Statistical Inference for Estimation in Data Science

DTSA 5002 offered on Coursera

by the University of Colorado, Boulder

Instructor: J.N. Corcoran

Week One Lesson Two

Joint Distributions

Let X and Y be discrete random variables. Often , we want to talk about the probability that $X = x$ and $Y = y$ at the same time. This is a *joint probability* which can be written as

$$P(X = x, Y = y).$$

(The comma here is read as "and".)

In DTSA 5001 (or any other beginner probability course), you would have seen several "event"-based probabilities for events denoted by letters like A, B, and C. For example, if we roll a fair 6-sided die, we might let A be the event that the outcome is an even number and B be the event that the outcome is greater than or equal to 4. We have

$$A = \{2, 4, 6\} \qquad \text{and} \qquad B = \{4, 5, 6\}.$$

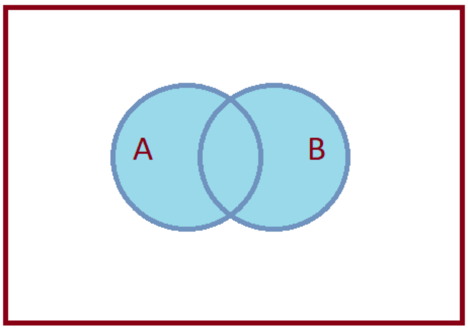
We also have

$$A \cup B = \{2, 4, 5, 6\} \qquad \text{and} \qquad A \cap B = \{4, 6\}.$$

There are many rules of probability such as

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \tag{1}$$

that can be easily recalled by thinking of probability as area in a Venn diagram containing circles representing A and B. In the following figure, $A \cup B$ is shaded.



Note that the area of $A \cup B$ is the area of A plus the area of B minus the area of the intersection since it got double-counted. This is how we can remember Equation (1).

Getting back to $P(X=x, Y=y)$, note that we can apply all of our "event" based probability rules here. We can think of this probability as $P(A \cap B)$ where

$$\begin{array}{rcl} A & = & \text{the event that } X \text{ equals } x \\ B & = & \text{the event that } Y \text{ equals } y \end{array}$$

Discrete Joint Distributions

There are not many joint distributions that have names. Most often, they consist of probabilities given in tabular form as in the following example.

		x		
		1	2	3
y	-5	0.2	0.1	0.05
	6	0.15	0.3	0.2

Here, the entries are probabilities corresponding to row and column heading values for X and Y. For example,

$$P(X=2, Y=6) = 0.3.$$

Suppose now that we just want to find the probability that $X=2$ and are not concerned with the value of Y? There are 2 ways this can happen. We can either have $X=2$ and $Y=-5$ or we can have $X=2$ and $Y=6$. Since these are disjoint events, we can simply add the two probabilities just as we can add two disjoint pieces of area in a Venn diagram. We have

$$P(X=2) = P(X=2, Y=-5) + P(X=2, Y=6) = 0.1 + 0.3 = 0.4.$$

Note that this and other probabilities for "X alone" or "Y alone" can be found in the margins of the tables if we include row and column sums.

		x			
		1	2	3	
y	-5	0.2	0.1	0.05	0.35
	6	0.15	0.3	0.2	0.65
		0.35	0.4	0.25	

This is why the separate distributions for X and Y alone:

x	1	2	3
$P(X=x)$	0.35	0.4	0.25

and

y	-5	6
$P(Y=y)$	0.35	0.65

are known as marginal distributions!

Analogous to the one-dimensional case, We will typically use $f(x, y)$ to denote the **joint probability mass function**

$$f(x, y) = P(X = x, Y = y).$$

The marginal pmf for X is obtained by summing out y:

$$f_X(x) = \sum_y f(x, y).$$

(We have used this subscript notation on the left-hand side so that we may use the letter f again for the other marginal pmf.)

Similarly, the marginal pmf for Y is obtained by summing out x:

$$f_Y(y) = \sum_x f(x, y).$$

Continuous Joint Distributions

Suppose now that X and Y are continuous random variables. As per our discussion in the univariate case, a joint probability $P(X = x, Y = y)$ will always be zero.

In the one-dimensional setting, the probability density function (pdf) was a curve under which area represented probability. In the two-dimensional setting, the joint pdf for X and Y might be called $f(x, y)$ and would give a surface under which *volume* represents probability. We must have

$$\int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1.$$

We can compute probabilities involving X and Y by integrating over the appropriate region. For example,

$$P(X \leq 1,\ 2 < Y < 6) = \int_{-\infty}^{1} \int_{2}^{6} f(x,y)\, dy\, dx$$

and

$$P(X \leq Y) = \int_{-\infty}^{\infty} \int_{x}^{\infty} f(x,y)\, dy\, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{y} f(x,y)\, dx\, dy.$$

(It may help you to draw the region where $x \leq y$ in the xy-plane in order to find the limits of integration needed.)

Analogous to the discrete case, we can get to the pdfs for X or Y alone by integrating out the unwanted variable. These *marginal pdfs* are

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

and

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx.$$

It may be the case that the joint pdf is zero in a lot of places and the integrals from $-\infty$ to ∞ may be equivalent to integration over a finite region. Examples are given in the video associated with this lesson.