Differential Geometry Learning Seminar - Curves

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Abstract

This talk is about curves. We want to answer the following questions and see some examples along the way. First, the basic ones: what is a parametrized curve; what is its arc-length, and why do we use the arc-length parametrization? Then we investigate the properties of curves. How do we assign a numerical value to curvature? What is torsion for a space curve? Finally, we will look at the Frenet-Serret equations and see why unit-speed space curves are determined by their curvature and torsion.

Outline

1. Define a curve
2. Arc-Length parametrization; unit-speed curve
3. Caution: not all parametrizations of a curve have a unit-speed reparametrization
4. Curvature as the distance that the curves pulls away from the tangent vector at a point
5. Frenet-Serret equations
6. unit-speed space curves are determined by curvature and torsion

1 Basics

Definition 1 (Regular parametrized curve)**.** A regular parametrized curve is a continuously differentiable function,

$$\gamma : I \subseteq \mathbb{R} \to \mathbb{R}^n,$$

such that γ' is nowhere zero, for some interval I.

Definition 2 (Regular curve)**.** A family of regular parametrized curve, with the equivalence relation of orientation preserving parameter transformation. (bijective function that is smooth (continuously differentiable) and has non-zero derivative everywhere).

Recall what a tangent vector is. A caveat: it is necessary to define tangent vectors at parameter values rather than at points on the curve.

Example 1 (Limaçon curve)**.** (See Figure 1 and Example 1.1.7 in Pressley.)

$$\gamma(t) = (1 + 2\cos t)(\cos t, \sin t)$$

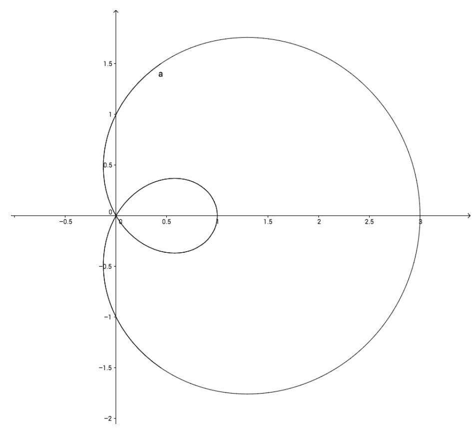

Figure 1: Limaçon

The curve self intersects at $(0,0)$ for $t = 2\pi/3$ and $t = 4\pi/3$. But the tangent vectors at the two points are different.

$$\begin{aligned}
\gamma'(t) &= (-\sin t - 4\cos t \sin t, \cos t - 2\sin^2 t + 2\cos^2 t) \\
&= (-\sin t - 2\sin 2t, \cos t + 2\cos 2t) \\
\gamma'(2\pi/3) &= (\sqrt{3}/2, -3/2) \\
\gamma'(4\pi/3) &= (-\sqrt{3}/2, -3/2)
\end{aligned}$$

Moral of this example: a point on a curve corresponds to a value in the parameter in $\mathbb{R}$, not a geometric point in $\mathbb{R}^n$.

Definition 3 (Tangent vector)**.** If γ is a regular parametrized curve, then the tangent vector of γ at the parameter value t is the first derivative $\gamma'(t)$.

A unit-speed parametrization is important.

Proposition 1. *The acceleration vector is perpendicular to the velocity vector for a unit-speed curve.*

Definition 4 (Arc-Length)**.** The arc-length of γ starting at $\gamma(t_0)$ is the function

$$s(t) = \int_{t_0}^{t} \|\gamma'(u)\| du$$

Definition 5 (Reparametrization of a curve)**.** $\tilde{\gamma} : \tilde{I} \to \mathbb{R}^n$ is a reparametrization of $\gamma : I \to \mathbb{R}^n$ if there is a smooth (continuously differentiable) bijection

$$\phi : \tilde{I} \to I$$

such that $\tilde{\gamma}(\tilde{t}) = \gamma(\phi(\tilde{t}))$ for all $\tilde{t} \in \tilde{I}$ and $\phi' > 0$.

Note that since ϕ is bijective, reparametrization is symmetric. It is actually an equivalence relation.

Consider the following arc-length parametrization of a curve γ.

$$\begin{aligned}
\tilde{\gamma}(s(t)) &= \gamma(t) \\
\tilde{\gamma}'(s(t))s'(t) &= \gamma'(t) \\
\|\tilde{\gamma}'\|s' &= \|\gamma'\| \\
\|\tilde{\gamma}'\|s' &= s' \\
\|\tilde{\gamma}'\| &= 1.
\end{aligned}$$

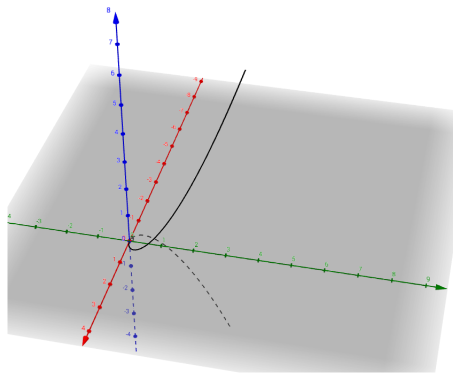

Figure 2: Twisted cubic

This shows that the arc-length parametrization is unit-speed, which will be helpful due to the reason mentioned above.

It can be difficult to write down a unit-speed reparametrization.

Example 2 (Twisted cubic)**.**

$$\gamma(t) = (t, t^2, t^3), t \in \mathbb{R}$$
$$\gamma'(t) = (1, 2t, 3t^2)$$

The arc-length function starting at 0 is

$$s(t) = \int_0^t \sqrt{1 + 4u^2 + 9u^4}du.$$

This is an example of an elliptic integral. It cannot be evaluated using usual integration techniques.

Another caveat: a regular curve can have a parametrization (this depends on the definition you use) that is not regular. Not all parametrizations of a curve has a unit-speed reparametrization.

Example 3 (Pathological parabola)**.**

$$\gamma(t) = (t^3, t^6)$$

2 Properties

Define curvature as the distance that a curve pulls away from its tangent vector.

Definition 6 (Curvature)**.** The curvature of a plane curve γ is the norm of its acceleration $\kappa = \|\gamma''\|$.

If we look at how far γ moves away from its tangent line at $\gamma(t)$ as $t \mapsto t + \Delta t$ in the direction of the principal normal $n = \gamma''/\|\gamma''\|$ at $\gamma(t)$, we see that the curvature is the coefficient of the second order term in this expression. See Figure 3.

$$\begin{aligned}
&(\gamma(t + \Delta t) - \gamma(t)) \cdot n \\
=&(\gamma(t) + \gamma'(t)\Delta t + \frac{1}{2}\gamma''(t)(\Delta t)^2 + O((\Delta t)^3) - \gamma(t)) \cdot n \\
=&\frac{1}{2}\kappa(\Delta t)^2 + O((\Delta t)^3)
\end{aligned}$$

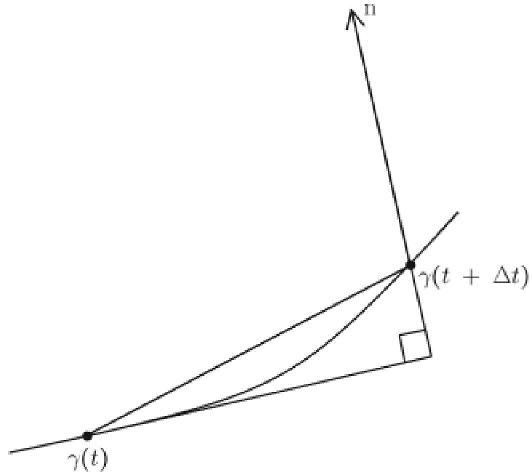

Figure 3: Curvature motivation

Since the curve is unit speed, $\|\gamma'\| = 1$, $\gamma' \perp n$, so $\gamma' \cdot n = 0$.

Express curvature in terms of the curve.

Proposition 2.

$$\kappa = \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3}$$

Definition 7 (Principal normal)**.** Let $\gamma(s)$ be a unit-speed parametrized curve in $\mathbb{R}^3$. Denote the tangent vector by $\mathbf{t} = \dot{\gamma}$. If the curvature $\kappa(s)$ is non-zero, then the principal normal of γ at $\gamma(s)$ is defined to be

$$\mathbf{n}(s) = \frac{1}{\kappa(s)}\dot{\mathbf{t}}(s).$$

Note that

$$\begin{aligned}\|\mathbf{n}(s)\| &= \frac{1}{|\kappa(s)|}\|\dot{\mathbf{t}}(s)\| \\ &= \frac{1}{\kappa(s)}\kappa(s) \\ &= 1.\end{aligned}$$

Since γ is a unit-speed parametrization, $\mathbf{t} \cdot \mathbf{n} = 0$. So they are perpendicular vectors.

Definition 8 (Binormal)**.** The binormal of γ at the point $\gamma(s)$ is defined to be

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}$$

The binormal is a unit vector perpendicular to both the tangent and principal normal vectors.

Remark 1. The set $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a right-handed orthonormal basis for $\mathbb{R}^3$.

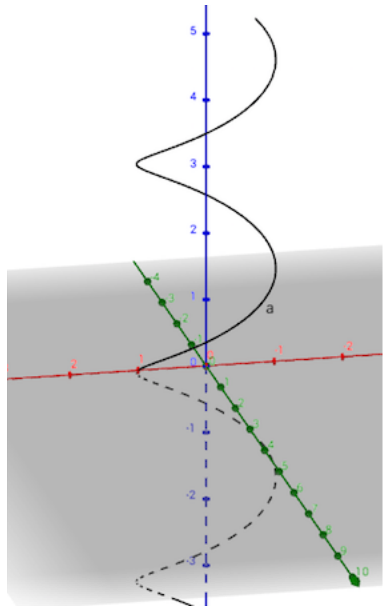

Figure 4: Circular helix

Remark 2. Since the binormal is a unit vector, we have $\mathbf{b} \cdot \dot{\mathbf{b}} = 0$. Also by product rule of cartesian/wedge product,

$$\begin{aligned}\frac{d}{ds}(\mathbf{t} \times \mathbf{n}) &= \frac{d\mathbf{t}}{ds} \times \mathbf{n} + \mathbf{t} \times \frac{d\mathbf{n}}{ds} \\ &= \dot{\mathbf{t}} \times \mathbf{n} + \mathbf{t} \times \dot{\mathbf{n}} \\ &= \kappa \mathbf{n} \times \mathbf{n} + \mathbf{t} \times \dot{\mathbf{n}} \\ &= \mathbf{t} \times \dot{\mathbf{n}}.\end{aligned}$$

Then $\dot{\mathbf{b}}$ is perpendicular to both $\mathbf{t}$ and $\mathbf{b}$, which implies it is a constant multiple of $\mathbf{n}$.

Definition 9 (Torsion)**.** We attribute a special meaning to this constant. Let the torsion of γ be the constant τ such that

$$\dot{\mathbf{b}} = -\tau \mathbf{n}.$$

Note that the torsion will only be defined if the curvature is non-zero. For a non unit-speed curve, define its torsion to be the one given to its unit reparametrization.

We can also express torsion in terms of the curve gamma, like we did for curvature.

Proposition 3.

$$\tau = \frac{\dddot{\gamma} \cdot (\dot{\gamma} \times \ddot{\gamma})}{\|\dot{\gamma} \times \ddot{\gamma}\|^2}$$

Example 4 (Circular helix)**.** Compute the tangent vector, principal normal, binormal, curvature and torsion of the circular helix. See Figure 4.

$$\gamma(\theta) = (a\cos\theta, a\sin\theta, b\theta)$$

First, the first, second and third derivatives of γ are,

$$\begin{aligned}
\dot{\gamma} &= (-a\sin\theta, a\cos\theta, b) \\
\ddot{\gamma} &= (-a\cos\theta, -a\sin\theta, 0) \\
\dddot{\gamma} &= (a\sin\theta, -a\cos\theta, 0).
\end{aligned}$$

Then the cross product,

$$\dot{\gamma} \times \ddot{\gamma} = (ab\sin\theta, -ab\cos\theta, a^2).$$

Therefore

$$\begin{aligned}
\mathbf{t} &= \frac{1}{\sqrt{a^2+b^2}}(-a\sin\theta, a\cos\theta, b) \\
\mathbf{n} &= (-\cos\theta, -\sin\theta, 0) \\
\mathbf{b} &= \frac{1}{\sqrt{a^2+b^2}}(b\sin\theta, -b\cos\theta, a).
\end{aligned}$$

(There may be a sign difference due to reparametrization from unit-speed parametrization).
The curvature is

$$\begin{aligned}
\kappa &= \frac{\|\ddot{\gamma} \times \dot{\gamma}\|}{\|\dot{\gamma}\|^3} \\
&= \frac{\sqrt{a^2b^2+a^4}}{\sqrt{a^2+b^2}^3} \\
&= \frac{|a|\sqrt{a^2+b^2}}{\sqrt{a^2+b^2}^3} \\
&= \frac{|a|}{a^2+b^2}.
\end{aligned}$$

And the torsion is

$$\begin{aligned}
\tau &= \frac{\dddot{\gamma} \cdot (\dot{\gamma} \times \ddot{\gamma})}{\|\dot{\gamma} \times \ddot{\gamma}\|^2} \\
&= \frac{a^2b\sin^2\theta + a^2b\cos^2\theta}{a^2b^2\sin^2\theta + a^2b^2\cos^2\theta + a^4} \\
&= \frac{a^2b}{a^2b^2+a^4} \\
&= \frac{b}{a^2+b^2}.
\end{aligned}$$

Note that the torsion becomes zero if $b = 0$, in which case the helix is just a circle in the xy-plane.

Torsion measures how much the plane spanned by the tangent vector and the principal normal changes.

Proposition 4. *Let γ be a regular curve with non-zero curvature everywhere (so that it has a unit-speed reparametrization and torsion is defined everywhere). Then the image of γ is contained in a plane if and only if the torsion is zero at every point of the curve.*

Remark 3. When $a = r, b = 0$, we see that the curvature of a circle is $1/r$, which agres with our intuition of curvature.

3 Frenet-Serret equations and applications

Theorem 5 (Frenet-Serret equations)**.** *Let γ be a unit-speed curve with nowhere vanishing curvature. Then the Frenet-Serret equations are*

$$\begin{aligned}\dot{\mathbf{t}} &= \kappa\mathbf{n}\\ \dot{\mathbf{n}} &= -\kappa\mathbf{t}+\tau\mathbf{b}\\ \dot{\mathbf{b}} &= -\tau\mathbf{n}.\end{aligned}$$

We get a skew-symmetric matrix that expresses $\dot{\mathbf{t}}, \dot{\mathbf{n}}, \dot{\mathbf{b}}$ in terms of $\mathbf{t}, \mathbf{n}, \mathbf{b}$ (it is equal to the negative of its transpose).

$$\begin{bmatrix} 0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}$$

Proposition 6. *A unit-speed curve in $\mathbb{R}^3$ with constant curvature and zero torsion is the parametrization of (part of) a circle.*

Proposition 7. *A unit-speed curve in $\mathbb{R}^3$ with constant curvature and nonzero constant torsion is the parametrization of (part of) a circular helix.*

Unit-speed space curves are fully determined by curvature and torsion.

Theorem 8. *Let $\gamma, \tilde{\gamma}$ be unit-speed regular parametrized curves. Let $\kappa, \tilde{\kappa}$ be the curvature and $\tau, \tilde{\tau}$ be the torsion of $\gamma, \tilde{\gamma}$ respectively.*

1. *If they have the same curvature and torsion everywhere, ie for all s,*

$$\begin{aligned}\kappa(s) &= \tilde{\kappa}(s)\\ \tau(s) &= \tilde{\tau}(s),\end{aligned}$$

then there exists an isometry (a rotation and translation) M such that $\tilde{\gamma} = M(\gamma)$.

2. *If $k > 0, t$ are real smooth functions, then there exists a unit-speed regular parametrized curve γ which has curvature k and torsion t.*

Proof. 1. Let $\mathbf{t}, \mathbf{n}, \mathbf{b}$ and $\tilde{\mathbf{t}}, \tilde{\mathbf{n}}, \tilde{\mathbf{b}}$ be the tangent, normal and binormal functions of γ and $\tilde{\gamma}$ respectively. Fix $s_0 \in \mathbb{R}$. Since both $\{\mathbf{t}(s_0), \mathbf{n}(s_0), \mathbf{b}(s_0)\}$ and $\left\{\tilde{\mathbf{t}}(s_0), \tilde{\mathbf{n}}(s_0), \tilde{\mathbf{b}}(s_0)\right\}$ are positively oriented orthonormal bases, we can apply some rotations, composed as ρ, that send the first to the second. Consider

$$A(s) = \tilde{\mathbf{t}}(s)\cdot\rho(\mathbf{t}(s)) + \tilde{\mathbf{n}}(s)\cdot\rho(\mathbf{n}(s)) + \tilde{\mathbf{b}}(s)\cdot\rho(\mathbf{b}(s))$$

By construction, $A(s_0) = 3$. If we show that $A'(s) = 0$, then $A(s) = 3$. Also ρ is an isometry means that $\rho(\mathbf{t}), \rho(\mathbf{n}), \rho(\mathbf{b})$ are all unit vectors. The dot product between two unit vectors is at most one. It is equal to one if the vectors are equal. This implies if $A'(s) = 0$, then each of the summands of A is one, or the dot product of two equal vectors. In particular,

$$\begin{aligned}\tilde{\mathbf{t}} &= \rho(\mathbf{t})\\ \tilde{\gamma}' &= \rho(\gamma')\\ \tilde{\gamma}' &= \rho(\gamma)',\end{aligned}$$

because ρ is a linear transformation independent of s, so the rotated tangent vector is equal to the tangent vector of the rotated curve. Therefore $\tilde{\gamma}' - \rho(\gamma)' = 0$. This implies the two curves only differ by a constant vector, or $\tilde{\gamma} - \gamma = c$. So we define the translation $T(x) = x + (\tilde{\gamma}(s_0) - \gamma(s_0))$ to be the translation by the difference between the curves at s_0. Finally, if we can show $A'(s) = 0$, it means $M = T \circ \rho$ is the isometry that takes γ to $\tilde{\gamma}$. Now it remains to show that $A' = 0$.

$$\begin{aligned} A' &= (\tilde{\mathbf{t}}' \cdot \rho(\mathbf{t}) + \tilde{\mathbf{t}} \cdot \rho(\mathbf{t}')) + (\tilde{\mathbf{n}}' \cdot \rho(\mathbf{n}) + \tilde{\mathbf{n}} \cdot \rho(\mathbf{n}')) + (\tilde{\mathbf{b}}' \cdot \rho(\mathbf{b}) + \tilde{\mathbf{b}} \cdot \rho(\mathbf{b}')) \\ &= (\tilde{\kappa}\tilde{\mathbf{n}} \cdot \rho(\mathbf{t}) + \tilde{\mathbf{t}} \cdot \rho(\kappa\mathbf{n})) + ((-\tilde{\kappa}\tilde{\mathbf{t}} + \tilde{\tau}\tilde{\mathbf{b}}) \cdot \rho(\mathbf{n}) + \tilde{\mathbf{n}} \cdot \rho(-\kappa\mathbf{t} + \tau\mathbf{b})) + (-\tilde{\tau}\tilde{\mathbf{n}} \cdot \rho(\mathbf{b}) + \tilde{\mathbf{b}} \cdot \rho(-\tau\mathbf{n})) \end{aligned}$$

by the Frenet-Serret equations.

$$\begin{aligned} A' &= \tilde{\kappa}(\tilde{\mathbf{n}} \cdot \rho(\mathbf{t})) + \kappa(\tilde{\mathbf{t}} \cdot \rho(\mathbf{n})) - \tilde{\kappa}(\tilde{\mathbf{t}} \cdot \rho(\mathbf{n})) + \tilde{\tau}(\tilde{\mathbf{b}} \cdot \rho(\mathbf{n})) - \kappa(\tilde{\mathbf{n}} \cdot \rho(\mathbf{t})) + \tau(\tilde{\mathbf{n}} \cdot \rho(\mathbf{b})) - \tilde{\tau}(\tilde{\mathbf{n}} \cdot \rho(\mathbf{b})) - \tau(\tilde{\mathbf{b}} \cdot \rho(\mathbf{n})) \\ &= 0. \end{aligned}$$

2. Let $\mathbf{t}, \mathbf{n}, \mathbf{b}$ be functions that satisfy the following differential equations such that $\mathbf{t}(0) = e_1, \mathbf{n}(0) = e_2, \mathbf{b}(0) = e_3$.

$$\begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}' = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & t \\ 0 & -t & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}$$

By the theory of ordinary differential equations, this has a unique solution and $\mathbf{t}, \mathbf{n}, \mathbf{b}$ are orthonormal for all s. Now we let

$$\gamma(s) = \int_0^s \mathbf{t}(u)du.$$

Then $\gamma' = \mathbf{t}$. So γ is unit-speed. Next $\gamma'' = \mathbf{t}' = k\mathbf{n}$. Since $\mathbf{n}$ is a unit vector, it must be the principal normal, and k is the curvature. We already know that $\mathbf{b}$ is one of $\pm\mathbf{t} \times \mathbf{n}$, because it is perpendicular to $\mathbf{t}, \mathbf{n}$, but we need to check that it has the right sign. But since $\mathbf{b}(0) = \mathbf{t}(0) \times \mathbf{n}(0)$ and $\mathbf{b}$ is smooth, we must have $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. Then $\mathbf{b}' = -tn$ means the torsion of γ is t by definition.

□

4 Generalizations

We can define Frenet curvatures and Frenet equations for curves in $\mathbb{R}^n$. For details, refer to Kuhnel's Differential Geometry.