Independence and uncorrelation

INDEPENDENCE: Iff jointly-distributed RVs are independent, then

$$f_X(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n)$$

- Means that joint pdf factors into product of marginal pdfs (not generally true)
  - "The particular value of the random variable X_1 has no impact on what value we would obtain for the random variable X_2"... no nonlinear or linear relationship

UNCORRELATED: Two jointly-distributed RVs X_1 and X_2 are uncorrelated iff

$$\mathrm{cov}(X_1, X_2) = \mathbb{E}[(X_1 - \bar{x}_1)(X_2 - \bar{x}_2)^T] = 0$$

- Can show that this implies that $\mathbb{E}[X_1 X_2^T] = \mathbb{E}[X_1]\mathbb{E}[X_2^T]$: Means that expectation of a product factors into product of expectations (not generally true)
- Implies $\rho_{12} = 0$; uncorrelated means there is no linear relationship between RVs

Independence versus uncorrelation

- In general, condition for RVs to be uncorrelated is much weaker than for them to be independent
- If jointly-distributed RVs X_1 and X_2 are independent they must also be uncorrelated: independence implies uncorrelation
  - However, uncorrelated RVs are not necessarily independent
- If joint normally (Gaussian) distributed RVs are uncorrelated, are also independent
  - This is a (very, very) special case
  - This is one of the reasons we assume that all pdfs in the sequential-probabilistic-inference solution are Gaussian

Conditional probability

- We can define a conditional pdf

$$f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y)$$

  as the likelihood that $X = x$ given that $Y = y$ has happened
- Note: Marginal pdf $f_Y(y)$ may be calculated as

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x$$

  - For each y, integrate out effect of X
- Conditional pdf = joint pdf / marginal pdf (of RV being conditioned on)

Bayes' rule

- As a direct extension of the definition of a conditional pdf,

$$\begin{aligned} f_{X,Y}(x,y) &= f_{X|Y}(x|y)\,f_Y(y) \\ &= f_{Y|X}(y|x)\,f_X(x) \end{aligned}$$

- Therefore, we can solve for one conditional pdf in terms of other

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)\,f_X(x)}{f_Y(y)}$$

- This is known as Bayes' rule
- Relates posterior probability to prior probability, forms a key step in KF derivation

Conditional expectation

- Can define conditional expectation as what we expect the value of X to be given that $Y = y$ has happened

$$\mathbb{E}[X = x|Y = y] = \mathbb{E}[X|Y] = \int_{-\infty}^{\infty} x\,f_{X|Y}(x|Y)\,\mathrm{d}x$$

- Conditional expectation is critical
- KF is an algorithm to compute $\mathbb{E}[x_k \mid \mathbb{Y}_k]\ldots$ expected value of model state vector given complete set of measurements made

Central limit theorem

- If $Y = \sum_i X_i$, and X_i are independent, identically distributed (IID), and have finite mean and variance: then Y is approx. normally distributed: approximation improves as more RVs are summed
  - Special case: linear combination of Gaussian RVs results in a Gaussian RV
- Since state of dynamic system adds up effects of lots of independent random inputs, reasonable to assume that distribution of state tends to normal distribution
- Leads to key assumptions for derivation of KF, as we will see:
  - Assume state x_k, process noise w_k, sensor noise v_k are normally distributed RVs
  - Assume w_k and v_k are uncorrelated with each other
- Even when these assumptions are broken in practice, KF works quite well

Summary

- If two RVs are independent, joint pdf equals product of marginal pdfs and knowing value of one RV cannot be used to help predict value of other in any way
- If two RVs are uncorrelated, expected value of product equals product of expected values and knowing value of one RV cannot be used with linear equation to help predict value of other
- In most cases, RVs we will work with are correlated, so conditional expectation will help us to predict value of one RV given values of other(s)
- Central limit theorem justifies assumption that RVs are Gaussian
- Uncorrelated Gaussian RVs are also independent: a very special case