1 Introduction

The Black-Scholes model[1] concerns with the option pricing problems and has achieved great success, especially in stock option pricing. It derived a parabolic second order PDE for the value $u(\mathrm{s}, t)$ of an option on stocks. However, it is a well-known fact that in actual markets the Black-Scholes assumptions are violated. The most apparent violation is that the volatility implied from traded options, the implied volatility, is not constant but exhibits a strike dependency and a term structure. This strike dependency is usually referred to as the implied volatility smile. The stochastic properties of interest rate r and volatility σ often lead to more complicated approaches because analytic solutions are usually absent.

A stochastic volatility model that has been particularly successful at explaining the implied volatility. It was first introduced to traditional model in the 1980's when Hull and White[2] and others generalized the Black-Scholes model. Heston model[3] was later presented in 1993 which offered an analytic formula in semi closed-form for the price of a vanilla option. A virtu of the Heston model is that, contrary to e.g. local volatility which is important when pricing forward skew dependent claims. It make the Heston model a prominent candidate for valuing and hedging exotic options. Moreover, a closed-form solution exists and the calculations for Greeks are more straightforward.

Heston model is a nice benchmark in testing numerical schemes dealing with parabolic partial differential equations(PDE). If there are no exists exact solution of PDE, then we can approximation by numerical computation.

2 The Heston model

Many empirical studies have indicated that the long-return of underlying asset, such as stock, is not always normal distributed. At the same time, the return and volatility are negative correlated. Those facts cannot be sufficiently reflected by the Black-Scholes model. In contrast, the Heston model is much more appropriate, since it can present many different distributions. The reason for presenting many distributions is the parameter ρ. It is the correlation between the two dependent Brownian motions, also representing the relationship between the return and the volatility of underlying asset.

In the following we describe stock price and variance processes, a partial differential inequality, an initial value and boundary conditions for the European option pricing model. This section defines a problem whose numerical solution is studied in the consecutive sections.

The Heston model concern with cases where volatility is stochastic. Assume that the spot index follows

$$dS(t) = S(t)[\mu dt + \sqrt{\nu(t)} dW_1(t)], \tag{1}$$

where $S(t)$ is spot price of equity, $\nu(t)$ is variance process and $W_1(t)$ is a Wiener stochastic process. The parameter mu is the deterministic growth rate of the stock price and $\sqrt{\nu(t)}$ is the standard deviation (the volatility) of the stock returns dS/S.

In Heston model, the volatility process follows

$$d\nu(t) = \kappa(\theta - \nu(t))dt + \sigma\sqrt{\nu(t)} dW_2(t), \tag{2}$$

where

$$\begin{aligned} \kappa &: \text{Mean-reversion rate of volatility} \\ \theta &: \text{Reversion level of variance} \\ \sigma &: \text{Volatility of variance} \\ W_2(t) &: \text{Another Wiener stochastic process} \end{aligned}$$

There two Wiener stochastic processes W_1 and W_2 are correlated such that $dW_1 \cdot dW_2 = \rho dt$ where $\rho \in [-1, 1]$. The stochastic volatility model (2) for the variance is related to the square-root process for Feller[4] and Cox, Ingersoll and Ross[5]. For the square-root process (2) the variance is always positive and if $2\kappa\theta > \sigma^2$ with zero boundary condition, which is known as the Feller condition, then it cannot reach zero. Note that the deterministic part of volatility process is asymptotically stable if $\kappa > 0$. Clearly, that equilibrium point is $\nu(t) = \theta$.

Heston's stochastic volatility model implies that the price U of contingent claims obeys the Garman's partial differential equation, which is applying the Ito lemma and standard arbitrage arguments

$$\begin{aligned} \frac{1}{2}\nu s^2 \frac{\partial^2 U}{\partial s^2} + \rho\sigma\nu s \frac{\partial^2 U}{\partial s \partial \nu} + \frac{1}{2}\sigma^2 \nu \frac{\partial^2 U}{\partial \nu^2} + rs \frac{\partial U}{\partial s} \\ + [\kappa(\theta - \nu(t)) - \lambda(s, \nu, t)] \frac{\partial U}{\partial \nu} - rU + \frac{\partial U}{\partial t} = 0 \end{aligned} \tag{3}$$

where λ is the market price of volatility risk that is independent of particular asset. It can be obtained theoretically from any asset depending on volatility. The Heston PDE (3) can be viewed as a time-dependent convection-diffusion-reaction equation, on an unbounded two dimensional spatial domain.

For a European call option as an example. Assume that the exercise price to be K where $K > 0$ and expiring time T. The price is considered in the rectangular area of $[0, \infty] \times [0, \infty]$ and on time horizon is $[0, T]$. The option price obeys equation (3) with boundary

$$
\begin{aligned}
&U(s,\nu,T) = max(0, s-K), \\
&U(0,\nu,t) = 0 \qquad (0 \le t \le T), \\
&\frac{\partial U}{\partial s}(\infty,\nu,t) = 1, \\
&rs\frac{\partial U}{\partial s}(s,0,t) + \kappa\eta\frac{\partial U}{\partial \nu}(s,0,t) - rU(s,0,t) + \frac{\partial U}{\partial t}(s,0,t) = 0, \\
&U(s,\infty,t) = s.
\end{aligned}
\tag{4}
$$

The first of above equations is the payoff yield at expiration, second term is a boundary condition at $s = 0$ holds and below equation is a boundary condition of s. Clearly, second and fifth are of Dirichlet type, whereas third is of Neumann type.

Note that an initial condition at $\nu = 0$ is specified which is PDE boundary.

In order to use finite difference approximations for space variables, we truncate it into a finite size computational domain

$$
(s,\nu,t) \ \in \ [0,S] \times [0,V] \times [0,T] \tag{5}
$$

where S and V are sufficiently large. We reduce the domain by finite, this yields a negligible modeling error.

2.1 The closed-form of the Heston PDE

Now we find the solution of the Heston PDE(3) two difference ways. First of ways using characteristic functions by Heston and the other one used the Green's function by Shaw, it referred to his working paper [6].

2.1.1 Heston's approach for exact solution

Heston solved above PDE (3) given boundary condition same as (4). Suppose that the solution to the Heston PDE is like the form of the Black-Scholes formula then we can analogize that

$$
U(s,\nu,t) = sP_1 - KP(t,T)P_2 \tag{6}
$$

where the first term P_1 is the present value of the spot asset upon optimal exercise, and the second term P_2 is the conditional risk neutral probability that the asset price will greater than strike price K and $P(t,T) = e^{-r(T-t)}$ represents the risk-free discounting asset. P_1 and P_2 are what we are going to find. Both of these terms must satisfy the Heston PDE (3). Make the transform $x = \ln(s)$ and substituting the proposed solution (6) into the Heston PDE. Then we will get

$$\begin{aligned}\frac{1}{2}\nu\frac{\partial^2 P_j}{\partial x^2} + \rho\sigma\nu\frac{\partial^2 P_j}{\partial x\partial\nu} + \frac{1}{2}\sigma^2\nu\frac{\partial^2 P_j}{\partial\nu^2} + (r + u_j\nu)\frac{\partial P_j}{\partial x} \\ +(a - b_j\nu)\frac{\partial P_j}{\partial\nu} + \frac{\partial P_j}{\partial t} = 0 \qquad \text{for} \quad j = 1, 2\end{aligned} \tag{7}$$

where $u_1 = \frac{1}{2},\ \ u_2 = -\frac{1}{2},\ \ a = \kappa\eta,\ \ b_1 = \kappa + \lambda - \rho\sigma,\ \ b_2 = \kappa + \lambda.$

Considering the payoff of the option, they are subject to the initial condition

$$P_j(x, \nu, T; \ln K) = \chi_{\{x \geq \ln K\}} \tag{8}$$

P_j are the conditional probability that the option expires in-the-money :

$$P_j(x, \nu, T; \ln K) = \text{Pr}[x(T) \geq \ln K \mid x(t) = x, \nu(t) = \nu].$$

Heston showed that the characteristic function of P_j, $f_j(x, \nu,\ T;\ \varphi)$, respectively, satisfies the PDE (7).

The terminal condition is

$$f_j(x,\ \nu,\ T;\ \varphi) = e^{i\varphi x}.$$

So the solution of characteristic function is

$$f_j(x,\ \nu,\ t;\ \varphi) = \exp[C(T-t;\ \varphi) + D(T-t;\ \varphi)\nu + i\varphi x] \tag{9}$$

where (Define $\tau = T - t$)

$$\begin{aligned} C(\tau;\ \varphi) &= r\varphi i\tau + \frac{a}{\sigma^2}(b_j - \rho\sigma\varphi i + d)\tau - 2\ln[f1 - ge^{d\tau}1 - g], \\ D(\tau;\ \varphi) &= \frac{b_j - \rho\sigma\varphi i}{\sigma^2}[\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}] \end{aligned}$$

and

$$g = \frac{b_j - \rho\sigma\varphi i + d}{b_j - \rho\sigma\varphi i - d},$$
$$d = \sqrt{(\rho\sigma\varphi i)^2\ \sigma^2(2u_j\varphi i - \varphi^2)}.$$

Provided that characteristic functions φ_1, φ_2 are known the term P_1, P_2 are defined via the inverse Fourier transformation

$$P_j(x, \nu, T; \ln K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\frac{e^{-i\varphi \ln K} f_j(x, \nu, T; \varphi)}{i\varphi}]d\varphi \tag{10}$$

Then solution of Heston PDE is a direct result. Numerical integration is available because the fast decay of the integrand.

2.1.2 Shaw's approach for exact solution

Another method is the Green's function, which is more useful way in computing the Greeks. Start from the Heston PDE with given boundary conditions (4), we make transformation :

$$\tau\ =\ T - t$$
$$x = \ln s + r\tau$$
$$W = Ve^{r\tau}$$

So $V(x, \nu, \tau) = U(s, \nu, t) = U(e^{x-\nu\tau}, \nu,\ T - \tau)$ Then use the Fourier transform, let

$$W(x,\ \nu,\ \tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega x}\tilde{W}(\omega, \nu, \tau)d\omega$$
$$\tilde{W}(\omega,\ \nu,\ \tau) = \int_{-\infty}^{\infty} e^{i\omega x} W(x,\ \nu,\ \tau)dx$$

The initial condition of $\tilde{W}(\omega, \nu, \tau)$ for European call is

$$\begin{aligned}\tilde{W}(\omega, \nu, 0) &= \int_{-\infty}^{\infty} e^{i\omega x}W(x, \nu, 0)dx\ =\ \int_{-\infty}^{\infty} e^{i\omega x}V(x, \nu, 0)dx\\ &= \int_{-\infty}^{\infty} e^{i\omega x}\max(e^x - K, 0)dx\ =\ \int_{gK}^{\infty} e^{i\omega x}(e^x - K)dx\\ &= \int_{gK}^{\infty}(e^{(1+i\omega)x} - Ke^{i\omega x})dx)\ =\ \frac{K^{1+i\omega}}{i\omega - \omega^2}\end{aligned}$$

when $\mathbf{Im}(\omega) > 1$.

The Fourier transform satisfies Heston PDE

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\nu(\frac{\partial^2 W}{\partial x^2} - \frac{\partial W}{\partial x}) + \rho\sigma\nu\frac{\partial^2 W}{\partial x \partial \nu} + \frac{1}{2}\sigma^2\nu\frac{\partial^2 W}{\partial \nu^2} + [\kappa(\eta - \nu) - \lambda]\frac{\partial W}{\partial \nu}$$

So,

$$\frac{\partial \tilde{W}}{\partial \tau} = \frac{1}{2}\sigma^2\nu\frac{\partial^2 \tilde{W}}{\partial \nu^2} + [\kappa(\eta - \nu) - i\omega\sigma\rho\nu]\frac{\partial \tilde{W}}{\partial \nu} - \frac{1}{2}\nu(\omega^2 - i\omega)\tilde{W}$$

Now introduce the Green's function $G(\omega, \nu, \tau) = \frac{\tilde{W}(\omega,\nu,\tau)}{\tilde{W}(\omega,\nu,0)}$. It satisfies the above PDE i.e.

$$\frac{\partial G}{\partial \tau} = \frac{1}{2}\sigma^2\nu\frac{\partial^2 G}{\partial \nu^2} + [\kappa(\eta - \nu) - i\omega\sigma\rho\nu]\frac{\partial G}{\partial \nu} - \frac{1}{2}\nu(\omega^2 - i\omega)G \tag{11}$$

and it has specific initial condition $G(\omega, \nu, 0) = 1$. We suppose that the Green's function has form

$$G(\omega, \nu, \tau) = e^{C(\tau,\omega)+\nu D(\tau,\omega)}$$

in which

$$C(\tau;\ \omega) = \frac{\kappa\eta}{\sigma^2}(\kappa + \lambda + \rho\sigma\omega i + d)\tau - 2\ln[\frac{1 - ge^{d\tau}}{1 - g}],$$
$$D(\tau;\ \omega) = \frac{\kappa + \lambda + \rho\sigma\omega i + d}{\sigma^2}[\frac{1 - e^{d\tau}}{1 - ge^{d\tau}}]$$

and

$$g = \frac{\kappa + \lambda + \rho\sigma\omega i + d}{\kappa + \lambda + \rho\sigma\omega i - d},$$
$$d = \sqrt{(\kappa + \lambda + \rho\sigma\omega i)^2\sigma^2(\omega i - \omega^2)}.$$

Finally the price of contingent claims is

$$U = \frac{1}{2\pi}\exp -r\tau \int_{ic-\infty}^{ic+\infty} e^{-i\omega x}\tilde{W}(\omega, \nu, 0)G(\omega, \nu, \tau)d\omega, \tag{12}$$

where

$x = \log S + r\tau, \qquad \tau = T - t$

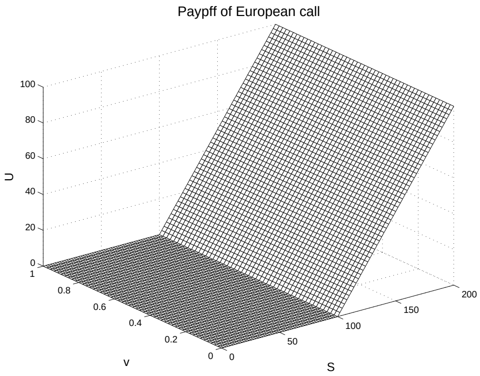
$\tilde{W}(\omega, \nu, \tau)$: the Fourier transformation of $W(\omega, \nu, \tau) = Ve^{r\tau}$

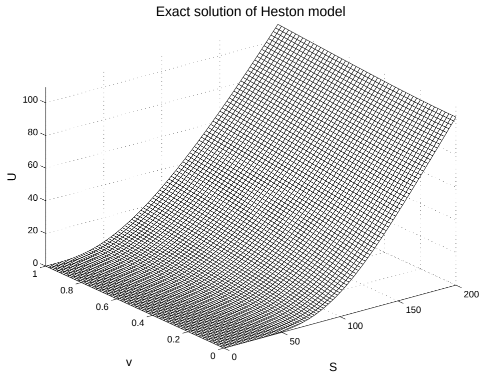
$G(\omega, \nu, \tau)$: the Green's function

The advantage of Shaw's approach is that the computation of the Greeks is straightforward.

Then the closed-form of the Heston model be used to calibrate the finite difference schemes.

Follow the specific parameter quantities chosen in [7]. The correlation factor $\rho = 0.8$, interest rate $r = 0.03$, reversion level $\eta = 0.2$, reversion rate $\kappa = 2$, and the volatility factor of volatility $\sigma = 0.3$. The maturity is $T = 1$ and strike price $K = 100$. Then the payoff and the price computed are as following :





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