Toward improving numeric robustness

- Within KF, covariance matrices $\Sigma_{\tilde{x},k}^{-}$ and $\Sigma_{\tilde{x},k}^{+}$ must remain
  1. Symmetric, and
  2. Positive definite (all eigenvalues strictly positive)

  at every time step

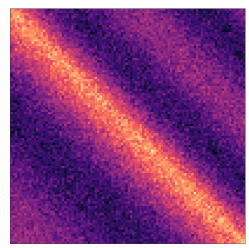
- It is possible for both conditions to be violated due to round-off errors in a computer implementation
- We wish to find ways to limit or eliminate these problems

Searching for cause of loss of symmetry

- The cause of covariance matrices becoming asymmetric or nonpositive definite must be due to either the time update or measurement update equations of the filter
- Consider first the time update equation:

$$\Sigma_{\tilde{x},k}^{-} = A\Sigma_{\tilde{x},k-1}^{+}A^{T} + \Sigma_{\widetilde{w}}$$

  - Because we are adding two positive-definite quantities together, the result must be positive definite
  - A "suitable implementation" of the products of the matrices will avoid loss of symmetry in the final result



Dealing with loss of symmetry

- Consider next the measurement update equation:

$$\Sigma_{\tilde{x},k}^{+} = \Sigma_{\tilde{x},k}^{-} - L_k C_k \Sigma_{\tilde{x},k}^{-}$$

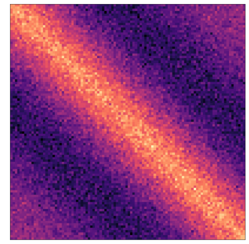
- Theoretically, result is positive definite, but subtraction operation enables round-off errors in an implementation to result in nonpositive-definite solution
- Better to use Joseph-form covariance update

$$\Sigma_{\tilde{x},k}^{+} = [I - L_k C_k]\,\Sigma_{\tilde{x},k}^{-}\,[I - L_k C_k]^{T} + L_k \Sigma_{\tilde{v}} L_k^{T}$$

  - (Proof: `http://mocha-java.uccs.edu/ECE5550/ECE5550-Notes05.pdf`)
- Because subtraction occurs in "squared" term, Joseph form guarantees positive-definite result

Dealing with loss of symmetry

- If we still compute nonpositive-definite matrix (numerics), can find “nearest” symmetric positive semidefinite matrix[1]
- Omitting the details, the procedure is:
  - Calculate singular-value decomposition: $\Sigma = USV^T$
  - Compute $H = VSV^T$
  - Replace Σ with $(\Sigma + \Sigma^T + H + H^T)/4$
- There are still improvements that may be made; can:
  - Generalize to handle correlated noises
  - Process measurements sequentially for multi-output systems
  - Improve numeric precision with “square-root” KF



[1] Nicholas J. Higham, “Computing a Nearest Symmetric Positive Semidefinite Matrix,” *Linear Algebra and its Applications*, 103, 103–118, 1988

Summary

- If covariance matrices become nonpositive-definite, KF will diverge and “error bounds” will become meaningless
- Theoretically “impossible” but often happens in practice due to numeric “round-off” errors in floating-point operations
- Can minimize likelihood of problem using Joseph-form covariance update
- Can further improve using Higham’s method, guaranteeing at least positive semi-definite matrices

Credits

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