Mathematical Modeling I, Fall 2018: Assignment 4
Due Monday Nov 12, 2018

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This homework covers fluid flow, balance laws, and Galerkin finite element method for solving PDEs. You can work in teams of up to three people. Submit one report per team.

1. [12 points] We consider (single-phase) fluid flow in a porous medium. We define porosity ϕ of the medium as the ratio of voids (in the medium) to total volume of the medium. While porosity can in general vary in space, in this problem, we assume porosity is a constant scalar.

   Consider a rod with a constant cross-sectional area, made of a porous material; assuming fluid density is uniform in y and z dirctions, $\rho = \rho(t, x)$, we can perform the analysis in one-space dimension, as done for related problems in the class. Consider a small volume

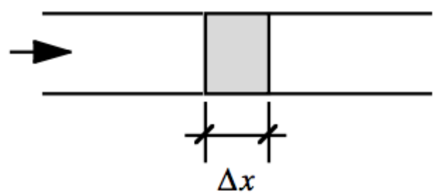

Figure 1: A portion of a one-dimensional rod model.

   element between x and $x + \Delta$; see Figure 1. Assuming no mass (of fluid) is created or destroyed in the volume element, we have the balance relation

$$\left\{\begin{array}{l}\text{rate change of mass}\\\text{in element}\end{array}\right\} = \left\{\begin{array}{l}\text{rate of mass}\\\text{entering face } x\end{array}\right\} - \left\{\begin{array}{l}\text{rate of mass}\\\text{leaving face } x+\Delta x\end{array}\right\}$$

   Using this, and assuming a constant cross-sectional area A, derive the corresponding continuity equation. [Hint: the rate of change of mass in the rod element is $\frac{\partial}{\partial t}(\phi A \Delta x \rho)$].

   From the balance relation, we have

$$\frac{\partial}{\partial t}(\phi A \Delta x \rho) = \mu^{\text{tot}} A\Big|_x - \mu^{\text{tot}} A\Big|_{x+\Delta x}.$$

   Dividing both sides by Δx and taking the limit as Δx goes to zero, we have

$$\frac{\partial}{\partial t}(\phi A \rho) = -\frac{\partial}{\partial x}\left(\mu^{\text{tot}} A\right).$$

   There is no diffusion, so $\mu^{\text{tot}} = \mu^{\text{bulk}} = \rho v$, so

$$\frac{\partial}{\partial t}(\phi A \rho) = -\frac{\partial}{\partial x}(A \rho v).$$

   Since A is constant, we have

$$\frac{\partial}{\partial t}(\phi \rho) = -\frac{\partial}{\partial x}(\rho v)$$

   which is the continuity equation.

2. [10 points] Generalize the derivation in the previous problem to a 3D domain by following the basic steps of formulating a balance law and using the divergence theorem to derive the governing PDE. For this problem, assume a mass source/sink function $f(t, \boldsymbol{x})$.

$$\begin{Bmatrix} \text{rate change of mass} \\ \text{in element } V \end{Bmatrix} = \begin{Bmatrix} \text{net flux through} \\ \text{the boundary } \partial V \end{Bmatrix} - \begin{Bmatrix} \text{net creation/} \\ \text{destruction in } V \end{Bmatrix}$$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \int_V \frac{\mathrm{d}}{\mathrm{d}t}(\phi\rho)\ \mathrm{d}x = -\int_{\partial V} \mu(t,x) \cdot n\ \mathrm{d}s + \int_V f(t,x)\ \mathrm{d}x.$$

By the divergence theorem,

$$\int_V \frac{\mathrm{d}}{\mathrm{d}t}(\phi\rho)\ \mathrm{d}x = -\int_V \nabla \cdot \mu\ \mathrm{d}x + \int_V f(t,x)\ \mathrm{d}x.$$

So,

$$\int_V \frac{\mathrm{d}}{\mathrm{d}t}(\phi\rho) + \nabla \cdot \mu - f\ \mathrm{d}x = 0$$

for all $V \subseteq U$, so

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi\rho) + \nabla \cdot (\rho v) = f$$

which is the continuity equation.

3. [10 points] A simple model for fluid flow is obtained by invoking Darcy's law, which says that the fluid velocity $\boldsymbol{v}$,[1] neglecting gravitational forces, is related to fluid pressure p through

$$\boldsymbol{v} = -\frac{1}{\eta}\mathbf{K}\nabla p \tag{1}$$

where η is fluid viscosity (the more usual notation for viscosity is μ, but I do not use it here to avoid confusing with flux notation from class) and $\mathbf{K}$ is the absolute permeability tensor (a matrix valued function).

Note: in the present context, the flux function we take in the continuity equation is of the form $\mu(t, \boldsymbol{x}) = \rho\boldsymbol{v}$, where $\boldsymbol{v}$ is the so called Darcy velocity, which is given by Darcy's law (1). This velocity is called by many names in the literature: Darcy velocity, Superficial Darcy velocity, Darcy flux, etc.

(a) For simplicity, assume an isotropic medium[2] in which case, $\mathbf{K} = \kappa\mathbf{I}$, where κ is a scalar function and $\mathbf{I}$ is the 3×3 identity matrix. Incorporate this assumption in Darcy's law and combine Darcy's law with the continuity equation obtained in the previous problem, to obtain an equation involving fluid density and pressure.

Combining Darcy's law (which in this case is $v = -\frac{\kappa}{\eta}\nabla p$) with the continuity equation from 2, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi\rho) + \nabla \cdot (-\rho\frac{\kappa}{\eta}\nabla p) = f$$

[1] The velocity $\boldsymbol{v}$ as defined by Darcy's law is called the Darcy velocity.

[2] Roughly speaking, a medium is called isotropic, if its physical properties are independent of its orientation.

Rearraging this, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi\rho) - \frac{\kappa}{\eta}\nabla\cdot(\rho\nabla p) = f.$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi\rho) - \frac{\kappa}{\eta}\Big[\rho\nabla^2 p + \nabla\rho\nabla p\Big] = f.$$

(b) Assume the fluid is incompressible; that is density ρ is a constant (recall that ϕ is also a constant). Write down the simplified equation, now for fluid pressure only, in this case. Cancel out ρ completely out of the equations, by scaling things approriately.

Since ρ (and also ϕ) is a constant, the terms $\frac{\mathrm{d}}{\mathrm{d}t}(\phi\rho)$ and $\nabla\rho\nabla p$ equal zero, so we get

$$-\frac{\kappa}{\eta}\rho\nabla^2 p = f.$$

Define $F($ as $F(t,\boldsymbol{x}) = \frac{1}{\rho}f(t,\boldsymbol{x})$. Then

$$-\frac{\kappa}{\eta}\nabla^2 p = F.$$

4. [8 points] Consider a traffic model where $\rho(t,x)$ is the density of cars per unit length of road, $\mu(t,x)$ is the flux function (note, I am using the term flux function as we did in the context of deriving PDEs from balance laws, in the classroom; in this one-dimensional setting, the rate at which cars flow past a point x can be more correctly termed as a flow rate), and v is the speed at which they are moving.

(a) In low density traffic, one can assume cars maintain a constant speed of v_0. Write down the differential equation modeling traffic flow in this case. Assuming $\rho(0,x) = \psi(x)$, show the solution of this differential equation is given by $\rho(t,x) = \psi(x - v_0 t)$.

The differential equation modeling traffic flow is

$$\frac{\partial\rho}{\partial t} + \frac{\partial\mu}{\partial x} = 0.$$

Since $\mu = v_0\rho$, we have

$$\frac{\partial\rho}{\partial t} = -v_0\frac{\partial\rho}{\partial x}.$$

To show that the solution to the differential equation is given by $\rho(t,x) = \psi(x - v_0 t)$, we use the chain rule to note that

$$\frac{\partial}{\partial t}\left(\psi(x - v_0 t)\right) = -v_0\ \psi'(x - v_0 t)$$

and

$$\frac{\partial}{\partial x}\left(\psi(x - v_0 t)\right) = \psi'(x - v_0 t).$$

Then

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t}\left(\psi(x - v_0 t)\right) = -v_0\ \psi'(x - v_0 t) = -v_0\frac{\partial}{\partial x}\left(\psi(x - v_0 t)\right) = -v_0\frac{\partial\rho}{\partial x}.$$

Also,

$$\rho(0,x) = \psi(x - v_0 0) = \psi(x).$$

(b) For heavy traffic, a car's speed can be modeled with the relation

$$v(\rho) = \alpha(1 - \rho/\beta),$$

where α is the maximum speed in light traffic. Determine a differential equation quantifying the flow of traffic under these conditions.

Note that in general, the differential equation modeling traffic flow is

$$\frac{\partial \rho}{\partial t} + \frac{\partial v\rho}{\partial t} = 0,$$

so in this case we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}\left(\alpha(1 - \rho/\beta)\rho\right) = \frac{\partial \rho}{\partial t} + \alpha\Big(1 - \frac{\rho}{\beta}\Big)\frac{\partial \rho}{\partial x} - \frac{\alpha}{\beta}\rho\frac{\partial \rho}{\partial x} = 0.$$

Rearranging this we have

$$\frac{\partial \rho}{\partial t} = \alpha\Big(\frac{2}{\beta}\rho - 1\Big)\frac{\partial \rho}{\partial x}.$$

5. [15 points] Consider the following problem

$$\begin{aligned} -\nabla \cdot (k\nabla u) + \alpha u &= f \quad \text{in } \Omega, \\ k\nabla u \cdot \boldsymbol{n} &= g \quad \text{on } \partial\Omega. \end{aligned}$$

Here Ω is a bounded domain in $\mathbb{R}^3$, $f \in L^2(\Omega)$, α is a positive constant, and $g \in L^2(\partial\Omega)$.

(a) Derive the weak formulation of the given problem. Be precise about the choice of the function space V for the weak solution and the test functions.

Since $-\nabla \cdot (k\nabla u) + \alpha u = f$ in Ω,

$$\int_\Omega (-\nabla \cdot (k\nabla u) + \alpha u)v \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x \quad \text{for all } v \in V = H_0^1(\Omega).$$

Using multivariate integration by parts, this becomes

$$\int_\Omega k\nabla u \cdot \nabla v \, \mathrm{d}x - \int_{\partial\Omega} vk\nabla u \cdot \boldsymbol{n} \, \mathrm{d}s + \int_\Omega \alpha uv \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x \quad \text{for all } v \in V.$$

By the boundary condition, this is

$$\int_\Omega k\nabla u \cdot \nabla v \, \mathrm{d}x - \int_{\partial\Omega} vg \, \mathrm{d}s + \int_\Omega \alpha uv \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x \quad \text{for all } v \in V.$$

This can be rearranged to be in the form $a(u,v) = \ell(v)$:

$$\int_\Omega k\nabla u \cdot \nabla v \, \mathrm{d}x + \alpha\int_\Omega uv \, \mathrm{d}x = \int_{\partial\Omega} vg \, \mathrm{d}s + \int_\Omega fv \, \mathrm{d}x \quad \text{for all } v \in V.$$

(b) Show that this weak formulation coincides with the Euler–Lagrange equation for the variational minimization problem

$$\min_{u \in V} J(u) = \frac{1}{2}\int_\Omega k\nabla u \cdot \nabla u \, dx + \frac{\alpha}{2}\int_\Omega u^2 \, dx - \int_\Omega fu \, dx - \int_{\partial\Omega} gu \, ds,$$

by computing $\delta J(u,v)$ and setting $\delta J(u,v) = 0$ for all $v \in V$.

$$\begin{aligned}
\delta J(u,v) &= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} J(u+\epsilon v) \\
&= \frac{\mathrm{d}}{\mathrm{d}\epsilon}\Big|_{\epsilon=0} \Bigg[\frac{1}{2}\int_\Omega k\nabla(u+\epsilon v)\cdot\nabla(u+\epsilon v)\ \mathrm{d}x + \frac{\alpha}{2}\int_\Omega (u+\epsilon v)^2\ \mathrm{d}x \\
&\quad - \int_\Omega f(u+\epsilon v)\ \mathrm{d}x - \int_{\partial\Omega} g(u+\epsilon v)\ \mathrm{d}s\Bigg] \\
&= \int_\Omega k\nabla u \cdot \nabla v\ \mathrm{d}x + \alpha\int_\Omega uv\ \mathrm{d}x - \int_\Omega fv\ \mathrm{d}x - \int_{\partial\Omega} gv\ \mathrm{d}s.
\end{aligned}$$

Note that $\delta J(u,v) = 0$ for all $v \in V$ is a necessary condition for u to be a solution to the variational minimization problem. Setting $\delta J(u,v)$ equal to zero for all $v \in V$ we get

$$\int_\Omega k\nabla u \cdot \nabla v\ \mathrm{d}x + \alpha\int_\Omega uv\ \mathrm{d}x = \int_{\partial\Omega} vg\ \mathrm{d}s + \int_\Omega fv\ \mathrm{d}x \quad \text{for all } v \in V,$$

which is the weak form found in part (a).

(c) The weak formulation you obtained in part (a) can be written in the form

$$a(u,v) = \ell(v), \quad \forall v \in V.$$

Computing the weak solution, in the Galerkin finite-element method, involves seeking an approximate solution $u_h \in V_h = \operatorname{span}\{\phi_i\}_{i=1}^N \subset V$, represented as $u_h(\boldsymbol{x}) = \sum_{i=1}^N u_i\phi(\boldsymbol{x})$ that satisfies,

$$a(u_h, v) = \ell(v), \quad v \in V_h.$$

We saw in class that this leads to a linear system of equations $\mathbf{A}\boldsymbol{U} = \boldsymbol{F}$, with $A_{ij} = a(\phi_i, \phi_j)$, $i,j = 1,\dots,N$, and $F_i = \ell(\phi_i)$, $i = 1,\dots,N$, and where $\boldsymbol{U} = (u_1, u_2, \dots, u_N)^T$. Show that the same linear system is obtained by finding $u_h \in V_h$ that solves

$$\min_{u_h \in V_h} J(u_h) = \frac{1}{2}a(u_h, u_h) - \ell(u_h).$$

[Hint: Note that each $u_h \in V_h$ can be represented as $u_h = \sum_{i=1}^N u_i\phi_i$; therefore, we can identify u_h with the vector of coefficients $(u_1, u_2, \dots, u_N)^T$].

We want to minimize

$$\begin{aligned}\frac{1}{2}a(u_h, u_h) - \ell(u_h) &= \frac{1}{2}a\left(\sum_{i=1}^{N} u_i\phi_i, \sum_{j=1}^{N} u_j\phi_j\right) - \ell\left(\sum_{i=1}^{N} u_i\phi_i\right) \\ &= \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a(\phi_i, \phi_j)u_i u_j - \sum_{i=1}^{N}\ell(\phi_i)u_i \\ &= \frac{1}{2}\boldsymbol{U}^T\mathbf{A}\boldsymbol{U} - \boldsymbol{U}^T\boldsymbol{F}.\end{aligned}$$

To minimize $\frac{1}{2}U^TAU - U^TF$, a necessary condition is that the gradient equal zero. So we need U such that

$$\mathbf{A}\boldsymbol{U} - \boldsymbol{F} = 0.$$

This is the same linear system $\mathbf{A}\boldsymbol{U} = \boldsymbol{F}$ which we derived from the weak solution.

Extra credit: [5 points] The following result was used when deriving PDEs describing balance laws. You are asked to prove it in this exercise. Let U be a region in $\mathbb{R}^3$. Assume that $f : U \to \mathbb{R}$ is a continuous function. Suppose

$$\int_V f(\boldsymbol{x})\, d\boldsymbol{x} = 0,$$

for all bounded open subsets V of U. Then, $f \equiv 0$ on U.

Assume $f(\boldsymbol{u}) \neq 0$ for some point $\boldsymbol{u} \in U$. Without loss of generality, let $f(\boldsymbol{u}) > 0$. Then there exists some $\delta > 0$ such that for all $y \in B(f(\boldsymbol{u}), \delta)$, $y > 0$ (where $B(a, r)$ is the open ball around a of radius r). Since f is continuous, there exists an $\epsilon > 0$ such that $f\big(B(\boldsymbol{u}, \epsilon)\big) \subseteq B(f(\boldsymbol{u}), \delta)$. Let us denote $\big(B(\boldsymbol{u}, \epsilon) \cap U\big)^o$ by V^*. We will assume that U has nice enough properties that V^* does not have measure zero. Note that $f(V^*) \subseteq f\big(B(\boldsymbol{u}, \epsilon)\big) \subseteq B(f(\boldsymbol{u}), \delta)$. Thus for all points $\boldsymbol{z} \in V^*$, $f(\boldsymbol{z}) > 0$. This implies $\int_{V^*} f(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} > 0$. V^* is a bounded open subset of U. Thus, it is not the case that $\int_V f(\boldsymbol{x})\, d\boldsymbol{x} = 0$ for all bounded open subsets V of U. In other words, if $\int_V f(\boldsymbol{x})\, d\boldsymbol{x} = 0$ for all bounded open subsets V of U, then $f \equiv 0$ on U.