## Approximate full solution: Derivation

- Fading memory can be added with modified cost function

$$\chi^2_{\text{FMAWTLS}} = \sum_{i=1}^{N} \gamma^{N-i} \frac{(y_i - \widehat{Q}x_i)^2}{(1+\widehat{Q}^2)^2} \left( \frac{\widehat{Q}^2}{\sigma^2_{x_i}} + \frac{1}{\sigma^2_{y_i}} \right)$$

- The Jacobian is

$$\frac{\partial \chi^2_{\text{FMAWTLS}}}{\partial \widehat{Q}} = \frac{2}{(\widehat{Q}^2+1)^3} \sum_{i=1}^{N} \gamma^{N-i} \Bigg[ \widehat{Q}^4 \left( \frac{x_i y_i}{\sigma^2_{x_i}} \right) + \widehat{Q}^3 \left( \frac{2x_i^2}{\sigma^2_{x_i}} - \frac{x_i^2}{\sigma^2_{y_i}} - \frac{y_i^2}{\sigma^2_{x_i}} \right)$$
$$+ \widehat{Q}^2 \left( \frac{3x_i y_i}{\sigma^2_{y_i}} - \frac{3x_i y_i}{\sigma^2_{x_i}} \right) + \widehat{Q} \left( \frac{x_i^2 - 2y_i^2}{\sigma^2_{y_i}} + \frac{y_i^2}{\sigma^2_{x_i}} \right) + \left( \frac{-x_i y_i}{\sigma^2_{y_i}} \right) \Bigg]$$





## Recursive solution

- Jacobian can be rewritten in terms of recursive sums

$$\frac{\partial \chi^2_{\text{FMAWTLS}}}{\partial \widehat{Q}} = \frac{2}{(\widehat{Q}^2+1)^3} \Big( \tilde{c}_5 \widehat{Q}^4 + (-\tilde{c}_1 + 2\tilde{c}_4 - \tilde{c}_6)\widehat{Q}^3$$
$$+ (3\tilde{c}_2 - 3\tilde{c}_5)\widehat{Q}^2 + (\tilde{c}_1 - 2\tilde{c}_3 + \tilde{c}_6)\widehat{Q} - \tilde{c}_2 \Big)$$

where

$$\tilde{c}_{1,n} = \gamma \tilde{c}_{1,n-1} + x_n^2/\sigma^2_{y_n}; \quad \tilde{c}_{3,n} = \gamma \tilde{c}_{3,n-1} + y_n^2/\sigma^2_{y_n}; \quad \tilde{c}_{5,n} = \gamma \tilde{c}_{5,n-1} + x_n y_n/\sigma^2_{x_n}$$
$$\tilde{c}_{2,n} = \gamma \tilde{c}_{2,n-1} + x_n y_n/\sigma^2_{y_n}; \quad \tilde{c}_{4,n} = \gamma \tilde{c}_{4,n-1} + x_n^2/\sigma^2_{x_n}; \quad \tilde{c}_{6,n} = \gamma \tilde{c}_{6,n-1} + y_n^2/\sigma^2_{x_n}$$

- Roots of the quartic equation, below, are candidate solutions for $\widehat{Q}$

$$\tilde{c}_5 \widehat{Q}^4 + (2\tilde{c}_4 - \tilde{c}_1 - \tilde{c}_6)\widehat{Q}^3 + (3\tilde{c}_2 - 3\tilde{c}_5)\widehat{Q}^2 + (\tilde{c}_1 - 2\tilde{c}_3 + \tilde{c}_6)\widehat{Q} - \tilde{c}_2 = 0$$





## Initializing recursive solution

- Initialized by setting

$$x_0 = 1 \text{ and } y_0 = Q_{\text{nom}}$$
$$\sigma^2_{y_0} = \text{uncertainty in } Q_{\text{nom}} \text{ versus } Q$$

and $\sigma^2_{x_0} = \sigma^2_{y_0}$

- Therefore

$$\tilde{c}_{1,0} = 1/\sigma^2_{y_0}; \quad \tilde{c}_{3,0} = Q^2_{\text{nom}}/\sigma^2_{y_0}; \quad \tilde{c}_{5,0} = Q_{\text{nom}}/\sigma^2_{y_0}$$
$$\tilde{c}_{2,0} = Q_{\text{nom}}/\sigma^2_{y_0}; \quad \tilde{c}_{4,0} = 1/\sigma^2_{y_0}; \quad \tilde{c}_{6,0} = Q^2_{\text{nom}}/\sigma^2_{y_0}$$

## Determining which solution to choose

- Again, can be solved using Ferrari method, for example; the four candidate solutions must be checked against the cost function to determine which is optimal
- The cost function in terms of these variables is

$$\chi^2_{\text{FMAWTLS}} = \frac{1}{(\widehat{Q}^2+1)^2}\left(\tilde{c}_4\widehat{Q}^4 - 2\tilde{c}_5\widehat{Q}^3 + (\tilde{c}_1+\tilde{c}_6)\widehat{Q}^2 - 2\tilde{c}_2\widehat{Q} + \tilde{c}_3\right)$$

- The Hessian is

$$\frac{\partial^2 \chi^2_{\text{FMAWTLS}}}{\partial \widehat{Q}^2} = \frac{2}{(\widehat{Q}^2+1)^4}\Big(-2\tilde{c}_5\widehat{Q}^5 + (3\tilde{c}_1-6\tilde{c}_4+3\tilde{c}_6)\widehat{Q}^4 + (-12\tilde{c}_2+16\tilde{c}_5)\widehat{Q}^3 + (-8\tilde{c}_1+10\tilde{c}_3+6\tilde{c}_4-8\tilde{c}_6)\widehat{Q}^2 + (12\tilde{c}_2-6\tilde{c}_5)\widehat{Q} + (\tilde{c}_1-2\tilde{c}_3+\tilde{c}_6)\Big)$$





## Scaling for better approximation

- AWTLS and WTLS cost functions not equal when $\sigma_{x_i} = k\sigma_{y_i}$
- This can be remedied easily: define $\tilde{y}_i = k y_i$; then $\sigma_{\tilde{y}_i} = \sigma_{x_i}$
- Invoke the AWTLS or FMAWTLS methods to find total capacity estimate $\widehat{Q}$ and Hessian $H$ using input sequences comprised of the original $\mathbf{x}$ vector and the scaled $\tilde{\mathbf{y}}$ vector (i.e., $(x_i, \tilde{y}_i)$ with corresponding variances $(\sigma^2_{x_i}, k^2\sigma^2_{y_i})$)
- True slope estimate found as $\widehat{Q}_{\text{corrected}} = \widehat{Q}/k$, and Hessian as $H_{\text{corrected}} = H/k^2$
- This is the method used next week, where $k$ is estimated as $k = \sigma_{x_1}/\sigma_{y_1}$
- This scaling improves results even when $\sigma_{y_i}$ and $\sigma_{x_i}$ not proportionally related, if $k$ chosen to give "order of magnitude" ratio between uncertainties of $x_i$ and $y_i$





## Summary

- AWTLS solutions share the nice properties of WLS solution:
  1. Closed-form solution for $\widehat{Q}$: No iteration or advanced algorithms are required—only simple mathematical operations.
  2. Solution can be computed very easily in a recursive manner: keep track of six running sums $c_{1,n}$ through $c_{6,n}$; when additional data become available, update sums and compute updated total-capacity estimate
  3. Fading memory can be added easily to allow $\widehat{Q}$ to place greater emphasis on more recent measurements than on earlier measurements, allowing adaptation of $\widehat{Q}$ to adjust for true cell total-capacity changes
  4. Furthermore, this method is superior to the TLS solution since it allows individual weighting on $x_i$ and $y_i$ data points