Discrete Random Variable

$$E[X] = (1)(1/4) + (2)(1/4) + (3)(1/2)$$
$$= 9/4 = 2.25$$

$$E[X] = \sum_{x} x P(X = x)$$
 (discrete)

Continuous analogue:

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

Notation:

$$\mu = E[X]$$
 or
$$\mu_X = E[X], \quad \mu_Y = E[Y]$$

Example:

$$X \sim N(\mu, \sigma^{2})$$

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^{2}}} e^{-\frac{1}{2\sigma^{2}}(x-\mu)^{2}} dx$$

$$= \cdots = \mu$$

Example:

$$X \sim \exp(\lambda), \lambda > 0$$

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_{-\infty}^{0} x \cdot 0 \, dx + \int_{0}^{\infty} x \cdot \lambda e^{-\lambda x} \, dx$$

- $= 1/\lambda$
- X = interarrival time
- arrival rate of 2 per hour $(\lambda = 2)$
- average/mean time between arrivals is 1/2 hour

"Rate" Versus "Mean" Parameterization of the Exponential Distribution

$$X \sim \exp(\lambda)$$
 means

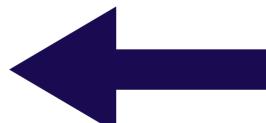
λ is a rate parameter

$$f(x) = \lambda e^{-\lambda x I_{(0,\infty)}(x)}$$
 and

$$E[X] = 1/\lambda$$

$$X \sim \exp(\text{rate} = \lambda)$$

 $X \sim \exp(\lambda)$ means



λ is a mean

$$f(x) = \frac{1}{\lambda} e^{-x/\lambda} I_{(0,\infty)}(x) \text{ and } \lambda \text{ is a mean parameter}$$

$$E[X] = \lambda$$

$$X \sim \exp(mean = \lambda)$$

- X with pdf $f_X(x)$
- g a function

Find E[g(X)]

Let Y=g(X). The pdf for Y is:

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

So,
$$E[g(X)] = E[Y] = \int_{-\infty}^{\infty} y \cdot f_{Y}(y) dy$$

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$$= \int_{-\infty}^{\infty} y \cdot f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right| dy$$

Let
$$x = g^{-1}(y)$$
. Then $dx = \frac{d}{dy}g^{-1}(y) dy$.

$$\Rightarrow E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

"Law of the Unconscious Statistician"

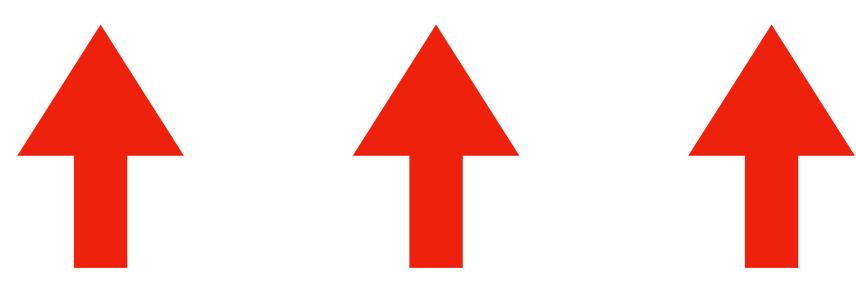


$$E[g(X)] = E[Y] = \int_{-\infty}^{\infty} y \cdot f_{Y}(y) dy$$

$$= \int_{-\infty}^{\infty} y \cdot f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right|$$

Let
$$x = g^{-1}(y)$$
. Then $dx = \frac{d}{dy}g^{-1}(y)$.

$$\Rightarrow E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$



"Law of the Unconcious Statistician"

Discrete Example Revisited

$$E[X^2] = (1^2)(1/4) + (2^2)(1/4) + (3^2)(1/2)$$

$$= 7/2 = 3.5$$

Continuous Example

$$X \sim \exp(\lambda)$$

$$E[X^{2}] = \int_{-\infty}^{\infty} x^{2} f_{X}(x) dx$$

$$= \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

$$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dy dx$$

Expectation is a Linear Operator:

$$E[aX + bY] = aE[X] + bE[Y]$$

Proof:

$$E[aX + bY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (ax + by) f_{X,Y}(x, y) dy dx$$

$$= a \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y}(x,y) dy dx + b \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{X,Y}(x,y) dy dx$$

$$\int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dy = f_X(x) \Rightarrow \begin{cases} \text{first term is} \\ f_X(x) \end{cases}$$

$$a \int_{-\infty}^{\infty} x f_X(x) dx = a E[X]$$

Expectation and Independence:

X and Y independent
$$\implies$$
 E[XY] = E[X]E[Y]

Proof:

$$E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X,Y}(x, y) dy dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{X}(x) f_{Y}(y) dy dx$$
 (indep)
$$= \int_{-\infty}^{\infty} x f_{X}(x) \int_{-\infty}^{\infty} y f_{Y}(y) dy dx$$

$$= E[Y] \int_{-\infty}^{\infty} x f_{X}(x) dx = E[Y]E[X]$$

Things:

If X and Y are independent

$$E[g(X)h(Y)] = E[g(X)]E[h(Y)]$$

- E[XY] = E[X]E[Y]

does not imply that X and Y are independent.

Example:

$$E[X] = 0 \Rightarrow E[X]E[Y] = 0$$
Define $Y = X^2$

Y is dependent on X yet,

$$E[XY] = E[X^3]$$

$$= (-1)^3(1/4) + (0)^3(1/2) + (1^3)(1/4)$$

$$E[XY] = E[X]E[Y]$$