Overview of vector random (stochastic) processes

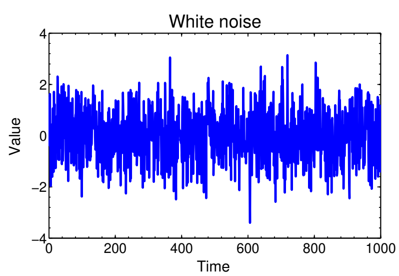
- A stochastic or random process is a family of random vectors indexed by a parameter set ("time" in our case)
- For example, we might refer to a random process X_k for generic k
  - Value of random process at any specific time $k = m$ is a random variable X_m
- Usually assume stationarity
  - The statistics (*i.e.,* pdf) of the RV are time-shift invariant
  - Therefore, $\mathbb{E}[X_k] = \bar{x}$ for all k and $\mathbb{E}[X_{k_1} X_{k_2}^T] = R_X(k_1 - k_2)$

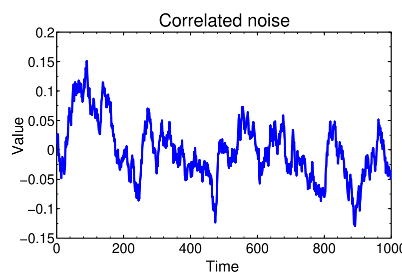
Properties and important points

1. Autocorrelation: $R_X(k_1, k_2) = \mathbb{E}[X_{k_1} X_{k_2}^T]$. If stationary,
$$R_X(\tau) = \mathbb{E}[X_k X_{k+\tau}^T]$$
   - Provides a measure of correlation between elements of the process having time displacement τ
   - $R_X(0) = \sigma_X^2$ for zero-mean X
   - $R_X(0)$ is always the maximum value of $R_X(\tau)$
2. Autocovariance: $C_X(k_1, k_2) = \mathbb{E}[(X_{k_1} - \mathbb{E}[X_{k_1}])(X_{k_2} - \mathbb{E}[X_{k_2}])^T]$. If stationary,
$$C_X(\tau) = \mathbb{E}[(X_k - \bar{x})(X_{k+\tau} - \bar{x})^T]$$

White noise

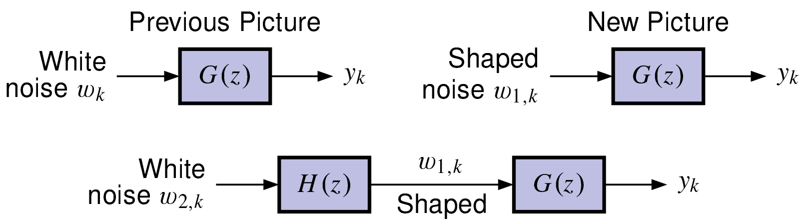
- White noise: Some processes have a unique autocorrelation:
  1. Zero mean
  2. $R_X(\tau) = \mathbb{E}[X_k X_{k+\tau}^T] = S_X \delta(\tau)$ where $\delta(\tau)$ is the Dirac delta. $\delta(\tau) = 0 \; \forall \; \tau \neq 0$
  - Therefore, the process is uncorrelated in time
  - Clearly an abstraction, but proves to be a very useful one





Shaping filters: Idea

- Will assume noise inputs to dynamic systems are white
  - Limiting assumption, but one that can be easily fixed
    - ➠ Can use second linear system to "shape" the noise as desired.



  - Can drive our linear system with noise that has desired characteristics by introducing shaping filter $H(z)$ that itself is driven by white noise

Shaping filters: Model

- Combined system $GH(z)$ looks exactly the same as before, *but* $G(z)$ is not driven by pure white noise any more
  - Analysis *augments* original system model with filter states. Original system has

$$x_{k+1} = Ax_k + B_w w_{1,k}$$
$$y_k = Cx_k$$

  - Shaping filter with white input and desired output statistics has

$$x_{s,k+1} = A_s x_{s,k} + B_s w_{2,k}$$
$$w_{1,k} = C_s x_{s,k}$$

  - Combine into larger-order augmented system driven by white noise:

$$\begin{bmatrix} x_{k+1} \\ x_{s,k+1} \end{bmatrix} = \begin{bmatrix} A & B_w C_s \\ 0 & A_s \end{bmatrix} \begin{bmatrix} x_k \\ x_{s,k} \end{bmatrix} + \begin{bmatrix} 0 \\ B_s \end{bmatrix} w_{2,k}$$

$$y_k = \begin{bmatrix} C & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{s,k} \end{bmatrix}$$

Gaussian processes

- We will work with Gaussian noises to a large extent
  - Uniquely defined by the first- and second central moments of the statistics ➠ Gaussian assumption not essential
  - Our filters will always track only the first two moments.

NOTATION: Until now, we have always used capital letters for random variables

- State of system driven by random process is an RV, so we could call it X_k
- More common to retain standard notation x_k and understand from context that we are discussing an RV

Summary

- Random process is a family of RVs indexed by time
- Will assume our random processes are stationary
- Autocorrelation and autocovariance measure self-predictability of a signal at different time offsets
- White noise is zero mean signal, completely uncorrelated with self (“completely random”)
  - An abstraction, but a very useful one
- If noises in a system of interest are not white, can filter white noise to create same general characteristics
- From now on, unless stated otherwise, all noise signals will be white and Gaussian