MA 574 – PROJECT 3

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(1) Consider a rod of length L with a fixed end at $x = 0$ as depicted in Figure 1. Let ρ, Y, c respectively denote the density, Young's modulus and internal damping coefficient and let $u(t, x)$ denote the longitudinal displacements. The rod decreases from a thickness $2h$ at $x = 0$ to a thickness of h at the free end and has uniform width b. Moreover, the rod has a mass m_L at $x = L$ along with a restraining spring with stiffness k_L that exerts a restoring force proportional to displacements $u(t, L)$.

(a) Balance forces to obtain a strong formulation of the model. Be sure to specify boundary conditions.

Solution:

First we balance forces using Newton's second law.

$$\int_x^{x+\Delta x} \rho A u_{tt}(t,s)ds = N(t, x+\Delta x) - N(t,x) + \int_x^{x+\Delta x} f(t,s)ds$$

Then, dividing by Δx and taking the limit at Δx approaches zero, we obtain

$$\rho A u_{tt} = N_x + f$$

Now, note that $N = \sigma A$, $\epsilon = \frac{\partial u}{\partial x}$, and $\sigma = Y\epsilon + c\frac{\partial \epsilon}{\partial t}$. Thus,

$$N = [Y\epsilon + c\epsilon_t]A = [Yu_x + cu_{xt}]A$$

$$\Rightarrow N_x = [Yu_{xx} + cu_{x^2t}]A + [Yu_x + cu_{xt}]A'.$$

The area is described by the equation

$$A(x) = b\left(2h - \frac{h}{L}x\right) = bh\left(2 - \frac{1}{L}x\right),$$

so $A'(x) = -\frac{bh}{L}$.
Thus, the strong formulation of the model is

$$\rho A u_{tt} = \left[Yu_{xx} + cu_{x^2t}\right]A + \left[Yu_x + cu_{xt}\right]\left(-bh/L\right) + f.$$

The boundary conditions are

$$u(t, 0) = 0$$

and

$$N(t, L) = -k_\ell u(t, L) - m_\ell u_{tt}(t, L).$$

The first boundary condition is known because the rod is fixed at $x = 0$. The second boundary condition is obtained by calculating the (resultant) force due ot the mass and the spring at the free endpoint.

(b) Integrate by parts to obtain a corresponding weak formulation. Specify the space of test functions.

Solution:

The space of test functions is

$$V = \{\Phi = (\phi, \varphi) \in X | \phi \in H^1(0, L), \phi(0) = 0, \phi(L) = \varphi\}$$

(where X is the same as in the notes) with inner product

$$\langle \Phi_1, \Phi_2 \rangle_V = \int_0^L YA\phi_1' \phi_2' dx + k_L \varphi_1 \varphi_2.$$

We begin by multiplying the strong formulation by a test function and integrating.

$$\int_0^L \rho A u_{tt} \phi dx - \int_0^L N_x \phi dx = \int_0^L f\phi dx$$

Then, we integrate by parts to obtain

$$\int_0^L \rho A u_{tt} \phi dx + \int_0^L N \phi_x dx - \Big[N\phi \Big]_0^L = \int_0^L f\phi dx.$$

Since $\phi(0) = 0$, $N(t, L) = -k_L u(t, L) - m_L u_{tt}(t, L)$, and $N(t, x) = [Yu_x + cu_{xt}]A$, this becomes

$$\int_0^L \rho A u_{tt} \phi dx + \int_0^L [YAu_x + cAu_{xt}]\phi_x dx = \int_0^L f\phi dx - [k_L u(t, L) + m_L u_{tt}(t, L)]\phi_L$$

(c) Determine an appropriate basis and finite element solution to (b). Determine the resulting matrix vector system in second-order and first-order form. You should specify the components in your mass and stiffness matrices but you do not need to numerically evaluate the integrals.

Solution:

Taking a local basis, we have

$$u(t, x) = \varphi^T(x)\mathbf{S}\mathbf{u}(t)$$

where

$$\mathbf{S} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{h} & \frac{1}{h} \end{bmatrix}, \qquad \varphi^T(x) = \begin{bmatrix} 1 \\ x \end{bmatrix}, \quad \text{and} \qquad \mathbf{u}(t) = u_l(t)u_r(t).$$

Then, putting this into the weak form from part (b), we obtain

$$\int_0^L \rho A \varphi^T(x)\mathbf{S}\ddot{\mathbf{u}}(t)\phi(x)dx + \int_0^L YA\varphi'^T(x)\mathbf{S}\mathbf{u}(t)\phi'(x)dx + \int_0^L cA\varphi'^T(x)\mathbf{S}\dot{\mathbf{u}}(t)\phi'(x)dx$$
$$= \int_0^L f\phi dx - [k_L u(t, L) + m_L u_{tt}(t, L)]\phi(L).$$

The right hand side is equal to zero if the outside forces and resultant at the boundary are equal to zero. That is,

$$M\ddot{\mathbf{u}}(t) + Q\dot{\mathbf{u}}(t) + K\mathbf{u}(t) = 0.$$

where

$$M = \int_0^L \rho A \varphi^T(x) \mathbf{S} \phi(x) dx, \qquad Q = \int_0^L cA\varphi'^T(x) S \phi'(x) dx, \quad \text{and} \qquad K = \int_0^L YA\varphi'^T(x) S\phi'(x) dx.$$

In first order form, this is

$$\dot{\mathbf{z}}(t) = A\mathbf{z}(t)$$

where

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}Q \end{bmatrix} \text{ and } \mathbf{z}(t) = \begin{bmatrix} \mathbf{u}(t) \\ \dot{\mathbf{u}}(t) \end{bmatrix}.$$

(d) Consider the undamped rod, $c = 0$, with $m_L = k_L = 0$. Determine the kinetic and potential energies and use Hamiltonian (energy) principles to derive a weak formulation of the model. How does it compare with the model you derived in (b)?

Solution:

The kinetic energy is described by

$$K = \frac{1}{2} \int_0^L \rho A u_t^2(t,x) dx$$

and the potential energy is described by

$$V = \frac{1}{2} \int_0^L YA u_x^2(t,x) dx.$$

Thus the action integral is

$$\mathcal{A}[u] = \int_{t_0}^{t_1} \mathcal{L} dt$$

where

$$\mathcal{L} = \frac{1}{2} \int_0^L [\rho A u_t^2 - YA u_x^2] dx.$$

Then the admissable variations are

$$\hat{u}(t,x) = u(t,x) + \epsilon \eta(t) \phi(x)$$

where

$$\text{(i) } \eta(t_0) = \eta(t_1) = 0$$
$$\text{(ii) } \phi \in V = H_0^1(0,L).$$

Now, by Hamilton's principle,

$$\begin{aligned} 0 = \frac{\partial}{\partial t}[\mathcal{A} + \epsilon \Phi]\Big|_{\epsilon=0} &= \frac{1}{2}\frac{\partial}{\partial \epsilon} \int_{t_0}^{t_1} \int_0^L \left[\rho A(u_t + \epsilon \dot{\eta}\phi)^2 - YA(u_x + \epsilon \eta \phi')^2\right] dx dt \Big|_{\epsilon=0} \\ &= \int_{t_0}^{t_1} \int_0^L \left[\rho A(u_t + \epsilon \dot{\eta}\phi)\dot{\eta}\phi - YA(u_x + \epsilon\eta\phi')\eta\phi'\right] dx dt \Big|_{\epsilon=0} \\ &= \int_{t_0}^{t_1} \int_0^L \left[\rho A u_t \dot{\eta}\phi - YA u_x \eta \phi'\right] dx dt. \end{aligned}$$

Using integration by parts, we obtain

$$0 = \frac{\partial}{\partial t}[\mathcal{A} + \epsilon\Phi]\bigg|_{\epsilon=0} = -\int_{t_0}^{t_1} \eta(t) \int_0^L \left[\rho A u_{tt}\phi + Y A u_x \phi'\right] dx dt.$$

Since this holds true for all t_0, t_1, this gives us the weak formulation

$$\int_0^L \rho A u_{tt}\phi dx + \int_0^L Y A u_x \phi' dx = 0.$$

This is equivalent to the weak form from part (b) when $f = c = k_L = m_L = 0$.

(2) Consider the weak formulation

$$\int_0^\ell \rho A \frac{\partial^2 u}{\partial t^2}\phi dx + \int_0^\ell \left[YA\frac{\partial u}{\partial x} + CA\frac{\partial^2 u}{\partial x \partial t}\right]\frac{\partial \phi}{\partial x}dx = -\left[k_\ell u(t,\ell) + c_\ell \frac{\partial u}{\partial t}(t,\ell) + m_\ell \frac{\partial^2 u}{\partial t^2}(t,\ell)\right]\phi(\ell)$$

which holds for all $(\phi, \phi(\ell) \in V$. Using the linear hat function defined in class, this yields the first order system

$$\frac{dz}{dt} = Az(t)$$
$$z(0) = z_0.$$

Construct the matrix A and compute its eigenvalues. Do all of them have negative real part? Now construct the matrix A when the right hand side is positive and discuss the sign of the eigenvalues.

Solution:

First, we will determine the correct units for the parameters of the weak formualtion. When a choice of units exists (e.g., g versus kg), we will chose to use SI units. The proposed units are as follows:

We know that the units of u, which measures displacement, must be meters (m). The units of the test function ϕ, must be the same as the units of u. The units of the remaining parameters are

$$\rho : \frac{kg}{m^3} \qquad A : m^2 \qquad C : Pa \cdot s \qquad Y : \frac{N}{m^2} \qquad k_\ell : \frac{N}{m} \qquad c_\ell : \frac{N \cdot s}{m} \qquad m_\ell : kg \qquad \ell : m$$

The units are verified by entering them into the weak formulation equation and then simplifying.

$$\frac{kg}{m^3} \cdot m^2 \cdot \frac{m}{s^2} \cdot m \cdot m + \left[\frac{N}{m^2} \cdot m^2 \cdot \frac{m}{m} + Pa \cdot s \cdot m^2 \cdot \frac{m}{m \cdot s}\right]\frac{m}{m} = \left[\frac{N}{m} \cdot m + \frac{N \cdot s}{m} \cdot \frac{m}{s} + kg \cdot \frac{m}{s^2}\right] m$$

$$\Rightarrow \frac{kg \cdot m^2}{s^2} + \left[N + Pa \cdot m^2 \right] m = \left[N + N + kg \cdot \frac{m}{s^2}\right] m$$

Since

$$N = \frac{kg \cdot m}{s^2} \quad \text{and} \quad Pa = \frac{N}{m^2},$$

we have

$$N \cdot m + \left[N + N \right] m = [N + N + N]m$$
$$\Rightarrow N \cdot m = N \cdot m.$$

Thus, the units are correct.

Next, the matrix, A, was constructed.

Since the values given for the parameters were not always in SI coefficients, we converted the given values to the units reported herein, where applicable. Additionaly, some values were adjusted so that the eigenvalues of A took on the appropriate values. Thus, the following values were used then constructing the matrix A:

$$
\begin{array}{llll}
\rho : 2.7 \times 10^3 \ \dfrac{kg}{m^3} & A : 10^{-4}\ m^2 & C : 10^3\ Pa \cdot s & Y : 7 \times 10^{10}\ \dfrac{N}{m^2} \\
k_\ell : 10^7\ \dfrac{N}{m} & c_\ell : 10^3\ \dfrac{N \cdot s}{m} & m_\ell : 10^{-3}\ kg & \ell : 10^{-1}\ m
\end{array}
$$

To obtain a system in one variable (time), the linear hat functions, ϕ_i, were substituted in place of the test function, ϕ, and the approximation $\sum_{j=1}^{N} u_j \phi_j$ was sustituted in place of u:

$$
\begin{aligned}
\int_0^\ell \rho A \sum_{j=1}^{N} \ddot{u}_j \phi_j \phi_i dx + \int_0^\ell \left[YA \sum_{j=1}^{N} u_j \phi_j' + CA \sum_{j=1}^{N} \dot{u}_j \phi_j' \right] \phi_i' dx \\
= - \left[k_\ell \sum_{j=1}^{N} u_j \phi_j(\ell) + c_\ell \sum_{j=1}^{N} \dot{u}_j \phi_j(\ell) + m_\ell \sum_{j=1}^{N} \ddot{u}_j \phi_j(\ell) \right] \phi_i(\ell)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \rho A \sum_{j=1}^{N} \ddot{u}_j \int_0^\ell \phi_j \phi_i dx + YA \sum_{j=1}^{N} u_j \int_0^\ell \phi_j' \phi_i' dx + CA \sum_{j=1}^{N} \dot{u}_j \int_0^\ell \phi_j' \phi_i' dx \\
= - \left[k_\ell u_N \phi_N(\ell) + c_\ell \dot{u}_N \phi_N(\ell) + m_\ell \ddot{u}_N \phi_N(\ell) \right] \phi_N(\ell)
\end{aligned}
$$

Since $\phi_N(\ell) = 1$,

$$
\Rightarrow \rho A \sum_{j=1}^{N} \ddot{u}_j \int_0^\ell \phi_j \phi_i dx + YA \sum_{j=1}^{N} u_j \int_0^\ell \phi_j' \phi_i' dx + CA \sum_{j=1}^{N} \dot{u}_j \int_0^\ell \phi_j' \phi_i' dx = - \left[k_\ell u_N + c_\ell \dot{u}_N + m_\ell \ddot{u}_N \right].
$$

When we evaluate the integrals, we obtain

$$
0 = M\ddot{\mathbf{u}}(t) + Q\dot{\mathbf{u}}(t) + K\mathbf{u}(t) \qquad \mathbf{u}(0) = u_0 \qquad \mathbf{u}'(0) = u_0'
$$

with

$$
M = \rho A h \begin{bmatrix} 2/3 & 1/6 & & & \\ 1/6 & 2/3 & 1/6 & & \\ & \ddots & \ddots & \ddots & \\ & & 1/6 & 2/3 & 1/6 \\ & & & 1/6 & 1/3 + \frac{m_\ell}{h} \end{bmatrix}, \qquad Q = \frac{CA}{h} \begin{bmatrix} 2 & -1 & & & \\ 2 & -1 & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & 2 & -1 & 2 \\ & & & 2 & 1 + hc_\ell \end{bmatrix},
$$

and

$$
K = \frac{YA}{h} \begin{bmatrix} 2 & -1 & & & \\ 2 & -1 & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & 2 & -1 & 2 \\ & & & 2 & 1 + hk_\ell \end{bmatrix}.
$$

Now as a first order equation, we have

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}Q \end{bmatrix}$$

with

$$\mathbf{z}(t) = \begin{bmatrix} \mathbf{u}(t) \\ \mathbf{u}'(t) \end{bmatrix} \quad \text{and} \quad \mathbf{z_0} = \begin{bmatrix} \mathbf{u}_0 \\ \mathbf{u}_0' \end{bmatrix}.$$

Note that when the right hand side of the weak form is positive, the matrix A remains the same, except that $M(N,N) = 1/3 - \frac{m_\ell}{h}$, $Q(N,N) = 1 - hc_\ell$, and $K(N,N) = 1 - hk_\ell$. The eigenvalues of the matrix A were calculated using Matlab for both positive and negative right hand sides of the weak formulation. Using a step-size h of 2^{-6}, we found that the eigenvalues of A all had negative real part when the right hand side of the weak formulation is negative. However, using the same step-size, when the right hand side is positive, two of the 128 eigenvalues of A had positive real part. These eigenvalues were approximately $10^8 \times 0.0003666 \pm 6.872i$. When the right hand side of the weak formulation is negative, the first order system in t is stiff, because all eigenvalues have negative real parts, and because the stiffness ratio is large ($Re(\lambda_1)/Re(\lambda_N) \approx 163$.) When the right hand side of the weak formulation is positive, the system is not stiff because not all eigenvalues of A have negative real parts.