Optimizing the WTLS cost function

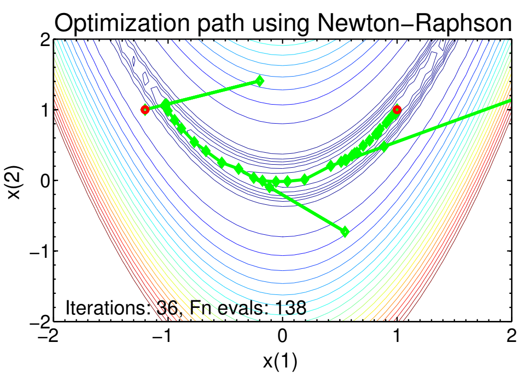
- In the previous lesson, found WTLS cost function to optimize
- To find $\widehat{Q}$ to minimize this cost function, set $\partial \chi^2_{\mathrm{WTLS}}/\partial \widehat{Q} = 0$

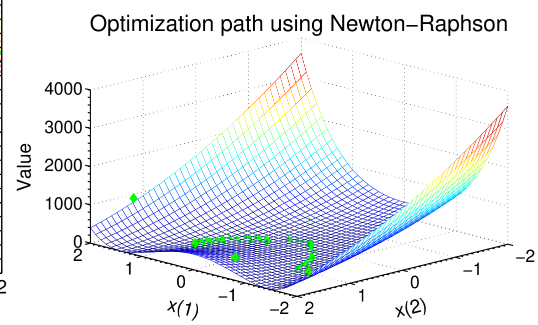
$$\frac{\partial \chi^2_{\mathrm{WTLS}}}{\partial \widehat{Q}} = \sum_{i=1}^{N} \frac{2(\widehat{Q}x_i - y_i)(\widehat{Q}y_i\sigma^2_{x_i} + x_i\sigma^2_{y_i})}{(\widehat{Q}^2\sigma^2_{x_i} + \sigma^2_{y_i})^2} = 0 \qquad \text{(via Mathematica)}$$

- Unfortunately, this solution has none of the nice properties of the WLS solution:
  1. There is no closed-form solution in the general case
  2. There is no recursive update in the general case
  3. There is no fading memory recursive update

1. No closed-form solution

- A numerical method must be used to find $\widehat{Q}$





- One possibility is to perform a Newton–Raphson search for $\widehat{Q}$

Newton–Raphson search

- Every time new (x_i, y_i) available, perform several iterations of

$$\widehat{Q}_k = \widehat{Q}_{k-1} - \left(\partial \chi^2_{\mathrm{WTLS}}/\partial \widehat{Q}\right) \Big/ \left(\partial^2 \chi^2_{\mathrm{WTLS}}/\partial \widehat{Q}^2\right)$$

- Numerator is “Jacobian” of original cost function, given by earlier equation
- Denominator is “Hessian” of original cost function, which can be shown to be

$$\frac{\partial^2 \chi^2_{\mathrm{WTLS}}}{\partial \widehat{Q}^2} = 2\sum_{i=1}^{N} \frac{\sigma^4_{y_i}x_i^2 + \sigma^4_{x_i}(3\widehat{Q}^2y_i^2 - 2\widehat{Q}^3x_iy_i) - \sigma^2_{x_i}\sigma^2_{y_i}(3\widehat{Q}^2x_i^2 - 6\widehat{Q}x_iy_i + y_i^2)}{(\widehat{Q}^2\sigma^2_{x_i} + \sigma^2_{y_i})^3}$$

- Search initialized with prior WLS estimate of $\widehat{Q}$, and is guaranteed to converge to global solution since cost function χ^2_{WTLS} is convex
- Number of significant figures in the solution doubles with each iteration
  - Around four iterations produce double-precision results

2. No recursive update

- There is no recursive update in the general case: this has storage implications and computational implications
  - To use WTLS, the entire vector $\mathbf{x}$ and $\mathbf{y}$ must be stored, which implies increasing storage as the number of measurements N increases
  - Furthermore, the number of computations grows as N grows
- Not well suited for an embedded-system application that must run in real time with limited storage capabilities

3. No fading-memory recursive update

- No fading memory recursive update (no recursive update)
- *Non-recursive* fading memory cost function may be defined

$$\chi^2_{\mathrm{FMWTLS}} = \sum_{i=1}^{N} \gamma^{N-i} \frac{(y_i - \widehat{Q}x_i)^2}{\widehat{Q}^2\sigma^2_{x_i} + \sigma^2_{y_i}}$$

- The Jacobian of this cost function can be found to be

$$\frac{\partial \chi^2_{\mathrm{FMWTLS}}}{\partial \widehat{Q}} = 2\sum_{i=1}^{N} \gamma^{N-i} \frac{(\widehat{Q}x_i - y_i)(\widehat{Q}y_i\sigma^2_{x_i} + x_i\sigma^2_{y_i})}{(\widehat{Q}^2\sigma^2_{x_i} + \sigma^2_{y_i})^2}$$

3. No fading-memory recursive update

- The Hessian of the cost function can be found to be

$$\frac{\partial^2 \chi^2_{\mathrm{FMWTLS}}}{\partial \widehat{Q}^2} = 2\sum_{i=1}^{N} \gamma^{N-i} \left(\frac{\sigma^4_{y_i}x_i^2 + \sigma^4_{x_i}(3\widehat{Q}^2y_i^2 - 2\widehat{Q}^3x_iy_i)}{(\widehat{Q}^2\sigma^2_{x_i} + \sigma^2_{y_i})^3} - \frac{\sigma^2_{x_i}\sigma^2_{y_i}(3\widehat{Q}^2x_i^2 - 6\widehat{Q}x_iy_i + y_i^2)}{(\widehat{Q}^2\sigma^2_{x_i} + \sigma^2_{y_i})^3} \right)$$

- Can use Newton–Raphson search to minimize cost function to find $\widehat{Q}$

Summary

- WTLS has none of the nice properties of the WLS solution:
  1. There is no closed-form solution in the general case
  2. There is no recursive update in the general case
  3. There is no fading memory recursive update
- Can use Newton–Raphson search, but requires growing memory and computation
- Next week, you will learn special case of WTLS that gives a closed-form solution, with recursive update, and fading memory
- Leads to approximate solution to general WTLS that also has these nice properties
- But first, we consider an important property of both the WLS and WTLS solutions