Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with pdf $f(x;\theta)$.

Let Θ be the parameter space.

Consider testing

$$H_0 : \theta \in \Theta_0 \quad \text{vs} \quad H_1 : \theta \in \Theta \backslash \Theta_0$$

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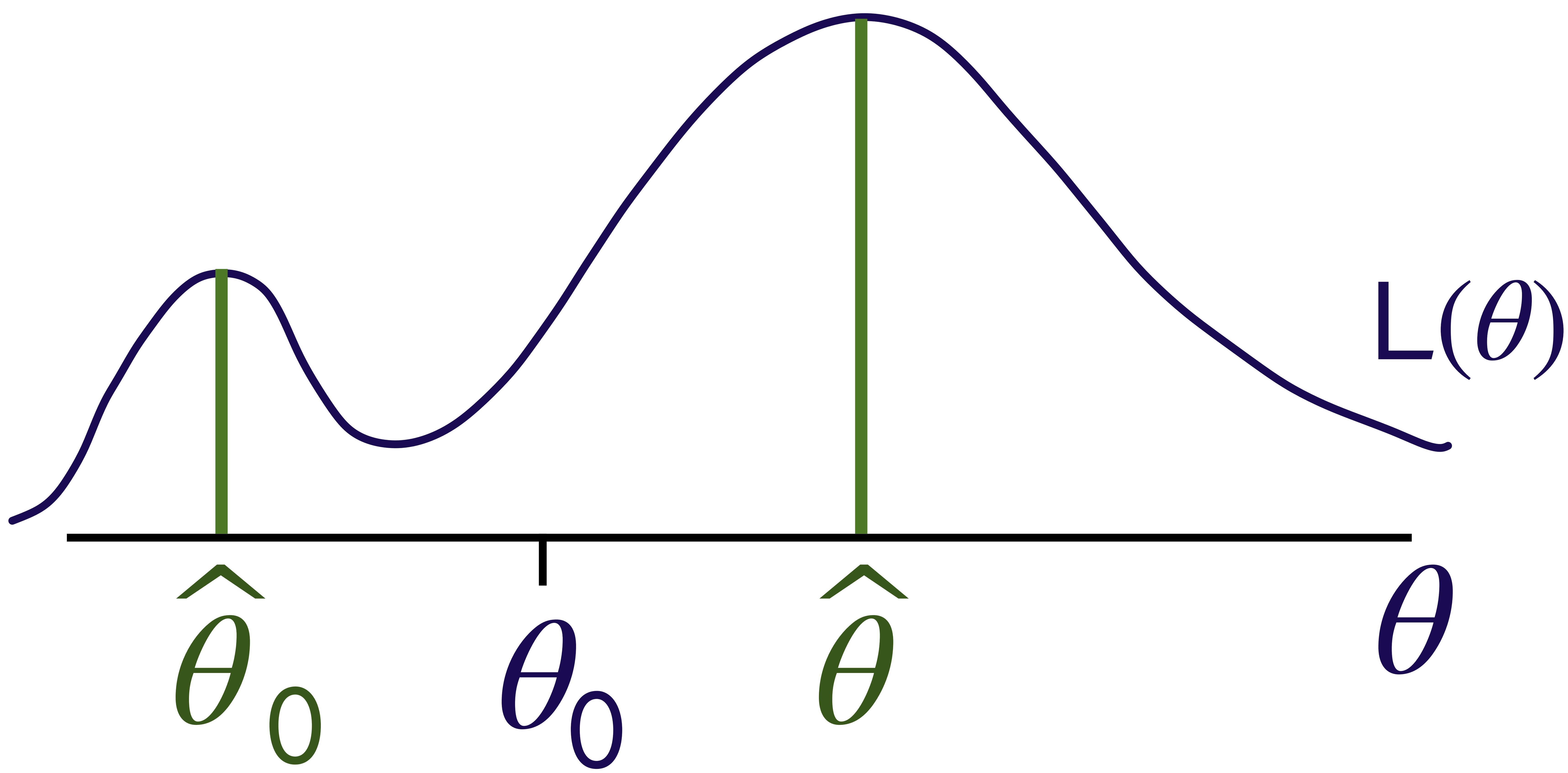
- Let $\widehat{\theta}$ be the maximum likelihood estimator for θ.
- Let $\widehat{\theta}_0$ be restricted MLE.

  $\widehat{\theta}_0$ is the MLE for θ if we assume that H_0 is true.

Consider, for example

$$H_0 : \theta \leq \theta_0 \quad \text{vs} \quad H_1 : \theta > \theta_0$$

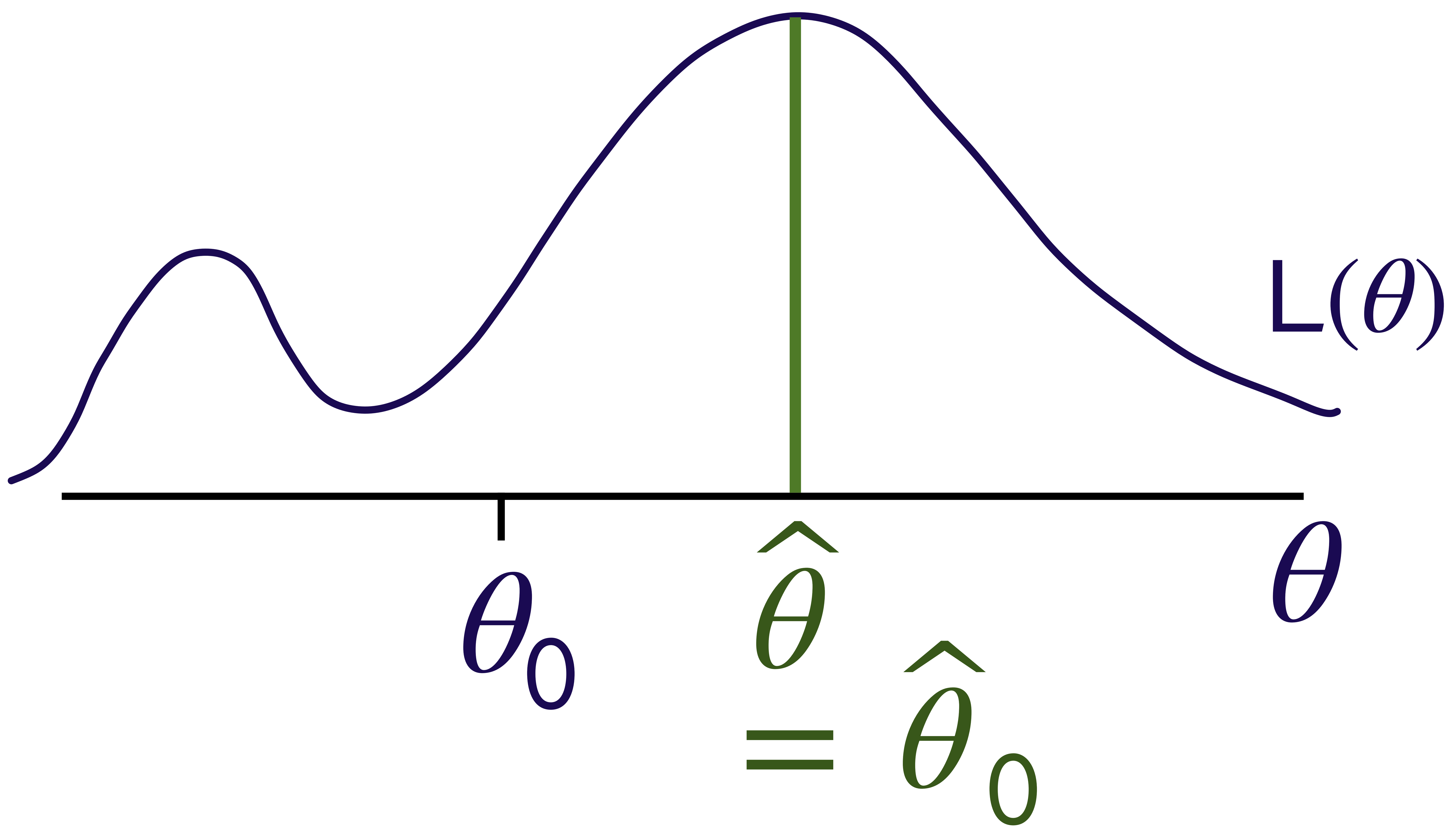
Suppose that the likelihood looks like this.



Consider, for example

$$H_0 : \theta \geq \theta_0 \quad \text{vs} \quad H_1 : \theta < \theta_0$$

Suppose that the likelihood looks like this.



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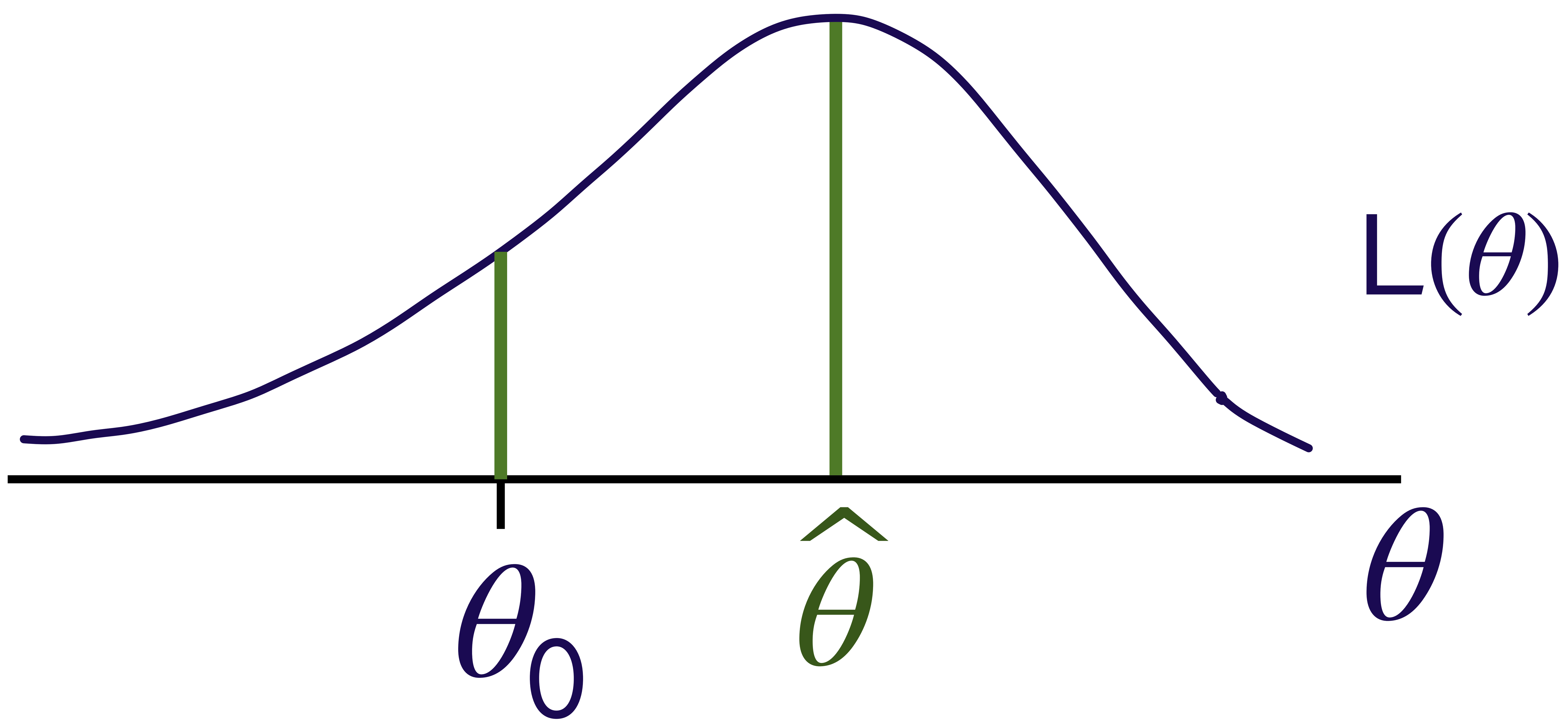

$$\hat{\theta}_0 = \theta_0$$

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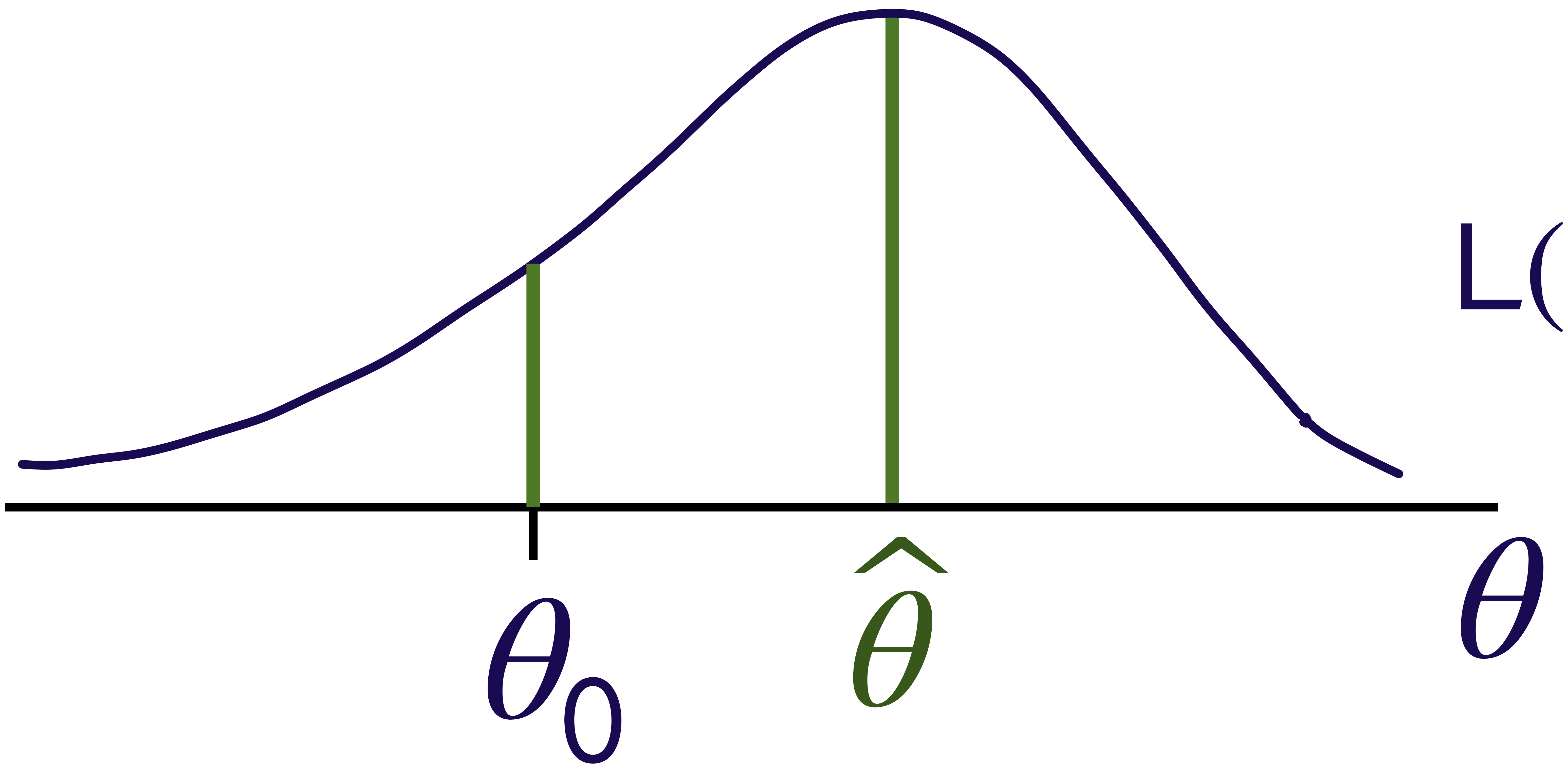

$$\hat{\theta}_0 = \theta_0$$

Definition:

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with pdf $f(x;\theta)$.

Consider testing

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta \neq \theta_0$$

Let $L(\theta)$ be a likelihood function.

The generalized likelihood ratio (GLR) is

$$\lambda(\overrightarrow{X}) = \frac{L(\widehat{\theta}_0)}{L(\widehat{\theta})}$$

Definition:

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with pdf $f(x; \theta)$.

Consider testing

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta \neq \theta_0$$

Let $L(\theta)$ be a likelihood function.

The generalized likelihood ratio test (GLRT) says to reject H_0, in favor of H_1, if

$$\lambda(\overrightarrow{X}) \leq c$$

Example:

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the continuous Pareto distribution with pdf

$$f(x;\gamma) = \begin{cases} \dfrac{\gamma}{(1+x)^{\gamma+1}}, & x > 0 \\ 0, & \textbf{otherwise} \end{cases}$$

Here, $\gamma > 0$ is a parameter.

Find the GLRT of size α for

$$H_0 : \gamma = \gamma_0 \quad \text{vs} \quad H_1 : \gamma \neq \gamma_0$$

A likelihood is

$$L(\gamma) = \frac{\gamma^n}{\left[\prod_{i=1}^n (1 + x_i)\right]^{\gamma+1}}$$

- The MLE for γ is

$$\hat{\gamma} = \frac{n}{\sum_{i=1}^n \ln(1 + X_i)}$$

- The restricted MLE for γ is

$$\hat{\gamma}_0 = \gamma_0$$

The GLR is

$$\lambda(\overrightarrow{X}) = \frac{L(\widehat{\gamma}_0)}{L(\widehat{\gamma})}$$

- The form of the test is to reject H_0, in favor of H_1, if $\lambda(\overrightarrow{X}) \leq c$, where c is determined by solving

$$P(\lambda(\overrightarrow{X}) \leq c;\ \gamma_0) = \alpha$$

or, equivalently

$$P(g(\overrightarrow{X}) \quad ? \quad c_1;\ \gamma_0) = \alpha$$

Yuck!