MA/CSC 580
Homework #2
Lindsay Eddy

1. Find $||x||_p, \ p = 1, 2, \infty$ for the following vectors

   (a) $\mathbf{x} = (3, -4, 0, -3/2)^T$.

$$||x||_1 = |3| + |-4| + |0| + |-3/2| = 17/2$$
$$||x||_2 = \left(3^2 + (-4)^2 + 0^2 + (-3/2)^2\right)^{-1/2} = \sqrt{109}/2$$
$$||x||_\infty = \max\{|3|, |-4|, |0|, |-3/2|\} = 4$$

   (b) $\mathbf{x} = (\sin k, \cos k, 2^k)^T$ for a fixed positive integer k.

$$||x||_1 = |\sin k| + |\cos k| + 2^k$$
$$||x||_2 = \sqrt{\sin^2 k + \cos^2 k + (2^k)^2} = \sqrt{1 + 2^{2k}}$$
$$||x||_\infty = \max\{|\sin k|, |\cos k|, |2^k|\} = 2^k \quad \text{(Since } k > 0, 2^k > 1.\ |\sin k|, |\cos k| \leq 1 \text{ for all } k.)$$

   (c) $\mathbf{x} = (4/(k+1), -2/k^2, k^2 e^{-k})^T$ for a fixed positive integer k.

$$||x||_1 = \frac{4}{k+1} + \frac{2}{k^2} + k^2 e^{-k}$$
$$||x||_2 = \left(\frac{16}{(k+1)^2} + \frac{4}{k^4} + k^4 e^{-2k}\right)^{\frac{1}{2}}$$
$$||x||_\infty = \max\left\{\frac{4}{k+1}, \frac{2}{k^2}, k^2 e^{-k}\right\}$$

2. (a): Find $||A||_p,\ p = 1, 2, \infty$ for the following matrices:

$$\begin{bmatrix} 1 & 2 \\ 0 & -3 \end{bmatrix}$$

$$||A||_\infty = \max\{|1| + |2|, |0| + |-3|\} = \max\{3, 3\} = 3$$
$$||A||_1 = \max\{|1| + |0|, |2| + |-3|\} = \max\{1, 5\} = 5$$
$$||A||_2 = \max_i \sqrt{\lambda_i(A^H A)} :$$
$$\det(A^H A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 13-\lambda \end{vmatrix} = (1-\lambda)(13-\lambda) - 4 = \lambda^2 - 14\lambda + 9 = 0$$
$$\lambda = \frac{14 \pm \sqrt{196 - 36}}{2} = 7 \pm 2\sqrt{10}$$
$$||A||_2 = \max_i \sqrt{\lambda_i(A^H A)} = \max_i \left\{ \sqrt{7 + 2\sqrt{10}}, \sqrt{7 - 2\sqrt{10}} \right\} = \sqrt{7 + 2\sqrt{10}}$$

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$||A||_\infty = \max\{|-2| + |1|, |1| + |-2|\} = \max\{3, 3\} = 3$$
$$||A||_1 = \max\{|-2| + |1|, |1| + |-2|\} = \max\{3, 3\} = 3$$
$$||A||_2 = \max_i \sqrt{\lambda_i(A^H A)} :$$
$$\det(A^H A - \lambda I) = \begin{vmatrix} 5-\lambda & -4 \\ -4 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 16 = \lambda^2 - 10\lambda + 9 = (\lambda - 1)(\lambda - 9) = 0$$
$$\lambda_1 = 1 \quad \lambda_2 = 9$$
$$||A||_2 = \max_i \sqrt{\lambda_i(A^H A)} = \max\{\sqrt{1}, \sqrt{9}\} = 3$$

(b): Assume that $A \in R^{n,n}$. Show that $||A||_1 = \max_{0 \le j \le n} \left\{ \sum_{i=1}^n |a_{ij}| \right\}$.

Let $M = \max_{0 \le j \le n} \{\sum_{i=1}^n |a_{ij}|\}$.

First we will show that $||A||_1 \le M$.

$$||A||_1 = \max_{||x||=1} ||Ax||_1$$

For all x where $||x||_1 = 1$,

$$||Ax||_1 = \sum_{i=1}^n \left| \sum_{j=1}^n a_{ij} x_j \right| \le \sum_{i=1}^n \sum_{j=1}^n |a_{ij}||x_j| = \sum_{j=1}^n \sum_{i=1}^n \Big(|a_{ij}||x_j| \Big)$$
$$= \sum_{j=1}^n \left[|x_j| \left(\sum_{i=1}^n |a_{ij}| \right) \right] \le \sum_{j=1}^n |x_j| M = M \sum_{j=1}^n |x_j| = M.$$

Thus $||A||_1 = \max_{||x||=1} ||Ax||_1 \leq M$.

Next we will show that $||A||_1 \geq M$. To do this, we will show that there exists an x^* with $||x^*||_1 = 1$ such that $||Ax^*||_1 \geq M$.

Denote the column of A with the largest sum of the magnitude of entries as the j^*th column. Let $x^* = \mathbf{e}_{j^*}$, the vector in $\mathbb{R}^n$ with 1 in the j^*th entry and 0 in all the other entries. Note that $||x^*|| = 1$. Then

$$\left(Ax^*\right)_i = \sum_{j=1}^{n} a_{ij} x_j^* = a_{ij^*}.$$

Therefore $||Ax^*||_1 = \sum_{i=1}^{n} |a_{ij*}| = \max_{0 \leq j \leq n} \{\sum_{i=1}^{n} |a_{ij}|\} = M$.

Thus $||A||_1 = \max_{||x||=1} ||Ax||_1 \geq ||Ax^*||_1 = M$.

We have shown that $||A||_1 \leq M$ and $||A||_1 \geq M$, thus, $||A||_1 = M = \max_{0 \leq j \leq n} \{\sum_{i=1}^{n} |a_{ij}|\}$.

3. (a) Show that $\|x\|_\infty$ is equivalent to $\|x\|_2$. That is to find constants C and c such that $c \le \|x\|_\infty \le \|x\|_2 \le C\|x\|_\infty$. Note that you need to determine such constants that the equalities are true for some particular x.

$$||x||_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \le \left(\sum_{i=1}^{n} ||x||_\infty^2 \right)^{\frac{1}{2}} = \left(n\ ||x||_\infty^2\right)^{\frac{1}{2}} = \sqrt{n}\ ||x||_\infty$$

$$||x||_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} \ge \left(||x||_\infty^2\right)^{\frac{1}{2}} = ||x||_\infty$$

That is,

$$||x||_\infty \le ||x||_2 \le \sqrt{n}\ ||x|_\infty.$$

(b) Show that $\|Qx\|_2 = \|x\|_2$ if Q is an orthogonal matrix ($Q^H Q = I$, $QQ^H = I$).

$$||Qx||_2^2 = (Qx)^H Qx = x^H Q^H Qx = x^H Ix = ||x||_2^2$$

(c) Show that $\|AB\| \le \|A\|\|B\|$ for any natural matrix norm, and $\|QA\|_2 = \|A\|_2$.

$$||AB|| = \max_{||x|| \ne 0} \frac{||ABx||}{||x||} \le \max_{||x|| \ne 0} \frac{||A||\ ||Bx||}{||x||} = ||A|| \max_{||x|| \ne 0} \frac{||Bx||}{x} = ||A||\ ||B||$$

$$||QA||_2 = \max_i \sqrt{\lambda_i\big((QA)^H QA\big)} = \max_i \sqrt{\lambda_i(A^H Q^H QA)} = \max_i \sqrt{\lambda_i(A^H A)} = ||A||_2$$

4. Given

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \end{bmatrix}$$

(a) Use Gaussian elimination with the partial pivoting to find the matrix decomposition $PA = LU$. This is a paper problem and you are asked to use exact calculations (use fractions if necessary).

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{P_{12}} \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} \xrightarrow{L_1} \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 3 & -2/3 & -1/3 \\ 0 & 2 & 8/3 & -2/3 \end{bmatrix} \xrightarrow{P_{23}} \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 3 & -2/3 & -1/3 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 8/3 & -2/3 \end{bmatrix}$$

$$\xrightarrow{L_2} \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 3 & -2/3 & -1/3 \\ 0 & 0 & 20/9 & 28/9 \\ 0 & 0 & 28/9 & -4/9 \end{bmatrix} \xrightarrow{P_{34}} \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 3 & -2/3 & -1/3 \\ 0 & 0 & 28/9 & -4/9 \\ 0 & 0 & 20/9 & 28/9 \end{bmatrix} \xrightarrow{L_3} \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 3 & -2/3 & -1/3 \\ 0 & 0 & 28/9 & -4/9 \\ 0 & 0 & 0 & 24/7 \end{bmatrix}$$

where

$$L_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2/3 & 0 & 1 & 0 \\ -1/3 & 0 & 0 & 1 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1/3 & 1 & 0 \\ 0 & -2/3 & 0 & 1 \end{bmatrix} \qquad L_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -5/7 & 1 \end{bmatrix}$$

and P_{ij} is the elementary permutation matrix where for all A, $P_{ij}A$ is identical to the matrix A except that rows i and j are switched.

We have

$$L_3 P_{34} L_2 P_{23} L_1 P_{12} A = U$$

where

$$U = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 3 & -2/3 & -1/3 \\ 0 & 0 & 28/9 & -4/9 \\ 0 & 0 & 0 & 24/7 \end{bmatrix}$$

Note that

$$U = L_3 P_{34} L_2 P_{23} L_1 P_{12} A = L_3 \tilde{L}_2 \tilde{\tilde{L}}_1 P_{34} P_{23} P_{12} A$$

where

$$\tilde{\tilde{L}}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2/3 & 1 & 0 & 0 \\ -1/3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \tilde{L}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2/3 & 1 & 0 \\ 0 & -1/3 & 0 & 1 \end{bmatrix}$$

Now we can get $PA = LU$ if we take

$$L = \tilde{\tilde{L}}_1^{-1} \tilde{L}_2^{-1} L_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 1 & 0 & 0 \\ 1/3 & 2/3 & 1 & 0 \\ 0 & 1/3 & 5/7 & 1 \end{bmatrix} \quad \text{and} \quad P = P_{34} P_{23} P_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

To verify, we perform the calculation

$$PA = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 2 & 3 & 0 & 1 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 1 & 0 & 0 \\ 1/3 & 2/3 & 1 & 0 \\ 0 & 1/3 & 5/7 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 3 & -2/3 & -1/3 \\ 0 & 0 & 28/9 & -4/9 \\ 0 & 0 & 0 & 24/7 \end{bmatrix} = LU$$

(b) Find the determinant of the matrix A.

$$\det(P) = (-1)^3 = -1 \qquad \det(L) = 1 \qquad \det(U) = 3(3)(28/9)(24/7) = 96$$

and

$$\begin{aligned} \det(P)\det(A) &= \det(L)\det(U) \\ \Rightarrow \det(A) &= \frac{\det(L)\det(U)}{\det(P)} = -96 \end{aligned}$$

(c) Use the factorization to solve $Ax = b$.

$Ax = b$ and $PA = LU$ so $LUx = PAx = Pb$. Note that $Pb = [6 \;\; 6 \;\; 6 \;\; 6]^T$. To find x, we will first solve $Ly = Pb$ for y using forward substitution, and then solve $Ux = y$ for y using backward substitution.

$\underline{Ly = Pb}$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2/3 & 1 & 0 & 0 \\ 1/3 & 2/3 & 1 & 0 \\ 0 & 1/3 & 5/7 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \\ 6 \end{bmatrix}$$

$$\begin{aligned} & & y_1 &= 6 \\ \frac{2}{3}(6) + y_2 &= 6 & \Rightarrow y_2 &= 2 \\ \frac{1}{3}(6) + \frac{2}{3}(2) + y_3 &= 6 & \Rightarrow y_3 &= \frac{8}{3} \\ \frac{1}{3}(2) + \frac{5}{7}\left(\frac{8}{3}\right) + y_4 &= 6 & \Rightarrow y_4 &= \frac{24}{7} \end{aligned}$$

$\underline{Ux = y}$:

$$\begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 3 & -2/3 & -1/3 \\ 0 & 0 & 28/9 & -4/9 \\ 0 & 0 & 0 & 24/7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 8/3 \\ 24/7 \end{bmatrix}$$

$$\begin{aligned} \frac{24}{7}x_4 &= \frac{24}{7} & \Rightarrow x_4 &= 1 \\ \frac{28}{9}x_3 - \frac{4}{9}(1) &= \frac{8}{3} & \Rightarrow x_3 &= 1 \\ 3x_3 - \frac{2}{3}(1) - \frac{1}{3}(1) &= 2 & \Rightarrow x_2 &= 1 \\ 3x_1 + 1(1) + 2(1) &= 6 & \Rightarrow x_1 &= 1 \end{aligned}$$

$$\therefore x = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$$

5. Consider solving $AX = B$ for X, $A \in R^{n,n}$, $B \in R^{n,m}$. There are two obvious algorithms. The first one is to get $A = PLU$ using Gaussian elimination, and then to solve for each column of X by forward and backward substitution. The second algorithm is to compute A^{-1} using Gaussian elimination and then to multiply $A^{-1}B$ to get X. Count the number of operations by each algorithm and determine which one is faster.

In the method where we get $A = PLU$ using Gaussian elimination:

First we will consider how many operations are needed to obtain the matrix U using Gaussian elimination.
As the kth step, we take $n - k$ divisions (to compute a_{ik}/a_{kk}), and $(n - k)^2$ multiplications and subtractions each (for updating the modified elements during the row operation). This results in the total number of operations for Gaussian elimination equal to

$$\sum_{k=1}^{n-1} 2k^2 + k = \frac{(n-1)n(2n-1)}{3} + \frac{n(n-1)}{2} = \frac{2}{3}n^3 + O(n^2).$$

The pivoting does not use any additions, subtractions, multiplications, or divisions, although it does affect computational speed. Additionally, finding the matricies P and L from the elementary permutation matrices and the lower triangular matrices used in each step of the Gaussian elimination does not use any addition/multiplications/etc, since we have methods to determine them without the standard operations (addition/multiplication/etc) or matrix multiplication.

Now we consider the cost of forward and backward substitution. First we use forward substitution to get each column of Y in $PLY = B$. Computing PL doesn't use any operations since we can just swap rows.
We have one standard forward substitution for each of the m columns of Y. This proceeds the same way as a forward substitution of the form $Ly = b$ where y and b are vectors in $\mathbb{R}^n$.

At the ith step, we have one division, $i - 1$ multiplications and $i - 1$ subtractions. Since there are n steps, we have a total of $\frac{(n-1)n}{2}$ additions and subtractions and $\frac{(n+1)n}{2}$ multiplications and divisions. This gives us a total of n^2 operations for forward substitution for each column of Y. Since there are m columns of Y, we have a total of $n^2 m$ operations for forward substitution.

Next we use backward substitution to get each column of X in $UX = Y$. Backward subtitution takes the same number of steps as forward substitution, so forward and backward substitution combined require a total of $2n^2 m$ operations.

In the method where we compute A^{-1}:

First we will consider how many operations are needed to find A^{-1} using Gaussian elimination.
At the kth step, we need $n - k$ divisions (to compute a_{ik}/a_{kk}) and $(2n - k)(n - k)$ multiplications and subtractions each (to update the modified elements on both sides of the augmented matrix using row reductions). Thus the total number of operations at the kth step is

$$n - k + 2(2n - k)(n - k) = (n - k)(4n - 2k + 1) = 4n^2 + n - 6nk + 2k^2 + k.$$

The total overall number of operations for computing A^{-1} is

$$\begin{aligned}\sum_{k=1}^{n-1} 4n^2 + n - 6nk + 2k^2 + k &= (n-1)(4n^2+n) - (6n+1)\sum_{k=1}^{n-1} k + 2\sum_{k=1}^{n-1} k^2 \\ &= (n-1)(4n^2+n) - (6n+1)\frac{n(n-1)}{2} + 2\frac{(n-1)n(2n-1)}{6} \\ &= \frac{5}{3}n^3 + O(n^2).\end{aligned}$$

Once we've computed A^{-1}, we still have to compute X by multipying $A^{-1}B$.
For each row in A^{-1}, for each column in B, we have n multiplications and $n-1$ additions. Since there are n rows in A^{-1} and m columns in B, this gives us n^2m multiplications and $(n-1)nm = n^2m - nm$ additions. Thus we use $2n^2m - nm$ total operations multiplying $A^{-1}B$.

In summary, the number of operations to use Gaussian elimination to get $A = PLU$ is $\frac{2}{3}n^3 + O(n^2)$, and the number of operations needed to compute X using forward and backward substitution is $2n^2m$. Thus the total number of operations used in the Gaussian elimination with partial pivoting method is $\frac{2}{3}n^3 + 2n^2m + O(n^2)$.
The number of operations to compute A^{-1} is $\frac{5}{3}n^3 + O(n^2)$. The number of operations needed to compute X by multiplying $A^{-1}B$ is $n^2 - nm$. Thus the total number of operations used in the A^{-1} method is $\frac{5}{3}n^3 + 2n^2m - nm + O(n^2)$.

Therefore the Gaussian elimination method is faster than the A^{-1} method. When we compare the coefficient of the leading term (in front of n^3), it is $\frac{5}{3}$ in the A^{-1} method and $\frac{2}{3}$ in the Gaussian elimination method. The coefficient of the leading term which includes m (n^2m) is 2 for both methods, so they both have about the same speed in that regard. It is possible that for very small n and very large m, the $-nm$ term in the equation for the A^{-1} method would start to over come the leading n^3 term. In this case, the A^{-1} method might be faster than the Gaussian elimination method. It is also worth noting that the Gaussian elimination involves certain procedures which do not use operations such as addition/multiplication/etc, but do take computational time. For example, when doing partial pivoting, the size of the entries along the columns of A are compared, and rows the the matrix A are switched. This could also slow down the Gaussian elimination method.

6. (Programming Part) Given a sequence of data

$$(x_1, y_1),\ (x_2, y_2),\ \cdots,\ (x_m, y_m), (x_{m+1}, y_{m+1}),$$

write a program to interpolate the data using the following model

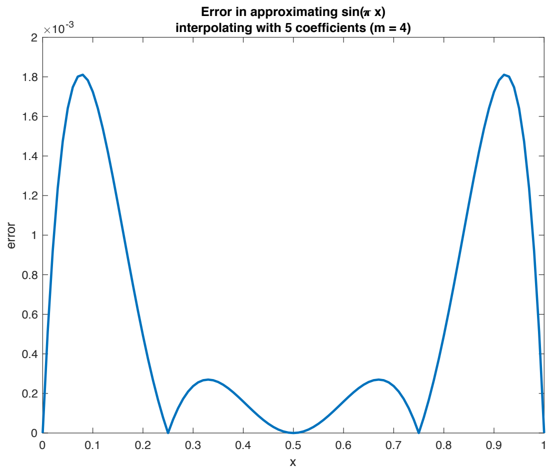
$$y(x) = a_0 + a_1 x + \cdots + a_{m-1} x^{m-1} + a_m x^m.$$

(a) Derive the linear system of equations for the interpolation problem.

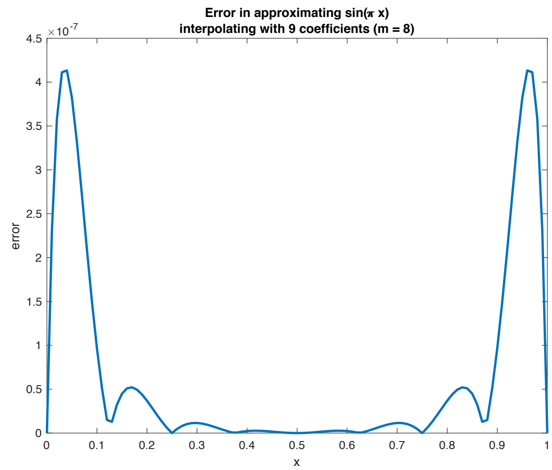
$$\begin{aligned}
y_1 &= a_0 + a_1 x_1 + \cdots + a_{m-1} x_1^{m-1} + a_m x_1^m \\
y_2 &= a_0 + a_1 x_2 + \cdots + a_{m-1} x_2^{m-1} + a_m x_2^m \\
&\vdots \\
y_{m+1} &= a_0 + a_1 x_{m+1} + \cdots + a_{m-1} x_{m+1}^{m-1} + a_m x_{m+1}^m
\end{aligned}$$

(b) Let $x_i = (i-1)h$, $i = 1, 2, \cdots, m+1$, $h = 1/m$, $y_i = \sin \pi x_i$, write a computer code using the Gaussian elimination with column partial pivoting to solve the problem. Test your code with $m = 4, 8, 16, 32, 64$ and plot the error $|y(x) - \sin \pi x|$ with 100 or more points between 0 and 1, that is, predict the function at more points in addition to the sample points. For example, you can set $h1 = 1/100;\ x1 = 0 : h1 : 1,\quad y1(i) = a_0 + a_1 x1(i) + \cdots + a_{m-1}(x1(i))^{m-1} + a_m(x1(i))^m$, $y2(i) = \sin(\pi x1(i)),\ plot(x1, y1 - y2)$.

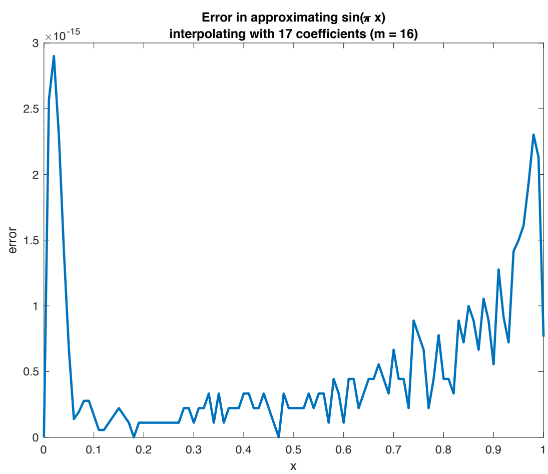
Code uploaded to Moodle. The error plots for each m are below.



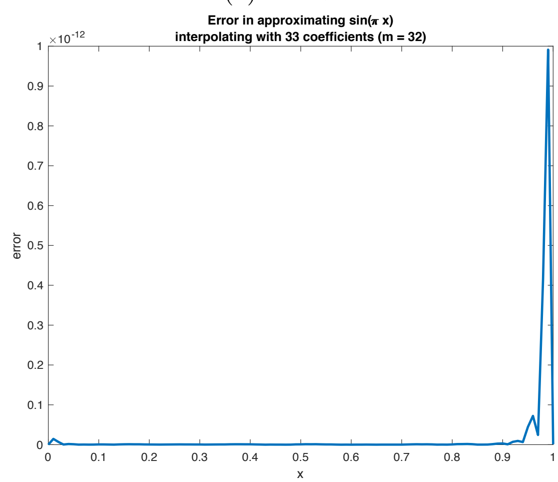
(a) $m = 4$



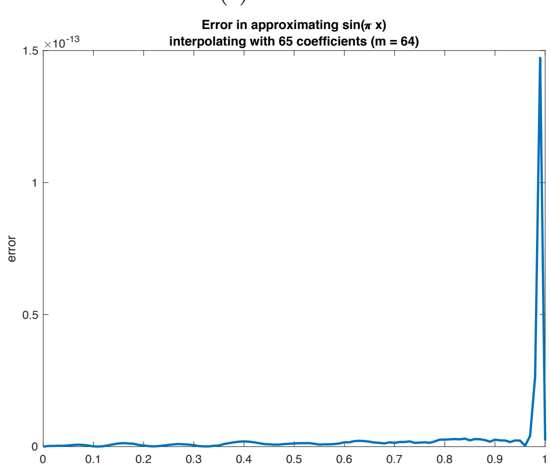
(b) $m = 8$



(c) $m = 16$



(d) $m = 32$



(e) $m = 64$

Figure 1: Error of interpolation using Gaussian Elimination with partial pivoting interpolating at points (x_1, y_1), (x_2, y_2), $\cdots$, (x_m, y_m), (x_{m+1}, y_{m+1})

(c) Record the CPU time (in Matlab type *help cputime*) for $m = 50, 100, 150, 200, \cdots, 350, 400$. Plot the CPU time versus m. Then use the Matlab function polyfit $z = polyfit(m, cputime(m), 3)$ to find a cubic fitting of the CPU time versus m. Write down the polynomial and analyze your result. Does it look like a cubic function?

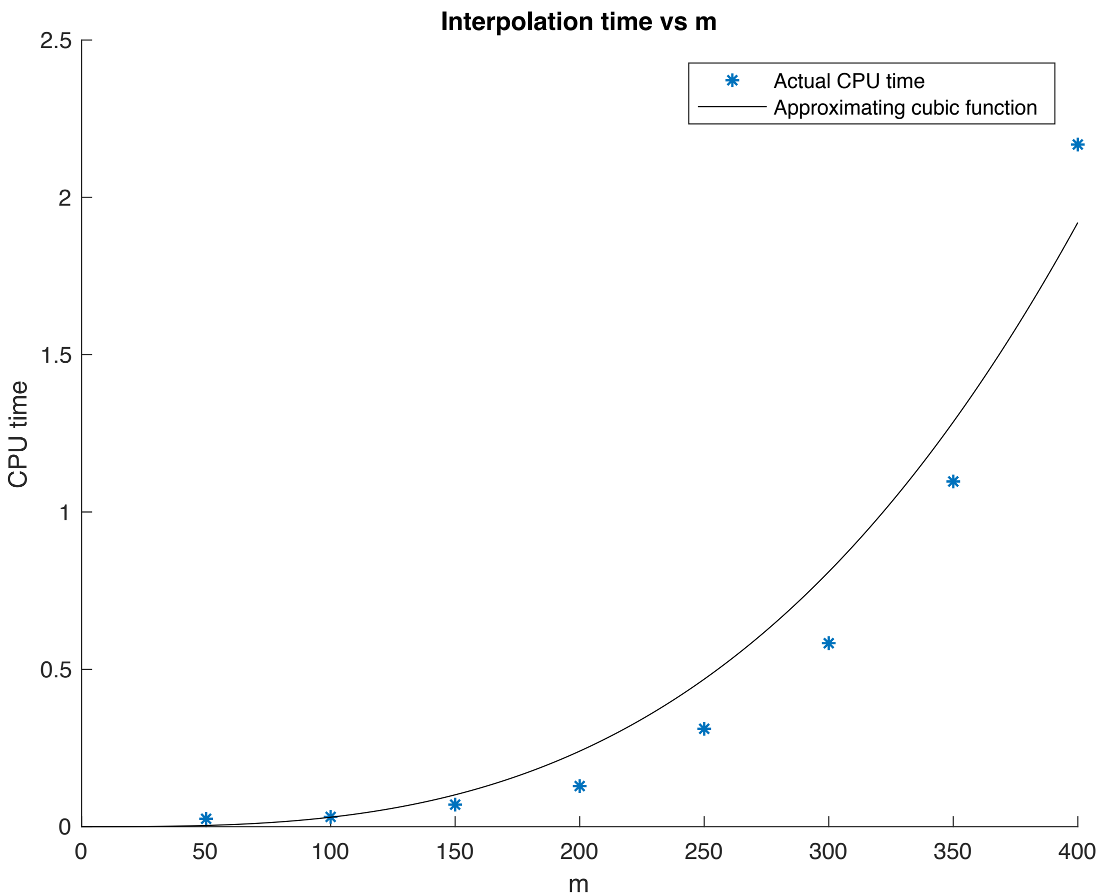


Figure 2: CPU time at $m = 50, 100, 150, \cdots, 400$, and approximating cubic function $3 \times 10^{-8}x^3$.

The polynomial given by $polyfit(m, cputime(m), 3)$ is

$$f(x) = (1.3 \times 10^{-7})x^3 - (5.6 \times 10^{-5})x^2 + (8.2 \times 10^{-3})x - 2.8 \times 10^{-1}.$$

The result does look like a cubic function. It is close to the approximating cubic function $3 \times 10^{-8}x^3$. This makes sense, since the operation count for Gaussian Elimination is $\frac{2}{3}O(n^3)$.

7. **Extra Credit:** Choose **one** from the following (Note: please do not ask the instructor about the solution since it is extra credit):

   (a) Let $A \in R^{n\times n}$. Show that $\|A\|_2 = \max\limits_{1\le i\le n} \sqrt{\lambda_i(A^TA)}$ and $\|A^{-1}\|_2 = \dfrac{1}{\min\limits_{1\le i\le n} \sqrt{\lambda_i(A^TA)}}$, where $\lambda_i(A^TA)$, $i = 1, 2, \cdots, n$ are the eigenvalues of A^TA. Show further that $cond_2(A) = \dfrac{\sigma_{max}}{\sigma_{min}}$, where $\sigma_{max}, \sigma_{min}$ are the largest and smallest nonzero singular values of A.

   (b) Show that if A is a symmetric positive definite matrix, then after one step of Gaussian elimination (without pivoting), then reduced matrix A_1 in

$$A \Longrightarrow \begin{bmatrix} a_{11} & * \\ 0 & A_1 \end{bmatrix}$$

   must be symmetric positive definite. Therefore no pivoting is necessary.