## The extended Kalman filter (EKF)

- EKF makes two simplifying assumptions when adapting general sequential inference equations to a nonlinear system:
  - When computing estimates of the output of a nonlinear function, EKF assumes $\mathbb{E}[\text{fn}(x)] \approx \text{fn}(\mathbb{E}[x])$, which is not true in general
  - When computing covariance estimates, EKF uses Taylor-series expansion to linearize the system equations around the present operating point
- In this lesson, you will learn how to apply these approximations and assumptions to derive the EKF prediction steps
- In the next lesson, you will learn how to derive the EKF update steps





## EKF step 1a: State prediction time update

- The state prediction step is approximated as

$$\begin{aligned}\hat{x}_k^- &= \mathbb{E}[f(x_{k-1}, u_{k-1}, w_{k-1}) \mid \mathbb{Y}_{k-1}] \\ &\approx f(\hat{x}_{k-1}^+, u_{k-1}, \bar{w}_{k-1}),\end{aligned}$$

  where $\bar{w}_{k-1} = \mathbb{E}[w_{k-1}]$. (Often, $\bar{w}_{k-1} = 0$.)
- That is, we approximate the expected value of the new state by assuming that it is reasonable to simply propagate $\hat{x}_{k-1}^+$ and $\bar{w}_{k-1}$ through the state equation.





## EKF step 1b: Error covariance time update (1)

- We start with an expression for $\tilde{x}_k^-$:
  $\tilde{x}_k^- = x_k - \hat{x}_k^- = f(x_{k-1}, u_{k-1}, w_{k-1}) - f(\hat{x}_{k-1}^+, u_{k-1}, \bar{w}_{k-1})$
- Approx. first term via Taylor series around prior operating point $\{\hat{x}_{k-1}^+, u_{k-1}, \bar{w}_{k-1}\}$

$$\begin{aligned}x_k \approx\ & f(\hat{x}_{k-1}^+, u_{k-1}, \bar{w}_{k-1}) + \underbrace{\left.\frac{\mathrm{d}f(x_{k-1}, u_{k-1}, w_{k-1})}{\mathrm{d}x_{k-1}}\right|_{x_{k-1}=\hat{x}_{k-1}^+}}_{\text{Defined as } \hat{A}_{k-1}} (x_{k-1} - \hat{x}_{k-1}^+) \\ &+ \underbrace{\left.\frac{\mathrm{d}f(x_{k-1}, u_{k-1}, w_{k-1})}{\mathrm{d}w_{k-1}}\right|_{w_{k-1}=\bar{w}_{k-1}}}_{\text{Defined as } \hat{B}_{k-1}} (w_{k-1} - \bar{w}_{k-1})\end{aligned}$$

- This gives $\tilde{x}_k^- \approx \left(\hat{A}_{k-1}\tilde{x}_{k-1}^+ + \hat{B}_{k-1}\widetilde{w}_{k-1}\right)$

## EKF step 1b: Error covariance time update (2)

- Substituting this to find the prediction-error covariance:
  $\Sigma_{\tilde{x},k}^{-} = \mathbb{E}[(\tilde{x}_k^{-})(\tilde{x}_k^{-})^T] \approx \hat{A}_{k-1}\Sigma_{\tilde{x},k-1}^{+}\hat{A}_{k-1}^T + \hat{B}_{k-1}\Sigma_{\widetilde{w}}\hat{B}_{k-1}^T$
- Note, by the chain rule of total differentials,

$$\mathrm{d}f(x_{k-1},u_{k-1},w_{k-1}) = \frac{\partial f(x_{k-1},u_{k-1},w_{k-1})}{\partial x_{k-1}}\mathrm{d}x_{k-1} + \frac{\partial f(x_{k-1},u_{k-1},w_{k-1})}{\partial u_{k-1}}\mathrm{d}u_{k-1} + \frac{\partial f(x_{k-1},u_{k-1},w_{k-1})}{\partial w_{k-1}}\mathrm{d}w_{k-1}$$

$$\frac{\mathrm{d}f(x_{k-1},u_{k-1},w_{k-1})}{\mathrm{d}x_{k-1}} = \frac{\partial f(x_{k-1},u_{k-1},w_{k-1})}{\partial x_{k-1}} + \frac{\partial f(x_{k-1},u_{k-1},w_{k-1})}{\partial u_{k-1}}\cancelto{0}{\frac{\mathrm{d}u_{k-1}}{\mathrm{d}x_{k-1}}} + \frac{\partial f(x_{k-1},u_{k-1},w_{k-1})}{\partial w_{k-1}}\cancelto{0}{\frac{\mathrm{d}w_{k-1}}{\mathrm{d}x_{k-1}}} = \frac{\partial f(x_{k-1},u_{k-1},w_{k-1})}{\partial x_{k-1}}$$





## EKF step 1c: Output estimate

- Similarly, $\frac{\mathrm{d}f(x_{k-1},u_{k-1},w_{k-1})}{\mathrm{d}w_{k-1}} = \frac{\partial f(x_{k-1},u_{k-1},w_{k-1})}{\partial w_{k-1}}$
- Distinction between total and partial differential not yet critical
- Will be in fourth course in specialization

- System output is estimated to be

$$\begin{aligned}\hat{y}_k &= \mathbb{E}[h(x_k,u_k,v_k) \mid \mathbb{Y}_{k-1}] \\ &\approx h(\hat{x}_k^{-},u_k,\bar{v}_k),\end{aligned}$$

  where $\bar{v}_k = \mathbb{E}[v_k]$
- That is, it is assumed that propagating $\hat{x}_k^{-}$ and the mean sensor noise is the best approximation to estimating the output





## Summary

- EKF makes two fundamental assumptions to generalize KF
  1. Assume $\mathbb{E}[\text{fn}(x)] \approx \text{fn}(\mathbb{E}[x])$, which is not true in general
  2. Assume Taylor-series expansion to linearize system equations for covariances
- Applying these to generic-probabilistic-inference solution yields first three EKF steps (final three in next lesson)
- In general must be careful to distinguish between total and partial differentials (although not critical here)
- First half of EKF is now derived