Suppose that

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \Gamma(\alpha, \beta)$$

How can we estimate α ?

$$E[X_1] = \frac{\alpha}{\beta}$$

- We could estimate the true mean α/β with the sample mean $\overline{X}$, but we still can't get at α if we don't know β.

- One possibility is to use

  Method of Moments Estimation.

Method of Moments Estimators:

Recall that the "moments" of a distribution are defined as $E[X], E[X^2], E[X^3], \ldots$ These are distribution or "population" moments.

- $\mu = E[X]$ is a probability weighted average of the values in the population.

- $\overline{X}$ is the average of the values in the sample.

It was natural for us to think about estimating μ with the average in our sample.

- $E[X^2]$ is a probability weighted average of the squares of the values in the population.

It is intuitively nice to estimate it with the average of the squared values in the sample:

$$\frac{1}{n}\sum_{i=1}^{n} X_i^2$$

The kth population moments:

$$\mu_k = E[X^k] \quad k = 1, 2, 3, \ldots$$

The kth population moments:

$$\mu_k = E[X^k] \quad k = 1, 2, 3, \ldots$$

The kth sample moments:

$$M_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k \quad k = 1, 2, 3, \ldots$$

Method of Moments Estimators (MMEs)

Idea: Equate population and sample moments and solve for the unknown parameters.

Example:

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \exp(\text{rate} = \lambda)$$

First population moment:

$$\mu_1 = \mu = E[X] = \frac{1}{\lambda}$$

First sample moment:

$$M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}$$

Equate: $\dfrac{1}{\lambda} = \overline{X}$

WARNING: This makes no sense! On the left side we have a constant and on the right side we have a random variable!

It is just an intermediate step to facilitate clean algebra.

Equate: $\frac{1}{\lambda} = \overline{X}$

Solve for the unknown parameter...

$$\lambda = \frac{1}{\overline{X}}$$

... and “put a hat on it”!

The MME is $\boxed{\hat{\lambda} = \frac{1}{\overline{X}}}$

We left the “hat” off originally just to facilitate the algebra. Technically, it should have been there from the beginning.

Question:

Is this an unbiased estimator of λ?

Check:

$$E[\hat{\lambda}] = E\left[\frac{1}{\overline{X}}\right] = E\left[\frac{n}{\sum_{i=1}^{n} X_i}\right]$$

It is super important to note that:

$$\boxed{E\left[\frac{1}{\overline{X}}\right] \neq \frac{1}{E[\overline{X}]}} \qquad \boxed{E\left[\frac{1}{X}\right] \neq \frac{1}{E[X]}}$$

$$E\left[\frac{1}{X}\right] = \int_{-\infty}^{\infty} \frac{1}{x} f_X(x)\, dx \neq \frac{1}{\int_{-\infty}^{\infty} x\, sf_X(x)\, dx}$$

Question:

Is this an unbiased estimator of λ?

Check:

$$E[\hat{\lambda}] = E\left[\frac{1}{\overline{X}}\right] = E\left[\frac{n}{\sum_{i=1}^{n} X_i}\right]$$

Super Important Notes:

$$\boxed{E\left[\frac{1}{\overline{X}}\right] \neq \frac{1}{E[\overline{X}]}} \qquad \boxed{\frac{1}{\sum_{i=1}^{n} X_i} \neq \sum_{i=1}^{n} \frac{1}{X_i}}$$

$$E\left[\frac{1}{X}\right] = \int_{-\infty}^{\infty} \frac{1}{x} f_X(x)\, dx \neq \frac{1}{\int_{-\infty}^{\infty} x \cdot f_X(x)\, dx}$$

$$E[\hat{\lambda}] = E\left[\frac{1}{\overline{X}}\right] = E\left[\frac{n}{\sum_{i=1}^{n} X_i}\right]$$

$$= n\,E\left[\frac{1}{Y}\right] \quad \text{where} \quad Y \sim \Gamma(\alpha, \beta)$$

$$= n \int_{-\infty}^{\infty} \frac{1}{y} f_Y(y)\,dy = n \int_0^{\infty} \frac{1}{y} \cdot \frac{1}{\Gamma(n)} \lambda^n y^{n-1} e^{-\lambda y}\,dy$$

$$= n \int_0^{\infty} \frac{1}{\Gamma(n)} \lambda^n \underbrace{y^{n-2} e^{-\lambda y}}\,dy$$

Looks like a $\Gamma(n-1, \lambda)$ pdf

$$= n\lambda \frac{\Gamma(n-1)}{\Gamma(n)} \underbrace{\int_0^{\infty} \frac{1}{\Gamma(n-1)} \lambda^{n-1} y^{n-2} e^{-\lambda y}\,dy}_{=1}$$

$$= n\lambda \frac{\Gamma(n-1)}{\Gamma(n)}$$

$$E[\hat{\lambda}] = E\left[\frac{1}{\overline{X}}\right] = n\lambda\frac{\Gamma(n-1)}{\Gamma(n)}$$

$$= n\lambda\frac{\Gamma(n-1)}{(n-1)\Gamma(n-1)} = \frac{n}{n-1}\lambda \neq \lambda$$

So, MMEs are not necessarily unbiased estimators.

Can we find an unbiased estimator of λ for the exponential distribution based on $\overline{X}$?

$$E\left[\frac{1}{\overline{X}}\right] = \frac{n}{n-1}\lambda \Rightarrow E\left[\frac{n-1}{n}\frac{1}{\overline{X}}\right] = \lambda$$

Unbiased estimator of λ: $\boxed{\hat{\lambda} = \frac{n-1}{\sum_{i=1}^{n} X_i}}$

Example:

$$X_1, X_2, \ldots, X_n \overset{iid}{\sim} \Gamma(\alpha, \beta)$$

First population moment:

$$\mu_1 = \mu = E[X] = \frac{\alpha}{\beta}$$

First sample moment:

$$M_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \overline{X}$$

Equate: $\dfrac{\alpha}{\beta} = \overline{X}$

One equation, two unknowns. Go for second moments!

Second Population Moment:

$$\mu_2 = E[X^2] \;\; = Var[X] + (E[X])^2$$

$$= \frac{\alpha}{\beta} + (\frac{\alpha}{\beta})^2$$

Second sample moment:

$$M_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

Equate:

$$\frac{\alpha}{\beta^2} + \left(\frac{\alpha}{\beta}\right)^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$\frac{\alpha}{\beta} = \overline{X}$$

$$\frac{\alpha}{\beta^2} + \left(\frac{\alpha}{\beta}\right)^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$$

$$\hat{\beta} = \frac{\overline{X}}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2}$$

$$\hat{\alpha} = \frac{\overline{X^2}}{\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \overline{X}^2}$$