Statistical Inference and Hypothesis Testing in Data Science Applications

DTSA 5003 offered on Coursera

by the University of Colorado, Boulder

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Week One Lesson Six

A First Test

A Generalization

This isn't really, as the title suggests, our first test. It is a generalization of the test in the previous lesson.

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the $N(\mu, \sigma^2)$ distribution where σ^2 is known.

Let's construct a test of size (level of significance) α for

$$H_0 : \mu = \mu_0 \qquad \text{versus} \qquad H_1 : \mu = \mu_1$$

where μ_0 and μ_1 are fixed and known.

Since μ_0 and μ_1 are known, we can tell which is smaller and which is larger. Here, let's first suppose that $\mu_0 < \mu_1$.

Step One: Choose a statistic on which to base the test.

Since this is a test concerning a population mean, we will choose to base this test on the sample mean $\overline{X}$.

Step Two: Give the "form" of the test.

Since the distribution under H_1 has the larger mean, we expect values sampled from the distribution to be larger when H_1 is true. In particular, we expect that the sample mean will be larger than it would be if H_0 were true. We will

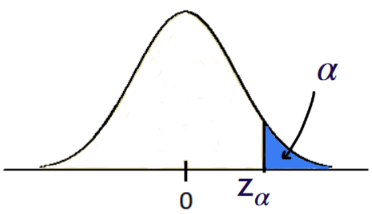
"Reject H_0, in favor of H_1, if $\overline{X} > c$ for some c to be determined."

Step Three: Find c.

$$\begin{aligned}
\alpha &= P(\text{ Type I Error }) \\
&= P(\text{ Reject } H_0 \text{ when it's true }) \\
&= P(\overline{X} > c \text{ when } \mu = \mu_0 \,) \\
&= P\left(\frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}} > \frac{c-\mu_0}{\sigma/\sqrt{n}} \text{ when } \mu = \mu_0 \right) \\
&= P\left(Z > \frac{c-\mu_0}{\sigma/\sqrt{n}} \right)
\end{aligned}$$

where $Z \sim N(0,1)$.

What number is Z greater than with probability α? This is our definition of the critical value z_α.



That is, we must have

$$\frac{c-\mu_0}{\sigma/\sqrt{n}} = z_\alpha,$$

which gives us that

$$c = \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}.$$

Step Four: Conclusion

"We reject H_0, in favor of H_1 if $\overline{X} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$."

If we were testing

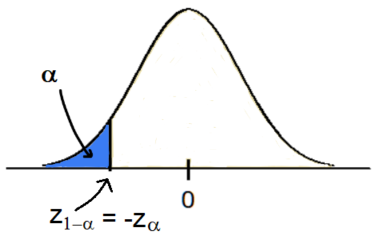
$$H_0 : \mu = \mu_0 \qquad \text{versus} \qquad H_1 : \mu = \mu_1$$

where $\mu_0 > \mu_1$, the alternative hypothesis would seem to be true if $\overline{X}$ is "small". That is, we would want to reject H_0, in favor of H_1 if $\overline{X} < c$ for some c.

Step Three of the test would give us that

$$\alpha = P\left(Z < \frac{c - \mu_0}{\sigma/\sqrt{n}}\right)$$

The critical value that captures area α to the left for a standard normal curve will capture area $1-\alpha$ to the right. Using our already established notation, it is called $z_{1-\alpha}$. However, by symmetry of the $N(0,1)$ distribution about 0, we have that $z_{1-\alpha} = -z_\alpha$.



Our test of size (level of significance) α of

$$H_0 : \mu = \mu_0 \qquad \text{versus} \qquad H_1 : \mu = \mu_1$$

where $\mu_0 > \mu_1$ is to

"Reject H_0, in favor of H_1 if $\overline{X} < \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}}$."