Joint/dual state and parameter estimation

- Last lesson, you learned that it is possible to estimate a system's state and parameter vectors simultaneously via joint and dual estimation
- This lesson does not contain much teaching content, but serves primarily as a reference for how to implement joint and dual estimators using EKF and SPKF
- Joint estimation via SPKF uses a standard SPKF where state vector is augmented with parameters
- Similarly, joint estimation via EKF is straightforward, but we must implement a recursive calculation of $\mathrm{d}\mathcal{F}/\mathrm{d}\mathbb{X}$
- Dual estimation via SPKF uses two SPKFs that intermix signals
- Dual estimation via EKF is similar, but need to be careful when computing $\hat{C}_k^{\theta}$

Dual estimation using EKF

- When finding $\hat{C}_k^{\theta}$, total-differential expansion must be correct

$$\hat{C}_k^{\theta} = \left.\frac{\mathrm{d}h(\hat{x}_k^-, u_k, \theta)}{\mathrm{d}\theta}\right|_{\theta=\hat{\theta}_k^-}$$

$$\frac{\mathrm{d}h(\hat{x}_k^-, u_k, \theta)}{\mathrm{d}\theta} = \frac{\partial h(\hat{x}_k^-, u_k, \theta)}{\partial\theta} + \frac{\partial h(\hat{x}_k^-, u_k, \theta)}{\partial \hat{x}_k^-}\frac{\mathrm{d}\hat{x}_k^-}{\mathrm{d}\theta}$$

$$\frac{\mathrm{d}\hat{x}_k^-}{\mathrm{d}\theta} = \frac{\partial f(\hat{x}_{k-1}^+, u_{k-1}, \theta)}{\partial\theta} + \frac{\partial f(\hat{x}_{k-1}^+, u_{k-1}, \theta)}{\partial \hat{x}_{k-1}^+}\frac{\mathrm{d}\hat{x}_{k-1}^+}{\mathrm{d}\theta}$$

$$\frac{\mathrm{d}\hat{x}_{k-1}^+}{\mathrm{d}\theta} = \frac{\mathrm{d}\hat{x}_{k-1}^-}{\mathrm{d}\theta} - L_{k-1}^x \frac{\mathrm{d}h(\hat{x}_{k-1}^-, u_{k-1}, \theta)}{\mathrm{d}\theta}$$

- Assumes L_{k-1}^x not a function of θ (it is—weakly—not worth extra computation)
- The three total derivatives are computed recursively, initialized to zero

Summary

- The appendix to this lesson contains summary tables of joint and dual EKF and SPKF estimators
- The format of these tables is consistent
  - First, the assumed model structure is stated
  - Second, some definitions are made
  - Third, initialization statements are presented
  - Finally, the computation steps are listed (in order)
- Next lesson, we will discuss some challenges to joint/dual estimation and how these can be overcome

Joint EKF for state and parameter estimation (1)

State-space model:

$$\begin{bmatrix} x_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} f(x_{k-1}, u_{k-1}, w_{k-1}, \theta_{k-1}) \\ \theta_{k-1} + r_{k-1} \end{bmatrix} \quad \text{or} \quad \begin{aligned} \mathbb{X}_k &= \mathcal{F}(\mathbb{X}_{k-1}, u_{k-1}, \mathbb{W}_{k-1}) \\ d_k &= h(\mathbb{X}_k, u_k, v_k), \end{aligned}$$

$$d_k = h(x_k, u_k, v_k, \theta_k),$$

where w_k, r_k, and v_k are independent, Gaussian noise processes of covariance matrices $\Sigma_{\tilde{w}}$, $\Sigma_{\tilde{r}}$, and $\Sigma_{\tilde{v}}$, respectively

For brevity, we let $\mathbb{X}_k = \begin{bmatrix} x_k^T, & \theta_k^T \end{bmatrix}^T$, $\mathbb{W}_k = \begin{bmatrix} w_k^T, & r_k^T \end{bmatrix}^T$ and $\Sigma_{\tilde{\mathbb{W}}} = \operatorname{diag}(\Sigma_{\tilde{w}}, \Sigma_{\tilde{r}})$

Definitions:

$$\hat{A}_k = \left.\frac{\mathrm{d}\mathcal{F}(\mathbb{X}_k, u_k, \mathbb{W}_k)}{\mathrm{d}\mathbb{X}_k}\right|_{\mathbb{X}_k = \hat{\mathbb{X}}_k^+} \qquad \hat{B}_k = \left.\frac{\mathrm{d}\mathcal{F}(\mathbb{X}_k, u_k, \mathbb{W}_k)}{\mathrm{d}\mathbb{W}_k}\right|_{\mathbb{W}_k = \bar{\mathbb{W}}_k}$$

$$\hat{C}_k = \left.\frac{\mathrm{d}h(\mathbb{X}_k, u_k, v_k)}{\mathrm{d}\mathbb{X}_k}\right|_{\mathbb{X}_k = \hat{\mathbb{X}}_k^-} \qquad \hat{D}_k = \left.\frac{\mathrm{d}h(\mathbb{X}_k, u_k, v_k)}{\mathrm{d}v_k}\right|_{v_k = \bar{v}_k}.$$

Joint EKF for state and parameter estimation (2)

Initialization: For $k = 0$, set

$$\begin{aligned} \hat{\mathbb{X}}_0^+ &= \mathbb{E}\big[\mathbb{X}_0\big] \\ \Sigma_{\tilde{\mathbb{X}},0}^+ &= \mathbb{E}\big[(\mathbb{X}_0 - \hat{\mathbb{X}}_0^+)(\mathbb{X}_0 - \hat{\mathbb{X}}_0^+)^T\big] \end{aligned}$$

Computation: For $k = 1, 2, \ldots$ compute:

State estimate time update:	$\hat{\mathbb{X}}_k^- = \mathcal{F}(\hat{\mathbb{X}}_{k-1}^+, u_{k-1}, \bar{\mathbb{W}}_{k-1})$
Error covariance time update:	$\Sigma_{\tilde{\mathbb{X}},k}^- = \hat{A}_{k-1}\Sigma_{\tilde{\mathbb{X}},k-1}^+\hat{A}_{k-1}^T + \hat{B}_{k-1}\Sigma_{\tilde{\mathbb{W}}}\hat{B}_{k-1}^T$
Output estimate:	$\hat{d}_k = h(\hat{\mathbb{X}}_k^-, u_k, \bar{v}_k)$
Estimator gain matrix:	$L_k = \Sigma_{\tilde{\mathbb{X}},k}^-\hat{C}_k^T[\hat{C}_k\Sigma_{\tilde{\mathbb{X}},k}^-\hat{C}_k^T + \hat{D}_k\Sigma_{\tilde{v}}\hat{D}_k^T]^{-1}$
State estimate meas. update:	$\hat{\mathbb{X}}_k^+ = \hat{x}_k^- + L_k\big(d_k - \hat{d}_k\big)$
Error covariance meas. update:	$\Sigma_{\tilde{\mathbb{X}},k}^+ = \Sigma_{\tilde{\mathbb{X}},k}^- - L_k\Sigma_{\tilde{d},k}L_k^T$

Dual EKF for state and parameter estimation (1)

State-space model:

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, \theta_k, w_k) \\ y_k &= h(x_k, u_k, \theta_k, v_k) \end{aligned} \quad \text{and} \quad \begin{aligned} \theta_{k+1} &= \theta_k + r_k, \\ d_k &= h(x_k, u_k, \theta_k, e_k) \end{aligned}$$

where w_k, v_k, r_k and e_k are independent Gaussian noise processes with means $\bar{w}$, $\bar{v}$, $\bar{r}$, and $\bar{e}$ and covariance matrices $\Sigma_{\tilde{w}}$, $\Sigma_{\tilde{v}}$, $\Sigma_{\tilde{r}}$ and $\Sigma_{\tilde{e}}$, respectively

Definitions:

$$\hat{A}_k = \left.\frac{\mathrm{d}f(x_k, u_k, \hat{\theta}_k^-, w_k)}{\mathrm{d}x_k}\right|_{x_k = \hat{x}_k^+} \qquad \hat{B}_k = \left.\frac{\mathrm{d}f(x_k, u_k, \hat{\theta}_k^-, w_k)}{\mathrm{d}w_k}\right|_{w_k = \bar{w}}$$

$$\hat{C}_k^x = \left.\frac{\mathrm{d}h(x_k, u_k, \hat{\theta}_k^-, v_k)}{\mathrm{d}x_k}\right|_{x_k = \hat{x}_k^-} \qquad \hat{D}_k^x = \left.\frac{\mathrm{d}h(x_k, u_k, \hat{\theta}_k^-, v_k)}{\mathrm{d}v_k}\right|_{v_k = \bar{v}}$$

$$\hat{C}_k^\theta = \left.\frac{\mathrm{d}h(\hat{x}_k^-, u_k, \theta, e_k)}{\mathrm{d}\theta}\right|_{\theta = \hat{\theta}_k^-} \qquad \hat{D}_k^\theta = \left.\frac{\mathrm{d}h(\hat{x}_k^-, u_k, \theta, e_k)}{\mathrm{d}e_k}\right|_{e_k = \bar{e}}$$

Dual EKF for state and parameter estimation (2)

Initialization: For $k = 0$, set

$$\hat{\theta}_0^+ = \mathbb{E}[\theta_0], \qquad \Sigma_{\tilde{\theta},0}^+ = \mathbb{E}\big[(\theta_0 - \hat{\theta}_0^+)(\theta_0 - \hat{\theta}_0^+)^T\big].$$
$$\hat{x}_0^+ = \mathbb{E}[x_0], \qquad \Sigma_{\tilde{x},0}^+ = \mathbb{E}\big[(x_0 - \hat{x}_0^+)(x_0 - \hat{x}_0^+)^T\big].$$

Computation: For $k = 1, 2, \ldots$ compute:

Time update for weight filter:
$$\hat{\theta}_k^- = \hat{\theta}_{k-1}^+$$
$$\Sigma_{\tilde{\theta},k}^- = \Sigma_{\tilde{\theta},k-1}^+ + \Sigma_{\tilde{r}}$$

Time update for state filter:
$$\hat{x}_k^- = f(\hat{x}_{k-1}^+, u_{k-1}, \hat{\theta}_k^-, \bar{w})$$
$$\Sigma_{\tilde{x},k}^- = \hat{A}_{k-1}\Sigma_{\tilde{x},k-1}^+\hat{A}_{k-1}^T + \hat{B}_{k-1}\Sigma_{\tilde{w}}\hat{B}_{k-1}^T$$

Meas. update for state filter:
$$L_k^x = \Sigma_{\tilde{x},k}^-(\hat{C}_k^x)^T\big[\hat{C}_k^x\Sigma_{\tilde{x},k}^-(\hat{C}_k^x)^T + \hat{D}_k^x\Sigma_{\tilde{v}}(\hat{D}_k^x)^T\big]^{-1}$$
$$\hat{x}_k^+ = \hat{x}_k^- + L_k^x\big[z_k - h(\hat{x}_k^-, u_k, \hat{\theta}_k^-, \bar{v})\big]$$
$$\Sigma_{\tilde{x},k}^+ = \Sigma_{\tilde{x},k}^- - L_k^x\Sigma_{\tilde{z},k}(L_k^x)^T$$

Dual EKF for state and parameter estimation (3)

Computation (cont.): For $k = 1, 2, \ldots$ compute:

Meas. update for weight filter:
$$L_k^\theta = \Sigma_{\tilde{\theta},k}^-(\hat{C}_k^\theta)^T\big[\hat{C}_k^\theta\Sigma_{\tilde{\theta},k}^-(\hat{C}_k^\theta)^T + \hat{D}_k^\theta\Sigma_{\tilde{e}}(\hat{D}_k^\theta)^T\big]^{-1}$$
$$\hat{\theta}_k^+ = \hat{\theta}_k^- + L_k^\theta\big[z_k - h(\hat{x}_k^-, u_k, \hat{\theta}_k^-, \bar{e})\big]$$
$$\Sigma_{\tilde{\theta},k}^+ = \Sigma_{\tilde{\theta},k}^- - L_k^\theta\Sigma_{\tilde{z},k}(L_k^\theta)^T$$

Joint SPKF for state and parameter estimation (1)

State-space model:

$$\begin{bmatrix} x_k \\ \theta_k \end{bmatrix} = \begin{bmatrix} f(x_{k-1}, u_{k-1}, w_{k-1}, \theta_{k-1}) \\ \theta_{k-1} + r_{k-1} \end{bmatrix} \quad \text{or} \quad \begin{aligned} \mathbb{X}_k &= \mathcal{F}(\mathbb{X}_{k-1}, u_{k-1}, \mathbb{W}_{k-1}) \\ z_k &= h(\mathbb{X}_k, u_k, v_k), \end{aligned}$$
$$z_k = h(x_k, u_k, v_k, \theta_k),$$

where w_k, r_k, and v_k are independent, Gaussian noise processes with means $\bar{w}$, $\bar{r}$, and $\bar{v}$, and covariance matrices $\Sigma_{\tilde{w}}$, $\Sigma_{\tilde{r}}$, and $\Sigma_{\tilde{v}}$, respectively. For brevity, we let $\mathbb{X}_k = \big[x_k^T,\ \theta_k^T\big]^T$, $\mathbb{W}_k = \big[w_k^T,\ r_k^T\big]^T$ and $\Sigma_{\tilde{\mathbb{W}}} = \mathrm{diag}(\Sigma_{\tilde{w}}, \Sigma_{\tilde{r}})$

Definitions:

$$\mathbb{X}_k^a = \big[\mathbb{X}_k^T,\ \mathbb{W}_k^T,\ v_k^T\big]^T, \quad \mathcal{X}_k^a = \big[(\mathcal{X}_k^{\mathbb{X}})^T,\ (\mathcal{X}_k^{\mathbb{W}})^T,\ (\mathcal{X}_k^v)^T\big]^T, \quad p = 2 \times \dim(\mathbb{X}_k^a)$$

Joint SPKF for state and parameter estimation (2)

Initialization: For $k = 0$, set

$$\begin{aligned}
\hat{\mathbb{X}}_0^+ &= \mathbb{E}\big[\mathbb{X}_0\big] \\
\Sigma^+_{\tilde{\mathbb{X}},0} &= \mathbb{E}\big[(\mathbb{X}_0 - \hat{\mathbb{X}}_0^+)(\mathbb{X}_0 - \hat{\mathbb{X}}_0^+)^T\big] \\
\hat{\mathbb{X}}_0^{a,+} &= \mathbb{E}\big[\mathbb{X}_0^a\big] = \big[(\hat{\mathbb{X}}_0^+)^T,\ \bar{\mathbb{W}},\ \bar{v}\big]^T
\end{aligned}
\qquad
\begin{aligned}
\Sigma^{a,+}_{\tilde{\mathbb{X}},0} &= \mathbb{E}\big[(\mathbb{X}_0^a - \hat{\mathbb{X}}_0^{a,+})(\mathbb{X}_0^a - \hat{\mathbb{X}}_0^{a,+})^T\big] \\
&= \operatorname{diag}\big(\Sigma^+_{\tilde{\mathbb{X}},0},\, \Sigma_{\tilde{\mathbb{W}}},\, \Sigma_{\tilde{v}}\big)
\end{aligned}$$

Computation: For $k = 1, 2, \ldots$ compute:

State time update:
$$\begin{aligned}
\mathcal{X}^{a,+}_{k-1} &= \Big\{\hat{\mathbb{X}}^{a,+}_{k-1},\, \hat{\mathbb{X}}^{a,+}_{k-1} + \gamma\sqrt{\Sigma^{a,+}_{\tilde{\mathbb{X}},k-1}},\, \hat{\mathbb{X}}^{a,+}_{k-1} - \gamma\sqrt{\Sigma^{a,+}_{\tilde{\mathbb{X}},k-1}}\Big\} \\
\mathcal{X}^{\mathbb{X},-}_{k,i} &= \mathcal{F}(\mathcal{X}^{\mathbb{X},+}_{k-1,i}, u_{k-1}, \mathcal{X}^{\mathbb{W},+}_{k-1,i}) \\
\hat{\mathbb{X}}^-_k &= \textstyle\sum_{i=0}^p \alpha_i^{(\mathrm{m})} \mathcal{X}^{\mathbb{X},-}_{k,i}
\end{aligned}$$

Covariance time update:
$$\Sigma^-_{\tilde{\mathbb{X}},k} = \textstyle\sum_{i=0}^p \alpha_i^{(\mathrm{c})} \big(\mathcal{X}^{\mathbb{X},-}_{k,i} - \hat{\mathbb{X}}^-_k\big)\big(\mathcal{X}^{\mathbb{X},-}_{k,i} - \hat{\mathbb{X}}^-_k\big)^T$$

Joint SPKF for state and parameter estimation (3)

Computation (cont.): For $k = 1, 2, \ldots$ compute:

Output estimate:
$$\begin{aligned}
\mathcal{Z}_{k,i} &= h(\mathcal{X}^{\mathbb{X},-}_{k,i}, u_k, \mathcal{X}^{v,+}_{k-1,i}) \\
\hat{z}_k &= \textstyle\sum_{i=0}^p \alpha_i^{(\mathrm{m})} \mathcal{Z}_{k,i}
\end{aligned}$$

Estimator gain matrix:
$$\begin{aligned}
\Sigma_{\tilde{z},k} &= \textstyle\sum_{i=0}^p \alpha_i^{(\mathrm{c})} \big(\mathcal{Z}_{k,i} - \hat{z}_k\big)\big(\mathcal{Z}_{k,i} - \hat{z}_k\big)^T \\
\Sigma^-_{\tilde{\mathbb{X}}\tilde{z},k} &= \textstyle\sum_{i=0}^p \alpha_i^{(\mathrm{c})} \big(\mathcal{X}^{\mathbb{X},-}_{k,i} - \hat{\mathbb{X}}^-_k\big)\big(\mathcal{Z}_{k,i} - \hat{z}_k\big)^T \\
L_k &= \Sigma^-_{\tilde{\mathbb{X}}\tilde{z},k}\Sigma^{-1}_{\tilde{z},k}
\end{aligned}$$

State estimate meas. update: $\hat{\mathbb{X}}^+_k = \hat{\mathbb{X}}^-_k + L_k\big(z_k - \hat{z}_k\big)$

Error covariance meas. update: $\Sigma^+_{\tilde{\mathbb{X}},k} = \Sigma^-_{\tilde{\mathbb{X}},k} - L_k\Sigma_{\tilde{z},k}L_k^T$

Dual SPKF for state and parameter estimation (1)

State-space model:

$$\begin{aligned}
x_k &= f(x_{k-1}, u_{k-1}, w_{k-1}, \theta_{k-1}) \\
z_k &= h(x_k, u_k, v_k, \theta_k)
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
\theta_k &= \theta_{k-1} + r_{k-1}, \\
d_k &= h(x_k,\, u_k, \bar{v}_k, \theta_{k-1}, e_k)
\end{aligned}$$

where w_k, v_k, r_k and e_k are independent, Gaussian noise processes with means $\bar{w}$, $\bar{v}$, $\bar{r}$, and $\bar{e}$, and covariance matrices $\Sigma_{\tilde{w}}$, $\Sigma_{\tilde{v}}$, $\Sigma_{\tilde{r}}$ and $\Sigma_{\tilde{e}}$, respectively

Definitions:

$$x_k^a = \big[x_k^T,\ w_k^T,\ v_k^T\big]^T, \quad \mathcal{X}_k^a = \big[(\mathcal{X}_k^x)^T,\ (\mathcal{X}_k^w)^T,\ (\mathcal{X}_k^v)^T\big]^T, \quad p = 2\times\dim(x_k^a)$$

Dual SPKF for state and parameter estimation (2)

Initialization: For $k = 0$, set

$$\begin{aligned} \hat{\theta}_0^+ &= \mathbb{E}[\theta_0] \\ \hat{x}_0^+ &= \mathbb{E}[x_0] \\ \Sigma_{\tilde{x},0}^+ &= \mathbb{E}\big[(x_0 - \hat{x}_0^+)(x_0 - \hat{x}_0^+)^T\big] \\ \Sigma_{\tilde{\theta},0}^+ &= \mathbb{E}\big[(\theta_0 - \hat{\theta}_0^+)(\theta_0 - \hat{\theta}_0^+)^T\big] \end{aligned}$$

$$\begin{aligned} \hat{x}_0^{a,+} &= \mathbb{E}\big[x_0^a\big] = \big[(\hat{x}_0^+)^T,\ \bar{w},\ \bar{v}\big]^T \\ \Sigma_{\tilde{x},0}^{a,+} &= \mathbb{E}\big[(x_0^a - \hat{x}_0^{a,+})(x_0^a - \hat{x}_0^{a,+})^T\big] \\ &= \mathrm{diag}\big(\Sigma_{\tilde{x},0}^+, \Sigma_w, \Sigma_v\big) \end{aligned}$$

Computation: For $k = 1, 2, \ldots$ compute:

Parameter estimate time update: $\hat{\theta}_k^- = \hat{\theta}_{k-1}^+$

Parameter covariance time update: $\Sigma_{\tilde{\theta},k}^- = \Sigma_{\tilde{\theta},k-1}^+ + \Sigma_{\tilde{r}}$

Dual SPKF for state and parameter estimation (3)

Computation (cont.): For $k = 1, 2, \ldots$ compute:

State time update:

$$\begin{aligned} \mathcal{X}_{k-1}^{a,+} &= \Big\{\hat{x}_{k-1}^{a,+}, \hat{x}_{k-1}^{a,+} + \gamma\sqrt{\Sigma_{\tilde{x},k-1}^{a,+}}, \hat{x}_{k-1}^{a,+} - \gamma\sqrt{\Sigma_{\tilde{x},k-1}^{a,+}}\Big\} \\ \mathcal{X}_{k,i}^{x,-} &= f(\mathcal{X}_{k-1,i}^{x,+}, u_{k-1}, \mathcal{X}_{k-1,i}^{w,+}, \hat{\theta}_k^-) \\ \hat{x}_k^- &= \textstyle\sum_{i=0}^p \alpha_i^{(\mathrm{m})} \mathcal{X}_{k,i}^{x,-} \end{aligned}$$

Covariance time update: $\Sigma_{\tilde{x},k}^- = \sum_{i=0}^p \alpha_i^{(\mathrm{c})} \big(\mathcal{X}_{k,i}^{x,-} - \hat{x}_k^-\big)\big(\mathcal{X}_{k,i}^{x,-} - \hat{x}_k^-\big)^T$

Output, parameter filter:

$$\begin{aligned} \mathcal{W}_k &= \Big\{\hat{\theta}_k^-, \hat{\theta}_k^- + \gamma\sqrt{\Sigma_{\tilde{\theta},k}^-}, \hat{\theta}_k^- - \gamma\sqrt{\Sigma_{\tilde{\theta},k}^-}\Big\} \\ \mathcal{D}_{k,i} &= h(f(\hat{x}_{k-1}^+, u_{k-1}, \bar{w}_{k-1}, \mathcal{W}_{k,i}), u_k, \bar{v}_k, \mathcal{W}_{k,i}) \\ \hat{d}_k &= \textstyle\sum_{i=0}^p \alpha_i^{(\mathrm{m})} \mathcal{D}_{k,i} \end{aligned}$$

Output, state filter:

$$\begin{aligned} \mathcal{Z}_{k,i} &= h(\mathcal{X}_{k,i}^{x,-}, u_k, \mathcal{X}_{k-1,i}^{v,+}, \hat{\theta}_k^-) \\ \hat{z}_k &= \textstyle\sum_{i=0}^p \alpha_i^{(\mathrm{m})} \mathcal{Z}_{k,i} \end{aligned}$$

Dual SPKF for state and parameter estimation (4)

Computation (cont.): For $k = 1, 2, \ldots$ compute:

State filter gain matrix:

$$\begin{aligned} \Sigma_{\tilde{z},k} &= \textstyle\sum_{i=0}^p \alpha_i^{(\mathrm{c})} \big(\mathcal{Z}_{k,i} - \hat{z}_k\big)\big(\mathcal{Z}_{k,i} - \hat{z}_k\big)^T \\ \Sigma_{\tilde{x}\tilde{z},k}^- &= \textstyle\sum_{i=0}^p \alpha_i^{(\mathrm{c})} \big(\mathcal{X}_{k,i}^{x,-} - \hat{x}_k^-\big)\big(\mathcal{Z}_{k,i} - \hat{z}_k\big)^T \\ L_k^x &= \Sigma_{\tilde{x}\tilde{z},k}^- \Sigma_{\tilde{z},k}^{-1} \end{aligned}$$

Param. filter gain matrix:

$$\begin{aligned} \Sigma_{\tilde{d},k} &= \textstyle\sum_{i=0}^p \alpha_i^{(\mathrm{c})} \big(\mathcal{D}_{k,i} - \hat{d}_k\big)\big(\mathcal{D}_{k,i} - \hat{d}_k\big)^T \\ \Sigma_{\tilde{\theta}\tilde{d},k}^- &= \textstyle\sum_{i=0}^p \alpha_i^{(\mathrm{c})} \big(\mathcal{W}_{k,i} - \hat{\theta}_k^-\big)\big(\mathcal{D}_{k,i} - \hat{d}_k\big)^T \\ L_k^\theta &= \Sigma_{\tilde{\theta}\tilde{d},k}^- \Sigma_{\tilde{d},k}^{-1} \end{aligned}$$

State meas. update: $\hat{x}_k^+ = \hat{x}_k^- + L_k^x\big(z_k - \hat{z}_k\big)$

Covariance meas. update: $\Sigma_{\tilde{x},k}^+ = \Sigma_{\tilde{x},k}^- - L_k^x \Sigma_{\tilde{z},k} (L_k^x)^T$

Parameter meas. update: $\hat{\theta}_k^+ = \hat{\theta}_k^- + L_k^\theta\big(z_k - \hat{d}_k\big)$

Covariance meas. update: $\Sigma_{\tilde{\theta},k}^+ = \Sigma_{\tilde{\theta},k}^- - L_k^\theta \Sigma_{\tilde{d},k} (L_k^\theta)^T$