Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with pdf $f(x; \theta)$.

Let Θ be the parameter space. Assume that the parameter is one-dimensional.

Consider testing

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta \neq \theta_0$$

Let $\lambda(\overrightarrow{X}) = L(\theta_0)/L(\widehat{\theta})$ be the GLR for this test.

If the parameter does not define the support for the pdf, for example as in the unif$(0, \theta)$, we have...

Wilks' Theorem

Under the assumption that H_0 is true

$$-2 \ln \lambda(\overrightarrow{X}) \xrightarrow{d} \chi^2(1)$$

Note that

$$\alpha = P(\lambda(\overrightarrow{X}) \leq c; \theta_0)$$

$$= P(\ln \lambda(\overrightarrow{X}) \leq c_1; \theta_0)$$

$$= P(\underbrace{-2 \ln \lambda(\overrightarrow{X})}_{\substack{\text{approximately } \chi^2(1) \\ \text{for large sample size n}}} \geq c_2; \theta_0)$$

$$\approx P(W \geq c_2; \theta_0)$$

where $W \sim \chi^2(1)$

$$\alpha = P(W \geq c_2; \theta_0)$$

$$\Downarrow$$

$$c_2 = \chi^2_{\alpha,1}$$

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Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with pdf $f(x;\theta)$.

Suppose that θ is not involved in the support of f.

Suppose that n is "large".

An approximate GLRT of size α for testing

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta \neq \theta_0$$

is to reject H_0 if $-2 \ln \lambda(\overrightarrow{X}) > \chi^2_{\alpha,1}$.

Example:

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the continuous Pareto distribution with pdf

Find an approximate large sample GLRT of size $\alpha = 0.05$ for

$$H_0 : \gamma = 1.8 \quad \text{vs} \quad H_1 : \gamma \neq 1.8$$

A likelihood is

$$L(\gamma) = \frac{\gamma^n}{\left[\prod_{i=1}^{n} (1 + x_i)\right]^{\gamma+1}}$$

- The MLE for γ is

$$\hat{\gamma} = \frac{n}{\sum_{i=1}^{n} \ln(1 + X_i)}$$

- The restricted MLE for γ is

$$\hat{\gamma}_0 = 1.8$$

Compute $\lambda(\overrightarrow{X}) = L(\gamma_0)/L(\overrightarrow{X})$.

Compute $-2\ln\lambda(\overrightarrow{X})$.

Reject $H_0 : \gamma = 1.8$ if

$$-2\ln\lambda(\overrightarrow{X}) > \chi^2_{\alpha,1} = 3.841459$$