A path to solution

- General WTLS solution provides excellent results, but is impractical to implement in an embedded system
- Therefore, we search for cases that lead to simpler implementations
- Here, we look at an exact solution when the uncertainties on the x_i and y_i data points are proportional to each other for all i, which leads to a simple solution that can easily be implemented in an embedded system
- With insights from this solution we will next look at an approximate WTLS solution that also has nice implementation properties

Proportional confidence on x_i and y_i

- If $\sigma_{x_i} = k\sigma_{y_i}$, then WTLS cost function reduces to generalization of standard TLS cost function
  - Substitute $\sigma_{x_i} = k\sigma_{y_i}$ into χ^2_{WTLS} and associated results to get:

$$\chi^2_{\mathrm{TLS}} = \sum_{i=1}^{N} \frac{(x_i - X_i)^2}{k^2\sigma^2_{y_i}} + \frac{(y_i - Y_i)^2}{\sigma^2_{y_i}} = \sum_{i=1}^{N} \frac{(y_i - \widehat{Q}x_i)^2}{(\widehat{Q}^2k^2+1)\sigma^2_{y_i}}$$

- Furthermore, Jacobian of WTLS cost function reduces to (again, via $\sigma_{x_i} = k\sigma_{y_i}$)

$$\frac{\partial \chi^2_{\mathrm{TLS}}}{\partial \widehat{Q}} = 2\sum_{i=1}^{N} \frac{(\widehat{Q}x_i - y_i)(\widehat{Q}k^2y_i + x_i)}{(\widehat{Q}^2k^2+1)^2\sigma^2_{y_i}}$$

- This may be solved for an exact solution to $\widehat{Q}$, without requiring iteration to do so

Finding quadratic form

- We first collect terms

$$\frac{\partial \chi^2_{\mathrm{TLS}}}{\partial \widehat{Q}} = 2\sum_{i=1}^{N} \frac{(\widehat{Q}x_i - y_i)(\widehat{Q}k^2y_i + x_i)}{(\widehat{Q}^2k^2+1)^2\sigma^2_{y_i}} = 0$$

$$= \widehat{Q}^2 \underbrace{\sum_{i=1}^{N} k^2\frac{x_iy_i}{\sigma^2_{y_i}}}_{a=k^2c_{2,n}} + \widehat{Q}\underbrace{\sum_{i=1}^{N} \frac{x_i^2 - k^2y_i^2}{\sigma^2_{y_i}}}_{b=c_{1,n}-k^2c_{3,n}} + \underbrace{\sum_{i=1}^{N} \frac{-x_iy_i}{\sigma^2_{y_i}}}_{c=-c_{2,n}} = 0$$

where we define $c_{3,n} = \sum_{i=1}^{n} y_i^2/\sigma^2_{y_i}$

- Then, we can solve for $\widehat{Q}$ using the familiar quadratic equation solution

$$\widehat{Q} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Routh test to choose root of solution

- In terms of familiar recursive quantities, specific solution is

$$\widehat{Q}_n = \frac{-(c_{1,n} - k^2 c_{3,n}) \pm \sqrt{(c_{1,n} - k^2 c_{3,n})^2 + 4k^2 c_{2,n}^2}}{2k^2 c_{2,n}}$$

- Which of the two roots to choose? Can show via Routh array and test that this quadratic always has one positive root and one negative root

$$\begin{array}{c|cc} \widehat{Q}^2 & k^2 c_{2,n} & -c_{2,n} \\ \widehat{Q}^1 & c_{1,n} - k^2 c_{3,n} & 0 \\ \widehat{Q}^0 & -c_{2,n} & 0 \end{array}$$

- First column of the array always has exactly one sign change, so there is one root of the polynomial in the right-half plane (and other in left-half plane or on axis)

Total capacity must be nonnegative!

- So there is one root of the polynomial in the right-half plane (and other in left-half plane or on axis)
- By the fundamental theorem of algebra, because the coefficients $c_{1,n}$, $c_{2,n}$, and $c_{3,n}$ are real, the polynomial roots must either both be real or be complex conjugates
- The fact that they are in different halves of the complex plane shows that they cannot be complex conjugates, and therefore must both be real
- We choose larger root of the quadratic equation, which corresponds to positive root

$$\widehat{Q}_n = \frac{-(c_{1,n} - k^2 c_{3,n}) + \sqrt{(c_{1,n} - k^2 c_{3,n})^2 + 4k^2 c_{2,n}^2}}{2k^2 c_{2,n}}$$

Summary

- You have already learned that WLS is biased but general WTLS is computationally inefficient
- But, if uncertainties on x_i and y_i are proportional for all i, then WTLS has simple quadratic form that can be written in terms of recursively computed quantities
  - Notice that we refer to this specific condition as the TLS problem, as distinct from the earlier more-general WTLS problem
- Next lesson will summarize the TLS solution in more specific detail