Defining recursive quantities

- Solution to proportional-confidence TLS found to be

$$\widehat{Q}_n = \frac{-c_{1,n} + k^2 c_{3,n} + \sqrt{(c_{1,n} - k^2 c_{3,n})^2 + 4k^2 c_{2,n}^2}}{2k^2 c_{2,n}}, \quad \text{where}$$

$$c_{1,n} = c_{1,n-1} + x_n^2/\sigma_{y_i}^2; \quad c_{2,n} = c_{2,n-1} + x_n y_n/\sigma_{y_i}^2; \quad c_{3,n} = c_{3,n-1} + y_n^2/\sigma_{y_i}^2$$

- Initialized by setting $x_0 = 1$, $y_0 = Q_{\text{nom}}$, $\sigma_{y_i}^2$ to a value representing the uncertainty of total capacity with respect to nominal capacity
- Therefore, $c_{1,0} = 1/\sigma_{y_i}^2$, $c_{2,0} = Q_{\text{nom}}/\sigma_{y_i}^2$ and $c_{3,0} = Q_{\text{nom}}^2/\sigma_{y_i}^2$

Confidence interval of estimate

- The Hessian, required to compute estimate's uncertainty, may also be found in terms of the recursive parameters:

$$\frac{\partial^2 \chi_{\mathrm{TLS}}^2}{\partial \widehat{Q}^2} = \frac{(-4k^4 c_2)\widehat{Q}^3 + 6(k^4 c_3 - c_1 k^2)\widehat{Q}^2 + 12 c_2 k^2 \widehat{Q} + 2(c_1 - k^2 c_3)}{(\widehat{Q}^2 k^2 + 1)^3}$$

- One-sigma bounds on $\widehat{Q}$ are computed as $\left.\sqrt{2/(\partial^2 \chi_{\mathrm{TLS}}^2/\partial \widehat{Q}^2)}\right|_{Q=\widehat{Q}}$

Fading memory

- Fading memory may be easily incorporated recursively via

$$\widehat{Q}_n = \frac{-\tilde{c}_{1,n} + k^2 \tilde{c}_{3,n} + \sqrt{(\tilde{c}_{1,n} - k^2 \tilde{c}_{3,n})^2 + 4k^2 \tilde{c}_{2,n}^2}}{2k^2 \tilde{c}_{2,n}}, \quad \text{where}$$

$$\tilde{c}_{1,n} = \gamma \tilde{c}_{1,n-1} + x_n^2/\sigma_{y_i}^2; \quad \tilde{c}_{2,n} = \gamma \tilde{c}_{2,n-1} + x_n y_n/\sigma_{y_i}^2; \quad \tilde{c}_{3,n} = \gamma \tilde{c}_{3,n-1} + y_n^2/\sigma_{y_i}^2$$

- Initialization unchanged: $\tilde{c}_{1,0} = 1/\sigma_{y_i}^2$, $\tilde{c}_{2,0} = Q_{\text{nom}}/\sigma_{y_i}^2$ and $\tilde{c}_{3,0} = Q_{\text{nom}}^2/\sigma_{y_i}^2$
- Can obtain Hessian in terms of recursive parameters $\tilde{c}_1$ through $\tilde{c}_3$

$$\frac{\partial^2 \chi_{\mathrm{FMTLS}}^2}{\partial \widehat{Q}^2} = \frac{(-4k^4 \tilde{c}_2)\widehat{Q}^3 + 6(k^4 \tilde{c}_3 - \tilde{c}_1 k^2)\widehat{Q}^2 + 12 \tilde{c}_2 k^2 \widehat{Q} + 2(\tilde{c}_1 - k^2 \tilde{c}_3)}{(\widehat{Q}^2 k^2 + 1)^3}$$

Summary

- This TLS solution shares the nice properties of WLS solution:
  1. It gives a closed-form solution for $\widehat{Q}$
     - No iteration or advanced algorithms required—only simple math operations
  2. The solution can be very easily computed in a recursive manner
     - We keep track of the three running sums $c_{1,n}$, $c_{2,n}$ and $c_{3,n}$
     - When an additional data point becomes available, we update the sums and compute an updated total capacity estimate
  3. Fading memory is easily added.
- Unfortunately, this solution does not allow $\sigma_{x_i}^2$ and $\sigma_{y_i}^2$ to be arbitrary—they must be proportionally related by the scaling factor $\sigma_{x_i} = k\sigma_{y_i}$
- Next, we seek to approximate TLS to allow an arbitrary relationship