A Few Continuous Distributions:

- the normal distribution

  Two Parameters:

  - Mean : $-\infty < \mu < \infty$
  - Variance: $\sigma^2 > 0$

  The Probability Density Function:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$$

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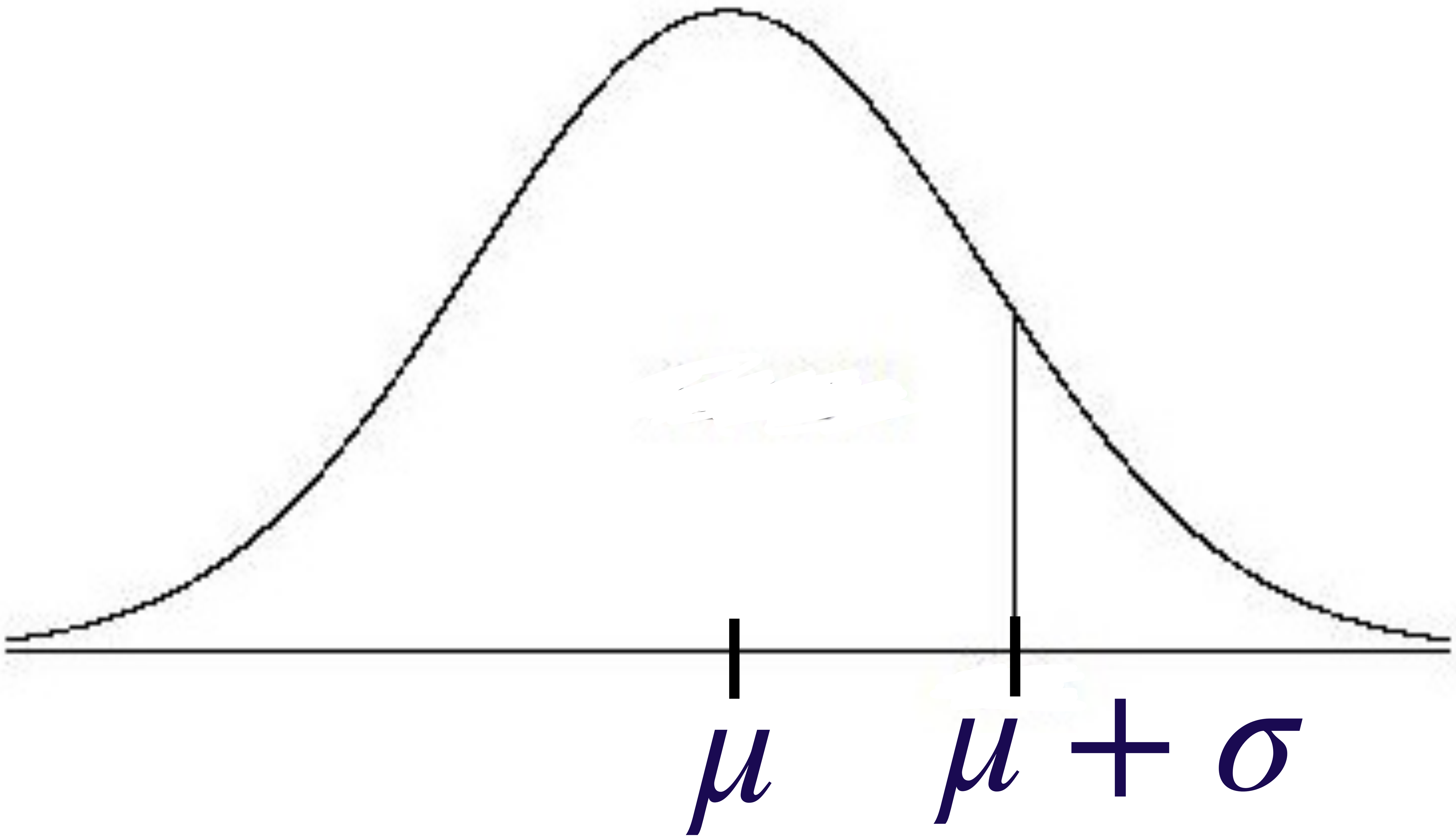


$$X \sim N(\mu, \sigma^2)$$

A Few Continuous Distributions:

- the exponential distribution

One Parameter:

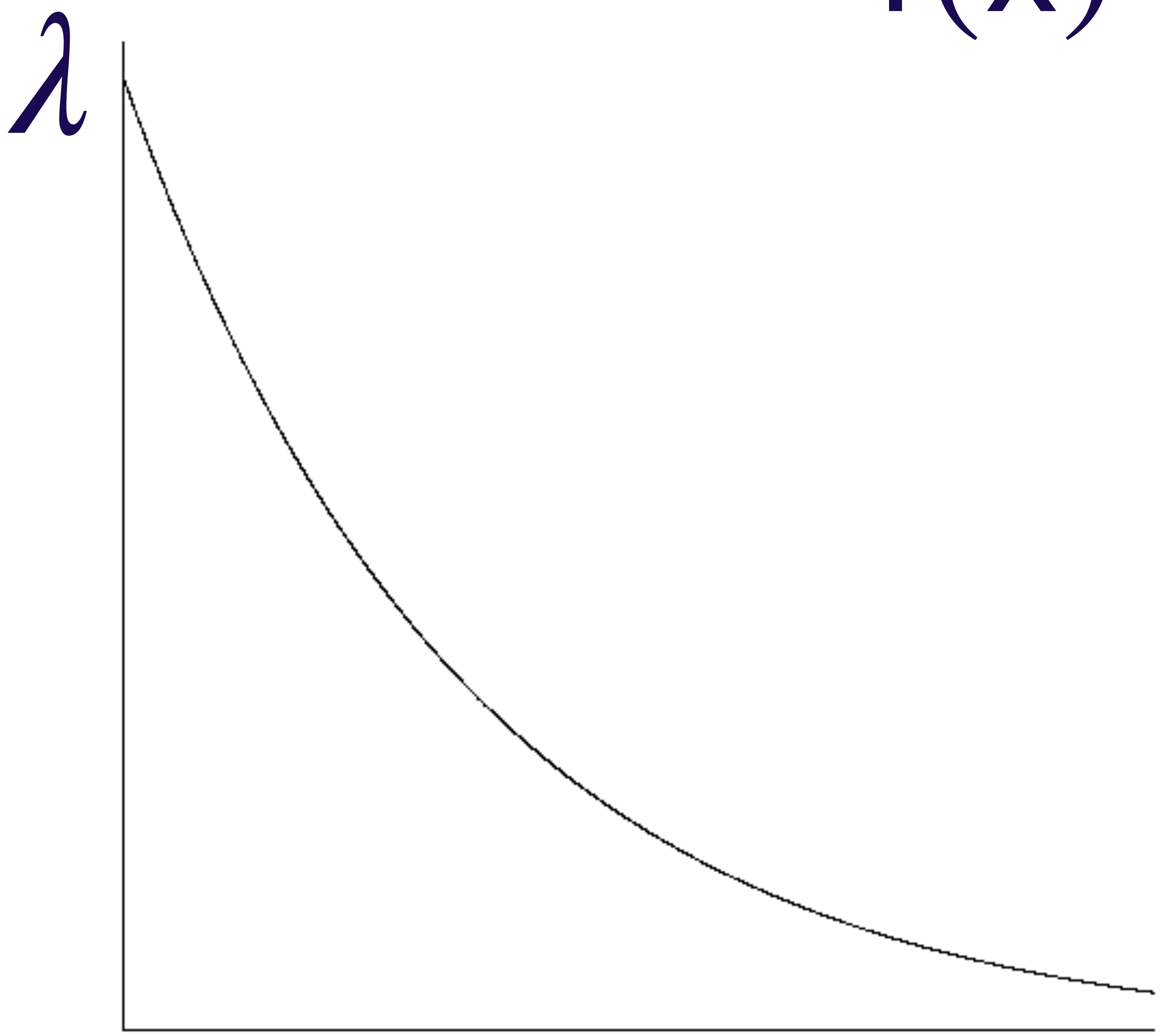
- Rate: $\lambda > 0$

The Probability Density Function:

$$f(x) = \lambda e^{-\lambda x}$$

for $x > 0$

$$f(x) = \lambda e^{-\lambda x}$$



- the exponential distribution
    - mean

$$\mu = E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{0}^{\infty} x\, \lambda\, e^{-\lambda x}\, dx$$

$$= \frac{1}{\lambda}$$

- the exponential distribution
  - Variance

$$\sigma^2 = \mathrm{Var}[X] = \mathrm{E}[(X - \mu)^2]$$

$$= \mathrm{E}[X^2] - (\mathrm{E}[X])^2$$

$$= \cdots = \frac{1}{\lambda^2}$$

- the exponential distribution

$$f(x) = \lambda e^{-\lambda x}$$

If we reparameterize this as

$$f(x) = \frac{1}{\lambda} e^{-x/\lambda}$$

the <u>mean</u> would be λ.

People write $X \sim \exp(\lambda)$.

- the exponential distribution

Our notation

$$\boxed{\begin{array}{c} X \sim \exp(\text{rate} = \lambda) \\ f(x) = \lambda e^{-\lambda x} \end{array}}$$

$$X \sim \exp(\text{mean} = \lambda)$$

$$f(x) = \frac{1}{\lambda} e^{-x/\lambda}$$

A Few Continuous Distributions:

- the gamma distribution

  Two Parameters:

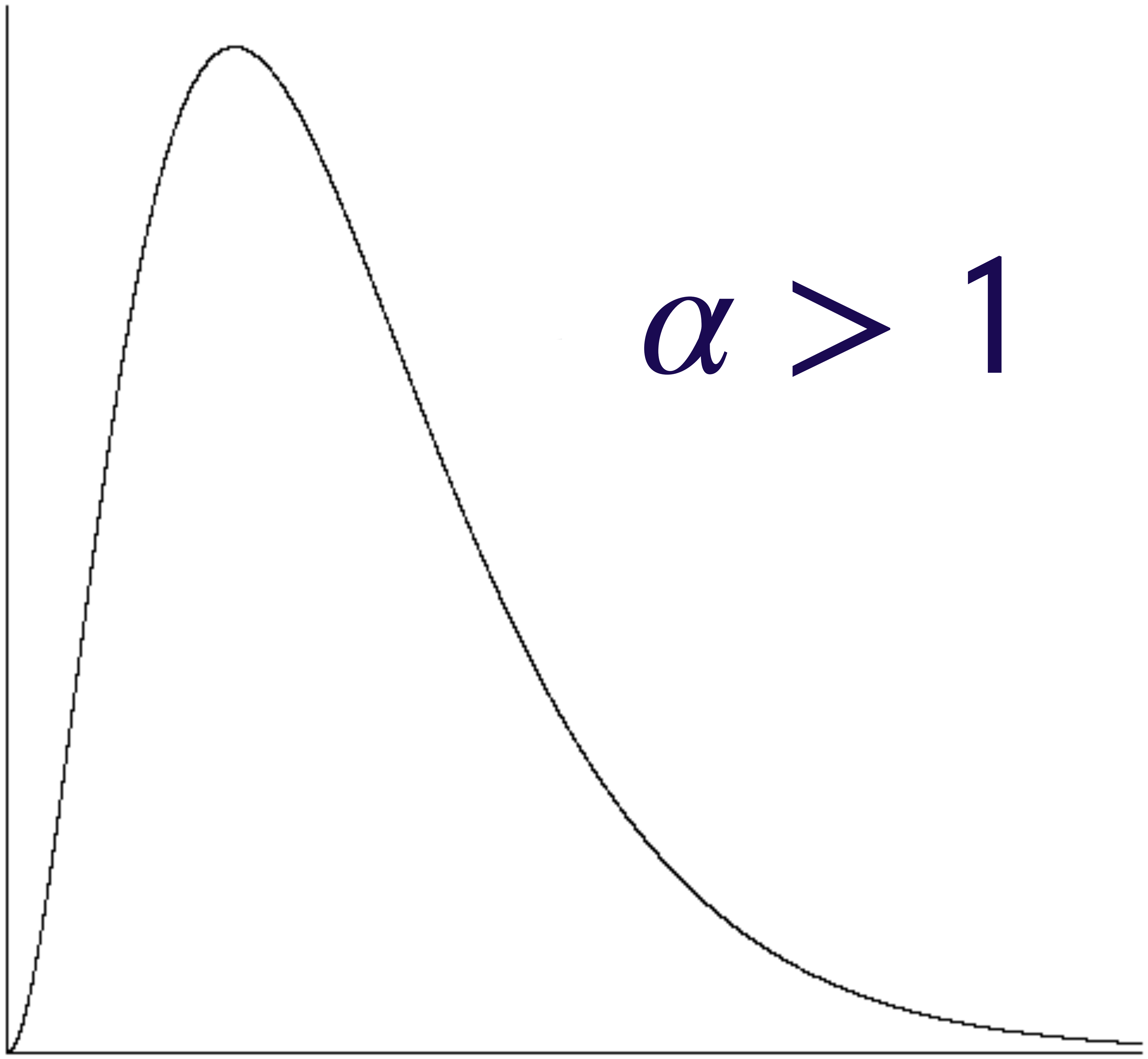
  - Shape: $\alpha > 0$
  - Inverse Scale: $\beta > 0$

  The Probability Density Function:

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{\alpha - 1} e^{-\beta x}$$

for $x > 0$

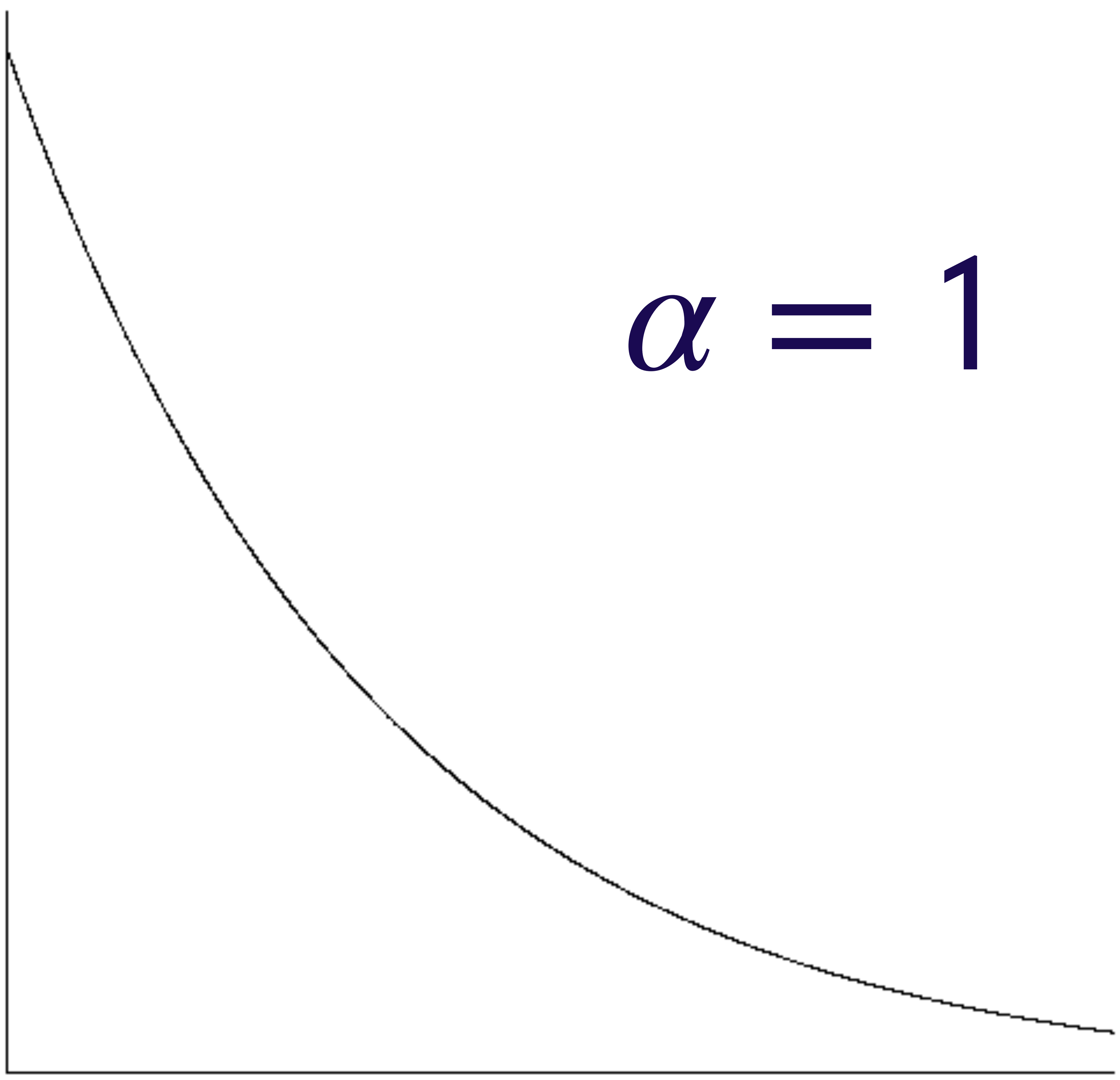
$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{\alpha-1} e^{-\beta x} \quad \text{for } x > 0$$



$\alpha > 1$

$X \sim \Gamma(\alpha, \beta)$

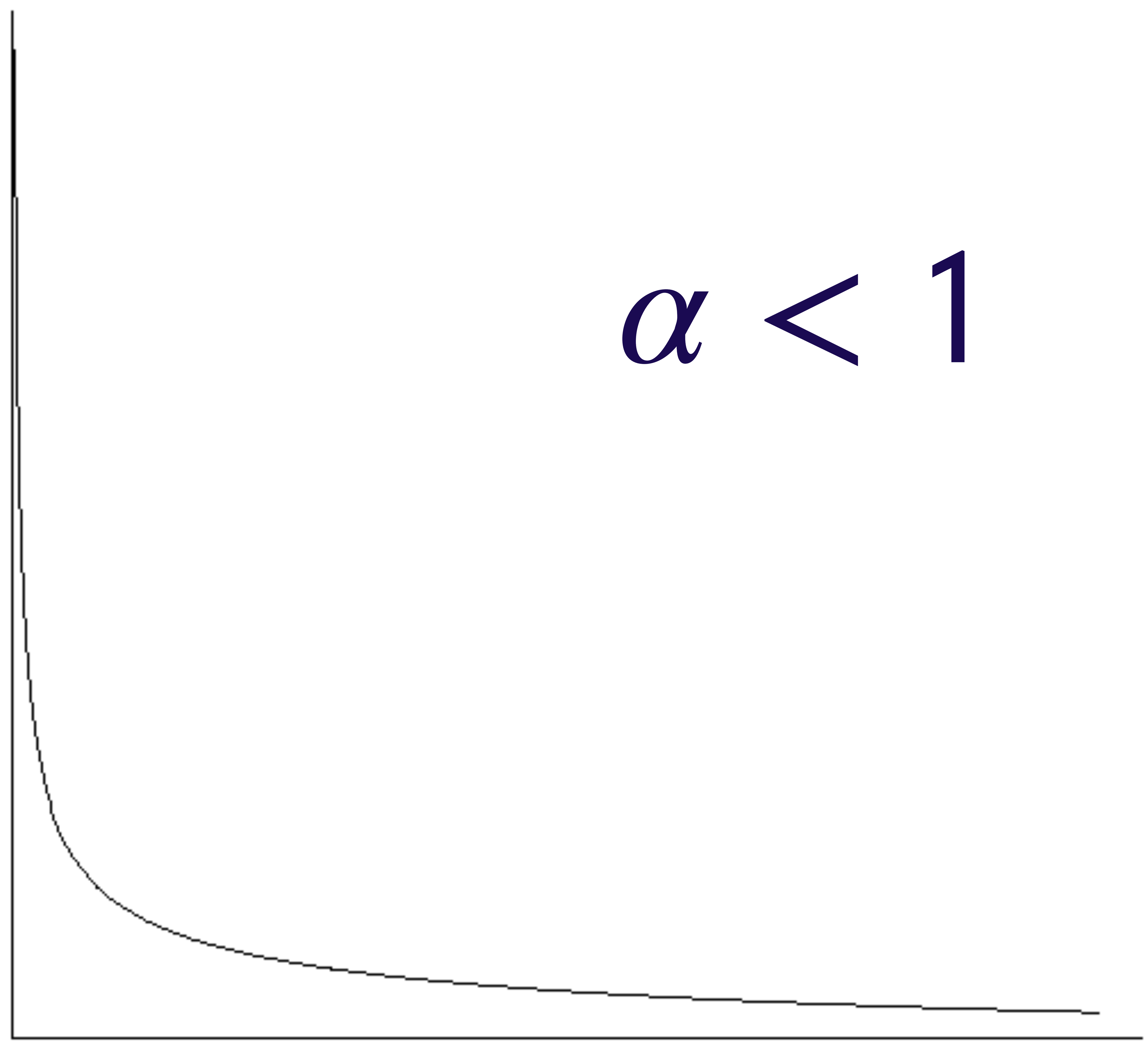
$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{\alpha-1} e^{-\beta x} \qquad \text{for } x > 0$$



$\alpha = 1$

$X \sim \Gamma(\alpha, \beta)$

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{\alpha-1} e^{-\beta x} \quad \text{for } x > 0$$



$\alpha < 1$

$X \sim \Gamma(\alpha, \beta)$

The Gamma Function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx$$

Properties:

- $\Gamma(1) = 1$
- $\Gamma(\alpha) = (\alpha - 1)\,\Gamma(\alpha - 1)$ for $\alpha > 1$
- $\Gamma(n) = (n - 1)!$ for an integer $n \geq 1$

$$X \sim \Gamma(\alpha, \beta)$$

- **mean**

$$\mu = E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{0}^{\infty} x \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{\alpha-1} e^{-\beta x}\, dx$$

$$= \frac{\alpha}{\beta}$$

$$\int_0^\infty x \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x} dx$$

$$= \int_0^\infty \frac{1}{\Gamma(\alpha)} \beta^\alpha \underbrace{x^\alpha e^{-\beta x} dx}_{\substack{\text{looks like} \\ \Gamma(\alpha+1,\beta) \text{ pdf}}}$$

$$= \frac{1}{\beta} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \int_0^\infty \underbrace{\frac{1}{\Gamma(\alpha+1)} \beta^{\alpha+1} x^\alpha e^{-\beta x} dx}_{\substack{\text{is a} \\ \Gamma(\alpha+1,\beta) \text{ pdf}}}$$

$$\mu = \mathrm{E}[X] = \int_0^\infty x \, \frac{1}{\Gamma(\alpha)} \beta^\alpha x^{\alpha-1} e^{-\beta x} \, dx$$

$$= \frac{1}{\beta} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \qquad = \frac{1}{\beta} \frac{\alpha \, \Gamma(\alpha)}{\Gamma(\alpha)}$$

$$= \frac{\alpha}{\beta}$$

$$\mathrm{X} \sim \Gamma(\alpha, \beta)$$

- **Variance**

$$\begin{aligned}\sigma^2 = \mathrm{Var}[\mathrm{X}] &= \mathrm{E}[(\mathrm{X}-\mu)^2] \\ &= \mathrm{E}[\mathrm{X}^2] - (\mathrm{E}[\mathrm{X}])^2 \\ &= \cdots = \frac{\alpha}{\beta^2}\end{aligned}$$

A Few Continuous Distributions:

- the chi-squared distribution

  One Parameter:

  - degrees of freedom: $n \geq 1$

    (n is an integer)

$$X \sim \chi^2(n)$$

is defined as $\Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$

$$X \sim \chi^2(n)$$

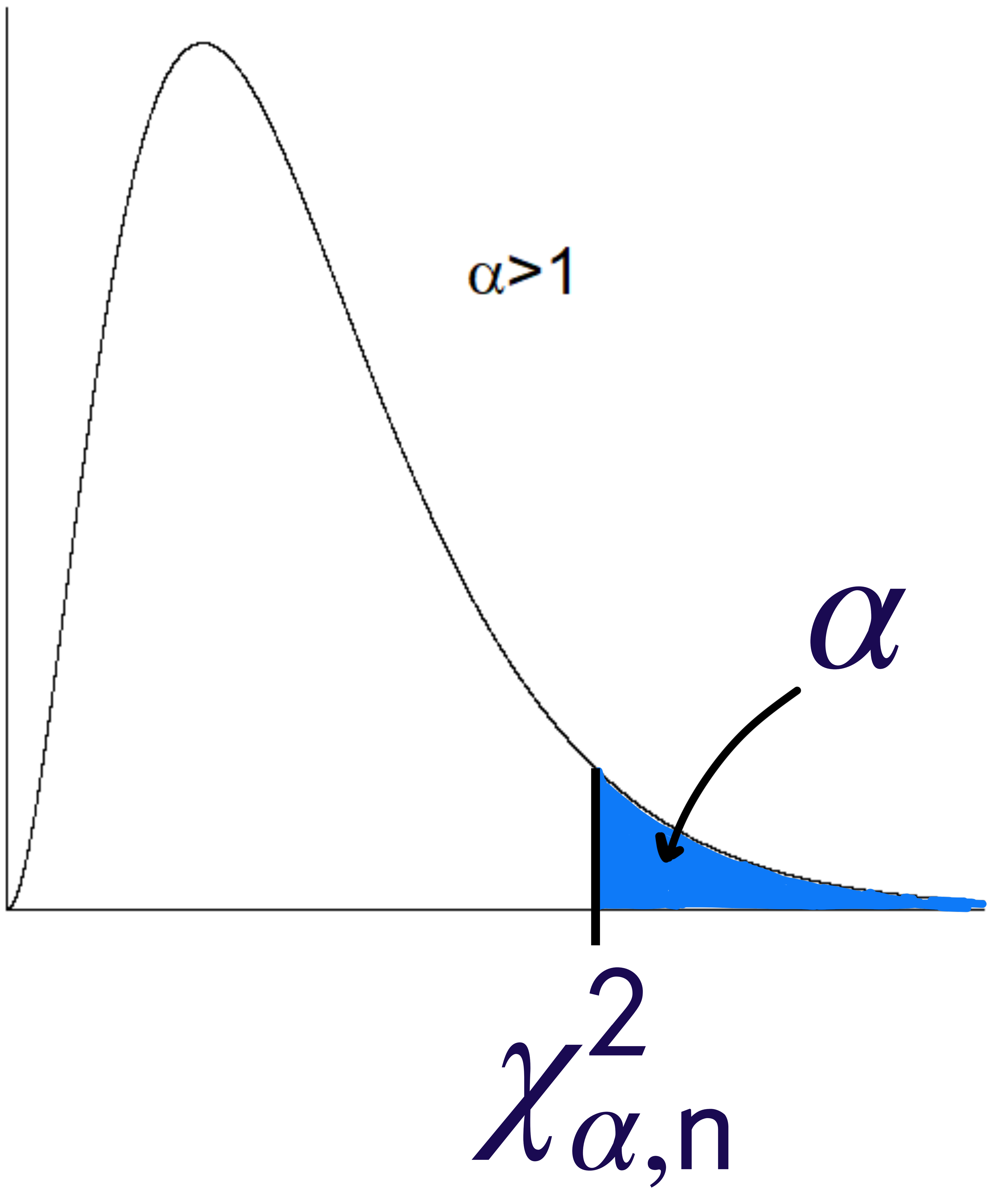
- mean

$$\mu = E[X] = n$$

- variance

$$\sigma^2 = Var[X] = 2n$$

$$X \sim \chi^2(n)$$



A Few Continuous Distributions:

- the t-distribution

Let $Z \sim N(0,1)$ and $W \sim \chi^2(n)$ be independent random variables.

Define

$$T = \frac{Z}{\sqrt{W/n}}$$

then T has pdf…

A Few Continuous Distributions:

- the t-distribution

$\boxed{\text{Write } X \sim t(n)}$

One Parameter:

- degrees of freedom: $n \geq 1$

(n is an integer)

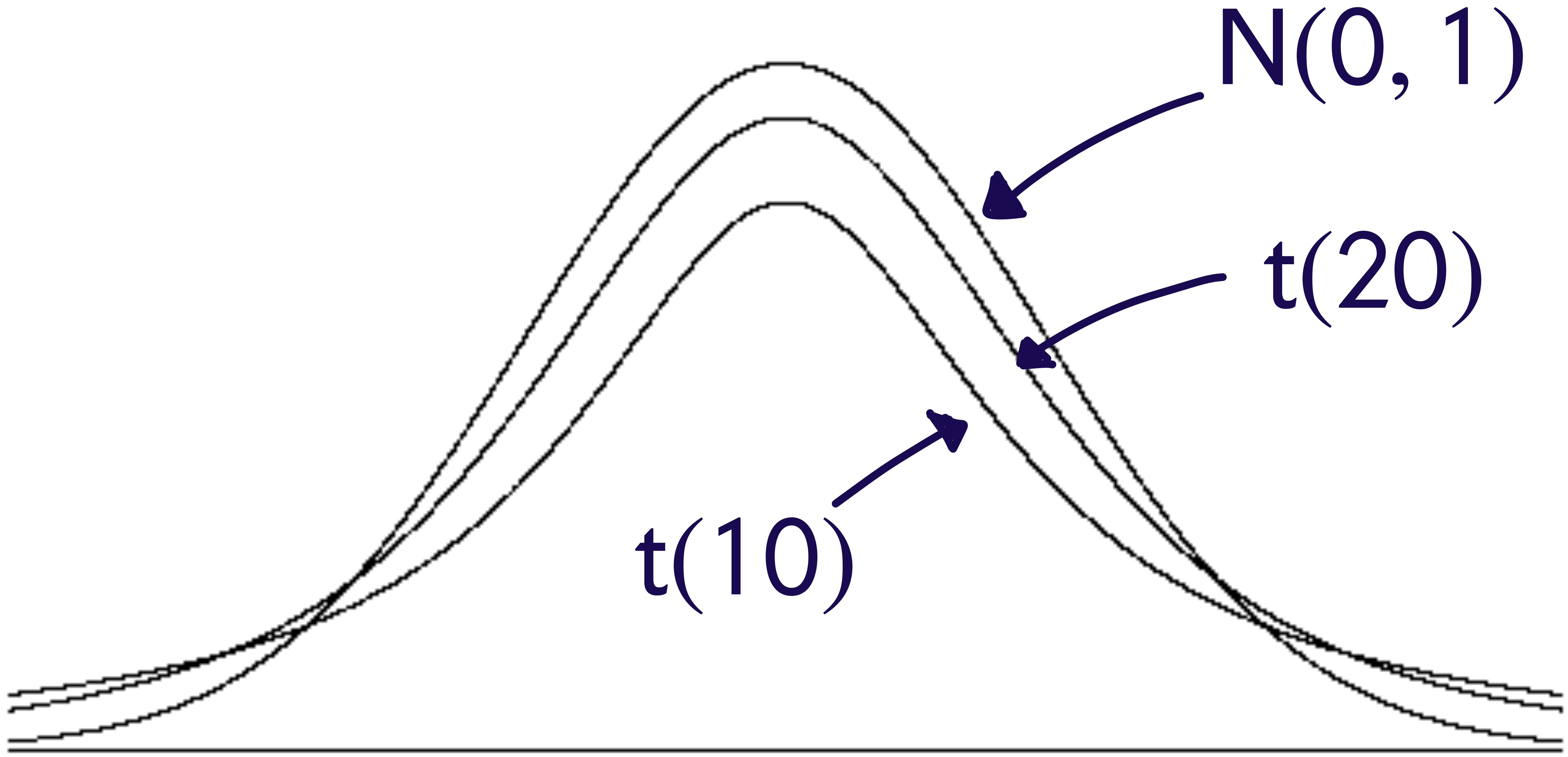
The pdf:

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\,\Gamma\left(\frac{n}{2}\right)}\left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}$$

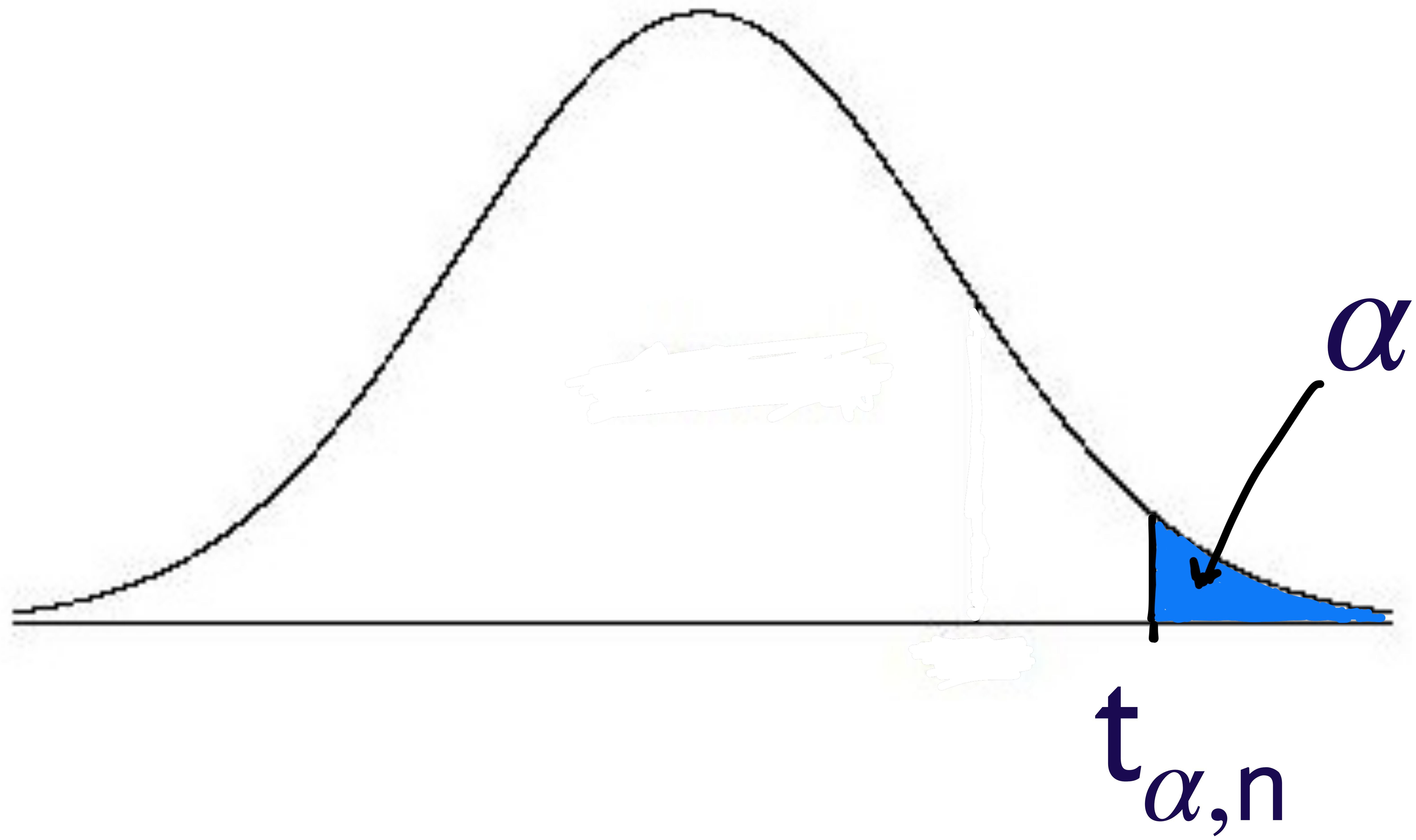
$$-\infty < x < \infty$$

Recall that

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$



$X \sim t(n)$



Fun Facts!

- $Z \sim N(0,1) \quad \Rightarrow \quad Z^2 \sim \chi^2(1)$

- $X_1, X_2, \ldots, X_k$ **independent with** $X_i \sim \chi^2(n_i)$

$$\sum_{i=1}^{k} X_i \sim \chi^2(n_1 + n_2 + \cdots + n_k)$$

In particular, $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \chi^2(1)$

$$\Rightarrow \quad \sum_{i=1}^{n} X_i \sim \chi^2(n)$$

Fun Facts!

- $X \sim \Gamma(\alpha, \beta)$ and $c > 0$

$$\Rightarrow \quad cX \sim \Gamma(\alpha, \beta/c)$$

- $X \sim \Gamma(\alpha, \beta)$

$$f(x) = \frac{1}{\Gamma(\alpha)} \beta^{\alpha} x^{\alpha-1} e^{-\beta x}$$

$$\alpha = 1 \quad \Rightarrow \quad X \sim \exp(\text{rate} = \beta)$$

Fun Facts!

- $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \exp(rate = \lambda)$

$$\Rightarrow \quad \sum_{i=1}^{n} X_i \sim \Gamma(n, \lambda)$$

- $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \Gamma(\alpha, \beta)$

$$\Rightarrow \quad \sum_{i=1}^{n} X_i \sim \Gamma(n\alpha, \beta)$$

Things we now know...

- $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \exp(\text{rate} = \lambda)$

$$\Rightarrow \quad \overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \sim \Gamma(n, n\lambda)$$

- $X_1, X_2, \ldots, X_n \overset{iid}{\sim} \Gamma(\alpha, \beta)$

$$\Rightarrow \quad \overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \sim \Gamma(n\alpha, n\beta)$$

Things we now know…

- $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} \exp(\text{rate} = \lambda)$

$$\Rightarrow \quad \overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim \Gamma(n, n\lambda)$$

$$\Rightarrow \quad 2n\lambda\,\overline{X} = \sim \Gamma\left(n, \frac{1}{2}\right)$$

$$= \Gamma\left(\frac{2n}{2}, \frac{1}{2}\right) = \chi^2(2n)$$