Numerical Analysis II: Homework Assignment 3
Spring 2018
Due Feb 26
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This homework assignment concerns interpolation theory. For the computer problems, please submit your computer codes as well.

1. [*10 points*] Problem 30 [1, pages 125].

Let x_∞ be the value that maximizes $\lambda_n(x)$ on the interval $[a, b]$; that is, $\|\lambda_n\|_\infty = \lambda_n(x_\infty)$.
Let $\varphi \in C[a, b]$ be a piecewise linear function such that $\varphi(x_i) = \text{sgn } l_i(x_\infty)$ for $i = 0, 1, \ldots, n$. Note that $\|\varphi\|_\infty = 1$.
Then

$$\begin{aligned}
\|\rho_n(\varphi; \cdot)\|_\infty &= \max_{a \le x \le b} \left| \sum_{i=0}^{n} \varphi(x_i) l_i(x) \right| \\
&\ge \left| \sum_{i=0}^{n} \varphi(x_i) l_i(x_\infty) \right| \\
&= \sum_{i=0}^{n} |l_i(x_\infty)| = \lambda_n(x_\infty) = \Lambda_n = \Lambda_n \|\varphi\|_\infty
\end{aligned}$$

.
We have shown that $\|\rho_n(\varphi; \cdot)\|_\infty \ge \Lambda_n \|\varphi\|_\infty$. Since $\varphi \in C[a, b]$, $\|\rho_n(\varphi; \cdot)\|_\infty \le \Lambda_n \|\varphi\|_\infty$. Thus, $\|\rho_n(\varphi; \cdot)\|_\infty \le \Lambda_n \|\varphi\|_\infty$.

2. [*5 points*] Problem 33 in [1, pages 126].

$$\begin{aligned}
f(x) - p_2(f; x) &= \frac{f^3(\epsilon)}{3!}(x - x_0)(x - x_1)(x - x_2) \qquad (\epsilon \text{ in the smallest closed interval containing } x_0, x_1, x_2, x) \\
&= \frac{2}{(3!)\epsilon^3}(x - x_0)(x - x_1)(x - x_2) \\
&= \frac{(x - x_0)(x - x_1)(x - x_2)}{3\epsilon^3}
\end{aligned}$$

$$|f(11.1) - p_2(f; 11.1)| = \left| \frac{(11.1 - 10)(11.1 - 11)(11.1 - 12)}{3\epsilon^3} \right| = \frac{.999}{3\epsilon^3} \qquad \text{where } \epsilon \in [10, 12]$$

$$\frac{.999}{3(12^3)} \le |f(11.1) - p_2(f; 11.1)| \le \frac{.999}{3(10^3)}$$

$$1.7361 \times 10^{-5} = \frac{.999}{3(12^3)11.1} \le \left| \frac{f(11.1) - p_2(f; 11.1)}{11.1} \right| \le \frac{.999}{3(10^3)11.1} = 3 \times 10^{-5}$$

3. [*10 points*] Problem 65 in [1, page 131].

(a)

Use Hermite interpolation to find the lowest degree polynomial satisfying $p(-1) = p'(-1) = 0$, $p(0) = 1$, $p(1) = p'(1) = 0$:

x	f				
ccline1-2 -1	$\boxed{0}$				
-1	0	$f'(-1) = \boxed{0}$			
0	1	$\frac{1-0}{0+1} = 1$	$\frac{1-0}{0+1} = \boxed{1}$		
1	0	$\frac{0-1}{1-0} =$ -1	$\frac{-1-1}{1+1} =$ -1	$\frac{-1-1}{1+1} = \boxed{-1}$	
1	0	$f'(1) = 0$	$\frac{0+1}{1-0} = 1$	$\frac{1+1}{1+1} = 1$	$\frac{1+1}{1+1} = \boxed{1}$

$$\begin{aligned} p(x) &= 0 + 0(x+1) + 1(x+1)^2 - 1(x+1)^2(x) + 1(x+1)^2(x)(x-1) \\ &= x^4 - 2x^2 + 1 = (x^2-1)^2 \end{aligned}$$

(b)

Let $f(x) = [\cos(\pi x/2)]^2$, which we approximate on $[-1, 1]$.

(b1)

$$e(x) = f(x) - p(x) = \frac{f^{(5)}(\epsilon)}{5!}(x+1)^2(x)(x-1)^2$$

$$f^{(5)}(x) = -\pi^5 \cos(\pi x/2)\sin(\pi x/2)$$

So

$$e(x) = \frac{-\pi^5 \cos(\pi\epsilon/2)\sin(\pi\epsilon/2)}{120} x(x-1)^2(x+1)^2$$

.

(b2) First, finding the max of $|\cos(\pi x/2)\sin(\pi x/2)|$ on $[-1, 1]$:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos(\pi x/2)\sin(\pi x/2)] = \frac{\pi(\cos(\pi x/2))^2}{2} - \frac{\pi(\sin(\pi x/2))^2}{2} = 0$$

$$\cos\left(\frac{\pi x}{2}\right) = \sin\left(\frac{\pi x}{2}\right)$$

$$\frac{\pi x}{2} = \frac{\pi}{4} \Rightarrow x = \frac{1}{2} \qquad\qquad \frac{\pi x}{2} = -\frac{\pi}{4} \Rightarrow x = -\frac{1}{2}$$

$$\left|\cos\left(\frac{\pi(1/2)}{2}\right)\sin\left(\frac{\pi(1/2)}{2}\right)\right| = \left|\cos\left(\frac{\pi(-1/2)}{2}\right)\sin\left(\frac{\pi(-1/2)}{2}\right)\right| = \frac{1}{2}$$

Now, find an upper bound for $|e(x)|$ for a fixed x in $[-1, 1]$:

$$\begin{aligned} |e(x)| &= \left|\frac{-\pi^5 \cos(\pi\epsilon/2)\sin(\pi\epsilon/2)}{120} x(x-1)^2(x+1)^2\right| \\ &= \frac{\pi^5}{120}\left|\cos\left(\frac{\pi\epsilon}{2}\right)\sin\left(\frac{\pi\epsilon}{2}\right)\right| \left|x(x-1)^2(x+1)^2\right| \\ &\le \frac{\pi^5}{120}\left(\frac{1}{2}\right)\left|x(x-1)^2(x+1)^2\right|. \end{aligned}$$

(b3) Finding the max of $|x(x-1)^2(x+1)^2|$ on $[-1,1]$:
(Denote $g(x) = x(x-1)^2(x+1)^2$.)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x(x-1)^2(x+1)^2\right] = 5x^4 - 6x^2 + 1 = (5x^2-1)(x^2-1) = 0$$

So $x = \pm\sqrt{1/5}$; $x = \pm 1$. Then $g(-\sqrt{1/5}) = g(\sqrt{1/5}) = \frac{16}{25\sqrt{5}}$ and $g(1) = g(-1) = 0$.
Thus max $|x(x-1)^2(x+1)^2| = \frac{16}{25\sqrt{5}}$.

Now, estimate $\max_{-1\le x\le 1}|e(x)|$:

$$|e(x)| \le \frac{\pi^5}{120}\left(\frac{1}{2}\right)\left|x(x-1)^2(x+1)^2\right|$$

so

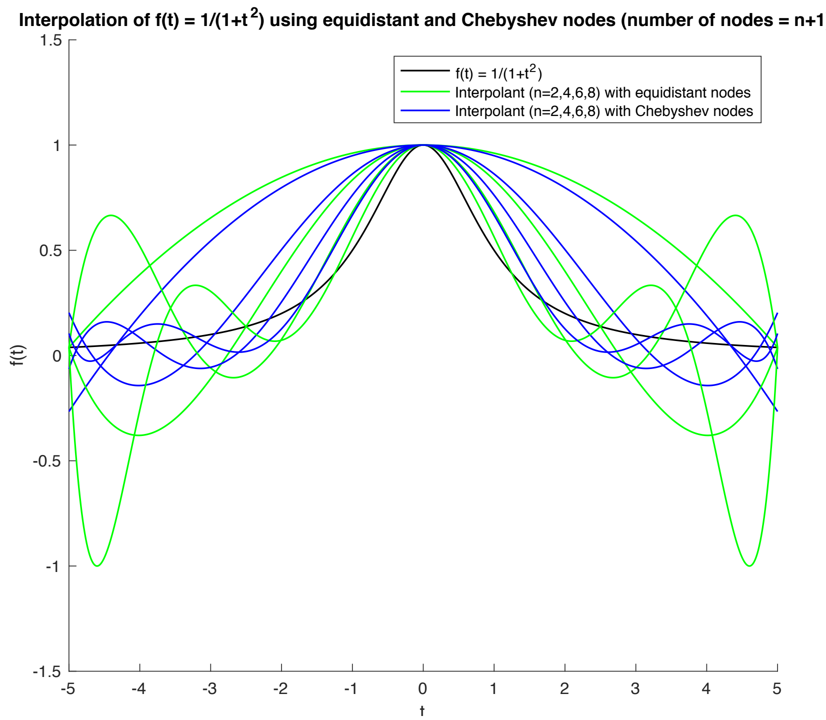
$$\max_{-1\le x\le 1}|e(x)| \le \frac{\pi^5}{120}\left(\frac{1}{2}\right)\left(\frac{16}{25\sqrt{5}}\right) = \frac{\pi^5}{375\sqrt{5}}$$

.

4. [*10 points*] Machine Assignment 7 in [1, page 137].[1] For part (b) Also repeat the experiment with Chebyshev nodes. Explain the numerical method implemented and comment on your results.

(a) See Matlab code ("interpol_ Newton.m").' Newton's method was implemented in "interpol_ Newton.m". First, the function was evaulated at the interpolation nodes (either Chebshev or equidistant). Then the Newton's coefficients were computed. To save memory, a $1 \times n+1$ array was used in the calculation of the coefficients. The array initially was populated with the function evaluated at each interpolation node (i.e., $f[x_0], f[x_1], \ldots, f[x_n]$). Then $f[x_0, x_1], f[x_1, x_2], \ldots f[x_{n-1}, x_n]$ were computed, and the 1st–$(n+1)$st elements of the array were replaced with these values. The same thing was done for $f[x_0, x_1, x_2], \ldots, f[x_{n-2}, x_{n-1}, x_n]$, etc. This process was repeated until the array contained only Newton's coefficients ($f[x_0], f[x_0, x_1], \ldots f[x_0, x_1, \ldots, x_n]$). The inerpolant was then evaulated at t by substituting the computed Newton's coefficients into Newton's formula.

(b) See Matlab code ("Q7_ script.m","plotInterpolant.m","interpol_ Newton.m").



The interpolating polynomials computed using Chebyshev nodes (in blue in the plot) approximate the function $f(t)$ (black) better than the interpolating polynomials computed using equidistant nodes (green). In particular, for a given number of nodes, the maximum $|p_n(f;t)-f(t)|$ is much larger for the interpolants calculated using equidistant nodes than for those calculated using Chebyshev nodes (especially for when a large amount of nodes, eg. 9, are used.) For the interpolating polynomials calculated using equidistant nodes, increasing the number of nodes doesn't seem to improve the approximation too much. However, for the interpolating polynomials calculated using Chebyshev nodes, increasing the number of nodes significantly improves the approximation.

[1]The solution for this problem is provided in the text. Feel free to build on the textbook's solution when you write your computer codes for this exercise.

5. [*10 points*] Consider the Newton divided difference $f[x_0, x_1, \ldots, x_n]$, for $f \in C^{(n+1)}([a, b])$, where $x_0, x_1, \ldots, x_n$ are distinct real numbers, and $[a, b]$ is an interval containing $x_0, x_1, \ldots, x_n$.

   (a) Consider
   $$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}, \tag{1}$$
   where
   $$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \quad f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$
   Prove directly (i.e., without any use of interpolation theory we have discussed in class) that
   $$f[x_0, x_1, x_2] = \frac{f''(\xi)}{2},$$
   for some ξ in the in the interior of the smallest interval containing x_0, x_1, x_2. [Hint: assume without loss of generality that $x_0 < x_1 < x_2$ and use Taylor expansion of f around x_1 in (1).]

   Without loss of generality, assume that $x_0 < x_1 < x_2$.
   $$f(x) = f(x_1) + f'(x_1)(x - x_1) + \frac{f''(\epsilon)}{2}(x - x_1)^2] \quad \text{for some } \epsilon \in [x_0, x_2].$$
   So
   $$f(x_2) - f(x_1) = f'(x_1)(x_2 - x_1) + \frac{f''(\epsilon_2)}{2}(x_2 - x_1)^2 \quad \text{for some } \epsilon_2 \in [x_1, x_2].$$
   and
   $$f(x_1) - f(x_0) = -f'(x_1)(x_0 - x_1) - \frac{f''(\epsilon_0)}{2}(x_0 - x_1)^2 \quad \text{for some } \epsilon_0 \in [x_0, x_1].$$
   Then
   $$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_1) + \frac{f''(\epsilon_2)}{2}(x_2 - x_1)$$
   and
   $$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = f'(x_1) + \frac{f''(\epsilon_0)}{2}(x_0 - x_1).$$
   So
   $$f[x_1, x_2] - f[x_0, x_1] = \frac{f''(\epsilon_2)}{2}(x_2 - x_1) + \frac{f''(\epsilon_0)}{2}(x_1 - x_0).$$
   Since $x_0 < x_1, x_2$, $(x_2 - x_1)/2 > 0$ and $(x_0 - x_1)/2 > 0$.
   Therefore, by proof in 1b of homework 1,
   $$f[x_1, x_2] - f[x_0, x_1] = \frac{(x_2 - x_1)}{2} f''(\epsilon_2) + \frac{(x_1 - x_0)}{2} f''(\epsilon_0) = f''(\xi)\left(\frac{x_2 - x_1}{2} + \frac{x_1 - x_0}{2}\right) = f''(\xi)\left(\frac{x_2 - x_0}{2}\right)$$
   for some $\xi \in [x_0, x_2]$. Finally,
   $$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{f''(\xi)}{2}$$
   for some $\xi \in [x_0, x_2]$.

   (b) Show that
   $$\frac{d}{dx} f[x_0, x_1, \ldots, x_n, x] = f[x_0, x_1, \ldots, x_n, x, x].$$
   You can use the result that $f[x_0, x_1, \ldots, x_n]$ is a continuous function of x_i, $i = 0, \ldots, n$. The latter is a consequence of Hermite–Genocchi Theorem; see e.g., [2, p. 189].

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} f[x_0, x_1, \ldots, x_n, x] &= \lim_{h \to 0} \frac{f[x_0, x_1, \ldots, x_n, x+h] - f[x_0, x_1, \ldots, x_n, x]}{x+h-x} \\
&= \lim_{h \to 0} \frac{f[x_0, x_1, \ldots, x_n, x+h] - f[x, x_0, x_1, \ldots, x_n]}{x+h-x} \\
&= \lim_{h \to 0} f[x, x_0, x_1, \ldots, x_n, x+h] \\
&= f[x, x_0, x_1, \ldots, x_n, x] = f[x_0, x_1, \ldots, x_n, x, x]
\end{aligned}
$$

6. [*10 points*] Let

$$
x_i^C = \cos\left(\frac{2i+1}{2n+2}\pi\right), \quad i = 0, 1, \ldots, n.
$$

be the Chebyshev points on $[-1, 1]$. Consider an interval $[a, b]$, and obtain the analogous Chebyshev points t_i^C on $[a, b]$. Then, prove the estimate

$$
\left|\prod_{i=0}^{n}(t - t_i^C)\right| \leq \frac{1}{2^n}\left(\frac{b-a}{2}\right)^{n+1}, \quad \text{for all } t \in [a, b].
$$

$$
t_i^C = \frac{a+b}{2} + \frac{b-a}{2}\, x_i^C
$$

And in general, there is a mapping from $x \in [-1, 1] \mapsto t \in [a, b]$:

$$
t = \frac{a+b}{2} + \frac{b-a}{2}\, x.
$$

Then

$$
\begin{aligned}
\left|\prod_{i=0}^{n}(t - t_i^C)\right| &= \left|\prod_{i=0}^{n}\left(\frac{a+b}{2} + \frac{b-a}{2}x - \left(\frac{a+b}{2} + \frac{b-a}{2}x_i^C\right)\right)\right| \\
&= \prod_{i=0}^{n}\left(\frac{b-a}{2}(x - x_i^C)\right) \\
&= \left(\frac{b-a}{2}\right)^{n+1}\left|\prod_{i=0}^{n}(x - x_i^C)\right| \\
&\leq \frac{1}{2^n}\left(\frac{b-a}{2}\right)^{n+1}.
\end{aligned}
$$

References

[1] Gautschi, Walter. Numerical analysis. Springer Science & Business Media, 2011.

[2] Hohmann, Andreas, and Deuflhard, Peter. Numerical analysis in modern scientific computing: an introduction. Vol. 43. Springer Science & Business Media, 2012.