Need to model *all* randomness with sigma points

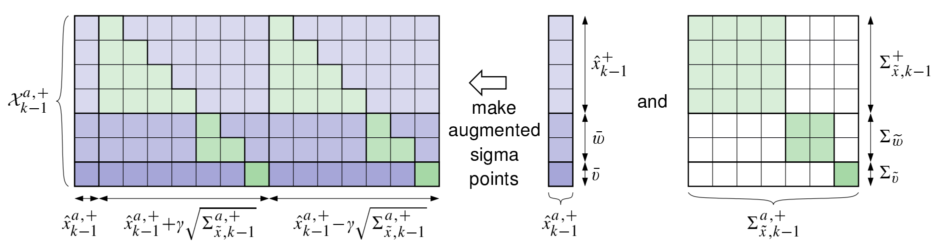
- We now apply sigma-point approach of propagating statistics through a nonlinear function to the state-estimation problem
- These sigma-points must jointly model *all* randomness:
  - Uncertainty of the state
  - Uncertainty of process noise
  - Uncertainty of sensor noise
- So we first define an augmented random vector x_k^a that combines these random factors at time index k

Step 1a: State estimate time update (1)

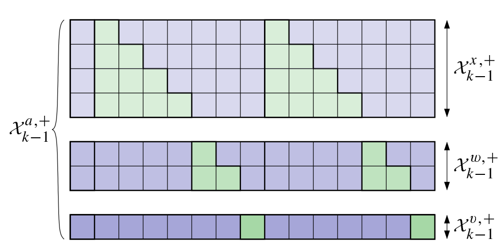
- First, form augmented prior state estimate, covariance:
  $\hat{x}_{k-1}^{a,+} = \left[(\hat{x}_{k-1}^{+})^T, \bar{w}, \bar{v}\right]^T$ and $\Sigma_{\tilde{x},k-1}^{a,+} = \operatorname{diag}\left(\Sigma_{\tilde{x},k-1}^{+}, \Sigma_{\widetilde{w}}, \Sigma_{\tilde{v}}\right)$
- These factors are used to generate the $p+1$ augmented sigma points

$$\mathcal{X}_{k-1}^{a,+} = \left\{ \hat{x}_{k-1}^{a,+},\ \hat{x}_{k-1}^{a,+} + \gamma\sqrt{\Sigma_{\tilde{x},k-1}^{a,+}},\ \hat{x}_{k-1}^{a,+} - \gamma\sqrt{\Sigma_{\tilde{x},k-1}^{a,+}} \right\}$$

- Can be organized in convenient matrix form:

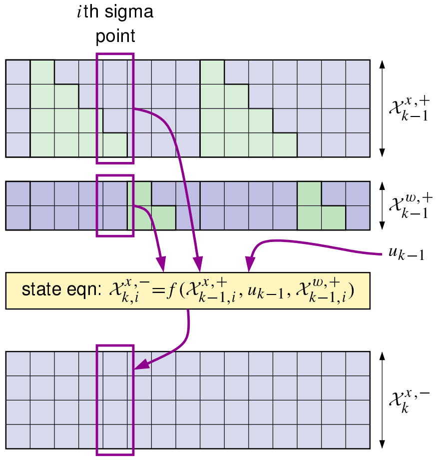


Step 1a: State estimate time update (2)



- Split augmented sigma points $\mathcal{X}_{k-1}^{a,+}$ into state portion $\mathcal{X}_{k-1}^{x,+}$, process-noise portion $\mathcal{X}_{k-1}^{w,+}$, and sensor-noise portion $\mathcal{X}_{k}^{v,+}$

Step 1a: State estimate time update (3)

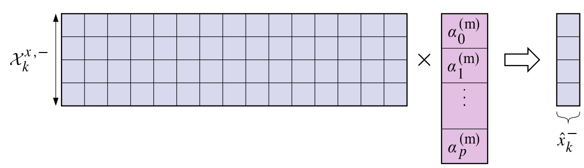


- Evaluate state equation using all pairs of $\mathcal{X}_{k-1,i}^{x,+}$ and $\mathcal{X}_{k-1,i}^{w,+}$ (where subscript i denotes that the i th vector is being extracted from the original set), yielding the prediction sigma points $\mathcal{X}_{k,i}^{x,-}$
- That is, compute $\mathcal{X}_{k,i}^{x,-} = f(\mathcal{X}_{k-1,i}^{x,+}, u_{k-1}, \mathcal{X}_{k-1,i}^{w,+})$

Step 1a–b: State estimate time update (4)

- Finally, present state prediction is computed as

$$\begin{aligned}
\hat{x}_k^- &= \mathbb{E}\big[f(x_{k-1}, u_{k-1}, w_{k-1}) \mid \mathbb{Y}_{k-1}\big] \\
&\approx \sum_{i=0}^{p} \alpha_i^{(\mathrm{m})} f(\mathcal{X}_{k-1,i}^{x,+}, u_{k-1}, \mathcal{X}_{k-1,i}^{w,+}) \\
&= \sum_{i=0}^{p} \alpha_i^{(\mathrm{m})} \mathcal{X}_{k,i}^{x,-} = \big[\mathcal{X}_k^{x,-}\big]\big[\alpha^{(\mathrm{m})}\big]
\end{aligned}$$



- Can compute with simple matrix multiply

- Then, covariance estimate is computed as

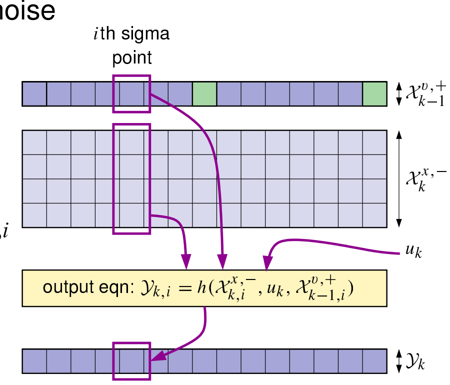
$$\begin{aligned}
\Sigma_{\tilde{x},k}^- &= \sum\nolimits_{i=0}^{p} \alpha_i^{(\mathrm{c})} \big(\mathcal{X}_{k,i}^{x,-} - \hat{x}_k^-\big)\big(\mathcal{X}_{k,i}^{x,-} - \hat{x}_k^-\big)^T \\
&= \big[\mathcal{X}_k^{x,-} - \hat{x}_k^-\big]\big[\mathrm{diag}(\alpha^{(\mathrm{c})})\big]\big[\mathcal{X}_k^{x,-} - \hat{x}_k^-\big]^T
\end{aligned}$$

Step 1c: Estimate system output y_k

- Output y_k estimated by evaluating model output equation using sigma points describing state and sensor noise
- First, compute points $\mathcal{Y}_{k,i} = h(\mathcal{X}_{k,i}^{x,-}, u_k, \mathcal{X}_{k-1,i}^{v,+})$
- Output estimate is then

$$\begin{aligned}
\hat{y}_k &= \mathbb{E}\big[h(x_k, u_k, v_k) \mid \mathbb{Y}_{k-1}\big] \\
&\approx \sum_{i=0}^{p} \alpha_i^{(\mathrm{m})} h(\mathcal{X}_{k,i}^{x,-}, u_k, \mathcal{X}_{k-1,i}^{v,+}) = \sum_{i=0}^{p} \alpha_i^{(\mathrm{m})} \mathcal{Y}_{k,i}
\end{aligned}$$

- Can be computed with simple matrix multiplication, as was done when calculating $\hat{x}_k^-$ at end of Step 1a



Step 2a: Estimator gain matrix L_k

- To find L_k, must first compute required covariance matrices

$$\Sigma_{\tilde{y},k} = \sum_{i=0}^{p} \alpha_i^{(c)} \big(\mathcal{Y}_{k,i} - \hat{y}_k\big)\big(\mathcal{Y}_{k,i} - \hat{y}_k\big)^T$$

$$\Sigma_{\tilde{x}\tilde{y},k}^{-} = \sum_{i=0}^{p} \alpha_i^{(c)} \big(\mathcal{X}_{k,i}^{x,-} - \hat{x}_k^{-}\big)\big(\mathcal{Y}_{k,i} - \hat{y}_k\big)^T$$

- These depend on sigma-point matrices $\mathcal{X}_k^{x,-}$ and $\mathcal{Y}_k$, already computed in Steps 1b and 1c, as well as $\hat{x}_k^{-}$ and $\hat{y}_k$, already computed in Steps 1a and 1c
- The summations can be performed using matrix multiplies, as we did in Step 1b
- Then, we simply compute $L_k = \Sigma_{\tilde{x}\tilde{y},k}^{-} \Sigma_{\tilde{y},k}^{-1}$

Step 2b–c: State, covariance measurement update

- Then the state estimate is computed as

$$\hat{x}_k^{+} = \hat{x}_k^{-} + L_k(y_k - \hat{y}_k)$$

- Finally, the estimation-error covariance matrix is calculated directly from the optimal formulation:

$$\Sigma_{\tilde{x},k}^{+} = \Sigma_{\tilde{x},k}^{-} - L_k \Sigma_{\tilde{y},k} L_k^T$$

Summary

- SPKF uses sigma-point method to propagate uncertainty of input RV to output of model's (possibly) nonlinear state and output equations
- Applying this procedure to generic-probabilistic-inference solution yields all six SPKF steps
- Matrices and vectors are convenient way to store all the sigma points and to compute means and covariances from the sigma points
- SPKF is now derived!