MA 574 – PROJECT 1

Lindsay Eddy
Partner: Alex Mendez

Due: Friday, February 1

(1) In class, we linearized about the static velocity $u_0 = 0$. For the 1-D case, derive Euler's equation, the continuity equation, and the equation of state if you linearize about $u_0 \neq 0$; i.e., consider perturbations $u(t,x) = u_0 + \hat{u}(t,x)$. Can you derive a wave equation posed solely in terms of $\hat{p}$, $\hat{u}$ or a potential ϕ?

Solution:

- Euler's equation

$$(\rho_0 + \hat{\rho})\left[\frac{\partial}{\partial t}(u_0 + \hat{u}) + (u_0 + \hat{u})\frac{\partial}{\partial x}(u_0 + \hat{u})\right] = -\frac{\partial}{\partial x}(p_0 + \hat{p})$$

Since u_0 and p_0 are constant,

$$(\rho_0 + \hat{\rho})\left[\frac{\partial \hat{u}}{\partial t} + (u_0 + \hat{u})\frac{\partial \hat{u}}{\partial x}\right] = -\frac{\partial \hat{p}}{\partial x}$$

$$\Rightarrow \rho_0\frac{\partial \hat{u}}{\partial t} + \rho_0 u_0\frac{\partial \hat{u}}{\partial x} + \underbrace{\rho_0\hat{u}\frac{\partial \hat{u}}{\partial x} + \hat{\rho}\frac{\partial \hat{u}}{\partial t} + \hat{\rho}u_0\frac{\partial \hat{u}}{\partial x} + \hat{\rho}\hat{u}\frac{\partial \hat{u}}{\partial x}}_{\text{H.O.T.s}} = -\frac{\partial \hat{p}}{\partial x}.$$

Removing higher order terms, we have the linearized Euler's equation:

$$\boxed{\rho_0\left[\frac{\partial \hat{u}}{\partial t} + u_0\frac{\partial \hat{u}}{\partial x}\right] = -\frac{\partial \hat{p}}{\partial x}}$$

- continuity equation

$$\frac{\partial}{\partial t}(\rho_0 + \hat{\rho}) + \frac{\partial}{\partial x}\left[(\rho_0 + \hat{\rho})(u_0 + \hat{u})\right] = 0$$

Since ρ_0 is constant with respect to time,

$$\frac{\partial \hat{\rho}}{\partial t} + \frac{\partial}{\partial x}\left[\rho_0 u_0 + \rho_0\hat{u} + \hat{\rho}u_0 + \underbrace{\hat{\rho}\hat{u}}_{\text{H.O.T.}}\right] = 0$$

Since $\hat{\rho}\hat{u}$ is a higher order term, and since u_0 is constant, we get the linearized continuity equation

$$\boxed{\frac{\partial \hat{\rho}}{\partial t} + u_0\left[\frac{\partial \rho_0}{\partial x} + \frac{\partial \hat{\rho}}{\partial x}\right] + \frac{\partial}{\partial x}(\rho_0\hat{u}) = 0}$$

Note that this is the same as

$$\frac{\partial \hat{\rho}}{\partial t} + u_0\frac{\partial \rho}{\partial x} + \frac{\partial}{\partial x}(\rho_0 + \hat{u}) = 0.$$

- equation of state

  Since u is not involved in the equation of state, the derivation is the same as we did in class:

$$\begin{aligned}
p &= f(\rho) \\
&= f(\rho_0 + \hat{\rho}) \\
&= f(\rho_0) + f'(\rho_0)\hat{\rho} + \text{H.O.T.s} \qquad \text{(Taylor expansion)} \\
&= p_0 + f'(\rho_0)\hat{\rho} + \text{H.O.T.s} \qquad \big(\text{since } p = f(\rho)\big)
\end{aligned}$$

  The equation of state is

$$\boxed{\hat{p} = c^2\hat{\rho}}$$

  where $c^2 \equiv f'(\rho_0)$ is the speed of sound in the material.

We were not able to derive a wave equation posed solely in terms of $\hat{p}$, $\hat{u}$, or a potential ϕ. We attempted to use the same methods use used in class for the 3D case where we assumed the static velocity was zero, but these methods failed in the case where the static velocity was non-zero.

- Density

  (State into Continuity:)

$$\frac{1}{c^2}\frac{\partial \hat{p}}{\partial t} + u_0\frac{\mathrm{d}\rho_0}{\mathrm{d}x} + \frac{\partial}{\partial x}(\rho_0\hat{u}) + \frac{u_0}{c^2}\frac{\partial \hat{p}}{\partial x} = 0$$

  (Differentiate with respect to time:)

$$\frac{1}{c^2}\frac{\partial^2 \hat{p}}{\partial t^2} + u_0\frac{\partial}{\partial x}\left[\frac{\partial \rho_0}{\partial t}\right] + \frac{\partial}{\partial x}\left[\frac{\partial(\rho_0\hat{u})}{\partial t}\right] + \frac{u_0}{c^2}\frac{\partial}{\partial x}\left[\frac{\partial \hat{p}}{\partial t}\right] = 0$$

  Since ρ_0 is constant with respect to time,

$$\frac{1}{c^2}\frac{\partial^2 \hat{p}}{\partial t^2} + \frac{\partial}{\partial x}\left[\rho_0\frac{\partial \hat{u}}{\partial t}\right] + \frac{u_0}{c^2}\frac{\partial^2 \hat{p}}{\partial x \partial t} = 0.$$

  We cannot substitute this into Euler because of the $\rho_0 u_0 \frac{\partial \hat{u}}{\partial x}$ term.

- Velocity

  (Differentiate Euler with respect to time:)

$$\rho_0\left[\frac{\partial^2 \hat{u}}{\partial t^2} + u_0\frac{\partial^2 \hat{u}}{\partial x \partial t}\right] = -\frac{\partial^2 \hat{p}}{\partial x \partial t}$$

  and
  (Differentiate state with respect to time:)

$$\begin{aligned}
\frac{\partial \hat{p}}{\partial t} &= c^2\frac{\partial \hat{\rho}}{\partial t} \\
&= -c^2\left[u_0\frac{\partial \rho_0}{\partial x} + \frac{\partial(\rho_0\hat{u})}{\partial x} + u_0\frac{\partial \hat{\rho}}{\partial x}\right] \qquad \text{(substitute differentiated Euler)}
\end{aligned}$$

$$\Rightarrow \rho_0\left[\frac{\partial^2 \hat{u}}{\partial t^2} + u_0\frac{\partial^2 \hat{u}}{\partial x \partial t}\right] = c^2\frac{\partial}{\partial x}\left[u_0\frac{\partial \rho_0}{\partial x} + \frac{\partial(\rho_0\hat{u})}{\partial x} + u_0\frac{\partial \hat{\rho}}{\partial x}\right]$$

Now, assume ρ_0 is constant:

$$\rho_0\left[\frac{\partial^2 \hat{u}}{\partial t^2}+u_0\frac{\partial^2 \hat{u}}{\partial x \partial t}\right]=c^2\frac{\partial}{\partial x}\left[\rho_0\frac{\partial \hat{u}}{\partial x}+u_0\frac{\partial \hat{\rho}}{\partial x}\right]$$
$$=c^2\left[\rho_0\frac{\partial^2 \hat{u}}{\partial x^2}+u_0\frac{\partial^2 \hat{\rho}}{\partial x^2}\right]$$

We now assume that $\partial^2\hat{u}/\partial x^2=0$ (the one-dimensional version of irrotational flow):

$$\rho_0\frac{\partial^2 \hat{u}}{\partial t^2}=c^2 u_0\frac{\partial^2 \hat{\rho}}{\partial x^2}$$

We still have terms with ρ in them. There is no reasonable assumption we can make that would remove the ρ terms.

- Potential

  We take the initial assumption that p_0 is constant.

  (Take the space derivative - 1D analog of curl - of Euler's equation:)

$$\rho_0\left[\frac{\partial}{\partial x}\left[\frac{\partial \hat{u}}{\partial t}\right]+u_0\frac{\partial^2 \hat{u}}{\partial x^2}\right]=-\frac{\partial^2 \hat{p}}{\partial x^2}$$

$$\Rightarrow \frac{\partial}{\partial t}\left[\frac{\partial \hat{u}}{\partial x}\right]=-u_0\frac{\partial^2 \hat{u}}{\partial x^2}$$

The time derivative is not zero, so even if we assume the initial vorticity is zero, it does not stay zero for all time.

(2) Here we are going to investigate some of the principles associated with a noise cancelling headset. As detailed in the paper "Engineering Silence: Active Noise Cancellation," headsets can eliminate low frequency noise by inverting acoustic waves measured by a microphone on the headset. This can be accomplished using a simple configuration of operational amplifiers (op-amps). We are going to numerically simulate aspects of this process.

Download the file `noise_data.mat` which contains a signal `y` sampled at a rate `Fs`. This signal is very familiar piece corrupted by noise comprised of two frequencies:

$$\text{Noise} = A_1 \sin(\omega_1 t + \phi) + A_2 \sin(\omega_2 t + \phi).$$

You need to determine the amplitudes A_1, A_2, frequencies ω_1, ω_2 and phase ϕ, and invert the signal to determine the piece which you can play using the command `sound(y,Fs)`.

You should start by using the sample rate Fs to determine an appropriate time vector. You can then estimate the phase by plotting the initial signal as a function of time. It is probably easiest to next determine the frequencies which can be accomplished using the MATLAB `fft` command. If you check the documentation for that command, you find the following example:

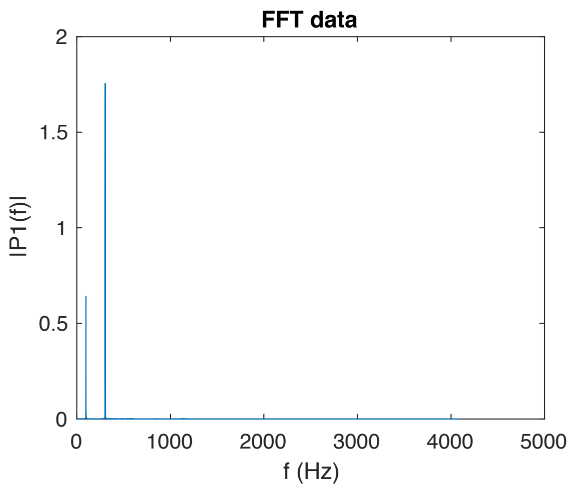
```
t = 0:0.001:0.6;
x = sin(2*pi*50*t)+sin(2*pi*120*t);
y = x + 2*randn(size(t));
plot(1000*t(1:50),y(1:50))
title('Signal Corrupted with Zero-Mean Random Noise')
xlabel('time (milliseconds)')

Y = fft(y,512);
Pyy = Y.* conj(Y)/512;
f = 1000*(0:256)/512;
figure(2)
plot(f,Pyy(1:257))
title('Frequency content of y')
xlabel('frequency (Hz)')
```

Here you are taking a 512-point FFT with a sample rate of 1000 (e.g., $dt = 0.001$). You should modify this for your sample rate `Fs` and determine the frequencies in your signal. You may need to numerically experiment to obtain the correct magnitudes A_1, A_2 associated with the frequencies ω_1, ω_2. Once you have done so, you can invert the signals and play the piece.

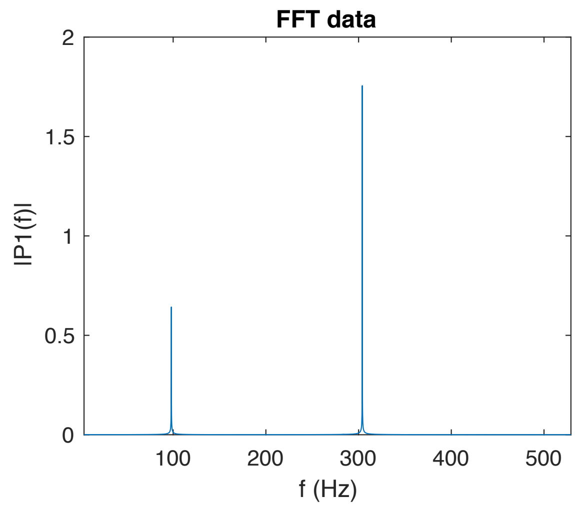
Solution:

The piece was Hallelujah Chorus from Handel's Messiah. To determine the parameters for the noise, we estimated the frequencies using Matlab's `fft` command, and then estimated the amplitudes and phase shift by eye. Finally, we refined our amplitudes and phase shift estimates by constructing a parameter estimation function. Once we had determined the parameters for the noise, we subracted the noise data from the signal data to play the piece.

First, we estimated the frequencies, ω_1 and ω_2 using Matlab's fast Fourier transform (Figure 1). We based our `fft` code off of examples in Matlab's documentation.



(a) Full data.

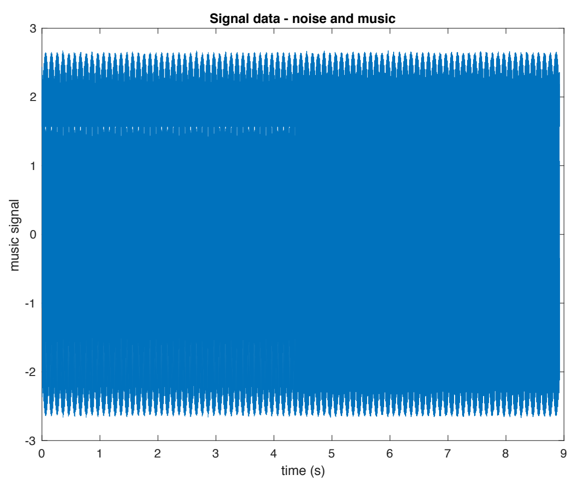


(b) Zoomed on f axis.

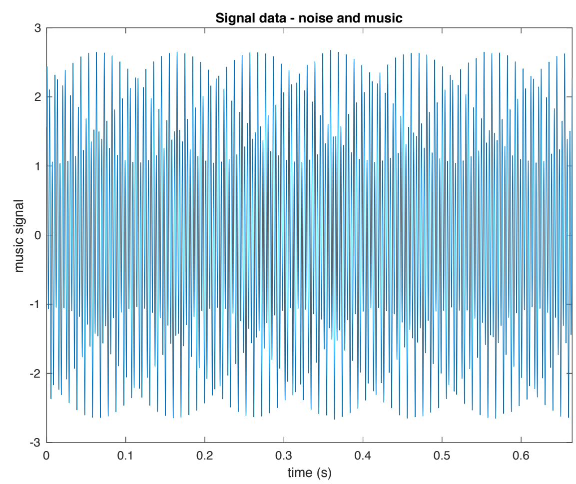
Figure 1: Fast Fourier transform data. The frequencies found were $w_1 = 304(2\pi)$ and $w_2 = 98(2\pi)$.

We found the frequencies to be $w_1 = 304(2\pi)$ and $w_2 = 98(2\pi)$.

Next we estimated the amplitudes A_1 and A_2 by looking at the plot of signal vs time (Figure 2).



(a) Full length of signal.

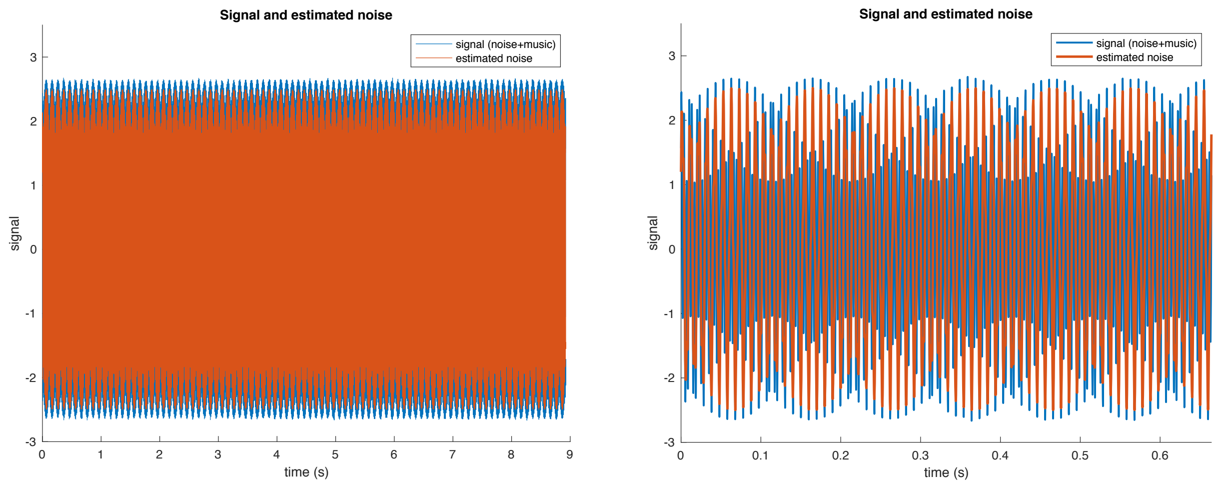


(b) Zoomed in on time axis.

Figure 2: The signal data, with both the music and the noise, plotted against time. This plot was used to estimate the amplitudes of the noise.

The total amplitude of the noise and music combined is about 2.6. After listening to the signal using the `sound` command, we assumed that the signal was comprised of mostly noise. Consequently, we estimated A_1 and A_2 to be about 2 and 0.5. We used these estimates in the initial iterate to our parameter estimation problem.

We then estimated the phase, ϕ, by guessing values in the range $[0, 2\pi)$, then computing the noise using the guessed phase value and our previously estimated amplitudes and frequencies. We then plotted the computed noise on a figure with our signal vs time plot. We found that $\phi = 1$ seemed to agree with the signal best (Figure 3). Since we eventually fine tune our phase estimatation using the parameter estimation optimization problem, it is not too important to have an exact value for ϕ.



(a) Full range of data. (b) Zoomed in on time axis.

Figure 3: Signal data, and estimated noise using $\phi = 1$.

Finally, we used Matlab's `lsqnonlin` function to find the minimizers A_1, A_2, and ϕ of:

$$\min_{(A_1,A_2,\phi)} \left|\left|F(A_1, A_2, \phi)\right|\right|_2^2$$

where

$$F_j(A_1, A_2, \phi) = y_j - \big(A_1 \sin(\omega_1 t_j + \phi) + A_2 \sin(\omega_2 t_j + \phi)\big)$$

where y_j is the jth component of the signal data. We used initial iterates $A_1 = 2$, $A_2 = 0.5$, and $\phi = 1$. We fixed the frequencies to those calculated using `fft` $\big(\omega_1 = 304(2\pi)$, $\omega_2 = 98(2\pi)\big)$. Our result was $A_1 = 1.8494$, $A_2 = 0.7928$, and $\phi = 0.6573$. We then computed the noise using the above parameters, subtracted the noise from the signal, and played the Hallelujah Chorus from Handel's Messiah.

(3) We are all familiar with the Doppler effect in which the perceived frequency of sound changes when the sound source or receiver are moving relative to the medium. Here we assume 1-D wave motion and let f_o and f_e respectively denote the observed and emitted frequencies. Let u and v denote the speed of the source and observer relative to the medium and $c = 343$ m/s denote the speed of sound at 20 ^{o}C. The positions of the source and receiver (observer) are denoted by S and R.

(a) Consider first the case when the receiver is stationary. Let S and R denote the positions of the source and receiver when both are fixed and let S' denote the position of the moving source. Show that $SR - S'R = u\Delta t$ from which it follows that $\lambda_e f_e \Delta t - \lambda_o f_e \Delta t = u\Delta t$ where λ_o and λ_e are the observed and emitted wavelengths. Use this to show that

$$f_o = f_e \lambda_e / \lambda_o = \frac{c}{c-u} f_e.$$

Solution:

First, note that

$$SR - S'R = SS'$$

Since u is the speed of S,

$$u = \frac{SS'}{\Delta t} \Rightarrow SR - S'R = u\Delta t$$

Now, we'll check the units of

$$\begin{aligned} \lambda_e f_e \Delta t - \lambda_o f_e \Delta t &= u\Delta t \\ (m \cdot s^{-1} \cdot s) - (m \cdot s^{-1} \cdot s) = m &= \left(\frac{m}{s} \cdot s\right). \end{aligned}$$

Also, note that we can think of $\lambda_e f_e \Delta t$ as the distance the sound has traveled in Δt (away from S), and we can think of $\lambda_o f_e \Delta t$ as the distance the sound has traveled in Δt relative to S'. (To see this, note that $f_e \Delta t$ describes the number of waves that are emitted in the time period described by Δt. Also, λ_e describes the length of those waves from the point of view of the source S, and λ_o describes the length of those waves from the point of view of S'.) The difference between those two distances is $SS' = u\Delta t$. Thus $\lambda_e f_e \Delta t - \lambda_o f_e \Delta t = u\Delta t$.

Now,

$$f_o = \frac{c}{\lambda_o} = f_e \frac{\lambda_e}{\lambda_o}.$$

Now we want to show that $f_e \frac{\lambda_e}{\lambda_o} = \frac{c}{c-u} f_e$. This is equivalent to showing $\lambda_o c = \lambda_e (c-u)$. This equivalence is shown below:

$$\begin{aligned} 0 &= \lambda_o c - \lambda_e c - (\lambda_o c - \lambda_e c) \\ &= \lambda_o c - \lambda_e c - (\lambda_o \lambda_e f_e - \lambda_e \lambda_e f_e) \\ &= \lambda_o c - \lambda_e c - \lambda_e (\lambda_o f_e - \lambda_e f_e) \\ &= \lambda_o c - \lambda_e c + \lambda_e u \quad \text{(since } u = \lambda_e f_e - \lambda_o f_e) \\ &= \lambda_o c - \lambda_e (c-u). \end{aligned}$$

(b) Use similar analysis to show that if the source is fixed and the receiver is moving with velocity v, then

$$f_o = \frac{c-v}{c} f_e.$$

When both are moving, the observed and emitted frequencies are related by the equation

$$f_o = \frac{c-v}{c-u} f_e.$$

Solution:

$SR' - SR = v\Delta t \Rightarrow \lambda_o f_o \Delta t - \lambda_e f_o \Delta t = v\Delta t.$

Now, we'll check the units of

$$\begin{aligned}\lambda_o f_o \Delta t - \lambda_e f_o \Delta t &= v\Delta t \\ (m \cdot s^{-1} \cdot s) - (m \cdot s^{-1} \cdot s) = m &= \left(\frac{m}{s} \cdot s\right).\end{aligned}$$

Now we want to show that $f_e \frac{\lambda_e}{\lambda_o} = \frac{c-v}{c} f_e$. This is equivalent to showing $\lambda_e c = \lambda_o (c-v)$. This equivalence is shown below:

$$\begin{aligned}0 &= \lambda_o c - \lambda_e c - (\lambda_o c - \lambda_e c) \\ &= \lambda_o c - \lambda_e c - (\lambda_o f_o \lambda_o - \lambda_e f_o \lambda_o) \\ &= \lambda_o c - \lambda_e c - \lambda_o(\lambda_o f_o - \lambda_e f_o) \\ &= \lambda_o c - \lambda_e c - \lambda_o v \quad (\text{since } v = \lambda_o f_o - \lambda_e f_o) \\ &= \lambda_o (c - v) - \lambda_e c.\end{aligned}$$

Now consider the case where both the source and the reciever are moving. We can make use of the previous two equations by thinking of X as a stationary "middle man" who first observes the waves from the source, and then "emits" them at the same frequency (f_x) at which he observed them. Then

$$f_x = \frac{c-v}{c} f_e \qquad \text{and} \qquad f_o = \frac{c}{c-u} f_x \Rightarrow f_x = \frac{c-u}{c} f_o.$$

Therefore

$$\frac{c-v}{c} f_e = \frac{c-u}{c} f_o$$

$$\therefore f_o = \frac{c-v}{c-u} f_e.$$

(c) At the winter olympics, you attend a ski jumping competition where you are able to observe from the end of the ramp. While there, you note with your perfect pitch that one skier's scream changes from D above Middle C to B flat below Middle C as he goes off the jump. How fast is he going if you take the speed of sound at -5 ^{o}C to be 328.25 m/s?

Solution:

When the skier is moving towards the observer (before the jump) their velocity (u), and the frequency observed pre-jump (f_{o_1}), are related by

$$f_{o_1} = \frac{c}{c-u} f_e.$$

When the skier is moving away from the observer (after the jump) their velocity (u), and the frequency observed post-jump (f_{o_2}), are related by

$$f_{o_2} = \frac{c}{c+u} f_e.$$

The frequency emitted remains constant, so

$$\frac{c-u}{c} f_{o_1} = \frac{c+u}{c} f_{o_2}.$$

Note that f_{o_1} is D above middle C, which is 293.66 s^{-1}, and f_{o_2} is B flat below middle C, which is 233.08 s^{-1}, and $c = 328.25$ m/s.

With some algebra, we obtain

$$u = 37.75.$$

(4) Write a MATLAB script to play the first four measures of Beethoven's 5th symphony based on tones of the notes with the duration of each note dictated by the rhythm. You can find the notes on Wikipedia.

Solution:

The first four measures of Beethoven's 5th are

$$\begin{array}{cccc|c|cccc|c} \text{rest} & \text{G} & \text{G} & \text{G} & \text{E} & \text{rest} & \text{F} & \text{F} & \text{F} & \text{D} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{2} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{2} \end{array}$$

The frequencies for these notes are

$$\begin{aligned} \text{G} &= 392.00 \text{ Hz} \\ \text{E} &= 329.63 \text{ Hz} \\ \text{F} &= 349.23 \text{ Hz} \\ \text{D} &= 293.66 \text{ Hz}. \end{aligned}$$

We use a frequency of 0 for the rests.

We modeled the sound by

$$\text{note}(t) = \sin\big(2\pi(\text{note frequency})t\big).$$

We used a tempo of 120 beats (quarter notes) per minute, because that is in the range of allegro, which the piece is written in. This is two quarter notes per second, which is one half note (the length of one measure here) per second. We chose a sampling rate of 1000 Hz.

The code `p4` is attached.