

Let X_1, X_2, \dots, X_n be a random sample from the normal distribution with mean μ and known variance σ^2 .

Consider testing the simple versus simple hypotheses

$$H_0 : \mu = \mu_0 \quad H_1 : \mu = \mu_1$$

where μ_0 and μ_1 are fixed and known.

Suppose that $\mu_0 < \mu_1$.

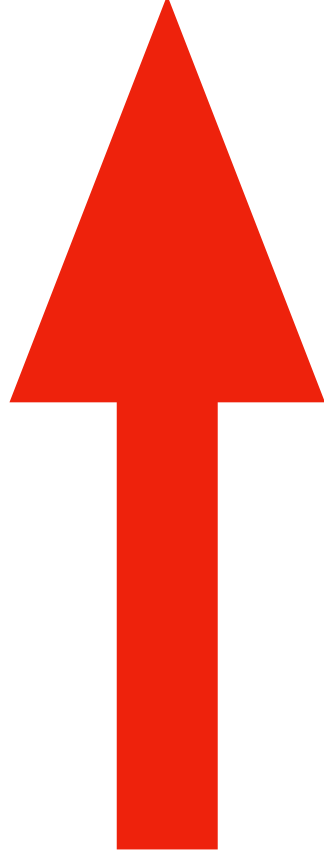
The Test:

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu = \mu_1$$

$$\mu_0 < \mu_1$$

Reject H_0 , in favor of H_1 if

$$\bar{X} > \mu_0 + z_{\alpha} \frac{\sigma}{\sqrt{n}}$$


came from the
probability of
making a Type I error

The Test:

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu = \mu_1$$

$$\mu_0 < \mu_1$$

Reject H_0 , in favor of H_1 if

$$\bar{X} > \mu_0 + z_{\alpha} \frac{\sigma}{\sqrt{n}}$$

Question:

What about the Type II error?

Type II Error

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu = \mu_1$$

$$\mu_0 < \mu_1$$

Your Decision

fail to reject H_0 reject H_0

H_0 true



Type I error

H_0 false

Type II error



Type II Error

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu = \mu_1$$

$$\mu_0 < \mu_1$$

- It is locked in!

$$\beta = P(\text{Type II Error})$$

$$= P(\text{Fail to Reject } H_0 \text{ when false})$$

$$= P\left(\bar{X} \leq \mu_0 + z_{\alpha} \frac{\sigma}{\sqrt{n}} \text{ when } \mu = \mu_1\right)$$

$$= P\left(\bar{X} \leq \mu_0 + z_{\alpha} \frac{\sigma}{\sqrt{n}}; \mu_1\right)$$

Type II Error

- It is locked in!

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu = \mu_1$$

$$\mu_0 < \mu_1$$

$$\beta = P \left(\bar{X} \leq \mu_0 + z_{\alpha} \frac{\sigma}{\sqrt{n}} ; \mu_1 \right)$$

$$\bar{X} \sim N(\mu_1, \sigma^2/n)$$

$$= P \left(\frac{\bar{X} - \mu_1}{\sigma/\sqrt{n}} \leq \frac{\mu_0 + z_{\alpha} \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma/\sqrt{n}} ; \mu_1 \right)$$

Type II Error

- It is locked in!

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu = \mu_1$$

$$\mu_0 < \mu_1$$

$$\beta = P \left(\frac{\bar{X} - \mu_1}{\sigma/\sqrt{n}} \leq \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma/\sqrt{n}} ; \mu_1 \right)$$

$$= P \left(Z \leq \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma/\sqrt{n}} \right)$$

Type II Error

- It is locked in!

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu = \mu_1$$

$$\mu_0 < \mu_1$$

$$\beta = P \left(Z \leq \underbrace{\frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma/\sqrt{n}}} \right)$$

This is a fixed number, so
compute the probability and
that's your β !

Type II Error

- It is locked in!

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu = \mu_1$$

$$\mu_0 < \mu_1$$

$$\beta = P \left(Z \leq \frac{\mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma/\sqrt{n}} \right)$$

We could create the entire test starting from the “ β point of view” and then α would be locked in.

Type II Error

- It is locked in!

$$H_0 : \mu = \mu_0$$

$$H_1 : \mu = \mu_1$$

$$\mu_0 < \mu_1$$



$$\beta = P \left(Z \leq \frac{\mu_0 + z_{\alpha} \frac{\sigma}{\sqrt{n}} - \mu_1}{\sigma / \sqrt{n}} \right)$$

If we want to set both α and β we would have to free up the sample size as another unknown. (c and n)

Type II Error

Your Decision

fail to reject H_0 reject H_0

H_0 true		Type I error
H_0 false	Type II error	

Note: $\beta \neq 1 - \alpha$

Composite vs Composite

$$X_1, X_2, \dots, X_n \sim N(\mu, \sigma^2), \quad \sigma^2 \text{ known}$$

$$H_0 : \mu \leq \mu_0 \quad \text{vs} \quad H_1 : \mu > \mu_0$$

Step One:

$$H_0 : \mu \leq \mu_0$$

$$H_1 : \mu > \mu_0$$

Choose an estimator for μ .

$$\hat{\mu} = \bar{X}$$

Step Two:

Give the “form” of the test.

Reject H_0 , in favor of H_1 if $\bar{X} > c$,
where c is to be determined.

Step Three:

$$H_0 : \mu \leq \mu_0$$

$$H_1 : \mu > \mu_0$$

Find c.

$$\alpha = P(\text{Type I Error})$$

$$= P(\text{Reject } H_0 \text{ when true})$$

$$= P(\bar{X} > c \text{ when } \mu \leq \mu_0)$$

= ?

The definitions we have used for α and β are for simple hypotheses only.

Definitions:

- The **level of significance** or “**size**” of a test is denoted by α and is defined by

$$\alpha = \max P(\text{Type I Error})$$

$$= \max_{\mu \in H_0} P(\text{Reject } H_0; \mu)$$

$$\beta = \max P(\text{Type II Error})$$

$$= \max_{\mu \in H_1} P(\text{Fail to Reject } H_0; \mu)$$

Definitions:

- $1 - \beta$ is known as the
power of the test

$$1 - \beta = 1 - \max_{\mu \in H_1} P(\text{Fail to Reject } H_0; \mu)$$

$$= \min_{\mu \in H_1} (1 - P(\text{Fail to Reject } H_0; \mu))$$

$$= \min_{\mu \in H_1} P(\text{Reject } H_0; \mu)$$

High power
is good!