Review of probability

- Sequential probabilistic inference (and hence Kalman filters) seek to find best state estimate in the presence of process and sensor noises on measurements
- By definition, noise is not deterministic—it is random in some sense
- So, to discuss the impact of noise on the system dynamics, we must understand (vector) "random variables" (RVs)
  - Can't predict exactly what we will get each time we measure the RV, but
  - Can characterize likelihood of different possible measurements by RV's "probability density function" (pdf)

Review of random vectors

- As review, define random vector X, sample vector x_0

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}, \quad x_0 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

where X_1 through X_n are themselves scalar RVs, and x_1 through x_n are scalar constant values these RVs can take on

- Random vector X described by (scalar function) joint pdf $f_X(x)$ of vector X
  - $f_X(x_0)$ means $f_X(X_1 = x_1, X_2 = x_2, \cdots, X_n = x_n)$
  - $f_X(x_0)\,\mathrm{d}x_1\,\mathrm{d}x_2\,\cdots\,\mathrm{d}x_n$ is probability that X is between x_0 and $x_0 + \mathrm{d}x$
  - $f_X(x_0)$ is scaled probability or "likelihood" of measuring sample vector x_0

Key properties of joint pdf of random vector

1. $f_X(x) \ge 0 \quad \forall \quad x$
2. $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n = 1$
3. $\displaystyle\bar{x} = \mathbb{E}[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} x f_X(x)\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n$
4. Correlation matrix (note *outer* product, not inner product):

$$\Sigma_X = \mathbb{E}[XX^T] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} xx^T f_X(x)\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n$$

5. Covariance matrix: Define $\widetilde{X} = X - \bar{x}$. Then,

$$\Sigma_{\widetilde{X}} = \mathbb{E}[(X-\bar{x})(X-\bar{x})^T] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} (x-\bar{x})(x-\bar{x})^T f_X(x)\,\mathrm{d}x_1\,\mathrm{d}x_2\cdots\mathrm{d}x_n$$

Properties of correlation and covariance

- $\Sigma_{\widetilde{X}}$ is symmetric and positive-semi-definite (psd). This means

$$y^T \Sigma_{\widetilde{X}} y \geq 0 \quad \forall \quad y$$

- Notice that correlation = covariance for zero-mean random vectors
- The covariance entries have specific meaning:

$$(\Sigma_{\widetilde{X}})_{ii} = \sigma^2_{X_i}$$
$$(\Sigma_{\widetilde{X}})_{ij} = \rho_{ij}\sigma_{X_i}\sigma_{X_j} = (\Sigma_{\widetilde{X}})_{ji}$$

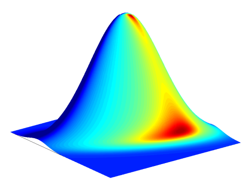
  - The diagonal entries are the variances of each vector component
  - Correlation coefficient ρ_{ij} is a measure of linear dependence between X_i and X_j; $|\rho_{ij}| \leq 1$

The multivariable Gaussian pdf

- There are infinite variety in pdfs; however, we assume only (multivariable) Gaussian pdf in KF
- All noises and the state vector itself are assumed to be Gaussian random vectors
- Gaussian or normal pdf is (we say "$X \sim \mathcal{N}(\bar{x}, \Sigma_{\widetilde{X}})$")

$$f_X(x) = \frac{1}{(2\pi)^{n/2}\left|\Sigma_{\widetilde{X}}\right|^{1/2}} \exp\left(-\frac{1}{2}(x-\bar{x})^T \Sigma_{\widetilde{X}}^{-1}(x-\bar{x})\right)$$

$|\Sigma_{\widetilde{X}}| = \det(\Sigma_{\widetilde{X}})$, $\quad \Sigma_{\widetilde{X}}^{-1}$ requires positive-definite $\Sigma_{\widetilde{X}}$



- Contours of constant $f_X(x)$ are hyper-ellipsoids, centered at $\bar{x}$, rotated via $\Sigma_{\widetilde{X}}$
- Good news... We won't need to work directly with this equation very much!

Summary

- To develop sequential-probabilistic-inference solution, must have background understanding of random variables (RVs)
- RVs are described by probability density functions (pdfs)
- These pdfs have important properties, which we have reviewed
- In particular, we will be making use of mean and covariance *a lot*
- The pdf we will assume for all RVs is multivariable Gaussian (or normal) distribution
- While this seems complicated at first, it turns out to simplify the math a lot later on