Real-Time Systems

Detailed Algebra for Rate Monotonic Least Upper Bound

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Rate Monotonic Least Upper Bound for 2 Tasks

Claim - General RM Least Upper Bound:

(Guarantees that all Service Releases Can meet Deadlines)

$$U = \sum_{i=1}^{m} (Ci / Ti) \le m(2^{\frac{1}{m}} - 1)$$

Goal - Derive RM LUB For 2 Tasks:

$$U = C_1 / T_1 + C_2 / T_2 \le 2(2^{\frac{1}{2}} - 1) \le 0.83$$

Free CPU

- MARGIN for errors!
- Idle time for best effort services

30% Free CPU Here

- Below the RM LUB
- Margin provide Safety
- Schedule works - feasible

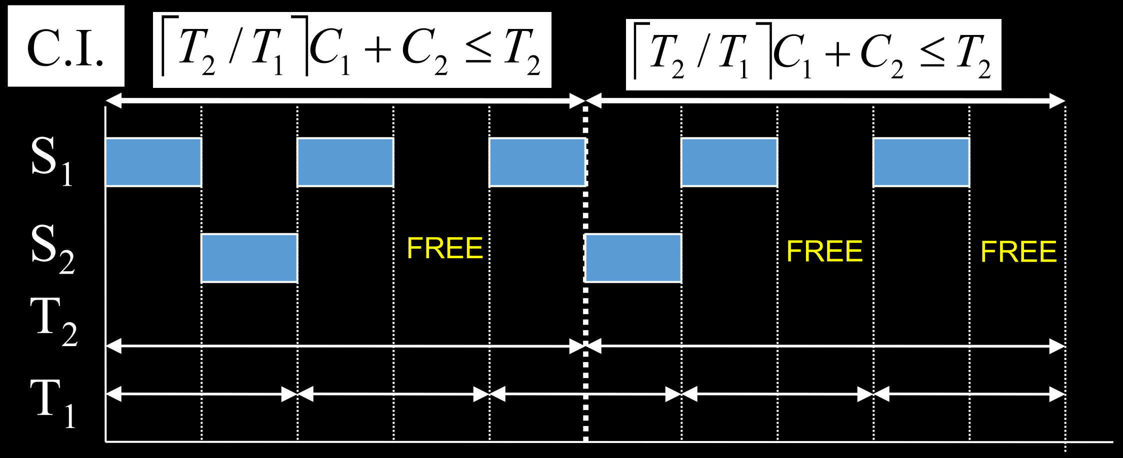
For a System, Can All C's fit in largest T over LCM time?

Given: Services S_1, S_2 with periods T_1 and T_2 and C_1 and C_2, Assume $T_2 > T_1$

E.g. T_1=2, T_2=5, C_1=1, C_2= 1, then if prio(S_1) > prio(S_2), we can see that ...

$$U = 1/2 + 1/5 = 0.7$$

$$U = 0.7 \le 2(2^{\frac{1}{2}} - 1) \le 0.83$$



Recall RM Assumptions & Constraints

- A1: All Services Requested on Periodic Basis, the Period is Constant
- A2: Completion-Time < Period
- A3: Service Requests are Independent (No Known Phasing)
- A4: Run-time is Known and Deterministic (WCET may be Used)
- C1: Deadline = Period by Definition
- C2: Fixed Priority, Preemptive, Run-to-Completion Scheduling
- Critical Instant: longest response time for a service occurs when all system services are requested simultaneously (maximum interference case for lowest priority service)
- No Other Shared Resources – Not in Paper, but key assumption – e.g. shared memory

Recall Part 1 - RM LUB Derivation

Given Cases 1 and 2 (T_1=2, T_2=5)

Case 1

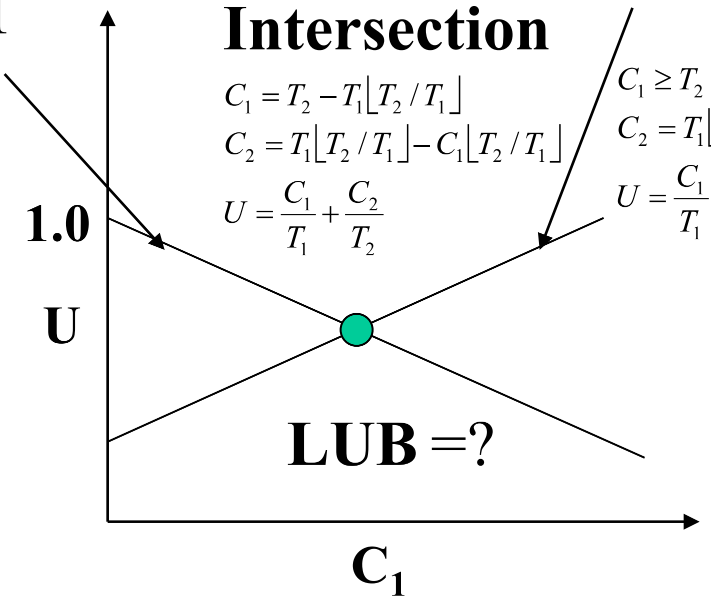
$$C_1 \le T_2 - T_1\lfloor T_2/T_1 \rfloor$$
$$C_2 = T_2 - C_1\lceil T_2/T_1 \rceil$$
$$U = \frac{C_1}{T_1} + \frac{C_2}{T_2}$$

Intersection

$$C_1 = T_2 - T_1\lfloor T_2/T_1 \rfloor$$
$$C_2 = T_1\lfloor T_2/T_1 \rfloor - C_1\lfloor T_2/T_1 \rfloor$$
$$U = \frac{C_1}{T_1} + \frac{C_2}{T_2}$$

Case 2

$$C_1 \ge T_2 - T_1\lfloor T_2/T_1 \rfloor$$
$$C_2 = T_1\lfloor T_2/T_1 \rfloor - C_1\lfloor T_2/T_1 \rfloor$$
$$U = \frac{C_1}{T_1} + \frac{C_2}{T_2}$$



T_1=	2	T_2=	5
Case 1		**Case 2**	
C_1	U	C_1	U
0	1	0	0.8
0.1	0.99	0.1	0.81
0.2	0.98	0.2	0.82
0.3	0.97	0.3	0.83
0.4	0.96	0.4	0.84
0.5	0.95	0.5	0.85
0.6	0.94	0.6	0.86
0.7	0.93	0.7	0.87
0.8	0.92	0.8	0.88
0.9	0.91	0.9	0.89
1	0.9	1	0.9

Case 1: $U = 1 + C_1\left[(1/T_1) - \frac{\lceil T_2/T_1 \rceil}{T_2}\right]$

Case 2: $U = (T_1/T_2)\lfloor T_2/T_1 \rfloor + C_1\left[(1/T_1) - (1/T_2)\lfloor T_2/T_1 \rfloor\right]$

Note: we want the LUB for any given T_2 and T_1, not the ones assumed here in particular, so the general LUB could be something other than 0.9, and must be found in terms of T_1 and T_2 only for general LUB.

Detailed Algebra Notes (p. 82)

Take simple equations at intersection

$$C_1 = T_2 - T_1\lfloor T_2/T_1 \rfloor$$

$$C_2 = T_2 - C_1\lceil T_2/T1 \rceil$$

$$U = \frac{C_1}{T_1} + \frac{C_2}{T_2}$$

Plug C_1 and C_2 into U equation

$$U = \frac{T_2 - T_1\lfloor T_2/T_1 \rfloor}{T_1} + \frac{T_2 - C_1\lceil T_2/T_1 \rceil}{T_2}$$

Substitute in definition of C_1

$$U = \frac{T_2 - T_1\lfloor T_2/T_1 \rfloor}{T_1} + \frac{T_2 - (T_2 - T_1\lfloor T_2/T_1 \rfloor)\lceil T_2/T_1 \rceil}{T_2}$$

$$U = \left(\frac{T_2}{T_1}\right) - \left\lfloor \frac{T_2}{T_1} \right\rfloor + (T_2/T_2) - \frac{T_2\lceil T_2/T_1 \rceil + T_1\lfloor T_2/T_1 \rfloor\lceil T_2/T_1 \rceil}{T_2}$$

$$U = \left(\frac{T_2}{T_1}\right) - \left\lfloor \frac{T_2}{T_1} \right\rfloor + 1 - \lceil T_2/T_1 \rceil + (T_1/T_2)\lfloor T_2/T_1 \rfloor\lceil T_2/T1 \rceil$$

Re-arrange the terms

$$U = 1 - \lceil T_2/T_1 \rceil + (T_1/T_2)\lfloor T_2/T_1 \rfloor\lceil T_2/T_1 \rceil + \left(\frac{T_2}{T_1}\right) - \left\lfloor \frac{T_2}{T_1} \right\rfloor$$

Pull out -(T_1/T_2)

$$U = 1 + -(T_1/T_2)\left((T_2/T_1)\lceil T_2/T_1 \rceil - \lfloor T_2/T_1 \rfloor\lceil T_2/T_1 \rceil - \left(\frac{T_2}{T_1}\right)^2 + \left(\frac{T2}{T1}\right)\left\lfloor \frac{T_2}{T_1} \right\rfloor\right)$$

Factor into

$$U = 1 - (T_1/T_2)\left[\lceil T_2/T_1 \rceil - (T_2/T_1)\right]\left[(T_2/T_1) - \lfloor T_2/T_1 \rfloor\right]$$

We get U in terms of T_1 and T_2 only

RM LUB Derivation

Intersection of Case 1 & 2 is Least Upper Bound

Plug In Intersection for C_1 and C_2 into U to get expression in terms of T_1 and T_2 only: $U = 1 - (T_1/T_2)[\lceil T_2/T_1 \rceil - (T_2/T_1)][(T_2/T_1) - \lfloor T_2/T_1 \rfloor]$

(Substitute for both C_1 and C_2 in U expression and simplify)

Let $I = \lfloor T_2/T_1 \rfloor$ and $f = (T_2/T_1) - \lfloor T_2/T_1 \rfloor$ so, $U = 1 - \left(\frac{f(1-f)}{(T_2/T_1)} \right)$

I is the integer number of times that T_1 occurs during T_2

f is the fractional time of the last release for T_1 during T2, noting that if ***f*** =0, then T_1 and T_2 are harmonic, and therefore U=1, an uninteresting ideal case.

Substituting ***I*** and ***f*** into the U expression above and simplifying, we get:

$U = 1 - (T_1/T_2)[\lceil T_2/T_1 \rceil - (T_2/T_1)][(T_2/T_1) - \lfloor T_2/T_1 \rfloor]$ (noting that 1+floor(N+/-0.d) = ceiling(N+/-0.d) when f non-zero)

$U = 1 - (T_1/T_2)[1 + \lfloor T_2/T_1 \rfloor - (T_2/T_1)][(T_2/T_1) - \lfloor T_2/T_1 \rfloor]$

$U = 1 - (T_1/T_2)[1 - ((T_2/T_1) - \lfloor T_2/T_1 \rfloor)][(T_2/T_1) - \lfloor T_2/T_1 \rfloor]$

So, $U = 1 - (T_1/T_2)(1-f)(f)$, Re-arranged to obtain: $U = 1 - \left(\frac{f(1-f)}{(T_2/T_1)} \right)$

RM LUB Derivation

$U = 1 - \left(\frac{f(1-f)}{(T_2/T_1)} \right)$ Can also be expressed as:

$U = 1 - \left(\frac{f(1-f)}{\lfloor T_2/T_1 \rfloor + (T_2/T_1) - \lfloor T_2/T_1 \rfloor} \right)$ By adding and subtracting the same denominator term to get:

$U = 1 - \left(\frac{f(1-f)}{(I+f)} \right)$ smallest I is 1, and LUB for U occurs when I is minimized, so: $U = 1 - \left(\frac{(f-f^2)}{(1+f)} \right)$

Now taking the derivative of U w.r.t. f, and solving for extreme, we get:

$$dU/df = \frac{(1+f)(1-2f) - (f-f^2)(1)}{(1+f)^2} = 0$$

Solving for *f*, we get: $f = (2^{1/2} - 1)$

And, plugging f back into U, we get: $U = 2(2^{1/2} - 1)$ The RM LUB!

Detailed Algebra Notes

1. From the RMA LUB derivation completed in class, we know utility is ideally 100% if there is no fractional interference, f=0, from:

   $U = 1 - \left(\frac{f(1-f)}{\left({}^{T_2}/_{T_1}\right)}\right), I = \left\lfloor {}^{T_2}/_{T_1} \right\rfloor, f = \left({}^{T_2}/_{T_1}\right) - \left\lfloor {}^{T_2}/_{T_1} \right\rfloor$

2. Knowing the above, we arrive at the equations below by the algebraic steps shown:

   $U = 1 - \left(\frac{f(1-f)}{(I+f)}\right)$ *can be derived from* $I, f,$ *and* U *above as follows*:

   $U = 1 - \left(\frac{f(1-f)}{\left({}^{T_2}/_{T_1}\right)}\right)$, *which is* $1 - \left(\frac{f(1-f)}{\left\lfloor {}^{T_2}/_{T_1} \right\rfloor + \left({}^{T_2}/_{T_1}\right) - \left\lfloor {}^{T_2}/_{T_1} \right\rfloor}\right)$, *which is* $1 - \left(\frac{f(1-f)}{(I+f)}\right)$

3. Furthermore, the m=2 LUB of: $f = \left(2^{1/2} - 1\right)$, *and* $U = 2\left(2^{1/2} - 1\right)$ is found by:

   $U = 1 - \left(\frac{f(1-f)}{(I+f)}\right), I = 1$ *for worst case, so* $U = 1 - \left(\frac{f(1-f)}{(1+f)}\right), \frac{dU}{df} = \frac{(1+f)(1-2f)-\left(f-f^2\right)(1)}{(1+f)^2} = 0$

   so $1 - 2f + f - 2f^2 - f + f^2 = 0$, *so* $1 - 2f - f^2 = 0$, *so* $1 = f^2 + 2f$, so $f^2 + 2f + 1 = 2$ and therefore $(f+1)^2 = 2$, *so* $f + 1 = \sqrt{2}$, *so* $f = \sqrt{2} - 1$

   so $U = 1 - \left(\frac{f(1-f)}{(1+f)}\right) = \left(\frac{1+f-f+f^2}{(1+f)}\right) = \left(\frac{1+f^2}{(1+f)}\right) = \frac{1+2-2\sqrt{2}+1}{1+\sqrt{2}-1} = \frac{4-2\sqrt{2}}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right) = \frac{4\sqrt{2}-4}{2} = 2(\sqrt{2}-1)$

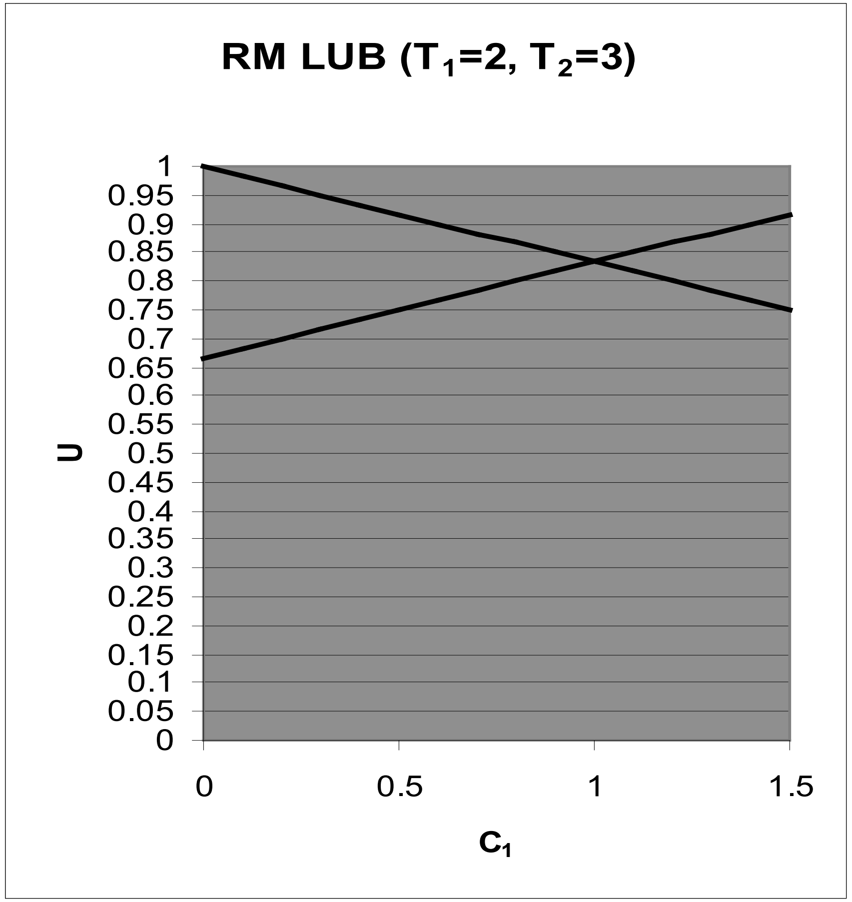
RM LUB Scenario for 2 Services

(With T_1=2, T_2=3, We Show U=0.83 Worst-Case Graphically)

Given Case 1: $U = (T_1 / T_2)\lfloor T_2 / T_1 \rfloor + C_1[(1/T_1) - (1/T_2)\lfloor T_2 / T_1 \rfloor]$

And Case 2: $U = 1 + C_1\left[(1/T_1) - \frac{\lceil T_2 / T_1 \rceil}{T_2}\right]$

T_1=	2		T_2=	3	
Case 1			Case 2		
C_1	U	C_2	C_1	U	C_2
0	1	3	0	0.666667	2
0.1	0.983333	2.8	0.1	0.683333	1.9
0.2	0.966667	2.6	0.2	0.7	1.8
0.3	0.95	2.4	0.3	0.716667	1.7
0.4	0.933333	2.2	0.4	0.733333	1.6
0.5	0.916667	2	0.5	0.75	1.5
0.6	0.9	1.8	0.6	0.766667	1.4
0.7	0.883333	1.6	0.7	0.783333	1.3
0.8	0.866667	1.4	0.8	0.8	1.2
0.9	0.85	1.2	0.9	0.816667	1.1
1	0.833333	1	1	0.833333	1
1.1	0.816667	0.8	1.1	0.85	0.9
1.2	0.8	0.6	1.2	0.866667	0.8
1.3	0.783333	0.4	1.3	0.883333	0.7
1.4	0.766667	0.2	1.4	0.9	0.6
1.5	0.75	0	1.5	0.916667	0.5



RM LUB Scenario for 2 Services

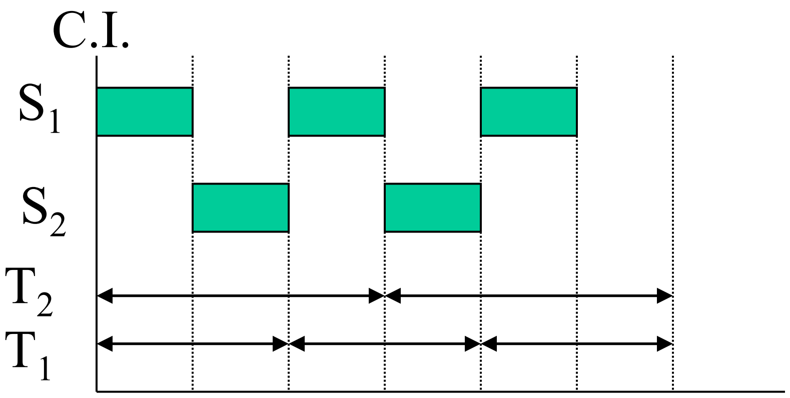
(Showing U=0.83 Worst-Case Timing Diagram)

<u>Given Cases 1 and 2 again:</u>

T_1=2, T_2=3, C_1=1, C_2=1 at U=0.83

Note: 5 out of 6 Time Units Used Over LCM

Such That U=0.83, the RM LUB!



RM FAQ

- Where in the paper does it say that T must equal D?
  - T=D is in section 4 above Theorem 1 in Liu and Layland paper
- Why does Floor(x.d)+1 = Ceiling(x.d)?, because if x=1 or any integer, this is not true.
  - Floor(x.d) + 1 = Ceiling(x.d) iff x is not an integer value, which is always true if T_2 is not a multiple of T_1
- If $T_2 > T_1$, but T_2 is a multiple of T_1 (if T_1 and T_2 are harmonic), then U=1, and doesn't this violate the RM LUB?
  - Yes, it does, but remember the RM LUB is only sufficient, not N&S, and therefore it will pessimistically fail service sets that can actually be scheduled
  - The RM LUB will correctly fail all service sets than in fact can't be scheduled, so it is sufficient
- Why Not Just Use EDF Scheduling Policy if it Can Achieve 100% Utility and Be Shown to Meet Deadlines?
  - What Happens When EDF is Overloaded?

Dynamic Priority

- EDF [Earliest Deadline First] – When Ready Queue is Updated, the Scheduler Must Re-assign all Priorities According to Each Service’s Time Remaining to Deadline

- LLF [Least Laxity First] - Ready Queue is Updated, the Scheduler Must Re-assign all Priorities According to Each Service’s (Time Remaining to Deadline - Execution Time Remaining)

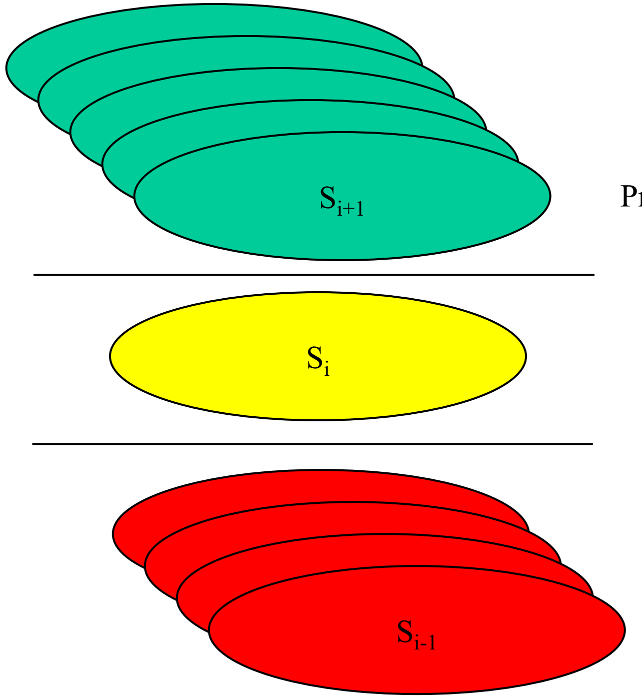
Dynamic Priority Issues

- Earliest Deadline First Policy (aka Deadline Driven Scheduling)
  - ***Can Be High Overhead*** - Every time the Ready Queue is Updated, the Scheduler Must Re-assign all Priorities According to Each Service's Time Remaining to Deadline and Perform a Priority Preemption.

  - ***Overload Failure Modes Non-Deterministic*** - In an Overload Scenario, EDF Can Have Difficult to Predict and Cascading Multi-Service Failures
    - Overload with RM Policy is Deterministic - the Lowest Priority Service Will Fail First and The Number of Services Failing From Lowest Priority Up Can be Determined by Duration of Over-run and Priority of Over-running Service
      - Over-run Interference Has Known Impact on Lower Priority Services
      - Over-runs Will Be Preempted By Higher Priority Services if Allowed to Continue
    - Overload with EDF Policy is Non-Deterministic
      - Time to Deadline for an Over-running Service is Zero or Negative?
      - Over-running Service Will Maintain High Priority Based on Time to Deadline
      - Additional Services that Over-run Due to Interference Also Have Elevated Priority
      - Chaining Priority Amplification and Over-runs Can Cause Cascading Failure
    - Over-run Termination Policy May Help (Worst Case Interference Margin)

  - ***Difficult to Debug Dynamic Priority When Deadlines Are Missed***

RM Fixed Prio Overload

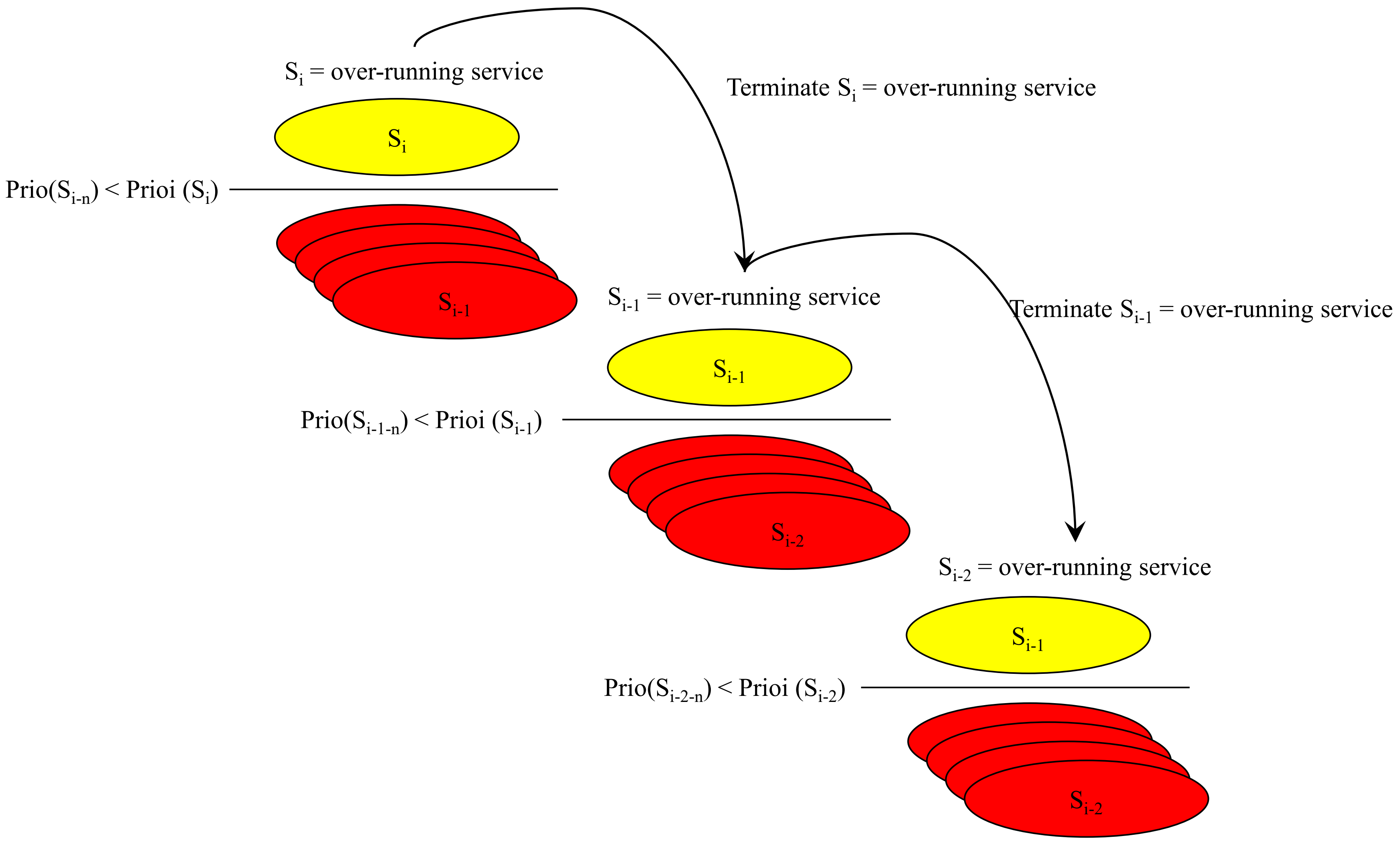


Prio(S_{i+n}) > Prioi (S_i)

S_i = over-running service

Prio(S_{i-n}) < Prioi (S_i)

EDF Dynamic Prio Overload



Dynamic Priority Issues Cont'd

- Least Laxity First Policy
  - ***Can Be High Overhead*** - Every time the Ready Queue is Updated, the Scheduler Must Re-assign all Priorities According to Each Service's (Time Remaining to Deadline - Execution Time Remaining) and Perform a Priority Preemption.
  - ***Execution Time Remaining Hard to Estimate*** - Compared to EDF, Least Laxity First is More Difficult to Implement Since Execution Time Remaining for All Services Must Be Estimated to Re-assign Priorities
  - ***Overload Failure Modes Better, But Still Non-Deterministic*** - In an Overload Scenario, LLF May Behave More Favorably Since it Encodes the Concept of Dispatching the Service with the Most Pressing Need for the CPU Resource (I.e. Remaining Execution Time is Factored In)
    - Over-run Invalidates Estimates of Execution Time Remaining
    - Suffers Same Potential Cascading Priority Amplification and Interference on Over-runs
    - Over-run Termination Policy May Help
  - ***Difficult to Debug Dynamic Priority When Deadlines Are Missed***

Example #0
(Lots of Slack Time)

Example 0	T1	2	C1	1	U1	0.5	LCM =	30							
	T2	10	C2	1	U2	0.1									
	T3	15	C3	2	U3	0.133333	Utot =	0.733333							
RM Schedule															
S1															
S2															
S3															
EDF Schedule															
S1															
S2															
S3															
TTD															
S1	2	X	2	X	2	X	2	X	2	X	2	X	2	X	2
S2	10	9	X	X	X	X	X	X	X	X	X	X	X	X	X
S3	15	14	13	12	11	10	X	X	X	X	X	X	X	X	X
LLF Schedule															
S1															
S2															
S3															
Laxity															
S1	1	X	1	X	1	X	1	X	1	X	1	X	1	X	1
S2	9	8	X	X	X	X	X	X	X	X	X	X	X	X	X
S3	13	12	11	10	10	9	X	X	X	X	X	X	X	X	X

- ***RM LUB = $3*(2^{1/3}-1) = 0.78$***

Example #1
(RM Policy Failure)

Example 1	T1	2	C1	1	U1	0.5	LCM =	70		
	T2	5	C2	1	U2	0.2				
	T3	7	C3	2	U3	0.285714	Utot =	0.985714		
RM Schedule										
S1							????????			
S2						????????				
S3								LATE		
EDF Schedule										
S1										
S2										
S3										
TTD										
S1	2	X	2	X	2	X	2	X	2	X
S2	5	4	X	X	X	5	4	3	X	X
S3	7	6	5	4	3	2	X	7	6	5
LLF Schedule										
S1										
S2										
S3										
Laxity										
S1	1	X	1	X	1	X	1	X	1	X
S2	4	3	X	X	X	4	3	2	X	X
S3	5	4	3	2	2	1	X	5	4	3

Example #2
(Highly Loaded – RM S4 Over-runs)

Example 2	T1	2	C1	1	U1	0.5	LCM =	70						
	T2	5	C2	1	U2	0.2								
	T3	7	C3	1	U3	0.142857								
	T4	13	C4	2	U4	0.153846	Utot =	0.996703						
RM Schedule														
S1													????????	
S2														
S3														
S4														FAILURE
EDF Schedule														
S1														
S2														
S3														
S4														
TTD														
S1	2	X	2	X	2	X	2	X	2	X	2	X	2	X
S2	5	4	X	X	X	5	X	X	X	X	5	4	3	2
S3	7	6	5	4	X	X	X	7	6	5	4	3	X	X
S4	13	12	11	10	9	8	7	6	5	4	X	X	X	X
LLF Schedule														
S1														
S2														
S3														
S4														
Laxity														
S1	1	X	1	X	1	X	1	X	1	X	1	X	1	X
S2	4	3	X	X	X	4	X	X	X	X	4	3	2	1
S3	6	5	4	3	X	X	X	6	5	4	X	X	X	X
S4	11	10	9	8	7	6	5	4	4	3	2	1	X	X

Example #3
(3 Policies, 3 Common Schedules)

Example 3	T1	3	C1	1	U1	0.33	LCM =	15							
	T2	5	C2	2	U2	0.4									
	T3	15	C3	3	U3	0.2	Utot =	0.93							
RM Schedule															
S1															
S2															
S3															
EDF Schedule															
S1															
S2															
S3															
TTD															
S1	3	X	X	3	X	X	3	X	X	3	X	X	3	X	X
S2	5	4	3	X	X	5	4	3	X	X	5	4	X	X	X
S3	15	14	13	12	11	10	9	8	7	6	5	4	3	2	X
LLF Schedule															
S1															
S2															
S3															
Laxity															
S1	2	X	X	2	X	X	2	X	X	2	X	X	2	X	X
S2	3	2	2	X	X	3	3	2	X	X	3	3	X	X	X
S3	12	11	10	9	8	8	7	6	5	5	4	3	3	2	X

Example #4
(Harmonic 1x, 4x, 8x – $\sum U=1.0$)

Example 4	T1	2	C1	1	U1	0.5	LCM =	16								
	T2	4	C2	1	U2	0.25										
	T3	16	C3	4	U3	0.25	Utot =	1								
RM Schedule																
S1																
S2																
S3																
EDF Schedule																
S1																
S2																
S3																
TTD																
S1	2	X	2	X	2	X	2	X	2	X	2	X	2	X	2	X
S2	4	3	X	X	4	3	X	X	4	3	X	X	4	3	X	X
S3	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1
LLF Schedule																
S1																
S2																
S3																
Laxity																
S1	1	X	1	X	1	X	1	X	1	X	1	X	1	X	1	X
S2	3	2	X	X	3	2	X	X	3	2	X	X	3	2	X	X
S3	12	11	10	9	9	8	7	6	6	5	4	3	3	2	1	0

Example #5
(Harmonic 1x, 2x, 5x – ∑U=1.0)

Example 5	T1	2	C1	1	U1	0.5	LCM =	10		
	T2	5	C2	2	U2	0.4				
	T3	10	C3	1	U3	0.1	Utot =	1		
RM Schedule										
S1										
S2										
S3										
EDF Schedule										
S1										
S2										
S3										
TTD										
S1	2	X	2	X	2	X	2	X	2	X
S2	5	4	3	2	X	5	4	3	X	X
S3	10	9	8	7	6	5	4	3	2	1
LLF Schedule										
S1										
S2										
S3										
Laxity										
S1	1	X	1	X	1	X	1	X	1	X
S2	3	2	2	1	X	3	3	2	X	X
S3	9	8	7	6	5	4	3	2	1	0

Recall - Harmonics

A **harmonic** of a wave is a component frequency of the signal that is an integer multiple of the fundamental frequency, i.e. if the fundamental frequency is *f*, the harmonics have frequencies 2*f*, 3*f*, 4*f*, . . . etc. The harmonics have the property that they are all periodic at the fundamental frequency, therefore the sum of harmonics is also periodic at that frequency. As multiples of the fundamental frequency, successive harmonics can be found by repeatedly adding the fundamental frequency. For example, if the fundamental frequency (first harmonic) is 25 Hz, the frequencies of the next harmonics are: 50 Hz (2nd harmonic), 75 Hz (3rd harmonic), 100 Hz (4th harmonic) etc.

https://en.wikipedia.org/wiki/Harmonic