MA 574 – PROJECT 2

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(1) The geometry of the PDE (namely, the circle and square) was created using the `pdetool` GUI.

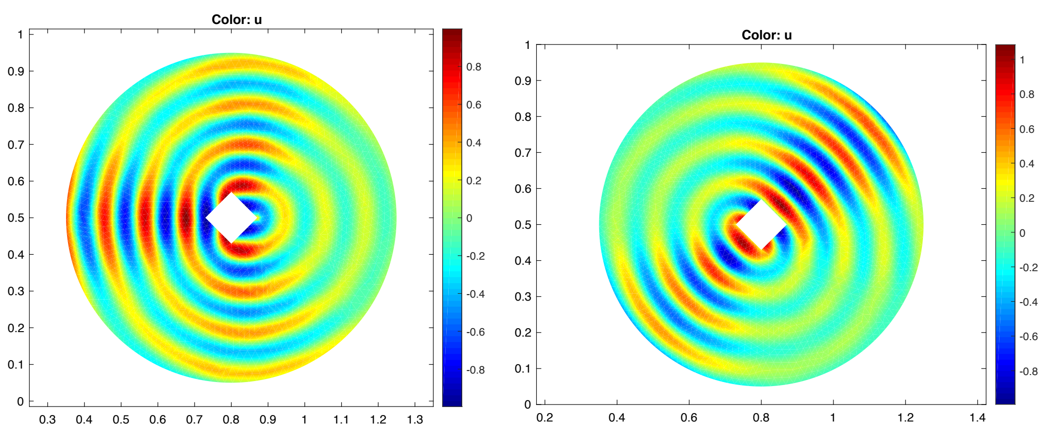
For the circle, a Neumann boundary condition was used, with $q = ik = 60i$, and $g = 0$.

In the first case, the incident wave is in the negative x direction, so $\vec{a} = (-1, 0)$, and $\vec{a} \cdot \vec{x} = -x$. Thus, in the first case, the square has a Dirichlet boundary condition with $h = 1$ and $r = -e^{-60ix}$.

In the second case, the incident was is coming from $\pi/4$ towards the center, so $\vec{a} = (-1/\sqrt{2}, -1/\sqrt{2})$. Thus, $\vec{a} \cdot \vec{x} = -\frac{1}{\sqrt{2}}(x + y)$. Therefore, in the second case, the square has a Dirchlet boundary condition with $h = 1$ and $r = -e^{-60i(x+y)/\sqrt{2}}$.

The mesh was generated and then refined twice. The PDE was elliptic with coefficients $c = 1$, $a = -k^2 = -3600$, and $f = 0$.

The solutions are shown in Figure 1.



(a) Case i: Incident wave in the negative x direction.

(b) Case ii: Incident wave from $\pi/4$ toward the center.

Figure 1: Solution to PDE modeling acoustic scattering around a rectangular solid.

(2) (a) In the reference volume, the mass and flux are

$$\begin{aligned}\text{mass} &:= \overline{\rho}\Delta x b h(t,x)\\ \text{flux} &:= \overline{\rho} u(t,x)\end{aligned}$$

To use conservation of mass, we implement the relation

$$\left\{\begin{matrix}\text{Change}\\ \text{in mass}\end{matrix}\right\} = \left\{\begin{matrix}\text{Mass}\\ \text{in}\end{matrix}\right\} - \left\{\begin{matrix}\text{Mass}\\ \text{out}\end{matrix}\right\}.$$

That is,

$$\begin{aligned}\frac{\partial}{\partial t}\big(\overline{\rho} b h(t,x)\Delta x\big) &= \overline{\rho} u(t,x) b h(t,x) - \overline{\rho} u(t,x+\Delta x) b h(t,x+\Delta x)\\ \Rightarrow \overline{\rho} b \Delta x \frac{\partial h}{\partial t}(t,x) &= \overline{\rho} b\big[u(t,x)h(t,x) - u(t,x+\Delta x)h(t,x+\Delta x)\big]\\ \Rightarrow \Delta x \frac{\partial h}{\partial t}(t,x) &= u(t,x)h(t,x) - u(t,x+\Delta x)h(t,x+\Delta x).\end{aligned}$$

Finally, we divide by Δx and take the limit as Δx approches zero to obtain

$$\frac{\partial h}{\partial t} + \frac{\partial(uh)}{\partial x} = 0.$$

(b) In order to conserve momentum, it must be the case that

$$\left\{\begin{matrix}\text{Rate of}\\ \text{accumulated}\\ \text{momentum}\end{matrix}\right\} = \left\{\begin{matrix}\text{Rate of}\\ \text{momentum in}\end{matrix}\right\} - \left\{\begin{matrix}\text{Rate of}\\ \text{momentum out}\end{matrix}\right\} + \left\{\begin{matrix}\text{Sum of forces}\\ \text{acting on system}\end{matrix}\right\}.$$

More specifically,

$$\begin{aligned}\frac{\partial}{\partial t}\int_x^{x+\Delta x} b h(t,x)\overline{\rho} u(t,x)dx =& b\overline{\rho}\big[h(t,x)u^2(t,x) - h(t,x+\Delta x)u^2(t,x+\Delta x)\big]\\ &+ b\big[h(t,x)p(t,x) - h(t,x+\Delta x)p(t,x+\Delta x)\big].\end{aligned}$$

Now we can divide by b, and also divide by Δx and take the limit as Δx approaches zero to obtain a differential equation incorporating pressue.

$$\overline{\rho}\frac{\partial(hu)}{\partial t} + \overline{\rho}\frac{\partial(hu^2)}{\partial x} + \frac{\partial(hp)}{\partial x} = 0$$

Finally, we will check our units. The units of the height, velocity, density, and pressure are

$$h: m, \quad u: \frac{m}{s}, \quad \overline{\rho}: \frac{kg}{m^3}, \quad p: \frac{N}{m^2}\left(\text{equivalenty, } \frac{kg}{m\cdot s^2}\right).$$

Thus, the units of our modeling equation are

$$\frac{kg}{m^3}\cdot\left(m\cdot\frac{m}{s}\right)/s + \frac{kg}{m^3}\left(m\cdot\left(\frac{m}{s}\right)^2\right)/m + \left(m\cdot\frac{kg}{m\cdot s^2}\right)/m$$

$$\Rightarrow \frac{kg}{m\cdot s^2} + \frac{kg}{m\cdot s^2} + \frac{kg}{m\cdot s^2}.$$

(c) Note that pressure is force per unit area ($p = F/A$), so $F = pA$. Thus, the total force at (t, x) is

$$\in_0^{h(t,x)} \bar{\rho} g y b dy = \frac{1}{2}\bar{\rho} g h^2 b.$$

Thus the total pressure, P, is

$$P(t, x) = \frac{F(t, x)}{A(t, x)} = \frac{\frac{1}{2}\bar{\rho} g h^2 b}{hb} = \frac{1}{2}\bar{\rho} g h.$$

Consequently, the reformulated momentum equation is

$$\bar{\rho}\frac{\partial(hu)}{\partial t} + \bar{\rho}\frac{\partial(hu^2)}{\partial x} = \frac{1}{2}\bar{\rho} g \frac{\partial(h^2)}{\partial x} = 0.$$

(d) The expanded momentum equation is

$$h_t u + h u_t + (hu)_x u + (hu) u_x + \frac{1}{2} g(2h) h_x = 0.$$

By the continutity equation, $h_t + (uh)_x = 0$, so we can take out the $[h_t + (uh)_x]u$ term and divide by h to get

$$u_t + u u_x + g h_x = 0.$$

Since $u u_x = \left(\frac{1}{2}u^2\right)_x$ and $g h_x = (gh)_x$, the momentum equation can be written the the form

$$u_t + \left(\frac{1}{2}u^2 + gh\right)_x = 0.$$

Together, the continutity and momentum equations form the first order system

$$\begin{bmatrix} h \\ u \end{bmatrix}_t + \begin{bmatrix} uh \\ \frac{1}{2}u^2 + gh \end{bmatrix}_x = 0.$$

(e) Along the sides of the stream, the water is still and has heigt zero, therefore

$$h\Big|_{\text{bd}} = u\Big|_{\text{bd}} = 0.$$