

Functional Analysis Course Notes

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1 Point set topology

Definition 1 (Topology). A topology on a set X is a collection τ of subsets of X such that

1. $\emptyset, X \in \tau$
2. $U, V \in \tau \implies U \cap V \in \tau$
3. $\{U_\alpha\}_{\alpha \in A} \subset \tau \implies \cup_\alpha U_\alpha \in \tau$

The elements of τ are called open sets.

Example 1. 1. Metric spaces

2. Discrete topology $\tau = \mathcal{P}(X)$, the power set of X
3. Trivial topology
4. Let $(X, <)$ be a total order. $\tau = \{\cup_\alpha (x_\alpha, y_\alpha) : x_\alpha < y_\alpha\}$.
5. Induced topology (subspace topology) of (X, τ) on $Y \subset X$ is $\sigma = \{U \cap Y : U \in \tau\}$.

Definition 2 (Complement, interior, closure notations). • Complement: $F^c = X \setminus F$

• Interior: $A^0 = \cup_{U \in \{U : U \in \tau, U \subset A\}} U$

• Closure: $\bar{A} = \cap_{F \in \{F : F^c \in \tau, A \subset F\}} F$

Proposition 1. • The collection of closed sets is closed under infinite intersection and finite union.

- $x \in \bar{A}$ iff every open $U \ni x, U \cap A \neq \emptyset$.

- $\bar{A} = ((A^c)^0)^c$

Proposition 2. Let $S \subset \mathcal{P}(X)$ be a subset of the power set on X . Then there exists a weakest topology τ that contains S , namely

$$\tau = \{\cup_{\alpha} (S_{\alpha,1} \cap \cdots \cap S_{\alpha,n_{\alpha}})\}$$

Definition 3 (Base, sub-base). • S is a base of τ if every set $U \in \tau$ is the union of sets in S .

- S is a sub-base of τ if it generates τ as in the above proposition.

Example 2. 1. If (X, d) is a metric space, then $\{b_r(x) : x \in X, r > 0\}$ is a base for the topology.

2. $\{(r, s) : r < s, r, s \in \mathbb{Q}\}$ is a base for the standard topology on \mathbb{R} .

3. Let $X = C[0, 1]$. For $\lambda \in \mathbb{C}, a \in [0, 1], r > 0$, let

$$U(\lambda, a, r) = \{g : |g(a) - \lambda| < r\}$$

$$V(\lambda_1, \dots, \lambda_n, a_1, \dots, a_n, r) = \{g : |g(a_i) - \lambda_i| < r \forall i\}.$$

Then the U 's form a sub-base for X and the V 's form a base for X .

Definition 4 (Strength of a topology). Let X be a set. Let σ, τ be topologies on X . We say σ is weaker than τ or τ is stronger than σ if $\sigma \subset \tau$.

Example 3. 1. $(\mathbb{R}, \text{trivial}) \subset (\mathbb{R}, d) \subset (\mathbb{R}, \text{discrete})$

2. $(C[0, 1], \text{pointwise topology}) \subset (C[0, 1], d), d(f, g) = \sup_x |f(x) - g(x)| = \|f - g\|_{\infty}$

Definition 5. (X, τ) is separable if there is a countable dense subset. ie $A = \{x_n\}_{n \in \mathbb{N}}, \bar{A} = X$.

(X, τ) is first countable if for all $x \in X$, there is a countable neighbourhood base at x . ie $\exists B = \{U_n : x \in U_n \in \tau\}$ such that if $x \in U \in \tau$, then $\exists U_n \in B$ such that $U_n \subset U$.

(X, τ) is second countable if there is a countable base.

Example 4. • Suppose (X, d) is a metric space, then X is first countable. Then $\{b_{1/n}(x) : n \in \mathbb{N}\}$ is a countable local base.

- For $(X, \text{discrete})$, a base must contain (be) $\{\{x\} : x \in X\}$. So X is second countable iff X is countable. Also X is separable iff X is countable.

- X is compact means X is separable. (Check for correctness)

Proposition 3. If X is a metric space, then being separable implies being second countable. (Is converse true?)

Proof. (\Rightarrow) Let $\{x_n\}_{n \in \mathbb{N}}$ be dense. Let $S = \{b_{1/k}(x_n) : k, n \in \mathbb{N}\}$, which is countable. Let U be an open neighbourhood of $x \in X$. Then there is $r > 0$ such that $b_r(x) \subset U$. Pick $1/n < r/2$. Then $b_{2/n}(x) \subset b_r(x) \subset U$. Pick $x_k \in b_{1/n}(x)$, which is allowed because the x_k 's are dense. Then $x \in b_{1/n}(x_k) \subseteq b_{2/n}(x) \subset U$. Therefore $U = \cup \{b_{1/n}(x_k)\}$ implies S is a countable base. \square

Definition 6. $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous if for all $V \in \sigma$, $f^{-1}(V) \in \tau$.
 f is a homeomorphism if f is a bijection and f, f^{-1} are continuous.

Example 5. • Let $f : (\mathbb{R}, \text{discrete}) \rightarrow (\mathbb{R}, d), g : (\mathbb{R}, d) \rightarrow (\mathbb{R}, \text{trivial})$ be identity functions. Then f, g are both bijective and continuous, but f^{-1}, g^{-1} are not continuous.

- $f : (X, \text{discrete}) \rightarrow (Y, \sigma)$ is always continuous.
- $f : (X, \text{trivial}) \rightarrow (Y, d)$ is continuous iff f is constant.

Definition 7. (X, τ) is Hausdorff if $\forall x \neq y \in X$, there is an open neighbourhood of x , U , and an open neighbourhood of y , V , such that $U \cap V = \emptyset$.

Example 6. 1. Metric spaces are Hausdorff.

2. Let $X = \{0, 1\}, \tau = \{\emptyset, \{0\}, X\}$. Cannot separate 0, 1, so not Hausdorff.
3. Let $X = [0, 1) \cup \{a, b\}$. If $U \subseteq [0, 1)$ is open in (X, d) , then $U \in \tau$. Also $(r, 1) \cup \{a\}, (r, 1) \cup \{b\}$ are open. But

$$((r, 1) \cup \{a\}) \cap ((s, 1) \cup \{b\}) = (\max(r, s), 1) \neq \emptyset$$

means it is not Hausdorff.