MA 574 – PROJECT 4

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Problem 1:

(a) The neutral line is where the stress and strain are zero and no tension or compression of the material occurs. Here, since the material of the beam is the same throughout, the neutral line is in the geometric center. That is,

$$n(x) = \frac{h}{2}\Big(1 - \frac{x}{L}\Big).$$

A visual representation of the neutral line on the beam is shown in Figure 1.

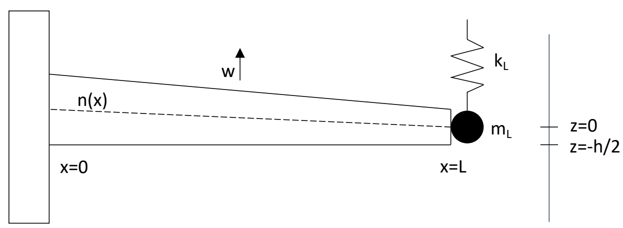

Figure 1: The neutral line, $n(x)$, where no tension or compression occurs.

(b) The linear density is

$$\rho(x) = \hat{\rho} bh\Big(2 - \frac{x}{L}\Big).$$

To find the stiffness, we will first find the moment, M. In this case, we have significant internal damping, so the stress is modeled by the equation

$$\sigma = Y\varepsilon - c\dot{\varepsilon},$$

where $\varepsilon = k(z - n(x))$. Therefore

$$\begin{aligned} M &= b\int_{-\frac{h}{2}}^{\frac{3h}{2}-\frac{h}{L}x} \sigma\big(z - n(x)\big)dz \\ &= b\int_{-\frac{h}{2}}^{\frac{3h}{2}-\frac{h}{L}x} b\kappa\big(z - n(x)\big)^2 + bC\dot{\kappa}\big(z - n(x)\big)^2 dz \\ &= \kappa Y bh^3\frac{(2L-x)^3}{12L^3} + C\dot{\kappa} bh^3\frac{(2L-x)^3}{12L^3}. \end{aligned}$$

Thus, the stiffness is

$$YI(x) = Ybh^3\frac{(2L-x)^3}{12L^3}.$$

(c) Balancing forces, we obtain

$$\int_x^{x+\Delta x} \rho(x)\frac{\partial^2 w}{\partial t^2}(t,s)ds = Q(t,x+\Delta x) - Q(t,x) - \gamma\int_x^{x+\Delta x}\frac{\partial w}{\partial t}(t,s)ds.$$

Now we divide by Δx and take the limit as Δx approaches zero to obtain

$$\rho(x)\frac{\partial^2 w}{\partial t^2} = \frac{\partial Q}{\partial x} - \gamma\frac{\partial w}{\partial t}.$$

Substituting in the equation for linear density from part (b) results in

$$\hat{\rho} bh\Big(2 - \frac{x}{L}\Big)\frac{\partial^2 w}{\partial t^2} + \gamma\frac{\partial w}{\partial t} = \frac{\partial Q}{\partial x}.$$

Now for the moment balance, we have

$$M(t,x+\Delta x) - M(t,x) - Q(t,x+\Delta x)\Delta x = 0.$$

Therefore

$$\frac{\partial M}{\partial x} = Q.$$

We have already determined the formula for the moment. Additionally, since we are modeling a thin beam, we note that $\kappa = -\frac{\partial^2 w}{\partial x^2}$. Thus

$$Q = -\frac{\partial}{\partial x}\left(YI(x)\frac{\partial^2 w}{\partial x^2} + CI(x)\frac{\partial^3 w}{\partial^2 x\partial t}\right).$$

We now obtain the strong formulation by combining the force and moment balances.

$$\hat{\rho} bh\Big(2 - \frac{x}{L}\Big)\frac{\partial^2 w}{\partial t^2} + \gamma\frac{\partial w}{\partial t} + \frac{\partial^2}{\partial x^2}\left(YI(x)\frac{\partial^2 w}{\partial x^2} + CI(x)\frac{\partial^3 w}{\partial^2 x\partial t}\right) = 0$$

(d) Since the beam is fixed at $x = 0$,

$$w(t,0) = \frac{\partial w}{\partial x}(t,0) = 0.$$

The beam is free at $x = L$, so

$$M(t,L) = 0.$$

At $x = L$, the beam is subject to forces from a mass and a spring, so

$$Q(t,L) = -k_L w(t,L) - m_L\frac{\partial^2 w}{\partial t^2}(t,L).$$

(e) Let the space of test functions be

$$V = \big\{\phi \in H^2(0,L)\big|\phi(0) = \phi'(0) = 0\big\}$$

with the inner product

$$\langle\psi,\phi\rangle_V = \int_0^L YI(x)\psi''\phi'' dx.$$

Let the test function $\phi \in V$. We multiply the strong formulation by the test function and integrate by parts.

$$\begin{aligned}
0 &= \int_0^L \hat{\rho} bh\Big(2 - \frac{x}{L}\Big)\frac{\partial^2 w}{\partial t^2}\phi dx + \gamma \int_0^L \frac{\partial w}{\partial t}\phi dx - \int_0^L \frac{\partial^2 M}{\partial x^2}\phi dx \\
&= \int_0^L \hat{\rho} bh\Big(2 - \frac{x}{L}\Big)\frac{\partial^2 w}{\partial t^2}\phi dx + \gamma \int_0^L \frac{\partial w}{\partial t}\phi dx \\
&\quad - \int_0^L M\frac{\partial^2 \phi}{\partial x^2} dx - Q(t,L)\phi(L) + Q(t,0)\phi(0) + M(t,L)\phi'(L) - M(t,0)\phi'(0)
\end{aligned}$$

The last three terms vanish, so

$$\begin{aligned}
\int_0^L \hat{\rho} bh\Big(2 - \frac{x}{L}\Big)\frac{\partial^2 w}{\partial t^2}\phi dx + \gamma \int_0^L \frac{\partial w}{\partial t}\phi dx + \int_0^L \Big(YI(x)\frac{\partial^2 w}{\partial x^2} + CI(x)\frac{\partial^3 w}{\partial^2 x \partial t}\Big)\frac{\partial^2 \phi}{\partial x^2} dx \\
+ \big[k_L w(t,L) + m_L \frac{\partial^2 w}{\partial t^2}(t,L)\big]\phi(L) = 0.
\end{aligned}$$

(f) Assume $\gamma = C = m_L = k_L = 0$.

The kinetic and potential energies, respectively, are

$$K = \frac{1}{2}\int_0^L \hat{\rho} bh\Big(2 - \frac{x}{L}\Big) w_t^2 dx$$

and

$$U = \frac{1}{2}\int_0^L YI(x) w_{xx}^2(t,x) dx.$$

The action integral is

$$A = \int_{t_0}^{t_1} [K - U] dt.$$

The admissible variations are

$$\hat{w}(t,x) = w(t,x) + \varepsilon \eta(t)\phi(x)$$

where

$$\begin{aligned}
&(i) \eta(t_0) = \eta(t_1) = 0 \\
&(ii) \phi \in V = H_0^1(0,L).
\end{aligned}$$

We now use Hamilton's principle to obtain the weak form.

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon} A[w + \varepsilon \Phi]\Big|_{\varepsilon=0} &= \frac{1}{2}\frac{\partial}{\partial \varepsilon}\int_{t_0}^{t_1}\int_0^L \hat{\rho} bh\Big(2 - \frac{x}{L}\Big)\big(w_t + \varepsilon \dot{\eta}\phi\big)^2 - YI(x)\big(w_{xx} + \varepsilon \eta \phi''\big)^2 dx dt\Big|_{\varepsilon=0} \\
&= \int_{t_0}^{t_1}\int_0^L \hat{\rho} bh\Big(2 - \frac{x}{L}\Big)\big(w_t + \varepsilon \dot{\eta}\phi\big)\dot{\eta}\phi - YI(x)\big(w_{xx} + \varepsilon \eta \phi''\big)\eta \phi'' dx dt\Big|_{\varepsilon=0} \\
&= \int_{t_0}^{t_1}\int_0^L \hat{\rho} bh\Big(2 - \frac{x}{L}\Big) w_t \dot{\eta}\phi - YI(x) w_{xx}\eta \phi'' dx dt \\
&= \int_0^L \hat{\rho} bh\Big(2 - \frac{x}{L}\Big) w_{tt}\phi + YI(x) w_{xx}\eta \phi'' dx = 0
\end{aligned}$$

This weak form is the same that the weak form which was derived in part (e), except, of course, with $\gamma = C = m_L = k_L = 0$.

Problem 3:

Fix a point $x \in [0, L]$ on the string. For any point $s \in [x, L]$, the radius is $r = s$ and the mass is $m = \rho \Delta s$. To find the tension of the string, we use the equation $F = mr\omega^2$ and integrate with respect to s from x to L.

$$\begin{aligned} T(x) &= \int_x^L \rho s \omega^2 ds \\ &= \frac{1}{2}\rho\omega^2 s^2 \Big|_x^L \\ &= \frac{1}{2}\rho(L^2 - x^2)\omega^2 \end{aligned}$$

By Newton's second law,

$$\int_x^{x+\Delta x} \rho u_{tt} dx = T(t, x + \Delta x) u_x(t, x + \Delta x) - T(t, x) u_x(t, x).$$

Now we divide by Δx and take the limit at Δx approaches zero, which results in the equation

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\Big(T \frac{\partial u}{\partial x}\Big).$$

We need two boundary conditions. Since the end at $x = 0$ is stationary, one of the boundary conditions is

$$u(t, 0) = 0.$$

Because the string is taut, the other boundary condtion is

$$u_x(t, L) = 0.$$

The initial conditions to the differential equation are

$$\begin{aligned} u(0, x) &= f(x) \\ u'(0, x) &= g(x). \end{aligned}$$