Numerical Analysis II: Homework 1
Spring 2018
Due Jan 26th
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This homework covers some of the mathematical preliminaries, including some basic results from calculus (Mean Value Theorem, Intermediate Value Theorem). For the computer problems, please submit your computer codes as well.

1. [10 points] Let $f \in C[a,b]$ and suppose $x_1, \ldots, x_n$ are in $[a,b]$.

   (a) Let $S = \frac{1}{n}\sum_{i=1}^{n} f(x_i)$. Show that there exists $\zeta \in [a,b]$ such that $f(\zeta) = S$.

   Let $\alpha = \min(\{f(x_i) : i = 1, \ldots, n\})$ and $\beta = \max(\{f(x_i) : i = 1, \ldots, n\})$.
   Let $x_\alpha, x_\beta \in \{x_1, x_2, \ldots, x_n\}$ such that $f(x_\alpha) = \alpha$ and $f(x_\beta) = \beta$.
   Then
   $$\alpha = \frac{1}{n}(n\alpha) \leq \frac{1}{n}\sum_{i=1}^{n} f(x_i) = S \qquad \text{(Since } \alpha \leq f(x_i)\,\forall\, i = 1, \ldots, n)$$
   $$\beta = \frac{1}{n}(n\beta) \geq \frac{1}{n}\sum_{i=1}^{n} f(x_i) = S \qquad \text{(Since } \beta \geq f(x_i)\,\forall\, i = 1, \ldots, n)$$
   Thus $\alpha \leq S \leq \beta$. Therefore, by the intermediate value theorem, there exists $\zeta \in [x_\alpha, x_\beta]$ (or $[x_\beta, x_\alpha]$ if $x_\beta < x_\alpha$) such that $f(\zeta) = S$. Since $x_\alpha, x_\beta \in [a,b]$, $\zeta \in [a,b]$.

   (b) Suppose $w_1, \ldots, w_n$ are positive real numbers. Using a similar argument as the previous part, show that there is a $\eta \in [a,b]$ such that
   $$\sum_{i=1}^{n} w_i f(x_i) = f(\eta)\Big(\sum_{i=1}^{n} w_i\Big).$$
   Let $\alpha = \min(\{f(x_i) : i = 1, \ldots, n\})$ and $\beta = \max(\{f(x_i) : i = 1, \ldots, n\})$.
   Let $x_\alpha, x_\beta \in \{x_1, x_2, \ldots, x_n\}$ such that $f(x_\alpha) = \alpha$ and $f(x_\beta) = \beta$.
   Define $F(x) = f(x)\sum_{i=1}^{n} w_i$.
   $$F(x_\alpha) = \sum_{i=1}^{n} w_i\alpha \leq \sum_{i=1}^{n} w_i f(x_i) \quad \text{(Since } \alpha \leq f(x_i)\,\forall\, i = 1, \ldots, n \text{ and } w_1, \ldots, w_n > 0.)$$
   $$F(x_\beta) = \sum_{i=1}^{n} w_i\beta \geq \sum_{i=1}^{n} w_i f(x_i) \quad \text{(Since } \beta \geq f(x_i)\,\forall\, i = 1, \ldots, n \text{ and } w_1, \ldots, w_n > 0.)$$

Thus $F(x_\alpha) \leq \sum_{i=1}^{n} w_i f(x_i) \leq F(x_\beta)$. Therefore, by the intermediate value theorem, there exists $\eta \in [x_\alpha, x_\beta]$ (or $[x_\beta, x_\alpha]$ if $x_\beta < x_\alpha$) such that

$$F(\eta) = f(\eta)(\sum_{i=1}^{n} w_i) = \sum_{i=1}^{n} w_i f(x_i).$$

Since $x_\alpha, x_\beta \in [a, b]$, $\eta \in [a, b]$.

(c) Assume w is a non-negative integrable function on $[a, b]$. Show that there exists $\theta \in [a, b]$ such that,

$$\int_a^b f(x)w(x)\, dx = f(\theta) \int_a^b w(x)\, dx.$$

Let α be the greatest lower bound of $f(x)$ on $[a, b]$ (where $L \in [a, b]$ such that $\alpha = f(L)$). Let β be the least upper bound of $f(x)$ on $[a, b]$ (where $U \in [a, b]$ such that $\beta = f(U)$).

Define a function F such that $F(t) = f(t) \int_a^b w(x)dx$.

Then

$$F(L) = \alpha \int_a^b w(x)dx = \int_a^b \alpha w(x)dx \leq \int_a^b f(x)w(x)dx$$

and

$$F(U) = \beta \int_z^b w(x)dx = \int_b^a \beta w(x)dx \geq \int_a^b f(x)w(x)dx$$

Thus $F(L) \leq \int_a^b f(x)w(x)dx \leq F(U)$. Therefore, the the intermediate value theorem, there exists a $\theta \in [L, U]$ (or $[U, L]$ is $U < L$) such that

$$F(\theta) = f(\theta) \int_a^b w(x)dx = \int_a^b f(x)w(x)dx.$$

Hint (for all parts): Intermediate Value Theorem.

2. [5 points] Let $f : [0, 1] \to \mathbb{R}$ be continuous, and assume $f(0) = f(1)$. Show there exists $\xi \in [0, 1/2]$ such that $f(\xi) = f(\xi + 1/2)$.

Define $f_1 : [0, \frac{1}{2}] \to \mathbb{R}$ such that $f_1(x) = f(x)$. Define $f_2 : [0, \frac{1}{2}] \to \mathbb{R}$ such that $f_2(x) = f(x + \frac{1}{2})$. Since f is continuous, f_1 and f_2 are likewise continuous.

Define $h : [0, \frac{1}{2}] \to \mathbb{R}$ such that $h(x) = f_1(x) - f_2(x)$. Since f_1 and f_2 are continuous, h is continuous.

Since $f_1(1/2) = f(1/2) = f_2(0)$ and $f_2(1/2) = f(1) = f(0) = f_1(0)$,

$$h(1/2) = f_1(1/2) - f_2(1/2) = f_2(0) - f_1(0) = -h(0).$$

Therefore, 0 lies between $h(0)$ and $h(1/2)$. Thus by the Intermediate Value Theorem, there exists some $\xi \in [0, 1/2]$ such that $h(\xi) = 0$.

Thus $f(\xi) - f(\xi + 1/2) = f_1(\xi) - f_2(\xi) = h(\xi) = 0$ – that is, $f(\xi) = f(\xi + 1/2)$.

3. [10 points] We say that a function $f : \mathbb{R} \to \mathbb{R}$ is (globally) Lipschitz continuous if there exists some $K \geq 0$ such that

$$|f(x) - f(y)| \leq K|x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

Suppose that f is differentiable at all $x \in \mathbb{R}$. Show that f is Lipschitz continuous if and only if its derivative f' is bounded (that is, for all x, $|f'(x)| \leq M$ for some $M \geq 0$)

- <u>If f is Lipschitz continuous, its derivative f' is bounded:</u>

$$|f'(x)| = \left| \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \right| = \lim_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|}$$

  Since f is Lipschitz continuous and $x, (x+h) \in \mathbb{R}$,

$$|f(x+h) - f(x)| \leq K|h| \quad \text{for all } x, h \in \mathbb{R} \text{ where } K \geq 0 \in \mathbb{R}$$

  thus

$$|f'(x)| = \lim_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|} \leq \lim_{h \to 0} K = K.$$

  Thus $|f'(x)| \leq K$ for all x, so f' is bounded.

- <u>If f' is bounded, f is Lipschitz continuous:</u>
  Let $x, y \in \mathbb{R}$. Without loss of generality, assume $x < y$. By the Mean Value Theorem, there exists some $c \in [x, y]$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| = |f'(c)|$$

  Since f' is bounded, $|f'(c)| < K$ for some $K \geq 0$. Thus

$$\left| \frac{f(x) - f(y)}{x - y} \right| = \frac{|f(x) - f(y)|}{|x - y|} \leq K$$

  and

$$|f(x) - f(y)| \leq K|x - y|.$$

4. [10 points] Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space, over base field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, equipped with the norm $\|x\| = \langle x, x \rangle^{1/2}$. Let $\{v_1, \ldots, v_n\}$ be an orthonormal set in V. Prove that the following statements are equivalent.

   (a) If $\langle x, v_i \rangle = 0$, for $i = 1, \ldots, n$, then $x = 0$.
   (b) The subspace spanned by $\{v_1, \ldots, v_n\}$ is the whole spave V.
   (c) If $x \in V$, the $x = \sum_{i=1}^n \langle x, v_i \rangle v_i$.
   (d) For x and y in V, $\langle x, y \rangle = \sum_{i=1}^n \langle x, v_i \rangle \langle v_i, y \rangle$.
   (e) For x in V, $\|x\|^2 = \sum_{i=1}^n |\langle x, v_i \rangle|^2$.

   [Hint: prove as follows (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (a).]

- (a) $\Rightarrow$ (b):
  Assume (a) is true. Assume $V =$ span$\{v_1, \dots v_m\}$ where $m \geq n$. Thus x can be written $x = \langle x, v_1\rangle v_1 + \cdots + \langle x, v_n\rangle v_n + \langle x, v_{n+1}\rangle v_{n+1} + \cdots + \langle x, v_m\rangle v_m$.
  Since V is closed under addition $y = \langle x_{n+1}, v_{n+1}\rangle v_{n+1} + \cdots + \langle x_m, v_m\rangle v_m \in V$. Since $\langle x, v_i\rangle = \langle y, v_i\rangle = 0$ for all $i = 1, \dots, n$, then $y = 0$.
  Therefore $y = \langle x_{n+1}, v_{n+1}\rangle v_{n+1} + \cdots + \langle x_m, v_m\rangle v_m = 0$. Since inner products are non-negative, $\langle x_{n+1}, v_n + 1\rangle = \cdots = \langle x_m, v_m\rangle = 0$. Thus $m = n$ and $V =$ span$\{x_1, \dots, x_n\}$.
- (b) $\Rightarrow$ (c):
  The subspace spanned by $\{v_1, \dots, v_n\}$ is the whole space V. Thus for all x in V, there exist $\alpha_1, \dots, \alpha_n$ such that $x = \alpha_1 v_1 + \cdots + \alpha_n v_n$.
  Let $x \in V$. For all $i = 1, \dots, n$,

  $$\langle x, v_i\rangle = \langle \alpha_1 v_1 + \cdots + \alpha_i v_i + \cdots + \alpha_n v_n, v_i\rangle = \alpha_1\langle v_1, v_i\rangle + \dots \alpha_i\langle v_i, v_i\rangle + \dots \alpha_n\langle v_n, v_i\rangle = \alpha_i$$

  Therefore

  $$x = \langle x, v_1\rangle v_1 + \cdots + \langle x, v_n\rangle v_n = \sum_{i=1}^{n}\langle x, v_i\rangle v_i.$$

- (c) $\Rightarrow$ (d):
  For x and y in V,

  $$\langle x, y\rangle = \left\langle \sum_{i=1}^{n}\langle x, v_i\rangle v_i, \sum_{j=1}^{n}\langle y, v_j\rangle v_j \right\rangle = \sum_{i=1}^{n}\langle x, v_i\rangle \left\langle v_i, \sum_{j=1}^{n}\langle y, v_j\rangle v_i \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n}\langle x, v_i\rangle\langle y, v_j\rangle\langle v_i, v_j\rangle$$

  Since $\{v_1, \dots, v_n\}$ are orthonormal,

  $$\langle v_i, v_j\rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases}$$

  So $\langle x, y\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n}\langle x, v_i\rangle\langle y, v_j\rangle\langle v_i, v_j\rangle = \sum_{i=1}^{n}\langle x, v_i\rangle\langle y, v_i\rangle = \sum_{i=1}^{n}\langle x, v_i\rangle\langle v_i, y\rangle$.
- (d) $\Rightarrow$ (e):

  $$\|x\|^2 = \sum_{i=1}^{n}\langle x, v_i\rangle\langle v_i, x\rangle = \sum_{i=1}^{n}|\langle x, v_i\rangle|^2.$$

- (e) $\Rightarrow$ (a):
  If $\langle x, v_i\rangle = 0 \ \forall\ i = 1, \dots, n$,

  $$\langle x, x\rangle^{1/2} = \|x\|^2 = \sum_{i=1}^{n} 0^2 = 0.$$

  $$\langle x, x\rangle^{1/2} = 0 \Rightarrow \langle x, x\rangle = 0 \Rightarrow x = 0$$

5. [10 points] For the functions in Table 1:

   (a) Find the Taylor polynomial p_n for the function f around a point a for a general positive integer n.

(b) By bounding the error term, find the minimum n needed to ensure that the absolute value of the remainder is less than the given tolerance level τ for every x in the given interval.

(c) Demonstrate numerically that the n you found in the previous part works. That is, provide a plot of $|p_n(x) - f(x)|$ for x in the given interval, and show that it is always less than the given τ.

Table 1: Functions for exercise 4.

Function	a	Interval	τ
$f(x) = e^x$	$a = 1$	$[0, 1]$	$\tau = 10^{-3}$
$f(x) = 1/(1-x)$	$a = 0$	$[\frac{-1}{2}, 0]$	$\tau = 10^{-4}$

- For $f(x) = e^x$:
  - p_n around 1:

$$p_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = \sum_{i=0}^{n} \frac{e^1}{i!}(x-1)^i = e \sum_{i=0}^{n} \frac{(x-1)^i}{i!}$$

  - Minimum n needed to ensure error is less than $\tau = 10^{-3}$:

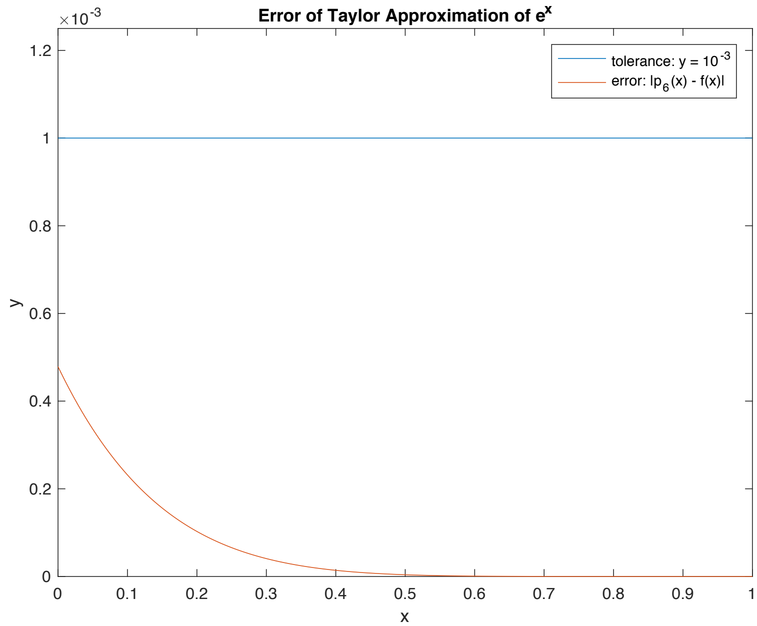
$$\text{error } = |R_{n+1}(x)| = \left| \frac{e^\xi}{(n+1)!}(x-1)^{(n+1)} \right| \text{ where } x \in [0,1] \text{ and } \xi \in [x, 1]$$

$$\left| \frac{e^\xi}{(n+1)!}(x-1)^{(n+1)} \right| = \frac{e^\xi}{(n+1)!}|x-1|^{n+1}$$

    The error term is maximized when $x = 0$ and $\xi = 1$, so

$$\text{error } = \frac{e^\xi}{(n+1)!}|x-1|^{n+1} \leq \frac{e^1}{(n+1)!}|-1|^{n+1} = \frac{e}{(n+1)!}.$$

    Thus, to ensure the error is less than the tolerance, we need $\frac{e}{(n+1)!} < 10^{-3}$. The smallest integer n that satisfies the inequality is $\boxed{n = 6}$.
  - Plot of $|p_n(x) - f(x)|$:



- For $f(x) = 1/(1-x)$:
  - p_n around 0:

$$p_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = \sum_{i=0}^{n} \frac{i!\,(1-0)^{-(i+1)}}{i!}(x-0)^i = \sum_{i=0}^{n} x^i$$

  - Minimum n needed to ensure error is less than $\tau = 10^{-4}$:

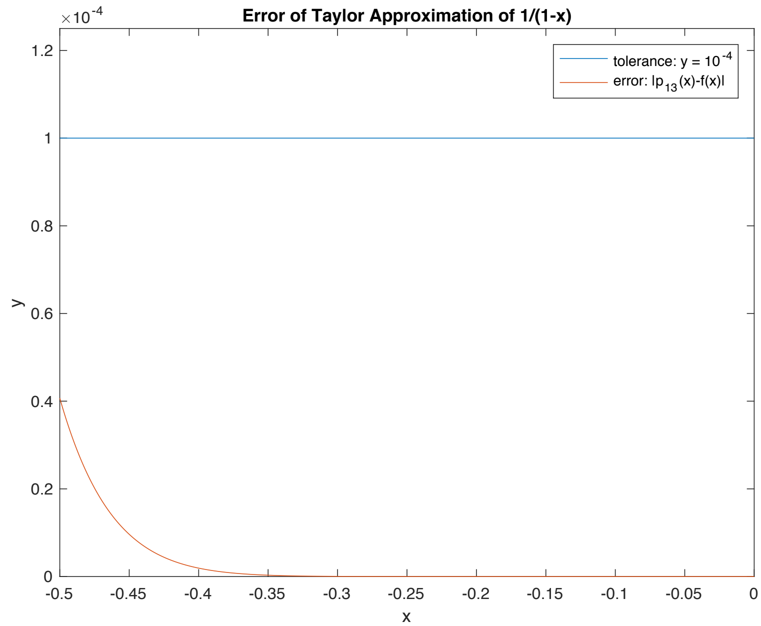
$$\begin{aligned} \text{error } &= |R_{n+1}(x)| = \left| \frac{(n+1)!\,(1-\xi)^{-(n+2)}}{(n+1)!} x^{n+1} \right| \\ &= \frac{|x|^{n+1}}{|1-\xi|^{n+2}} \text{ where } x \in [-\frac{1}{2}, 0] \text{ and } \xi \in [x, 0]. \end{aligned}$$

    The error term is maximized when the numerator is maximized and the denominator is minimized – in this case, when $x = -\frac{1}{2}$ and $\xi = 0$, so

$$\text{error } = \frac{|x|^{n+1}}{|1-\xi|^{n+2}} \leq \frac{|-1/2|^{n+1}}{|1-0|^{n+2}} = \frac{1}{2}^{n+1}.$$

    Thus, to ensure the error is less than the tolerance, we need $\frac{1}{2}^{n+1} < 10^{-4}$. The smallest integer n that satisfies the inequality is $\boxed{n = 13}$.

– Plot of $|p_n(x) - f(x)|$:



6. [10 points] This problem concerns Taylor's theorem for real-valued functions of several variables. [1]

   (a) Let $f : \mathbb{R}^n \to \mathbb{R}$ have continuous partial derivatives of order up to two. Consider the function $g(t) = f(\boldsymbol{x} + t\boldsymbol{d})$ with $t \in \mathbb{R}$ and where $\boldsymbol{d} \in \mathbb{R}^n$ is a fixed vector. Derive the expressions for $g'(t)$ and $g''(t)$.
   Let $y = \boldsymbol{x} + t\boldsymbol{d}$.

$$g'(t) = \nabla f(y)\,\mathbf{H}y(t_0) = \nabla f(\boldsymbol{x} + t\boldsymbol{d})\boldsymbol{d}$$
$$g''(t) = \mathbf{H}f(y_0)\,\mathbf{H}y(t_0)\boldsymbol{d} = \boldsymbol{d}^\mathsf{T}\mathbf{H}f(\boldsymbol{x} + t\boldsymbol{d})\boldsymbol{d}$$

   (b) Prove the following special case of the multivariate Taylor Theorem: let U be convex open set in $\mathbb{R}^n$. Consider a function $f : U \to \mathbb{R}$, and assume f has continous partial derivatives of up to order two. Let $\boldsymbol{x}_0 \in U$, and $\boldsymbol{x} \in U$. Then,

$$f(\boldsymbol{x}) = f(\boldsymbol{x}_0) + \nabla f(\boldsymbol{x}_0)^\mathsf{T}(\boldsymbol{x} - \boldsymbol{x}_0) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}_0)^\mathsf{T}\mathbf{H}(\boldsymbol{\xi})(\boldsymbol{x} - \boldsymbol{x}_0),$$

   where $\mathbf{H}$ is the Hessian matrix, $H_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$, and $\boldsymbol{\xi}$ is a point in the interior of the line segment connecting $\boldsymbol{x}_0$ and $\boldsymbol{x}$. (Hint: consider the function $g(t) = f\big(\boldsymbol{x}_0 + t(\boldsymbol{x} - \boldsymbol{x}_0)\big)$, $t \in [0, 1]$.)

   Let $g(t) = f\big(\boldsymbol{x}_0 + t(\boldsymbol{x} - \boldsymbol{x}_0)\big)$, $t \in [0, 1]$. By the Mean Value Theorem, $g(t) = g(0) + tg'(\xi')$ (for some ξ' between 0 and t). Then also by the Mean Value Theorem, $g(t) = g(0) + tg'(0) + \frac{t^2}{2}g''(\xi')$.
   Recall from part (a):

$$g'(t) = \nabla f(\boldsymbol{x_0} + t(\boldsymbol{x} - \boldsymbol{x_0}))^\mathsf{T}(\boldsymbol{x} - \boldsymbol{x_0}) \quad \text{and}$$

[1] In the class we focused on Taylor's theorem on $\mathbb{R}^n$. However, the result can be extended to infinite-dimensional Banach spaces as well; see e.g., the book, *Applied functional analysis, main principles and their application*, by E. Zeidler.

$$g''(t) = (\boldsymbol{x} - \boldsymbol{x_0})^{\mathsf{T}} \mathbf{H} f(\boldsymbol{x} + t(\boldsymbol{x} - \boldsymbol{x_0}))) \, (\boldsymbol{x} - \boldsymbol{x_0}).$$

Thus

$$g(t) = f(\boldsymbol{x_0}) + t \nabla f(\boldsymbol{x_0})^{\mathsf{T}} (\boldsymbol{x} - \boldsymbol{x_0}) + \frac{t^2}{2} (\boldsymbol{x} - \boldsymbol{x_0})^{\mathsf{T}} \mathbf{H} f(\boldsymbol{x} + \xi'(\boldsymbol{x} - \boldsymbol{x_0})) \, (\boldsymbol{x} - \boldsymbol{x_0}).$$

and

$$f(\boldsymbol{x}) = g(1) = f(\boldsymbol{x_0}) + \nabla f(\boldsymbol{x_0})^{\mathsf{T}} (\boldsymbol{x} - \boldsymbol{x_0}) + \frac{1}{2} (\boldsymbol{x} - \boldsymbol{x_0})^{\mathsf{T}} \mathbf{H} f(\boldsymbol{x} + \xi'(\boldsymbol{x} - \boldsymbol{x_0}))(\boldsymbol{x} - \boldsymbol{x_0}).$$

Let $\boldsymbol{\xi} = \boldsymbol{x} + \xi'(\boldsymbol{x} - \boldsymbol{x_0})$. Since ξ' is between 0 and t, $\boldsymbol{\xi}$ is in the interiotr of the line segment connecting $\boldsymbol{x_0}$ and $\boldsymbol{x}$, and

$$f(\boldsymbol{x}) f(\boldsymbol{x_0}) + \nabla f(\boldsymbol{x_0})^{\mathsf{T}} (\boldsymbol{x} - \boldsymbol{x_0}) + \frac{1}{2} (\boldsymbol{x} - \boldsymbol{x_0})^{\mathsf{T}} \mathbf{H} f(\boldsymbol{\xi})(\boldsymbol{x} - \boldsymbol{x_0}).$$