MA/CSC 580, Homework #6

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1. Let $w \neq 0 \in \mathbb{R}^n, A = w^T w, B = ww^T$.

   (a) A is a scalar and $B \in \mathbb{R}^{n\times n}$.

   (b) Since A is a scalar, $\|B\|_1 = \|B\|_2 = \|B\|_\infty = w^T w$.
   Note that the ith column of B is $w_i \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix}^T$ and the ith row of B is $w_i \begin{bmatrix} w_1 & w_2 & \dots & w_n \end{bmatrix}$ for $i = 1, \dots, n$. Thus

   $$\|B\|_1 = \|B\|_\infty = \left(\max_i |w_i|\right)\left(\sum_{i=1}^{n} |w_i|\right).$$

   Since B is symmetric,

   $$\|B\|_2 = \max|\lambda_i(B)| = \sum_{i=1}^{n} w_i^2 = w^T w = A.$$

   An explanation of how this eigenvalue was determined can be found in part (e).

   (c) A is a scalar, so rank(A)=1.
   Let the ith column of B be denoted by B_i. Note that

   $$B_i = \frac{w_i}{w_1} B_1 \qquad \text{(for all } i = 1, \dots, n).$$

   Thus, rank(B)=1.

   (d) Since A is a scalar, $cond_p(A) = 1$ for all $p = 1, 2, \infty$.
   For $n > 1$, B is singular since rank$(B) = 1 \neq n$. Thus $cond_p(B) = \infty$ for all $p = 1, 2, \infty$.
   For $n = 1$, B is also a scalar, so $cond_p(B) = 1$ for all $p = 1, 2, \infty$.

   (e) Since A is a scalar, it is its own eigenvalue, and all non-zeros vectors are eigenvectors.
   Since all columns of B are a scalar multiple of w, $Bx = cw$ for all x, where $c \in \mathbb{R}$. Note that c can equal zero. Thus the eigenvectors of B are w (and scalar multiples of w), as well as the null space of B.

   (f) The pseudo-inverse of A is $1/A$ since A is a scalar.
   The pseduo-inverse of B is $\frac{1}{\sigma_1^2} B$.

2. (a) Let $y = Sx$. Then, for $i = 1, \dots, n$,

   $$\begin{aligned} (A+E)x &= \lambda_i(A+E)x \\ S(A+E)S^{-1}Sx &= \lambda_i(A+E)Sx \\ (D + SES^{-1})y &= \lambda_i(A+E)y \\ (D - \lambda_i(A+E)I)y &= -SES^{-1}y \\ y &= (D - \lambda_i(A+E)I)^{-1}(-SES^{-1})y \end{aligned}$$

   so

   $$\begin{aligned} \Rightarrow \|y\| &= \left\|(D - \lambda_i(A+E)I)^{-1}(-SES^{-1})y\right\| \\ &\leq \left\|(D - \lambda_i(A+E)I)^{-1}\right\| \|S\| \left\|S^{-1}\right\| \|E\| \|y\| \\ &= \left\|(D - \lambda_i(A+E)I)^{-1}\right\| cond(S) \|E\| \|y\| \end{aligned}$$

Now consider $\left\|(D-\lambda_i(A+E)I)^{-1}\right\|$. First, recall that the inverse of a diagonal matrix is another diagonal matrix where each diagonal entry is inverted with respect to the original matrix. Since $(D-\lambda_i(A+E)I)^{-1}$ is a diagonal matrix, the norm is equal to the max of the absolute value of the diagonal entries. Since $D = diag(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$,

$$\left\|\left(D-\lambda_i(A+E)I\right)^{-1}\right\| = \max_i \left|\left(\lambda_i(A)-\lambda_i(A+E)\right)^{-1}\right| = \left(\min_i \left|\left(\lambda_i(A)-\lambda_i(A+E)\right)\right|\right)^{-1}.$$

Substituting this into our previous equation,

$$\|y\| \leq \left(\min_i \left|\left(\lambda_i(A)-\lambda_i(A+E)\right)\right|\right)^{-1} cond(S)\,\|E\|\,\|y\|\,.$$

Therefore

$$\min_i \left|\left(\lambda_i(A)-\lambda_i(A+E)\right)\right| \leq cond(S)\,\|E\|\,.$$

If $A = A^T$, then this is true for $\max_i \left|\left(\lambda_i(A)-\lambda_i(A+E)\right)\right|$, not just the min. In fact, if $A = A^T$,

$$\max_i \left|\left(\lambda_i(A)-\lambda_i(A+E)\right)\right| \leq \|E\|\,.$$

(b) Matrix A:
$\lambda_1 = 2$ and

$$2-1-\|E\| \leq \lambda_1 \leq 2+1+\|E\|$$

since A is symmetric.
Matrix B:
$\lambda_1 = 3, \lambda_2 = 2$

$$3-1-C\,\|E\|^{1/4} \leq \lambda_1 \leq 3+1+C\,\|E\|^{1/4}$$
$$2-1-C\,\|E\|^{1/2} \leq \lambda_2 \leq 2+1+C\,\|E\|^{1/2}$$

3. (a) The normal equation for this problem is

$$\begin{bmatrix} 0_1^T & B^T & 0_2^T \end{bmatrix} \begin{bmatrix} 0_1 \\ B \\ 0_2 \end{bmatrix} x = \begin{bmatrix} 0_1^T & B^T & 0_2^T \end{bmatrix} \begin{bmatrix} b_1 \\ b_1 \\ b_3 \end{bmatrix}$$
$$\begin{bmatrix} 0_3 \\ B^T B \\ 0_4 \end{bmatrix} x = \begin{bmatrix} 0_5 \\ B^T b_2 \\ 0_6 \end{bmatrix}$$

where 0_3 and 0_4 are zero matrices and 0_5 and 0_6 are zero vectors. Thus the solution to the normal equation (and hence the least squares problem) is the solution to $B^T B x = B^T b_2$. Since B is invertible, B^T is invertible, so the solution is $x^* = B^{-1} b_2$.

(b) First least squares problem:

$$B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad b_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

so

$$x^* = B^{-1}b_2 = \begin{bmatrix} 1/2 & 1/6 & 1/6 \\ 0 & -1/3 & 2/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The residual is $\|b - Ax^*\|_2$:

$$b - Ax^* = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so $b - Ax^* = \sqrt{3}$.

Second least squares problem:

$$B = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

so

$$x^* = B^{-1}b_2 = \begin{bmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

The residual is $\|b - Ax^*\|_2$:

$$b - Ax^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -8 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -8 \\ 1 \end{bmatrix}$$

so $\|b - Ax^*\|_2 = \sqrt{65}$.

4. First we find the QR decomposition of A. This is necessary for multiple parts.

We first will find P' such that $P'x = y$ where

$$x = \begin{bmatrix} 0 \\ 3 \\ -4 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}$$

$$\|y\|_2 = \|x\|_2 = \sqrt{9+16} = 5$$

Since the first component of P' is zero, we can choose the sign of α arbitrarily, so we choose $\alpha = 5$. To find P' we first compute

$$\omega = \frac{x-y}{\|x-y\|_2} = \begin{bmatrix} -5 \\ 3 \\ -4 \end{bmatrix} / \sqrt{50}$$

Thus

$$P' = I - \frac{2}{50} \begin{bmatrix} -5 \\ 3 \\ -4 \end{bmatrix} \begin{bmatrix} -5 & 3 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 3/5 & -4/5 \\ 3/5 & 16/25 & 12/25 \\ -4/5 & 12/25 & 9/25 \end{bmatrix}$$

, so

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3/5 & -4/5 \\ 0 & 0 & 3/5 & 16/25 & 12/25 \\ 0 & 0 & -4/5 & 12/25 & 9/25 \end{bmatrix}$$

where $PA = R$ (here $Q = P$) and

$$R = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(a) To solve the over-determined problem, we consider $PAx = Pb$:

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 10 \\ 0 \\ 0 \end{bmatrix}$$

Using backward substitution, we get the solution to the least squares problem

$$x^* = \begin{bmatrix} 3/2 \\ 0 \\ 2 \end{bmatrix}$$

The residual vector is

$$Ax^* - b = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 3/2 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 0 \\ 6 \\ -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

so the 2-norm of the residual vector is 0.

(b) Again, we consider $PAx = Pb$:

$$\begin{bmatrix} 2 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} .= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3/5 \\ -4/5 \end{bmatrix}$$

Using backwards substitution, we get the solution to the least squares problem:

$$x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The residual vector is

$$Ax^* - b = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix},$$

so the 2-norm of the residual vector is 1.

(c) We wish to find a Given's rotation which puts $a_{53} = 0$. Thus, we have:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the element in the 5th row, 3rd column equals zeros, it is necessary for

$$-3\sin\theta - 4\cos\theta = 0$$

Thus

$$\tan\theta = -4/3,$$

so $\tan^2\theta = 16/9$. Since

$$\tan^2\theta + 1 = 1/\cos\theta,$$

we have

$$\cos^2\theta = 9/25 \Rightarrow \cos\theta = 3/5.$$

Now we use the common trig identity $sin^2\theta + \cos^2\theta = 1$ to get

$$\sin^2\theta = 1 - \cos^2\theta = 16/25.$$

Since $\tan\theta < 0$, the sign of $\sin\theta$ should be opposite the sign of $\cos\theta$. Thus,

$$\sin\theta = -4/5.$$

The Given's rotation applied to A is then

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&3/5&-4/5\\0&0&0&4/5&3/5\end{bmatrix}\begin{bmatrix}2&2&0\\0&5&0\\0&0&0\\0&0&3\\0&0&-4\end{bmatrix}=\begin{bmatrix}2&2&0\\0&5&0\\0&0&0\\0&0&5\\0&0&0\end{bmatrix}.$$

We now apply the Given's rotation to $b = e_5$ to get

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&3/5&-4/5\\0&0&0&4/5&3/5\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\-4/5\\3/5\end{bmatrix}.$$

Now, to get the solution to $Ax = b$, we must solve the equation $GAx = Gb$:

$$\begin{bmatrix}2&2&0\\0&5&0\\0&0&0\\0&0&5\\0&0&0\end{bmatrix}x=\begin{bmatrix}0\\0\\0\\-4/5\\3/5\end{bmatrix}.$$

This system is easy to solve using backward substitution.

$$\begin{aligned}5x_3 &= -\frac{4}{5} \Rightarrow x_3 = -\frac{4}{25}\\ 5x_2 &= 0 \Rightarrow x_2 = 0\\ 2x_1 &= 0 \Rightarrow x_1 = 0\end{aligned}$$

Thus, the least squares solution is

$$x^* = \begin{bmatrix}0\\0\\-4/25\end{bmatrix}.$$

The residual vector is

$$Ax^* - b = \begin{bmatrix}2&2&0\\0&5&0\\0&0&0\\0&0&3\\0&0&-4\end{bmatrix}\begin{bmatrix}0\\0\\-4/25\end{bmatrix}-\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\-12/25\\16/25\end{bmatrix}-\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\\-12/25\\-9/25\end{bmatrix}.$$

Thus, the 2-norm of the residual vector is

$$\sqrt{\left(-\frac{12}{25}\right)^2+\left(-\frac{9}{25}\right)^2}=\sqrt{\frac{225}{625}}=\frac{3}{5}.$$

5. (a) Find the singular value decomposition for the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}:$$

First, we find the singular values of A, which are the eigenvalues of $A^H A$.

$$A^H A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

so $\sigma_1^2 = 2$, $\sigma_2^2 = 1$, and $\sigma_3^2 = 0$.
Note that x_1 is the eigenvector associated with the eigenvalue σ_1^2.

$$x_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

Indeed, we can see that

$$A^H A x_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{bmatrix} = 2 \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = 2x_1 = \sigma_1^2 x_1.$$

Now we use a Householder mastrix to expand x_1 to an orthonormal basis to get V. The first step is to find a P such that

$$P \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$\omega = \frac{x_1 - \boldsymbol{e}_1}{\|x_1 - \boldsymbol{e}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{2}} - 1 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \Big/ \sqrt{2 - \sqrt{2}}$$

Then

$$P\boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{2 - \sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} - 1 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} - 1 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Since $\omega^T \boldsymbol{e}_1 = 0$,

$$P\boldsymbol{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Now we compute

$$\begin{aligned} P\boldsymbol{e}_3 &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2 - \sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} - 1 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} - 1 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \frac{\sqrt{2}}{2 - \sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}} - 1 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}}. \end{bmatrix} \end{aligned}$$

Thus,

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Next we will compute U.

$$y_1 = Ax_1/\sigma_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Since y_1 and y_2 form an orthogonal basis, it is now easy to see that

$$y_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus,

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Finally, we note that the singular values of A (computed previously) are on the diagonal of Σ, so the SVD decomposition of A is

$$A = U\Sigma V^H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

(b) Use your result to solve the linear system of equations $Ax = b$, where $b = \begin{bmatrix} 2 & 1 \end{bmatrix}^T$:
First we find the pseudo-inverse of A

$$A^+ = V\Sigma^+U^H = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix}.$$

Then, the solution to the system $Ax = b$ is

$$x = A^+b = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

(c) Explain the meaning of the solution:
Since $Ax = b$ is underdetermined, there are multiple solutions, and $x = A^+b$ is the solution with the minimal 2-norm.

6. (a)

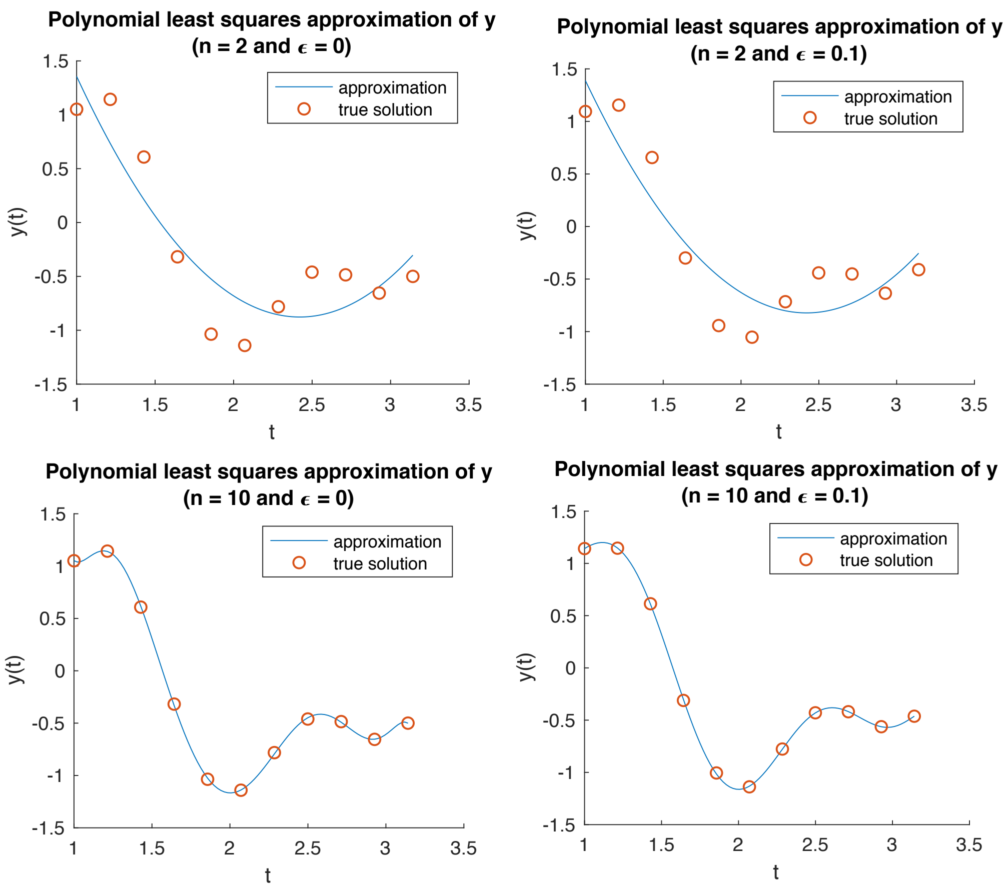

Figure 1: Least squares solution vs data for $y(t)$ from part (a), with $n = 2, 10$ and $\epsilon = 0, 0.1$.

(b)

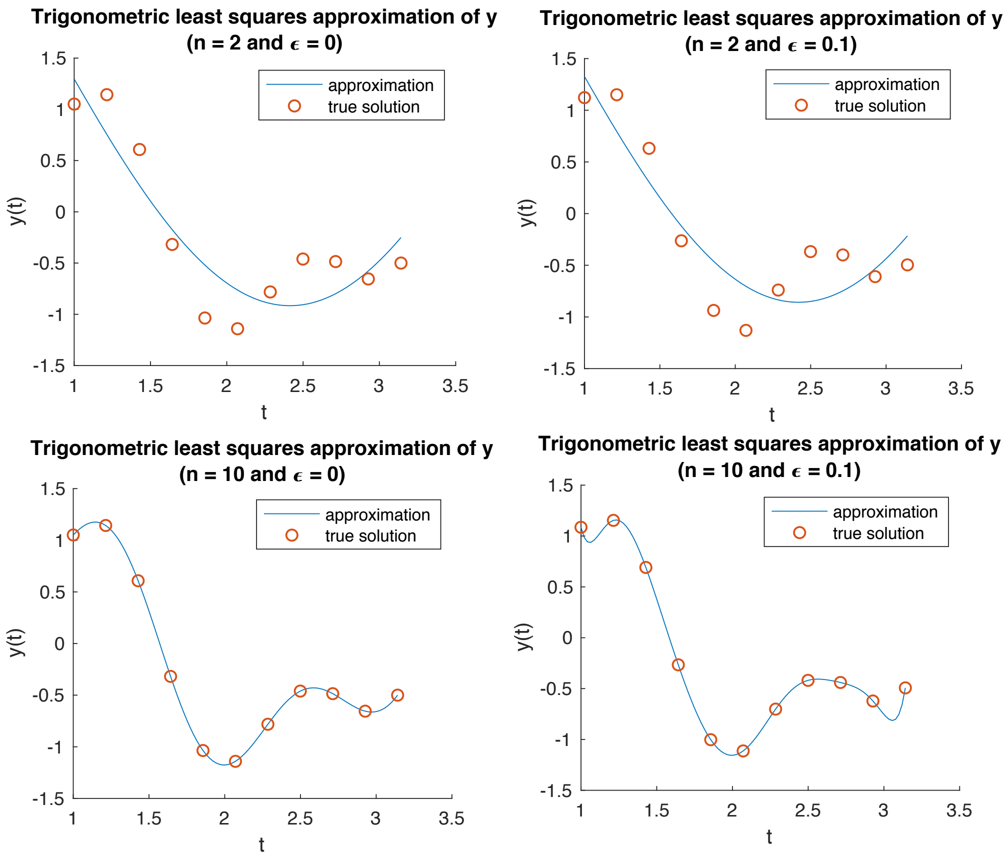

Figure 2: Least squares solution vs data for $y(t)$ from part (b), with $n = 2, 10$ and $\epsilon = 0, 0.1$.

(c)

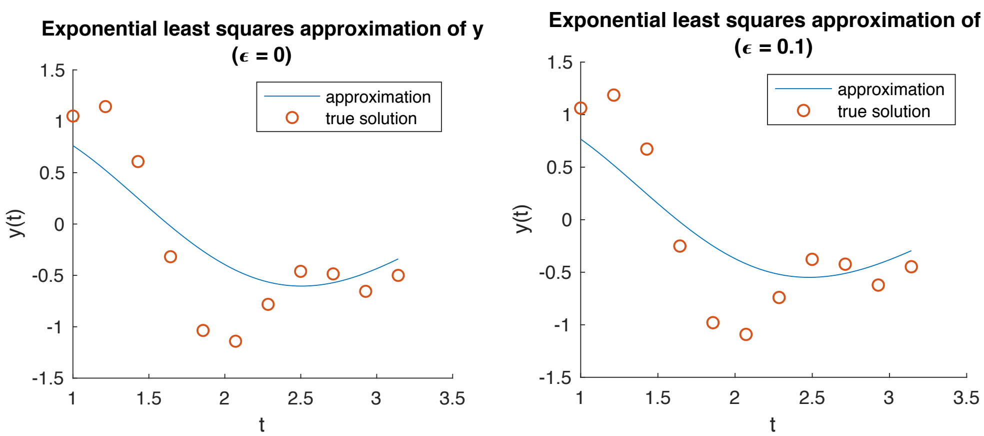

Figure 3: Least squares solution vs data for $y(t)$ from part (a), with $\epsilon = 0, 0.1$.