Notation/Terminology:

"Random Sample"

$$X_1, X_2, \ldots, X_n$$

- variables before they are sampled, observed, and "locked in"

- they are assumed to be independent and identically distributed (iid)

random sample = iid

More Notation:

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the gamma distribution with parameters α and β.

We write

$$\boxed{X_1, X_2, \ldots, X_n \overset{\boxed{\text{iid}}}{\sim} \Gamma(\alpha, \beta)}$$

More Notation:

θ will denote a generic parameter

For example, $\theta = \mu$ $\quad \theta = p$

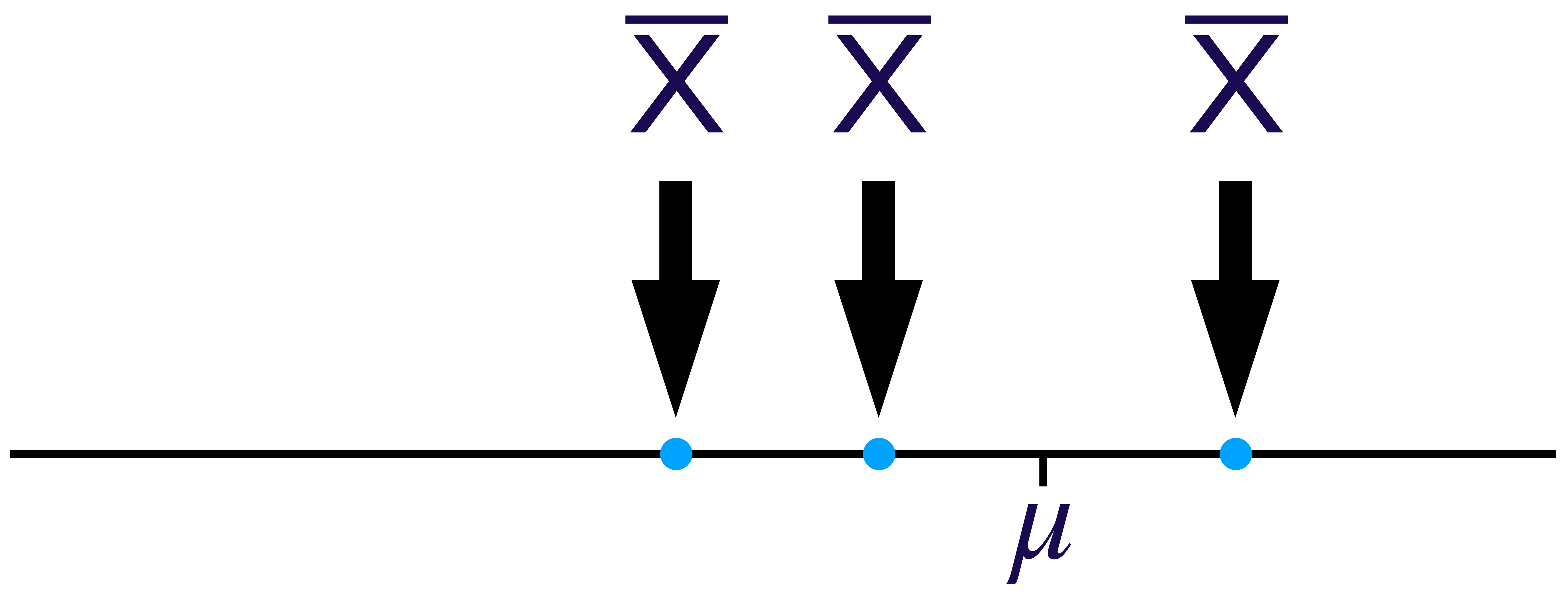
$\theta = \lambda$ $\quad \theta = (\alpha, \beta)$

- Estimator: $\hat{\theta}$ = a random variable

  Example: $\hat{\theta} = \overline{X}$

- Estimate: $\hat{\theta}$ = an observation/number

  Example: $\hat{\theta} = \overline{x} = 42.8$



- We want our estimator of μ to be correct "on average.
- $\overline{X}$ is a random variable with its own distribution and its own mean or expected value.

We would like $E[\overline{X}] = \mu$.

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If this is true, we say that $\overline{X}$ is an unbiased estimator of μ.

In general, $\hat{\theta}$ is an unbiased estimator of θ if:

$$\boxed{E[\hat{\theta}] = \theta}$$

Let $X_1, X_2, \ldots, X_n$ be a random sample from any distribution with mean μ.

That is, $E[X_i] = \mu$ for $i = 1, 2, \ldots, n$.

Then

$$E\left[\overline{X}\right] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} E\left[X_i\right]$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mu = \frac{1}{n}(\mu + \mu + \cdots + \mu) = \frac{1}{n} n\mu = \mu$$

$$E[aX + bY] = aE[X] + bE[Y]$$

$\boxed{\text{We have shown that, no matter what distribution we are working with, if the mean is } \mu, \overline{X} \text{ is an unbiased estimator for } \mu.}$

Example:

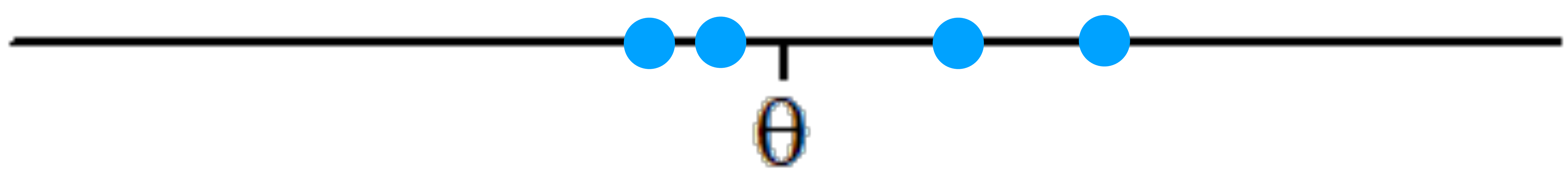
Suppose that

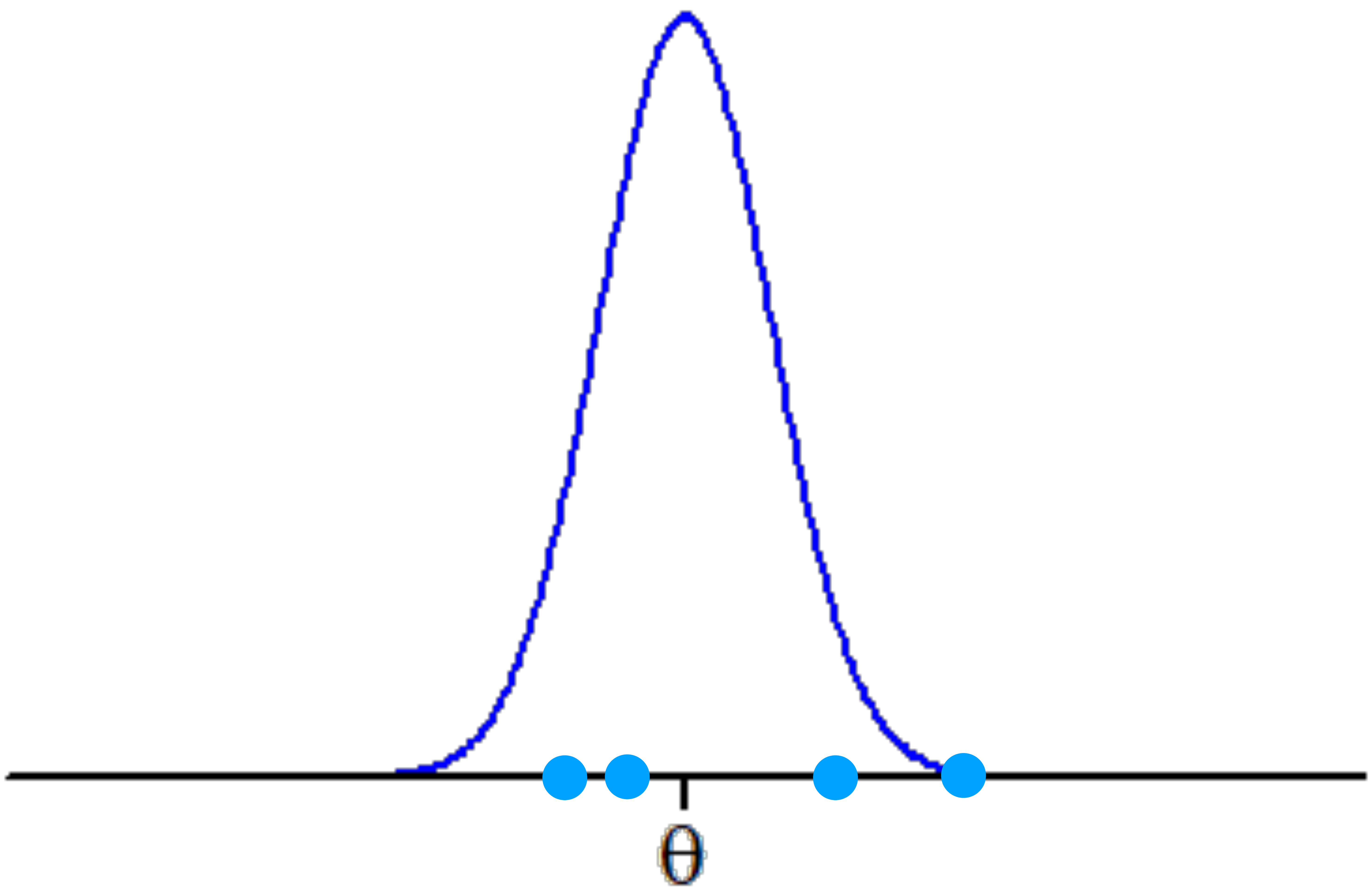
$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \exp(\text{rate} = \lambda)$$

Let $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ be the sample mean.

We know, for the exponential distribution, that $E[X_i] = 1/\lambda$.

$\boxed{\text{Then } E[\overline{X}] = 1/\lambda.}$


θ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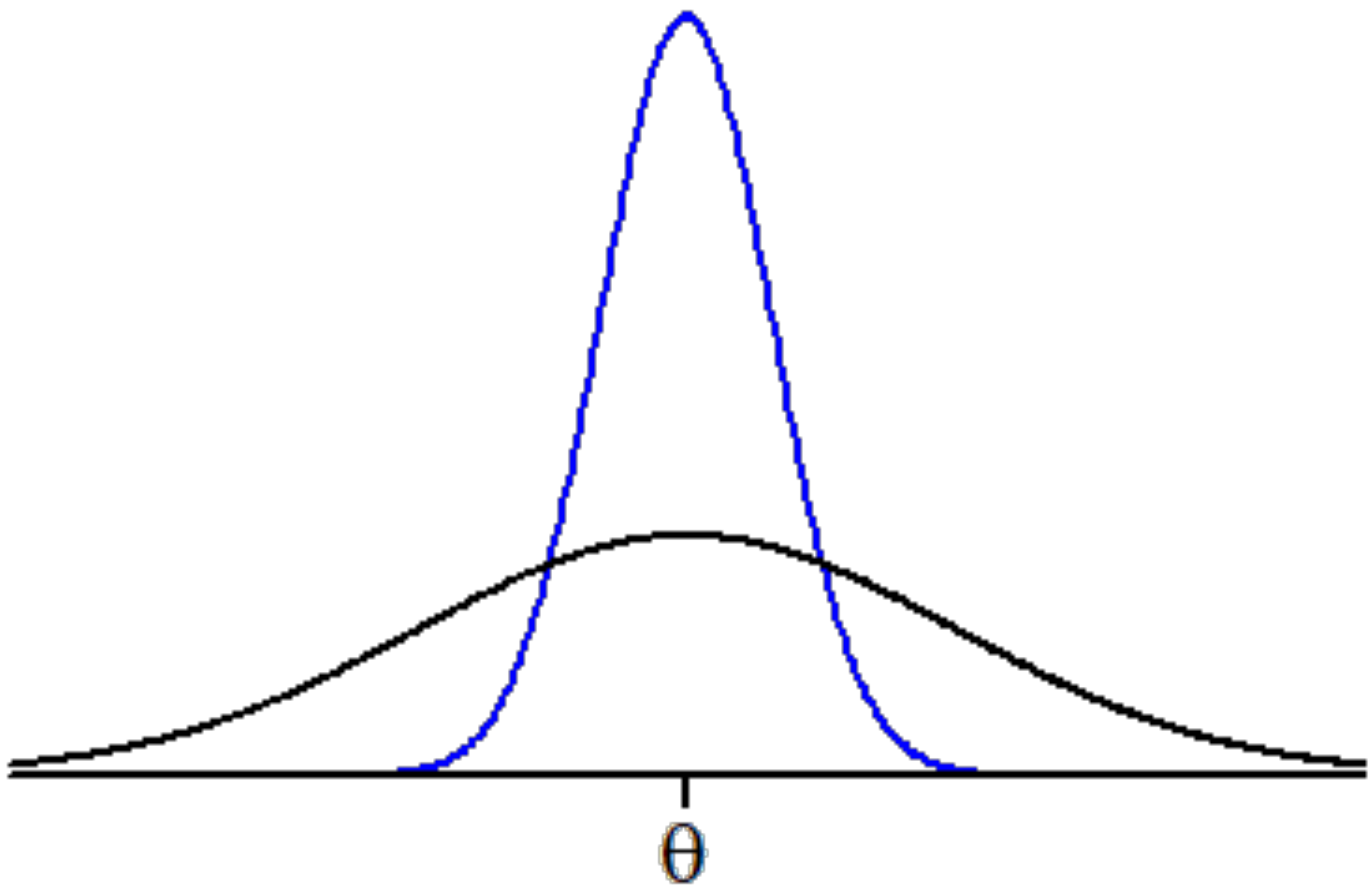
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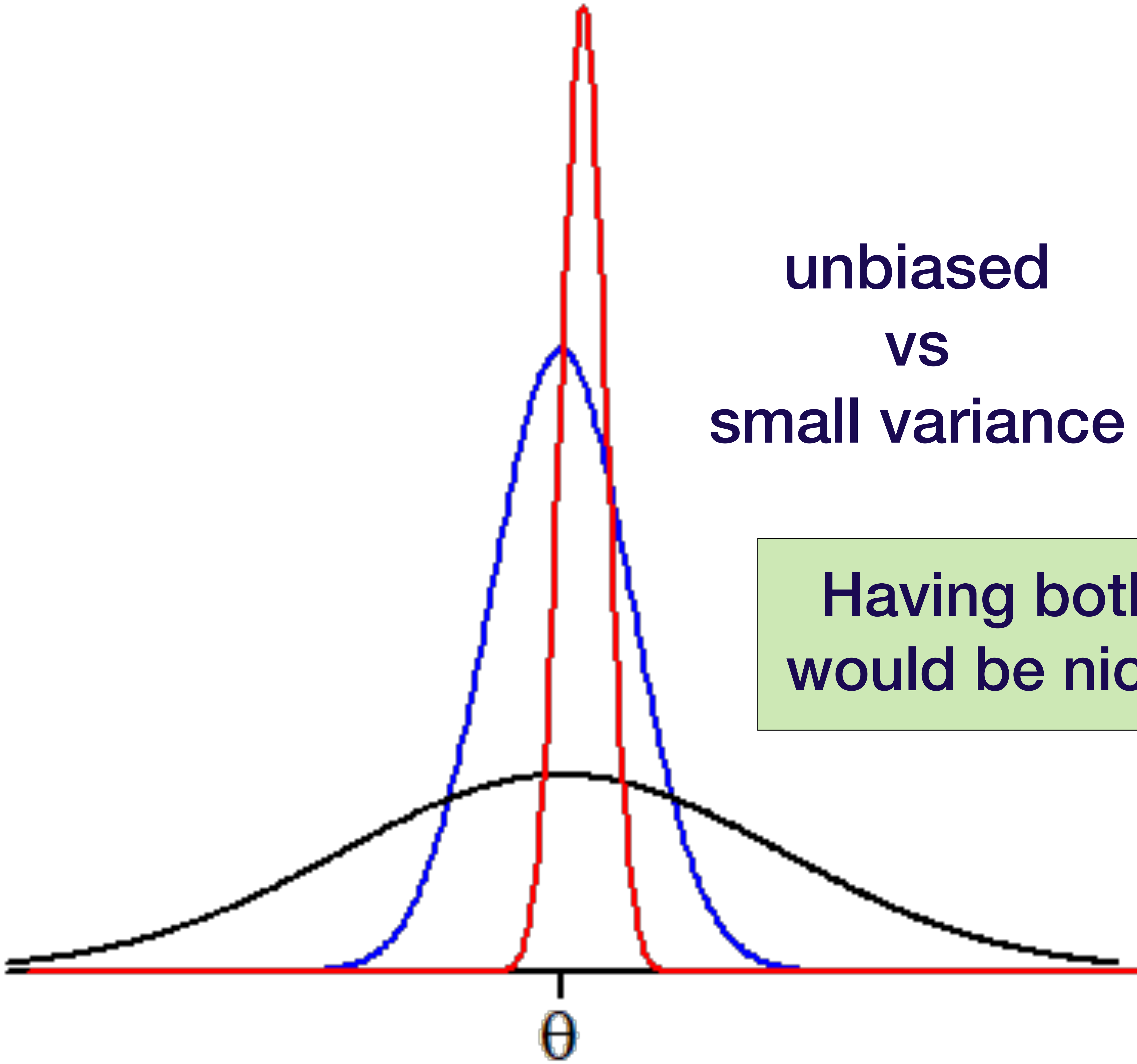

unbiased
vs
small variance

Having both
would be nice!

Let $X_1, X_2, \ldots, X_n$ be a random sample from any distribution with mean μ and variance σ^2.

- We already know that $\overline{X}$ is an unbiased estimator for μ.

- What can we say about the variance of $\overline{X}$?

$$\mathrm{Var}\left[\overline{X}\right] = \mathrm{Var}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]$$

$$= \frac{1}{n^2}\mathrm{Var}\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}\left[X_i\right]$$

by independence

$$
\begin{aligned}
\mathrm{Var}[\overline{X}] &= \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}\left[X_i\right] \\
&= \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 \\
&= \frac{1}{n^2}\, n\, \sigma^2 \\
&= \frac{\sigma^2}{n}
\end{aligned}
$$