Discrete Random Variable

x	1	2	3
$P(X = x)$	1/4	1/4	1/2

$$E[X] = (1)(1/4) + (2)(1/4) + (3)(1/2)$$

$$= 9/4 = 2.25$$

$$\boxed{E[X] = \sum_x x \cdot P(X = x)}$$ (**discrete**)

$f(x)$

Continuous analogue:

$$\boxed{E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx}$$

Notation:

$$\mu = E[X]$$

or $\mu_X = E[X]$, $\mu_Y = E[Y]$

Example:

$$X \sim N(\mu, \sigma^2)$$

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

$$= \cdots = \mu$$

Example:

$$X \sim \exp(\lambda), \quad \lambda > 0$$

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{0} x \cdot 0\, dx + \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x}\, dx$$

$$\vdots$$

$$= 1/\lambda$$

- X = interarrival time
- arrival rate of 2 per hour $(\lambda = 2)$
- average/mean time between arrivals is 1/2 hour

“Rate” Versus “Mean” Parameterization of the Exponential Distribution

$X \sim \exp(\lambda)$ means

$f(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$ and

$E[X] = 1/\lambda$

λ is a rate parameter

$\boxed{X \sim \exp(\text{rate} = \lambda)}$

versus

$X \sim \exp(\lambda)$ means

$f(x) = \dfrac{1}{\lambda} e^{-x/\lambda} I_{(0,\infty)}(x)$ and

$E[X] = \lambda$

λ is a mean parameter

$\boxed{X \sim \exp(\text{mean} = \lambda)}$

- X with pdf $f_X(x)$
- g a function

$\boxed{\text{Find } E[g(X)]}$

Let Y=g(X). The pdf for Y is:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

So, $E[g(X)] = E[Y] = \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy$

$$E[g(X)] = E[Y] = \int_{-\infty}^{\infty} y \cdot f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty} y \cdot f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right| \, dy$$

Let $x = g^{-1}(y)$. Then $dx = \frac{d}{dy} g^{-1}(y)\,dy$.

$$\Rightarrow \boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx}$$

"Law of the Unconscious Statistician"


Quiet...
Stats in
Progress!

$$E[g(X)] = E[Y] = \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} y \cdot f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

Let $x = g^{-1}(y)$. Then $dx = \frac{d}{dy} g^{-1}(y)$.

$$\Rightarrow \quad \boxed{E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx}$$

“Law of the Unconcious Statistician”

Discrete Example Revisited

x	1	2	3
$P(X = x)$	1/4	1/4	1/2

$$E[X^2] = (1^2)(1/4) + (2^2)(1/4) + (3^2)(1/2)$$

$$= 7/2 = 3.5$$

Continuous Example

$$X \sim \exp(\lambda)$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_{0}^{\infty} x^2 \lambda e^{-\lambda x}\, dx$$

$$= \frac{2}{\lambda^2}$$

$$E[g(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)\, f_{X,Y}(x, y)\, dy\, dx$$

Expectation is a Linear Operator:

$$E[aX + bY] = a\,E[X] + b\,E[Y]$$

Proof:

$$E[aX + bY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ax + by)\, f_{X,Y}(x, y)\, dy\, dx$$

$$= a \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{X,Y}(x, y)\, dy\, dx + b \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{X,Y}(x, y)\, dy\, dx$$

$$\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy = f_X(x) \quad \Rightarrow$$

first term is

$$a \int_{-\infty}^{\infty} x\, f_X(x)\, dx = a\,E[X]$$

Expectation and Independence:

X and Y independent $\Rightarrow$ $E[XY] = E[X]E[Y]$

Proof:

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\,dy\,dx \quad \textbf{(indep)}$$

$$= \int_{-\infty}^{\infty} x\, f_X(x) \underbrace{\int_{-\infty}^{\infty} y\, f_Y(y)\,dy}_{E[Y]}\,dx$$

$$= E[Y]\int_{-\infty}^{\infty} x\, f_X(x)\,dx \quad = E[Y]E[X] \checkmark$$

Things:

- **If X and Y are independent**

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

- $E[XY] = E[X]E[Y]$

  does not imply that X and Y are independent.

Example:

x	-1	0	1
$P(X = x)$	$1/4$	$1/2$	$1/4$

$$E[X] = 0 \quad \Rightarrow \quad \boxed{E[X]E[Y] = 0}$$

Define $Y = X^2$

Y is dependent on X yet,

$$\boxed{E[XY]} = E[X^3]$$

$$= (-1)^3(1/4) + (0)^3(1/2) + (1^3)(1/4)$$

$$\boxed{= 0}$$

$$E[XY] = E[X]E[Y] \checkmark$$