\mathcal{J}$ are also ordered lexicographically.

Proof. Heuristically, the vector space $T_{\mathbf{\Psi}}$ is constructed to be the infinite tensor product

$$[\omega'] \otimes \left(L(\tilde{Y}_{q^{2R-1}})\right)^{\otimes a_R} \otimes \left(L(\tilde{Y}_{q^{2R-3}})\right)^{\otimes a_{R-1}} \otimes \cdots \left(L(\tilde{Y}_{q^{-1}})\right)^{\otimes a_0} \otimes \cdots \left(L(\tilde{Y}_{q^{-2r-1}})\right)^{\otimes a_{-r}} \otimes \cdots$$

Hence it is generated by the pure tensors:

$$\begin{aligned} v_{2R-1}^{\pm(1)} \otimes v_{2R-1}^{\pm(2)} \otimes \cdots v_{2R-1}^{\pm(a_R)} \otimes v_{2R-3}^{\pm(1)} \otimes \cdots v_{2R-3}^{\pm(a_{R-1})} \otimes \cdots v_{-1}^{\pm(1)} \otimes v_{-1}^{\pm(a_0)} \otimes \cdots \\ \otimes v_{-2r-1}^{\pm(1)} \otimes \cdots v_{-2r-1}^{\pm(a_{-r})} \otimes \cdots , \end{aligned} \tag{1.3.1.11}$$

with a finite number of lowest weight components $(-)$.

Formally, as in (1.3.1.2), let $T_{\boldsymbol{\Psi}}$ be the vector space

$$T_{\boldsymbol{\Psi}} := \bigoplus_{J \in \mathcal{P}_f(\mathcal{J})} \mathbb{C} v_J.$$

As before, as the *infinite pure tensors* have a finite number of lowest weight components, and thanks to the approximate coproduct formulas (1.3.1.3) and (1.3.1.4), the action of $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$ on the finite tensor products $\tilde{S}_N = [\omega'] \bigotimes_{r=-N}^{R} \left(L(\tilde{Y}_{q^{2r-1}})\right)^{\otimes a_r}$ stabilizes as $N \to +\infty$ and the limit can be taken as the action of $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$ on $T_{\boldsymbol{\Psi}}$.

With the decomposition in the proof of Lemma 1.3.6, one can see that the action of the ℓ-Cartan subalgebra $\mathcal{U}_q(\mathfrak{b})^0$ on the v_J's is upper triangular, for the order on $\mathcal{P}_f(\mathcal{J})$ defined in (1.3.13).

Then, one has, for all $J \in \mathcal{P}_f(\mathcal{J})$, $r \geqslant 1$,

$$h_r.v_J \in \lambda_{r,J} v_J + \bigoplus_{K \prec J, K \subset \mathrm{Conv}(J)} \mathbb{C} v_K,$$

where the $(\lambda_{r,J})_{r \geqslant 1}$ satisfy

$$\sum_{r \geqslant 1} \lambda_{r,J} z^r = \frac{1}{q - q^{-1}} \log(\Phi_J(z)),$$

with

$$\Phi_J(z) = \sum_{m \geqslant 0} \phi_{m,J} z^m, \quad \text{and } (\phi_{m,J})_{m \geqslant 0} = \boldsymbol{\Psi} \prod_{(r,k) \in J} A_{q^{2r}}^{-1}.$$

Hence the vector space $T_{\boldsymbol{\Psi}}$ has a basis of ℓ-weight vectors. Let us write:

$$T_{\boldsymbol{\Psi}} = \bigoplus_{J \in \mathcal{P}_f(J)} \mathbb{C} w_J,$$

where, for all $J \in \mathcal{P}_f(J)$, w_J is an ℓ-weight vector of ℓ-weight $\boldsymbol{\Psi} \prod_{(r,k) \in J} A_{q^{2r}}^{-1}$ (different w_J can contribute to the same ℓ-weight space).

Thus, $T_{\boldsymbol{\Psi}}$ has finite dimensional ℓ-weight spaces and its q-character is

$$\chi_q(T_{\boldsymbol{\Psi}}) = \sum_{J \in \mathcal{P}_f(J)} \boldsymbol{\Psi} \prod_{(r,k) \in J} A_{q^{2r}}^{-1} = [\boldsymbol{\Psi}] \cdot \chi_{\boldsymbol{\Psi}}^{\infty}.$$

□

1.3.2 Construction of induced modules

As stated in Remark 1.3.10, to obtain a $\mathcal{U}_q(\mathfrak{b})$-module structure on the infinite tensor product, one needs to extend these modules. This is why our asymptotical standard modules will be obtained by induction.

Let M be a $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$-module. Define the $\mathcal{U}_q(\mathfrak{b})$-module M^c induced from M:

$$M^c = \mathcal{U}_q(\mathfrak{b}) \otimes_{\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})} M \cong \mathcal{U}_q(\mathfrak{b})^- \otimes_{\mathbb{C}} M.$$

This induction preserves the q-character of the module. More precisely, we have Proposition 1.3.18, which is obtained from Lemma 1.3.16 bellow.

First of all, we use the following Lemma on the structure of $\mathcal{U}_q(\mathfrak{b})^-$.

Lemma 1.3.14. *[Dam93] The elements* $\left(x^-_{m_1}x^-_{m_2}x^-_{m_3}\cdots x^-_{m_s}\right)$*, where* $s \geqslant 0$ *and* $1 \leqslant m_1 \leqslant m_2 \leqslant m_3 \leqslant \cdots \leqslant m_s$*, form a basis of* $\mathcal{U}_q(\mathfrak{b})^-$*.*

Remark 1.3.15. This result can also be obtained by seeing the $\langle x^-_m\rangle_{1\leqslant m\leqslant N} \subset \mathcal{U}_q(\mathfrak{b})^-$ as successive Ore extensions. Then, with the methods used in [Kas95], we see that the elements $(x^-_1)^{i_1}(x^-_2)^{i_2}\cdots(x^-_N)^{i_N}$ form a basis of $\langle x^-_m\rangle_{1\leqslant m\leqslant N}$.

Let us recall relation (1.1.1.4) and relation (1.1.1.1) in this context:

$$x^-_{m+1}x^-_l - q^{-2}x^-_l x^-_{m+1} = q^{-2}x^-_m x^-_{l+1} - x^-_{l+1}x^-_m, \text{ for all } m,l \geqslant 1, \tag{1.3.2.1}$$

$$[h_r, x^-_m] = -\frac{[2r]_q}{r}x^-_{m+r}, \text{ for all } r,m \geqslant 1. \tag{1.3.2.2}$$

Note that the algebra $\mathcal{U}_q(\mathfrak{b})^-$ has a natural $\mathbb{N}$-graduation, which is different from the graduation coming from the Q-graduation on $\mathcal{U}_q(\mathfrak{g})$ (1.1.1.3). We note:

$$\deg(x^-_{m_1}x^-_{m_2}x^-_{m_3}\cdots x^-_{m_s}) = \sum_{i=1}^{s} m_i. \tag{1.3.2.3}$$

Thanks to the relation (1.3.2.1), this is a well-defined graduation on $\mathcal{U}_q(\mathfrak{b})^-$.

Lemma 1.3.16. *Let* $v \in M^c$ *be an eigenvector of* h_r *for a certain* $r \geqslant 1$*. Then* v *belongs to the subspace* $1 \otimes M$*.*

Proof. Let us write, with the result of Lemma 1.3.14:

$$v = \sum_{s\geqslant 0}\sum_{1\leqslant m_1\leqslant m_2\leqslant\cdots\leqslant m_s} x^-_{m_1}x^-_{m_2}\cdots x^-_{m_s}\otimes u_{m_1,m_2,\ldots,m_s},$$

where all but finitely many of the $u_{m_1,m_2,\ldots,m_s} \in M$ are 0. By (1.3.2.2), one has, for all $s \geqslant 1$, and all $1 \leqslant m_1 \leqslant m_2 \leqslant \cdots \leqslant m_s$:

$$h_r\left(x^-_{m_1}x^-_{m_2}\cdots x^-_{m_s}\right) = x^-_{m_1}x^-_{m_2}\cdots x^-_{m_s}h_r - \frac{[2r]_q}{r}\sum_{j=1}^{s} x^-_{m_1}\cdots x^-_{m_j+r}\cdots x^-_{m_s}.$$

Thus, from the definition of the induced module,

$$\begin{aligned} h_r\cdot v = 1\otimes h_r u_{1,\ldots,1} + \sum_{s\geqslant 1}\Bigg(\sum_{1\leqslant m_1\leqslant m_2\leqslant\cdots\leqslant m_s}&\Big(x^-_{m_1}x^-_{m_2}\cdots x^-_{m_s}\otimes h_r u_{m_1,m_2,\ldots,m_s}\\ &-\frac{[2r]_q}{r}\sum_{j=1}^{s} x^-_{m_1}\cdots x^-_{m_j+r}\cdots x^-_{m_s}\otimes u_{m_1,m_2,\ldots,m_s}\Big)\Bigg).\end{aligned}$$

By hypothesis, there exists $\lambda \in \mathbb{C}$ such that $h_r\cdot v = \lambda v$. Thus, from the Q-graduation (1.1.1.3) on $\mathcal{U}_q(\mathfrak{b})^-$, one has $1\otimes h_r u_{1,\ldots,1} = \lambda 1\otimes u_{1,\ldots,1}$, and, for all $s \geqslant 1$,

$$\begin{aligned}\sum_{1\leqslant m_1\leqslant m_2\leqslant\cdots\leqslant m_s}&\Big(x^-_{m_1}x^-_{m_2}\cdots x^-_{m_s}\otimes h_r u_{m_1,m_2,\ldots,m_s}\\ &-\frac{[2r]_q}{r}\sum_{j=1}^{s} x^-_{m_1}\cdots x^-_{m_j+r}\cdots x^-_{m_s}\otimes u_{m_1,m_2,\ldots,m_s}\Big)\\ &= \lambda\sum_{1\leqslant m_1\leqslant m_2\leqslant\cdots\leqslant m_s} x^-_{m_1}x^-_{m_2}\cdots x^-_{m_s}\otimes u_{m_1,m_2,\ldots,m_s}.\end{aligned}$$

Moreover, from the $\mathbb{N}$-graduation (1.3.2.3) on $\mathcal{U}_q(\mathfrak{b})^-$, one has, for all $s \geqslant 1$, for all $N \geqslant 1$,

$$\sum_{\substack{1\leqslant m_1\leqslant m_2\leqslant\cdots\leqslant m_s\\ \sum m_i=N+r}} x^-_{m_1}x^-_{m_2}\cdots x^-_{m_s}\otimes(h_r-\lambda)u_{m_1,m_2,\ldots,m_s} - \frac{[2r]_q}{r}\sum_{\substack{1\leqslant m_1\leqslant m_2\leqslant\cdots\leqslant m_s\\ \sum m_i=N}}\sum_{j=1}^{s} x^-_{m_1}\cdots x^-_{m_j+r}\cdots x^-_{m_s}\otimes u_{m_1,m_2,\ldots,m_s}=0. \quad (1.3.2.4)$$

Let us note that the $x^-_{m_1}\cdots x^-_{m_j+r}\cdots x^-_{m_s}$ are not necessarily in the basis of $\mathcal{U}_q(\mathfrak{b})^-$ of Lemma 1.3.14, but from (1.3.2.1) they can be written as a linear combination of these basis elements. Moreover, with Remark 1.3.15, $x^-_{m_1}\cdots x^-_{m_j+r}\cdots x^-_{m_s}\in\langle x^-_m\rangle_{1\leqslant m\leqslant\overline{m}}\subset\mathcal{U}_q(\mathfrak{b})^-$, where $\overline{m}=\max(m_j+r,m_s)$, and can be decomposed in the basis of this subalgebra (note that m_j+r is possibly greater than m_s).

Now, suppose that v does not belong to the subspace $1\otimes M$. Then, for all $s\geqslant 1$ and all $N\geqslant 1$ such that there exists $(m_1,m_2,\ldots,m_s)$ with $\sum m_i=N$ and $u_{m_1,m_2,\ldots,m_s}\neq 0$, consider

$$\overline{m_s}=\max\left\{m_s\in\mathbb{N}\mid \exists s\geqslant 1,\exists(m_1,m_2,\ldots,m_{s-1}),\sum m_i=N,u_{m_1,\ldots,m_{s-1},\overline{m_s}}\neq 0\right\}. \quad (1.3.2.5)$$

Then, we decompose the left-hand term of relation (1.3.2.4) into sums of pure tensors whose first factor is in the basis of Lemma 1.3.14. Then we extract from this sum the terms whose first factor is of the form $x^-_{m_1}\cdots x^-_{m_{s-1}}x^-_{\overline{m_s}+r}$. We get a sum equal to 0, from relation (1.3.2.4). Let us explicit the terms we obtain. As $r>0$, the first term of the left-hand-side of (1.3.2.4) does not contribute. The remaining terms have $-\frac{[2r]_q}{r}$ as a scalar factor. The terms obtained from $j=s$ are of the form

$$\sum_{\substack{1\leqslant m_1\leqslant m_2\leqslant\cdots\leqslant m_{s-1}\leqslant\overline{m_s}\\ \sum m_i=N}} x^-_{m_1}\cdots x^-_{m_{s-1}}x^-_{\overline{m_s}+r}\otimes u_{m_1,\ldots,m_{s-1},\overline{m_s}},$$

where the first factors of each terms are elements of the considered basis, so no permutation of the x^-_m is needed.

By rewriting (1.3.2.1),

$$x^-_{\overline{m_s}+r}x^-_{\overline{m_s}}=q^{-2}x^-_{\overline{m_s}}x^-_{\overline{m_s}+r}+\underbrace{q^{-2}x^-_{\overline{m_s}+r-1}x^-_{\overline{m_s}+1}-x^-_{\overline{m_s}+1}x^-_{\overline{m_s}+r-1}}_{\text{does not contribute}},$$

thus, the terms obtained from $j=s-1$ are of the form

$$\sum_{\substack{1\leqslant m_1\leqslant\cdots\leqslant m_{s-2}\leqslant m_{s-1}=\overline{m_s}\\ \sum m_i=N}} q^{-2}x^-_{m_1}\cdots x^-_{m_{s-2}}x^-_{\overline{m_s}}x^-_{\overline{m_s}+r}\otimes u_{m_1,\ldots,m_{s-2},\overline{m_s},\overline{m_s}},$$

as only the terms for which $m_{s-1}=\overline{m_s}$ contribute to the considered sum.

Recursively, the terms obtained from j=1 are of the form

$$q^{-2(s-1)}\left(x^-_{\overline{m_s}}\right)^{s-1}x^-_{\overline{m_s}+r}\otimes u_{\overline{m_s},\overline{m_s},\ldots,\overline{m_s}},$$

if $N=s\overline{m_s}$, and 0 otherwise. Thus, the resulting formula is:

$$0=\sum_{\substack{1\leqslant m_1\leqslant\cdots\leqslant m_{s-1}\leqslant\overline{m_s}\\ \sum m_i=N}} C_{m_1,\ldots,m_{s-1}}x^-_{m_1}\cdots x^-_{m_{s-1}}x^-_{\overline{m_s}+r}\otimes u_{m_1,\ldots,m_{s-1},\overline{m_s}}, \quad (1.3.2.6)$$

where

$$C_{m_1,\ldots,m_{s-1}} = \sum_{k=0}^{\mathcal{N}} q^{-2k} \text{ , with } \mathcal{N} = \sharp\{m_i \mid m_i = \overline{m_s}\}. \tag{1.3.2.7}$$

As q is not a root of unity, these $C_{m_1,\ldots,m_{s-1}}$ are non-zero. This implies that all the $u_{m_1,\ldots,m_{s-1},\overline{m_s}}$ are 0, which is a contradiction with the hypothesis taken in the definition of $\overline{m_s}$ in (1.3.2.5). □

Remark 1.3.17. Recall the definition of ℓ-weight vectors from Section 1.1.2: v in the $\mathcal{U}_q(\mathfrak{b})$-module V is an ℓ-weight vector if there is an ℓ-weight $\mathbf{\Psi}$ and $p \in \mathbb{N}$ such that:

$$\forall i \in I, \forall m \geqslant 0, (\phi_{i,m}^+ - \psi_{i,m})^p v = 0.$$

Proposition 1.3.18. *The ℓ-weights vectors of M^c are exactly the $1 \otimes u$, where u is an ℓ-weight vector of M.*

Proof. First of all, if u is an ℓ-weight vector of M, then by construction of M^c, $1 \otimes u$ is an ℓ-weight vector of M^c.

Conversely, let us write:

$$M^c = (1 \otimes M) \oplus M_{\geqslant 1}, \tag{1.3.2.8}$$

where $M_{\geqslant 1}$ is generated by the pure tensors for which the first factor is of non-zero Q-degree. Notice that from (1.3.2.2), $M_{\geqslant 1}$ is a sub-$\mathcal{U}_q(\mathfrak{b})^0$-module.

For all ℓ-weight vector v of M^c, there exists:

$$p_v = \min\left\{p \in \mathbb{N}^+ \mid \exists \psi_1 \in \mathbb{C}, (\phi_1^+ - \psi_1)^p v = 0\right\}.$$

Now suppose there exists non-zero ℓ-weight vectors in M^c which is not in $1 \otimes M$. Then its projection on $M_{\geqslant 1}$ in the decomposition (1.3.2.8) is a non-zero ℓ-weight vector in $M_{\geqslant 1}$. Consider such a v_0 which minimizes p_v. Let $p_0 = p_{v_0}$. There is $\psi_1 \in \mathbb{C}$ such that $(\phi_1^+ - \psi_1)^{p_0} v_0 = 0$. By definition,

$$w = (\phi_1^+ - \psi_1)^{p_0 - 1} v_0 \neq 0.$$

But, $\phi_1^+ w = \psi_1 w$, as $h_1 = k_1^{-1}\phi_1^+$, and v_0 and w are weight vectors, then w is an eigenvector of h_1. By Lemma 1.3.16, $w \in 1 \otimes M$. However, $w \in M_{\geqslant 1}$ as $M_{\geqslant 1}$ is stabilized by $\mathcal{U}_q(\mathfrak{b})^0$. Hence $w = 0$, which is a contradiction with the definition of v_0. □

Now everything is in place to define the asymptotical standard modules. Let $\mathbf{\Psi}$ be a negative ℓ-weight such that $L(\mathbf{\Psi})$ is in $\mathcal{O}_{\mathbb{Z}}^-$. Define the induced $\mathcal{U}_q(\mathfrak{b})$-module from the $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$-module $T_{\mathbf{\Psi}}$ constructed in the previous section:

$$T_{\mathbf{\Psi}}^c := \mathcal{U}_q(\mathfrak{b}) \otimes_{\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})} T_{\mathbf{\Psi}}.$$

Theorem 1.3.19. *The vector space $T_{\mathbf{\Psi}}^c$ is a $\mathcal{U}_q(\mathfrak{b})$-module, such that its ℓ-weight spaces are finite-dimensional. As such, $T_{\mathbf{\Psi}}^c$ satisfies the Conjecture 1.2.7 for $\mathbf{\Psi}$. One has:*

$$\chi_q(T_{\mathbf{\Psi}}^c) = [\mathbf{\Psi}] \cdot \chi_{\mathbf{\Psi}}^{\infty}.$$

Proof. From Proposition 1.3.18 we know that the ℓ-weight vectors of $T_{\mathbf{\Psi}}^c$ are exactly the $1 \otimes u$, where u is an ℓ-weight vector of $T_{\mathbf{\Psi}}$. Thus, with the result of Theorem 1.3.12, we know that $T_{\mathbf{\Psi}}^c$ has finite-dimensional ℓ-weight spaces which satisfy:

$$\chi_q(T_{\mathbf{\Psi}}^c) = \chi_q(T_{\mathbf{\Psi}}) = [\mathbf{\Psi}] \cdot \chi_{\mathbf{\Psi}}^{\infty}.$$

Furthermore, let us look at the way $T_{\mathbf{\Psi}}^{\mathfrak{c}}$ is built to see that it satisfies Conjecture 1.2.7. Recall the standard modules defined in Section 1.2.2 and used in Section 1.3.1. For all $N \geqslant 1$, let

$$S_N = \overrightarrow{\bigotimes_{r=-N}^{R}} \left(V_{q^{-2r-1}}^{\otimes a_r} \right).$$

Then the sum of ℓ-weights spaces of $T_{\mathbf{\Psi}}^{\mathfrak{c}}$ is the $\mathcal{U}_q^{\geqslant 0}(\mathfrak{b})$-module $T_{\mathbf{\Psi}}$, which contains for all $N \geqslant 1$ a sub-$\mathcal{U}_q(\mathfrak{b})^+$-modules isomorphic to S_N. □

Remark 1.3.20. The space $T_{\mathbf{\Psi}}^{\mathfrak{c}}$ contains submodules without ℓ-weight vectors. For example, for $\mathbf{\Psi} = \mathbf{\Psi}_1^{-1}$, the submodule $M =< x_r^- \otimes w_\varnothing - 1 \otimes w_{\{0\}} \mid r \in \mathbb{N}^* >$ does not contain any ℓ-weight vectors.

From now on, for all ℓ-weights $\mathbf{\Psi}$ such that $L(\mathbf{\Psi})$ is in the category $\mathcal{O}_{\mathbb{Z}}^-$,

$$M(\mathbf{\Psi}) := T_{\mathbf{\Psi}}^{\mathfrak{c}},$$

will denote the generalized standard module associated to the ℓ-weight $\mathbf{\Psi}$.

1.4 Decomposition of the q-character of asymptotical standard modules

As stated in the introduction, for the category $\mathscr{C}$ of finite-dimensional representations of $\mathcal{U}_q(\mathfrak{b})$, it is known ([Nak01a]) that the classes of the standard modules $[M(m)]$ form a second basis of the Grothendieck ring $K(\mathscr{C})$ (in addition to the classes of simple modules). Moreover, the two bases are triangular with respect to Nakajima's partial ordering of dominant monomials (see [Nak04])):

$$[M(m)] = [L(m)] + \sum_{m' < m} P_{m',m}[L(m')], \tag{1.4.0.1}$$

where the coefficients $P_{m',m} \in \mathbb{Z}$ are non-negative.

The aim of this Section is to show that the q-characters of the modules we have just built have similar decomposition into sum of q-characters of simple modules.

1.4.1 Partial order on $P_\ell^{\mathfrak{r}}$

For a formula of the type (1.4.0.1) to make sense, one needs to define a partial order on $P_\ell^{\mathfrak{r}}$, which is the index set of both the simple modules and the standard modules in our context.

We draw our inspiration from the partial order defined in the proof of [HL16b, Lemma 6.4], which is itself a generalization of the order Nakajima used in [Nak04], recalled in (1.2.2.3). The following definition is valid for $\mathfrak{g}$ any untwisted affine Kac-Moody algebra.

Definition 1.4.1. Let $\mathbf{\Psi}, \mathbf{\Psi}' \in P_\ell^{\mathfrak{r}}$. we say that $\mathbf{\Psi}' \leqslant \mathbf{\Psi}$ if $\mathbf{\Psi}'(\mathbf{\Psi})^{-1}$ is a monomial in the $A_{i,a}^{-1}$, with $i \in I, a \in \mathbb{C}^\times$.

Remark 1.4.2. Contrary to the finite-dimensional case, every ℓ-weight has an infinite number of *lower* ℓ-weights.

1.4.2 Decomposition for $M(\mathbf{\Psi}_1^{-1})$

Let us recall the q-character of the $\mathcal{U}_q(\mathfrak{b})$-module $M(\mathbf{\Psi}_1^{-1})$:

$$\chi_q(M(\mathbf{\Psi}_1^{-1})) = \sum_{J \in \mathcal{P}_f(\mathbb{N})} [\mathbf{\Psi}_1^{-1}] \prod_{j \in J} A^{-1}_{q^{-2j}}.$$

Theorem 1.4.3. *The q-character of the $\mathcal{U}_q(\mathfrak{b})$-module $M(\mathbf{\Psi}_1^{-1})$ has a decomposition into a sum of q-characters of simple modules. More precisely,*

$$\chi_q(M(\mathbf{\Psi}_1^{-1})) = \sum_{m=0}^{+\infty} \sum_{\substack{1 \leqslant r_1 < r_2 < \cdots < r_m \\ r_{i+1} > r_i + 1}} \chi_q(L(\mathbf{\Psi}_1^{-1} A^{-1}_{q^{-2r_1}} A^{-1}_{q^{-2r_2}} \cdots A^{-1}_{q^{-2r_m}})).$$

Remark 1.4.4. This formula is multiplicity-free.

Proof. Let $m \in \mathbb{N}^*$ and $(r_1, r_2, \dots, r_m) \in (\mathbb{N}^*)^m$, satisfying $r_{i+1} > r_i + 1$ for all $1 \leqslant i \leqslant m$. One has

$$\begin{aligned}\mathbf{\Psi}_1^{-1} A^{-1}_{q^{-2r_1}} A^{-1}_{q^{-2r_2}} \cdots A^{-1}_{q^{-2r_m}} = [(-r_m+1)\omega_1] \left(Y_{q^{-1}} \cdots Y_{q^{-2r_1+3}}\right) \left(Y_{q^{-2r_1-3}} \cdots Y_{q^{-2r_2+3}}\right) \cdots \\ \cdots \left(Y_{q^{-2r_{m-1}-3}} \cdots Y_{q^{-2r_m+3}}\right) \mathbf{\Psi}^{-1}_{q^{-2r_m-2}},\end{aligned}$$

The q-sets $\{q^{-1}, q^{-3}, \dots, q^{-2r_1+3}\}$, $\{q^{-2r_i-3}, q^{-2r_i-5}, \dots, q^{-2r_{i+1}+3}\}$, for $1 \leqslant i \leqslant m-1$, and $\{q^{-2r_m-1}, q^{-2r_m-3}, \dots\}$ are pairwise in general position. Hence, following [HL16b, Theorem 7.9], the following tensor product is simple:

$$L(Y_{q^{-1}} \cdots Y_{q^{-2r_1+3}}) \otimes L(Y_{q^{-2r_1-3}} \cdots Y_{q^{-2r_2+3}}) \otimes \cdots \otimes L(Y_{q^{-2r_{m-1}-3}} \cdots Y_{q^{-2r_m+3}}) \otimes L(\mathbf{\Psi}^{-1}_{q^{-2r_m-2}})$$

and of highest ℓ-weight $\mathbf{\Psi}_1^{-1} A^{-1}_{q^{-2r_1}} A^{-1}_{q^{-2r_2}} \cdots A^{-1}_{q^{-2r_m}}$. Thus,

$$\begin{aligned}L(\mathbf{\Psi}_1^{-1} A^{-1}_{q^{-2r_1}} A^{-1}_{q^{-2r_2}} \cdots A^{-1}_{q^{-2r_m}}) = [(-r_m+1)\omega_1] \otimes L(Y_{q^{-1}} \cdots Y_{q^{-2r_1+3}}) \otimes \\ L(Y_{q^{-2r_1-3}} \cdots Y_{q^{-2r_2+3}}) \otimes \cdots \otimes L(Y_{q^{-2r_{m-1}-3}} \cdots Y_{q^{-2r_m+3}}) \otimes L(\mathbf{\Psi}^{-1}_{q^{-2r_m-2}}).\end{aligned}$$

One can then compute the q-character of $L(\mathbf{\Psi}_1^{-1} A^{-1}_{q^{-2r_1}} A^{-1}_{q^{-2r_2}} \cdots A^{-1}_{q^{-2r_m}})$.

$$\begin{aligned}[L(\mathbf{\Psi}_1^{-1} A^{-1}_{q^{-2r_1}} A^{-1}_{q^{-2r_2}} \cdots A^{-1}_{q^{-2r_m}})] &= [(-r_m+1)\omega_1][L(Y_{q^{-1}} \cdots Y_{q^{-2r_1+3}})] \\ &\quad [L(Y_{q^{-2r_1-3}} \cdots Y_{q^{-2r_2+3}})] \cdots [L(Y_{q^{-2r_{m-1}-3}} \cdots Y_{q^{-2r_m+3}})][L(\mathbf{\Psi}^{-1}_{q^{-2r_m-2}})] \\ &= [\mathbf{\Psi}_1^{-1}] \sum_{\substack{J \in \mathcal{P}_f(\mathbb{N}) \\ (\star)}} \prod_{j \in J} A^{-1}_{q^{-2j}}.\end{aligned}$$

where in $(\star)$ we consider the finite subsets of $\mathbb{N}$ with m connected components, starting with $r_1, r_2, \dots, r_m$ respectively, and with $m+1$ connected components, starting with $0, r_1, r_2, \dots, r_m$ respectively.

Thus,

$$\begin{aligned}&\sum_{m=0}^{+\infty} \sum_{\substack{1 \leqslant r_1 < r_2 < \cdots < r_m \\ r_{i+1} > r_i + 1}} [L(\mathbf{\Psi}_1^{-1} A^{-1}_{q^{-2r_1}} A^{-1}_{q^{-2r_2}} \cdots A^{-1}_{q^{-2r_m}})] \\ &\qquad = [\mathbf{\Psi}_1^{-1}] \sum_{J \in \mathcal{P}_f(\mathbb{N})} \prod_{j \in J} A^{-1}_{q^{-2j}} = \chi_q(M(\mathbf{\Psi}_1^{-1}))\end{aligned}$$

$\square$

Remark 1.4.5. One has indeed

$$\chi_q(M(\mathbf{\Psi}_1^{-1})) \in \sum_{\mathbf{\Psi} \leqslant \mathbf{\Psi}_1^{-1}} \mathbb{N}[L(\mathbf{\Psi})],$$

for the partial order on $P_\ell^{\mathfrak{r}}$ defined in Section 1.4.1.

1.4.3 General decomposition

Consider a negative ℓ-weight $\mathbf{\Psi}$, as in Section 1.3.1, such that $L(\mathbf{\Psi})$ is in $\mathcal{O}_{\mathbb{Z}}^-$. It can be written as a finite product

$$\mathbf{\Psi} = [\omega] \times m \times \prod \mathbf{\Psi}_{q^{2r}}^{-v_r}, \tag{1.4.3.1}$$

where $\lambda \in P_{\mathbb{Q}}$, m is a monomial in $\mathbb{Z}[Y_{q^{2r-1}}]_{r\in\mathbb{Z}}$ and $v_r \in \mathbb{N}$. First we need the following Lemma.

Lemma 1.4.6. *For $\mathbf{\Psi}^1, \mathbf{\Psi}^2$, negative ℓ-weights as in (1.4.3.1), one has*

$$\chi_q(M(\mathbf{\Psi}^1\mathbf{\Psi}^2)) = \chi_q(M(\mathbf{\Psi}^1))\chi_q(M(\mathbf{\Psi}^2)). \tag{1.4.3.2}$$

Proof. If $\mathbf{\Psi}^1$ and $\mathbf{\Psi}^2$ are monomials in $\mathbb{Z}[Y_{q^{2r-1}}]_{r\in\mathbb{Z}}$, then the standard modules $M(\mathbf{\Psi}^1)$ and $M(\mathbf{\Psi}^2)$ are finite dimensional and the result is known.

Else, as in (1.3.1.10), their normalized q-characters are limits of normalized q-characters of finite dimensional standard modules:

$$\tilde{\chi}_q(M(\mathbf{\Psi}^i)) = \lim_{N\to+\infty} \tilde{\chi}_q(M(m_N^i)), \forall i \in \{1,2\}.$$

The result is obtained by taking the products of the limits. $\square$

Thus, the q-character of the standard module $M(\mathbf{\Psi})$ is a product,

$$\chi_q(M(\mathbf{\Psi})) = [\omega] \cdot \chi_q(m) \cdot \prod \chi_q(\mathbf{\Psi}_a)^{-v_a},$$

for which each term has a decomposition into a sum of q-characters of simple modules, corresponding to *lower* highest ℓ-weights.

Finally, using the following lemma, which is straightforward from the definition of the order.

Lemma 1.4.7. *The order on the negative ℓ-weights is compatible with the product. More precisely, for $\mathbf{\Psi}^1, \mathbf{\Psi}^2, \mathbf{\Psi}$ and $\mathbf{\Psi}'$ some negative ℓ-weights,*

$$\left(\mathbf{\Psi} \leqslant \mathbf{\Psi}^1 \text{ and } \mathbf{\Psi}' \leqslant \mathbf{\Psi}\mathbf{\Psi}^2\right) \Rightarrow \mathbf{\Psi}' \leqslant \mathbf{\Psi}^1\mathbf{\Psi}^2.$$

One can finally conclude,

Corollary 1.4.8. *For every negative ℓ-weight $\mathbf{\Psi}$ such that $L(\mathbf{\Psi})$ is in $\mathcal{O}_{\mathbb{Z}}^-$, one has*

$$\chi_q(M(\mathbf{\Psi})) \in \sum_{\mathbf{\Psi}' \leqslant \mathbf{\Psi}} \mathbb{N}\chi_q(L(\mathbf{\Psi}')). \tag{1.4.3.3}$$

Remark 1.4.9. We showed that the coefficients in the decomposition (1.4.3.3) are non-negative. It would be interesting to show that these coefficients can be interpreted as dimensions, which would explain their non-negativity.

Chapter 2

Quantum Grothendieck rings as quantum cluster algebras.

This chapter is an adapted version of [Bit19], preprint arXiv:1902.00502.

Abstract.

We define and construct a quantum Grothendieck ring for a certain monoidal subcategory of the category $\mathcal{O}$ of representations of the quantum loop algebra introduced by Hernandez-Jimbo. We use the cluster algebra structure of the Grothendieck ring of this category to define the quantum Grothendieck ring as a quantum cluster algebra. When the underlying simple Lie algebra is of type A, we prove that this quantum Grothendieck ring contains the quantum Grothendieck ring of the category of finite-dimensional representations of the associated quantum affine algebra. In type A_1, we identify remarkable relations in this quantum Grothendieck ring.

2.1 Introduction

Let $\mathfrak{g}$ be a simple Lie algebra of Dynkin type A, D or E (also called simply laced types), and let $L\mathfrak{g} = \mathfrak{g}\otimes\mathbb{C}[t^{\pm1}]$ be the loop algebra of $\mathfrak{g}$. For q a generic complex number, Drinfeld [Dri88] introduced a q-deformation of the universal enveloping algebra $U(L\mathfrak{g})$ of $L\mathfrak{g}$ called the *quantum loop algebra* $\mathcal{U}_q(L\mathfrak{g})$. It is a $\mathbb{C}$-algebra with a Hopf algebra structure, and the category $\mathscr{C}$ of its finite-dimensional representations is a monoidal category. The category $\mathscr{C}$ was studied extensively, in particular to build solutions to the quantum Yang-Baxter equation with spectral parameter (see [Jim89] for a detailed review).

Using the so-called "Drinfeld-Jimbo" presentation of the quantum loop algebra, one can define a *quantum Borel subalgebra* $\mathcal{U}_q(\mathfrak{b})$, which is a Hopf subalgebra of $\mathcal{U}_q(L\mathfrak{g})$. We are here interested in studying a category $\mathcal{O}$ of representations of $\mathcal{U}_q(\mathfrak{b})$ introduced by Hernandez-Jimbo [HJ12]. The category $\mathcal{O}$ contains all finite-dimensional $\mathcal{U}_q(\mathfrak{b})$-modules, as well as some infinite-dimensional representations, albeit with finite-dimensional weight spaces. In particular, this category $\mathcal{O}$ contains the *prefundamental representations*. These are a family of infinite dimensional simple $\mathcal{U}_q(\mathfrak{b})$-modules, which first appeared in the work of Bazhanov, Lukyanov, Zamolodchikov [BLZ99] for $\mathfrak{g} = \mathfrak{sl}_2$ under the name *q-oscillator representations*.

These prefundamental representations were also used by Frenkel-Hernandez [FH15] to prove Frenkel-Reshetikhin's conjecture on the spectra of quantum integrable systems [FR99]. More precisely, quantum integrable systems are studied via a partition function Z, which in turns can be scaled down to the study of the eigenvalues λ_j of the transfer matrix T. For the 6-vertex (and 8-vertex) models, [Bax72] showed that the eigenvalues of T have the following remarkable form:

$$\lambda_j = A(z)\frac{Q_j(zq^{-2})}{Q_j(z)} + D(z)\frac{Q_j(zq^{2})}{Q_j(z)}, \tag{2.1.0.1}$$

where q and z are parameters of the model, the functions $A(z)$, $D(z)$ are universal, and Q_j is a polynomial. This relation is called the *Baxter relation*. In the context of representation theory, relation (2.1.0.1) can be categorified as a relation in the Grothendieck ring of the category $\mathcal{O}$. For $\mathfrak{g} = \mathfrak{sl}_2$, if V is the two-dimensional simple representation of $\mathcal{U}_q(L\mathfrak{g})$ of highest loop-weight $Y_{aq^{-1}}$, then

$$[V \otimes L_a^+] = [\omega_1][L_{aq^{-2}}^+] + [-\omega_1][L_{aq^2}^+], \tag{2.1.0.2}$$

where $[\pm\omega_1]$ are one-dimensional representations of weight $\pm\omega_1$ and L_a^+ denotes the positive prefundamental representation of quantum parameter a.

Frenkel-Reshetikhin's conjecture stated that for more general quantum integrable systems, constructed via finite-dimensional representations of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{g}})$ (of which the quantum loop algebra is a quotient), the spectra had a similar form as relation (2.1.0.1).

Let t be an indeterminate. The Grothendieck ring of the category $\mathscr{C}$ has an interesting t-deformation called the *quantum Grothendieck ring*, which belongs to a non-commutative quantum torus $\mathcal{Y}_t$. The quantum Grothendieck ring was first studied by Nakajima [Nak04] and Varagnolo-Vasserot [VV03] in relation with quiver varieties. Inside this ring, one can define for all simple modules L classes $[L]_t$, called (q,t)-*characters*. Using these classes, Nakajima was able to compute the characters of the simple modules L, which were completely inaccessible in general, thanks to a Kazhdan-Lusztig type algorithm.

One would want to extend these results to the context of the category $\mathcal{O}$, with the ultimate goal of (algorithmically) computing characters of all simple modules in $\mathcal{O}$. In order to do that, one first needs to build a quantum Grothendieck ring $K_t(\mathcal{O})$ inside which the classes $[L]_t$ can be defined.

Another interesting approach to this category $\mathcal{O}$ is its *cluster algebra* structure (see below). Hernandez-Leclerc [HL13] first noticed that the Grothendieck ring of a certain monoidal subcategory $\mathscr{C}_1$ of the category $\mathscr{C}$ of finite-dimensional $\mathcal{U}_q(L\mathfrak{g})$-modules had the structure of a cluster algebra. Then, they proved [HL16b] that the Grothendieck ring of a certain monoidal subcategory $\mathcal{O}_{\mathbb{Z}}^+$ of the category $\mathcal{O}$ had a cluster algebra structure, of infinite rank, for which one can take as initial seed the classes of the positive prefundamental representations (the category $\mathcal{O}_{\mathbb{Z}}^+$ contains the finite-dimensional representations and the positive prefundamental representations whose spectral parameter satisfies an integrality condition). Moreover, some exchange relations coming from cluster mutations appear naturally. For example, the Baxter relation (2.1.0.2) is an exchange relation in this cluster algebra.

To construct a quantum Grothendieck ring for the category $\mathcal{O}$, the approaches used previously are not applicable anymore. The geometrical approach of Nakajima and Varagnolo-Vasserot (in which the t-graduation naturally comes from the graduation of cohomological complexes) requires a geometric interpretation of the objects in the category $\mathcal{O}$, which has not yet been found. The more algebraic approach consisting of realizing the (quantum) Grothendieck ring as an invariant under a sort of Weyl symmetry, which allowed Hernandez to define a quantum Grothendieck ring of finite-dimensional representations in non-simply laced types, is again not relevant for the category $\mathcal{O}$. Only the cluster algebra approach yields results in this context.

In this Chapter, we propose to build the quantum Grothendieck of the category $\mathcal{O}_{\mathbb{Z}}^+$ as a *quantum cluster algebra*. Quantum cluster algebras are non-commutative versions of cluster algebras, they live inside a quantum torus, generated by the initial variables, together with t-commuting relations:

$$X_i * X_j = t^{\Lambda_{ij}} X_j * X_i. \tag{2.1.0.3}$$

First of all, one has to build such a quantum torus, and check that it contains the quantum torus $\mathcal{Y}_t$ of the quantum Grothendieck ring of the category $\mathscr{C}$. This is proven as the first result of this chapter (Proposition 2.3.3).

Next, one has to show that this quantum torus is compatible with a quantum cluster algebra structure based on the same quiver as the cluster algebra structure of the Grothendieck ring $K_0(\mathcal{O}_{\mathbb{Z}}^+)$. In order to do that, we exhibit (Proposition 2.4.10) a *compatible pair*. From then, the quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ is defined as the quantum cluster algebra defined from this compatible pair.

We then conjecture (Conjecture 2.5.2) that this quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ contains the quantum Grothendieck ring $K_t(\mathscr{C}_{\mathbb{Z}})$. We propose to demonstrate this conjecture by proving that $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ contains the (q,t)-characters of the fundamental representations $[L(Y_{i,q^r})]_t$, as they generate $K_t(\mathscr{C}_{\mathbb{Z}})$. We state in Conjecture 2.5.7 that these objects can be obtained in $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ as quantum cluster variables, by following the same finite sequences of mutations used in the classical cluster algebra $K_0(\mathcal{O}_{\mathbb{Z}}^+)$ to obtain the $[L(Y_{i,q^r})]$.

Naturally, Conjecture 2.5.7 implies Conjecture 2.5.2. Finally, we prove Conjecture 2.5.7 (and thus Conjecture 2.5.2) in the case where the underlying simple Lie algebra $\mathfrak{g}$ is of type A (Theorem 2.6.1). The proof is based on the thinness property of the fundamental representations in this case. When $\mathfrak{g} = \mathfrak{sl}_2$, some explicit computations are possible. For example, we give a quantum version of the Baxter relation (2.6.2.1), for all $r \in \mathbb{Z}$,

$$[V_{q^{2r-1}}]_t * [L^+_{1,q^{2r}}]_t = t^{-1/2}[\omega_1][L^+_{1,q^{2r-2}}]_t + t^{1/2}[-\omega_1][L^+_{1,q^{2r+2}}]_t.$$

Additionally, we realize a part of the quantum cluster algebra we built as a quotient of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. This is a reminiscence of the result of Qin [Qin16], who constructed $\mathcal{U}_q(\mathfrak{g})$ as a quotient of the Grothendieck ring arising from certain cyclic quiver varieties.

The Chapter is organized as follows. In Section 2.2, we review some results for the category $\mathcal{O}$, its subcategories $\mathcal{O}^\pm$ and $\mathcal{O}^\pm_\mathbb{Z}$ and their Grothendieck rings. In Section 2.3, after recalling the definition of the quantum torus $\mathcal{Y}_t$, we define the quantum torus $\mathcal{T}_t$ in which $K_t(\mathcal{O}^+_\mathbb{Z})$ will be constructed and we prove the inclusion of the quantum tori. In Section 2.4, we prove that we have all the elements to build a quantum cluster algebra and we define the quantum Grothendieck ring $K_t(\mathcal{O}^+_\mathbb{Z})$. In Section 2.5, we state some properties of the quantum Grothendieck ring. We present the two conjectures regarding the inclusion of the quantum Grothendieck rings in Section 2.5.2. Finally, in Section 2.6 we prove these conjectures in type A, and we prove finer properties specific to the case when $\mathfrak{g} = \mathfrak{sl}_2$.

2.2 Category $\mathcal{O}$ of representations of quantum loop algebras

We use the notations from Appendix A. We first recall the definitions of the quantum loop algebra and its Borel subalgebra, before introducing the Hernandez-Jimbo category $\mathcal{O}$ of representations, as well as some known results on the subject. We will sporadically use concepts and notations from the two previous sections.

2.2.1 Quantum loop algebra and Borel subalgebra

Fix a nonzero complex number q, which is not a root of unity, and $h \in \mathbb{C}$ such that $q = e^h$. Then for all $r \in \mathbb{Q}, q^r := e^{rh}$ is well-defined. Since q is not a root of unity, for $r, s \in \mathbb{Q}$, we have $q^r = q^s$ if and only if $r = s$.

We will use the following standard notations.

$$[m]_z = \tfrac{z^m - z^{-m}}{z - z^{-1}}, \quad [m]_z! = \textstyle\prod_{j=1}^m [j]_z, \quad \begin{bmatrix} r \\ s \end{bmatrix}_z = \tfrac{[r]_z!}{[s]_z![r-s]_z!}$$

Definition 2.2.1. One defines the *quantum loop algebra* $\mathcal{U}_q(L\mathfrak{g})$ as the $\mathbb{C}$-algebra generated by $e_i, f_i, k_i^{\pm 1}, 0 \leqslant i \leqslant n$, together with the following relations, for $0 \leqslant i, j \leqslant n$:

$$\begin{gathered} k_i k_j = k_j k_i, \quad k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_0^{a_0} k_1^{a_1} \cdots k_n^{a_n} = 1, \\ [e_i, f_j] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q - q^{-1}}, \\ k_i e_j k_i^{-1} = q^{C_{ij}} e_j, \quad k_i f_j k_i^{-1} = q^{-C_{ij}} e_j, \\ \sum_{r=0}^{1-C_{ij}} (-1)^r e_i^{(1-C_{ij}-r)} e_j e_i^{(r)} = 0, \ (i \neq j), \\ \sum_{r=0}^{1-C_{ij}} (-1)^r f_i^{(1-C_{ij}-r)} f_j f_i^{(r)} = 0, \ (i \neq j), \end{gathered} \tag{2.2.1.1}$$

where $x_i^{(r)} = x_i^r/[r]_q!$, $(x_i = e_i, f_i)$.

Definition 2.2.2. The *Borel algebra* $\mathcal{U}_q(\mathfrak{b})$ is the subalgebra of $\mathcal{U}_q(L\mathfrak{g})$ generated by the $e_i, k_i^{\pm 1}$, for $0 \leqslant i \leqslant n$.

Both the quantum loop algebra and its Borel subalgebra are Hopf algebras.

From now on, except when explicitly stated otherwise, we are going to consider representations of the Borel algebra $\mathcal{U}_q(\mathfrak{b})$. Particularly, we consider the action of the ℓ-Cartan subalgebra $\mathcal{U}_q(\mathfrak{b})^0$, a commutative subalgebra of $\mathcal{U}_q(\mathfrak{b})$ generated by the so-called Drinfeld generators:

$$\mathcal{U}_q(\mathfrak{b})^0 := \left\langle k_i^{\pm 1}, \phi_{i,r}^+ \right\rangle_{i \in I, r > 0}.$$

2.2.2 Highest ℓ-weight modules

Let V be a $\mathcal{U}_q(\mathfrak{b})$-module and $\omega \in P_{\mathbb{Q}}$ a weight. One defines the *weight space* of V of weight ω by

$$V_\omega := \{v \in V \mid k_i v = q^{\omega(\alpha_i^\vee)} v, 1 \leqslant i \leqslant n\}.$$

The vector space V is said to be *Cartan diagonalizable* if $V = \bigoplus_{\omega \in P_{\mathbb{Q}}} V_\omega$.

Definition 2.2.3. A series $\boldsymbol{\Psi} = (\psi_{i,m})_{i \in I, m \geqslant 0}$ of complex numbers, such that $\psi_{i,m} \in q^{\mathbb{Q}}$ for all $i \in I$ is called an *ℓ-weight.* The set of ℓ-weights is denoted by P_ℓ. One identifies the ℓ-weight $\boldsymbol{\Psi}$ to its generating series :

$$\boldsymbol{\Psi} = (\psi_i(z))_{i \in I}, \quad \psi_i(z) = \textstyle\sum_{m \geqslant 0} \psi_{i,m} z^m.$$

Let us define some particular ℓ-weights which are important in our context.

For $\omega \in P_{\mathbb{Q}}$, let $[\omega]$ be defined as

$$([\omega])_i(z) = q^{\omega(\alpha_i^\vee)}, 1 \leqslant i \leqslant n. \tag{2.2.2.1}$$

For $i \in I$ and $a \in \mathbb{C}^\times$, let

- $\boldsymbol{\Psi}_{i,a}$ be defined as

$$\left(\boldsymbol{\Psi}_{i,a}\right)_j(z) = \begin{cases} 1 - az & \text{if } j = i \\ 1 & \text{if } j \neq i \end{cases}. \tag{2.2.2.2}$$

- $Y_{i,a}$ be defined as

$$\left(Y_{i,a}\right)_j(z) = \begin{cases} q\frac{1 - aq^{-1}z}{1 - aqz} & \text{if } j = i \\ 1 & \text{if } j \neq i \end{cases}. \tag{2.2.2.3}$$

The sets $P_{\mathbb{Q}}$ and P_ℓ have group structures (the elements of P_ℓ are invertible formal series) and one has a surjective group morphism $\varpi : P_\ell \to P_{\mathbb{Q}}$ which satisfies $\psi_i(0) = q^{\varpi(\boldsymbol{\Psi})(\alpha_i^\vee)}$, for all $\boldsymbol{\Psi} \in P_\ell$ and all $i \in I$.

Let V be $\mathcal{U}_q(\mathfrak{b})$-module and $\boldsymbol{\Psi} \in P_\ell$ an ℓ-weight. One defines the *ℓ-weight space* of V of ℓ-weight $\boldsymbol{\Psi}$ by

$$V_{\boldsymbol{\Psi}} := \{v \in V \mid \exists p \geqslant 0, \forall i \in I, \forall m \geqslant 0, (\phi_{i,m}^+ - \psi_{i,m})^p v = 0\}.$$

Remark 2.2.4. With the usual convention $\phi_{i,0}^+ = k_i$, one has $V_{\boldsymbol{\Psi}} \subset V_{\varpi(\boldsymbol{\Psi})}$.

Definition 2.2.5. Let V be a $\mathcal{U}_q(\mathfrak{b})$-module. It is said to be *of highest ℓ-weight* $\mathbf{\Psi} \in P_\ell$ if there is $v \in V$ such that $V = \mathcal{U}_q(\mathfrak{b})v$,

$$e_i v = 0, \forall i \in I \quad \text{and} \quad \phi^+_{i,m} v = \psi_{i,m} v, \ \forall i \in I, m \geqslant 0.$$

In this case, the ℓ-weight $\mathbf{\Psi}$ is entirely determined by V, it is called the ℓ-weight of V, and v is a highest ℓ-weight vector of V.

Proposition 2.2.6. *[HJ12] For all $\mathbf{\Psi} \in P_\ell$ there is, up to isomorphism, a unique simple highest ℓ-weight module of ℓ-weight $\mathbf{\Psi}$, denoted by $L(\mathbf{\Psi})$.*

Example 2.2.7. For $\omega \in P_{\mathbb{Q}}$, $L([\omega])$ is a one-dimensional representation of weight ω. We also denote it by $[\omega]$ (tensoring by this representation is equivalent to shifting the weights by ω).

2.2.3 Definition of the category $\mathcal{O}$

As explained in the Introduction, our focus here is a category $\mathcal{O}$ of representations of the Borel algebra, which was first defined in [HJ12], mimicking the usual definition of the BGG category $\mathcal{O}$ for Kac-Moody algebras. Here, we are going to use the definition in [HL16b], which is slightly different.

For all $\lambda \in P_{\mathbb{Q}}$, define $D(\lambda) := \{\omega \in P_{\mathbb{Q}} \mid \omega \leqslant \lambda\}$.

Definition 2.2.8. A $\mathcal{U}_q(\mathfrak{b})$-module V is in the category $\mathcal{O}$ if :

1. V is Cartan diagonalizable,
2. For all $\omega \in P_{\mathbb{Q}}$, one has $\dim(V_\omega) < \infty$,
3. There is a finite number of $\lambda_1, \ldots, \lambda_s \in P_{\mathbb{Q}}$ such that all the weights that appear in V are in the cone $\bigcup_{j=1}^{s} D(\lambda_j)$.

The category $\mathcal{O}$ is a monoidal category.

Example 2.2.9. All finite dimensional $\mathcal{U}_q(\mathfrak{b})$-modules are in the category $\mathcal{O}$.

Let $P_\ell^{\mathfrak{r}}$ be the set of ℓ-weights Ψ such that, for all $i \in I$, $\Psi_i(z)$ is rational. We will use the following result.

Theorem 2.2.10. *[HJ12] Let $\Psi \in P_\ell$. Simple objects in the category $\mathcal{O}$ are highest ℓ-weight modules. The simple module $L(\Psi)$ is in the category $\mathcal{O}$ if and only if $\Psi \in P_\ell^{\mathfrak{r}}$. Moreover, if V is in the category $\mathcal{O}$ and $V_\Psi \neq 0$, then $\Psi \in P_\ell^{\mathfrak{r}}$.*

Example 2.2.11. For all $i \in I$ and $a \in \mathbb{C}^\times$, define the *prefundamental representations* $L^{\pm}_{i,a}$ as

$$L^{\pm}_{i,a} := L(\mathbf{\Psi}^{\pm}_{i,a}), \tag{2.2.3.1}$$

for $\mathbf{\Psi}_{i,a}$ defined in (2.2.2.2). Then from Theorem 2.2.10, the prefundamental representations belong to the category $\mathcal{O}$.

2.2.4 Connection to finite-dimension $\mathcal{U}_q(L\mathfrak{g})$-modules

Throughout this chapter, we will use results already known for finite-dimensional representations of the quantum loop algebra $\mathcal{U}_q(L\mathfrak{g})$ with the purpose of generalizing some of them to the context of the category $\mathcal{O}$ of representations of the Borel subalgebra $\mathcal{U}_q(\mathfrak{b})$. Let us first recognize that this approach is valid.

Let $\mathscr{C}$ be the category of all (type 1) finite-dimensional $\mathcal{U}_q(L\mathfrak{g})$-modules.

Proposition 2.2.12. *[Bow07][CG05, Proposition 2.7] Let V be a simple finite-dimensional $\mathcal{U}_q(L\mathfrak{g})$-module. Then V is simple as a $\mathcal{U}_q(\mathfrak{b})$-module.*

Using this result and the classification of finite-dimensional simple module of quantum loop algebras in [CP95a], one has

Proposition 2.2.13. *For all $i \in I$, let $P_i(z) \in \mathbb{C}[z]$ be a polynomial with constant term 1. Let $\mathbf{\Psi} = (\Psi_i)_{i\in I}$ be the ℓ-weight such that*

$$\Psi_i(z) = q^{\deg(P_i)} \frac{P_i(zq^{-1})}{P_i(zq)}, \quad \forall i \in I. \tag{2.2.4.1}$$

Then $L(\mathbf{\Psi})$ is finite-dimensional.

Moreover the action of $\mathcal{U}_q(\mathfrak{b})$ can be uniquely extended to an action of $\mathcal{U}_q(L\mathfrak{g})$, and any simple object in the category $\mathscr{C}$ is of this form.

Hence, the category $\mathscr{C}$ is a subcategory of the category $\mathcal{O}$ and the inclusion functor preserves simple objects.

Example 2.2.14. For all $i \in I$ and $a \in \mathbb{C}^\times$, consider the simple $\mathcal{U}_q(\mathfrak{b})$-module $L(\mathbf{\Psi})$ of highest ℓ-weight $Y_{i,a}$, as in (2.2.2.3), then by Proposition 2.2.13, $L(Y_{i,a})$ is finite-dimensional. This module is called a *fundamental representation* and will be denoted by

$$V_{i,a} := L(Y_{i,a}). \tag{2.2.4.2}$$

In general, simple modules in $\mathscr{C}$ are indexed by monomials in the variables $(Y_{i,a})_{i\in I, a\in\mathbb{C}^\times}$, called *dominant monomials*. Frenkel-Reshetikhin [FR99] defined a *q-character morphism* χ_q (see Section 2.2.8) on the Grothendieck ring of $\mathscr{C}$. It is an injective ring morphism

$$\chi_q : K_0(\mathscr{C}) \to \hat{\mathcal{Y}} := \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^\times}. \tag{2.2.4.3}$$

Example 2.2.15. In the continuity of Example 2.2.14, for $\mathfrak{g} = \mathfrak{sl}_2$, one has, for all $a \in \mathbb{C}^\times$,

$$\chi_q(L(Y_{1,a})) = Y_{1,a} + Y_{1,aq^2}^{-1}. \tag{2.2.4.4}$$

2.2.5 Categories $\mathcal{O}^\pm$

Let us now recall the definitions of some subcategories of the category $\mathcal{O}$, introduced in [HL16b]. These categories are interesting to study for different reasons, here we are particularly interested in the cluster algebra structure of their Grothendieck rings.

Definition 2.2.16. An ℓ-weight of $P_\ell^{\mathfrak{r}}$ is said to be *positive* (resp. *negative*) if it is a monomial in the following ℓ-weights :

- the $Y_{i,a} = q\mathbf{\Psi}_{i,aq}^{-1}\mathbf{\Psi}_{i,aq^{-1}}$, where $i \in I$ and $a \in \mathbb{C}^\times$,
- the $\mathbf{\Psi}_{i,a}$ (resp. $\mathbf{\Psi}_{i,a}^{-1}$), where $i \in I$ and $a \in \mathbb{C}^\times$,
- the $[\omega]$, where $\omega \in P_{\mathbb{Q}}$.

Definition 2.2.17. The category $\mathcal{O}^+$ (resp. $\mathcal{O}^-$) is the category of representations in $\mathcal{O}$ whose simple constituents have a positive (resp. negative) highest ℓ-weight.

The category $\mathcal{O}^+$ (resp. $\mathcal{O}^-$) contains the category of finite-dimensional representations, as well as the positive (resp. negative) prefundamental representations $L_{i,a}^+$ (resp. $L_{i,a}^-$), defined in (2.2.11), for all $i \in I, a \in \mathbb{C}^\times$.

Theorem 2.2.18. *[HL16b] The categories $\mathcal{O}^+$ and $\mathcal{O}^-$ are monoidal categories.*

2.2.6 The category $\mathcal{O}^+_{\mathbb{Z}}$

Recall the infinite quiver Γ from Section A.4 of the Appendix and its set of vertices $\hat{I}$.

In [HL10], Hernandez and Leclerc defined a subcategory $\mathscr{C}_{\mathbb{Z}}$ of the category $\mathscr{C}$. $\mathscr{C}_{\mathbb{Z}}$ is the full subcategory whose objects satisfy: for all composition factor $L(\mathbf{\Psi})$, for all $i \in I$, the roots of the polynomials P_i, as in Proposition 2.2.13 are of the form q^{r+1}, such that $(i, r) \in \hat{I}$.

This subcategory is interesting to study because each simple object in $\mathscr{C}$ can be written as a tensor product of simple objects which are essentially in $\mathscr{C}_{\mathbb{Z}}$ (see [HL10, Section 3.7]). Thus, the study of simple modules in $\mathscr{C}$ is equivalent to the study of simple modules in $\mathscr{C}_{\mathbb{Z}}$.

Consider the same type of restriction on the category $\mathcal{O}$.

Definition 2.2.19. Let $\mathcal{O}_{\mathbb{Z}}$ be the subcategory of representations of $\mathcal{O}$ whose simple components have a highest ℓ-weight $\mathbf{\Psi}$ such that the roots and poles of $\Psi_i(z)$ are of the form q^r, such that $(i, r) \in \hat{I}$.

We also define $\mathcal{O}^{\pm}_{\mathbb{Z}}$ as the subcategory of $\mathcal{O}^{\pm}$ whose simple components have a highest ℓ-weight $\mathbf{\Psi}$ such that the roots and poles of $\Psi_i(z)$ are of the form q^r, such that $(i, r) \in \hat{I}$.

2.2.7 The Grothendieck ring $K_0(\mathcal{O})$

Hernandez and Leclerc showed that the Grothendieck rings of the categories $\mathcal{O}^{\pm}_{\mathbb{Z}}$ have some interesting cluster algebra structures.

First of all, define $\mathcal{E}$ as the additive group of maps $c : P_{\mathbb{Q}} \to \mathbb{Z}$ whose support is contained in a finite union of sets of the form $D(\mu)$. For any $\omega \in P_{\mathbb{Q}}$, define $[\omega] \in \mathcal{E}$ as the δ-function at ω (this is compatible with the notation in Example 2.2.7). The elements of $\mathcal{E}$ can be written as formal sums

$$c = \sum_{\omega \in \mathrm{supp}(c)} c(\omega)[\omega]. \tag{2.2.7.1}$$

$\mathcal{E}$ can be endowed with a ring structure, where the product is defined by

$$[\omega] \cdot [\omega'] = [\omega + \omega'], \quad \forall \omega, \omega' \in P_{\mathbb{Q}}.$$

If $(c_k)_{k \in \mathbb{N}}$ is a countable family of elements of $\mathcal{E}$ such that for any $\omega \in P_{\mathbb{Q}}$, $c_k(\omega) = 0$ except for finitely many $k \in \mathbb{N}$, then $\sum_{k \in \mathbb{N}} c_k$ is a well-defined map from $P_{\mathbb{Q}}$ to $\mathbb{Z}$. In that case, we say that $\sum_{k \in \mathbb{N}} c_k$ is a *countable sum of elements* in $\mathcal{E}$.

The Grothendieck ring of the category $\mathcal{O}$ can be viewed as a ring extension of $\mathcal{E}$. Similarly to the case of representations of a simple Lie algebra (see [Kac90], Section 9.6), the multiplicity of an irreducible representation in a given representation of the category $\mathcal{O}$ is well-defined. Thus, the Grothendieck ring of the category $\mathcal{O}$ is formed of formal sums

$$\sum_{\mathbf{\Psi} \in P^{\mathfrak{r}}_{\ell}} \lambda_{\mathbf{\Psi}} [L(\mathbf{\Psi})], \tag{2.2.7.2}$$

such that the $\lambda_{\mathbf{\Psi}} \in \mathbb{Z}$ satisfy:

$$\sum_{\mathbf{\Psi} \in P^{\mathfrak{r}}_{\ell}, \omega \in P_{\mathbb{Q}}} |\lambda_{\mathbf{\Psi}}| \dim(L(\mathbf{\Psi})_{\omega})[\omega] \in \mathcal{E}.$$

In this context, $\mathcal{E}$ is identified with the Grothendieck ring of the category of representations of $\mathcal{O}$ with constant ℓ-weight.

A notion of countable sum of elements in $K_0(\mathcal{O})$ is defined exactly as for $\mathcal{E}$.

Now consider the cluster algebra $\mathcal{A}(\Gamma)$ defined by the infinite quiver Γ of Section A.4 of the Appendix, with infinite set of coordinates denoted by

$$\boldsymbol{z} = \left\{ z_{i,r} \mid (i,r) \in \hat{I} \right\}. \tag{2.2.7.3}$$

By the *Laurent Phenomenon* (see B.4.2 in the Appendix), the cluster algebra $\mathcal{A}(\Gamma)$ is contained in the ring $\mathbb{Z}[z_{i,r}^{\pm 1}]_{(i,r)\in\hat{I}}$. Define a $\mathcal{E}$-algebra homomorphism $\chi : \mathbb{Z}[z_{i,r}^{\pm 1}] \otimes_{\mathbb{Z}} \mathcal{E} \to \mathcal{E}$ by

$$\chi(z_{i,r}^{\pm 1}) = \left[\left(\frac{\mp r}{2} \right) \omega_i \right], \quad ((i,r) \in \hat{I}). \tag{2.2.7.4}$$

The map χ is defined on $\mathcal{A}(\Gamma) \otimes_{\mathbb{Z}} \mathcal{E}$, and for each $A \in \mathcal{A}(\Gamma) \otimes_{\mathbb{Z}} \mathcal{E}$, one can write $\chi(A) = \sum_{\omega \in P_{\mathbb{Q}}} A_\omega \otimes [\omega]$, and $|\chi|(A) = \sum_{\omega \in P_{\mathbb{Q}}} |A_\omega| \otimes [\omega]$.

Consider the completed tensor product

$$\mathcal{A}(\Gamma) \hat{\otimes}_{\mathbb{Z}} \mathcal{E}, \tag{2.2.7.5}$$

of countable sums $\sum_{k\in\mathbb{N}} A_k$ of elements $A_k \in \mathcal{A}(\Gamma) \otimes_{\mathbb{Z}} \mathcal{E}$, such that $\sum_{k\in\mathbb{N}} |\chi|(A_k)$ is a countable sum of elements of $\mathcal{E}$, as defined above.

Theorem 2.2.20. *[HL16b, Theorem 4.2] The category $\mathcal{O}^+_{\mathbb{Z}}$ is monoidal, and the identification*

$$z_{i,r} \otimes \left[\frac{r}{2} \omega_i \right] \equiv [L^+_{i,q^r}], \quad \left((i,r) \in \hat{I} \right), \tag{2.2.7.6}$$

defines an isomorphism of $\mathcal{E}$-algebras

$$\mathcal{A}(\Gamma) \hat{\otimes}_{\mathbb{Z}} \mathcal{E} \simeq K_0(\mathcal{O}^+_{\mathbb{Z}}). \tag{2.2.7.7}$$

Example 2.2.21. We mentioned in the introduction that the Baxter relation (2.1.0.2) was an exchange relation for this cluster algebra structure. Let us detail this. For $\mathfrak{g} = \mathfrak{sl}_2$, the quiver Γ is:

$$\begin{array}{c} \vdots \\ \uparrow \\ (1,2) \\ \uparrow \\ (1,0) \\ \uparrow \\ (1,\text{-}2) \\ \uparrow \\ \vdots \end{array}$$

If we mutate at the node $(1,0)$, the new cluster variable obtained is

$$z'_{1,0} = \frac{1}{z_{1,0}} \left(z_{1,2} + z_{1-2} \right).$$

Thus, via the identification (2.2.7.6),

$$z'_{1,0} = \frac{1}{[L^+_{1,1}]} \left([-\omega_1][L^+_{1,q^2}] + [\omega_1][L^+_{1,q^{-2}}] \right).$$

We indeed recognize the Baxter relation.

Moreover, the new cluster variable $z'_{1,0}$ identifies to a fundamental representation:

$$z'_{1,0} = [L(Y_{1,q^{-1}})]. \tag{2.2.7.8}$$

Remark 2.2.22. An analog theorem could be written for $K_0(\mathcal{O}^-_{\mathbb{Z}})$, as these are isomorphic as $\mathcal{E}$-algebras ([HL16b, Theorem 5.17]).

2.2.8 The q-character morphism

Here we detail the notion of *q-character* on the category $\mathcal{O}$. This notion extends the q-character morphism on the category of finite-dimensional $\mathcal{U}_q(L\mathfrak{g})$-modules mentioned in Section 2.2.4.

Similarly to Section 2.2.7, consider $\mathcal{E}_\ell$, the additive group of maps $c : P_\ell^{\mathfrak{r}} \to \mathbb{Z}$ such that the image by ϖ of its support is contained in a finite union of sets of the form $D(\mu)$, and for any $\omega \in P_{\mathbb{Q}}$, the set $\mathrm{supp}(c) \cap \varpi^{-1}(\{\omega\})$ is finite. The map ϖ extends naturally to a surjective morphism $\varpi : \mathcal{E}_\ell \to \mathcal{E}$. For $\mathbf{\Psi} \in P_\ell^{\mathfrak{r}}$, define the delta function $[\mathbf{\Psi}] = \delta_{\mathbf{\Psi}} \in \mathcal{E}_\ell$.

The elements of $\mathcal{E}_\ell$ can be written as formal sums

$$c = \sum_{\mathbf{\Psi} \in P_\ell^{\mathfrak{r}}} c(\mathbf{\Psi})[\mathbf{\Psi}]. \tag{2.2.8.1}$$

Endow $\mathcal{E}_\ell$ with a ring structure given by

$$(c \cdot d)(\mathbf{\Psi}) = \sum_{\mathbf{\Psi}'\mathbf{\Psi}'' = \mathbf{\Psi}} c(\mathbf{\Psi}')d(\mathbf{\Psi}''), \quad (c, d, \in \mathcal{E}_\ell, \mathbf{\Psi} \in P_\ell^{\mathfrak{r}}). \tag{2.2.8.2}$$

In particular, for $\mathbf{\Psi}, \mathbf{\Psi}' \in P_\ell^{\mathfrak{r}}$,

$$[\mathbf{\Psi}] \cdot [\mathbf{\Psi}'] = [\mathbf{\Psi}\mathbf{\Psi}']. \tag{2.2.8.3}$$

For V a module in the category $\mathcal{O}$, define the q-character of V as in [FR99], [HJ12]:

$$\chi_q(V) := \sum_{\mathbf{\Psi} \in P_\ell^{\mathfrak{r}}} \dim(V_{\mathbf{\Psi}})[\mathbf{\Psi}]. \tag{2.2.8.4}$$

By definition of the category $\mathcal{O}$, $\chi_q(V)$ is an object of the ring $\mathcal{E}_\ell$.

The following result extends the one from [FR99] to the context of the category $\mathcal{O}$.

Proposition 2.2.23 ([HJ12])**.** *The q-character map*

$$\begin{array}{rccl} \chi_q : & K_0(\mathcal{O}) & \to & \mathcal{E}_\ell \\ & [V] & \mapsto & \chi_q(V), \end{array} \tag{2.2.8.5}$$

is an injective ring morphism.

Example 2.2.24. For any $a \in \mathbb{C}^\times$, $i \in I$, one has [HJ12, FH15],

$$\chi_q(L_{i,a}^+) = [\mathbf{\Psi}_{i,a}]\, \chi_i, \tag{2.2.8.6}$$

where $\chi_i = \chi(L_{i,a}^+) \in \mathcal{E}$ does not depend on a.

For example, if $\mathfrak{g} = \mathfrak{sl}_2$,

$$\chi_1 = \chi = \sum_{r \geqslant 0} [-2r\omega_1]. \tag{2.2.8.7}$$

2.3 Quantum tori

Let t be an indeterminate. The aim of this section is to build a non-commutative quantum torus $\mathcal{T}_t$ which will contain the quantum Grothendieck ring for the category $\mathcal{O}$. For the category $\mathscr{C}$ of finite-dimensional $\mathcal{U}_q(L\mathfrak{g})$-modules, such a quantum torus already exists, denoted by $\mathcal{Y}_t$ here. Thus one natural condition on $\mathcal{T}_t$ is for it to contain $\mathcal{Y}_t$. We show it is the case in Proposition 2.3.3.

We start this section by recalling the definition and some properties of $\mathcal{Y}_t$. Here we use the same quantum torus as in [Her04], which is slightly different from the one used in [Nak04] and [VV03].

2.3.1 The torus $\mathcal{Y}_t$

In this section, we consider $\mathcal{U}_q(L\mathfrak{g})$-modules and no longer $\mathcal{U}_q(\mathfrak{b})$-modules. We have seen in Section 2.2.4 that for finite-dimension representations, these settings were not too different.

As seen in (2.2.4.3), the Grothendieck ring of $\mathscr{C}$ can be seen as a subring of a ring of Laurent polynomials

$$K_0(\mathscr{C}) \subseteq \hat{\mathcal{Y}} = \mathbb{Z}[Y_{i,a}]_{i\in I, a\in\mathbb{C}^\times}$$

In order to define a t-deformed non-commutative version of this Grothendieck ring, one first needs a non-commutative, t-deformed version of $\hat{\mathcal{Y}}$, denoted by $\mathcal{Y}_t$.

Following [Her04], we define

$$\mathcal{Y} := \mathbb{Z}[Y_{i,q^r}^{\pm} \mid (i,r) \in \hat{I}], \tag{2.3.1.1}$$

the Laurent polynomial ring generated by the commuting variables Y_{i,q^r}.

Let $(\mathcal{Y}_t, *)$ be the $\mathbb{Z}(t)$-algebra generated by the $(Y_{i,q^r}^{\pm})_{(i,r)\in\hat{I}}$, with the t-commutations relations:

$$Y_{i,q^r} * Y_{j,q^s} = t^{\mathcal{N}_{i,j}(r-s)} Y_{j,q^s} * Y_{i,q^r}, \tag{2.3.1.2}$$

where $\mathcal{N}_{i,j} : \mathbb{Z} \to \mathbb{Z}$ is the antisymmetrical map, defined by

$$\mathcal{N}_{i,j}(m) = \tilde{C}_{i,j}(m+1) - \tilde{C}_{i,j}(m-1), \quad \forall m \geqslant 0, \tag{2.3.1.3}$$

using the notations from Section A.2 of the Appendix.

Example 2.3.1. If we continue Example A.2.3 from the Appendix, for $\mathfrak{g} = \mathfrak{sl}_2$, in this case, $\hat{I} = (1, 2\mathbb{Z})$, for $r \in \mathbb{Z}$, one has

$$Y_{1,2r} * Y_{1,2s} = t^{2(-1)^{s-r}} Y_{1,2s} * Y_{1,2r}, \quad \forall s > r > 0. \tag{2.3.1.4}$$

The $\mathbb{Z}(t)$-algebra $\mathcal{Y}_t$ is viewed as a quantum torus of infinite rank.

Let us extend this quantum torus $\mathcal{Y}_t$ by adjoining a fixed square root $t^{1/2}$ of t:

$$\mathbb{Z}(t^{1/2}) \otimes_{\mathbb{Z}(t)} \mathcal{Y}_t. \tag{2.3.1.5}$$

By abuse of notation, the resulting algebra will still be denoted $\mathcal{Y}_t$.

For a family of integers with finitely many non-zero components $(u_{i,r})_{(i,r)\in\hat{I}}$, define the *commutative monomial* $\prod_{(i,r)\in\hat{I}} Y_{i,q^r}^{u_{i,r}}$ as

$$\prod_{(i,r)\in\hat{I}} Y_{i,q^r}^{u_{i,r}} := t^{\frac{1}{2}\sum_{(i,r)<(j,s)} u_{i,r}u_{j,s}\mathcal{N}_{i,j}(r,s)} \overrightarrow{\ast}_{(i,r)\in\hat{I}} Y_{i,q^r}^{u_{i,r}}, \tag{2.3.1.6}$$

where on the right-hand side an order on $\hat{I}$ is chosen so as to give meaning to the sum, and the product $*$ is ordered by it (notice that the result does not depend on the order chosen).

The commutative monomials form a basis of the $\mathbb{Z}(t^{1/2})$-vector space $\mathcal{Y}_t$.

2.3.2 The torus $\mathscr{T}_t$

We now want to extend the quantum torus $\mathcal{Y}_t$ to a larger non-commutative algebra $\mathscr{T}_t$ which would contain at least all the ℓ-weights, and possibly all the candidates for the (q,t)-characters of the modules in the category $\mathcal{O}_{\mathbb{Z}}^+$.

In particular, $\mathscr{T}_t$ contains the $\mathbf{\Psi}_{i,q^r}$, for $(i,r) \in \hat{I}$, and these t-commutes with a relation compatible with the t-commutation relation between the $Y_{i,q^{r+1}}$ (2.3.1.2).

Remark 2.3.2. One notices that there is a shift of parity between the powers of q in the Y's and the $\mathbf{\Psi}$'s. From now on, we will consider the $\mathbf{\Psi}_{i,q^r}$ and the $Y_{i,q^{r+1}}$, for $(i,r) \in \hat{I}$.

We start as in Section 2.3.1. First of all, define

$$\mathcal{T} := \mathbb{Z}\left[z_{i,r}^{\pm} \mid (i,r) \in \hat{I}\right], \tag{2.3.2.1}$$

the Laurent polynomial ring generated by the commuting variables $z_{i,r}$.

Then, build a t-deformation T_t of $\mathcal{T}$, as the $\mathbb{Z}[t^{\pm 1}]$-algebra generated by the $z_{i,r}^{\pm}$, for $(i,r) \in \hat{I}$, with a non-commutative product $*$, and the t-commutations relations

$$z_{i,r} * z_{j,s} = t^{\mathcal{F}_{ij}(s-r)} z_{j,s} * z_{i,r}, \quad \left((i,r),(j,s) \in \hat{I}\right), \tag{2.3.2.2}$$

where, for all $i,j \in I$, $\mathcal{F}_{ij} : \mathbb{Z} \to \mathbb{Z}$ is a anti-symmetrical map such that, for all $m \geqslant 0$,

$$\mathcal{F}_{ij}(m) = - \sum_{\substack{k \geqslant 1 \\ m \geqslant 2k-1}} \tilde{C}_{ij}(m-2k+1). \tag{2.3.2.3}$$

Now, let

$$\mathcal{T}_t := \mathbb{Z}[t^{\pm 1/2}] \otimes_{\mathbb{Z}[t^{\pm 1}]} T_t. \tag{2.3.2.4}$$

Similarly, we define the commutative monomials in $\mathscr{T}_t$ as,

$$\prod_{(i,r) \in \hat{I}} z_{i,q^r}^{v_{i,r}} := t^{\frac{1}{2}\sum_{(i,r)<(j,s)} v_{i,r} v_{j,s} \mathcal{F}_{i,j}(r,s)} \overrightarrow{\ast}_{(i,r)\in\hat{I}} z_{i,q^r}^{v_{i,r}}. \tag{2.3.2.5}$$

This based quantum torus will be enough to define a structure of quantum cluster algebra, but for it to contain the quantum Grothendieck ring of the category $\mathcal{O}_{\mathbb{Z}}^+$, one needs to extend it. In order to do that, we draw inspiration from Section 2.2.7. Recall the definition of χ from (2.2.7.4). We extend it to the $\mathcal{E}$-algebra morphism $\chi : \mathcal{T}_t \otimes_{\mathbb{Z}} \mathcal{E} \to \mathcal{E}$ defined by imposing $\chi(t^{\pm 1/2}) = 1$, as well as

$$\chi(z_{i,r}^{\pm 1}) = \left[\left(\frac{\mp r}{2}\right)\omega_i\right], \quad ((i,r) \in \hat{I}).$$

As before, for $z \in \mathcal{T}_t \otimes_{\mathbb{Z}} \mathcal{E}$, one writes $\chi(z) = \sum_{\omega \in P_{\mathbb{Q}}} z_\omega[\omega]$ and $|\chi|(z) = \sum_{\omega \in P_{\mathbb{Q}}} |z_\omega|[\omega]$.

Define the completed tensor product

$$\mathscr{T}_t := \mathcal{T}_t \hat{\otimes}_{\mathbb{Z}[t^{\pm 1/2}]} \mathcal{E}, \tag{2.3.2.6}$$

of countable sums $\sum_{k \in \mathbb{N}} z_k$ of elements $z_k \in \mathcal{T}_t \otimes_{\mathbb{Z}} \mathcal{E}$, such that $\sum_{k \in \mathbb{N}} |\chi|(z_k)$ is a countable sum of $\mathcal{E}$, as in Section 2.2.7.

Consistently with the identification (2.2.7.6), and the character of the $z_{i,r}^{\pm 1}$, we use the following notation, for $(i,r) \in \hat{I}$,

$$[\mathbf{\Psi}_{i,q^r}^{\pm 1}] := z_{i,r}^{\pm 1}\left[\frac{\pm r}{2}\omega_i\right] \quad \in \mathscr{T}_t. \tag{2.3.2.7}$$

Proposition 2.3.3. *The identification*

$$\mathcal{J} : Y_{i,q^{r+1}} \mapsto z_{i,r} z_{i,r+2}^{-1} = [\omega_i][\boldsymbol{\Psi}_{i,q^r}][\boldsymbol{\Psi}_{i,q^{r+2}}^{-1}], \tag{2.3.2.8}$$

where the products on the right hand side are commutative, extends to a well-defined injective $\mathbb{Z}(t)$-algebra morphism $\mathcal{J} : \mathcal{Y}_t \to \mathscr{T}_t$.

Proof. One needs to check that the images of the $Y_{i,q^{r+1}}$ satisfy (2.3.1.2). Thus, we need to show that, for all $(i,r),(j,s) \in \hat{I}$,

$$\left(z_{i,r} z_{i,r+2}^{-1}\right) * \left(z_{j,s} z_{j,s+2}^{-1}\right) = t^{\mathcal{N}_{i,j}(s-r)} \left(z_{j,s} z_{j,s+2}^{-1}\right) * \left(z_{i,r} z_{i,r+2}^{-1}\right),$$

which is equivalent to checking that:

$$2\mathcal{F}_{i,j}(s-r) - \mathcal{F}_{i,j}(s-r+2) - \mathcal{F}_{i,j}(s-r-2) = \mathcal{N}_{i,j}(s-r). \tag{2.3.2.9}$$

Suppose $s \geqslant r+2$, let $m = s-r$.

$$\begin{aligned} 2\mathcal{F}_{i,j}(m) - \mathcal{F}_{i,j}(m+2) - \mathcal{F}_{i,j}(m-2) = &- \sum_{\substack{k\geqslant 1\\ m\geqslant 2k-1}} \tilde{C}_{ij}(m-2k+1) \\ &+ \sum_{\substack{k\geqslant 0\\ m\geqslant 2k-1}} \tilde{C}_{ij}(m-2k+1) + \sum_{\substack{k\geqslant 2\\ m\geqslant 2k-1}} \tilde{C}_{ij}(m-2k+1) \\ &= -\tilde{C}_{ij}(m-1) + \tilde{C}_{ij}(m+1). \end{aligned}$$

Thus $2\mathcal{F}_{i,j}(m) - \mathcal{F}_{i,j}(m+2) - \mathcal{F}_{i,j}(m-2) = \mathcal{N}_{i,j}(m)$, using (2.3.1.3).

If $s = r+1$, the left-hand side of (2.3.2.9) is equal to

$$3\mathcal{F}_{i,j}(1) - \mathcal{F}_{i,j}(3) = \tilde{C}_{ij}(2) = \mathcal{N}_{i,j}(1).$$

□

Example 2.3.4. Let us continue Examples A.2.3 and 2.3.1. For all $r \in \mathbb{Z}$. One has

$$z_{1,2r} * z_{1,2s} = t^{f(s-r)} z_{1,2s} * z_{1,2r} \ , \forall r,s \in \mathbb{Z}, \tag{2.3.2.10}$$

where $f : \mathbb{Z} \to \mathbb{Z}$ is antisymmetric and defined by

$$f_{|\mathbb{N}} : m \mapsto \frac{(-1)^m - 1}{2}. \tag{2.3.2.11}$$

And this is compatible with the relations (2.3.1.4).

Definition 2.3.5. Define the *evaluation at $t = 1$* as the $\mathcal{E}$-morphism

$$\pi : \mathscr{T}_t \to \mathcal{E}_\ell, \tag{2.3.2.12}$$

such that

$$\begin{aligned} \pi(z_{i,r}) &= \left[\frac{-r}{2}\omega_i\right][\boldsymbol{\Psi}_{i,q^r}], \\ \pi(t^{\pm 1/2}) &= 1. \end{aligned}$$

Remark 2.3.6. The identification (2.2.7.6) is between the element $z_{i,r}\,[r\omega_i/2]$ and the class of the prefundamental representation $[L_{i,q^r}^+]$. But this identification is not compatible with the character χ defined in (2.2.7.4), as the character of L_{i,q^r}^+ is χ_i, as in (2.2.8.6). Here, we choose to identify the variables $z_{i,r}$ with the highest ℓ-weights of the prefundamental representations (up to a shift of weight), in particular, this identification is compatible with the character morphism χ.

2.4 Quantum Grothendieck rings

The aim of this section is to build $K_t(\mathcal{O}^+_\mathbb{Z})$, a t-deformed version of the Grothendieck ring of the category $\mathcal{O}^+_\mathbb{Z}$. This ring will be built inside the quantum torus $\mathscr{T}_t$, as a quantum cluster algebra.

Let us summarize the existing objects in this context in a diagram:

$$\begin{tikzcd}
\mathscr{C}_\mathbb{Z} \arrow[r, phantom, "\subset"] \arrow[d, dashed] & \mathcal{O}^\pm_\mathbb{Z} \arrow[d, dashed] & \\
K_0(\mathscr{C}_\mathbb{Z}) \arrow[r, phantom, "\subset"] \arrow[d, squiggly] & K_0(\mathcal{O}^\pm_\mathbb{Z}) \arrow[r, phantom, "\simeq"] \arrow[d, squiggly] & \mathcal{A}(\Gamma)\hat{\otimes}_\mathbb{Z}\mathcal{E} \arrow[dl, red] \\
K_t(\mathscr{C}_\mathbb{Z}) \arrow[r, phantom, "\subset"] & \textbf{?} &
\end{tikzcd}$$

A natural idea to build a t-deformation of the Grothendieck ring $K_0(\mathcal{O}^+_\mathbb{Z})$ is to use its cluster algebra structure and define a t-deformed quantum cluster algebra, as in Section B.3 of the Appendix, with the same basis quiver. One has to make sure that the resulting object is indeed a subalgebra of the quantum torus $\mathscr{T}_t$.

2.4.1 The finite-dimensional case

We start this section with some reminders regarding the quantum Grothendieck ring of the category of finite-dimensional $\mathcal{U}_q(L\mathfrak{g})$-modules.

This object was first discussed by Nakajima [Nak04] and Varagnolo-Vasserot [VV03] in the study of perverse sheaves. Then Hernandez gave a more algebraic definition, using t-analogs of screening operators [Her03],[Her04]. This is the version we consider here, with the restriction to some specific tensor subcategory $\mathscr{C}_\mathbb{Z}$, as in [HL15].

Definition of the Quantum Grothendieck ring

As in Section 2.2.6, consider $\mathscr{C}_\mathbb{Z}$ the full subcategory of $\mathscr{C}$ whose simple components have highest ℓ-weights which are monomials in the Y_{i,q^r}, with $(i,r) \in \hat{I}$.

For $(i, r-1) \in \hat{I}$, define the commutative monomials

$$A_{i,r} := Y_{i,q^{r+1}} Y_{i,q^{r-1}} \prod_{j\sim i} Y^{-1}_{j,q^r} \quad \in \mathcal{Y}_t. \tag{2.4.1.1}$$

For all $i \in I$, let $K_{i,t}(\mathscr{C}_\mathbb{Z})$ be the $\mathbb{Z}(t^{1/2})$-subalgebra of $\mathcal{Y}_t$ generated by the

$$Y_{i,q^r}(1 + A^{-1}_{i,r+1}), \quad Y_{j,q^s} \quad \left((i,r),(j,s) \in \hat{I}, j \neq i\right). \tag{2.4.1.2}$$

Finally, as in [Her04], define

$$K_t(\mathscr{C}_\mathbb{Z}) := \bigcap_{i\in I} K_{i,t}(\mathscr{C}_\mathbb{Z}). \tag{2.4.1.3}$$

Remark 2.4.1. Frenkel-Mukhin's algorithm [FM01] allows for the computation of certain q-characters, in particular those of the fundamental representations. In [Her04], Hernandez introduced a t-deformed version of this algorithm to compute the (q,t)-characters of the fundamental representations, and thus to characterized the quantum Grothendieck ring as the subring of $\mathcal{Y}_t$ generated for those (q,t)-characters:

$$K_t(\mathscr{C}_{\mathbb{Z}}) = \left\langle [L(Y_{i,q^r})]_t \mid (i,r) \in \hat{I} \right\rangle. \tag{2.4.1.4}$$

(q,t)-characters in $K_t(\mathscr{C}_{\mathbb{Z}})$

Let us recall some more detailed results about the theory of (q,t)-characters for the modules in the category $\mathscr{C}_{\mathbb{Z}}$.

Let $\mathcal{M}$ be the set of monomials in the variables $(Y_{i,q^{r+1}})_{(i,r)\in\hat{I}}$, also called *dominant monomials*. From [Her04] we know that for all dominant monomial m, there is a unique element $F_t(m)$ in $K_t(\mathscr{C}_{\mathbb{Z}})$ such that m occurs in $F_t(m)$ with multiplicity 1, and no other dominant monomial occurs in $F_t(m)$. These $F_t(m)$ form a $\mathbb{C}(t^{1/2})$-basis of $K_t(\mathscr{C}_{\mathbb{Z}})$.

For all dominant monomial $m = \prod_{(i,r)\in\hat{I}} Y_{i,q^{r+1}}^{u_{i,r}(m)} \in \mathcal{M}$, define

$$[M(m)]_t := t^{\alpha(m)} \overleftarrow{\mathop{\ast}_{r\in\mathbb{Z}}} F_t\left(\prod_{i\in I} Y_{i,q^{r+1}}^{u_{i,r}(m)}\right) \quad \in K_t(\mathscr{C}_{\mathbb{Z}}), \tag{2.4.1.5}$$

where $\alpha(m) \in \frac{1}{2}\mathbb{Z}$ is fixed such that m appears with coefficient 1 in the expansion of $[M(m)]_t$ on the basis of the commutative monomials. The specialization at $t = 1$ of $[M(m)]_t$ recovers the q-character $\chi_q(M(m))$ of the standard module $M(m)$.

Consider the bar-involution $\overline{}$, the anti-automorphism of $\mathcal{Y}_t$ defined by:

$$\overline{t^{1/2}} = t^{-1/2}, \quad \overline{Y_{i,q^{r+1}}} = Y_{i,q^{r+1}}, \quad \left((i,r)\in\hat{I}\right). \tag{2.4.1.6}$$

Remark 2.4.2. The commutative monomials, defined in Section 2.3.1, are clearly bar invariant, as well as the subring $K_t(\mathscr{C}_{\mathbb{Z}})$.

There is a unique family $\{[L(m)]_t \in K_t(\mathscr{C}_{\mathbb{Z}}) \mid m \in \mathcal{M}\}$ such that

(i)
$$\overline{[L(m)]_t} = [L(m)]_t, \tag{2.4.1.7}$$

(ii)
$$[L(m)]_t \in [M(m)]_t + \sum_{m'<m} t^{-1}\mathbb{Z}[t^{-1}][M(m')]_t, \tag{2.4.1.8}$$

where $m' \leqslant m$ means that $m(m')^{-1}$ is a product of $A_{i,r}$.

Lastly, we recall this result from Nakajima, proven using the geometry of quiver varieties.

Theorem 2.4.3. *[Nak04] For all dominant monomial $m \in \mathcal{M}$, the specialization at $t = 1$ of $[L(m)]_t$ is equal to $\chi_q(L(m))$.*

Moreover, the coefficients of the expansion of $[L(m)]_t$ as a linear combination of products of $Y_{i,r}^{\pm1}$ belong to $\mathbb{N}[t^{\pm1}]$.

Thus to all simple modules $L(\boldsymbol{\Psi})$ in $\mathscr{C}_{\mathbb{Z}}$ is associated an object $[L(m)]_t \in K_t(\mathscr{C}_{\mathbb{Z}})$, called the (q,t)-*character*. It is compatible with the q-character of the representation.

Remark 2.4.4. With the cluster algebra approach, we shed a new light on this last positivity result. We interpret the (q,t)-characters of the fundamental modules (and actually all simple modules which are realized as cluster variables in $K_0(\mathcal{O}_{\mathbb{Z}}^+)$) as quantum cluster variables (Conjecture 2.5.7). Thus using Theorem B.6.1 of the Appendix, we recover the fact that the coefficients of their expansion on the commutative monomials in the $(Y_{i,r}^{\pm 1})$ belong to $\mathbb{N}[t^{\pm 1}]$.

Remark 2.4.5. In order to fully extended this picture to the context of the category $\mathcal{O}$, and implement a Kazhdan-Lusztig type algorithm to compute the (q,t)-characters of all simple modules, one would need an equivalent of the standard modules in this category. These do not exist in general. This question was tackled in Chapter 1, in which equivalents of standard modules were defined when $\mathfrak{g} = \mathfrak{sl}_2$.

2.4.2 Compatible pairs

We now begin the construction of $K_t(\mathcal{O}_{\mathbb{Z}}^+)$.

First of all, to define a quantum cluster algebra, one needs a *compatible pair*, as in Section B.2 in the Appendix. The basis quiver we consider here is the same quiver Γ as Section A.4 in the Appendix.

Explicitly, the corresponding exchange matrix is the $\hat{I} \times \hat{I}$ skew-symmetric matrix $\tilde{B}$ such that, for all $((i,r),(j,s)) \in \hat{I}^2$,

$$\tilde{B}_{((i,r),(j,s))} = \begin{cases} 1 & \text{if} \quad i = j \text{ and } s = r+2 \\ & \quad\ \text{or } i \sim j \text{ and } s = r-1, \\ -1 & \text{if} \quad i = j \text{ and } s = r-2 \\ & \quad\ \text{or } i \sim j \text{ and } s = r+1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4.2.1}$$

Let Λ be the $\hat{I} \times \hat{I}$ skew-symmetric infinite matrix encoding the t-commutation relations (2.3.2.2). Precisely, for $((i,r),(j,s)) \in \hat{I}^2$ such that $s > r$,

$$\Lambda_{(i,r),(j,s)} = \mathcal{F}_{i,j}(s-r) = -\sum_{\substack{k \geqslant 1 \\ m \geqslant 2k-1}} \tilde{C}_{ij}(m-2k+1). \tag{2.4.2.2}$$

Remark 2.4.6. In [HL16b], it is noted that one can use sufficiently large finite subseed of Γ instead of an infinite rank cluster algebra. For our purpose, the same statement stays true, but one has to check that the subquiver still forms a compatible pair with the torus structure. Hence, we have to give a more precise framework for the restriction to finite subseeds.

For all $N \in \mathbb{N}^*$, define Γ_N, which is a finite slice of Γ of length $2N+1$, containing an upper and lower row of frozen vertices. More precisely, define $\hat{I}_N$ and $\tilde{I}_N$ as

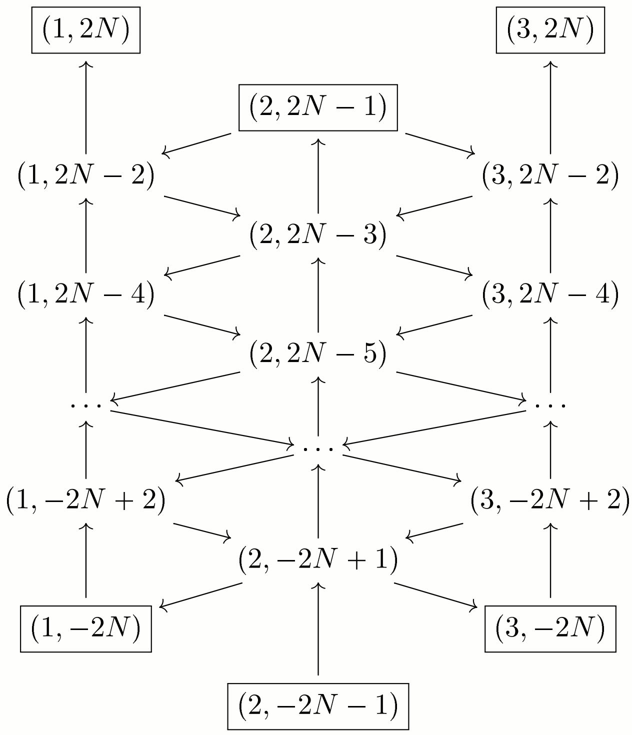
$$\hat{I}_N := \left\{ (i,r) \in \hat{I} \mid -2N+1 \leqslant r < 2N-1 \right\}, \tag{2.4.2.3}$$

$$\tilde{I}_N := \left\{ (i,r) \in \hat{I} \mid -2N-1 \leqslant r < 2N+1 \right\}. \tag{2.4.2.4}$$

Then Γ_N is the subquiver of Γ with set of vertices $\tilde{I}_N$, where the vertices is $\tilde{I}_N \backslash \hat{I}_N$ are frozen (thus the vertices in $\hat{I}_N$ are the exchangeable vertices).

This way, all cluster variables of $\mathcal{A}(\Gamma)$ obtained from the initial seed after a finite sequence of mutations are cluster variables of the finite rank cluster algebra $\mathcal{A}(\Gamma_N)$, for N large enough. With the same index restrict on $\tilde{B}$, we will be able to define a size increasing family of finite rank quantum cluster algebras.

Example 2.4.7. Recall from Example A.4.1 of the Appendix the infinite quiver Γ when $\mathfrak{g} = \mathfrak{sl}_4$. Then the quiver Γ_N is the following:

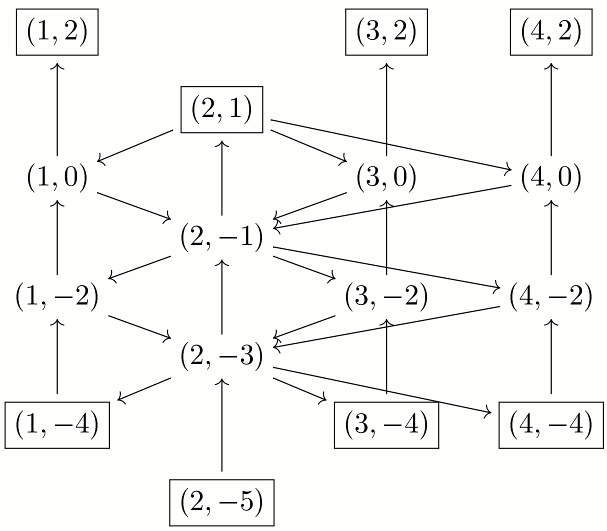


where the boxed vertices are frozen.

For $N \in \mathbb{N}^*$, let $\tilde{B}_N$ be the corresponding exchange matrix. It is the $\tilde{I}_N \times \hat{I}_N$ submatrix of $\tilde{B}$, thus its coefficients are as in (2.4.2.1).

For all $N \in \mathbb{N}^*$, let Λ_N be the $\tilde{I}_N \times \tilde{I}_N$ submatrix of Λ. It is a finite $(n(2N+1))^2$ skew-symmetric matrix, where n is the rank of the simple Lie algebra $\mathfrak{g}$.

Example 2.4.8. For $\mathfrak{g}$ of type D_4, let us explicit a finite slice of Γ of length 4, containing an upper and lower row of frozen vertices (which is thus not Γ_1, of length 3, nor Γ_2, of length 5):



If the set $\tilde{I} = \{(i, r) \in \hat{I} \mid i \in [\![1, 4]\!], -5 \leqslant r \leqslant 2\}$ is ordered lexicographically by r then i

(reading order), the quiver is represented by the following exchange matrix:

$$\tilde{B} := \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 & 1 & 1 & 1 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.4.2.5}$$

The principal part B of $\tilde{B}$ is the square submatrix obtained by omitting the first 4 columns and the last 4 columns. One notices that B is skew-symmetric.

Moreover, using Formula (2.4.2.2), one can compute the corresponding matrix Λ. We get the following 16×16 skew-symmetric matrix (with the same order of $\tilde{I}$ as before):

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 & 2 & 2 & 2 & 4 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 \\ -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ -2 & -2 & -2 & -3 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -2 & -1 & -1 & -2 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & -1 & -2 & -1 & -1 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -2 & -2 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & -2 & -4 & -2 & -2 & -2 & -3 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.4.2.6}$$

From here, it is easy to check that the product $\tilde{B}^T \Lambda$ is of the form:

$$\tilde{B}^T \Lambda = \begin{pmatrix} 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.4.2.7}$$

Thus, $(\Lambda, \tilde{B})$ is a compatible pair.

Remark 2.4.9. Notice the coefficients appearing of the diagonal here are all negative. From the definition of compatible pair from [BZ05], for $(\Lambda, \tilde{B})$ to be a compatible pair, these should be positive. That is why in Definition B.2.1 of the Appendix, the coefficients are of constant sign (the properties of quantum cluster algebras are still satisfied by taking negative coefficients in the compatibility condition). This allows for our t-commutation to be coherent with that of the quantum torus $\mathcal{Y}_t$.

We show that this result is true in general. Furthermore, the specific form we obtain in Equation (2.4.2.7) is what we get in general.

Proposition 2.4.10. *The pairs* $(\Lambda, \tilde{B})$ *and* $\left((\Lambda_N, \tilde{B}_N)\right)_{N\in\mathbb{N}^*}$ *are compatible pairs, in the sense of structure condition for quantum cluster algebras.*

Moreover,

$$\tilde{B}^T\Lambda = -2\,\mathrm{Id}_{\hat{I}},$$

$$\tilde{B}_N^T\Lambda_N = -2\left(\begin{array}{c|c|c} (0) & \mathrm{Id}_{\hat{I}_N} & (0) \end{array}\right).$$

Proof. Let $((i,r),(j,s)) \in \hat{I}^2$. Let us compute:

$$\left(\tilde{B}^T\Lambda\right)_{(i,r),(j,s)} = \sum_{(k,u)\in\hat{I}} b_{(k,u),(i,r)}\lambda_{(k,u),(j,s)}. \tag{2.4.2.8}$$

This is a finite sum, as each vertex in Γ is adjacent to a finite number of other vertices.

Suppose first that $r \neq s$. Without loss of generality, we can assume that $r < s$. Then, using the definition of the matrix Λ in (2.4.2.2) and the coefficients of $\tilde{B}$ in (2.4.2.1), we obtain

$$\left(\tilde{B}^T\Lambda\right)_{(i,r),(j,s)} = -\tilde{C}_{ij}(s-r-1) - \tilde{C}_{ij}(s-r+1) + \sum_{k\sim i}\tilde{C}_{kj}(s-r). \tag{2.4.2.9}$$

Now recall from Lemma A.2.4 of the Appendix, for all $(i,j) \in I^2$,

$$\tilde{C}_{ij}(m-1) + \tilde{C}_{ij}(m+1) - \sum_{k\sim i}\tilde{C}_{kj}(m) = 0, \quad \forall m \geqslant 1.$$

Thus, for all$(i,j) \in I^2$ and $r < s$, equation (2.4.2.9) gives:

$$\left(\tilde{B}^T\Lambda\right)_{(i,r),(j,s)} = 0. \tag{2.4.2.10}$$

Suppose now that $r = s$. In this case,

$$\left(\tilde{B}^T\Lambda\right)_{(i,r),(j,r)} = -2\tilde{C}_{ij}(1) = -2\delta_{i,j},$$

using the other result from Lemma A.2.4. Thus,

$$\tilde{B}^T\Lambda = -2\,\mathrm{Id}_{\hat{I}}. \tag{2.4.2.11}$$

Now, for all $N \in \mathbb{N}^*$, let $(i,r) \in \hat{I}_N$ and $(j,s) \in \tilde{I}_N$. Let us write:

$$\left(\tilde{B}^T\Lambda\right)_{(i,r),(j,s)} = \sum_{(k,u)\in\tilde{I}_N} b_{(k,u),(i,r)}\lambda_{(k,u),(j,s)}. \tag{2.4.2.12}$$

As $(i,r) \in \hat{I}_N$ is not a frozen variable, the $(j,s) \in \hat{I}$ such that $b_{(k,u),(i,r)} \neq 0$ are all in $\tilde{I}_N$. Hence, the rest of the reasoning is still valid, and the result follows. ☐

2.4.3 Definition of $K_t(\mathcal{O}_{\mathbb{Z}}^+)$

Everything is now in place to define $K_t(\mathcal{O}_{\mathbb{Z}}^+)$. Recall the based quantum torus $\mathscr{T}_t$, defined in Section 2.3.2. By construction, the associated skew-symmetric bilinear form Λ identifies with the infinite skew-symmetric $\hat{I} \times \hat{I}$-matrix from the previous section:

$$\Lambda(\mathbf{e}_{(i,r)}, \mathbf{e}_{(j,s)}) = \Lambda_{(i,r),(j,s)} = \mathcal{F}_{ij}(s-r), \quad \left((i,r),(j,s) \in \hat{I}\right) \tag{2.4.3.1}$$

where $\left(\mathbf{e}_{(i,r)}\right)_{(i,r)\in\hat{I}}$ is the standard basis of $\mathbb{Z}^{(\hat{I})}$.

Let $\mathcal{F}$ be the skew-field of fractions of $\mathscr{T}_t$. We define the toric frame $M : \mathbb{Z}^{(\hat{I})} \to \mathcal{F}\backslash\{0\}$ by setting

$$M(\mathbf{e}_{(i,r)}) = z_{i,r} \quad \in \mathcal{F}, \quad \forall (i,r) \in \hat{I}. \tag{2.4.3.2}$$

Then the infinite rank matrix Λ_M satisfies

$$\Lambda_M = \Lambda. \tag{2.4.3.3}$$

From the result of Proposition 2.4.10,

$$\mathcal{S} = \left(M, \tilde{B}\right) \tag{2.4.3.4}$$

is a quantum seed.

Definition 2.4.11. Let $\mathcal{A}_t(\Gamma)$ be the quantum cluster algebra associated to the mutation-equivalence class of the quantum seed $\mathcal{S}$.

Remark 2.4.12. One could note that this is an infinite rank quantum cluster algebra, which it not covered by the definition given in Appendix B. However, we have a sequence of quantum cluster algebras $(\mathcal{A}_t(\Gamma_N))_{N\in\mathbb{N}^*}$, built on the finite quivers $(\Gamma_N)_{N\in\mathbb{N}^*}$. As the mutation sequences are finite, one can always assume we are working in the quantum cluster algebra $\mathcal{A}_t(\Gamma_N)$, with N large enough.

Fix $N \in \mathbb{N}^*$. Let $m = (2N+1) \times n$, where n is the rank of the simple Lie algebra $\mathfrak{g}$.

Consider L_N, the sub-lattice of $\mathscr{T}_t$ generated by the $z_{i,r}$, with $(i,r) \in \tilde{I}_N$ (recall the definition of $\tilde{I}_N$ in (2.4.2.4)). L_N is of rank m. Consider the toric frame M_N which is the restriction of M to L_N. In that case,

$$\Lambda_{M_N} = \Lambda_N, \qquad \text{from the previous section.}$$

Thus,

$$\mathcal{S}_N := \left(M_N, \tilde{B}_N\right) \tag{2.4.3.5}$$

is a quantum seed.

Definition 2.4.13. Let $\mathcal{A}_t(\Gamma_N)$ be the quantum cluster algebra associated to the mutation-equivalence class of the quantum seed $\mathcal{S}_N$.

Remark 2.4.14. This is an explicit construction of an infinite rank quantum cluster algebra as a (co)limit of finite rank quantum cluster algebras. Thus we constructed explicitly the result of [GG18] in our situation.

Definition 2.4.15. Define

$$K_t(\mathcal{O}_{\mathbb{Z}}^+) := \mathcal{A}_t(\Gamma)\hat{\otimes}\mathcal{E}, \tag{2.4.3.6}$$

where the tensor product is completed as in (2.3.2.6). The ring $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ is a $\mathcal{E}[t^{\pm1/2}]$-subalgebra of $\mathscr{T}_t$.

For $N \in \mathbb{N}^*$, with the same completion of the tensor product, define

$$K_t(\mathcal{O}_{\mathbb{Z},N}^+) := \mathcal{A}_t(\Gamma_N)\hat{\otimes}\mathcal{E}. \tag{2.4.3.7}$$

2.5 Properties of $K_t(\mathcal{O}_{\mathbb{Z}}^+)$

2.5.1 The bar involution

The bar involution defined on $\mathcal{Y}_t$ (see Section 2.4.1) has a counterpart on the larger quantum torus $\mathcal{T}_t$. Besides, as (q,t)-character of simple modules in $\mathscr{C}_{\mathbb{Z}}$ are bar-invariant by definition, it is natural for (q,t)-characters of simple modules in $\mathcal{O}_{\mathbb{Z}}^+$ to also be bar-invariant.

There is unique $\mathcal{E}$-algebra anti-automorphism of $\mathcal{T}_t$ such that

$$\overline{t^{1/2}} = t^{-1/2}, \quad \overline{z_{i,r}} = z_{i,r}, \quad \text{and } \overline{[\omega_i]} = [\omega_i], \quad ((i,r) \in \hat{I}).$$

What is crucial to note here is that this definition is compatible with the bar-involution defined in general on the quantum torus of any quantum cluster algebra (see [BZ05, Section 6]). However, this bar-involution has an important property: all cluster variables are invariant under the bar involution.

Proposition 2.5.1. *All elements of $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ of the form $\chi_t \otimes 1$, where $\chi_t \in \mathcal{A}_t(\Gamma)$ is a cluster variable, are invariant under the bar-involution.*

2.5.2 Inclusion of quantum Grothendieck rings

As stated earlier, one natural property we would want to be satisfied by the quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ is to include the already-existing quantum Grothendieck $K_t(\mathscr{C}_{\mathbb{Z}})$ of the category $\mathscr{C}_{\mathbb{Z}}$.

Note that these rings are contained in quantum tori, which are included in one another by the injective morphism $\mathcal{J}$ from Proposition 2.3.3:

$$\begin{tikzcd} K_t(\mathscr{C}_{\mathbb{Z}}) \arrow[d, dashed] & \subset & \mathcal{Y}_t \arrow[d, hook, "\mathcal{J}"] \\ K_t(\mathcal{O}_{\mathbb{Z}}^+) & \subset & \mathcal{T}_t. \end{tikzcd}$$

Thus it is natural to formulate the following Conjecture:

Conjecture 2.5.2. *The injective morphism $\mathcal{J}$ restricts to an inclusion of the quantum Grothendieck rings*

$$\mathcal{J} : K_t(\mathscr{C}_{\mathbb{Z}}) \subset K_t(\mathcal{O}_{\mathbb{Z}}^+). \tag{2.5.2.1}$$

Recall that the quantum Grothendieck ring $K_t(\mathscr{C}_{\mathbb{Z}})$ is generated by the classes of the fundamental representations $[L(Y_{i,q^{r+1}})]_t$, for $(i,r) \in \hat{I}$ (see Section 2.4.1). Hence, in order to prove Conjecture 2.5.2, it is enough to show that the images of these $[L(Y_{i,q^{r+1}})]_t$ belong to $K_t(\mathcal{O}_{\mathbb{Z}}^+)$.

In Example 2.2.21 we saw how, when the $\mathfrak{g} = \mathfrak{sl}_2$, the class of the fundamental representation $[L(Y_{1,q^{-1}})]$ could be obtained as a cluster variable in $\mathcal{A}(\Gamma)$ after one mutation in direction $(1,0)$.

This fact is actually true in more generally, as seen in [HL16b], in the proof of Proposition 6.1. Let us recall this process precisely.

Fix $(i,r) \in \hat{I}$. We first define a specific sequence of vertices in Γ, as in [HL16a]. Recall the definition of the dual Coxeter number $h^\vee$.

$\mathfrak{g}$	A_n	D_n	E_6	E_7	E_8
$h^\vee$	$n+1$	$2n-2$	12	18	30

Let $h' = \lceil h^\vee/2 \rceil$. Fix an ordering $(j_1, \ldots, j_n)$ of the vertices of the Dynkin diagram of $\mathfrak{g}$ by taking first $j_1 = i$, then all vertices which appear with the same oddity as i in $\hat{I}$ (the j such that $(j,r) \in \hat{I}$), then the vertices which appear with a different oddity ($(j, r+1) \in \hat{I}$). For all $k \in \{2, \ldots, h'\}$, $j \in \{1, \ldots, n\}$, define the sequence $S_{j,k}$ of k vertices of the column j of Γ in decreasing order:

$$S_{j,k} = (j, r + 2h' - \epsilon), (j, r + 2h' - \epsilon - 2), \ldots, (j, r + 2h' - \epsilon - 2k + 2), \tag{2.5.2.2}$$

where $\epsilon \in \{0, 1\}$, depending of the oddity. Then define

$$S_k = \overrightarrow{\bigcup_j} S_{j,k}, \tag{2.5.2.3}$$

with the order defined before. Finally, let:

$$S = S_{h'} \cdots S_2 \ (i, r + 2h'),$$

by reading left to right and adding one last $(i, r + 2h')$ at the end.

Example 2.5.3. For $\mathfrak{g}$ of type D_4, and $(i,r) = (1,0)$, the sequence S is

$$\begin{aligned} S = &(1,6)\ (1,4)\ (1,2)\ \ (3,6)\ (3,4)\ (3,2) \\ &(4,6)\ (4,4)\ (4,2)\ \ (2,5)\ (2,3)\ (2,1) \\ &(1,6)\ (1,4)\ \ (3,6)\ (3,4)\ \ (4,6)\ (4,4) \\ &(2,5)\ (2,3)\ \ \ (1,6) \end{aligned}$$

Using [HL16a, Theorem 3.1] and elements from the proof of Proposition 6.1 in [HL16b], one gets the following result.

Proposition 2.5.4. *Let $\chi_{i,r}$ be the cluster variable of $\mathcal{A}(\Gamma)$ obtained at the vertex $(i, r+2h')$ after following the sequence of mutations S, then, via the identification (2.2.7.6)*

$$\chi_{i,r} \equiv [L(Y_{i,q^{r+1}})]. \tag{2.5.2.4}$$

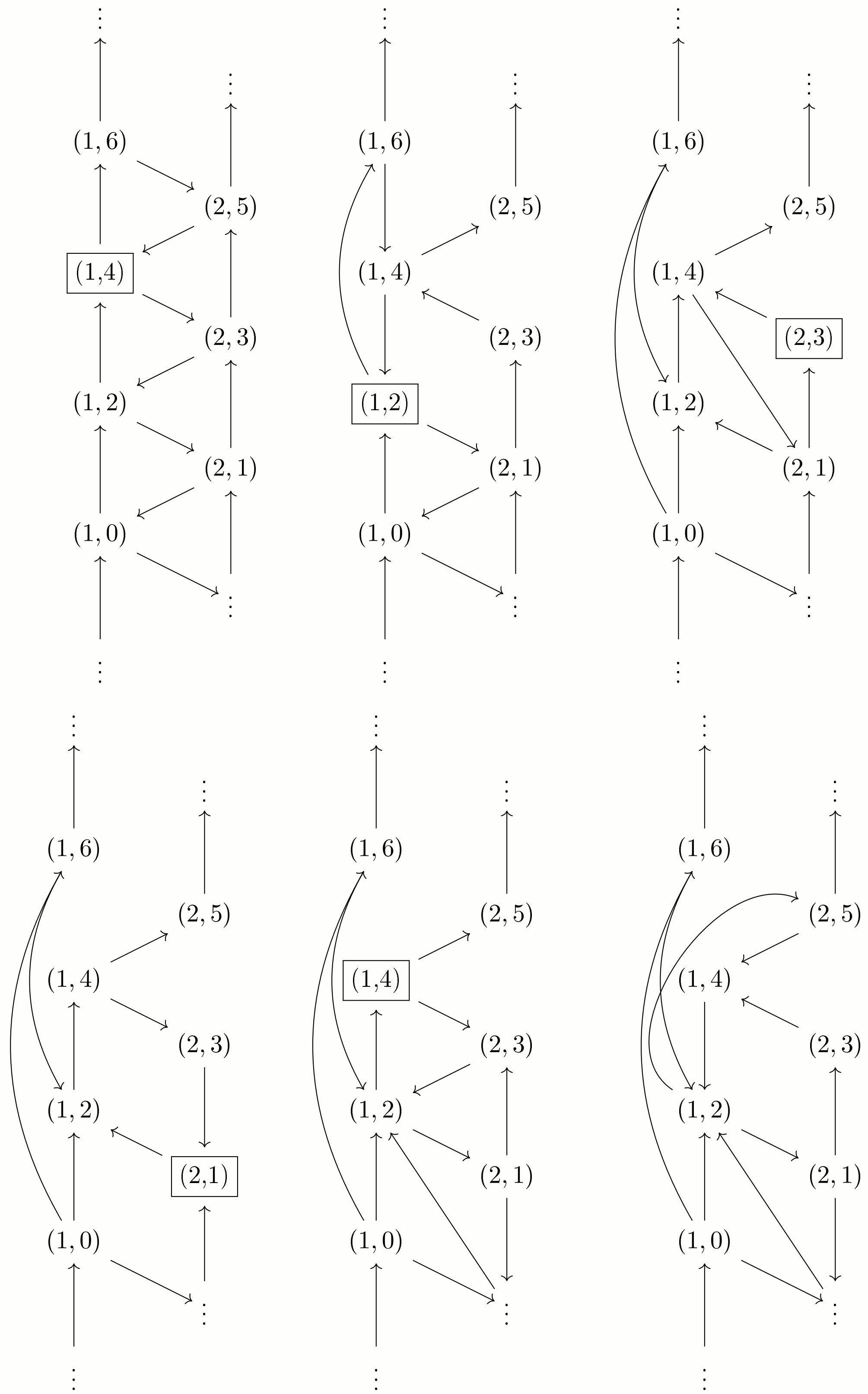
To see this result differently, if one writes $\chi_{i,r}$ as a Laurent polynomial in the variables $(z_{j,s})$, then $\chi_{i,r}$ is in the image of $\mathcal{J}$, and

$$\chi_{i,r} = \mathcal{J}(\chi_q(L(Y_{i,q^{r+1}})). \tag{2.5.2.5}$$

Example 2.5.5. Let $\mathfrak{g} = \mathfrak{sl}_3$ and $(i,r) = (1,0)$. The sequence of vertices S is

$$S = (1,4)\ (1,2)\ \ (2,3)\ (2,1)\ \ (1,4). \tag{2.5.2.6}$$

Let us compute the sequence of mutations S:


(1,6)
(2,5)
(1,4)
(2,3)
(1,2)
(2,1)
(1,0)
(1,6)
(2,5)
(1,4)
(2,3)
(1,2)
(2,1)
(1,0)
(1,6)
(2,5)
(1,4)
(2,3)
(1,2)
(2,1)
(1,0)
(1,6)
(2,5)
(1,4)
(2,3)
(1,2)
(2,1)
(1,0)
(1,6)
(2,5)
(1,4)
(2,3)
(1,2)
(2,1)
(1,0)
(1,6)
(2,5)
(1,4)
(2,3)
(1,2)
(2,1)
(1,0)

The associated cluster variables are:

$$\begin{aligned}
z_{1,4}^{(1)} &= z_{1,2}z_{1,4}^{-1}z_{2,5} + z_{1,4}^{-1}z_{1,6}z_{2,3},\\
z_{1,2}^{(1)} &= z_{1,0}z_{1,4}^{-1}z_{2,5} + z_{1,0}z_{1,2}^{-1}z_{1,4}^{-1}z_{1,6}z_{2,3} + z_{1,2}^{-1}z_{1,6}z_{2,-1},\\
z_{2,3}^{(1)} &= z_{2,1}z_{2,3}^{-1} + z_{1,2}z_{1,4}^{-1}z_{2,5}z_{2,3}^{-1} + z_{1,4}^{-1}z_{1,6},\\
z_{1,4}^{(2)} &= z_{1,0}z_{1,2}^{-1} + z_{1,2}^{-1}z_{1,4}z_{2,1}z_{2,3}^{-1} + z_{2,3}^{-1}z_{2,5}.
\end{aligned}$$

Thus, $\chi_{1,0} = z_{1,4}^{(2)}$ is in the image of $\mathcal{J}$, and

$$\chi_{1,0} = \mathcal{J}(Y_{1,q} + Y_{1,q^3}^{-1}Y_{2,q^2} + Y_{2,q^4}^{-1}) = \mathcal{J}(\chi_q(L(Y_{1,q}))). \tag{2.5.2.7}$$

Notice also that $z_{2,3}^{(1)}$ was already in the image of $\mathcal{J}$ and that $z_{2,3}^{(1)} = \mathcal{J}(\chi_q(L(Y_{2,q^2})))$.

Thus, for each $(i,r) \in \hat{I}$, consider the quantum cluster variables $\tilde{\chi}_{i,r} \in K_t(\mathcal{O}_{\mathbb{Z}}^+)$ obtained from the initial quantum seed $(\boldsymbol{z}, \Lambda)$ via the sequence of mutations S.

Example 2.5.6. Suppose $\mathfrak{g} = \mathfrak{sl}_2$. Consider the quiver Γ_1 a well as the skew-symmetric matrix Λ_1,

$$\Gamma_1 = \begin{array}{c} \boxed{(1,2)} \\ \uparrow \\ (1,0) \\ \uparrow \\ \boxed{(1,-2)} \end{array}, \quad \Lambda_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

As seen in Example 2.2.21 (with a shift of quantum parameters), the fundamental representation $[L(Y_{1,q^{-1}})]$ is obtained in $K_0(\mathcal{O}_Z^+)$ after one mutation at $(1,0)$ (here $S = (1,0)$).

The quantum cluster variable obtained after a quantum mutation at $(1,0)$, written with commutative monomials, is

$$\begin{aligned}
\tilde{\chi}_{1,-2} &= z_{1,-2}z_{1,0}^{-1} + z_{1,2}z_{1,0}^{-1} = \mathcal{J}(Y_{1,q^{-1}} + Y_{1,q}^{-1}) = \mathcal{J}([L(Y_{1,q^{-1}})]_t),\\
&= \mathcal{J}\left(Y_{1,q^{-1}}(1 + A_{1,1}^{-1})\right) \quad \in \mathcal{J}(K_t(\mathscr{C}_{\mathbb{Z}})),
\end{aligned}$$

Thus, we note that in this particular case, the quantum cluster variable $\tilde{\chi}_{1,-2}$ recovers the (q,t)-character $[L(Y_{1,q^{-1}})]_t$ of the fundamental representation $L(Y_{1,q^{-1}})$.

In particular, Conjecture 2.5.2 is satisfied in this case.

This example incites us to formulate another conjecture.

Conjecture 2.5.7. *For all $(i,r) \in \hat{I}$, the quantum cluster variable $\tilde{\chi}_{i,r}$ recovers, via the morphism $\mathcal{J}$, the (q,t)-character of the fundamental representation $L(Y_{i,q^{r+1}})$:*

$$\tilde{\chi}_{i,r} = \mathcal{J}\left([L(Y_{i,q^{r+1}})]_t\right). \tag{2.5.2.8}$$

Remark 2.5.8. Notice that Conjecture 2.5.7 implies Conjecture 2.5.2, and that Conjecture 2.5.7 is also satisfied when $\mathfrak{g} = \mathfrak{sl}_2$, from Example 2.5.6.

What can be said, in general, of the quantum cluster variables $\tilde{\chi}_{i,r}$?

Proposition 2.5.9. *For all $(i,r) \in \hat{I}$, the quantum cluster variable $\tilde{\chi}_{i,r}$ satisfies the following properties:*

(i) invariant under the bar involution:

$$\overline{\tilde{\chi}_{i,r}} = \tilde{\chi}_{i,r}. \tag{2.5.2.9}$$

(ii) the coefficients of its expansion as a Laurent polynomial in the initial quantum cluster variables $\{z_{i,r}\}$ are Laurent polynomials in $t^{1/2}$ with non-negative integers coefficients:

$$\tilde{\chi}_{i,r} \in \bigoplus_{\boldsymbol{u}=u_{i,r}\in\mathbb{Z}^{(\hat{I})}} \mathbb{N}[t^{\pm 1/2}]\boldsymbol{z}^{\boldsymbol{u}}. \tag{2.5.2.10}$$

with $\boldsymbol{z}^{\boldsymbol{u}} = \prod_{(i,r)\in\hat{I}} z_{i,r}^{u_{i,r}}$ denoting the commutative monomial.

(iii) its evaluation at $t = 1$ (as seen in (2.3.2.12)), recovers the q-character of the fundamental representation $L(Y_{i,q^{r+1}})$:

$$\pi(\tilde{\chi}_{i,r}) = \chi_q(L(Y_{i,q^{r+1}})). \tag{2.5.2.11}$$

Proof. The first property is a direct consequence of Proposition 2.5.1 and the second is a direct consequence of the positivity result of Theorem B.6.1 in the Appendix.

For the third property, notice we have used two evaluation maps so far, with the same notation.

- The evaluation map defined in (B.5.1) on the bases quantum torus of a quantum cluster algebra:
$$\pi : \mathcal{A}_t(M, \tilde{B}) \to \mathbb{Z}[\tilde{\mathbf{X}}^{\pm 1}],$$
- The evaluation map defined in (2.3.2.12) on $\mathscr{T}_t$:
$$\pi : \mathscr{T}_t \to \mathcal{E}_\ell.$$

These notations are coherent because the map π from (2.3.2.12) is the evaluation map defined on a based quantum torus (of infinite rank) of a quantum cluster algebra, extended to a $\mathcal{E}$-morphism on $\mathscr{T}_t$. In this case, the Laurent polynomial ring $\mathbb{Z}[\tilde{\mathbf{X}}^{\pm 1}]$ is $\mathbb{Z}[z_{i,r}^{\pm 1} \mid (i,r) \in \hat{I}]$, which becomes $\mathcal{E}[\boldsymbol{\Psi}_{i,r}^{\pm 1}]$ after extension to a $\mathcal{E}$-morphism and via the identification (2.2.7.6).

Thus we can apply Corollary B.5.3 of the Appendix to this map π. As $\tilde{\chi}_{i,r}$ is a quantum cluster variable, its evaluation by π is the cluster variable $\chi_{i,r}$, which is obtained from the initial seed $\mathbf{z}$, via the same sequence of mutations S (the initial seed and quantum seeds are fixed and identified by the evaluation π on the quantum torus $\mathscr{T}_t$). By Proposition 2.5.4,

$$\pi(\tilde{\chi}_{i,r}) = \chi_{i,r} = \chi_q(L(Y_{i,q^{r+1}})). \tag{2.5.2.12}$$

□

These two properties imply that the $\tilde{\chi}_{i,r}$ are good candidates for the (q,t)-characters of the fundamental representations, as stated in Conjecture 2.5.2.

2.5.3 (q,t)-characters for positive prefundamental representations

Recall the q-characters of the positive prefundamental representations in (2.2.8.6), for all $i \in I, a \in \mathbb{C}^\times$,

$$\chi_q(L_{i,a}^+) = [\boldsymbol{\Psi}_{i,a}]\chi_i,$$

where $\chi_i \in \mathcal{E}$ is the (classical) character of $L_{i,a}^+$.

Definition 2.5.10. For $(i, r) \in \hat{I}$, define

$$[L^+_{i,q^r}]_t := [\boldsymbol{\Psi}_{i,q^r}] \otimes \chi_i \quad \in K_t(\mathcal{O}^+_{\mathbb{Z}}), \tag{2.5.3.1}$$

using the notation from (2.3.2.7).

Remark 2.5.11. It is the quantum cluster variable obtained from the initial quantum seed, via the same sequence of mutations used to obtain $[L^+_{i,q^r}]$ in $K_0(\mathcal{O}^+_{\mathbb{Z}})$, which in this case, is no mutation at all.

In particular, the evaluation of $[L^+_{i,q^r}]_t$ recovers the q-character of $[L^+_{i,q^r}]$:

$$\pi([L^+_{i,q^r}]_t) = [\boldsymbol{\Psi}_{i,q^r}] \otimes \chi_i = \chi_q(L^+_{i,a}) \quad \in \mathcal{E}_\ell. \tag{2.5.3.2}$$

2.6 Results in type A

Suppose in this section that the underlying simple Lie algebra $\mathfrak{g}$ is of type A.

2.6.1 Proof of the conjectures

In this case, the situation of Example 2.5.6 generalizes.

Theorem 2.6.1. *Conjecture 2.5.7 is satisfied in this case.*

In this case, the key ingredient is the following well-known result (see for exemple [FR96, Section 11], and references therein).

Theorem 2.6.2. *When $\mathfrak{g}$ is of type A, all ℓ-weight spaces of all fundamental representations $L(Y_{i,a})$ are of dimension 1.*

Proof. Fix $(i, r) \in \hat{I}$. From the second property of Proposition 2.5.9, we know that $\tilde{\chi}_{i,r}$ can be written as

$$\tilde{\chi}_{i,r} = \sum_{\boldsymbol{u} \in \mathbb{Z}^{(\hat{I})}} P_{\boldsymbol{u}}(t^{1/2}) \boldsymbol{z}^{\boldsymbol{u}}, \tag{2.6.1.1}$$

where the $P_{\boldsymbol{u}}(t^{1/2})$ are Laurent polynomials with non-negative integer coefficients. Using the third property of Proposition 2.5.9, we deduce the evaluation at $t = 1$ of equality (2.6.1.1):

$$\chi_q(L(Y_{i,q^{r-1}})) = \sum_{\boldsymbol{u} \in \mathbb{Z}^{(\hat{I})}} P_{\boldsymbol{u}}(1) \prod_{(i,r) \in \hat{I}} ([\boldsymbol{\Psi}_{i,q^r}][-r\omega_i/2])^{u_{i,r}} \quad \in \mathcal{E}_\ell. \tag{2.6.1.2}$$

From the aforementioned theorem, this decomposition is multiplicity-free. Thus, the non-zero coefficients $P_{\boldsymbol{u}}(t^{1/2})$ are of the form $t^{k/2}$, with $k \in \mathbb{Z}$. Finally, as $\chi_{i,r}$ is bar-invariant, from the first property of Proposition 2.5.9, and the $\boldsymbol{z}^{\boldsymbol{u}}$ are also bar-invariant as commutative monomials, we know that the Laurent polynomials $P_{\boldsymbol{u}}(t^{1/2})$ are even functions:

$$P_{\boldsymbol{u}}(-t^{1/2}) = P_{\boldsymbol{u}}(t^{1/2}). \tag{2.6.1.3}$$

Thus the variable $t^{1/2}$ does not explicitly appear in the decomposition (2.6.1.1), and so:

$$\begin{aligned}\chi_{i,r} &= \sum_{\boldsymbol{u} \in \mathbb{Z}^{(\hat{I})}} P_{\boldsymbol{u}}(1) \boldsymbol{z}^{\boldsymbol{u}}, \\ &= \mathcal{J}\left(\chi_q(L(Y_{i,q^{r-1}}))\right).\end{aligned}$$

Moreover, with the same arguments, as $[L(Y_{i,q^{r-1}})]_t$ is bar-invariant by definition,

$$[L(Y_{i,q^{r-1}})]_t = \chi_q(L(Y_{i,q^{r-1}})), \tag{2.6.1.4}$$

written in the basis of the commutative monomials.

Hence, we recover the fact that the quantum cluster variable $\chi_{i,r}$ is equal, via the inclusion map $\mathcal{J}$, to the (q,t)-character of $L(Y_{i,q^{r-1}})$ and Conjecture 2.5.7 is satisfied. □

2.6.2 A remarkable subalgebra in type A_1

When $\mathfrak{g} = \mathfrak{sl}_2$, we can make explicit computations. Recall the formula (2.3.2.10) of the quantum torus from Example 2.3.4:

$$z_{1,2r} * z_{1,2s} = t^{f(s-r)} z_{1,2s} * z_{1,2r} \ , \forall r,s \in \mathbb{Z},$$

where $f : \mathbb{Z} \to \mathbb{Z}$ is antisymmetric and defined by

$$f_{|\mathbb{N}} : m \mapsto \frac{(-1)^m - 1}{2}.$$

For all $r \in \mathbb{Z}$, the (q,t)-character of the prefundamental representation $L^+_{1,q^{2r}}$ defined in (2.5.3.1) is

$$[L^+_{1,q^{2r}}]_t = [\mathbf{\Psi}_{1q^{2r}}]\chi_1.$$

Proposition 2.6.3. *With these (q,t)-characters, we can write a t-deformed version of the Baxter relation (2.1.0.2), for all $r \in \mathbb{Z}$,*

$$[L(Y_{1,q^{2r-1}})]_t * [L^+_{1,q^{2r}}]_t = t^{-1/2}[\omega_1][L^+_{1,q^{2r-2}}]_t + t^{1/2}[-\omega_1][L^+_{1,q^{2r+2}}]_t. \tag{2.6.2.1}$$

We call this relation the *quantized Baxter relation.*

Remark 2.6.4. If we identify the variables $Y_{1,q^{2r}}$ and their images through the injection $\mathcal{J}$, this relation is actually the exchange relation related to the quantum mutation in Example 2.5.6 (for a generic quantum parameter q^{2r}).

Now consider the quantum cluster algebra $\mathcal{A}(\Lambda_1, \Gamma_1)$, with notations from Section 2.4.2 (Λ_1 and Γ_1 are given explicitly in Example 2.5.6).

It is a quantum cluster algebra of finite type (if we remove the frozen vertices from the quiver, we get just one vertex, which is a quiver of type A_1). It has two quantum clusters, containing the two frozen variables $z_{1,2}, z_{1,-2}$ and the mutable variables $z_{1,0}$ and $z_{1,0}^{(1)}$, respectively. Thus, it is generated as a $\mathbb{C}(t^{1/2})$-algebra by

$$\begin{aligned} E &:= [L(Y_{1,q^{-1}})]_t \ (= z_{1,0}^{(1)}), & F &:= [L^+_{1,1}]_t \ (= z_{1,0}), \\ K &:= [\omega_1][L^+_{1,q^{-2}}]_t \ (= z_{1,-2}), & K' &:= [-\omega_1][L^+_{1,q^2}]_t \ (= z_{1,2}). \end{aligned} \tag{2.6.2.2}$$

This algebra is a quotient of a well-known $\mathbb{C}(t^{1/2})$-algebra.

Let q be a formal parameter. The quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ can be seen as the quotient

$$\mathcal{U}_q(\mathfrak{sl}_2) = \mathfrak{D}_2 / \left\langle KK' = 1 \right\rangle, \tag{2.6.2.3}$$

where $\mathfrak{D}_2$ is the $\mathbb{C}(q)$-algebra with generators E, F, K, K' and relations:

$$\begin{aligned} KE &= q^2 EK, & K'E &= q^{-2}EK' \\ KF &= q^{-2}FK, & K'F &= q^2 FK' \\ KK' &= K'K, & &\text{and } [E,F] = (q-q^{-1})(K-K'). \end{aligned} \tag{2.6.2.4}$$

Remark 2.6.5.
- As in [SS19, Remark 3.1], notice that the last relation in (2.6.2.4) is not the usual relation
$$[e, f] = \frac{K - K'}{q - q^{-1}}.$$
But both presentations are equivalent, given the change of variables
$$E = (q - q^{-1})e, \quad F = (q - q^{-1})f.$$
- $\mathfrak{D}_2$ is the *Drinfeld double* [Dri87] of the Borel subalgebra of $\mathcal{U}_q(\mathfrak{sl}_2)$ (the subalgebra generated by K, E).

Proposition 2.6.6. *The $\mathbb{C}(t^{1/2})$-algebra $\mathcal{A}(\Lambda_1, \Gamma_1)$ is isomorphic to the quotient of the Drinfeld double $\mathfrak{D}_2$ of parameter $-t^{1/2}$,*
$$\mathcal{A}(\Lambda_1, \Gamma_1) \xrightarrow{\sim} \mathfrak{D}_2 / C_{-t^{1/2}}, \tag{2.6.2.5}$$
where $C_{-t^{1/2}}$ is the quantized Casimir element:
$$C_{-t^{1/2}} := EF - t^{1/2}K - t^{-1/2}K'. \tag{2.6.2.6}$$

Proof. One has, in $\mathcal{A}(\Lambda_1, \Gamma_1)$,
$$E * F = t^{-1/2}K + t^{1/2}K'. \tag{2.6.2.7}$$
This is the quantized Baxter relation (2.6.2.1). Thus,
$$[E, F] = (-t^{1/2} + t^{-1/2})(K - K').$$
We check that the other relations in (2.6.2.4) are also satisfied using the structure of the quantum torus $\mathscr{T}_t$ (which is given explicitly in Example 2.3.4).

Hence the map
$$\mathcal{A}(\Lambda_1, \Gamma_1) \xrightarrow{\theta} \mathfrak{D}_2,$$
sending generators to generators, is well-defined and descends onto the quotient
$$\mathcal{A}(\Lambda_1, \Gamma_1) \to \mathfrak{D}_2 / C_{-t^{1/2}}.$$
Moreover, from [CIKLFP13], the cluster monomials in a given cluster in a cluster algebra are linearly independent. In this case, the quantum cluster algebra $\mathcal{A}(\Lambda_1, \Gamma_1)$ is of type A_1 (without frozen variables), thus of finite-type. It has two (quantum) clusters: (E, K, K') and (F, K, K'). Thus, the set of commutative quantum cluster monomials
$$\left\{ E^\alpha K^\beta K'^\gamma \mid \alpha, \beta, \gamma \in \mathbb{Z} \right\} \cup \left\{ K^\beta K'^\gamma F^\alpha \mid \alpha, \beta, \gamma \in \mathbb{Z} \right\}, \tag{2.6.2.8}$$
forms a $\mathbb{C}(t^{1/2})$-basis of $\mathcal{A}(\Lambda_1, \Gamma_1)$.

Consider the PBW basis of $\mathfrak{D}_2$:
$$\left\{ E^\alpha K^\beta K'^\gamma F^\delta \mid \alpha, \beta, \gamma, \delta \in \mathbb{Z} \right\}. \tag{2.6.2.9}$$
From the expression of the Casimir element $C_{-t^{1/2}}$ (2.6.2.6), we deduce a $\mathbb{C}(t^{1/2})$-basis of $\mathfrak{D}_2 / C_{-t^{1/2}}$, of the same form as (2.6.2.8):
$$\left\{ E^\alpha K^\beta K'^\gamma \mid \alpha, \beta, \gamma \in \mathbb{Z} \right\} \cup \left\{ K^\beta K'^\gamma F^\alpha \mid \alpha, \beta, \gamma \in \mathbb{Z} \right\}, . \tag{2.6.2.10}$$
Hence, the map θ sends a basis to a basis, and thus it is isomorphic.

□

This result should be compared with the recent work of Schrader and Shapiro [SS19], in which they recognize the same structure of $\mathfrak{D}_2$ in an algebra built on a quiver, with some quantum $\mathcal{X}$-cluster algebra structure. In their work, they generalized this result in type A (Theorem 4.4). Ultimately, they obtain an embedding of the whole quantum group $\mathcal{U}_q(\mathfrak{sl}_n)$ into a quantum cluster algebra. The result of Proposition 2.6.6, together with their results, gives hope that one could find a realization of the quantum group $\mathcal{U}_q(\mathfrak{g})$ as a quantum cluster algebra, related to the representation theory of $\mathcal{U}_q(L\mathfrak{g})$.

Furthermore, define in this case $\mathcal{O}_1^+$, the subcategory of $\mathcal{O}_{\mathbb{Z}}^+$ of objects whose image in the Grothendieck ring $K_0(\mathcal{O}_{\mathbb{Z}}^+)$ belongs to the subring generated by $[L^+_{1,q^{-2}}], [L^+_{1,1}], [L^+_{1,q^2}]$ and $[L(Y_{1,q^{-1}})]$. Then $\mathcal{O}_1^+$ is a monoidal category.

From the classification of simple modules when $\mathfrak{g} = \mathfrak{sl}_2$ in [HL16b, Section 7], we know that the only prime simple modules in $\mathcal{O}_1^+$ are

$$L^+_{1,q^{-2}}, L^+_{1,1}, L^+_{1,q^2}, L(Y_{1,q^{-1}}). \tag{2.6.2.11}$$

Moreover, a tensor product of those modules is simple if and only if it does not contain both a factor $L^+_{1,1}$ and a factor $L(Y_{1,q^{-1}})$ (the others are in so-called pairwise *general position*). Thus, in this situation, the simple modules are in bijection with the cluster monomials:

$$\begin{array}{ccc}
\left\{ \begin{array}{c} \text{simple modules} \\ \text{in } \mathcal{O}_1^+ \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \text{commutative quantum cluster} \\ \text{monomials in } \mathcal{A}(\Lambda_1, \Gamma_1) \end{array} \right\} \\
\left(L^+_{1,q^{-2}}\right)^{\otimes\alpha} \otimes \left(L^+_{1,1}\right)^{\otimes\beta} \otimes \left(L^+_{1,q^2}\right)^{\otimes\gamma} & \mapsto & K^{\alpha} F^{\beta} K'^{\gamma}, \\
\left(L^+_{1,q^{-2}}\right)^{\otimes\alpha'} \otimes \left(L^+_{1,q^2}\right)^{\otimes\beta'} \otimes L(Y_{1,q^{-1}})^{\otimes\gamma'} & \mapsto & K^{\alpha'} K'^{\beta'} E^{\gamma'}.
\end{array}$$

Chapter 3

Quantum cluster algebra algorithm and (q,t)-characters

Abstract.
We establish a quantum cluster algebra structure on the quantum Grothendieck ring of a certain monoidal subcategory of the category of finite-dimensional representations of a simply-laced quantum affine algebra. Moreover, the (q,t)-characters of certain irreducible representations, among which fundamental representations, are obtained as quantum cluster variables. This approach gives a new algorithm to compute these (q,t)-characters. As an application, we prove that the quantum Grothendieck ring of a larger category of representations of the Borel subalgebra of the quantum affine algebra, defined in a previous work as a quantum cluster algebra, contains indeed the well-known quantum Grothendieck ring of the category of finite-dimensional representations. Finally, we display our algorithm on a concrete example.

3.1 Introduction

Finite-dimensional representations of quantum affine algebras have been classified by Chari-Pressley [CP95b] with a quantum analog of Cartan's highest weight classification of finite-dimensional representations of simple Lie algebras. Combining this classification with the notations from Frenkel-Reshetikhin q-character [FR99] theory, one gets the following.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and $\mathcal{U}_q(\hat{\mathfrak{g}})$ be the quantum affine algebra. Irreducible finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$ are indexed by monomials in the infinite set of variables $\{Y_{i,a}\}_{i\in I, a\in\mathbb{C}^\times}$, where $I = \{1,\dots,n\}$ are the vertices of the Dynkin diagram of $\mathfrak{g}$. For such a monomial m, the corresponding simple $\mathcal{U}_q(\hat{\mathfrak{g}})$-module is denoted by $L(m)$. If the monomial is just one term $m = Y_{i,a}$, the corresponding simple module $L(Y_{i,a})$ is called a *fundamental module*. Chari-Pressley classification result also implies that every simple module can be obtained as a subquotient of a tensor product of such fundamental modules.

This classification is a major result in the way of obtaining information of the finite-dimensional representations of quantum affine algebras. However, it gives limited information of the structure of the modules in themselves. For that purpose, Frenkel and Reshetikhin have developed a theory of q-characters, giving the decomposition of the modules into generalized eigenspaces for the action of a large commutative subalgebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$. Frenkel-Mukhin established an algorithm to compute those q-characters [FM01]. This algorithm is guaranteed to work on fundamental modules, but not on all irreducible $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules [NN11].

Then, when $\mathfrak{g}$ is of simply-laced type, Nakajima [Nak01b] used the input from geometry, and more precisely perverse sheaves on quiver varieties, to construct t-deformations of these q-characters, called (q,t)-characters, as elements of a *quantum Grothendieck ring*. He introduced a second base for the Grothendieck ring of the category of finite-dimensional representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$, also indexed by the monomials in the variables $\{Y_{i,a}\}_{i\in I, a\in\mathbb{C}^\times}$, formed by the *standard modules*. Geometrically, these standard modules correspond to constant sheaves, but algebraically, for each monomial m, the standard module $M(m)$ can be seen as the tensor product of the fundamental modules corresponding to each of the factors in m, in a particular order (see also [VV02b]).

He first used a t-deformed version of Frenkel-Mukhin's algorithm to compute (q,t)-characters for the fundamental modules, then extended the (q,t)-characters to the standard modules, denoted $[M(m)]_t$. Next, he defined (q,t)-characters for the simple modules as some unique family of elements $[L(m)]_t$ of the quantum Grothendieck ring satisfying some invariant property, as well as having a decomposition of the form

$$[L(m)]_t = [M(m)]_t + \sum_{m'<m} Q_{m',m}(t)[M(m')]_t, \tag{3.1.0.1}$$

where $<$ is a partial order on the set of Laurent monomials in the variables $\{Y_{i,a}\}_{i\in I, a\in\mathbb{C}^\times}$, defined by Nakajima, and $Q_{m',m}(t) \in \mathbb{Z}[t^{\pm 1}]$ is a Laurent polynomial.

Nakajima then showed that these (q,t)-characters were indeed t-deformations of the q-characters, in the sense that the evaluation of the (q,t)-characters at $t=1$ recovers the q-characters. Finally, inverting the unitriangular decomposition (3.1.0.1), one gets an algorithm, of the Kazhdan-Lusztig type, to compute the (q,t)-characters, and so the q-characters of all simple finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules.

This algorithm is theoretically computable, but as noted in [Nak10], trying to compute it in reality can easily exceed the size of computer memory available. The first step of the algorithm is to compute the (q,t)-characters of the fundamental representations, and for example, for $\mathfrak{g}$ of type E_8, the 5th fundamental representation requires 120Go of memory to compute.

In [HL10] Hernandez and Leclerc introduced a new point of view on representations of quantum affine algebras, using the theory of cluster algebras that was developed by Fomin and Zelevinsky in the early 2000's [FZ02], [FZ03a], [BFZ05],[FZ07]. In [HL16a] they established a new algorithm to compute q-characters of a particular class of irreducible modules, called *Kirillov-Reshetikhin modules*, which include the fundamental modules, using the cluster algebra structure of the Grothendieck ring of a subcategory of the category of finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules. The picture is completed when put into the broader context of the category $\mathcal{O}^+$ of representations of $\mathcal{U}_q(\hat{\mathfrak{g}})$, introduced by Hernandez-Jimbo in [HJ12]. In [HL16b], Hernandez and Leclerc showed that the Grothendieck ring of this category, which contains the finite-dimensional representations, is isomorphic to a cluster algebra built on an infinite quiver, while explicitly giving the identification.

In Chapter 2, the author defined the quantum Grothendieck ring for this category $\mathcal{O}^+$ of representations as a quantum cluster algebra, as defined by Berenstein and Zelevinsky [BZ05]. However, the question of whether this quantum Grothendieck ring contained the quantum Grothendieck of the category of finite-dimensional representation, as used by Nakajima, was only proven in type A, and remained conjectural for other types.

In this Chapter we propose to show that, when $\mathfrak{g}$ is of simply-laced type, the quantum Grothendieck ring of a certain monoidal subcategory of the category of finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules has a quantum cluster algebra structure (Proposition 3.6.7). The proof relies heavily on a family of relations satisfied by the (q,t)-characters of the Kirillov-Reshetikhin modules called *quantum T-systems* proved in [Nak03a]. These relations are t-deformations of the T-systems relations, first stated in [KNS94]. These relations have not been generalized to non-simply-laced types, except for type B_n in [HO19]. This is the main reason why the results of this chapter are limited to ADE types. This quantum cluster algebra approach gives a new algorithm to compute the (q,t)-characters of the Kirillov-Reshetikhin modules, and in particular of the fundamental modules (see Proposition 3.5.2). This algorithm seems more efficient, at least in terms of number of steps, than the Frenkel-Mukhin algorithm (see Remark 3.6.10).

For certain subcategories of the category of finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules generated by a finite number of fundamental modules, Qin obtained in [Qin16] in a different context results similar to some results whose direct proofs are given here (see Remark 3.4.7). In this present work, we give explicit sequences of mutations to obtain (q,t)-characters of fundamental modules.

Next, we use this new result to prove a conjecture that was stated by the author in Chapter 2. This previous work dealt with a category $\mathcal{O}^+$ of representations of the Borel subalgebra of the quantum affine algebra, which contains the finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules. The quantum Grothendieck ring of this category was defined as a quantum cluster algebra, and it was conjectured that this ring contained the quantum Grothendieck ring of the category of finite-dimensional representations. Here, we show that the quantum cluster algebra considered in Chapter 2 can be seen as a twisted version (in the sense of [GL14]) of the quantum cluster algebra occurring in the finite-dimensional case (see Proposition 3.6.3). As an application, the (q,t)-characters of the fundamental modules are obtained as quantum cluster variables in the quantum Grothendieck ring of the category $\mathcal{O}^+$ (Proposition 3.6.7), and the inclusion of quantum Grothendieck rings conjectured in Chapter 2 follows naturally (Theorem 3.6.5 and Corollary 3.6.9).

Note that these results extend the algorithm to compute (q,t)-characters of some simple modules in the category $\mathcal{O}^+$. However, for this category of representations, the question of defining analogs of standard modules remains open. The author tackled this question in Chapter 1, and gave a complete answer when the underlying simple Lie algebra is $\mathfrak{g} = \mathfrak{sl}_2$. This work is also a partial answer to the first point of Nakajima's "to do" list from [Nak11].

The author would also like to note that in type A, parallel results to the ones presented here were proven in [Tur18], via a different approach. In this work, (q,t)-characters of Kirillov-Reshetikhin modules are also obtained as quantum cluster variables in some quantum cluster algebra, the method uses a generalization of the tableaux-sum notations introduced by Nakajima in [Nak03b].

Finally, we use this algorithm to explicitly compute, when $\mathfrak{g}$ is of type D_4, the (q,t)-character of the fundamental representation at the trivalent node.

This chapter is organized as follows. In Section 3.3, we recall results regarding the t-deformation of Grothendieck rings, such as (q,t)-characters and quantum T-systems. In Section 3.4 we prove the existence of a quantum cluster algebra $\mathcal{A}_t$ with t-commutations relations coherent with the framework of (q,t)-characters. Then, in Section 3.5 we prove that this quantum cluster algebra is isomorphic to the quantum Grothendieck ring of a certain monoidal subcategory of the category of finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules; in this process, we established an algorithm to compute (q,t)-characters of Kirillov-Reshetikhin modules. Section 3.6 is devoted to the category $\mathcal{O}^+$, and to the proof of the inclusion Conjecture of Chapter 2. Finally, the explicit computation mentioned just above is done in Section 3.7.

3.2 Finite-dimensional representations of quantum affine algebras

We use the notations from Appendix A.

In this section, we recall the notations and different results regarding quantum affine algebras and finite-dimensional representations of quantum affine algebras.

3.2.1 Quantum affine algebra

Let $\hat{\mathfrak{g}}$ be the untwisted affine Lie algebra corresponding to $\mathfrak{g}$.

Fix an nonzero complex number q, which is not a root of unity, and $h \in \mathbb{C}$ such that $q = e^h$. Then for all $r \in \mathbb{Q}, q^r := e^{rh}$. Since q is not a root of unity, for $r, s \in \mathbb{Q}$, we have $q^r = q^s$ if and only if $r = s$.

Let $\mathcal{U}_q(\hat{\mathfrak{g}})$ be the *quantum enveloping algebra* of the Lie algebra $\hat{\mathfrak{g}}$ (see [CP95a]), it is a $\mathbb{C}$-Hopf algebra.

3.2.2 Finite-dimensional representations

Let $\mathscr{C}$ be the category of all (type 1) finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules. As $\mathcal{U}_q(\hat{\mathfrak{g}})$ is a Hopf algebra, $\mathscr{C}$ is a tensor category. The simple modules in $\mathscr{C}$ have been classified by Chari-Pressley ([CP95a]), in terms of Drinfeld polynomials.

The simple finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules are indexed by the monomials in the infinite set of variables $(Y_{i,a})_{i\in I, a\in\mathbb{C}^\times}$, called *dominant monomials* ([FR99]). For such a monomial m, let $L(m)$ denote the corresponding simple $\mathcal{U}_q(\hat{\mathfrak{g}})$-module.

We define the following sets of dominant monomials:

$$\mathcal{M} = \left\{ \prod_{\text{finite}} Y_{i,q^r}^{n_{i,r}} \mid (i,r) \in \hat{I}, n_{i,r} \in \mathbb{Z}_{\geqslant 0}, n_{i,r} = 0 \text{ except for a finite number of } (i,r) \right\},$$

$$\mathcal{M}^- = \left\{ \prod_{\text{finite}} Y_{i,q^r}^{n_{i,r}} \mid (i,r) \in \hat{I}^-, n_{i,r} \in \mathbb{Z}_{\geqslant 0}, n_{i,r} = 0 \text{ except for a finite number of } (i,r) \right\}.$$

Definition 3.2.1. Let $\mathscr{C}_\mathbb{Z}$ be the full subcategory of $\mathscr{C}$ of objects whose composition factors are of the form $L(m)$, with $m \in \mathcal{M}$.

Let $\mathscr{C}_\mathbb{Z}^-$ be the full subcategory of $\mathscr{C}$ of objects whose composition factors are of the form $L(m)$, with $m \in \mathcal{M}^-$.

The category $\mathscr{C}_\mathbb{Z}$ is a tensor category and $\mathscr{C}_\mathbb{Z}^-$ is a monoidal category ([HL10, 5.2.4] and [HL16a, Proposition 3.10]).

Remark 3.2.2. Every simple object in $\mathscr{C}$ can be written as a tensor product of simple objects which are essentially in $\mathscr{C}_\mathbb{Z}$ (see [HL10, Section 3.7]). Thus, the description of the simple objects of $\mathscr{C}$ reduces to the description of the simple objects of $\mathscr{C}_\mathbb{Z}$.

Let us introduce some particular irreducible finite-dimensional representations.

Definition 3.2.3. For all $(i,r) \in \hat{I}$, $V_{i,r} := L(Y_{i,r})$ is called a *fundamental module.*

For all $(i,r) \in \hat{I}, k \in \mathbb{Z}_{>0}$, let

$$m_{k,r}^{(i)} := Y_{i,r}Y_{i,r+2}\cdots Y_{i,r+2k-2}, \tag{3.2.2.1}$$

the corresponding irreducible module $L(m_{k,r}^{(i)})$ is called a *Kirillov-Reshetikhin module*, or *KR-module*, and denote by

$$W_{k,r}^{(i)} := L(m_{k,r}^{(i)}). \tag{3.2.2.2}$$

Note that fundamental module are particular KR-modules, for $k = 1, m_{1,r}^{(i)} = Y_{i,r}$.

3.2.3 q-characters and truncated q-characters

Frenkel and Reshetikhin introduced in [FR99] an injective ring morphism, called the *q-character* morphism, on the Grothendieck ring $K_0(\mathscr{C})$ of the category $\mathscr{C}$:

$$\chi_q : K_0(\mathscr{C}) \to \mathbb{Z}\left[Y_{i,a}^{\pm 1} \mid i \in I, a \in \mathbb{C}^\times\right]. \tag{3.2.3.1}$$

Moreover, the q-character $\chi_q(V)$ of a $\mathcal{U}_q(\hat{\mathfrak{g}})$-module V gives information about the decomposition into Jordan subspaces for the action of a large commutative subalgebra of $\mathcal{U}_q(\hat{\mathfrak{g}})$.

Here, as we restrict ourselves to the study of the category $\mathscr{C}_{\mathbb{Z}}$, the q-character will only involve variables $Y_{i,q^r}^{\pm 1}$, for $(i,r) \in \hat{I}$. Hence, for simplicity of notation we denote them by: $Y_{i,r} := Y_{i,q^r}$. The q-character we are interested in is the injective ring morphism:

$$\chi_q : K_0(\mathscr{C}_{\mathbb{Z}}) \to \mathcal{Y} := \mathbb{Z}\left[Y_{i,r}^{\pm 1} \mid (i,r) \in \hat{I}\right]. \tag{3.2.3.2}$$

We use the usual notation [FR99],

$$A_{i,r} := Y_{i,r-1}Y_{i,r+1}\left(\prod_{j\sim i} Y_{j,r,}\right)^{-1}. \tag{3.2.3.3}$$

for all $(i,r) \in \hat{J}$ (see (A.4.3) in the Appendix). Note that $A_{i,r}$ is a Laurent monomial in the variables $Y_{j,s}$, with $(j,s) \in \hat{I}$. The monomials $A_{i,r}$ are analogs of the simple roots.

Proposition 3.2.4. *[FM01, Theorem 4.1] For m a dominant monomial in $\mathcal{M}$, the q-character of the finite-dimensional irreducible representation $L(m)$ is of the form*

$$\chi_q(L(m)) = m\left(1 + \sum_p M_p\right), \tag{3.2.3.4}$$

where M_p is a monomial in the variables $A_{i,r}^{-1}$, with $(i,r) \in \hat{J}$.

Let us recall Nakajima's partial order on monomials. For m and m' Laurent monomials in $\mathcal{Y}$,

$$m \leqslant m' \iff m'm^{-1} \text{ is a product of } A_{i,r}, \text{ with } (i,r) \in \hat{I}. \tag{3.2.3.5}$$

Remark 3.2.5. Note that Proposition 3.2.4 can be translated as follows: for all dominant monomials m, the monomials occurring in the q-character of the finite-dimensional irreducible representation $L(m)$ are lower than m, for Nakajima's partial order.

We also recall the *truncated q-characters* from [HL10]. For m a monomial in $\mathcal{M}^-$, the q-character $\chi_q(L(m))$ may contain Laurent monomials in which variables $Y_{i,r}$, with $(i,r) \in \hat{I}\backslash\hat{I}^-$ occur. Let $\chi_q^-(L(m))$ be the Laurent polynomial obtained from $\chi_q(L(m))$ by removing any such Laurent monomial. By definition

$$\chi_q^-(L(m)) \in \mathbb{Z}\left[Y_{i,r}^{\pm 1} \mid (i,r) \in \hat{I}^-\right]. \tag{3.2.3.6}$$

Example 3.2.6. For $\mathfrak{g}$ of type A_2, one has

$$\begin{aligned}
\chi_q^-(L(Y_{1,0})) &= Y_{1,0},\\
\chi_q^-(L(Y_{1,-2})) &= Y_{1,-2} + Y_{1,0}^{-1}Y_{2,-1},\\
\chi_q^-(L(Y_{1,-4})) &= Y_{1,-4} + Y_{1,-2}^{-1}Y_{2,-3} + Y_{2,-1}^{-1} = \chi_q(L(Y_{1,-4})).
\end{aligned}$$

Proposition 3.2.7. *[HL16a, Proposition 3.10] The assignment $[L(m)] \mapsto \chi_q^-(L(m))$ extends to an injective ring homomorphism*

$$K_0(\mathscr{C}_{\mathbb{Z}}^-) \to \mathbb{Z}\left[Y_{i,r}^{\pm 1} \mid (i,r) \in \hat{I}^-\right]. \tag{3.2.3.7}$$

As such, all simple modules in $\mathscr{C}_{\mathbb{Z}}^-$ are identified with their isoclasses through the truncated q-character morphism.

3.2.4 Cluster algebra structure

One of the main ingredient we want to use in this work is the *cluster algebra* structure of the Grothendieck ring of the category $\mathscr{C}_{\mathbb{Z}}^-$. See Appendix B for notations and results for cluster algebras and quantum cluster algebras.

Consider the cluster algebra $\mathcal{A} := \mathcal{A}(\boldsymbol{u}, G^-)$, with initial seed $(\boldsymbol{u}, G^-)$, where

- $\boldsymbol{u}$ are initial cluster variables indexed by $\hat{I}^-$, $\boldsymbol{u} = \left\{u_{i,r} \mid (i,r) \in \hat{I}^-\right\}$,
- G^- is the semi-infinite quiver with vertex set $\hat{I}^-$ defined in the previous section.

Consider the identification, for all $(i,r) \in \hat{I}^-$,

$$u_{i,r} \equiv \prod_{\substack{k \geqslant 0 \\ r+2k \leqslant 0}} Y_{i,r+2k}. \tag{3.2.4.1}$$

This identification makes sense, as these monomials are algebraically independent.

From the *Laurent phenomenon*, we know that all cluster variables of $\mathcal{A}$ are Laurent polynomials in the variables $u_{i,r}$. Thus, via the identification (3.2.4.1), $\mathcal{A}$ is seen as a subring of $\mathbb{Z}\left[Y_{i,r}^{\pm 1} \mid (i,r) \in \hat{I}^-\right]$.

Theorem 3.2.8. *[HL16a, Theorem 5.1] The injective ring homomorphism χ_q^- is an isomorphism between the Grothendieck ring of the category $\mathscr{C}_{\mathbb{Z}}^-$ and the cluster algebra $\mathcal{A}$, after identification (3.2.4.1):*

$$\chi_q^- : K_0(\mathscr{C}_{\mathbb{Z}}^-) \xrightarrow{\sim} \mathcal{A}. \tag{3.2.4.2}$$

Moreover, truncated q-characters of Kirillov-Reshetikhin modules can be obtained as cluster variables, via the identification of initial seed (3.2.4.1), it is the main result of [HL16a].

3.3 Quantum Grothendieck rings

We will recall in this section the definition of the quantum Grothendieck ring of the category $\mathscr{C}_{\mathbb{Z}}$, introduce that of the category $\mathscr{C}_{\mathbb{Z}}^-$, and study those rings.

Let t be an indeterminate. The quantum Grothendieck rings of the categories $\mathscr{C}_{\mathbb{Z}}$ and $\mathscr{C}_{\mathbb{Z}}^-$ are non-commutative t-deformations of the Grothendieck rings.

3.3.1 Quantum torus

Let $\mathbf{Y_t}$ be the $\mathbb{Z}[t^{\pm 1}]$-algebra generated by the variables $Y_{i,r}^{\pm 1}$, for $(i,r) \in \hat{I}$, and the t-commutations relations:

$$Y_{i,r} * Y_{j,s} = t^{\mathcal{N}_{i,j}(r-s)} Y_{j,s} * Y_{i,r},$$

where $\mathcal{N}_{i,j} : \mathbb{Z} \to \mathbb{Z}$ is the antisymmetrical map:

$$\mathcal{N}_{i,j}(m) = \mathbf{C}_{i,j}(m+1) - \mathbf{C}_{i,j}(m-1), \quad \forall m \in \mathbb{Z},$$

using the notations from Section A.2 of the Appendix.

Remark 3.3.1. Here we work with the quantum torus of [Her04] and [HL15], which is slightly different from the original quantum torus used to define the quantum Grothendieck ring in [Nak04] and [VV03].

Example 3.3.2. If we use Example A.2.3 of the Appendix, for $\mathfrak{g} = \mathfrak{sl}_2$, in this case, $\hat{I} = (1, 2\mathbb{Z})$, for $r \in \mathbb{Z}$, one has

$$Y_{1,2r} * Y_{1,2s} = t^{2(-1)^{s-r}} Y_{1,2s} * Y_{1,2r}, \quad \forall s > r > 0. \tag{3.3.1.1}$$

The $\mathbb{Z}(t)$-algebra $\mathbf{Y_t}$ is viewed as a quantum torus of infinite rank.

We extend this quantum torus by adjoining a fixed square root $t^{1/2}$ of t:

$$\mathcal{Y}_t := \mathbb{Z}[t^{1/2}] \otimes_{\mathbb{Z}[t]} \mathbf{Y_t}. \tag{3.3.1.2}$$

Let $\mathcal{Y}_t^-$ be the quantum torus defined exactly the same way, except by only taking as generators the $Y_{i,r}^{\pm 1}$, for $(i,r) \in \hat{I}^-$.

Let us denote by

$$\pi : \mathcal{Y}_t \to \mathcal{Y}_t^-, \tag{3.3.1.3}$$

the projection of $\mathcal{Y}_t$ onto $\mathcal{Y}_t^-$,

$$\pi(Y_{i,r}) = 0 \text{ if } (i,r) \in \hat{I} \backslash \hat{I}^-. \tag{3.3.1.4}$$

Remark 3.3.3. Even if $\mathcal{Y}_t^-$ is an infinite rank quantum torus, it can be seen as a limit of finite rank quantum tori. As finite rank quantum tori are of polynomial growth, they are Ore domains (see [IM94]). Moreover, the Ore condition being local (any pair of elements of $\mathcal{Y}_t^-$ belongs to some sufficiently larger finite rank quantum torus), $\mathcal{Y}_t^-$ is an Ore domain. Hence we can consider its skew field of fractions $\mathcal{F}_t$.

3.3.2 Commutative monomials

For a family of integers with finitely many non-zero components $(u_{i,r})_{(i,r)\in\hat{I}}$, define the *commutative monomial* $\prod_{(i,r)\in\hat{I}} Y_{i,r}^{u_{i,r}}$ as

$$\prod_{(i,r)\in\hat{I}} Y_{i,r}^{u_{i,r}} := t^{\frac{1}{2}\sum_{(i,r)<(j,s)} u_{i,r}u_{j,s}\mathcal{N}_{i,j}(r,s)} \overrightarrow{\ast}_{(i,r)\in\hat{I}} Y_{i,r}^{u_{i,r}}, \tag{3.3.2.1}$$

where on the right-hand side an order on $\hat{I}$ is chosen so as to give meaning to the sum, and the product $*$ is ordered by it (notice that the result does not depend on the order chosen).

The commutative monomials form a basis of the $\mathbb{Z}[t^{1/2}]$-vector space $\mathcal{Y}_t$.

The non-commutative product of two commutative monomials m_1 and m_2 in $\mathcal{Y}_t$ is given by:

$$m_1 * m_2 = t^{D(m_1,m_2)} m_2 * m_1 = t^{\frac{1}{2}D(m_1,m_2)} m_1 m_2, \tag{3.3.2.2}$$

where $m_1 m_2$ denotes the commutative product of the monomials, and

$$D(m_1, m_2) = \sum_{(i,r),(j,s)\in\hat{I}} u_{i,r}(m_1) u_{j,s}(m_2) \mathcal{N}_{i,j}(r,s). \tag{3.3.2.3}$$

3.3.3 Quantum Grothendieck ring $K_t(\mathscr{C}_{\mathbb{Z}})$

We define the quantum Grothendieck ring $K_t(\mathscr{C}_{\mathbb{Z}})$ of the category $\mathscr{C}_{\mathbb{Z}}$ as in [HL15, Section 5.4] (see Remark 3.3.1 for original references).

For all $(i,r) \in \hat{J}$, let $A_{i,r}$ denote the commutative monomial in $\mathcal{Y}_t$ defined as in (3.2.3.3):

$$A_{i,r} := Y_{i,r-1}Y_{i,r+1}\left(\prod_{j\sim i} Y_{j,r}\right)^{-1}.$$

For all $i \in I$, define $K_{i,t}$ the subring of $\mathcal{Y}_t$ generated by the

$$Y_{i,r}\left(1 + A_{i,r+1}^{-1}\right), \quad Y_{j,s}^{\pm 1} \quad \left((i,r),(j,s) \in \hat{I}, j \neq i\right). \tag{3.3.3.1}$$

In [Her03], the $K_{i,t}$ are defined as kernels of t-deformed screening operators, motivated by the results in [FR99]. Let us detail this, as it will be important in the proof of the main result. For all $i \in I$, define the free $\mathcal{Y}_t$-modules

$$\mathcal{Y}_{t,i}^{l} := \sum_{r\in\mathbb{Z}|(i,r)\in\hat{I}} \mathcal{Y}_t \cdot S_{i,r}.$$

Then let $\mathcal{Y}_{t,i}$ be the quotient of $\mathcal{Y}_{t,i}^{l}$ by the left-$\mathcal{Y}_t$-module generated by the elements

$$Q_{i,r} := A_{i,r+1}^{-1}S_{i,r+2} - t^2 S_{i,r}, \quad \forall (i,r) \in \hat{I}.$$

Lemma 3.3.4. *For all $i \in I$, the module $\mathcal{Y}_{t,i}$ is free.*

Proof. The elements $Q_{i,r}$ are linearly independent and for all r_0 such that $(i,r_0) \in \hat{I}$ fixed, the set $\{Q_{i,r}, S_{i,r_0} \mid (i,r) \in I\}$ forms a basis of $\mathcal{Y}_{t,i}^{l}$.

Hence $\mathcal{Y}_{t,i}$ is a quotient of a free module by a submodule generated by elements of a basis, thus it is free. □

From [Her03], for all $i \in I$, there exists a $\mathbb{Z}[t^{\pm 1/2}]$-linear map

$$S_{i,t} : \mathcal{Y}_t \to \mathcal{Y}_{i,t}, \tag{3.3.3.2}$$

which is a derivation and such that

$$K_{i,t} = \ker(S_{i,t}). \tag{3.3.3.3}$$

Finally, let

$$K_t(\mathscr{C}_{\mathbb{Z}}) := \bigcap_{i\in I} K_{i,t}. \tag{3.3.3.4}$$

From [Nak01b] and [Her04] we know that for all dominant monomials $m \in \mathcal{M}$, there exists a unique element $F_t(m) \in K_t(\mathscr{C}_{\mathbb{Z}})$ such that m occurs in $F_t(m)$ with multiplicity 1 and no other dominant monomial occurs in $F_t(m)$. Thus, all elements of $K_t(\mathscr{C}_{\mathbb{Z}})$ are characterized by the coefficients of their dominant monomials.

The $F_t(m)$ linearly generate $K_t(\mathscr{C}_{\mathbb{Z}})$.

Remark 3.3.5. For all $(i,r) \in \hat{I}^{-}$,

$$F_t(Y_{i,r}) = [L(Y_{i,r})]_t. \tag{3.3.3.5}$$

The $[L(Y_{i,r})]_t$ generate $K_t(\mathscr{C}_{\mathbb{Z}}^{-})$ algebraically.

3.3.4 (q,t)-characters

For a dominant monomial $m \in \mathcal{M}$, write it as a commutative monomial in $\mathcal{Y}_t$:

$$m = \prod_{(i,r)\in\hat{I}} Y_{i,r}^{n_{i,r}(m)} \quad \in \mathcal{Y}_t. \tag{3.3.4.1}$$

Define

$$[M(m)]_t := t^{\alpha(m)} \overleftarrow{\circledast}_{r\in\mathbb{Z}} F\left(\prod_{i\in I} Y_{i,r}^{n_{i,r}(m)}\right) \quad \in K_t(\mathscr{C}_{\mathbb{Z}}), \tag{3.3.4.2}$$

where $\alpha(m) \in \frac{1}{2}\mathbb{Z}$ is fixed such that m occurs with multiplicity one in the expansion of $[M(m)]_t$ on the basis of the commutative monomials of $\mathcal{Y}_t$, and the product $\overleftarrow{\circledast}$ is taken with decreasing $r \in \mathbb{Z}$.

In particular, from (3.3.3.5), for all $(i,r) \in \hat{I}$,

$$[L(Y_{i,r})]_t = [M(Y_{i,r})]_t. \tag{3.3.4.3}$$

One has, for all $m \in \mathcal{M}$,

$$[M(m)]_t \xrightarrow{t=1} \chi_q(M(m)). \tag{3.3.4.4}$$

This result is a direct consequence of the definition of $[M(m)]_t$, as it is satisfied for the fundamental modules $[L(Y_{i,r})]_t = F_t(Y_{i,r})$. Thus $[M(m)]_t$ is called the *(q,t)-character* of the standard module $M(m)$.

As in [Nak04], we consider the $\mathbb{Z}$-algebra anti-automorphism $\overline{}$ of $\mathcal{Y}_t$ defined by:

$$\overline{t^{1/2}} = t^{-1/2}, \quad \overline{Y_{i,r}} = Y_{i,r}, \quad \left((i,r) \in \hat{I}\right). \tag{3.3.4.5}$$

This map is called the *bar-involution*.

Theorem 3.3.6. *[Nak04] There exists a unique family $\{[L(m)]_t\}_{m\in\mathcal{M}}$ of elements of $K_t(\mathscr{C}_{\mathbb{Z}})$ such that, for all $m \in \mathcal{M}$,*
- $\overline{[L(m)]_t} = [L(m)]_t$,
- $[L(m)]_t \in [M(m)]_t + \sum_{m'<m} t^{-1}\mathbb{Z}[t^{-1}][M(m')]_t$, *where $m' < m$ for Nakajima's partial order (3.2.3.5).*

The following Theorem extends (3.3.4.4), but more importantly gives an algorithm, similar to the Kazhdan-Lusztig algorithm, to compute the (q,t)-characters (and thus the q-characters) of the simple modules.

Theorem 3.3.7. *[Nak04, Corollary 3.6] The evaluation at $t = 1$ of the (q,t)-characters recovers the q-characters. For all $m \in \mathcal{M}$,*

$$[L(m)]_t \xrightarrow{t=1} \chi_q(L(m)) \quad \in \mathcal{Y}.$$

Moreover, the coefficients of the expansion of $[L(m)]_t$ as a linear combination of Laurent monomials in the variables $(Y_{i,r})_{(i,r)\in\hat{I}}$ belong to $\mathbb{N}[t^{\pm 1}]$.

Note that the positivity result of this Theorem has only been proven for ADE types as yet.

3.3.5 Truncated (q,t)-characters and quantum Grothendieck ring $K_t(\mathscr{C}_{\mathbb{Z}}^-)$

As in Section 3.2.3, one can define truncated versions of the (q,t)-characters.

For all dominant monomials m in $\mathcal{M}^-$, let $[L(m)]_t^-$ be the Laurent polynomial obtained from $[L(m)]_t$ by removing any term in which a variable $Y_{i,r}$, with $(i,r) \in \hat{I}\backslash\hat{I}^-$ occurs:

$$[L(m)]_t^- = \pi\left([L(m)]_t\right) \quad \in \mathcal{Y}_t^-, \tag{3.3.5.1}$$

where π is the projection defined in (3.3.1.3).

Define $K_t(\mathscr{C}_{\mathbb{Z}}^-)$ as the $\mathbb{Z}[t^{1/2}]$-submodule of $\mathcal{Y}_t^-$ generated by the truncated (q,t)-characters $[L(m)]_t^-$ of the simple finite-dimensional modules $L(m)$ in the category $\mathscr{C}_{\mathbb{Z}}^-$.

Lemma 3.3.8. *The quantum Grothendieck ring $K_t(\mathscr{C}_{\mathbb{Z}}^-)$ is actually a subalgebra of $\mathcal{Y}_t^-$. Moreover, it is algebraically generated by the truncated (q,t)-characters of the fundamental modules:*

$$K_t(\mathscr{C}_{\mathbb{Z}}^-) = \left\langle [L(Y_{i,r})]_t^- \mid (i,r) \in \hat{I}^- \right\rangle. \tag{3.3.5.2}$$

Proof. For every dominant monomials $m_1, m_2 \in \mathcal{M}$, one can write:

$$[L(m_1)]_t * [L(m_2)]_t = \sum_{m \in \mathcal{M}} c_{m1,m2}^m(t^{1/2})[L(m)]_t. \tag{3.3.5.3}$$

Hence the image of (3.3.5.3) by the projection π of (3.3.1.3) is:

$$[L(m_1)]_t^- * [L(m_2)]_t^- = \sum_{m \in \mathcal{M}} c_{m1,m2}^m(t^{1/2})[L(m)]_t^-.$$

Thus $K_t(\mathscr{C}_{\mathbb{Z}}^-)$ is stable by products.

By definition the truncated (q,t)-characters of the fundamental modules $L(Y_{i,r})$, for $(i,r) \in \hat{I}^-$ belong to $K_t(\mathscr{C}_{\mathbb{Z}}^-)$.

Reciprocally, the (q,t)-characters of the fundamental modules $L(Y_{i,r})$, for all $(i,r) \in \hat{I}$, algebraically generate the quantum Grothendieck ring $K_t(\mathscr{C}_{\mathbb{Z}})$ (see remark 3.3.5). Hence the truncated (q,t)-characters $[L(Y_{i,r})]_t^-$, for all $(i,r) \in \hat{I}$, algebraically generate $K_t(\mathscr{C}_{\mathbb{Z}}^-)$.

From Proposition 3.2.4 and Theorem 3.3.6, for all dominant monomials $m \in \mathcal{M}$, the (q,t)-character of the simple representation $L(m)$ is of the form

$$[L(m)]_t = m\left(1 + \sum_p M_p\right),$$

where M_p is a monomial in the variables $(A_{i,r}^{-1})_{i,r)\in\hat{J}}$, with coefficients in $\mathbb{Z}[t^{\pm 1}]$. Thus,

$$\pi\left([L(Y_{i,r})]_t\right) = 0, \text{ if } (i,r) \in \hat{I}\backslash\hat{I}^-.$$

Hence, $K_t(\mathscr{C}_{\mathbb{Z}}^-)$ is algebraically generated by the $[L(Y_{i,r})]_t^-$, with $(i,r) \in \hat{I}^-$. $\square$

$K_t(\mathscr{C}_{\mathbb{Z}}^-)$ is a t-deformed version of the Grothendieck ring of the category $\mathscr{C}_{\mathbb{Z}}^-$, in the sense that the evaluation $[L(m)]_t^- \xrightarrow{t=1} \chi_q^-(L(m))$ extends to a ring homomorphism

$$K_t(\mathscr{C}_{\mathbb{Z}}^-) \xrightarrow{t=1} K_0(\mathscr{C}_{\mathbb{Z}}^-), \tag{3.3.5.4}$$

where $K_0(\mathscr{C}_{\mathbb{Z}}^-)$ is identified with its image under the truncated q-character (3.2.3.7), which is an injective map.

3.3.6 Quantum T-systems

The (q,t)-characters of the Kirillov-Reshetikhin modules satisfy some algebraic relations called *quantum T-systems*. Those are t-deformed versions of the T-system relations, which are satisfied by the q-characters of the KR-modules [KNS94, Nak04, Her06].

Proposition 3.3.9. *[Nak03a][HL15, Proposition 5.6] For all $(i,r) \in \hat{I}$ and $k \in \mathbb{Z}_{>0}$, the following relation holds in $K_t(\mathscr{C}_\mathbb{Z})$:*

$$[W_{k,r}^{(i)}]_t * [W_{k,r+2}^{(i)}]_t = t^{\alpha(i,k)}[W_{k-1,r+2}^{(i)}]_t * [W_{k+1,r}^{(i)}]_t + t^{\gamma(i,k)} \mathop{\ast}_{j\sim i}[W_{k,r+1}^{(j)}]_t, \tag{3.3.6.1}$$

where

$$\alpha(i,k) = -1 + \frac{1}{2}\left(\tilde{C}_{ii}(2k-1) + \tilde{C}_{ii}(2k+1)\right), \quad \gamma(i,k) = \alpha(i,k) + 1. \tag{3.3.6.2}$$

Remark 3.3.10. First of all, one notices that the dominant monomials of $W_{k-1,r+2}^{(i)}$ and $W_{k+1,r}^{(i)}$ commute:

$$m_{k-1,r+2}^{(i)} * m_{k+1,r}^{(i)} = m_{k+1,r}^{(i)} * m_{k-1,r+2}^{(i)}. \tag{3.3.6.3}$$

Moreover, the tensor product of the KR-modules $W_{k-1,r+2}^{(i)} \otimes W_{k+1,r}^{(i)}$ is irreducible (this result is proved in [Cha02] and also by explicit computation of its (q,t)-character in [Nak03a]). Thus their respective (q,t)-characters t-commute (see [HL15, Corollary 5.5]). As their dominant monomials commute, these (q,t)-characters in fact commute and their product can be written as a commutative product, as in Section 3.3.2.

By the same arguments, for $j \sim i$, the (q,t)-characters $[W_{k,r+1}^{(j)}]_t$ commute so the order of the factors in $\ast_{j\sim i}$ in (3.3.6.1) does not matter.

By taking the image of (3.3.6.1) through the projection π of (3.3.1.3), one obtains the following relation in $K_t(\mathscr{C}_\mathbb{Z}^-)$. For all $(i,r) \in \hat{I}$ and $k \in \mathbb{Z}_{>0}$,

$$[W_{k,r}^{(i)}]_t^- * [W_{k,r+2}^{(i)}]_t^- = t^{\alpha(i,k)}[W_{k-1,r+2}^{(i)}]_t^- \cdot [W_{k+1,r}^{(i)}]_t^- + t^{\gamma(i,k)} \prod_{j\sim i}[W_{k,r+1}^{(j)}]_t^-, \tag{3.3.6.4}$$

where $\alpha(i,k)$ and $\gamma(i,k)$ are defined in (3.3.6.2).

Note that in (3.3.6.4), the products appearing on the left-hand side are commutative products, which are well-defined from Remark 3.3.10. Hence the change of notations since (3.3.6.1).

3.4 Quantum cluster algebra structure

We define in this section the quantum cluster algebra structure built within the quantum torus $\mathcal{Y}_t^-$.

3.4.1 A compatible pair

For all $(i,r) \in \hat{I}^-$, the variable $u_{i,r}$, written as in (3.2.4.1), can be seen as commutative monomial in $\mathcal{Y}_t^-$. Define

$$U_{i,r} := \prod_{\substack{k\geqslant 0 \\ r+2k\leqslant 0}} Y_{i,r+2k} \quad \in \mathcal{Y}_t^-.$$

They satisfy the following t-commutation relations. For all $((i,r),(j,s)) \in (\hat{I}^-)^2$,

$$U_{i,r} * U_{j,s} = t^{L((i,r),(j,s))} U_{j,s} * U_{i,r}, \tag{3.4.1.1}$$

where

$$L\left((i,r),(j,s)\right) = \sum_{\substack{k \geqslant 0 \\ r+2k \leqslant 0}} \sum_{\substack{l \geqslant 0 \\ s+2l \leqslant 0}} \mathcal{N}_{ij}(s+2l-r-2k). \tag{3.4.1.2}$$

Let B_- be the $\hat{I}^- \times \hat{I}^-$-matrix encoding the quiver G^-, for all $((i,r),(j,s)) \in (\hat{I}^-)^2$:

$$B_-\left((i,r),(j,s)\right) = |\{ \text{ arrows } (i,r) \to (j,s) \text{ in } G^- \}| - |\{ \text{ arrows } (j,s) \to (i,r) \text{ in } G^- \}|. \tag{3.4.1.3}$$

Let L be the $\hat{I}^- \times \hat{I}^-$ skew-symmetric matrix

$$L := \left(L\left((i,r),(j,s)\right)\right)_{((i,r),(j,s)) \in (\hat{I}^-)^2}. \tag{3.4.1.4}$$

The pair of $\hat{I}^- \times \hat{I}^-$-matrices (L, B_-) forms a *compatible pair*, in the sense of quantum cluster algebras.

More precisely, we prove the following.

Proposition 3.4.1. *For all* $((i,r),(j,s)) \in (\hat{I}^-)^2$,

$$\left(B_-^T L\right)\left((i,r),(j,s)\right) = \begin{cases} -2 & \text{if } (i,r) = (j,s) \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.1.5}$$

Remark 3.4.2. In [BZ05], by definition a pair of $J \times J$-matrices (Λ, B) forms a compatible pair if ${}^T BL$ is a diagonal matrix with positive integer coefficients. But as explained in Chapter 2, quantum cluster algebras can be built exactly the same way given as data a pair (Λ, B) such that ${}^T BL$ is a diagonal matrix with integer coefficients with constant signs.

Proof. Fix $((i,r),(j,s)) \in (\hat{I}^-)^2$, there are different cases to consider.

- If $r \leqslant -2$, one has:

$$\begin{aligned}\left(B_-^T L\right)\left((i,r),(j,s)\right) &= L\left((i,r-2),(j,s)\right) - L\left((i,r+2),(j,s)\right) \\ &\quad + \sum_{k \sim i} \left(L\left((k,r+1),(j,s)\right) - L\left((k,r-1),(j,s)\right)\right).\end{aligned}$$

One has

$$\begin{aligned}L\left((i,r-2),(j,s)\right) - L\left((i,r+2),(j,s)\right) &= -\mathbf{C}_{ij}(s-r-1) - \mathbf{C}_{ij}(s-r+1) \\ &\quad + \mathbf{C}_{ij}(-r+3-\xi_j) + \mathbf{C}_{ij}(-r+1-\xi_j),\end{aligned}$$

where $\xi : I \to \{0,1\}$ is the height function on the Dynkin diagram of $\mathfrak{g}$ fixed in Section A.3 of the Appendix.
On the other hand, for all $k \sim i$, one has

$$L\left((k,r+1),(j,s)\right) - L\left((k,r-1),(j,s)\right) = \mathbf{C}_{kj}(s-r) - \mathbf{C}_{kj}(-r+2-\xi_j).$$

Thus, with the reformulation (A.2.6) of Lemma A.2.4 of the Appendix,

$$\left(B_-^T L\right)\left((i,r),(j,s)\right) = \begin{cases} -2\delta_{i,j} & \text{if } s = r, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.1.6}$$

- If $r = -1$, one has:

$$\left(B_-^T L\right)((i,-1),(j,s)) = L((i,-3),(j,s)) + \sum_{k\sim i}\left(L((k,0),(j,s)) - L((k,-2),(j,s))\right).$$

However,

$$\begin{aligned} L((i,-3),(j,s)) &= \sum_{\substack{l\geqslant 0\\ s+2l\leqslant 0}} \left(\mathcal{N}_{ij}(s+2l+3)+\mathcal{N}_{ij}(s+2l+1)\right), \\ &= \mathbf{C}_{ij}(4-\xi_j) + \mathbf{C}_{ij}(2-\xi_j) - \mathbf{C}_{ij}(s) - \mathbf{C}_{ij}(s+2). \end{aligned}$$

And, for all $k \sim i$,

$$L((k,0),(j,s)) - L((k,-2),(j,s)) = \mathbf{C}_{ij}(s+1) - \mathbf{C}_{ij}(3-\xi_j).$$

Thus, with relation (A.2.6),

$$\left(B_-^T L\right)((i,-1),(j,s)) = \begin{cases} -2\delta_{i,j} & \text{if } s=-1, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.1.7}$$

- If $r = 0$, one has

$$\left(B_-^T L\right)((i,0),(j,s)) = L((i,-2),(j,s)) - \sum_{k\sim i} L((k,-1),(j,s)).$$

However,

$$L((i,-2),(j,s)) = \mathbf{C}_{ij}(3-\xi_j) + \mathbf{C}_{ij}(1-\xi_j) - \mathbf{C}_{ij}(s+1) - \mathbf{C}_{ij}(s-1).$$

And, for all $k \sim i$,

$$L((k,-4),(j,s)) = -\mathbf{C}_{ij}(s) + \mathbf{C}_{ij}(2-\xi_j).$$

Thus, with relation (A.2.6),

$$\left(B_-^T L\right)((i,0),(j,s)) = \begin{cases} -2\delta_{i,j} & \text{if } s=0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4.1.8}$$

The combination of the results (3.4.1.6),(3.4.1.7) and (3.4.1.8) gives the general expression (3.4.1.5). □

3.4.2 The quantum cluster algebra $\mathcal{A}_t$

Definition 3.4.3. Let T be the based quantum torus with generators $\{\, u_{i,r} \mid (i,r) \in \hat{I}^- \}$ satisfying the quasi-commutation relations (3.4.1.1):

$$u_{i,r} * u_{j,s} = t^{L((i,r),(j,s))} u_{j,s} * u_{i,r}.$$

Let $\mathcal{F}$ be the skew-field of fractions of T.

As (L, B_-) forms a compatible pair, it defines a quantum seed in $\mathcal{F}$. Let $\mathcal{S}$ be the mutation equivalence class of the quantum seed (L, B_-).

Definition 3.4.4. Let $\mathcal{A}_t$ be the quantum cluster algebra defined by the quantum seed $\mathcal{S}$, as in [BZ05].

By definition, $\mathcal{A}_t$ is a $\mathbb{Z}[t^{\pm 1/2}]$-subalgebra of $\mathcal{F}$. However, by the quantum Laurent phenomenon, $\mathcal{A}_t$ is actually a $\mathbb{Z}[t^{\pm 1/2}]$-subalgebra of the quantum torus T.

Lemma 3.4.5. *The map*

$$\eta : \begin{array}{ccc} T & \longrightarrow & \mathcal{Y}_t^- \\ u_{i,r} & \longmapsto & U_{i,r}, \end{array} \tag{3.4.2.1}$$

where the variables $U_{i,r}$ are defined in (3.2.4.1), is an isomorphism of quantum tori.

Proof. First of all, this map is well-defined because the variables $u_{i,r}$ t-commute exactly as the variables $U_{i,r}$, by definition of the matrix L (3.4.1.1).

Secondly, this map is invertible, with inverse:

$$\eta^{-1} : \begin{array}{ccc} \mathcal{Y}_t^- & \longrightarrow & T \\ Y_{i,r} & \longmapsto & \begin{cases} u_{i,r}u_{i,r+2}^{-1} & \text{if } r+2 \leqslant 0, \\ u_{i,r} & \text{otherwise,} \end{cases} \end{array} .$$

□

With this lemma, we know that the quantum cluster algebra $\mathcal{A}_t$ belongs to the quantum torus $\mathcal{Y}_t^-$. The following result is the main result of this chapter, it extends Theorem 3.2.8 to the quantum setting.

Theorem 3.4.6. *The image of the quantum cluster algebra $\mathcal{A}_t$ by the injective ring morphism η is the quantum Grothendieck ring of the category $\mathscr{C}_{\mathbb{Z}}^-$,*

$$\eta\restriction_{\mathcal{A}_t} : \mathcal{A}_t \xrightarrow{\sim} K_t(\mathscr{C}_{\mathbb{Z}}^-). \tag{3.4.2.2}$$

Moreover, the truncated (q,t)-characters of the Kirillov-Reshetikhin modules which are in $\mathscr{C}_{\mathbb{Z}}^-$ are obtained as quantum cluster variables.

The proof of this Theorem will be developed in the following section. It is mainly based on Proposition 3.5.2.

Remark 3.4.7. In [Qin16, Theorem 8.4.3], for certain subcategories of $\mathscr{C}$ generated by a finite number of fundamental modules, Qin proved that there existed an isomorphism between the quantum Grothendieck ring of the category and a quantum cluster algebra, which identifies classes of Kirillov-Reshetikhin modules to cluster variables. It is our understanding that this identification coincides with the truncated (q,t)-characters in our work. Here, the isomorphism is given explicitly, and we obtain directly the truncated (q,t)-characters.

3.5 Quantum cluster algebras and quantum Grothendieck ring

We prove in this section that the quantum Grothendieck ring of the category $\mathscr{C}_{\mathbb{Z}}^-$ is isomorphic to the quantum cluster algebra we have just defined.

3.5.1 A note on the bar-involution

The bar-involution $\overline{}$, as defined in (3.3.4.5), is a $\mathbb{Z}$-algebra anti-automorphism of the quantum torus $\mathcal{Y}_t$. The commutative monomials are invariant under this involution.

On the other hand, the quantum cluster algebra $\mathcal{A}_t$ is also equipped with a $\mathbb{Z}$-linear bar-involution morphism $\overline{}$ on its quantum torus T (see [BZ05, Section 6]), which satisfies

$$\overline{t^{1/2}} = t^{-1/2},$$
$$\overline{u_{i,r}} = u_{i,r}.$$

As noted in Section 2.5.1, these definitions are compatible. In our case, they define exactly the same involution on $\mathcal{Y}_t^-$; the following diagram is commutative:

$$\begin{tikzcd} \mathcal{Y}_t^- \arrow[r, "\overline{}"] \arrow[d, "\eta^{-1}"'] & \mathcal{Y}_t^- \\ T \arrow[r, "\overline{}"'] & T \arrow[u, "\eta"'] \end{tikzcd} \quad . \tag{3.5.1.1}$$

From [BZ05, Remark 6.4], all cluster variables are invariant under the bar-involution. Thus, the images of the quantum cluster variables in $\mathcal{A}_t$ are bar-invariant elements of $\mathcal{Y}_t^-$.

We will use the terminology "commutative products", as in Section 3.3.2 for bar-invariant elements of the quantum torus T.

3.5.2 A sequence of vertices

In [HL16a] Hernandez and Leclerc exhibited a particular sequence of mutations in the cluster algebra $\mathcal{A}(\boldsymbol{u}, G^-)$ (see Section 3.2.4) in order to obtain the truncated q-characters of all the KR-modules, up to a shift of spectral parameter.

The key idea we used was that at each step of this sequence, the exchange relation was a T-system equation.

We will recall this sequence of mutations and show that, if applied to the quantum cluster algebra $\mathcal{A}_t$, the quantum exchange relations at each step are in fact quantum T-systems relations, as in (3.3.6).

Recall the height function $\xi : I \to \{0, 1\}$ fixed on the Dynkin diagram of $\mathfrak{g}$ in Section A.3 of the Appendix.

First, fix an order on the columns of G^-:

$$i_1, i_2, \dots, i_n, \tag{3.5.2.1}$$

such that if $k \leqslant l$ then $\xi_{i_k} \leqslant \xi_{i_l}$ (select first the vertices i such that $\xi_i = 0$ then the others).

Then, the sequence $\mathscr{S}$ is defined by reading each column, from top to bottom, in this order.

Example 3.5.1. We follow Examples A.4.1 and A.4.3 of the Appendix, $\mathfrak{g} = \mathfrak{sl}_4$ and

$$\hat{I}^- = (1, 2\mathbb{Z}_{\leqslant 0}) \cup (2, 2\mathbb{Z}_{\leqslant 0} - 1) \cup (3, 2\mathbb{Z}_{\leqslant 0}).$$

We fix the following order on the columns: $1, 3, 3$. Then the sequence $\mathscr{S}$ is

$$\mathscr{S} = (1,0), (1,-2), (1,-4), \dots, (3,0), (3,-2), (3,-4), \dots, (2,-1), (2,-3), (2,-5), \dots. \tag{3.5.2.2}$$

3.5.3 Truncated (q,t)-characters as quantum cluster variables

As in [HL16a], let $\mu_{\mathscr{S}}$ be the sequence of quantum cluster mutations in $\mathcal{A}_t$ indexed by the sequence of vertices $\mathscr{S}$.

For all $m \geqslant 1$, let $u_{i,r}^{(m)}$ be the quantum cluster variable obtained at vertex (i,r) after applying m times the sequence of mutations $\mu_{\mathscr{S}}$ to the quantum cluster algebra $\mathcal{A}_t$ with initial seed $\{ u_{i,r} \mid (i,r) \in \hat{I}^- \}$. By the quantum Laurent phenomenon, the $u_{i,r}^{(m)}$ belong to the quantum torus T. Let

$$w_{i,r}^{(m)} := \eta(u_{i,r}^{(m)}) \quad \in \mathcal{Y}_t^-, \tag{3.5.3.1}$$

where $\eta : T \to \mathcal{Y}_t^-$ is the isomorphism defined in Lemma 3.4.5.

The following result is an extension of Theorem 3.1 from [HL16a] to the quantum setting.

Proposition 3.5.2. *For all $(i,r) \in \hat{I}^-$ and $m \geqslant 0$,*

$$w_{i,r}^{(m)} = [W_{k_{i,r},r-2m}^{(i)}]_t^-, \tag{3.5.3.2}$$

where $k_{i,r}$ is defined in (A.4.4) in the Appendix.

In particular, if $2m \geqslant h$, this truncated (q,t)-character is equal to its (q,t)-character and

$$w_{i,r}^{(m)} = [W_{k_{i,r},r-2m}^{(i)}]_t. \tag{3.5.3.3}$$

Remark 3.5.3. The sequence of vertices $\mathscr{S}$ is infinite. However, in order to compute one fixed truncated (q,t)-character, one only has to compute a finite number of mutations in the infinite sequence $\mu_{\mathscr{S}}$. In Section 2.5.2, the exact finite sequence needed to compute the (q,t)-characters of the fundamental representations $V_{i,r}$ is given explicitly, we will also recall it in Section 3.6.

Proof. We prove this Proposition by induction on m, the number of times the mutation sequence $\mu_{\mathscr{S}}$ is applied on the initial quantum cluster variables $\{ u_{i,r} \mid (i,r) \in \hat{I}^- \}$.

The base step is given noting, as in [HL16a], that the images by the isomorphism η of the initial quantum cluster variables are indeed truncated (q,t)-characters. For all $(i,r) \in \hat{I}^-$,

$$w_{i,r}^{(0)} = \eta(u_{i,r}) = U_{i,r} = \prod_{\substack{k \geqslant 0 \\ r+2k \leqslant 0}} Y_{i,r+2k} \quad \in \mathcal{Y}_t^-.$$

Thus

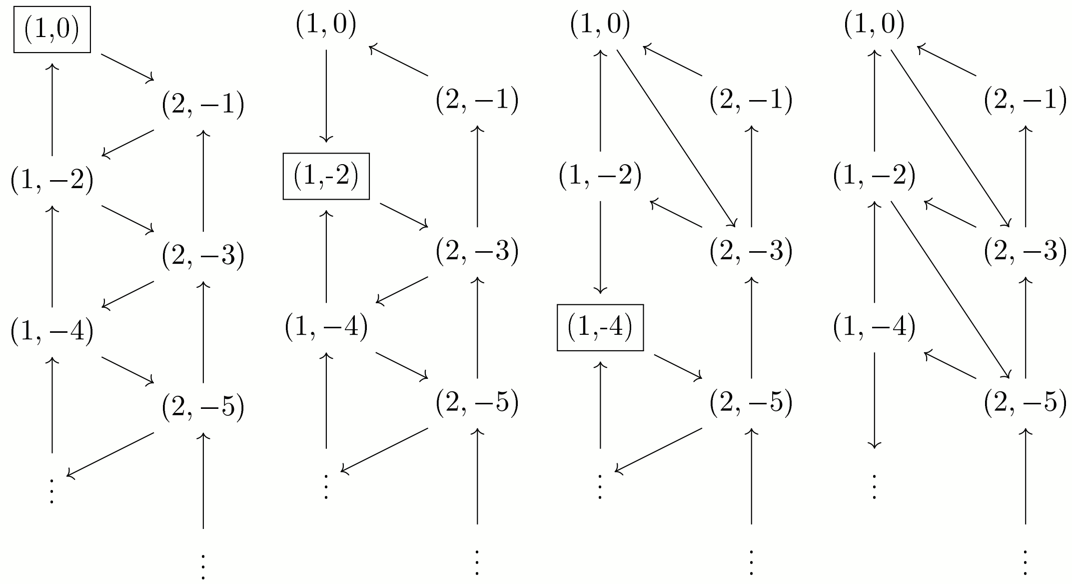
$$w_{i,r}^{(0)} = [W_{k_{i,r},r}^{(i)}]_t^-. \tag{3.5.3.4}$$

Let $m \geqslant 0$ and $(i,r) \in \hat{I}^-$. Supposed we have applied m times the mutation sequence $\mu_{\mathscr{S}}$, and a $(m+1)$th time on all vertices preceding (i,r) in the sequence $\mathscr{S}$, and that all those previous vertices satisfy (3.5.3.2).

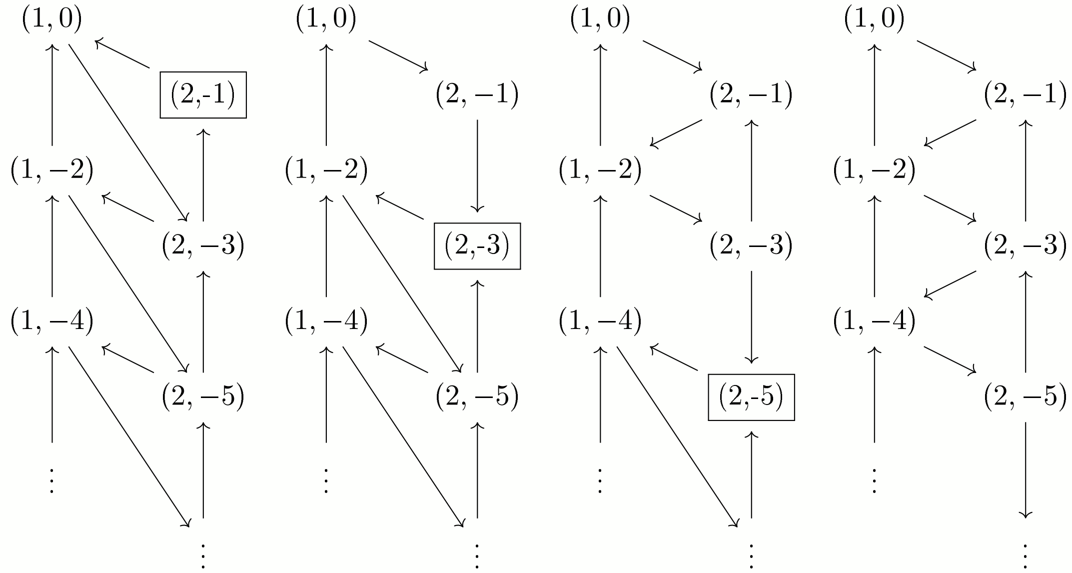
We want to write the quantum exchange relation corresponding to the mutation at vertex (i,r). From the proof of Theorem 3.1 in [HL16a], we know the shape of the quiver just before this mutation ($\mathcal{A}_t$ and $\mathcal{A}(\boldsymbol{u}, G^-)$ are defined on the same initial quiver and the mutation process on the quiver is the same for classical and quantum cluster algebras).

As explained in [HL16a, Section 3.2.3], for a general simply laced Lie algebra $\mathfrak{g}$, the mutation process takes place at vertices (i,r) having two (or one if $r = -\xi_i$) in-going arrows from $(i, r \pm 2)$ and outgoing arrows to vertices (j,s), with $j \sim i$. Thus the effect of the mutation sequence $\mu_{\mathscr{S}}$ on two fixed columns of the quiver is the same as the effect of an iteration of the mutation sequence on the corresponding quiver of rank 2.

Let us recall the mutation process on the quiver when $\mathfrak{g}$ is of type A_2.



After an infinite number of mutations (or a sufficiently large one), we start mutating on the second column.



Thus, in general, the quantum exchange relation has the form:

$$u_{i,r}^{(m+1)} = u_{i,r+2}^{(m+1)} u_{i,r-2}^{(m)} \left(u_{i,r}^{(m)}\right)^{-1} + \prod_{j\sim i_0} u_{j,r-\epsilon_i}^{(m+\xi_i)} \left(u_{i,r}^{(m)}\right)^{-1}, \tag{3.5.3.5}$$

where $u_{i,r+2}^{(m)} = 1$ if $r+2 \geqslant 0$ and ϵ_i is defined in the Appendix in (A.3.2), and both terms are commutative products. This relation can also be written:

$$u_{i,r}^{(m+1)} * u_{i,r}^{(m)} = t^{\alpha} u_{i,r+2}^{(m+1)} u_{i,r-2}^{(m)} + t^{\beta} \prod_{j\sim i_0} u_{j,r-\epsilon_i}^{(m+\xi_i)}, \tag{3.5.3.6}$$

where $\alpha, \beta \in \frac{1}{2}\mathbb{Z}$.

If we apply η to (3.5.3.6), and use the induction hypothesis, we get the following relation in $\mathcal{Y}_t^-$:

$$\begin{aligned}\eta(u_{i,r}^{(m+1)}) * [W_{k_{i,r},r-2m}^{(i)}]_t^- = t^{\alpha}[W_{k_{i,r+2},r-2m}^{(i)}]_t^- [W_{k_{i,r-2},r-2-2m}^{(i)}]_t^- \\ + t^{\beta}\prod_{j\sim i}[W_{k_{j,r-\epsilon_i},r-1-2m}^{(j)}]_t^- .\end{aligned} \tag{3.5.3.7}$$

Whereas, the corresponding (truncated) quantum T-system relation (3.3.6.4) is

$$\begin{aligned}[W_{k_{i,r},r-2m-2}^{(i)}]_t^- * [W_{k_{i,r},r-2m}^{(i)}]_t^- = t^{\alpha'}[W_{k_{i,r}-1,r-2m}^{(i)}]_t^- [W_{k_{i,r}+1,r-2-2m}^{(i)}]_t^- \\ + t^{\beta'}\prod_{j\sim i}[W_{k_{i,r},r-1-2m}^{(j)}]_t^- ,\end{aligned} \tag{3.5.3.8}$$

where $\alpha', \beta' \in \frac{1}{2}\mathbb{Z}$ are given in (3.3.6.2). Note that one has indeed

$$k_{i,r+2} = k_{i,r} - 1, \quad k_{i,r-2} = k_{i,r} + 1, \quad \text{and } k_{j,r-\epsilon_i} = k_{i,r}, \text{ for } j \sim i. \tag{3.5.3.9}$$

Let $k = k_{i,r}$ and $r' = r - 2m$. Let us precise how to obtain the coefficients α and β. From (3.5.3.5) and (3.5.3.6), α and β are such that the terms

$$t^{\alpha}[W_{k-1,r'}^{(i)}]_t^- [W_{k+1,r'-2}^{(i)}]_t^- * \left([W_{k,r'}^{(i)}]_t^-\right)^{-1}, \tag{3.5.3.10}$$

and

$$t^{\beta}\prod_{j\sim i}[W_{k,r'-1}^{(j)}]_t^- * \left([W_{k,r'}^{(i)}]_t^-\right)^{-1}, \tag{3.5.3.11}$$

are bar-invariant. Thus, if one takes only the dominant monomials of (3.5.3.10) and (3.5.3.11), they are bar-invariant:

$$\begin{aligned}t^{\alpha} m_{k-1,r'}^{(i)} m_{k+1,r'-2}^{(i)} * \left(m_{k,r'}^{(i)}\right)^{-1} &= t^{-\alpha}\left(m_{k,r'}^{(i)}\right)^{-1} * m_{k-1,r'}^{(i)} m_{k+1,r'-2}^{(i)}, \\ t^{\beta}\prod_{j\sim i} m_{k,r'-1}^{(j)} * \left(m_{k,r'}^{(i)}\right)^{-1} &= t^{-\beta}\left(m_{k,r'}^{(i)}\right)^{-1} * \prod_{j\sim i} m_{k,r'-1}^{(j)}.\end{aligned}$$

This enables us to compute α and β.

$$\begin{aligned}\alpha &= \frac{1}{2}\sum_{l=0}^{k-1}\left(\sum_{p=0}^{k-2}\mathcal{N}_{i,i}(2l-2p) + \sum_{p=0}^{k}\mathcal{N}_{i,i}(2l-2p+2)\right) \\ &= \frac{1}{2}\sum_{l=0}^{k-1}\left(\mathbf{C}_{ii}(2l+1) - \mathbf{C}_{ii}(2l-2k+3) + \mathbf{C}_{ii}(2l+3) - \mathbf{C}_{ii}(2l-2k+1)\right) \\ &= \frac{1}{2}\left(\mathbf{C}_{ii}(2k+1) + \mathbf{C}_{ii}(2k-1)\right) - \mathbf{C}_{ii}(1).\end{aligned}$$

Thus $\alpha = \alpha(i,k) = -1 + \frac{1}{2}\left(\tilde{C}_{ii}(2k-1) + \tilde{C}_{ii}(2k+1)\right)$. And

$$\begin{aligned}\beta &= \frac{1}{2}\sum_{j\sim i}\sum_{l=0}^{k-1}\sum_{p=0}^{k-1}\mathcal{N}_{i,j}(2l-2p+1) = \frac{1}{2}\sum_{j\sim i}\sum_{l=0}^{k-1}\left(\mathbf{C}_{ij}(2l+2) - \mathbf{C}_{ij}(2l-2k+2)\right) \\ &= \frac{1}{2}\sum_{j\sim i}\left(\mathbf{C}_{ij}(2k) - \underbrace{\mathbf{C}_{ij}(0)}_{=0}\right) = \frac{1}{2}\left(\mathbf{C}_{ii}(2k+1) + \mathbf{C}_{ii}(2k-1)\right), \quad \text{using (A.2.6).}\end{aligned}$$

Hence $\beta = \gamma(i,k) = \frac{1}{2}\left(\tilde{C}_{ii}(2k-1) + \tilde{C}_{ii}(2k+1)\right)$.

Thus $\alpha = \alpha'$ and $\beta = \beta'$, and

$$\eta(u_{i,r}^{(m+1)}) * [W_{k,r'}^{(i)}]_t^- = [W_{k,r'-2}^{(i)}]_t^- * [W_{k,r'}^{(i)}]_t^- . \tag{3.5.3.12}$$

However, $[W_{k,r'}^{(i)}]_t^-$ is invertible in the skew-field of fractions $\mathcal{F}_t$ of the quantum torus $\mathcal{Y}_t^-$ (see Remark 3.3.3). Thus

$$\eta(u_{i,r}^{(m+1)}) = [W_{k_{i,r},r-2-2m}^{(i)}]_t^- . \tag{3.5.3.13}$$

This concludes the induction.

Finally, from [FM01, Corollary 6.14], we know that for all $(i,r) \in \hat{I}^-$, the monomials m occurring in the q-character $\chi_q(W_{k,r}^{(i)})$ of the KR-module $W_{k,r}^{(i)}$ are products of $Y_{j,r+s}^{\pm 1}$, with $0 \leqslant s \leqslant 2k+h$. Moreover, from Theorem 3.3.6, if one writes the (q,t)-character $[W_{k,r}^{(i)}]_t$ of this KR-module as a linear combination of Laurent monomials in the variables $Y_{i,s}$, with coefficients in $\mathbb{Z}[t^{\pm 1}]$, all monomials which occur in this expansion also occur in its q-character. Thus, if $r + 2k + h \leqslant 0$, the truncated (q,t)-character $[W_{k,r}^{(i)}]_t^-$ is equal to the (q,t)-character $[W_{k,r}^{(i)}]_t$. In particular, for $m \geqslant h$,

$$w_{i,r}^{(m)} = [W_{k_{i,r},r-2m}^{(i)}]_t.$$

□

3.5.4 Proof of Theorem 3.4.6

We can now prove Theorem 3.4.6. This proof is a quantum analog of the proof of [HL16a, Theorem 5.1]. Naturally, there are technical difficulties brought forth by the non-commutative quantum tori structure. For example, in our situation, the quantum cluster algebra $\mathcal{A}_t$ is isomorphic to the truncated quantum Grothendieck ring $K_t(\mathscr{C}_{\mathbb{Z}}^-)$ only via the isomorphism of quantum tori η (3.4.2.1). In particular, we need the following result.

Lemma 3.5.4. *The identification*

$$\eta' : u_{i,r} \longmapsto [W_{k_{i,r},r}^{(i)}]_t. \tag{3.5.4.1}$$

extends to a well-defined injective $\mathbb{Z}[t^{\pm 1/2}]$-algebras morphism

$$\eta' : T \to \mathcal{F}_t,$$

where $\mathcal{F}_t$ is the skew-field of fractions of $\mathcal{Y}_t$ (see Remark 3.3.3).

Moreover, the restriction of η' to the quantum cluster algebra $\mathcal{A}_t$ has its image in the quantum torus $\mathcal{Y}_t$ and the $\mathbb{Z}[t^{\pm 1/2}]$-algebra morphisms η, η' and π satisfy the following commutative diagram:

$$\begin{tikzcd} \mathcal{A}_t \arrow[rr, "\eta"] \arrow[dr, "\eta'"'] & & \mathcal{Y}_t^- \\ & \mathcal{Y}_t \arrow[ur, "\pi"'] & \end{tikzcd} . \tag{3.5.4.2}$$

Proof. From Proposition 3.5.2, for all $(i,r) \in \hat{I}^-$, the *full* (q,t)-character $[W^{(i)}_{k_{i,r},r-2h}]_t$ is obtained as the image of the cluster variable sitting at vertex (i,r) after applying h times the mutation sequence $\mathscr{S}$ (which is locally a finite sequence of mutations):

$$\eta(u^{(h)}_{i,r}) = [W^{(i)}_{k_{i,r},r-2h}]_t. \tag{3.5.4.3}$$

In particular, for any two vertices $(i,r),(j,s)$, the variables $u^{(h)}_{i,r}$ and $u^{(h)}_{j,s}$ belong to a common cluster and t-commute. Thus the (q,t)-characters $[W^{(i)}_{k_{i,r},r-2h}]_t$ t-commute. As the quantum torus $\mathcal{Y}_t$ is invariant by shift of quantum parameters $(Y_{j,s} \mapsto Y_{j,s+2})$, the (q,t)-characters $[W^{(i)}_{k_{i,r},r}]_t$ also t-commute for the product $*$. Their t-commutation relations are determined by their dominant monomials, which are $\eta(u_{i,r})$. Thus the $[W^{(i)}_{k_{i,r},r}]_t$ satisfy exactly the same t-commutations relations are the $u_{i,r}$. This proves the first part of the lemma.

Let X be a cluster variable of $\mathcal{A}_t$ obtained from the initial seed $\boldsymbol{u} = \{u_{i,r}\}$ via finite sequence of mutations σ. We want to show that $\eta'(X) \in \mathcal{Y}_t$. As the sequence of mutations σ is finite, it will only involve a finite number of cluster variables. Now apply h times the mutation sequence $\mu_{\mathscr{S}}$ to the initial seed so as to replace each cluster variable considered by $u^{(h)}_{i,r}$ (again, we only need a finite number of mutations). Let us summarize:

$$\eta'(u_{i,r}) = [W^{(i)}_{k_{i,r},r}]_t, \quad \eta(u^{(h)}_{i,r}) = [W^{(i)}_{k_{i,r},r-2h}]_t.$$

Let X' be the cluster variable obtained by applying to this new seed the sequence of mutations σ. By construction, $\eta(X')$ is equal to $\eta'(X)$, up to the downward shift of spectral parameters by $2h$: every variable $Y^{\pm 1}_{j,s}$ is replaced by $Y^{\pm 1}_{j,s-2h}$. In particular, $\eta'(X) \in \mathcal{Y}_t$.

Next, the commutation of diagram (3.5.4.2) is verified as it is satisfied on the initial seed $\boldsymbol{u} = \{u_{i,r}\}$. □

Let R_t be the image of the quantum cluster algebra $\mathcal{A}_t$

$$R_t := \eta(\mathcal{A}_t) \quad \in \mathcal{Y}_t^-. \tag{3.5.4.4}$$

The inclusion $K_t(\mathscr{C}_{\mathbb{Z}}^-) \subset R_t$ is essentially contained in Proposition 3.5.2. For the reverse inclusion, the main idea is to use the characterization of the quantum Grothendieck ring as the intersection of kernel of operators, called deformed screening operators. We show by induction on the length of a sequence of mutations that the images of all cluster variables belong to those kernels. The images of the initial cluster variables $u_{i,r}$ clearly belong to the quantum Grothendieck ring, and the screening operators being derivations, the exchange relations force the newly created cluster variables to be in the intersection of the kernels too. let us prove this in details.

Proof. Recall from Lemma 3.3.8 that the quantum Grothendieck ring $K_t(\mathscr{C}_{\mathbb{Z}}^-)$ is algebraically generated by the truncated (q,t)-characters of the fundamental modules:

$$K_t(\mathscr{C}_{\mathbb{Z}}^-) = \left\langle [L(Y_{i,r})]_t^- \mid (i,r) \in \hat{I}^- \right\rangle.$$

By Proposition 3.5.2, for all $(i,r) \in \hat{I}^-$,

$$[L(Y_{i,r})]_t^- = w_{i,\xi_i}^{(\frac{r-\xi_i}{2})} = \eta\left(u_{i,\xi_i}^{(\frac{r-\xi_i}{2})}\right) \quad \in \eta(\mathcal{A}_t). \tag{3.5.4.5}$$

Which proves the first inclusion:

$$K_t(\mathscr{C}_{\mathbb{Z}}^-) \subset R_t. \tag{3.5.4.6}$$

We prove the reverse inclusion as explained just above.

As explained in Section 3.3.3, Hernandez proved in [Her03] that for all $i \in I$ there exists operators $S_{i,t} : \mathcal{Y}_t \to \mathcal{Y}_{i,t}$, where $\mathcal{Y}_{i,t}$ is a $\mathcal{Y}_t$-module, which are $\mathbb{Z}[t^{\pm 1}]$-linear and derivations, such that

$$\bigcap_{i \in I} \ker(S_{i,t}) = K_t(\mathscr{C}_{\mathbb{Z}}). \tag{3.5.4.7}$$

Notice that these operators characterize the quantum Grothendieck ring $K_t(\mathscr{C}_{\mathbb{Z}})$ and not $K_t(\mathscr{C}_{\mathbb{Z}}^-)$. Hence the need for Lemma 3.5.4.

Let us prove by induction that all cluster variables Z in $\mathcal{A}_t$ satisfy $\eta'(Z) \in K_t(\mathscr{C}_{\mathbb{Z}})$.

Let Z be a quantum cluster variable in $\mathcal{A}_t$. If Z belongs to the initial cluster variables, $Z = u_{i,r}$ and

$$\eta'(Z) = \eta'(u_{i,r}) = [W_{k_{i,r},r}]_t \quad \in K_t(\mathscr{C}_{\mathbb{Z}}). \tag{3.5.4.8}$$

If not, then by induction on the length of the sequence μ, one can assume that Z is obtained via a quantum exchange relation

$$Z * Z_1 = t^{\alpha} M_1 + t^{\beta} M_2, \tag{3.5.4.9}$$

where Z_1 is a quantum cluster variable of $\mathcal{A}_t$, M_1 and M_2 are quantum cluster monomials of $\mathcal{A}_t$ and

$$\eta'(Z_1), \eta'(M_1), \eta'(M_2) \in K_t(\mathscr{C}).$$

Apply η' to (3.5.4.9):

$$\eta'(Z) * \eta'(Z_1) = t^{\alpha}\eta'(M_1) + t^{\beta}\eta'(M_2). \tag{3.5.4.10}$$

For all $i \in I$, apply the derivation $S_{i,t}$:

$$\begin{aligned} S_{i,t}\left(\eta'(Z) * \eta'(Z_1)\right) &= S_{i,t}\left(\eta'(Z)\right) * \eta'(Z_1) + \eta'(Z) * S_{i,t}\left(\eta'(Z_1)\right) \\ &= t^{\alpha} S_{i,t}\left(\eta'(M_1)\right) + t^{\beta} S_{i,t}\left(\eta'(M_2)\right). \end{aligned}$$

However, by hypothesis,

$$\begin{aligned} S_{i,t}\left(\eta'(Z_1)\right) &= 0, \\ S_{i,t}\left(\eta'(M_1)\right) &= 0, \\ S_{i,t}\left(\eta'(M_1)\right) &= 0. \end{aligned}$$

Moreover, $\eta'(Z_1) \neq 0$ and the images of the screening operator is in a free module over $\mathcal{Y}_t$ by Lemma 3.3.4. Thus $S_{i,t}\left(\eta'(Z)\right) = 0$, for all $i \in I$. Hence

$$\eta'(Z) \in K_t(\mathscr{C}_{\mathbb{Z}}^-),$$

which concludes the induction. We have proven

$$\eta'(\mathcal{A}_t) \subset K_t(\mathscr{C}_{\mathbb{Z}}).$$

Then, by the commutation of the diagram (3.5.4.2) in Lemma 3.5.4,

$$R_t = \eta(\mathcal{A}_t) \subset K_t(\mathscr{C}_{\mathbb{Z}}^-).$$

Which concludes the proof of the theorem.

□

3.6 Application to the proof of an inclusion conjecture

In this section, we use the quantum cluster algebra structure of the Grothendieck ring $\mathscr{C}_{\mathbb{Z}}^{-}$ to prove that the quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}}^{+})$ defined in Section 2.4.3 contains $K_t(\mathscr{C}_{\mathbb{Z}}^{-})$. In other words, we prove Conjecture 2.5.2 in Chapter 2. The result was already proven in Chapter 2 in type A, but the core argument used was different.

3.6.1 The quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}}^{+})$

The quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}}^{+})$ is defined as a quantum cluster algebra on the full infinite quiver Γ, of which the semi-infinite quiver G^- is a subquiver.

Let us recall some notations. For all $i,j \in I$, $\mathcal{F}_{ij} : \mathbb{Z} \to \mathbb{Z}$ is a anti-symmetrical map such that, for all $m \geqslant 0$,

$$\mathcal{F}_{ij}(m) = - \sum_{\substack{k \geqslant 1 \\ m \geqslant 2k-1}} \tilde{C}_{ij}(m-2k+1). \tag{3.6.1.1}$$

Let $\mathcal{T}_t$ be the quantum torus defined as the $\mathbb{Z}[t^{\pm 1/2}]$-algebra generated by the variables $z_{i,r}^{\pm}$, for $(i,r) \in \hat{I}$, with a non-commutative product $*$, and the t-commutations relations

$$z_{i,r} * z_{j,s} = t^{\mathcal{F}_{ij}(s-r)} z_{j,s} * z_{i,r}, \quad \left((i,r),(j,s) \in \hat{I}\right). \tag{3.6.1.2}$$

Recall also from Proposition 2.3.3 the inclusion of quantum tori $\mathcal{J}$ (with a slight shift of parameters on the $z_{i,r}$):

$$\mathcal{J} : \left\{ \begin{array}{ccc} \mathcal{Y}_t & \longrightarrow & \mathcal{T}_t, \\ Y_{i,r} & \longmapsto & z_{i,r}\left(z_{i,r+2}\right)^{-1}. \end{array} \right. . \tag{3.6.1.3}$$

Let Λ be the infinite skew-symmetric $\hat{I} \times \hat{I}$-matrix:

$$\Lambda_{(i,r),(j,s)} = \mathcal{F}_{ij}(s-r), \quad \left((i,r),(j,s) \in \hat{I}\right). \tag{3.6.1.4}$$

From proposition 2.4.10, the quiver Γ and the skew-symmetric matrix Λ form a compatible pair. Let $\mathcal{A}_t(\Gamma,\Lambda)$ be the associated quantum cluster algebra. Then, as in Definition 2.4.15,

$$K_t(\mathcal{O}_{\mathbb{Z}}^{+}) := \mathcal{A}_t(\Gamma,\Lambda)\hat{\otimes}\mathcal{E}, \tag{3.6.1.5}$$

where $\mathcal{E}$ is a commutative ring and the completion allows for certain countable sums.

3.6.2 Intermediate quantum cluster algebras

The general idea is to see the quantum cluster algebra $\mathcal{A}_t$ as a "sub-quantum cluster algebra" of $\mathcal{A}_t(\Gamma,\Lambda)$ (this term is not well-defined). However, as in Section 3.5.4 and contrary to the aforementioned proof, as we are dealing with quantum cluster algebras in our setting, this is not done trivially. Mainly, one notices that the map

$$\begin{array}{ccc} T & \longrightarrow & \mathcal{T}_t \\ u_{i,r} & \longmapsto & z_{i,r}, \end{array}$$

is not a well-defined inclusion of quantum tori, as the generators $u_{i,r}$ and $z_{i,r}$ do not satisfy the same t-commutation relations.

First, consider the subquiver Γ^- of Γ of index set

$$\hat{I}^-_{\leqslant 2} := \hat{I} \cap \{(i,r) \mid i \in I, r \leqslant 2\}, \tag{3.6.2.1}$$

such that the vertices (i,r), with $r > 0$ are frozen.

To summarize, for $\xi_i = 0$ and $j \sim i$:

$$\Gamma = \begin{tikzcd} \vdots & \\ & \vdots \\ (i,2) \arrow[uu] \arrow[dr] & \\ & (j,1) \arrow[uu] \arrow[dl] \\ (i,0) \arrow[uu] \arrow[dr] & \\ & (j,-1) \arrow[uu] \arrow[dl] \\ (i,-2) \arrow[uu] \arrow[dr] & \\ & \vdots \arrow[uu] \\ \vdots \arrow[uu] & \end{tikzcd}, \quad \Gamma^- = \begin{tikzcd} \boxed{\text{(i,2)}} & \\ & \boxed{\text{(j,1)}} \arrow[dl] \\ (i,0) \arrow[uu] \arrow[dr] & \\ & (j,-1) \arrow[uu] \arrow[dl] \\ (i,-2) \arrow[uu] \arrow[dr] & \\ & \vdots \arrow[uu] \\ \vdots \arrow[uu] & \end{tikzcd}, \quad G^- = \begin{tikzcd} (i,0) \arrow[dr] & \\ & (j,-1) \arrow[dl] \\ (i,-2) \arrow[uu] \arrow[dr] & \\ & \vdots \arrow[uu] \\ \vdots \arrow[uu] & \end{tikzcd}.$$

The quivers Γ and Γ^- are only connected by coefficients (as in [GG18, Definition 4.1]), thus by [GG18, Theorem 4.5] the inclusion of seeds $\Gamma^-, \Lambda \subset \Gamma, \Lambda$ induces an inclusion of the quantum cluster algebra $\mathcal{A}_t(\Gamma^-, \Lambda)$ into the quantum cluster algebra of $\mathcal{A}_t(\Gamma, \Lambda)$.

Now we need to link the quantum cluster algebras $\mathcal{A}_t$ and $\mathcal{A}_t(\Gamma^-, \Lambda)$.

In order to do that, we use a result from [GL14], which deals with *graded* cluster algebras.

Definition 3.6.1. [GL14] A quantum cluster seed (B, Λ) is *graded* if there exists an integer column vector G such that, for all mutable indices k, the kth row of B, B_j satisfies $B_j G = 0$.

Then, the G-degree of the initial variables are set by the vector G: for all cluster variables X_k in the initial cluster $\bar{X}$, $\deg_G(X) = G_i$.

The grading condition is equivalent to the following, for all mutable variables X_k, the sum of the degrees of all variables with arrows to X_k is equal to the sum of the degrees of all variables with arrows coming from X_k, i.e. exchange relations are homogeneous. Hence, each cluster variable has a well-defined degree.

Let us start by considering the quantum cluster algebra $\mathcal{A}_t(\Gamma^-, L)$, built on the quiver Γ^-, with coefficients 1 on the frozen vertices. This quantum cluster algebra is clearly isomorphic to $\mathcal{A}_t$, and is a graded quantum cluster algebra of grading $G = 0$.

Next, for all $i \in I$, we apply the process of [GL14, Theorem 4.6] to add coefficients f_i, while twisting the t-commutation relations.

Let

$$\underline{u}_i(j,s) = \delta_{i,j}, \quad \left((j,s) \in \hat{I}^-_{\leqslant 2}\right) \tag{3.6.2.2}$$

and

$$\underline{t}_i(j,s) = -\mathcal{F}_{ij}(2 - s - \xi_i), \quad \left((j,s) \in \hat{I}^-_{\leqslant 2}\right), \tag{3.6.2.3}$$

with $\xi : I \to \{0,1\}$ the height function, fixed in Section A.3 of the Appendix. Then

Lemma 3.6.2. *For all $i \in I$, $\underline{u}_i$ and $\underline{t}_i$ are gradings for the ice quiver Γ^-.*

Proof. For all $(j,s) \in \hat{I}^-$, let $B_{(j,s)}$ be the (j,s)th "row" of the $\hat{I}^-_{\leqslant 2} \times \hat{I}^-_{\leqslant 2}$-skew-symmetric matrix encoding the adjacency of the ice quiver Γ^-.

For all $i \in I$,

$$\begin{aligned} B_{(j,s)}\underline{u}_i &= \sum_{(k,r)\in \hat{I}^-_{\leqslant 2}} B_{(j,s)}(k,r)\underline{u}_i(k,r) \\ &= \underline{u}_i(j,s+2) - \underline{u}_i(j,s-2) + \sum_{k\sim j} (\underline{u}_i(k,s-1) - \underline{u}_i(k,s+1)) \\ &= \delta_{i,j} - \delta_{i,j} + \sum_{k\sim j}(\delta_{i,k} - \delta_{i,k}) = 0. \end{aligned}$$

And

$$\begin{aligned} B_{(j,s)}\underline{t}_i &= \sum_{(k,r)\in \hat{I}^-_{\leqslant 2}} B_{(j,s)}(k,r)\underline{t}_i(k,r) \\ &= \underline{t}_i(j,s+2) - \underline{t}_i(j,s-2) + \sum_{k\sim j} (\underline{t}_i(k,s-1) - \underline{t}_i(k,s+1)) \\ &= -\mathcal{F}_{ij}(-s-\xi_i) + \mathcal{F}_{ij}(4-s-\xi_i) + \sum_{k\sim j}(-\mathcal{F}_{ik}(3-s-\xi_i) + \mathcal{F}_{ik}(1-s-\xi_i)) \\ &= -\tilde{C}_{ij}(3-s-\xi_i) - \tilde{C}_{ij}(1-s-\xi_i) + \sum_{k\sim j}\tilde{C}_{ik}(2-s-\xi_i) = 0, \end{aligned}$$

from Lemma A.2.4 of the Appendix. □

Then, by [GL14, Theorem 4.6] (applied n times), the following is a valid set of initial data for a (multi-)graded quantum cluster algebra $\mathcal{A}_t(\tilde{\boldsymbol{u}}, \tilde{B}, \tilde{L}, \tilde{G})$, where

- or all $(i,r) \in \hat{I}^-_{\leqslant 2}$,

$$\tilde{u}_{i,r} = \begin{cases} u_{i,r}f_i & \text{if } r \leqslant 0, \\ f_i & \text{otherwise} \end{cases}. \tag{3.6.2.4}$$

  and $\tilde{\boldsymbol{u}} = \{\tilde{u}_{i,r}\}_{(i,r)\in \hat{I}^-_{\leqslant 2}}$.
- $\tilde{B} = B$, the $\hat{I}^-_{\leqslant 2} \times \hat{I}^-_{\leqslant 2}$-skew-symmetric adjacency matrix of the quiver Γ^-.
- $\tilde{L}$ encodes the t-commutations relations, such that, for all $(i,r) \in \hat{I}^-$, and $j \in I$,

$$f_j * u_{i,r} = t^{\underline{t}_j(i,r)} u_{i,r} * f_j, \tag{3.6.2.5}$$

  and the f_j pairwise commute.
- $\tilde{G}$ is a multi-grading, i.e. instead of being an integer column vector, each entry in $\tilde{G}$ is in the lattice $\mathbb{Z}^I$. It is defined by, for all $(i,r) \in \hat{I}^-_{\leqslant 2}$,

$$\tilde{G}(\tilde{u}_{i,r}) = e_i \quad \in \mathbb{Z}^I, \tag{3.6.2.6}$$

  or $\deg_i = \underline{u}_i$, for all $i \in I$.
  This is indeed the construction of [GL14], with the initial grading on $\mathcal{A}_t(\Gamma^-, L)$ being $G \equiv 0$. The new quantum cluster algebra is denoted by $\mathcal{A}_t^{\underline{u},\underline{t}}(\Gamma^-, L)$, to show that it is a twisted version of $\mathcal{A}_t(\Gamma^-, L)$.

Proposition 3.6.3. *The quantum cluster algebra $\mathcal{A}_t(\tilde{\mathbf{u}}, \tilde{B}, \tilde{L})$ is isomorphic to the quantum cluster algebra $\mathcal{A}_t(\Gamma^-, \Lambda)$.*

Proof. The rest of the data being the same, we only have to check that the $\hat{I}^-_{\leqslant 2} \times \hat{I}^-_{\leqslant 2}$-skew-symmetric matrices $\tilde{L}$ and $\Lambda_{|\hat{I}^-_{\leqslant 2}}$ are equal.

From (3.6.1.4), for all $\left((i,r),(j,s) \in \hat{I}^-_{\leqslant 2}\right)$,

$$\Lambda\left((i,r),(j,s)\right) = \mathcal{F}_{ij}(s-r).$$

And $\tilde{L}$ is defined as

$$\tilde{u}_{i,r} * \tilde{u}_{j,s} = t^{\tilde{L}((i,r),(j,s))} \tilde{u}_{j,s} * \tilde{u}_{i,r}. \tag{3.6.2.7}$$

Hence,

$$\tilde{L}\left((i,r),(j,s)\right) = \begin{cases} L\left((i,r),(j,s)\right) + \underline{t}_i(j,s) - \underline{t}_j(i,r) & , \text{ if } (i,r),(j,s) \in \hat{I}^-, \\ \underline{t}_i(j,s) & , \text{ if } \left[\begin{array}{l} (i,r) = (i, -\xi_i + 2) \\ (j,s) \in \hat{I}^- \end{array}\right. , \\ 0 & , \text{ if } \left[\begin{array}{l} (i,r) = (i, -\xi_i + 2) \\ (j,s) = (j, -\xi_j + 2) \end{array}\right. . \end{cases}$$

First, notice that, for all $i, j \in I$, $m \in \mathbb{Z}$,

$$\mathcal{N}_{ij}(m) = 2\mathcal{F}_{ij}(m) - \mathcal{F}_{ij}(m+2) - \mathcal{F}_{ij}(m-2). \tag{3.6.2.8}$$

This result is proven in Chapter 2, in the course of the proof of Proposition 2.3.3.

Thus, for all $(i,r),(j,s) \in \hat{I}^-$,

$$\begin{aligned} L\left((i,r),(j,s)\right) &= \sum_{\substack{k \geqslant 0 \\ r+2k \leqslant 0}} \sum_{\substack{l \geqslant 0 \\ s+2l \leqslant 0}} \mathcal{N}_{ij}(s+2l-r-2k) \\ &= \sum_{\substack{k \geqslant 0 \\ r+2k \leqslant 0}} \left(\mathcal{F}_{ij}(s-r-2k) - \mathcal{F}_{ij}(s-r-2k-2)\right. \\ &\left. + \mathcal{F}_{ij}(-\xi_i - r - 2k) - \mathcal{F}_{ij}(-\xi_j - r - 2k + 2)\right) \\ &= \mathcal{F}_{ij}(s-r) - \mathcal{F}_{ij}(s+\xi_i-2) + \underbrace{\mathcal{F}_{ij}(-\xi_j+\xi_i)}_{=0} - \mathcal{F}_{ij}(-\xi_j - r + 2) \\ &= \mathcal{F}_{ij}(s-r) - \underline{t}_i(j,s) + \underline{t}_j(i,r). \end{aligned}$$

And of course, for all $i \in I$, $(j,s) \in \hat{I}^-$,

$$\Lambda\left((i, -\xi_i + 2),(j,s)\right) = \mathcal{F}_{ij}(s + \xi_i - 2) = \underline{t}_i(j,s).$$

Thus, one had indeed,

$$\tilde{L} = \Lambda_{|\hat{I}^-_{\leqslant 2}}. \tag{3.6.2.9}$$

□

From now on, we will use the notations $\boldsymbol{z} = \{z_{i,r}, f_j\}_{(i,r) \in \hat{I}^-, j \in I}$ for the initial clusters variables of both $\mathcal{A}_t(\Gamma^-, \Lambda)$ and $\mathcal{A}_t^{u,t}(\Gamma^-, L)$.

Remark 3.6.4. • This result is natural, if we look at what the different initial cluster variables mean in terms of ℓ-weights. From (3.2.4.1), for all $(i,r) \in \hat{I}^-$, the cluster variable $u_{i,r}$ can be identified with the (commutative) dominant monomial $U_{i,r} = \prod_{\substack{k\geqslant 0 \\ r+2k\leqslant 0}} Y_{i,r+2k}$. Whereas, the quantum tori $\mathcal{Y}_t$ and $\mathcal{T}_t$ are compared via the inclusion $\mathcal{J}$ of (3.6.1.3)

$$\mathcal{J} : Y_{i,r} \longmapsto z_{i,r}\left(z_{i,r+2}\right)^{-1}, \quad \forall (i,r) \in \hat{I}.$$

Thus, the link between the variables $u_{i,r}$ and $z_{i,r}$ is the following:

$$u_{i,r} \equiv z_{i,r}\left(f_i\right)^{-1}, \tag{3.6.2.10}$$

with the previous convention $f_i = z_{i,-\xi_i+2}$. More precisely, there are different maps of quantum tori:

$$\begin{tikzcd} T \arrow[r, dotted] \arrow[d, "\eta"', "\sim"] & \mathcal{T}_t \\ \mathcal{Y}_t^- \arrow[r, hook] & \mathcal{Y}_t \arrow[u, "\mathcal{J}"'] \end{tikzcd}, \tag{3.6.2.11}$$

where identification (3.6.2.10) is the resulting dotted map, which we will denote by ρ:

$$\rho : T \to \mathcal{T}_t. \tag{3.6.2.12}$$

The quantum cluster algebra $\mathcal{A}_t^{u,t}(\Gamma^-, L)$ was built with this identification in mind.

• This process could also be seen as a quantum version of the multi-grading homogenization process of the seed $(\boldsymbol{u}, G^-)$, as in [Gra15, Lemma 7.1], where the multi-grading is defined by (3.6.2.6).

3.6.3 Inclusion of quantum Grothendieck rings

In the section, we prove that the quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}}^+)$, or more precisely, the quantum cluster algebra $\mathcal{A}_t(\Gamma, \Lambda)$ contains the quantum Grothendieck ring $K_t(\mathscr{C}_{\mathbb{Z}})$, which is the statement of Conjecture 2.5.2 in Chapter 2. Recalled that in Section 2.4.3, the ring $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ was defined as a completion of the quantum cluster algebra $\mathcal{A}_t(\Gamma, \Lambda)$, but the aim was to see it as a quantum Grothendieck ring for the category of representations $\mathcal{O}_{\mathbb{Z}}^+$ from [HJ12] and [HL16b]. As this category contains the category $\mathscr{C}_{\mathbb{Z}}$, it was expected for the quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ to contain $K_t(\mathscr{C}_{\mathbb{Z}})$.

In order to prove this result we will actually prove Conjecture 2 of Chapter 2, which is a stronger result. We state it as follows.

Theorem 3.6.5. *The (q,t)-characters of all fundamental representations in $\mathscr{C}_{\mathbb{Z}}$ are obtained as quantum cluster variables in the quantum cluster algebra $\mathcal{A}_t(\Gamma, \Lambda)$.*

More precisely, we show that for all $i \in I$, there exists a specific finite sequence of mutations S_i in $\mathcal{A}_t(\Gamma, \Lambda)$ such that, if applied to the initial seed $\{z_{j,s}\}_{(j,s)\in\hat{I}}$, the cluster variable sitting at vertex $(i, -\xi_i)$ is the image by $\mathcal{J}$ (of (3.6.1.3)) of the (q,t)-character of the fundamental module $[L(Y_{i,-\xi_i-2h'})]_t$, where

$$h' = \lceil \frac{h}{2} \rceil, \tag{3.6.3.1}$$

with h the Coxeter number of the simple Lie algebra $\mathfrak{g}$.

Let us define the sequence S_i. Let $(i_1, i_2, \dots, j_n)$ be an ordering of the columns of Γ as in (3.5.2.1), such that $i_1 = i$ (take first all columns j such that $\xi_j = \xi_i$). The sequence S_i is a sequence of vertices of Γ, and more precisely of Γ^-, defined as follows. First read all vertices (i_1, r) for $-2h'+2 \leqslant r \leqslant 0$, from top to bottom, then all (i_2, r), with $-2h' \leqslant r \leqslant 0$, and so on, then read again all vertices (i_1, r) for $-2h'+4 \leqslant r \leqslant 0$, and continue browsing the columns successively, until at the last step you only read the vertex $(i, -\xi_i)$.

Note that applying this sequence S_i of mutations on the quivers Γ^- or G^- has exactly the same effect on the cluster variable sitting at vertex $(i, -\xi_i)$ than applying h' times the infinite sequence $\mathscr{S}$ from Section 3.5.2.

Example 3.6.6. For $\mathfrak{g}$ of type D_4, $h = 6$, then $h' = 3$. Let us give explicitly the sequence S_2 starting on column 2. For simplicity of notations, we assume that the height function is chosen such that $\xi_2 = 0$. Then the sequence S_2 has 15 steps:

$$\begin{aligned} S_2 = (2,0), (2,-2), (2,-4), (1,-1), (1,-3), (3,-1), (3,-3), (4,-1), (4,-3), \\ (2,0), (2,-2), (1,-1)(3,-1)(4,-1)(2,0). \end{aligned} \tag{3.6.3.2}$$

Let $r_0 = -\xi_i - 2h'$. Let $\tilde{\chi}_{i,r_0} \in \mathcal{T}_t$ be the quantum cluster variable obtained at vertex $(i, -\xi_i)$ after applying the sequence of mutations S_i to the quantum cluster variable $\mathcal{A}_t(\Gamma^-, \Lambda)$ with initial seed $\{z_{j,s}, f_k\}_{(j,s)\in\hat{I}^-, k\in I}$.

Proposition 3.6.7. *As an element of the quantum torus $\mathcal{T}_t$, $\tilde{\chi}_{i,r_0}$ belongs to the image of the inclusion morphism $\mathcal{J}$.*

Moreover,

$$\tilde{\chi}_{i,r_0} = \mathcal{J}\left([L(Y_{i,r_0})]_t\right). \tag{3.6.3.3}$$

Proof. The cluster variable $\tilde{\chi}_{i,r_0}$ is a variable of the quantum cluster algebra $\mathcal{A}_t(\Gamma^-, \Lambda)$, which is isomorphic to $\mathcal{A}_t^{u,t}(\Gamma^-, L)$ from Proposition 3.6.3. By [GL14, Corollary 4.7], there is a bijection between the quantum cluster variables of $\mathcal{A}_t^{u,t}(\Gamma^-, L)$ and those of $\mathcal{A}_t(\Gamma^-, L)$.

With notations from Section 3.5.3, $u_{i,-\xi_i}^{(h')}$ is the cluster variable of $\mathcal{A}_t(\Gamma^-, L)$ obtained at vertex $(i, -\xi_i)$ after applying the mutations of the sequence S_i. Also from [GL14, Corollary 4.7], we know that there exists integers $a_j \in \mathbb{Z}$ such that

$$\tilde{\chi}_{i,r_0} = \rho\left(u_{i,-\xi_i}^{(h')}\right) \prod_{j\in I} f_j^{a_j}, \tag{3.6.3.4}$$

written as a commutative product (both $\tilde{\chi}_{i,r_0}$ and $u_{i,-\xi_i}^{(h')}$ are bar-invariant), with ρ defined in (3.6.2.12). The term $u_{i,-\xi_i}^{(h')}$ is a Laurent polynomial in the variables $u_{j,s}$, which satisfy $\rho(u_{j,s}) = z_{j,s}(f_j)^{-1}$ from (3.6.2.10). Thus expression (3.6.3.4) is a way of writing $\tilde{\chi}_{i,r_0}$ as a Laurent polynomial in the initial variables $\{z_{j,s}, f_k\}$. However, one can write $\tilde{\chi}_{i,r_0} = N/D$, where N is the Laurent polynomial in the cluster variables $\{z_{j,s}, f_k\}$, with coefficients in $\mathbb{Z}[t^{\pm 1/2}]$ and not divisible by any of the f_k, and D is a monomial in the non-frozen variables $\{z_{j,s}\}$. Thus $\prod_{j\in I} f_j^{a_j}$ is the smallest monomial such that

$$\rho\left(u_{i,-\xi_i}^{(h')}\right) \prod_{j\in I} f_j^{a_j}$$

contains only non-negative powers of the frozen variables f_k.

Moreover, from Proposition 3.5.2,

$$\eta(u_{i,-\xi_i}^{(h')}) = w_{i,-\xi_i}^{(h')} = [L(Y_{i,r_0})]_t. \tag{3.6.3.5}$$

However, all Laurent monomials occurring in $[L(Y_{i,r_0})]_t$ already occurred in the q-character $\chi_q(L(Y_{i,r_0}))$, as the (q,t)-character $[L(Y_{i,r_0})]_t$ has positive coefficients. Indeed, the (q,t)-characters of fundamental modules have been explicitly computed and have been found to have non-negative coefficients (in [Nak03b] for types A and D and [Nak10] for type E).

From [FM01], all monomials in $\chi_q(L(Y_{i,r_0}))$ are products of $Y_{j,s}^{\pm 1}$, with $s \leqslant r_0 + h$, but by definition of r_0, $r_0 + h \leqslant 0$, and the term with the highest quantum parameter being the anti-dominant monomial $Y_{\bar{i},r_0+h}^{-1}$, where $\bar{\ }: I \to I$ is the involutive map such that $\omega_0(\alpha_j) = -\alpha_{\bar{j}}$, with ω_0 the longest element of the Weyl group of $\mathfrak{g}$ (no relation with the bar-involution of Section 3.5.1).

Consider the change of variables, for $(j,s) \in \hat{I}^-$,

$$y_{j,s} = \eta^{-1}(Y_{j,s}) = \begin{cases} \frac{u_{j,s}}{u_{j,s+2}}, & \text{if } s+2 \leqslant 0, \\ u_{j,s}, & \text{otherwise} \end{cases}. \tag{3.6.3.6}$$

Thus

$$\rho(y_{j,s}) = \begin{cases} \frac{z_{j,s}}{z_{j,s+2}}, & \text{if } s+2 \leqslant 0, \\ \frac{z_{j,s}}{f_j}, & \text{otherwise} \end{cases}. \tag{3.6.3.7}$$

All monomials occurring in $[L(Y_{i,r_0})]_t$ are commutative monomials in the variables $Y_{j,s}^{\pm 1}$, with $s \leqslant r_0 + h \leqslant 0$. Moreover, the only monomials in which the variables $Y_{j,s}^{\pm 1}$, with $s + 2 \geqslant 0$, occur are the anti-dominant monomial $Y_{\bar{i},r_0+h}^{-1}$, and any possible monomial in which some variable Y_{j,r_0+h-1} occurs. But for such a monomial m, the variable $Y_{j,s}^{\pm 1}$ with the highest s in m occurs with a negative power in m (the monomial m is "right-negative", as from [FM01, Lemma 6.5]), thus the variable Y_{j,r_0+h-1} also occurs with a negative power.

The image by $\rho \circ \eta^{-1}$ of any monomial in the variables $Y_{j,s}^{\pm 1}$, with $s + 2 \leqslant 0$ is a monomial in the variables $\{z_{j,s}^{\pm 1}\}$ (without frozen variables). Thus, the image

$$\rho\left(u_{i,-\xi_i}^{(h')}\right) = \rho \circ \eta^{-1}\left([L(Y_{i,r_0})]_t\right)$$

is a Laurent polynomial with only positive powers of the variables f_j.

Necessarily, $\prod_{j \in I} f_j^{a_j} = 1$, and (3.6.3.4) becomes

$$\tilde{\chi}_{i,r_0} = \rho\left(u_{i,-\xi_i}^{(h')}\right).$$

Finally, from the diagram (3.6.2.11),

$$\tilde{\chi}_{i,r_0} = \rho\left(u_{i,-\xi_i}^{(h')}\right) = \mathcal{J}\left(\eta\left(u_{i,-\xi_i}^{(h')}\right)\right) = \mathcal{J}\left([L(Y_{i,r_0})]_t\right),$$

which concludes the proof. □

Remark 3.6.8. At some point in the proof we used the fact that the (q,t)-characters of the fundamental modules had non-negative coefficients. Note that this part of the proof could easily be extended to non-simply laced types, as the (q,t)-characters of their fundamental representations have also be explicitly computed, and also have non-negative coefficients (in types B and C, the (q,t)-characters of all fundamental representations are equal to their respective q-character, and all coefficients are actually equal to 1 [Her05, Proposition 7.2], and see [Her04] for type G_2 and [Her05] for type F_4).

Corollary 3.6.9. *For all* $(i, r) \in \hat{I}$ *there exists a quantum cluster variable* $\tilde{\chi}_{i,r}$ *in the quantum cluster algebra* $\mathcal{A}_t(\Gamma, \Lambda)$ *such that*

$$\tilde{\chi}_{i,r} = \mathcal{J}\left([L(Y_{i,r})]_t\right).$$

Proof. For all $(i, r) \in \hat{I}$, let $\tilde{\chi}_{i,r}$ be the cluster variable of the quantum cluster algebra $\mathcal{A}_t(\Gamma, \Lambda)$ obtained at vertex $(i, r + 2h')$ after applying the sequence of mutations S_i, but starting at vertex $(i, r + 2h')$ instead of $(i, -\xi_i)$.

Consider the change of variables in $\mathcal{A}_t(\Gamma, \Lambda)$:

$$s : z_{j,s} \longmapsto z_{j,s+r_0-r}, \quad \forall (j, s) \in \hat{I}. \tag{3.6.3.8}$$

The quantum cluster algebra $\mathcal{A}_t(\Gamma, \Lambda)$ is invariant under this shift s, and this change of variables is clearly invertible ($s^{-1}(z_{j,s}) = z_{j,s+r-r_0}$). One has $s(z_{i,r}) = z_{i,r_0}$, and $s(z_{i,r+2h'}) = z_{i,-\xi_i}$, thus $s\left(\tilde{\chi}_{i,r}\right) = \tilde{\chi}_{i,r_0}$, and from Proposition 3.6.7,

$$\tilde{\chi}_{i,r} = s^{-1}\left(\tilde{\chi}_{i,r_0}\right) = s^{-1}\left(\mathcal{J}\left([L(Y_{i,r_0})]_t\right)\right).$$

However, from the definition of the map $\mathcal{J}$ in (3.6.1.3), the shift s also acts as a change of variables in the quantum torus $\mathcal{Y}_t$, $s : Y_{j,u} \longmapsto Y_{j,u+r_0-r}$. Hence,

$$\tilde{\chi}_{i,r} = \mathcal{J}\left([L(Y_{i,r})]_t\right).$$

□

Thus we have proven Theorem 3.6.5.

Remark 3.6.10.
- When Conjecture 2.5.7 was first formulated in Chapter 2, a recent positivity result of Davison [Dav18] was mentioned there. This work proves the so-called "positivity conjecture" for quantum cluster algebras, which states that the coefficients of the Laurent polynomials into which the cluster variables decompose from the Laurent phenomenon are in fact non-negative. This is an important result, but also a difficult one, and it is not actually needed in order to obtain our result.
- One can note from this proof that we know a close bound on the number of mutations needed in order to compute the (q, t)-character of a fundamental module. For $\mathfrak{g}$ a simple-laced simple Lie algebra of rank n and of (dual) Coxeter number h, if $h' = \lceil h/2 \rceil$, then the number of steps is lower than

$$n\frac{h'(h'+1)}{2}. \tag{3.6.3.9}$$

We have chosen to go into details on the number of steps required for this process because it made sense from an algorithmic point of view to know its complexity.

One can compare this algorithm to Frenkel-Mukhin [FM01] to compute q-characters. As explained in [Nak10], when trying to compute q-characters of fundamental representations of large dimension (for example, in type E_8, the q-character of the 5th fundamental representation has approximately 6.4×2^{30} monomials), one encounters memory issues. Indeed, this algorithm has to keep track of all the previously computed terms. This advantage of the cluster algebra approach is that one only had to keep the seed in memory.

3.7 Explicit computation in type D_4

For $\mathfrak{g}$ of type D_4, with the height function $\xi_2 = 0, \xi_1 = \xi_3 = \xi_4 = 1$, we compute the (q,t)-character of the fundamental representation $L(Y_{2,-6})$ as a quantum cluster variable of the quantum cluster algebra $\mathcal{A}_t(\Gamma^-, \Lambda)$, using the algorithm presented in the previous Section.

From Example 3.6.6, the sequence of mutation we have to apply to the initial seed is

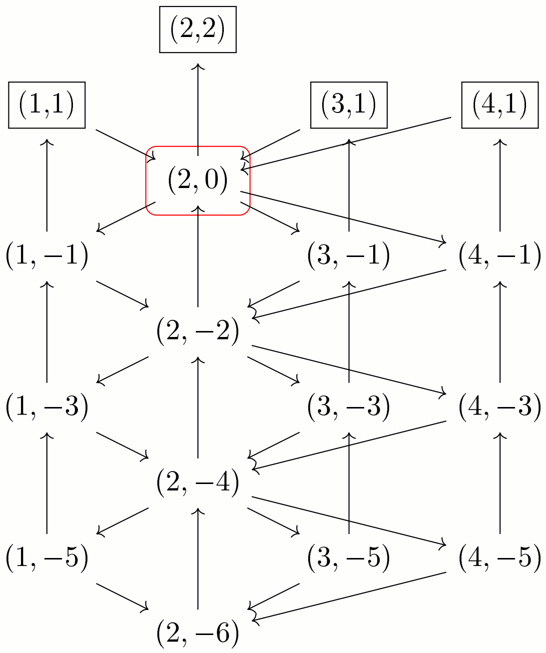
$$S_2 = (2,0), (2,-2), (2,-4), (1,-1), (1,-3), (3,-1), (3,-3), (4,-1), (4,-3),\\ (2,0), (2,-2), (1,-1)(3,-1)(4,-1)(2,0).$$

One can notice that the required number of steps is indeed lower than 24, which was the bound given in (3.6.3.9).

Let us give explicitly the mutations on the quiver Γ^- (which encodes more than G^-), as well as the quantum cluster variables obtained at (almost every) step. We give the quantum cluster variables as Laurent polynomials in the variables $\{z_{i,r}, f_j\}_{(i,r)\in\hat{I}, j\in I}$, as well as in the form (3.6.3.4). For completeness, we also compute the multi-degrees of the quantum cluster variables.

This computation was done thanks to the latest version (as of January 2019) of Bernhard Keller's wonderful quiver mutation applet:

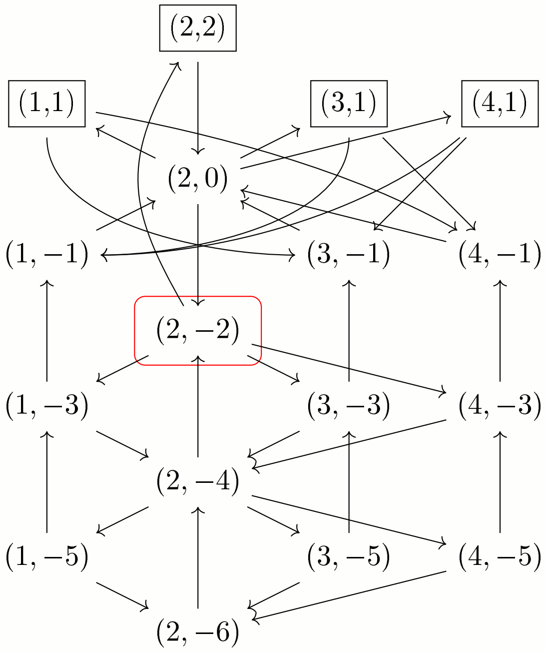
https://webusers.imj-prg.fr/~bernhard.keller/quivermutation/.



Mutation at $(2,0)$.

$\deg\left(z_{2,0}^{(1)}\right) = e_1 + e_3 + e_4.$

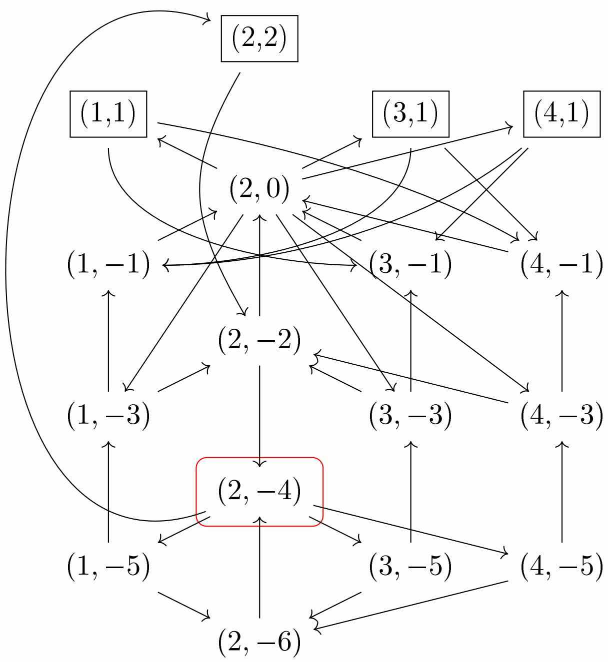
$$z_{2,0}^{(1)} = z_{2,-2}\left(z_{2,0}\right)^{-1} f_1 f_3 f_4 + z_{1,-1} z_{3,-1} z_{4,-1}\left(z_{2,0}\right)^{-1} f_2,$$
$$= \mathcal{J}\left(Y_{2,-2} + Y_{1,-1}Y_{3,-1}Y_{4,-1}Y_{2,0}^{-1}\right) f_1 f_3 f_4.$$



Mutation at $(2,-2)$.

$$\deg\left(z_{2,-2}^{(1)}\right) = e_1+e_3+e_4.$$

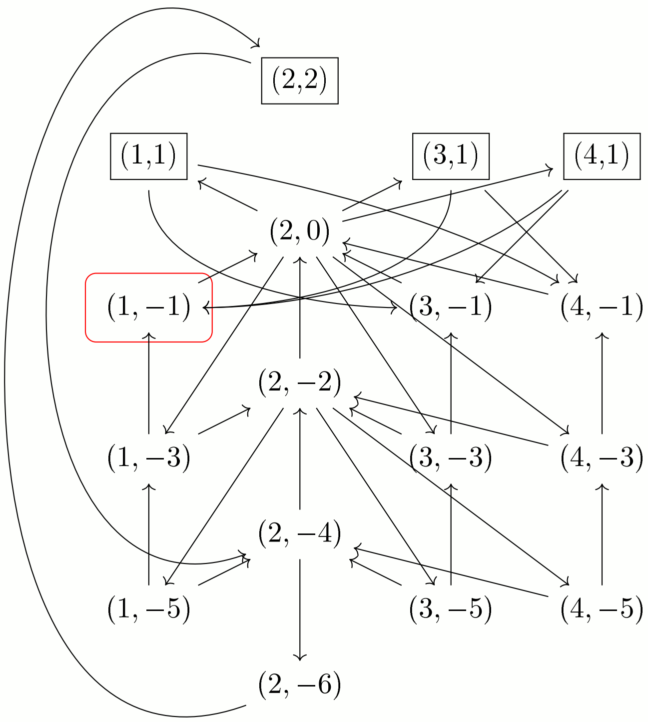
$$
\begin{aligned}
z_{2,-2}^{(1)} &= z_{2,-4}\left(z_{2,0}\right)^{-1} f_1 f_3 f_4 + z_{1,-1} z_{3,-1} z_{4,-1} z_{2,-4}\left(z_{2,-2}\right)^{-1}\left(z_{2,0}\right)^{-1} f_2 \\
&\quad + z_{1,-3} z_{3,-3} z_{4,-3}\left(z_{2,-2}\right)^{-1} f_2, \\
z_{2,-2}^{(1)} &= \mathcal{J}\Big(Y_{2,-4} Y_{2,-2} + Y_{1,-1} Y_{2,-4}\left(Y_{2,0}\right)^{-1} Y_{3,-1} Y_{4,-1} \\
&\quad + Y_{1,-3} Y_{1,-1}\left(Y_{2,-2}\right)^{-1}\left(Y_{2,0}\right)^{-1} Y_{3,-3} Y_{3,-1} Y_{4,-3} Y_{4,-1}\Big) f_1 f_3 f_4.
\end{aligned}
$$



Mutation at $(2,-4)$.

$$\deg\left(z_{2,-4}^{(1)}\right) = e_1+e_3+e_4. \tag{3.7.0.1}$$

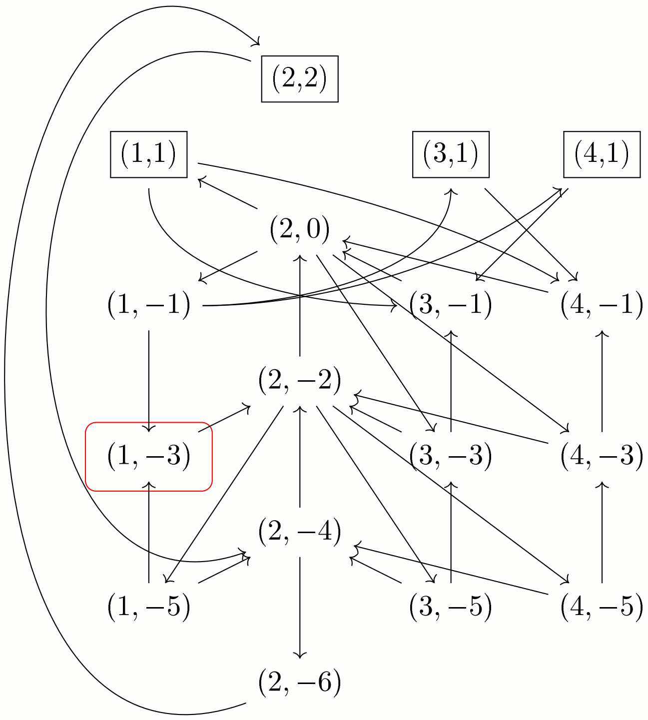
$$
\begin{aligned}
z^{(1)}_{2,-4} &= z_{2,-6}\left(z_{2,0}\right)^{-1} f_1 f_3 f_4 + z_{1,-1} z_{3,-1} z_{4,-1} z_{2,-6}\left(z_{2,-2}\right)^{-1}\left(z_{2,0}\right)^{-1} f_2 \\
&\quad + z_{1,-3} z_{3,-3} z_{4,-3} z_{2,-6}\left(z_{2,-4}\right)^{-1}\left(z_{2,-2}\right)^{-1} f_2 + z_{1,-5} z_{3,-5} z_{4,-5}\left(z_{2,-4}\right)^{-1} f_2, \\
z^{(1)}_{2,-4} &= \mathcal{J}\Big(Y_{2,-6} Y_{2,-4} Y_{2,-2} + Y_{1,-1} Y_{2,-6} Y_{2,-4}\left(Y_{2,0}\right)^{-1} Y_{3,-1} Y_{4,-1} \\
&\quad + Y_{1,-3} Y_{1,-1} Y_{2,-6}\left(Y_{2,-2}\right)^{-1}\left(Y_{2,0}\right)^{-1} Y_{3,-3} Y_{3,-1} Y_{4,-3} Y_{4,-1} \\
&\quad + Y_{1,-5} Y_{1,-3} Y_{1,-1}\left(Y_{2,-4}\right)^{-1}\left(Y_{2,-2}\right)^{-1} Y_{3,-5} Y_{3,-3} Y_{3,-1} Y_{4,-5} Y_{4,-3} Y_{4,-1}\Big) f_1 f_3 f_4.
\end{aligned}
$$



Mutation at $(1,-1)$.

$$\deg\left(z^{(1)}_{1,-1}\right) = e_3 + e_4.$$

$$
\begin{aligned}
z^{(1)}_{1,-1} &= z_{1,-3}\left(z_{1,-1}\right)^{-1} f_3 f_4 \\
&\quad + \left(z_{1,-1}\right)^{-1} z_{2,-2}\left(z_{2,0}\right)^{-1} f_1 f_3 f_4 + z_{3,-1} z_{4,-1}\left(z_{2,0}\right)^{-1} f_2 \\
z^{(1)}_{1,-1} &= \mathcal{J}\left(Y_{1,-3} + \left(Y_{1,-1}\right)^{-1} Y_{2,-2} + \left(Y_{2,0}\right)^{-1} Y_{3,-1} Y_{4,-1}\right) f_3 f_4.
\end{aligned}
$$

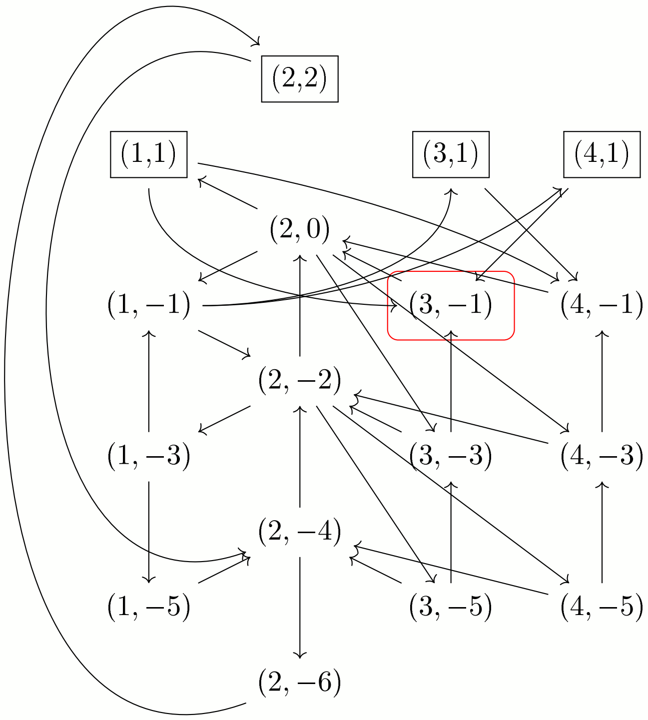


Mutation at $(1, -3)$.

$$\deg\left(z^{(1)}_{1,-3}\right) = e_3 + e_4.$$

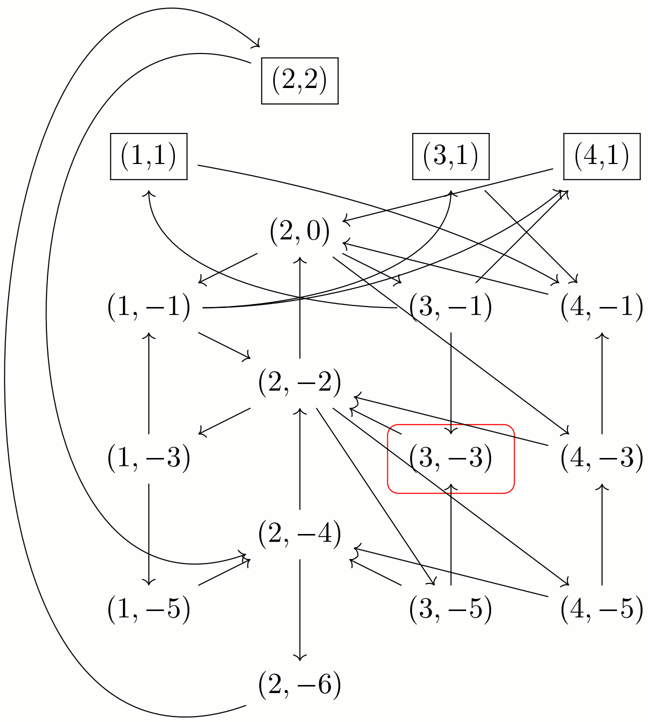
$$\begin{aligned}
z^{(1)}_{1,-3} &= z_{1,-5}\left(z_{1,-1}\right)^{-1} f_3 f_4 + z_{1,-5}\left(z_{1,-3}\right)^{-1}\left(z_{1,-1}\right)^{-1} z_{2,-2}\left(z_{2,0}\right)^{-1} f_1 f_3 f_4 \\
&\quad + z_{1,-5}\left(z_{1,-3}\right)^{-1}\left(z_{2,0}\right)^{-1} z_{3,-1} z_{4,-1} f_2 + \left(z_{1,-3}\right)^{-1} z_{2,-4}\left(z_{2,0}\right)^{-1} f_1 f_3 f_4 \\
&\quad + \left(z_{1,-3}\right)^{-1} z_{1,-1}\left(z_{2,-2}\right)^{-1}\left(z_{2,0}\right)^{-1} z_{3,-1} z_{4,-1} z_{2,-4} f_2 + \left(z_{2,-2}\right)^{-1} z_{3,-3} z_{4,-3} f_2, \\
z^{(1)}_{1,-3} &= \mathcal{J}\Big(Y_{1,-5}Y_{1,-3} + Y_{1,-5}\left(Y_{1,-1}\right)^{-1} Y_{2,-2} + Y_{1,-5}\left(Y_{2,0}\right)^{-1} Y_{3,-1}Y_{4,-1} \\
&\quad + \left(Y_{1,-3}\right)^{-1}\left(Y_{1,-1}\right)^{-1} Y_{2,-4}Y_{2,0} + \left(Y_{1,-3}\right)^{-1} Y_{2,-4}\left(Y_{2,0}\right)^{-1} Y_{3,-1}Y_{4,-1} \\
&\quad + \left(Y_{2,-2}\right)^{-1}\left(Y_{2,0}\right)^{-1} Y_{3,-3}Y_{3,-1}Y_{4,-3}Y_{4,-1}\Big) f_3 f_4.
\end{aligned}$$

The next steps of the mutation process are the mutations $(3,-1)$,$(3,-3)$ and $(4,-1)$, $(4,-3)$. However, the resulting cluster variables are equal $z^{(1)}_{1,-1}$ $z^{(1)}_{1,-3}$, up to the exchanges $1 \longleftrightarrow 3$ and $1 \longleftrightarrow 4$, so we will not give their expressions. We only show the mutation of the quiver, and the multi-degrees.



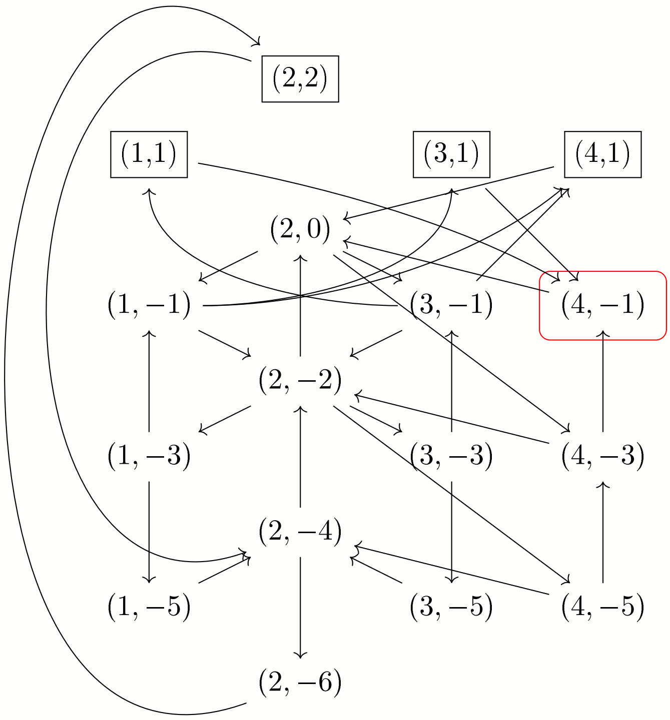
Mutation at $(3,-1)$.

$$\deg\left(z^{(1)}_{3,-1}\right) = e_1 + e_4.$$



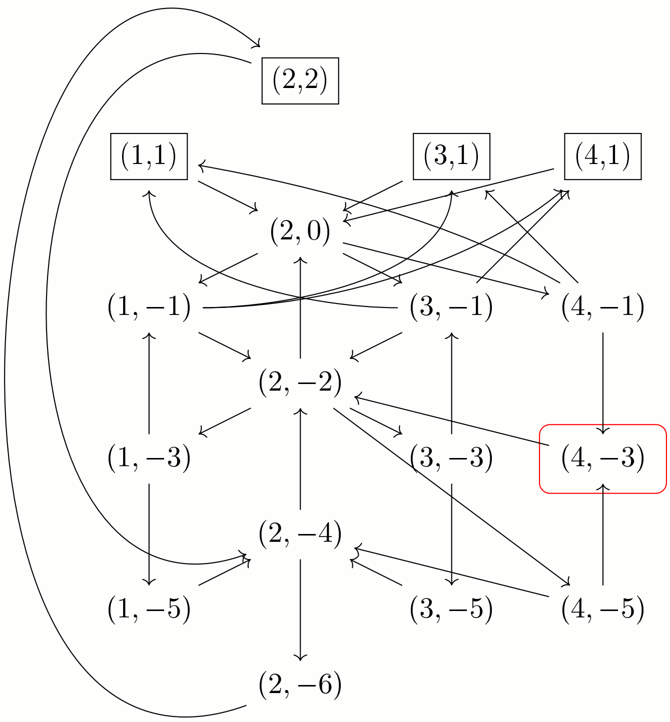
Mutation at $(3,-3)$.

$$\deg\left(z^{(1)}_{3,-3}\right) = e_1 + e_4.$$



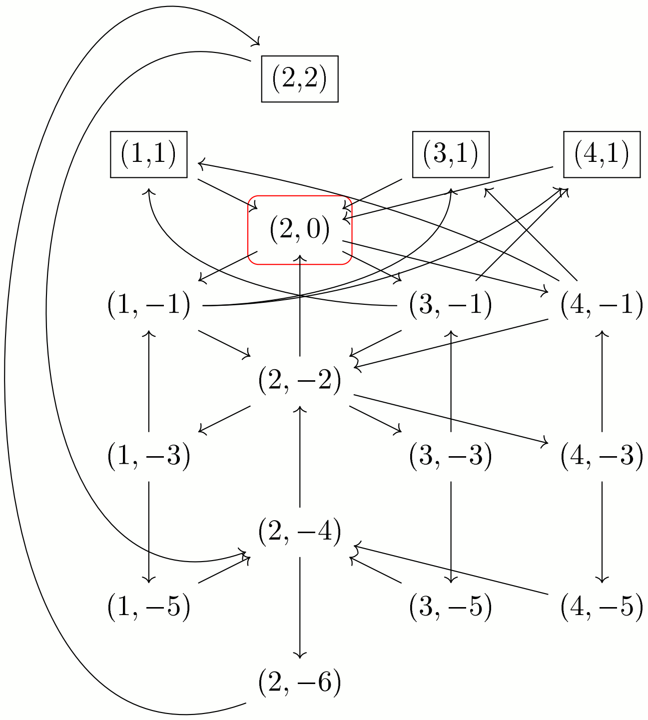
Mutation at $(4, -1)$.

$$\deg\left(z^{(1)}_{4,-1}\right) = e_1 + e_3.$$



Mutation at $(4, -3)$.

$$\deg\left(z^{(1)}_{4,-3}\right) = e_1 + e_3.$$

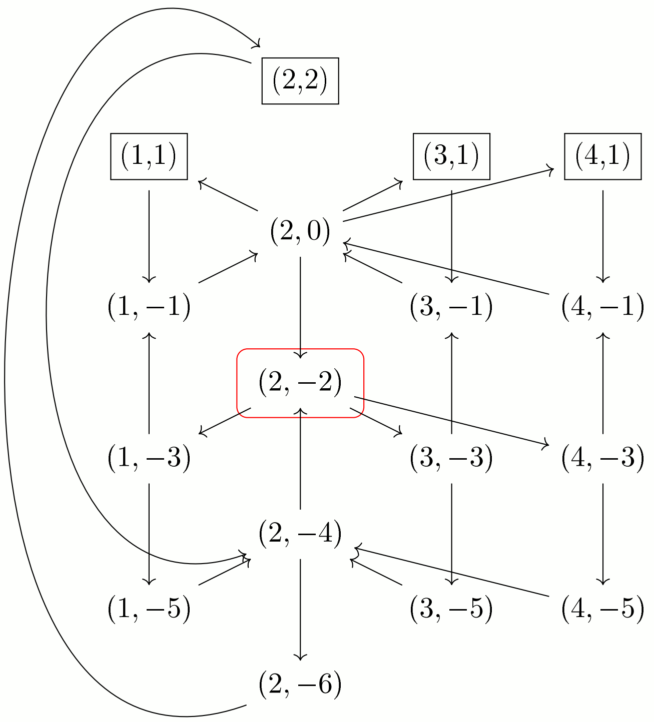


Mutation at $(2,0)$.

$\deg(z_{2,0}^{(2)}) = e_1 + e_3 + e_4$.

$$
\begin{aligned}
z_{2,0}^{(2)} = {} & z_{2,-4}\left(z_{2,-2}\right)^{-1} f_1 f_3 f_4 \\
& + z_{1,-3}\left(z_{1,-1}\right)^{-1}\left(z_{2,-2}\right)^{-1} z_{2,0} z_{3,-3}\left(z_{3,-1}\right)^{-1} z_{4,-3}\left(z_{4,-1}\right)^{-1} f_1 f_3 f_4 \\
& + \left(z_{1,-1}\right)^{-1} z_{3,-3}\left(z_{3,-1}\right)^{-1} z_{4,-3}\left(z_{4,-1}\right)^{-1} f_1^2 f_3 f_4 \\
& + z_{1,-3}\left(z_{1,-1}\right)^{-1}\left(z_{3,-1}\right)^{-1} z_{4,-3}\left(z_{4,-1}\right)^{-1} f_1 f_3^2 f_4 \\
& + z_{1,-3}\left(z_{1,-1}\right)^{-1} z_{3,-3}\left(z_{3,-1}\right)^{-1}\left(z_{4,-1}\right)^{-1} f_1 f_3 f_4^2 \\
& + \left(z_{1,-1}\right)^{-1} z_{2,-2}\left(z_{2,0}\right)^{-1}\left(z_{3,-1}\right)^{-1} z_{4,-3}\left(z_{4,-1}\right)^{-1} z_{4,1} f_1^2 f_3^2 f_4 \\
& + \left(z_{1,-1}\right)^{-1} z_{2,-2}\left(z_{2,0}\right)^{-1} z_{3,-3}\left(z_{3,-1}\right)^{-1}\left(z_{4,-1}\right)^{-1} f_1^2 f_3 f_4^2 \\
& + z_{1,-3}\left(z_{1,-1}\right)^{-1} z_{2,-2}\left(z_{2,0}\right)^{-1}\left(z_{3,-1}\right)^{-1}\left(z_{4,-1}\right)^{-1} f_1 f_3^2 f_4^2 \\
& + \left(z_{2,0}\right)^{-1} z_{4,-3} f_1 f_2 f_3 + \left(z_{2,0}\right)^{-1} z_{3,-3} f_1 f_2 f_4 + z_{1,-3}\left(z_{2,0}\right)^{-1} f_2 f_3 f_4 \\
& + \left(z_{1,-1}\right)^{-1} z_{2,-2}^2\left(z_{2,0}\right)^{-2}\left(z_{3,-1}\right)^{-1}\left(z_{4,-1}\right)^{-1} f_1^2 f_3^2 f_4^2 \\
& + \left(t + t^{-1}\right) z_{2,-2}\left(z_{2,0}\right)^{-2} f_1 f_2 f_3 f_4 + z_{1,-1}\left(z_{2,0}\right)^{-2} z_{3,-1} z_{4,-1} f_2^2,
\end{aligned}
$$

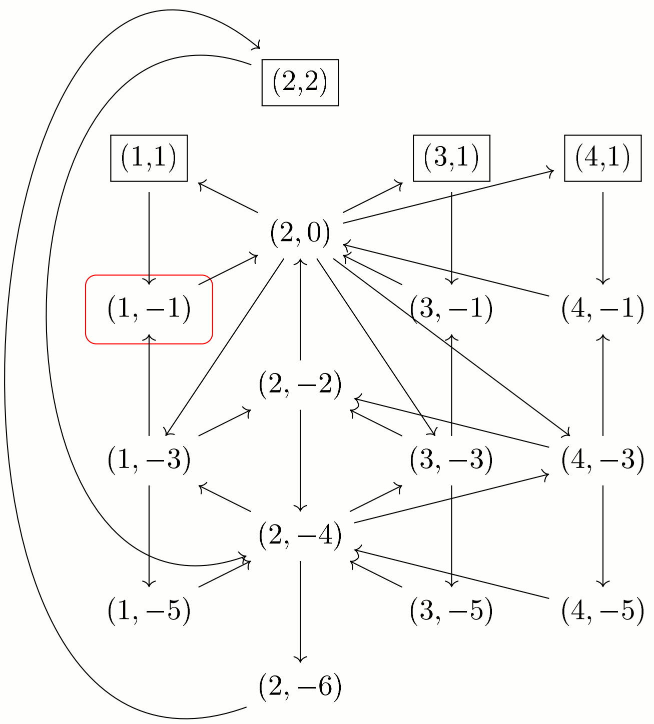
$$
\begin{aligned}
z_{2,0}^{(2)} = \mathcal{J}\Big(& Y_{2,-4} + Y_{1,-3}\left(Y_{2,-2}\right)^{-1} Y_{3,-3} Y_{4,-3} \\
& + \left(Y_{1,-1}\right)^{-1} Y_{3,-3} Y_{4,-3} + Y_{1,-3}\left(Y_{3,-1}\right)^{-1} Y_{4,-3} + Y_{1,-3} Y_{3,-3}\left(Y_{4,-1}\right)^{-1} \\
& + \left(Y_{1,-1}\right)^{-1} Y_{2,-2}\left(Y_{3,-1}\right)^{-1} Y_{4,-3} + \left(Y_{1,-1}\right)^{-1} Y_{2,-2} Y_{3,-3}\left(Y_{4,-1}\right)^{-1} \\
& + Y_{1,-3} Y_{2,-2}\left(Y_{3,-1}\right)^{-1}\left(Y_{4,-1}\right)^{-1} + \left(Y_{2,0}\right)^{-1} Y_{4,-3} Y_{4,-1} \\
& + \left(Y_{2,0}\right)^{-1} Y_{3,-3} Y_{3,-1} + Y_{1,-3} Y_{1,-1}\left(Y_{2,0}\right)^{-1} \\
& + \left(Y_{1,-1}\right)^{-1}\left(Y_{2,-2}\right)^2\left(Y_{3,-1}\right)^{-1}\left(Y_{4,-1}\right)^{-1} + \left(t + t^{-1}\right) Y_{2,-2}\left(Y_{2,0}\right)^{-1} \\
& + Y_{1,-1}\left(Y_{2,0}\right)^{-2} Y_{3,-1} Y_{4,-1}\Big) f_1 f_3 f_4.
\end{aligned}
$$



Mutation at $(2,-2)$.

$\deg(z^{(2)}_{2,-2}) = e_1+e_3+e_4$.

Here we do not write explicitly $z^{(2)}_{2,-2}$, because it has 92 terms.



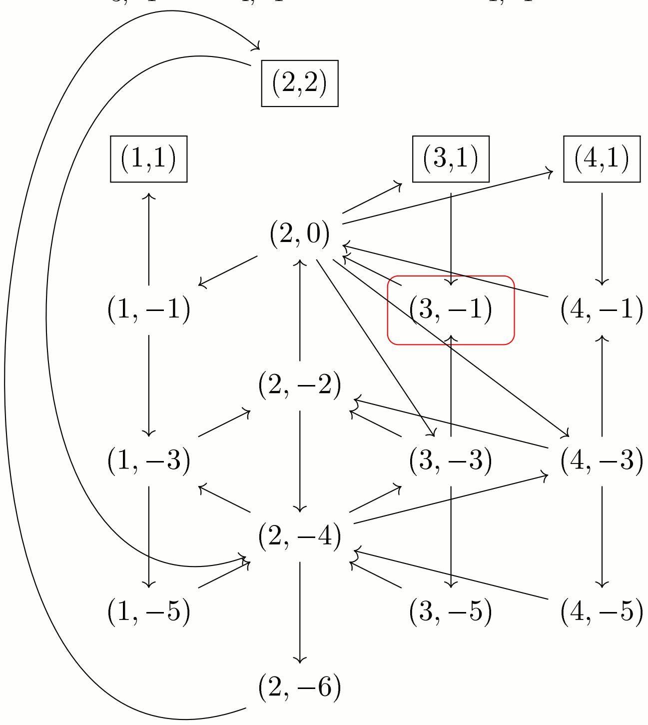
Mutation at $(1,-1)$.

$\deg(z^{(2)}_{1,-1}) = e_1$.

$$
\begin{aligned}
z^{(2)}_{1,-1} = {} & z_{1,-5}\left(z_{1,-3}\right)^{-1} f_1 + \left(z_{1,-3}\right)^{-1} z_{1,-1} z_{2,-4}\left(z_{2,-2}\right)^{-1} f_1 \\
& + \left(z_{2,-2}\right)^{-1} z_{2,0} z_{3,-3}\left(z_{3,-1}\right)^{-1} z_{4,-3}\left(z_{4,-1}\right)^{-1} f_1 \\
& + z_{3,-3}\left(z_{3,-1}\right)^{-1}\left(z_{4,-1}\right)^{-1} f_1 f_4 + \left(z_{3,-1}\right)^{-1} z_{4,-3}\left(z_{4,-1}\right)^{-1} f_1 f_3 \\
& + z_{2,-2}\left(z_{2,0}\right)^{-1}\left(z_{3,-1}\right)^{-1}\left(z_{4,-1}\right)^{-1} f_1 f_3 f_4 + z_{1,-1}\left(z_{2,0}\right)^{-1} f_2,
\end{aligned}
$$

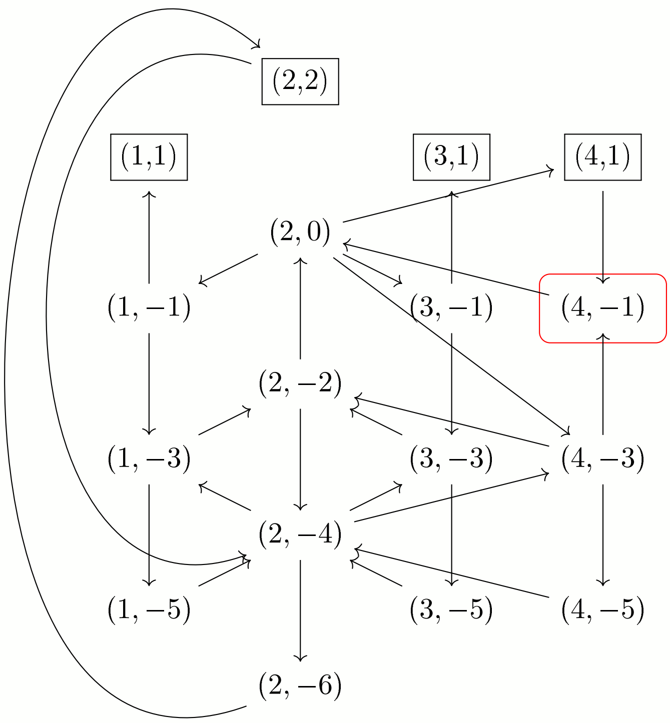
$$z_{1,-1}^{(2)} = \mathcal{J}\Big(Y_{1,-5} + (Y_{1,-3})^{-1} Y_{2,-4} + (Y_{2,-2})^{-1} Y_{3,-3}Y_{4,-3} + Y_{3,-3}(Y_{4,-1})^{-1} + (Y_{3,-1})^{-1} Y_{4,-3} + Y_{2,-2}(Y_{3,-1})^{-1}(Y_{4,-1})^{-1} + Y_{1,-1}(Y_{2,0})^{-1}\Big) f_1.$$

As before, $z_{3,-1}^{(2)}$ and $z_{4,-1}^{(2)}$ are similar to $z_{1,-1}^{(2)}$ so we do not write them.



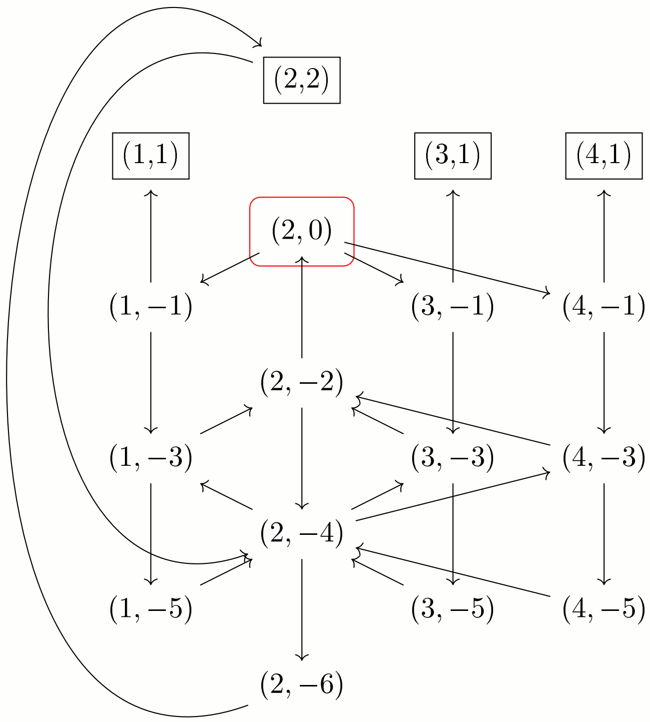
Mutation at $(3,-1)$.

$\deg(z_{3,-1}^{(2)}) = e_3$.



Mutation at $(4,-1)$.

$\deg(z_{4,-1}^{(2)}) = e_4$.



Mutation at $(2,0)$.

$\deg(z^{(3)}_{2,0}) = 0$.

$$
\begin{aligned}
z^{(3)}_{2,0} = {} & z_{2,-6}(z_{2,-4})^{-1} + z_{1,-5}(z_{1,-3})^{-1}(z_{2,-4})^{-1} z_{2,-2} z_{3,-5}(z_{3,-3})^{-1} z_{4,-5}(z_{4,-3})^{-1} \\
& + (z_{1,-3})^{-1} z_{1,-1} z_{3,-5}(z_{3,-3})^{-1} z_{4,-5}(z_{4,-3})^{-1} \\
& + z_{1,-5}(z_{1,-3})^{-1}(z_{3,-3})^{-1} z_{3,-1} z_{4,-5}(z_{4,-3})^{-1} \\
& + z_{1,-5}(z_{1,-3})^{-1} z_{3,-5}(z_{3,-3})^{-1}(z_{4,-3})^{-1} z_{4,-1} \\
& + (z_{1,-3})^{-1} z_{1,-1} z_{2,-4}(z_{2,-2})^{-1}(z_{3,-3})^{-1} z_{3,-1} z_{4,-5}(z_{4,-3})^{-1} \\
& + (z_{1,-3})^{-1} z_{1,-1} z_{2,-4}(z_{2,-2})^{-1} z_{3,-5}(z_{3,-3})^{-1}(z_{4,-3})^{-1} z_{4,-1} \\
& + z_{1,-5}(z_{1,-3})^{-1} z_{2,-4}(z_{2,-2})^{-1}(z_{3,-3})^{-1} z_{3,-1}(z_{4,-3})^{-1} z_{4,-1} \\
& + (z_{1,-3})^{-1} z_{1,-1}(z_{2,-4})^{2}(z_{2,-2})^{-2}(z_{3,-3})^{-1} z_{3,-1}(z_{4,-3})^{-1} z_{4,-1} \\
& + (z_{2,-2})^{-1} z_{2,0} z_{4,-5}(z_{4,-1})^{-1} + (z_{2,-2})^{-1} z_{2,0} z_{3,-5}(z_{3,-1})^{-1} \\
& + z_{1,-5}(z_{1,-1})^{-1}(z_{2,-2})^{-1} z_{2,0} + \left(t+t^{-1}\right) z_{2,-4}(z_{2,-2})^{-2} z_{2,0} \\
& + z_{1,-3}(z_{1,-1})^{-1}(z_{2,-2})^{-2}(z_{2,0})^{2} z_{3,-3}(z_{3,-1})^{-1} z_{4,-3}(z_{4,-1})^{-1} \\
& + z_{4,-5}(z_{4,-3})^{-1}(z_{4,-1})^{-1} f_4 + z_{3,-5}(z_{3,-3})^{-1}(z_{3,-1})^{-1} f_3 \\
& + z_{1,-5}(z_{1,-3})^{-1}(z_{1,-1})^{-1} f_1 + (z_{1,-3})^{-1} z_{2,-4}(z_{2,-2})^{-1} f_1 \\
& + z_{2,-4}(z_{2,-2})^{-1}(z_{3,-3})^{-1} f_3 + z_{2,-4}(z_{2,-2})^{-1}(z_{4,-3})^{-1} f_4 \\
& + (z_{1,-1})^{-1}(z_{2,-2})^{-1} z_{2,0} z_{3,-3}(z_{3,-1})^{-1} z_{4,-3}(z_{4,-1})^{-1} f_1 \\
& + z_{1,-3}(z_{1,-1})^{-1}(z_{2,-2})^{-1} z_{2,0}(z_{3,-1})^{-1} z_{4,-3}(z_{4,-1})^{-1} f_3 \\
& + z_{1,-3}(z_{1,-1})^{-1}(z_{2,-2})^{-1} z_{2,0} z_{3,-3}(z_{3,-1})^{-1}(z_{4,-1})^{-1} f_4 \\
& + (z_{1,-1})^{-1}(z_{3,-1})^{-1} z_{4,-3}(z_{4,-1})^{-1} f_1 f_3 \\
& + z_{1,-3}(z_{1,-1})^{-1}(z_{3,-1})^{-1}(z_{4,-1})^{-1} f_3 f_4 \\
& + (z_{1,-1})^{-1} z_{3,-3}(z_{3,-1})^{-1}(z_{4,-1})^{-1} f_1 f_4 \\
& + (z_{1,-1})^{-1} z_{2,-2}(z_{2,0})^{-1}(z_{3,-1})^{-1}(z_{4,-1})^{-1} f_1 f_3 f_4 + (z_{2,0})^{-1} f_2.
\end{aligned}
$$

$$\begin{aligned}
=&\mathcal{J}\Big(Y_{2,-6}+Y_{1,-5}\left(Y_{2,-4}\right)^{-1}Y_{3,-5}Y_{4,-5}\\
&+\left(Y_{1,-3}\right)^{-1}Y_{3,-5}Y_{4,-5}+Y_{1,-5}\left(Y_{3,-3}\right)^{-1}Y_{4,-5}+Y_{1,-5}Y_{3,-5}\left(Y_{4,-3}\right)^{-1}\\
&+\left(Y_{1,-3}\right)^{-1}Y_{2,-4}\left(Y_{3,-3}\right)^{-1}Y_{4,-5}+\left(Y_{1,-3}\right)^{-1}Y_{2,-4}Y_{3,-5}\left(Y_{4,-3}\right)^{-1}\\
&+Y_{1,-5}Y_{2,-4}\left(Y_{3,-3}\right)^{-1}\left(Y_{4,-3}\right)^{-1}+\left(Y_{1,-3}\right)^{-1}\left(Y_{2,-4}\right)^{2}\left(Y_{3,-3}\right)^{-1}\left(Y_{4,-3}\right)^{-1}\\
&+\left(Y_{2,-2}\right)^{-1}Y_{4,-5}Y_{4,-3}+\left(Y_{2,-2}\right)^{-1}Y_{3,-5}Y_{3,-3}+Y_{1,-5}Y_{1,-3}\left(Y_{2,-2}\right)^{-1}\\
&+\left(t+t^{-1}\right)Y_{2,-4}\left(Y_{2,-2}\right)^{-1}+Y_{1,-3}\left(Y_{2,-2}\right)^{-2}Y_{3,-3}Y_{4,-3}\\
&+Y_{4,-5}\left(Y_{4,-1}\right)^{-1}+Y_{3,-5}\left(Y_{3,-1}\right)^{-1}+Y_{1,-5}\left(Y_{1,-1}\right)^{-1}\\
&+\left(Y_{1,-3}\right)^{-1}\left(Y_{1,-1}\right)^{-1}Y_{2,-4}+Y_{2,-4}\left(Y_{3,-3}\right)^{-1}\left(Y_{3,-1}\right)^{-1}+Y_{2,-4}\left(Y_{4,-3}\right)^{-1}\left(Y_{4,-1}\right)^{-1}\\
&+\left(Y_{1,-1}\right)^{-1}\left(Y_{2,-2}\right)^{-1}Y_{3,-3}Y_{4,-3}+Y_{1,-3}\left(Y_{2,-2}\right)^{-1}\left(Y_{3,-1}\right)^{-1}Y_{4,-3}\\
&+Y_{1,-3}\left(Y_{2,-2}\right)^{-1}Y_{3,-3}\left(Y_{4,-1}\right)^{-1}+\left(Y_{1,-1}\right)^{-1}\left(Y_{3,-1}\right)^{-1}Y_{4,-3}\\
&+Y_{1,-3}\left(Y_{3,-1}\right)^{-1}\left(Y_{4,-1}\right)^{-1}+\left(Y_{1,-1}\right)^{-1}Y_{3,-3}\left(Y_{4,-1}\right)^{-1}\\
&+\left(Y_{1,-1}\right)^{-1}Y_{2,-2}\left(Y_{3,-1}\right)^{-1}\left(Y_{4,-1}\right)^{-1}+\left(Y_{2,0}\right)^{-1}\Big)\quad=\mathcal{J}\left([L(Y_{2,-6})]_t\right).
\end{aligned}$$

Remark 3.7.1. Note here that the coefficient of $Y_{2,-4}\left(Y_{2,-2}\right)^{-1}$ is $t+t^{-1}$. This coefficient is actually the only coefficient of $[L(Y_{2,-6})]_t$ to not be equal to 1. The quantum cluster variables being bar-invariant, all coefficients in the decomposition of the cluster variables into sums of commutative polynomials in the variables $Y_{i,r}^{\pm 1}$ are symmetric polynomials in $t^{\pm 1}$ ($P(t^{-1})=P(t)$). Thus this is the only coefficient that could have been different from the corresponding one in $[L(Y_{2,-6})]_t$, as it could also have been equal to 2, or any t^k+t^{-k}.

Chapter 4

Further directions

This chapter contains some leads for future work. We present evidence for further developments in the form of partial results.

4.1 Further results in the $\mathfrak{sl}_2$ case

In this section, we restrict ourselves to the case where $\mathfrak{g} = \mathfrak{sl}_2$, and we study representations of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$ or its Borel subalgebra $\mathcal{U}_q(\hat{\mathfrak{b}})$ (see previous chapters for details). The situation of $\mathfrak{sl}_2$ is particular because we have direct and explicit access to a lot of information which is otherwise less accessible.

4.1.1 (q,t)-characters for negative prefundamental representations

In Chapter 2, we defined the quantum Grothendieck ring for the category $\mathcal{O}_{\mathbb{Z}}^{+}$ of representations of $\mathcal{U}_q(\hat{\mathfrak{b}})$ and the (q,t)-characters of the positive prefundamental representations, but the chapter does not deal with the category $\mathcal{O}_{\mathbb{Z}}^{-}$ nor the (q,t)-characters of the negative prefundamental representations. The (q,t)-characters of the positive prefundamental representations are defined as elements of a quantum torus $\mathscr{T}_t$, which are both bar-invariant and recover the q-characters when evaluated at $t=1$ by the evaluation morphism π.

Recall from Chapter 2 that $\mathcal{E}$ is the ring of formal sums of weights, admitting countable sums. Let $\mathcal{E}[t^{\pm 1/2}]$ be the ring of Laurent polynomials in the variable $t^{1/2}$ with coefficients in $\mathcal{E}$. Next, recall that $\mathcal{T}_t$ is the $\mathbb{Z}[t^{\pm 1/2}]$-algebra generated by the variables $z_{2r}^{\pm 1}$, and the t-commutation relations

$$z_{2r} * z_{2s} = t^{f(s-r)} z_{2s} * z_{2r},$$

where $f : \mathbb{Z} \to \mathbb{Z}$ is the anti-symmetrical map defined by $f(m) = \frac{(-1)^m - 1}{2}$ for all $m \geqslant 0$. We have also define a $\mathcal{E}[t^{\pm 1/2}]$-algebra structure on $\mathcal{T}_t \otimes \mathcal{E}$, using the map $\chi : \mathcal{T}_t \otimes \mathcal{E} \to \mathcal{E}$ defined by $\chi(z_{2r}) = [-r\omega_1]$. Then $\mathscr{T}_t = \mathcal{T}_t \hat{\otimes} \mathcal{E}$, where tensor product is completed to allow some countable sums.

And, for all $r \in \mathbb{Z}$,

$$[L^+_{q^{2r}}]_t := [\mathbf{\Psi}_{q^{2r}}] \otimes \chi_1 \in K_t(\mathcal{O}^+_{\mathbb{Z}}) \subset \mathscr{T}_t,$$

where $\chi_1 \in \mathcal{E}$ is the character of $L^+_{q^{2r}}$, using the notation $[\mathbf{\Psi}_{q^{2r}}] = z_{2r} \otimes [r\omega_1]$.

When $\mathfrak{g} = \mathfrak{sl}_2$, the (q,t)-characters of the negative fundamental representations can be defined naturally, because the q-characters of those representations are explicitly known.

Recall that, for all $r \in \mathbb{Z}$,

$$\chi_q(L^-_{q^{2r}}) = [\mathbf{\Psi}^{-1}_{q^{2r}}] \left(1 + A^{-1}_{q^{2r}} + A^{-1}_{q^{2r}} A^{-1}_{q^{2r-2}} + A^{-1}_{q^{2r}} A^{-1}_{q^{2r-2}} A^{-1}_{q^{2r-4}} + \cdots \right) \quad \in \mathcal{E}_l.$$

As this element is multiplicity-free, there is only one element $[L^-_{2r}]_t$ in the quantum torus $\mathscr{T}_t$ with non-negative coefficients when written as a linear combination of commutative polynomials, and such that $\pi\left([L^-_{2r}]_t\right) = \chi_q(L^-_{q^{2r}})$, it is:

$$[L^-_{2r}]_t := [\mathbf{\Psi}^{-1}_{q^{2r}}] \left(1 + A^{-1}_{q^{2r}} + A^{-1}_{q^{2r}} A^{-1}_{q^{2r-2}} + A^{-1}_{q^{2r}} A^{-1}_{q^{2r-2}} A^{-1}_{q^{2r-4}} + \cdots \right), \tag{4.1.1.1}$$

where $[\mathbf{\Psi}^{-1}_{q^{2r}}] = [\mathbf{\Psi}_{q^{2r}}]^{-1} = z^{-1}_{2r}[-r\omega_1]$. Note that we identify the elements of the quantum torus $\mathcal{Y}_t$ with their images in $\mathscr{T}_t$ by the inclusion map $\mathcal{J}$.

This is indeed an element of the quantum torus $\mathscr{T}_t$ because

$$\begin{aligned} |\chi|\left([L^-_{2r}]_t\right) &= 1 + |\chi|(A^{-1}_{q^{2r}}) + |\chi|(A^{-1}_{q^{2r}} A^{-1}_{q^{2r-2}}) + |\chi|(A^{-1}_{q^{2r}} A^{-1}_{q^{2r-2}} A^{-1}_{q^{2r-4}}) + \cdots \\ &= 1 + [-2\omega_1] + [-4\omega_1] + [-6\omega_1] + \ldots = \chi(L^-_{q^{2r}}) = \chi(L^+_{q^{2r}}) = \chi, \end{aligned}$$

which is a countable sum in $\mathcal{E}$.

For any simple finite-dimensional module $L(m)$ of highest ℓ-weight m, define its *normalized* (q,t)*-character* as

$$\tilde{\chi}_{q,t}(L(m)) := m^{-1} \cdot [L(m)]_t, \tag{4.1.1.2}$$

where $\cdot$ is the commutative product between bar-invariant polynomials ($[L(m)]_t$ is bar-invariant by definition).

The prefundamental representations in the category $\mathcal{O}$ also have normalized (q,t)-characters:

$$\tilde{\chi}_{q,t}(L^+_{q^{2r}}) = \chi_1 = \sum_{k \in \mathbb{N}} [-2k\omega_1],$$

$$\tilde{\chi}_{q,t}(L^-_{q^{2r}}) = 1 + \sum_{m=1}^{+\infty} \prod_{k=0}^{m-1} A^{-1}_{q^{2r-2k}}.$$

We note that the normalized (q,t)-character $\tilde{\chi}_{q,t}(L^-_{q^{2r}})$ is the limit, as a formal power series in the variables $A^{-1}_{i,a}$, of the normalized (q,t)-characters of Kirillov-Reshetikhin modules, as in [Nak03a]. For all $r \in \mathbb{Z}$,

$$[L^-_{q^{2r}}]_t = \left(\mathbf{\Psi}_{q^{2r}}\right)^{-1} \lim_{N \to +\infty} \tilde{\chi}_{q,t}\left(W_{N,q^{2r-2N+1}}\right). \tag{4.1.1.3}$$

4.1.2 Quantum Grothendieck ring for the category $\mathcal{O}_{\mathbb{Z}}^-$

The finite-dimensional representations of the quantum affine algebra $\mathcal{U}_q(\hat{\mathfrak{sl}}_2)$ have been completely classified by Chari-Pressley [CP91]. Furthermore, this classification has been extended to the irreducible representations in the category $\mathcal{O}^+$ by Hernandez-Leclerc [HL16b]. Let us recall this result. We start with some definitions.

Definition 4.1.1. A *q-set* is a subset of $\mathbb{C}^\times$ of the form

$$\{aq^{2r_1}, aq^{2r_1+2}, aq^{2r_1+4}, \dots, aq^{2r_2}\}, \tag{4.1.2.1}$$

for some $a \in \mathbb{C}^\times$, and $-\infty \leqslant r_1 \leqslant r_2 \leqslant +\infty$. The Kirillov-Reshetikhin (KR)-modules $W_{k,a}$ and $W_{l,b}$ are said to be in *special position* if the union of the q-sets $\{a, aq^2, \dots, aq^{2k-2}\}$ and $\{b, bq^2, \dots, bq^{2l-2}\}$ is a q-set which contains them both properly. The KR-module $W_{k,a}$ and the positive prefundamental representation L_b^+ are said to be in *special position* if the union of the q-sets $\{a, aq^2, \dots, aq^{2k-2}\}$ and $\{bq, bq^3, bq^5, \dots\}$ is a q-set which contains them both properly. The KR-module $W_{k,a}$ and the negative prefundamental representation L_b^- are said to be in *special position* if the union of the q-sets $\{a, aq^2, \dots, aq^{2k-2}\}$ and $\{\dots, bq^{-5}, bq^{-3}, bq^{-1}\}$ is a q-set which contains them both properly. Two positive prefundamental representations are never in special position, nor are two negative prefundamental representations.

Two representations are in *general position* if they are not in special position.

Theorem 4.1.2. *[HL16b, Theorem 7.9] The prime simple objects in the categories $\mathcal{O}^\pm$ are the positive (resp. negative) prefundamental representations and the the KR-module (up to invertibles). Any simple representation in the categories $\mathcal{O}^\pm$ can be factorized in a unique way as a tensor product of positive (resp. negative) prefundamental representations and KR-modules (up to permutation of the factors and up to invertibles). Moreover, such a tensor product is simple if and only if all its factors are pairwise in general position.*

From Theorem 4.1.2 we deduce naturally that the Grothendieck ring of the category $\mathcal{O}_{\mathbb{Z}}^-$ is algebraically generated by the classes of the KR-modules in $\mathcal{O}_{\mathbb{Z}}^-$ and the negative prefundamental representations $[L_{q^{2r}}^-]$, for $r \in \mathbb{Z}$. Moreover, the classes of the KR-modules are algebraically generated by the classes of the fundamental modules. Thus, we define the quantum Grothendieck ring for the category $\mathcal{O}_{\mathbb{Z}}^-$ as follows.

Definition 4.1.3. Let K_t^- be the $\mathcal{E}[t^{\pm 1/2}]$-subalgebra of $\mathcal{T}_t \otimes \mathcal{E}$ generated by the elements $[L(Y_{q^{2r-1}})]_t$ and $[L_{q^{2s}}^-]_t$, for $k \geqslant 0$ and $r, s \in \mathbb{Z}$. Then

$$K_t(\mathcal{O}_{\mathbb{Z}}^-) := K_t^- \hat{\otimes}_{\mathbb{Z}[t^{\pm 1/2}]} \mathcal{E}, \tag{4.1.2.2}$$

where the tensor product is completed to allow countable sums, as for $\mathcal{T}_t$.

Inside this quantum Grothendieck ring, one can write a new quantum Baxter relation.

Proposition 4.1.4. *For all $r \in \mathbb{Z}$, the following relation is satisfied in $K_t(\mathcal{O}_{\mathbb{Z}}^-)$:*

$$[L(Y_{q^{2r+1}})]_t * [L_{q^{2r}}]_t = t^{1/2}[\omega_1][L_{q^{2r+2}}^-]_t + t^{-1/2}[-\omega_1][L_{q^{2r-2}}^-]_t. \tag{4.1.2.3}$$

Proof. For all $r \in \mathbb{Z}$,

$$\begin{aligned}
[L(Y_{q^{2r+1}})]_t * [L^-_{q^{2r}}]_t &= \left(Y_{q^{2r+1}}\left(1 + A^{-1}_{q^{2r+2}}\right)\right) * \left([\mathbf{\Psi}^{-1}_{q^{2r}}]\left(1 + A^{-1}_{q^{2r}} + A^{-1}_{q^{2r}}A^{-1}_{q^{2r-2}} + \cdots\right)\right) \\
&= t^{1/2}[\omega_1][\mathbf{\Psi}^{-1}_{q^{2r+2}}]\left(1 + t^{-1}A^{-1}_{q^{2r}}\left(1 + A^{-1}_{q^{2r-2}} + A^{-1}_{q^{2r-2}}A^{-1}_{q^{2r-4}} + \cdots\right)\right. \\
&\left. + A^{-1}_{q^{2r+2}} + A^{-1}_{q^{2r+2}}A^{-1}_{q^{2r}} + A^{-1}_{q^{2r+2}}A^{-1}_{q^{2r}}A^{-1}_{q^{2r-2}} + \cdots\right) \\
&= t^{1/2}[\omega_1][L^-_{q^{2r+2}}]_t + t^{-1/2}[-\omega_1][L^-_{q^{2r-2}}]_t.
\end{aligned}$$

□

Remark 4.1.5. The quantum Grothendieck ring $K_t(\mathcal{O}^+_{\mathbb{Z}})$ is defined in Chapter 2 as a quantum cluster algebra. However, we can see that both definitions account to the same ring. Let K^+_t be the $\mathcal{E}[t^{\pm 1/2}]$-subalgebra of $\mathcal{T}_t \otimes \mathcal{E}$ generated by the elements $[L(Y_{q^{2r-1}})]_t$ and $[L^+_{q^{2s}}]_t$, for $k \geqslant 0$ and $r, s \in \mathbb{Z}$. From Chapter 2, the classes of the fundamental modules $[L(Y_{q^{2r-1}})]_t$ are the cluster variables obtained at vertex $(1, 2r)$ after mutation in direction $(1, 2r)$, thus

$$\mathcal{E}[t^{\pm 1/2}]\left[[L(Y_{q^{2r-1}})]_t, [L^+_{q^{2s}}]_t \mid r, s \in \mathbb{Z}\right] \tag{4.1.2.4}$$

is the lower bound of the cluster algebra $\mathcal{A}_t(\Gamma, \Lambda)$. Besides, this cluster algebra is acyclic, as the quiver Γ is an infinite line:

$$\cdots \longrightarrow (1,-4) \longrightarrow (1,-2) \longrightarrow (1,0) \longrightarrow (1,2) \longrightarrow (1,4) \longrightarrow \cdots.$$

Thus, by [BZ05, Theorem 7.5], the cluster algebra $\mathcal{A}_t(\Gamma, \Lambda)$ is equal to its lower bound K^+_t, and

$$K_t(\mathcal{O}^+_{\mathbb{Z}}) := K^+_t \hat{\otimes}_{\mathbb{Z}[t^{\pm 1/2}]} \mathcal{E}. \tag{4.1.2.5}$$

Definition 4.1.6. Let $K_t(\mathcal{O}_{\mathbb{Z}})$ be the $\mathcal{E}[t^{\pm 1/2}]$-subalgebra of $\mathscr{T}_t$ generated by all elements in $K_t(\mathcal{O}^+_{\mathbb{Z}})$ and $K_t(\mathcal{O}^-_{\mathbb{Z}})$.

4.1.3 (q,t)-characters of simple modules

Recall the following result, valid for all simple finite-dimensional Lie algebra $\mathfrak{g}$ of simply-laced type.

Proposition 4.1.7. *Let L and L' be two simple finite-dimensional $\mathcal{U}_q(\hat{\mathfrak{g}})$-modules. If the tensor product $L \otimes L' \simeq L''$ is simple then there exists $k \in \mathbb{Z}/2$ such that*

$$[L]_t * [L']_t = t^{2k}[L']_t * [L]_t = t^k [L'']_t. \tag{4.1.3.1}$$

When the (q,t)-characters of two representation satisfy an equality such as (4.1.3.1), we say that they *t-commute*. This result was stated in this form in [HL15, Corollary 5.5], but is actually a direct consequence of the positivity of the structure constants proven in [VV03].

In this section, we will generalize Proposition 4.1.7 to the categories $\mathcal{O}^{\pm}_{\mathbb{Z}}$.

First, let us state a lemma, also satisfied for $\mathfrak{g}$ of any simply-laced type.

Lemma 4.1.8. *For all $(i,r) \in \hat{I}$, for all $(j,s) \neq (i,r)$, $[\mathbf{\Psi}_{i,r}]$ and $A^{-1}_{j,s}$ commute, and*

$$[\mathbf{\Psi}_{i,r}] * A^{-1}_{i,r} = t^{-2} A^{-1}_{i,r} * [\mathbf{\Psi}_{i,r}]. \tag{4.1.3.2}$$

Proof. For all $(j,s) \in \hat{I}$, the image of $A_{j,s}^{-1}$ by the map $\mathcal{J}$ is

$$\mathcal{J}(A_{j,s}^{-1}) = \mathcal{J}\left(Y_{j,s-1}^{-1}Y_{j,s+1}^{-1}\prod_{k\sim j}Y_{k,s}\right) = [\alpha_i][\mathbf{\Psi}_{j,s+2}][\mathbf{\Psi}_{j,s-2}^{-1}]\prod_{k\sim j}[\mathbf{\Psi}_{k,s-1}][\mathbf{\Psi}_{k,s+1}^{-1}].$$

Thus, for all $(i,r),(j,s)) \in \hat{I}^2$, $[\mathbf{\Psi}_{i,r}] * A_{j,s}^{-1} = t^{\alpha}A_{j,s}^{-1} * [\mathbf{\Psi}_{i,r}]$, with

$$\begin{aligned}\alpha &= \mathcal{F}_{ij}(s-r+2) - \mathcal{F}_{ij}(s-r-2) + \sum_{k\sim j}\left(\mathcal{F}_{ik}(s-r-1) - \mathcal{F}_{ik}(s-r+1)\right),\\ &= -\mathbf{C}_{ij}(s-r+1) - \mathbf{C}_{ij}(s-r-1) + \sum_{k\sim j}\mathbf{C}_{ik}(s-r),\\ &= -2\delta_{i,j} \quad \text{by Lemma } (A.2.6).\end{aligned}$$

□

We deduce the following.

Lemma 4.1.9. *If a KR-module and a positive (resp. negative) prefundamental representation are in general position, then their (q,t)-characters t-commute. Two positive (resp. negative) prefundamental representations always t-commute.*

Proof. The (q,t)-characters of positive and negative prefundamental representations being of very different forms, both cases will be treated separately.

- For the category $\mathcal{O}^+$: for all $r,s \in \mathbb{Z}$, as $[L_{q^{2r}}]_t = [\mathbf{\Psi}_{q^{2s}}]\chi_1$, the (q,t)-characters $[L_{q^{2r}}]_t$ and $[L_{q^{2s}}]_t$ necessarily t-commute.
  The KR-module $W_{k,q^{2r+1}}$ and the positive prefundamental representation $L_{q^{2s}}^+$ are in special position if and only if $r < s \leqslant r+k$. The commutative monomials appearing in $[W_{k,q^{2r+1}}]_t$ are of the form

$$Y_{q^{2r+1}}Y_{q^{2r+3}}\cdots Y_{q^{2r+2k-1}}\prod_{l=0}^{m-1}A_{q^{2r+2k-2l}}^{-1}, \tag{4.1.3.3}$$

  with $0 \leqslant m \leqslant k$.
  If $r+k < s$ or $s \leqslant r$, $[\mathbf{\Psi}_{q^{2s}}]$ commutes with $\prod_{l=0}^{m-1}A_{q^{2r+2k-2l}}^{-1}$ for all $0 \leqslant m \leqslant k$ by Lemma 4.1.8. Hence, $[W_{k,q^{2r+1}}]_t$ and $[L_{q^{2s}}^+]_t$ t-commute.
- For the category $\mathcal{O}^-$: The KR-module $W_{k,q^{2r+1}}$ and the negative prefundamental representation $L_{q^{2s}}^-$ are in special position if and only if $r \leqslant s < r+k$. The commutative monomials of $[L_{q^{2s}}^-]_t$ are of the form

$$[\mathbf{\Psi}_{q^{2s}}^{-1}]\prod_{p=0}^{q-1}A_{q^{2s-2p}}^{-1}, \tag{4.1.3.4}$$

  with $0 \leqslant q$.
  If $s < r$, all monomials $\prod_{l=0}^{m-1}A_{q^{2r+2-2l}}^{-1}$ commute with all monomials $\prod_{p=0}^{q-1}A_{q^{2s-2p}}^{-1}$, for $0 \leqslant m \leqslant k$ and $0 \leqslant q$. Moreover, the monomials $\prod_{l=0}^{m-1}A_{q^{2r+2-2l}}^{-1}$ commute with the highest ℓ-weight $[\mathbf{\Psi}_{q^{2s}}^{-1}]$ and all monomials $\prod_{p=0}^{q-1}A_{q^{2s-2p}}^{-1}$ commute with the highest ℓ-weight of $W_{k,q^{2r+1}}$. Hence, the (q,t)-characters $[W_{k,q^{2r+1}}]_t$ and $[L_{q^{2s}}^-]_t$ t-commute.

If $r+k \leqslant s$, we use the characterization of the normalized $[L^-_{q^{2s}}]_t$ as a limit of normalized (q,t)-characters of KR-modules of (4.1.1.3) :

$$[L^-_{q^{2s}}]_t = [\mathbf{\Psi}^{-1}_{q^{2s}}] \lim_{N\to+\infty} \tilde{\chi}_{q,t}(W_{N,q^{2s-2N+1}}).$$

For N large enough, the q-set

$$\{q^{2r+1}, q^{2r+3}, \ldots, q^{2r+2k-1}\}$$

is included in the q-set

$$\{q^{2s-2N+1}, q^{2s-2N+3}, \ldots, q^{2s-1}\}.$$

These q-sets are in general position and thus the tensor product of the KR-modules $W_{k,q^{2r+1}}$ and $W_{N,q^{2s-2N+1}}$ is simple by Theorem 4.1.2, and their (q,t)-characters t-commute by Proposition 4.1.7.
Now let $\alpha \in \mathbb{Z}$ be the t-commutation factor between the highest ℓ-weights of $W_{k,q^{2r+1}}$ and $L^-_{q^{2s}}$

$$m_{k,q^{2r+1}} * [\mathbf{\Psi}^{-1}_{q^{2s}}] = t^{\alpha}[\mathbf{\Psi}^{-1}_{q^{2s}}] * m_{k,q^{2r+1}}.$$

And form the difference

$$\begin{aligned}\Delta &= [W_{k,q^{2r+1}}]_t * [L^-_{q^{2s}}]_t - t^{\alpha}[L^-_{q^{2s}}]_t * [W_{k,q^{2r+1}}]_t,\\ &= t^{\alpha/2} m_{k,q^{2r+1}}[\mathbf{\Psi}^{-1}_{q^{2s}}]\mathsf{A},\end{aligned}$$

where A is a formal power series in the variables $A^{-1}_{q^{2p}}$, with coefficients in $\mathbb{Z}[t^{\pm 1/2}]$. Fix a degree d in those variables and let A_d be the component of A of degree d.
Proceed similarly with the (q,t)-characters $[W_{k,q^{2r+1}}]_t$ and $[W_{N,q^{2s-2N+1}}]_t$; let α_N be the t-commutation factor of their highest ℓ-weights, and form the difference

$$\begin{aligned}\Delta' &= [W_{k,q^{2r+1}}]_t * [W_{N,q^{2s-2N+1}}]_t - t^{\alpha_N}[W_{N,q^{2s-2N+1}}]_t * [W_{k,q^{2r+1}}]_t,\\ &= t^{\alpha_N/2} m_{k,q^{2r+1}} m_{N,q^{2s-2N+1}}\mathsf{B}.\end{aligned}$$

For N large enough, the component B_d of degree d of B is equal to A. However, for N large enough, the (q,t)-characters t-commute and $\Delta' = 0$. Thus, $\mathsf{A}_d = 0$ for all degree d and the (q,t)-characters $[W_{k,q^{2r+1}}]_t$ and $[L^-_{q^{2s}}]_t$ t-commute.
Moreover, for all $r, s \in \mathbb{Z}$, the (q,t)-characters of the negative prefundamental representations $L^-_{q^{2r}}$ and $L^-_{q^{2s}}$ t-commute, and this is shown exactly as above.

□

For all positive (resp. negative) ℓ-weight $\mathbf{\Psi}$, from the result of Theorem 4.1.2, write $L(\mathbf{\Psi})$ as a tensor product of KR-modules and positive (resp. negative) ℓ-weights in pairwise general position, with some possible weight:

$$L(\mathbf{\Psi}) = [\lambda] \otimes \bigotimes_{m=1}^{p} W_{k_m,q^{2r_m+1}} \otimes \bigotimes_{l=1}^{q} L^{\pm}_{q^{2s_l}}.$$

And the (q,t)-character of $L(\mathbf{\Psi})$ is defined as

$$[L(\mathbf{\Psi})]_t := [\lambda] \prod_{m=1}^{p} [W_{k_m,q^{2r_m+1}}]_t \prod_{l=1}^{q} [L^{\pm}_{q^{2s_l}}]_t, \qquad (4.1.3.5)$$

where we take the *commutative product*, which is well-defined because the factors pairwise t-commute from Lemma 4.1.9. Naturally,

$$\overline{[L(\mathbf{\Psi})]_t} = [L(\mathbf{\Psi})]_t.$$

As a direct consequence, one has the following.

Proposition 4.1.10. *Let L and L' be simple modules in the category $\mathcal{O}^+$ (resp. $\mathcal{O}^-$). If their tensor product $L \otimes L' \simeq L''$ is simple then their (q,t)-characters $[L]_t$ and $[L']_t$ t-commute.*

4.1.4 Quantum Wronskian relation

The quantum Baxter relations take place in the quantum Grothendieck rings $K_t(\mathcal{O}^{\pm}_{\mathbb{Z}})$. However, now that we have defined a full quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}})$, we can write relations which involve both positive and negative prefundamental representations. Among such relations, one can find as important examples the *quantum Wronskian relations* (for $\mathfrak{g} = \mathfrak{sl}_2$) or more generally the *$Q\tilde{Q}$-systems* (for more general $\mathfrak{g}$). The quantum Wronskian relation was described by Bazhanov-Lukyanov-Zamolodchikov in [BLZ99, BLZ03]. It encodes a famous system of relations called the *Bethe Ansatz equations*, which are important in the study of quantum integrable systems (see Introduction). In [KSZ18, Section 3.2], the link between the quantum Wronskian and the Bethe Ansatz was studied in more details. In [FH18], the quantum Wronskian relation was explicitly written as a relation in the Grothendieck ring $K_0(\mathcal{O})$. For all $a \in \mathbb{C}^\times$, one has

$$[L_a^+] \cdot [L_a^-] - [-\alpha_1][L_{aq^2}^+] \cdot [L_{aq^{-2}}^-] = \chi_1. \tag{4.1.4.1}$$

In [FH18], Frenkel and Hernandez wrote generalizations of the quantum Wronskian relation, in the form of $Q\tilde{Q}$-relations. The $Q\tilde{Q}$-relations were previously introduced by Masoero, Raimondo, and Valeri in [MRV16, MRV17], in the framework of affine opers, which are differential operators in one variable associated to the affine Kac–Moody algebra that is Langlands dual to $\hat{\mathfrak{g}}$, and were found to have links with the description of the spectra of quantum $\hat{g}$-KdV Hamiltonians. Frenkel and Hernandez proved analogs of those relations in the Grothendieck ring $K_0(\mathcal{O})$. For simply-laced types, they can be written as follows.

For all $i \in I$ and $a \in \mathbb{C}^\times$, consider the simple $\mathcal{U}_q(\hat{\mathfrak{g}})$-module

$$X_{i,a} := L(\mathbf{\Psi}_{i,a}^{-1} \prod_{j \sim i} \mathbf{\Psi}_{j,aq}). \tag{4.1.4.2}$$

From then, let Q and $\tilde{Q}$ be the following elements of $K_0(\mathcal{O})$, for all $i \in I$ and $a \in \mathbb{C}^\times$,

$$\begin{aligned} Q_{i,a} &= [L_{i,a}^+], \\ \tilde{Q}_{i,a} &= [X_{i,aq^{-2}}]\chi_i^{-1} \left[\frac{-\alpha_i}{2}\right]. \end{aligned}$$

Then, from [FH18, Theorem 3.2] in $K_0(\mathcal{O})$,

$$\left[\frac{\alpha_i}{2}\right] Q_{i,aq^{-1}}\tilde{Q}_{i,aq} - \left[\frac{-\alpha_i}{2}\right] Q_{i,aq}\tilde{Q}_{i,aq^{-1}} = \prod_{j \sim i} Q_{j,a}. \tag{4.1.4.3}$$

One note that when $\mathfrak{g} = \mathfrak{sl}_2$, $X_a = L_a^-$, thus $\tilde{Q}_a = [L^-_{aq^{-2}}]\chi_1^{-1}\left[\frac{-\alpha_1}{2}\right]$, and relation (4.1.4.3) reduces indeed to relation (4.1.4.1) in this case.

We have a quantum analog of relation (4.1.4.1) in our quantum Grothendieck ring $K_t(\mathcal{O})$.

Proposition 4.1.11. *For all $r \in \mathbb{Z}$,*

$$[L^+_{q^{2r}}]_t * [L^-_{q^{2r}}]_t - t^{-1}[-\alpha_1][L^+_{q^{2r+2}}]_t * [L^+_{q^{2r-2}}]_t = \chi_1. \tag{4.1.4.4}$$

Proof. For all $r \in \mathbb{Z}$,

$$\begin{aligned}[L^+_{q^{2r}}]_t * [L^-_{q^{2r}}]_t &= \left([\mathbf{\Psi}_{q^{2r}}]\chi_1\right) * \left([\mathbf{\Psi}^{-1}_{q^{2r}}](1 + A^{-1}_{q^{2r}} + A^{-1}_{q^{2r}}A^{-1}_{q^{2r-2}} + A^{-1}_{q^{2r}}A^{-1}_{q^{2r-2}}A^{-1}_{q^{2r-4}} + \cdots)\right)\\ &= \chi_1 + t^{-1}A^{-1}_{q^{2r}}\left(1 + A^{-1}_{q^{2r-2}} + A^{-1}_{q^{2r-2}}A^{-1}_{q^{2r-4}} + \cdots\right)\chi_1,\end{aligned}$$

by Lemma 4.1.8. And

$$\begin{aligned}[L^+_{q^{2r+2}}]_t * [L^-_{q^{2r-2}}]_t &= \left([\mathbf{\Psi}_{q^{2r+2}}]\chi_1\right) * \left([\mathbf{\Psi}^{-1}_{q^{2r-2}}](1 + A^{-1}_{q^{2r-2}} + A^{-1}_{q^{2r-2}}A^{-1}_{q^{2r-4}} + \cdots)\right)\\ &= [\alpha_1]A^{-1}_{q^{2r}}\left(1 + A^{-1}_{q^{2r-2}} + A^{-1}_{q^{2r-2}}A^{-1}_{q^{2r-4}} + \cdots\right)\chi_1.\end{aligned}$$

Thus

$$[L^+_{q^{2r}}]_t * [L^-_{q^{2r}}]_t = \chi_1 + t^{-1}[-\alpha_1][L^+_{q^{2r+2}}]_t * [L^-_{q^{2r-2}}]_t.$$

□

4.1.5 (q, t)-characters of asymptotical standard modules

In Chapter 1, we defined *asymptotical standard modules* as $\mathcal{U}_q(\hat{\mathfrak{b}})$-modules. These are meant to play the role of the standard modules studied by Nakajima [Nak04] and Varagnolo-Vasserot [VV02b] in the context of the category $\mathcal{O}^-$. In [Nak04], Nakajima used this standard modules to produce an algorithm (a type of "Kazhdan-Lusztig algorithm") to compute the (q, t)-characters of all the simple representations of the quantum affine algebra.

The aim of the construction of these asymptotical standard modules was to obtain a similar algorithm to compute (q, t)-characters of all irreducible $\mathcal{U}_q(\hat{\mathfrak{b}})$-modules in the category $\mathcal{O}^-$. Of course, for $\mathfrak{g} = \mathfrak{sl}_2$, the (q, t)-characters of the simple modules can be obtained via the classification Theorem 4.1.2 and the knowledge of the (q, t)-characters of the KR-modules and the negative prefundamental modules, as we did in (4.1.3.5). But with the goal in mind to generalize these results to other types than $\mathfrak{sl}_2$, it is interesting to complete the picture by producing (q, t)-characters for those asymptotical standard modules. Moreover, the idea behind the construction of the asymptotical standard modules was the limit character formula obtained in Chapter 1, Section 1.2. Thus, it is this formula we need to t-deform.

Recall that, for all $r \in \mathbb{Z}$,

$$\chi_q(M(\mathbf{\Psi}^{-1}_{q^{2r}})) = [\mathbf{\Psi}^{-1}_{q^{2r}}]\sum_{J \in \mathcal{P}_f(\mathbb{N})}\prod_{j \in J} A^{-1}_{q^{2r-2j}}, \tag{4.1.5.1}$$

where $\mathcal{P}_f(\mathbb{N})$ denotes the finite subsets of $\mathbb{N}$.

This q-character is multiplicity-free, hence there is only one way to define a bar-invariant (q,t)-character with positive coefficients which specializes at $\chi_q(M(\mathbf{\Psi}_{q^{2r}}^{-1}))$ at $t=1$. The main issue is to define a quantum torus containing such a formula.

Let $\mathcal{A}_{\mathcal{E}}$ be the non-commutative ring of formal series in variables $A_{q^{2r}}^{-1}$, with coefficients in $\mathcal{E}[t^{\pm 1/2}]$,

$$\mathcal{A}_{\mathcal{E}} := \mathcal{E}[t^{\pm 1/2}][[A_{q^{2r}}^{-1}]]_{r\in\mathbb{Z}}, \tag{4.1.5.2}$$

where the t-commutation relations are defined by:

$$A_{q^{2r}}^{-1} * A_{q^{2s}}^{-1} = t^{\alpha(r,s)} A_{q^{2s}}^{-1} * A_{q^{2r}}^{-1}, \quad \alpha(r,s) = \begin{cases} -2 & \text{if } s = r+1, \\ 2 & \text{if } s = r-1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1.5.3}$$

The inclusion of quantum tori $\mathcal{J}$ satisfies

$$\mathcal{J}(A_{q^{2r}}^{-1}) = z_{2r+2} z_{2r-2}^{-1}. \tag{4.1.5.4}$$

Thus, the non-commutative ring of polynomials $\mathcal{E}[t^{\pm 1/2}][A_{q^{2r}}^{-1}]_{r\in\mathbb{Z}}$ is included, via the injective map $\mathcal{J}$, in the $\mathcal{E}[t^{\pm 1/2}]$-algebra $\mathcal{T}_t \otimes \mathcal{E}$.

Finally, let

$$\hat{\mathscr{T}_t} := (\mathcal{T}_t \otimes \mathcal{E}) \otimes_{\mathcal{E}[t^{\pm 1/2}][A_{q^{2r}}^{-1}]} \mathcal{A}_{\mathcal{E}}. \tag{4.1.5.5}$$

One can define commutative monomials on $\hat{\mathscr{T}_t}$ exactly as in Chapter 2 Section 2.3.

For all $r \in \mathbb{Z}$, let

$$[M(\mathbf{\Psi}_{q^{2r}}^{-1})]_t := [\mathbf{\Psi}_{q^{2r}}^{-1}] \sum_{J\in\mathcal{P}_f(\mathbb{N})} \prod_{j\in J} A_{q^{2r-2j}}^{-1} \quad \in \hat{\mathscr{T}_t}, \tag{4.1.5.6}$$

written on the basis of the commutative monomials.

More generally, for all ℓ-weights $\mathbf{\Psi}$ such that $L(\mathbf{\Psi})$ is in $\mathcal{O}_{\mathbb{Z}}^-$. Write $\mathbf{\Psi}$ as a finite product

$$[\lambda] \times m \times \prod_{r\in\mathbb{Z}} [\mathbf{\Psi}_{q^{2r}}^{-v_r}], \tag{4.1.5.7}$$

where λ is a weight, m is a monomial in the variables $Y_{q^{2s}}$, and all but finitely many of the $v_r \in \mathbb{N}$ are 0, and the number of $Y_{q^{2r+1}}$ in the monomial m is minimal for such a writing.

Then the (q,t)-character of the standard module $M(\mathbf{\Psi})$, as defined in Chapter 1 is

$$[M(\mathbf{\Psi})]_t := t^{\alpha}[\lambda][M(m)]_t * \overleftarrow{\circledast}_{r\in\mathbb{Z}} [M(\mathbf{\Psi}_{q^{2r}}^{-1})]_t^{*v_r}, \tag{4.1.5.8}$$

where $a \in \frac{1}{2}\mathbb{Z}$ is fixed such that when the right-hand term is written on the basis of the commutative monomials, the coefficient of the highest ℓ-weight $\mathbf{\Psi}$ is 1.

Note that contrary to (4.1.3.5), the terms in this product do not necessarily t-commute, so an order has to be chosen, and we fix a decreasing order on $r \in \mathbb{Z}$.

4.2 Leads for other types

We have hope to be able to extend some of these results outside of the case where $\mathfrak{g} = \mathfrak{sl}_2$. However, as this an ongoing project, we only present here ideas for these generalizations.

The first level of generalization is to $\mathfrak{g}$ of any simply-laced type.

4.2.1 The (q,t)-characters of the negative prefundamental representations

Here, the simple Lie algebra $\mathfrak{g}$ is supposed to be of any simply-laced type.

Using what we did for $\mathfrak{sl}_2$ in Section 4.1.1, one could define the normalized (q,t)-character of the negative prefundamental representation L^-_{i,q^r} as the limit of normalized (q,t)-characters of KR-modules:

$$[L^-_{i,q^r}]_t := (\mathbf{\Psi}_{i,q^r})^{-1} \lim_{N\to+\infty} \tilde{\chi}_{q,t}\left(W^{(i)}_{N,q^{r-2N+1}}\right) \quad \in \mathscr{T}_t. \tag{4.2.1.1}$$

Indeed, Nakajima proved in [Nak03a] the existence of this limit. The advantage of this definition is that one can extend properties known in the finite-dimensional context to that of the category $\mathcal{O}^-_{\mathbb{Z}}$.

For example, when $\mathfrak{g}$ is simply-laced, Nakajima proved in [Nak04] that the coefficients of the (q,t)-characters of the simple modules were non-negative. Thus, as each term in the (q,t)-character of $[L^-_{i,q^r}]_t$ corresponds to a term in a (q,t)-character of a KR-modules, we deduce that

$$[L^-_{i,q^r}]_t \in (\mathbf{\Psi}_{i,q^r})^{-1}\left(1 + A^{-1}_{i,q^r}\mathbb{N}[t^{\pm 1/2}][[A^{-1}_{j,q^s}]]\right). \tag{4.2.1.2}$$

From there, one can hope that, similarly as Lemma 4.1.9, the (q,t)-characters of two negative prefundamental representations are t-commuting elements of the quantum torus $\mathscr{T}_t$.

Of course, for other types than $\mathfrak{sl}_2$, there is no classification theorem as complete as Theorem 4.1.2. However, it is known that a tensor product of two negative prefundamental representations is simple, so this conjectural result is a requirement for (4.2.1.2) to be an acceptable definition of the (q,t)-characters.

4.2.2 Quantum Grothendieck ring for the category $\mathcal{O}^-_{\mathbb{Z}}$

Our lead for the definition of the quantum Grothendieck ring $K_t(\mathcal{O}^-_{\mathbb{Z}})$ is based on our approach for the quantum Grothendieck ring $K_t(\mathcal{O}^+_{\mathbb{Z}})$ in Chapter 2. From [HL16b, Theorem 5.17], we know that the Grothendieck rings $K_0(\mathcal{O}^+_{\mathbb{Z}})$ and $K_0(\mathcal{O}^-_{\mathbb{Z}})$ are isomorphic as $\mathcal{E}$-algebras,

$$D : K_0(\mathcal{O}^+_{\mathbb{Z}}) \xrightarrow{\sim} K_0(\mathcal{O}^-_{\mathbb{Z}}), \tag{4.2.2.1}$$

such that

$$D([L^+_{i,q^r}]) = [L^-_{i,q^{-r}}]. \tag{4.2.2.2}$$

Hence, it would make sense for their quantum Grothendieck rings to be isomorphic as $\mathcal{E}[t^{\pm 1/2}]$-algebras. Thus,

$$K_t(\mathcal{O}^-_{\mathbb{Z}}) \approx \mathcal{A}_t(\Gamma, \Lambda), \tag{4.2.2.3}$$

as $\mathcal{E}[t^{\pm 1/2}]$-algebras. This identification provides the *structure* of the $\mathcal{E}[t^{\pm 1/2}]$-algebra $K_t(\mathcal{O}^-_{\mathbb{Z}})$, and the results of Chapter 3 show that

$$K_t(\mathscr{C}_{\mathbb{Z}}) \subset K_t(\mathcal{O}^-_{\mathbb{Z}}). \tag{4.2.2.4}$$

However, the identification (4.2.2.3) does not provide information about the (q,t)-characters, as the morphism D is not compatible with the q-character morphism (see [HL16b, Remark 5.15]).

A more interesting angle to study this quantum Grothendieck ring would be to take the (q,t)-characters of negative prefundamental representations (4.2.1.2) as initial seed of a quantum cluster algebra built on the quiver Γ, after showing that these elements of the quantum torus t-commute. Moreover, their t-commutations relations are fixed by their highest ℓ-weights, $\boldsymbol{\Psi}_{i,q^r}^{-1}$, and they are thus the same than between the $z_{i,-r}^{-1}$. This would give a new identification

$$K_t(\mathcal{O}_{\mathbb{Z}}^-) \approx \mathcal{A}_t(\Gamma, \Lambda), \tag{4.2.2.5}$$

which takes into account (q,t)-characters.

Remark 4.2.1. The difference of identifications (4.2.2.3) and (4.2.2.5) echoes the two identifications of $K_t(\mathscr{C}_{\mathbb{Z}}^-)$ in Chapter 3, by (q,t)-characters and truncated (q,t)-characters.

By performing quantum cluster mutations on the quantum cluster algebra $\mathcal{A}_t(\Gamma, \Lambda)$, one should still get the inclusion of quantum Grothendieck rings (4.2.2.4) with this new identification. However, recognizing (q,t)-characters of finite-dimensional modules formed by using Laurent series in (q,t)-characters of negative prefundamental representations should be as straightforward as in the positive case, although one may have to use some duality (see [HL16b, Example 5.14]).

Although this would give a new algorithm to obtain the (q,t)-characters of the fundamental modules, it is not something that makes sense from an algorithmic point of view. Indeed, it requires knowing the (q,t)-characters of the negative prefundamental representations, which are obtained as limits of (q,t)-characters of KR-modules, and are thus more difficult to access than the (q,t)-characters of the fundamental modules.

Once both the quantum Grothendieck rings $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ and $K_t(\mathcal{O}_{\mathbb{Z}}^-)$ have good realizations in terms of (q,t)-characters inside the quantum torus $\mathscr{T}_t$, one can define the full quantum Grothendieck ring $K_t(\mathcal{O}_{\mathbb{Z}})$, as the $\mathcal{E}[t^{\pm 1/2}]$-subalgebra of the quantum torus generated by the quantum Grothendieck rings $K_t(\mathcal{O}_{\mathbb{Z}}^+)$ and $K_t(\mathcal{O}_{\mathbb{Z}}^-)$.

As before, with this full quantum torus, one can write new relations involving both the positive and the negative prefundamental representations, such as the $Q\tilde{Q}$-systems.

4.2.3 Extension to non-simply-laced types

Most of the work done in this thesis (with the exception of some results in Chapter 1) is restricted to the simple Lie algebras $\mathfrak{g}$ of simply-laced types, but there is hope we will be able to extend it in the future.

First of all, one would need a generalization of the t-commutation function $\mathcal{F}$ from (2.3.2.2), satisfying a more general form of relation (I.2.2.2), while forming a compatible pair with the quiver Γ. This would allow for the definition of the quantum Grothendieck ring of the category $\mathcal{O}_{\mathbb{Z}}^+$ exactly as in Definition 2.4.15.

Next, we would want to extend Conjecture 2.5.2 and Conjecture 2.5.7 from Chapter 2, proved in Theorem 3.6.5 in Chapter 3. As the results from [HL16b] (there exists finite sequences of mutations to obtain the classes of the fundamental modules in the cluster algebra $\mathcal{A}(\Gamma)$) are not restricted to the simply-laced types, these conjectures extend verbatim.

Now, the proof of Theorem 3.6.5 relies heavily on the *quantum T-systems*, a family of relations in $K_t(\mathscr{C})$, which as of yet have only been stated in types ADE by Nakajima in [Nak03a] (see also [HL15, Proposition 5.6]), and by Hernandez-Oya in [HO19] in type B. Moreover, we have used in a few separate instances the fact that the (q,t)-characters of simple modules had non-negative coefficients. As before, this result is known in simply-laced types from [Nak04] and proven partially in type B in [HO19]. If the approach from Hernandez and Oya is generalized to all non-simply-laced types, then our proof could also be extended to all these types.

Appendix

Appendix A

Cartan data and quantum Cartan data

Let us fix some notations we use in Chapters 2, 3 and 4.

A.1 Root data

Let the $\mathfrak{g}$ be a simple Lie algebra of rank n and of type A,D or E. Let γ be the Dynkin diagram of $\mathfrak{g}$ and let $I := \{1, \dots, n\}$ be the set of vertices of γ.

The *Cartan matrix* of $\mathfrak{g}$ is the $n \times n$ matrix C such that

$$C_{i,j} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i \sim j \quad (\ i \text{ and } j \text{ are adjacent vertices of } \gamma \ ) , \\ 0 & \text{otherwise.} \end{cases}$$

Let us denote by $(\alpha_i)_{i \in I}$ the simple roots of $\mathfrak{g}$, $(\alpha_i^\vee)_{i \in I}$ the simple coroots and $(\omega_i)_{i \in I}$ the fundamental weights. We will use the usual lattices $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$, $Q^+ = \bigoplus_{i \in I} \mathbb{N}\alpha_i$ and $P = \bigoplus_{i \in I} \mathbb{Z}\omega_i$. Let $P_{\mathbb{Q}} = P \otimes \mathbb{Q}$, endowed with the partial ordering : $\omega \leqslant \omega'$ if and only if $\omega' - \omega \in Q^+$.

The Dynkin diagram of $\mathfrak{g}$ is numbered as in [Kac90], and let $a_1, a_2, \dots, a_n$ be the Kac labels ($a_0 = 1$).

Let h be the (dual) Coxeter number of $\mathfrak{g}$:

$\mathfrak{g}$	A_n	D_n	E_6	E_7	E_8
h	$n+1$	$2n-2$	12	18	30

(A.1.1)

A.2 Quantum Cartan matrix

Let z be an indeterminate.

Definition A.2.1. The *quantum Cartan matrix* of $\mathfrak{g}$ is the matrix $C(z)$ with entries,

$$C_{ij}(z) = \begin{cases} z + z^{-1} & \text{if } i = j, \\ -1 & \text{if } i \sim j, \\ 0 & \text{otherwise.} \end{cases}$$

Remark A.2.2. The evaluation $C(1)$ is the Cartan matrix of $\mathfrak{g}$. As $\det(C) \neq 0$, then $\det(C(z)) \neq 0$ and we can define $\tilde{C}(z)$, the inverse of the matrix $C(z)$. The entries of the matrix $\tilde{C}(z)$ belong to $\mathbb{Q}(z)$.

One can write

$$C(z) = (z + z^{-1})\,\mathrm{Id} - A,$$

where A is the adjacency matrix of γ. Hence,

$$\tilde{C}(z) = \sum_{m=0}^{+\infty} (z + z^{-1})^{-m-1} A^m.$$

Therefore, we can write the entries of $\tilde{C}(z)$ as power series in z. For all $i, j \in I$,

$$\tilde{C}_{ij}(z) = \sum_{m=1}^{+\infty} \tilde{C}_{i,j}(m) z^m \quad \in \mathbb{Z}[[z]]. \tag{A.2.1}$$

Example A.2.3. (i) For $\mathfrak{g} = \mathfrak{sl}_2$, one has

$$\tilde{C}_{11} = \sum_{n=0}^{+\infty} (-1)^n z^{2n+1} = z - z^3 + z^5 - z^7 + z^9 - z^{11} + \cdots \tag{A.2.2}$$

(ii) For $\mathfrak{g} = \mathfrak{sl}_3$, one has

$$\begin{aligned} \tilde{C}_{ii} &= z - z^5 + z^7 - z^{11} + z^{13} + \cdots, \quad 1 \leqslant i \leqslant 2 \\ \tilde{C}_{ij} &= z^2 - z^4 + z^8 - z^{10} + z^{14} + \cdots, \quad 1 \leqslant i \neq j \leqslant 2. \end{aligned}$$

We will need the following lemma :

Lemma A.2.4. *For all $(i, j) \in I^2$,*

$$\begin{aligned} \tilde{C}_{ij}(m-1) + \tilde{C}_{ij}(m+1) - \sum_{k \sim j} \tilde{C}_{ik}(m) = 0, \quad \forall m \geqslant 1, \\ \tilde{C}_{ij}(1) = \delta_{i,j}. \end{aligned}$$

Proof. By definition of $\tilde{C}$, one has

$$\tilde{C}(z) \cdot C(z) = \mathrm{Id} \quad \in \mathcal{M}_n(\mathbb{Q}(z)). \tag{A.2.3}$$

By writing $\tilde{C}(z)$ as a formal power series, and using the definition of $C(z)$, we obtain, for all $(i, j) \in I^2$,

$$\sum_{m=0}^{+\infty} \left(\tilde{C}_{ij}(m)(z^{m+1} + z^{m-1}) - \sum_{k \sim j} \tilde{C}_{ik}(m) z^m \right) = \delta_{i,j} \quad \in \mathbb{C}[[z]]. \tag{A.2.4}$$

Which is equivalent to

$$\begin{aligned} \tilde{C}_{ij}(m-1) + \tilde{C}_{ij}(m+1) - \sum_{k \sim j} \tilde{C}_{ik}(m) = 0, \quad \forall m \geqslant 1, \\ \tilde{C}_{ij}(1) - \sum_{k \sim j} \tilde{C}_{ij}(0) = \delta_{i,j}, \\ \tilde{C}_{ij}(0) = 0. \end{aligned}$$

□

Let us extend the functions $\tilde{C}_{ij}$ to symmetrical functions on $\mathbb{Z}$

$$\mathbf{C}_{i,j}(m) := \tilde{C}_{ij}(m) + \tilde{C}_{ij}(-m) \quad (m \in \mathbb{Z}), \tag{A.2.5}$$

with the usual convention $\tilde{C}_{ij}(m) = 0$ if $m \leqslant 0$.

Then Lemma A.2.4 translates as:

$$\mathbf{C}_{ij}(m-1) + \mathbf{C}_{ij}(m+1) - \sum_{k \sim j} \mathbf{C}_{ik}(m) = \begin{cases} 2\delta_{i,j} & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{A.2.6}$$

A.3 Height function

As $\mathfrak{g}$ is simply-laced, its Dynkin diagram γ is a bipartite graph. There is partition $I = I_0 \sqcup I_1$ such that every edge in γ connects a vertex of I_0 to a vertex of I_1.

Definition A.3.1. Define, for all $i \in I$,

$$\xi_i = \begin{cases} 0 & \text{if } i \in I_0 \\ 1 & \text{if } i \in I_1 \end{cases} \tag{A.3.1}$$

The map $\xi : I \to \{0, 1\}$ is called a *height function* on γ.

Remark A.3.2. In more generality, every function $\xi : I \to \mathbb{Z}$ satisfying

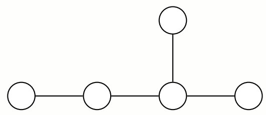
$$\xi(j) = \xi(i) \pm 1, \text{ when } j \sim i$$

is a height function on γ. It defines an orientation of the Dynkin diagram γ:

$$i \to j \text{ if } \xi(j) = i + 1.$$

Our particular choice of height function defines a *sink-source* orientation.

Example A.3.3. If $\mathfrak{g}$ if of type D_5, then γ is



and if we fix $\xi_1 = 0$, then

$$\begin{aligned} \xi_1 &= 1, & \xi_3 &= 1, \\ \xi_2 &= 0, & \xi_4 &= 0. \end{aligned}$$

From now on, we fix such a height function ξ.

We will also use the notation:

$$\epsilon_i := (-1)^{\xi_i} \quad \in \{\pm 1\} \quad (i \in I). \tag{A.3.2}$$

A.4 Infinite quiver

Let us define an infinite quiver Γ as in [HL16a].

First, let

$$\hat{I} := \bigcup_{i \in I} (i, 2\mathbb{Z} + \xi_i)\,. \tag{A.4.1}$$

Let Γ be the quiver with vertex set $\hat{I}$ and arrows

$$((i,r) \to (j,s)) \Longleftrightarrow (C_{i,j} \neq 0 \text{ and } s = r + C_{i,j})\,. \tag{A.4.2}$$

Example A.4.1. For $\mathfrak{g} = \mathfrak{sl}_4$, on choice of $\hat{I}$ is

$$\hat{I} = (1, 2\mathbb{Z}) \cup (2, 2\mathbb{Z}+1) \cup (3, 2\mathbb{Z}),$$

and Γ is the following:

$$\begin{tikzcd}
 & \vdots & \\
\vdots & (2,1) \arrow[u] \arrow[dl] \arrow[dr] & \vdots \\
(1,0) \arrow[u] \arrow[dr] & & (3,0) \arrow[u] \arrow[dl] \\
 & (2,-1) \arrow[uu] \arrow[dl] \arrow[dr] & \\
(1,-2) \arrow[uu] \arrow[dr] & & (3,-2) \arrow[uu] \arrow[dl] \\
 & (2,-3) \arrow[uu] \arrow[dl] \arrow[dr] & \\
(1,-4) \arrow[uu] & \vdots \arrow[u] & (3,-4) \arrow[uu] \\
\vdots \arrow[u] & & \vdots \arrow[u]
\end{tikzcd}$$

Definition A.4.2. Let

$$\hat{I}^- := \hat{I} \cap (I \times \mathbb{Z}_{\leqslant 0}).$$

And define G^- to be the semi-infinite subquiver of Γ of vertex set $\hat{I}^-$.

Let

$$\hat{J} := (I \times \mathbb{Z}) \backslash \hat{I}. \tag{A.4.3}$$

Example A.4.3. Following Example A.4.1, for $\mathfrak{g} = \mathfrak{sl}_4$, with the same choice of height function, one has

$$\hat{I}^- = (1, 2\mathbb{Z}_{\leqslant 0}) \cup (2, 2\mathbb{Z}_{\leqslant 0} - 1) \cup (3, 2\mathbb{Z}_{\leqslant 0}),$$

and G^- is the following:

$$\begin{tikzcd}
(1,0) \arrow[dr] & & (3,0) \arrow[dl] \\
 & (2,-1) \arrow[dl] \arrow[dr] & \\
(1,-2) \arrow[uu] \arrow[dr] & & (3,-2) \arrow[uu] \arrow[dl] \\
 & (2,-3) \arrow[uu] \arrow[dl] \arrow[dr] & \\
(1,-4) \arrow[uu] & \vdots \arrow[u] & (3,-4) \arrow[uu] \\
\vdots \arrow[u] & & \vdots \arrow[u]
\end{tikzcd}$$

Finally, we recall a useful notation from [HL16a]. For $(i, r) \in \hat{I}^-$, define

$$k_{i,r} := \frac{-r + \xi_i}{2}. \tag{A.4.4}$$

The vertex (i, r) is the $k_{i,r}$th vertex in its column in G^-, starting at the top.

Appendix B

Cluster algebras and quantum cluster algebras

Cluster algebras were defined by Fomin and Zelevinsky in the 2000s in a series of fundamental papers [FZ02],[FZ03a],[BFZ05] and [FZ07]. They were first introduced to study total positivity and canonical bases in quantum groups but soon applications to many different fields of mathematics were found.

In [BZ05], Berenstein and Zelevinsky introduced natural non-commutative deformations of cluster algebras called *quantum cluster algebras.*

In this section, we recall the definitions of these objects. The interested reader may refer fto the aforementioned papers for more details, or to reviews, such as [FZ03b].

B.1 Cluster algebras

Let $m \geqslant n$ be two positive integers and let $\mathcal{F}$ be the field of rational functions over $\mathbb{Q}$ in m independent commuting variables. Fix of subset $\mathbf{ex} \subset [\![1, m]\!]$ of cardinal n.

Definition B.1.1. A *seed* in $\mathcal{F}$ is a pair $(\tilde{\boldsymbol{x}}, \tilde{B})$, where

- $\tilde{\boldsymbol{x}} = \{x_1, \dots, x_m\}$ is an algebraically independent subset of $\mathcal{F}$ which generates $\mathcal{F}$.
- $\tilde{B} = (b_{i,j})$ of $\tilde{B}$ is a $m \times n$ integer matrix with rows labeled by $[\![1, m]\!]$ and columns labeled by $\mathbf{ex}$ such that
  1. the $n \times n$ submatrix $B = (b_{ij})_{i,j \in \mathbf{ex}}$ is skew-symmetrizable.
  2. $\tilde{B}$ has full rank n.

The matrix B is called the *principal part* of $\tilde{B}$, $\boldsymbol{x} = \{x_j \mid j \in \mathbf{ex}\} \subset \tilde{\boldsymbol{x}}$ is the *cluster* of the seed $(\tilde{\boldsymbol{x}}, \tilde{B})$, $\mathbf{ex}$ are the *exchangeable indices*, and $\boldsymbol{c} = \tilde{\boldsymbol{x}} \setminus \boldsymbol{x}$ is the set of *frozen variables.*

For all $k \in \mathbf{ex}$, define the *seed mutation* in direction k as the transformation from $(\tilde{\boldsymbol{x}}, \tilde{B})$ to $\mu_k(\tilde{\boldsymbol{x}}, \tilde{B}) = (\tilde{\boldsymbol{x}}', \tilde{B}')$, with

- $\tilde{B}' = \mu_k(\tilde{B})$ is the $m \times n$ matrix whose entries are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2} & \text{otherwise.} \end{cases} \tag{B.1.1}$$

  This operation is called *matrix mutation* in direction k. This matrix can also be obtained via the operation

$$\tilde{B}' = \mu_k(\tilde{B}) = E_k \tilde{B} F_k, \tag{B.1.2}$$

where E_k is the $m \times m$ matrix with entries

$$(E_k)_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k, \\ -1 & \text{if } j = i = k, \\ \max(0, -b_{ik}) & \text{if } i \neq j = k, \end{cases} \tag{B.1.3}$$

and F_k is the $n \times n$ matrix with entries

$$(F_k)_{ij} = \begin{cases} \delta_{ij} & \text{if } i \neq k, \\ -1 & \text{if } j = i = k, \\ \max(0, b_{kj}) & \text{if } i = k \neq j. \end{cases} \tag{B.1.4}$$

- $\tilde{\boldsymbol{x}}' = (\tilde{\boldsymbol{x}} \setminus \{x_k\}) \cup \{x'_k\}$, where $x'_k \in \mathcal{F}$ is determined by the *exchange relation*

$$x_k x'_k = \prod_{\substack{i \in [1,m] \\ b_{ik} > 0}} x_i^{b_{ik}} + \prod_{\substack{i \in [1,m] \\ b_{ik} < 0}} x_i^{-b_{ik}}. \tag{B.1.5}$$

Remark B.1.2. $(\tilde{\boldsymbol{x}}', \tilde{B}')$ is also a seed in $\mathcal{F}$ and the seed mutation operation is involutive: $\mu_k(\tilde{\boldsymbol{x}}', \tilde{B}') = (\tilde{\boldsymbol{x}}, \tilde{B})$. Thus, we have a equivalence relation: $(\tilde{\boldsymbol{x}}, \tilde{B})$ is *mutation-equivalent* to $(\tilde{\boldsymbol{x}}', \tilde{B}')$, denoted by $(\tilde{\boldsymbol{x}}, \tilde{B}) \sim (\tilde{\boldsymbol{x}}', \tilde{B}')$, if $(\tilde{\boldsymbol{x}}', \tilde{B}')$ can be obtained from $(\tilde{\boldsymbol{x}}, \tilde{B})$ by a finite sequence of seed mutations.

Graphically, if the matrix $\tilde{B}$ is skew-symmetric, it can be represented by a quiver and the matrix mutation by a simple operation on the quiver. Fix $\tilde{B}$ a skew-symmetric matrix. Define the quiver Q whose set of vertices is $[\![1, m]\!]$, where the vertices corresponding to $\boldsymbol{c}$ are usually denoted by a square $\square$ and called *frozen vertices*. For all $i \in [\![1, m]\!]$, $j \in \mathbf{ex}$, b_{ij} is the number of arrows from i to j (can be negative if the arrows are from j to i).

$$i \xrightarrow{b_{ij}} j$$

In this context, the operation of matrix mutation can be translated naturally to an operation on the quiver Q. For $k \in \mathbf{ex}$, the quiver $Q' = \mu_k(Q)$ is obtained from Q by the following operations:

- For each pair of arrows $i \to k \to j$ in Q, create an arrow from i to j.
- Invert all arrows adjacent to k.
- Remove all 2-cycles that were possibly created.

Definition B.1.3. Let $\mathcal{S}$ be a mutation-equivalence class of seeds in $\mathcal{F}$. The *cluster algebra* $\mathcal{A}(\mathcal{S})$ associated to $\mathcal{S}$ is the $\mathbb{Z}[\boldsymbol{c}^{\pm}]$-subalgebra of $\mathcal{F}$ generated by all the clusters of all the seeds in $\mathcal{S}$.

B.2 Compatible pairs

A quantum cluster algebra is a non-commutative version of a cluster algebra. Cluster variables will not commute anymore, but, if they are in the same cluster, commute up to some power of an indeterminate t. These powers can be encoded in a skew-symmetric matrix Λ. In order for the quantum cluster algebra to be well-defined, one needs to check that these t-commutation relations behave well with the exchange relations. This is made explicit via the notion of compatible pairs.

Definition B.2.1. Let $\tilde{B}$ be a $m \times n$ integer matrix, with rows labeled by $[\![1, m]\!]$ and columns labeled by $\mathbf{ex}$. Let $\Lambda = (\lambda_{ij})_{1 \leqslant i,j \leqslant m}$ be a skew-symmetric $m \times m$ integer matrix. We say that $(\Lambda, \tilde{B})$ forms a *compatible pair* if, for all $i \in \mathbf{ex}$ and $1 \leqslant j \leqslant m$, we have

$$\sum_{k=1}^{m} b_{ki}\lambda_{kj} = \delta_{i,j}d_i, \tag{B.2.1}$$

with $(d_i)_{i \in \mathbf{ex}}$ some integers, all positive or negative. Relation (B.2.1) is equivalent to saying that, up to reordering, if $\mathbf{ex} = [\![1, n]\!]$, the matrix $\tilde{B}^T\Lambda$ consists of two blocks, a diagonal $n \times n$ block, and a $n \times (m - n)$ zero block:

$$\left(\begin{array}{cccc|c} d_1 & & & & \\ & d_2 & & & \\ & & \ddots & & (0) \\ & & & d_n & \end{array}\right)$$

Fix a compatible pair $(\Lambda, \tilde{B})$ and fix $k \in \mathbf{ex}$. Define, in a similar way as in (B.1.2),

$$\Lambda' = \mu_k(\Lambda) := E_k^T \Lambda E_k, \tag{B.2.2}$$

with E_k from (B.1.3).

Proposition B.2.2 ([BZ05])**.** *The pair $(\Lambda', \tilde{B}')$ is compatible.*

We say that $(\Lambda', \tilde{B}')$ is the *mutation* in direction k of the pair $(\Lambda, \tilde{B})$, and we will use the notation:

$$\mu_k(\Lambda, \tilde{B}) := \Big(\mu_k(\Lambda), \mu_k(\tilde{B})\Big) = (\Lambda', \tilde{B}'). \tag{B.2.3}$$

Proposition B.2.3 ([BZ05])**.** *The mutation of a compatible pair is involutive. For any compatible pair $(\Lambda, \tilde{B})$ and any mutation direction $k \in$ **ex**, $\mu_k(\mu_k(\Lambda, \tilde{B})) = (\Lambda, \tilde{B})$.*

B.3 Definition of quantum cluster algebras

We now introduce the last notions we need in order to define quantum cluster algebras.

Let t be a formal variable. Consider $\mathbb{Z}[t^{\pm 1/2}]$, the ring of Laurent polynomials in the variable $t^{1/2}$.

Recall that any skew-symmetric integer matrix Λ of size $m \times m$ determines a skew-symmetric $\mathbb{Z}$-bilinear form on $\mathbb{Z}^m$, which will also be denoted by Λ:

$$\Lambda(\mathbf{e}_i, \mathbf{e}_j) := \lambda_{i,j}, \quad \forall i, j \in [\![1, m]\!], \tag{B.3.1}$$

where $\{\mathbf{e}_i \mid 1 \leqslant i \leqslant m\}$ is the standard basis of $\mathbb{Z}^m$.

Definition B.3.1. The *quantum torus* $\mathcal{T} = (\mathcal{T}(\Lambda), *)$ associated with the skew-symmetric bilinear form Λ is the $\mathbb{Z}[t^{\pm 1/2}]$-algebra generated by the $\{X^{\mathbf{e}} \mid \mathbf{e} \in \mathbb{Z}^m\}$, together with the t-commuting relations:

$$X^{\mathbf{e}} * X^{\mathbf{f}} = t^{\Lambda(\mathbf{e},\mathbf{f})/2} X^{\mathbf{e}+\mathbf{f}} = t^{\Lambda(\mathbf{e},\mathbf{f})} X^{\mathbf{f}} * X^{\mathbf{e}}, \quad \forall \mathbf{e}, \mathbf{f} \in \mathbb{Z}^m. \tag{B.3.2}$$

The quantum torus $\mathcal{T}(\Lambda)$ is an Ore domain (see details in [BZ05]), thus it is contained in its skew-field a fractions $\mathcal{F} = (\mathcal{F}, *)$. The field $\mathcal{F}$ is a $\mathbb{Q}(t^{1/2})$-algebra.

Definition B.3.2. A *toric frame* in $\mathcal{F}$ is a map $M : \mathbb{Z}^m \to \mathcal{F}\backslash\{0\}$ of the form

$$M(c) = \phi(X^{\eta(c)}), \quad \forall c \in \mathbb{Z}^m, \tag{B.3.3}$$

where $\phi : \mathcal{F} \to \mathcal{F}$ is a $\mathbb{Q}(t^{1/2})$-algebra automorphism and $\eta : \mathbb{Z}^m \to \mathbb{Z}^m$ is an isomorphism of $\mathbb{Z}$-modules.

For any toric frame M, define $\Lambda_M : \mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}$, a skew-symmetric bilinear form, by

$$\Lambda_M(\mathbf{e}, \mathbf{f}) = \Lambda(\eta(\mathbf{e}), \eta(\mathbf{f})), \quad \forall \mathbf{e}, \mathbf{f} \in \mathbb{Z}^m. \tag{B.3.4}$$

Then,

$$M(\mathbf{e}) * M(\mathbf{f}) = t^{\Lambda_M(\mathbf{e},\mathbf{f})/2} M(\mathbf{e} + \mathbf{f}) = t^{\Lambda_M(\mathbf{e},\mathbf{f})} M(\mathbf{f}) * M(\mathbf{e}). \tag{B.3.5}$$

Definition B.3.3. A *quantum seed* in $\mathcal{F}$ is a pair $(M, \tilde{B})$, where

- M is a toric frame in $\mathcal{F}$,
- $\tilde{B}$ is an $m \times \mathbf{ex}$ integer matrix,
- the pair $(\Lambda_M, \tilde{B})$ is compatible, as in Definition B.2.1.

Next, we need to define mutations of quantum seeds. Let $(M, \tilde{B})$ be a quantum seed, and fix $k \in \mathbf{ex}$. Define $M' : \mathbb{Z}^m \to \mathcal{F}\backslash\{0\}$ by setting

$$M'(\mathbf{f}) = \begin{cases} \sum_{p=0}^{k} \binom{f_k}{p}_{t^{d_k/2}} M(E_k\mathbf{f} + pb^k) & \text{if } f_k \geqslant 0, \\ M'(-\mathbf{f})^{-1} & \text{otherwise}, \end{cases}$$

where E_k is the matrix from (B.1.3), $b^k \in \mathbb{Z}^m$ is the kth column of $\tilde{B}$ and the t-binomial coefficient is defined by

$$\binom{r}{p}_t := \frac{(t^r - t^{-r})(t^{r-1} - t^{-r+1}) \cdots (t^{r-p+1}) - t^{-r+p-1}}{(t^p - t^{-p})(t^{p-1} - t^{-p+1}) \cdots (t - t^{-1})}, \quad \forall 0 \leqslant p \leqslant r. \tag{B.3.6}$$

Recall the definition of the mutated matrix $\tilde{B}' = \mu_k(\tilde{B})$ from Section B.1. Then the *mutation* in direction k of the quantum seed $(M, \tilde{B})$ is the pair $\mu_k(M, \tilde{B}) = (M', \tilde{B}')$

Proposition B.3.4 ([BZ05])**.** *(1) The pair $(M', \tilde{B}')$ is a quantum seed.*
(2) The mutation in direction k of the compatible pair $(\Lambda_M, \tilde{B})$ is the pair $(\Lambda_{M'}, \tilde{B}')$.

For a quantum seed $(M, \tilde{B})$, let $\tilde{\mathbf{X}} = \{X_1, \ldots, X_m\}$ be the free generating set of $\mathcal{F}$, given by $X_i := M(\mathbf{e}_i)$. Let $\mathbf{X} = \{X_i \mid i \in \mathbf{ex}\}$, we call it the *cluster* of the quantum seed $(M, \tilde{B})$, and let $\mathbf{C} = \tilde{\mathbf{X}}\backslash\mathbf{X}$.

For all $k \in \mathbf{ex}$, if $(M', \tilde{B}') = \mu_k(M, \tilde{B})$, then the $X'_i = M'(\mathbf{e}_i)$ are obtained by:

$$X'_i = \begin{cases} X_i & \text{if } i \neq k, \\ M\left(-\mathbf{e}_k + \sum_{b_{ik}>0} b_{ik}\mathbf{e}_i\right) + M\left(-\mathbf{e}_k - \sum_{b_{ik}<0} b_{ik}\mathbf{e}_i\right) & \text{if } i = k. \end{cases} \tag{B.3.7}$$

The mutation of quantum seeds, as the mutation of compatible pairs, is an involutive process: $\mu_k(M', \tilde{B}') = (M, \tilde{B})$. Thus, as before, we have an equivalence relation: two quantum seeds $(M_1, \tilde{B}_1)$ and $(M_2, \tilde{B}_2)$ are *mutation equivalent* if $(M_2, \tilde{B}_2)$ can be obtained from $(M_1, \tilde{B}_1)$ by a sequence of quantum seed mutations. From (B.3.7), the set $\mathbf{C}$ only depends on the mutation equivalence class of the quantum seed. The variables in $\mathbf{C}$, $(X_i)_{i \notin \mathbf{ex}}$, are called the *frozen variable* of the mutation equivalence class.

Definition B.3.5. Let $\mathcal{S}$ be a mutation equivalence class of quantum seeds in $\mathcal{F}$ and $\mathbf{C}$ the set of its frozen variables. The *quantum cluster algebra* $\mathcal{A}(\mathcal{S})$ associated with $\mathcal{S}$ is the $\mathbb{Z}[t^{1/2}]$-subalgebra of the skew-field $\mathcal{F}$ generated by the union of all clusters in all seeds in $\mathcal{S}$, together with the elements of $\mathbf{C}$ and their inverses.

B.4 Laurent phenomenon and quantum Laurent phenomenon

One of the main properties of cluster algebras is the so-called *Laurent Phenomenon* which was formulated in [BFZ05]. Quantum cluster algebras present a counterpart to this result called the *Quantum Laurent Phenomenon*.

Here, we follow [BZ05, Section 5]. In order to state this result, one needs the notion of *upper cluster algebras*.

Fix $(M, \tilde{B})$ a quantum seed, and $\tilde{\mathbf{X}} = \{X_1, \ldots, X_m\}$ given by $X_k = M(e_k)$. Let $\mathbb{ZP}[\mathbf{X}^{\pm 1}]$ denote the based quantum torus generated by the $(X_k)_{1 \leqslant k \leqslant m}$; it is a $\mathbb{Z}[t^{\pm 1/2}]$-subalgebra of $\mathcal{F}$ with basis $\{M(c) \mid c \in \mathbb{Z}^m\}$, such that the ground ring $\mathbb{ZP}$ is the ring of integer Laurent polynomials in the variables $t^{1/2}$ and $(X_j)_{j \notin \mathbf{ex}}$. For $k \in \mathbf{ex}$, let $(M_k, \tilde{B}_k)$ be the quantum seed obtained from $(M, \tilde{B})$ by mutation in direction k, and let $\mathbf{X}_k$ denote its cluster, thus:

$$\mathbf{X}_k = (\mathbf{X} \backslash \{X_k\}) \cup \{X'_k\}.$$

Define the quantum upper cluster algebra as the $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ given by

$$\mathcal{U}(M, \tilde{B}) := \mathbb{ZP}[\mathbf{X}^{\pm 1}] \cap \bigcap_{k \in \mathbf{ex}} \mathbb{ZP}[\mathbf{X}_k^{\pm 1}]. \tag{B.4.1}$$

Theorem B.4.1. *[BZ05, Theorem 5.1] The quantum upper algebra $\mathcal{U}(M, \tilde{B})$ depends only on the mutation-equivalence class of the quantum seed $(M, \tilde{B})$.*

Thus we use the notation: $\mathcal{U}(M, \tilde{B}) = \mathcal{U}(\mathcal{S})$, where $\mathcal{S}$ is the mutation-equivalence class of $(M, \tilde{B})$, one has:

$$\mathcal{U}(\mathcal{S}) = \bigcap_{(M, \tilde{B}) \in \mathcal{S}} \mathbb{ZP}[\mathbf{X}^{\pm 1}]. \tag{B.4.2}$$

Theorem B.4.1 has the following important corollary, which we refer to as the *quantum Laurent phenomenon.*

Corollary B.4.2. *[BZ05, Corollary 5.2] The cluster algebra $\mathcal{A}(\mathcal{S})$ is contained in $\mathcal{U}(\mathcal{S})$. Equivalently, $\mathcal{A}(\mathcal{S})$ is contained in the quantum torus $\mathbb{ZP}[\mathbf{X}^{\pm 1}]$ for every quantum seed $(M, \tilde{B}) \in \mathcal{S}$ of cluster $\mathbf{X}$.*

B.5 Specializations of quantum cluster algebras

Fix a quantum seed $(M, \tilde{B})$ and $\mathbf{X}$ its cluster. The based quantum torus $\mathbb{ZP}[\mathbf{X}^{\pm 1}]$ specializes naturally at $t = 1$, via the ring morphism:

$$\pi : \mathbb{ZP}[\mathbf{X}^{\pm 1}] \to \mathbb{Z}[\tilde{\mathbf{X}}^{\pm 1}], \tag{B.5.1}$$

such that

$$\begin{aligned} \pi(X_k) &= X_k, \quad (1 \leqslant k \leqslant m) \\ \pi(t^{\pm 1/2}) &= 1. \end{aligned}$$

If we restrict this morphism to the quantum cluster algebra $\mathcal{A}(\mathcal{S})$, it is not clear that we recover the (classical) cluster algebra $\mathcal{A}(\tilde{B})$. This question was tackled in a recent paper by Geiss, Leclerc and Schröer [GLS18].

Remark B.5.1. From a combinatorial point of view, the cluster algebras $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\tilde{B})$ are constructed on the same quiver $\tilde{B}$, and the mutations have the same effect on the quiver. Assume the initial seeds are fixed and identified, via the morphism (B.5.1). Then, each quantum cluster variable in $\mathcal{A}(\mathcal{S})$ is identified to a cluster variable in $\mathcal{A}(\tilde{B})$.

Proposition B.5.2. *[GLS18, Lemma 3.3] The restriction of π to $\mathcal{A}(\mathcal{S})$ is surjective on $\mathcal{A}(\tilde{B})$, and quantum cluster variables are sent to the corresponding cluster variables.*

They also conjectured that the specialization at $t = 1$ of the quantum cluster algebra is isomorphic to the classical cluster algebra, and gave a proof under some assumptions on the initial seed.

Nevertheless, by applying Proposition B.5.2 to different seeds (while keeping the identification (B.5.1) of the initial seeds), one gets:

Corollary B.5.3. *The evaluation morphism π sends all quantum cluster monomials to the corresponding cluster monomials.*

B.6 Positivity

Let us state a last general result on quantum cluster algebras: Davison's positivity theorem [Dav18].

We have recalled in Section B.4 that each (quantum) cluster variable can be written as a Laurent polynomial in the initial (quantum) cluster variables (and $t^{1/2}$). For classical cluster algebras, Fomin-Zelevinski conjectured that these Laurent polynomials have positive coefficients. The so-called *positivity conjecture* was proven by Lee-Schiffler in [LS15].

For quantum cluster algebras, the result is the following.

Theorem B.6.1. *[Dav18, Theorem 2.4] Let $\mathcal{A}$ be a quantum cluster algebra defined by a compatible pair $(\Lambda, \tilde{B})$. For a mutated toric frame M' and a quantum cluster monomial Y, let us write:*

$$Y = \sum_{\boldsymbol{e} \in \mathbb{Z}^m} a_{\boldsymbol{e}}(t^{1/2}) M'(\boldsymbol{e}), \tag{B.6.1}$$

with $a_{\boldsymbol{e}}(t^{1/2}) \in \mathbb{Z}[t^{\pm 1/2}]$. Then the coefficients $a_{\boldsymbol{e}}(t^{1/2})$ have positive coefficients.

Moreover, they can be written with in the form

$$a_{\boldsymbol{e}}(t^{1/2}) = t^{-\deg(b_{\boldsymbol{e}}(t))/2} b_{\boldsymbol{e}}(t), \tag{B.6.2}$$

where $b_{\boldsymbol{e}}(t) \in \mathbb{N}[q]$, i.e. each polynomial $a_{\boldsymbol{e}}(t^{1/2})$ contains only even or odd powers of $t^{1/2}$.

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