GENERAL INVESTIGATIONS OF CURVED SURFACES

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These lecture notes are adapted from Spivak's *A Comprehensive Introduction to Differential Geometry* for the *Differential Geometry Learning Seminar* at the University of Waterloo. The presentation balances modern and classical thought, and thus can simultaneously be read with the minimal background of multivariable calculus, or with an understanding of smooth manifold theory.

1. A brief review of curves

First we recall some facts about curves in $\mathbb{R}^3$, which were covered in much greater detail in last week's talk. We will work with smooth curves $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ such that $\gamma' \neq 0$ at every point in the open interval I. This condition on the derivative guarantees that if we compose the curve with a function giving its length, we will get a smooth unit speed curve $\|\tilde{\gamma}\| = 1$, specifically the image of the curve may have a kink and yet be a smooth function if the curve comes to a stop.

This reparametrization is called the arc length parametrization and is of interest because of the following calculation:

$$1 = \|\gamma'\|^2 = \langle \gamma', \gamma' \rangle$$
$$0 = \langle \gamma', \gamma' \rangle' = 2\langle \gamma', \gamma'' \rangle.$$

Hence a curved parametrized by arc length will always have an 'acceleration' vector that is orthogonal to its velocity.

We define a unit normal vector to a curve γ at t for which $\gamma''(t) \neq 0$ as a multiple of γ'', specifically under some orientation considerations there is a preferred of the two directions to define a length one vector $\boldsymbol{n}$. Then the curvature of a curve is a signed scalar quantity defined by: $\gamma'' = \kappa \boldsymbol{n}$, with $\kappa = 0$ when $\gamma'' = 0$. Curvature is given an interpretation by best approximating a curve at a given point by a circle - i.e. three non-co-linear points in space determine a unique circle and the limits of the center and radius of this circle as the the points lie on a curve approaching a given point are guaranteed to exist if $\gamma'' \neq 0$ at that point.

2. What they knew about surfaces before Gauss

Formally we will study surfaces that are immersed 2-submanifolds of $\mathbb{R}^3$. What this means for us is that the tangent space of surface M at point p has the interpretation of a plane in $\mathbb{R}^3$ that lies tangent to the M at that p, denoted M_p. We have two choices for a unit normal to M_p. Our results will be local of nature and thus we can assume all our surfaces to be orientable. Then the unit normal to a surface $\nu(p)$ is defined such that if (u, v) is a positively oriented basis for M_p, then $(\nu(p), u, v)$ is positively oriented in $\mathbb{R}^3$. We won't spend too much time on questions of orientation and take the function ν as given. Note also that the orientation on the surface defines the choice of unit normal on a curve in the surface even when $\gamma'' = 0$.

The study of surfaces began by investigating the curvature of curve cut out by a normal plane to the surface; that is, one containing the normal vector $\nu(p)$ and given by some tangent vector $X_p \in M_p$. The only significant result toward this end came from Euler in 1760, and was generalized by Meusnier in 1776, with Gauss's work coming some 50 years later.

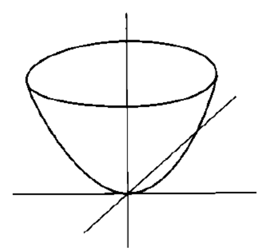
First consider the plane generated by $\nu(p)$ and $X \in M_p$ with $X \neq 0$, and give it an orientation such that $(\nu(p), X)$ is positive. Then let γ be the curve on the intersection of the plane and M parametrized by unit speed with $\gamma(0) = p$ and $\gamma'(0)$ a positive multiple of X_p. Then let the curvature of this curve be κ_X. Notice that $\kappa_X = \kappa_{-X}$ as both the direction of curve and orientation of the plane are changed.

Euler was able to prove:

Theorem 2.1. (Euler): *If κ_X is non-constant over $M_p \setminus \{0\}$, then there exist directions X_1 and X_2 such that κ_{X_1} and κ_{X_2} are the unique (to scaling) maximal and minimal curvatures. Moreover, if X is an arbitrary direction, then $\kappa_X = \kappa_{X_1} \cos^2\theta + \kappa_{X_2} \sin^2\theta$, where θ is the smallest positive angle formed between X and X_1.*

Proof. Curvature does not depend on orientation preserving rotations or translations, thus we pick co-ordinates such that $p = (0,0,0)$ and such that M_p is the xy plane. Then by the implicit function theorem, locally the surface is given by $\{(x, y, f(x, y))\}$.

$$f(0,0) = 0$$
$$\frac{\partial f}{\partial x}(0,0) = 0$$
$$\frac{\partial f}{\partial y}(0,0) = 0.$$



We will attempt to rotate the xy plane as to have $\frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$.

Such rotations are given by the matrix:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Then rotating through by an angle of θ gives surface $\{(x, y, f \circ R_{-\theta}(x, y))\}$, and we shall denote $f \circ R_{-\theta}$ by f_θ.

Now,

$$\begin{aligned}
f_\theta(x, y) &= f(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta) \\
\frac{\partial f_\theta}{\partial x} &= D_1 f(-,-)\cos\theta + D_2 f(-,-)\sin\theta \\
\frac{\partial^2 f_\theta}{\partial x \partial y} &= D_{11} f(-,-)(-\sin\theta\cos\theta) + D_{12} f(-,-)(\cos^2\theta - \sin^2\theta) \\
&\quad + D_{22} f(-,-)(\sin\theta\cos\theta) \\
\frac{\partial^2 f_\theta}{\partial x \partial y}(0,0) &= (\cos 2\theta) D_{12} f(0,0) + (\sin 2\theta)\frac{D_{22} f(0,0) - D_{11} f(0,0)}{2}
\end{aligned}$$

Then if $D_{22}f(0,0) = D_{11}f(0,0)$, let $\theta = \frac{\pi}{4}$, and otherwise set

$$\tan 2\theta = \frac{2D_{12}f(0,0)}{D_{22}f(0,0) = D_{11}f(0,0)}.$$

Thus we assume $\frac{\partial^2 f}{\partial x \partial y}(0,0) = 0$.

Let us now consider the cures cut out by the xz and yz planes. For the xz plane, out curve will be given by $c(t) = (t, f(t,0))$. We recall a formula for curvature of plane curves from the previous session:

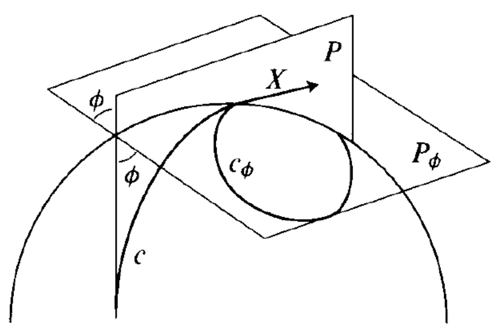
$$\kappa_x = \frac{\dot{c}_1 \ddot{c}_2 - \ddot{c}_1 \dot{c}_2}{(\dot{c}_1^2 + \dot{c}_2^2)^{3/2}} = \frac{\partial^2 f}{\partial x^2}(0,0)$$
$$\kappa_y = \frac{\partial^2 f}{\partial y^2}(0,0).$$

Now a plane that makes an angle of θ with the xz plane defines the curve $c(t) = (t, f(t\cos\theta, t\sin\theta))$, if κ is the curvature of this curve, then:

$$\kappa = \frac{d^2}{dt^2} f(t\cos\theta, t\sin\theta) = \kappa_x \cos^2\theta + \kappa_y \sin^2\theta$$

Thus we see that if κ is non-constant, the maximal curvatures arise in orthogonal directions and the formula we claimed holds. $\square$

What Meusnier was able to further prove is that if we rotate the normal plane about the axis defined by X through an angle ϕ, then $\kappa_\phi \cos\phi = \kappa_X$. The proof is omitted for lack of direct relevance to today's talk.



And that was all mathematicians were able to do until Gauss's paper of 1827, which set in motion the basic ideas of Riemannian geometry and sets a course we will follow to a motivated definition of the Riemannian curvature tensor and connections on both the tangent bundle and later on general principal bundles.

3. Preliminaries on surfaces

We begin by assuming that M is an oriented surface in $\mathbb{R}^3$ and will make use of the map $\nu : M \to S^2$ we have defined before, called the Gauss map. This surface can be given to us in one of three different ways. Locally, each approach is fully general.

- First, M could be the set $\{p : W(p) = 0\}$ for some smooth function $W : \mathbb{R}^3 \to \mathbb{R}$ which admits 0 as a regular value (that is if $W(p) = 0$, then $\nabla W(p) \neq 0$).

- Second, the surface can be presented as a chart giving local co-ordinates, that is the set $f(\mathbb{R}^2)$ for $f : M \to \mathbb{R}^2$ a diffeomorphism (bijective with smooth inverse).
- And thirdly, as a special case of both of the above two, the surface could be $\{(x, y, g(x, y))\}$ as in the previous theorem.

In this section we will find the direction $\nu(p)$ explicitly in each approach. We will do this both in modern language and classically. Note that $\nu(p)$ is defined to be the normal to M_p satisfying orientation conditions outlined in the previously.

Remark 3.1. **Classically** when a surface is defined as $W(q) = 0$ by the second method, then if (dx, dy, dz) is an infinitesimal displacement in $\mathbb{R}^3$, and $q = (x, y, z)$ then:

$$W(x + dx, y + dy, z + dz) = W(x, y, z) + \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz$$

Thus if (dx, dy, dz) is a displacement in M, then $dW = 0$ as W is constant on M. Thus for all displacements (dx, dy, dz) on the surface:

$$dW = \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz = 0,$$

and hence the vector normal to all infinitesimal displacements on the curve $\nu(q)$ is:

$$\nu = \text{normalized}\left(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y}, \frac{\partial W}{\partial z}\right).$$

Remark 3.2. **In modern language** the cotangent vector $dW \in \mathbb{R}^3_q$ acts on a tangent vector given generally as $c'(0)$ for a smooth curve with $c(0) = q$, by $dW(c'(0)) = \frac{\partial(W \circ c(t))}{\partial t}(0)$. Tangent vectors in M_q arise when $c \in C^\infty(M)$, but then $W \circ c = 0$, so $dW(X) = 0$ when $X \in M_p$. Because of how covariant tensors transform, we have the expansion:

$$dW = \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz.$$

And the conclusion above holds.

Remark 3.3. **Classically** when the surface is given as the image of $f : \mathbb{R}^2 \to M$, and say f is a function of s and t. Then:

$$f(s + ds, t + dt) = f(s, t) + \left(\frac{\partial f^1}{\partial s}ds + \frac{\partial f^1}{\partial t}dt, \frac{\partial f^2}{\partial s}ds + \frac{\partial f^2}{\partial t}dt, \frac{\partial f^3}{\partial s}ds + \frac{\partial f^3}{\partial t}dt\right).$$

So if (dx, dy, dz) is the displacement on M arising from a displacement (dt, dt), then:

$$dx = \frac{\partial f^1}{\partial s}ds + \frac{\partial f^1}{\partial t}dt$$
$$dx = \frac{\partial f^2}{\partial s}ds + \frac{\partial f^3}{\partial t}dt$$
$$dx = \frac{\partial f^2}{\partial s}ds + \frac{\partial f^3}{\partial t}dt$$

Thus as $\nu(q)$ is perpendicular to displacements on the curve from q:

$$\left(\nu^1\frac{\partial f^1}{\partial s} + \nu^2\frac{\partial f^2}{\partial s} + \nu^3\frac{\partial f^3}{\partial s}\right)ds + \left(\nu^1\frac{\partial f^1}{\partial t} + \nu^2\frac{\partial f^2}{\partial t} + \nu^3\frac{\partial f^3}{\partial t}\right)dt = 0$$

Which is true for any displacement (ds, dt), thus each term in parenthesise is zero.

Remark 3.4. **In modern language**, let $q = f(s,t)$, then if $f^{-1} = \chi$ is a co-ordinate system for the surface, M_q is spanned by tangent vectors $\frac{\partial}{\partial\chi^1}$ and $\frac{\partial}{\partial\chi^2}$. Because of how contravariant tensors transform (pick a C^∞ function to act on and use chain rule):

$$\frac{\partial}{\partial\chi^1} = \frac{\partial f^1}{\partial s}\frac{\partial}{\partial x} + \frac{\partial f^2}{\partial s}\frac{\partial}{\partial y} + \frac{\partial f^3}{\partial s}\frac{\partial}{\partial z}$$
$$\frac{\partial}{\partial\chi^2} = \frac{\partial f^1}{\partial t}\frac{\partial}{\partial x} + \frac{\partial f^2}{\partial t}\frac{\partial}{\partial y} + \frac{\partial f^3}{\partial t}\frac{\partial}{\partial z}.$$

Thus we have found two vectors that ν is perpendicular to as above. To finish both approaches, we find:

$$\nu = \text{normalized}\left(\frac{\partial f^1}{\partial s}, \frac{\partial f^2}{\partial s}, \frac{\partial f^3}{\partial s}\right) \times \left(\frac{\partial f^1}{\partial t}, \frac{\partial f^2}{\partial t}, \frac{\partial f^3}{\partial t}\right).$$

Remark 3.5. **Finally** if the surface is given as $\{(x, y, g(x, y))\}$, then this is clearly made into a function of two parameters, thus by the previous two remarks:

$$\nu = \text{normalized}\left(1, 0, \frac{\partial g}{\partial x}\right) \times \left(0, 1, \frac{\partial g}{\partial y}\right).$$

4. Curvature

In the previous section, we performed calculations to find information about the Gauss map ν. This function is the star player in the definition of curvature.

The **Gaussian curvature** of M at point q denoted $K(q)$, is defined by Gauss as the limit over small open neighborhoods A of q of the quotient of the area of $\nu(A)$ in S^2 by the area of A, with sign determined by whether ν is orientation preserving or reversing at q relative to the usual orientation of S^2 (his discussion of sign is a two page wall of text). It is not clear why this limit should exist, and is easy to see arguments against this by considering ellipses whose axes lie along directions of minimal and maximal curvature as in Euler's theorem with axis length falling off as $1/r^2$ and $1/r$. Gauss is able to produce formulas for the curvature notwithstanding these difficulties.

A modern definition comes out from the theory of integrating differential forms. Suppose σ' is the 2-form on S^2 which gives rise to the volume element that comes from $S^2 \subset \mathbb{R}^3$, and such that $\sigma'(v, w) \geq 0$ when (v, w) is positively oriented. Similarly, let dV be the 2-form on M with this property (such a representative of the volume element exists as M is assumed orientable, and here M has the induced Riemannian metric). Then:

$$\text{area}\,\nu(A) = \int_{\nu(A)} \sigma' = \pm\int_A \nu^*\sigma'$$

So Gauss would have:

$$K(q) = lim_{A\to q}\frac{\pm\int_A \nu^*\sigma'}{\int_A dV}.$$

Note that the division of 2-forms makes sense and we define:

$$K(q) = \frac{\pm\nu^*\sigma'}{dV}.$$