Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the exponential distribution with rate $\lambda > 0$.

Derive a uniformly most powerful hypothesis test of size α for

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda > \lambda_0$$

(Was $H_1 : \lambda = \lambda_1$ for $\lambda_1 > \lambda_0$)

The **uniformly most powerful** test of size α for testing

$$H_0 : \theta \in \Theta_0 \quad \text{vs.} \quad H_1 : \theta \in \Theta \backslash \Theta_0$$

is a test defined by a rejection region R^* such that

1. It has size α.

i.e. $$\max_{\theta \in \Theta_0} P(\overrightarrow{X} \in R^*; \theta) = \alpha$$

The **uniformly most powerful** test of size α for testing

$$H_0 : \theta \in \Theta_0 \quad \text{vs.} \quad H_1 : \theta \in \Theta \backslash \Theta_0$$

is a test defined by a rejection region R^* such that

2. It has higher power for all $\theta \in \Theta \backslash \Theta_0$

i.e. $\gamma_{R^*}(\theta) \geq \gamma_R(\theta)$ for all $\theta \in \Theta \backslash \Theta_0$

i.e. $P(\overrightarrow{X} \in R^*; \theta) \geq P(\overrightarrow{X} \in R; \theta)$

for all $\theta \in \Theta \backslash \Theta_0$

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from the exponential distribution with rate $\lambda > 0$.

Derive a uniformly most powerful hypothesis test of size α for

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda > \lambda_0$$

Step One:

Consider the simple versus simple hypotheses

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda = \lambda_1$$

for some fixed $\lambda_1 > \lambda_0$.

Steps Two, Three, and Four:

Find the best test of size α for

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda = \lambda_1$$

for some fixed $\lambda_1 > \lambda_0$.

This test is to reject H_0, in favor of H_1 if

$$\overline{X} < \frac{\chi^2_{1-\alpha,2n}}{2n\lambda_0}$$

This test is to reject H_0, in favor of H_1 if

$$\overline{X} < \frac{\chi^2_{1-\alpha,2n}}{2n\lambda_0}$$

Note that this test does not depend on the particular value of λ_1.

*** It does, however, depend on the fact that $\lambda_1 > \lambda_0$.

"Reject H_0, in favor of H_1, if

$$\left(\frac{\lambda_0}{\lambda_1}\right)^n e^{-(\lambda_0-\lambda_1)\sum_{i=1}^n x_i} \leq c$$

$$\vdots$$

$$-(\lambda_0-\lambda_1)\sum_{i=1}^n x_i \leq c_1$$

$$\sum_{i=1}^n x_i \leq c_2$$

“Reject H_0, in favor of H_1, if

$$\left(\frac{\lambda_0}{\lambda_1}\right)^n e^{-(\lambda_0-\lambda_1)\sum_{i=1}^n X_i} \leq c$$

$$\vdots$$

$$-(\lambda_0 - \lambda_1)\sum_{i=1}^{n} X_i \leq c_1$$

$$\sum_{i=1}^{n} X_i \geq c_2$$

if $\lambda_1 < \lambda_0$

The best (most powerful) test of

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda = \lambda_1$$

for $\lambda_1 > \lambda_0$ is to reject H_0, in favor of H_1 if

$$\overline{X} < \frac{\chi^2_{1-\alpha,2n}}{2n\lambda_0}$$

Note that this test does not depend on the particular value of λ_1 as long as $\lambda_1 > \lambda_0$.

It is the uniformly most powerful (best) test for

$$\boxed{H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda > \lambda_0}$$

The "UMP" test for

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda > \lambda_0$$

is to reject H_0, in favor of H_1 if

$$\overline{X} < \frac{\chi^2_{1-\alpha,2n}}{2n\lambda_0}$$

The "UMP" test for

$$H_0 : \lambda = \lambda_0 \quad \text{vs.} \quad H_1 : \lambda < \lambda_0$$

is to reject H_0, in favor of H_1 if

$$\overline{X} > \frac{\chi^2_{\alpha,2n}}{2n\lambda_0}$$

Does there exist a "UMP" test for

$$H_0 : \lambda = \lambda_0 \quad \textbf{vs.} \quad H_1 : \lambda \neq \lambda_0 \quad \textbf{?}$$

Answer: No!

For any $\lambda_1 \neq \lambda_0$,

- The best test if $\lambda_1 > \lambda_0$ is to reject H_0 if

$$\overline{X} < \frac{\chi^2_{1-\alpha,2n}}{2n\lambda_0}$$

- The best test if $\lambda_1 < \lambda_0$ is to reject H_0 if

$$\overline{X} > \frac{\chi^2_{\alpha,2n}}{2n\lambda_0}$$

There is no one best test that we can use for all $\lambda_1 \neq \lambda_0$!