Functional Analysis Course Notes

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Chapter 1

Point set topology

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1.1 Review

Definition 1.1.1 (Topology)**.** A topology on a set X is a collection τ of subsets of X such that

1. $\emptyset, X \in \tau$
2. $U, V \in \tau \implies U \cap V \in \tau$
3. $\{U_\alpha\}_{\alpha \in A} \subset \tau \implies \cup_\alpha U_\alpha \in \tau$

The elements of τ are called open sets.

Example 1.1.1.

1. Metric spaces
2. Discrete topology $\tau = \mathcal{P}(X)$, the power set of X
3. Trivial topology
4. Let $(X, <)$ be a total order. $\tau = \{\cup_\alpha (x_\alpha, y_\alpha) : x_\alpha < y_\alpha\}$.
5. Induced topology (subspace topology) of (X, τ) on $Y \subset X$ is $\sigma = \{U \cap Y : U \in \tau\}$.

Definition 1.1.2 (Complement, interior, closure notations)**.**

- Complement: $F^c = X \setminus F$
- Interior: $A^0 = \cup_{U \in \{U : U \in \tau, U \subset A\}} U$
- Closure: $\bar{A} = \cap_{F \in \{F : F^c \in \tau, A \subset F\}} F$

Proposition 1.1.1.

- *The collection of closed sets is closed under infinite intersection and finite union.*
- $x \in \bar{A}$ *iff every open* $U \ni x, U \cap A \neq \emptyset$.
- $\bar{A} = ((A^c)^0)^c$

Proposition 1.1.2. *Let $S \subset \mathcal{P}(X)$ be a subset of the power set on X. Then there exists a weakest topology τ that contains S, namely*

$$\tau = \{\cup_\alpha (S_{\alpha,1} \cap \cdots \cap S_{\alpha,n_\alpha})\}$$

Definition 1.1.3 (Base, sub-base)**.**

- S is a base of τ if every set $U \in \tau$ is the union of sets in S.

- S is a sub-base of τ if it generates τ as in the above proposition.

Example 1.1.2.
1. If (X, d) is a metric space, then $\{b_r(x) : x \in X, r > 0\}$ is a base for the topology.
2. $\{(r, s) : r < s, r, s \in \mathbb{Q}\}$ is a base for the standard topology on $\mathbb{R}$.
3. Let $X = C[0, 1]$. For $\lambda \in \mathbb{C}, a \in [0, 1], r > 0$, let

$$U(\lambda, a, r) = \{g : |g(a) - \lambda| < r\}$$
$$V(\lambda_1, \cdots, \lambda_n, a_1, \cdots, a_n, r) = \{g : |g(a_i) - \lambda_i| < r \forall i\}.$$

   Then the U's form a sub-base for X and the V's form a base for X.

Definition 1.1.4 (Strength of a topology)**.** Let X be a set. Let σ, τ be topologies on X. We say σ is weaker than τ or τ is stronger than σ if $\sigma \subset \tau$.

Example 1.1.3.
1. $(\mathbb{R}, \text{trivial}) \subset (\mathbb{R}, d) \subset (\mathbb{R}, \text{discrete})$
2. $(C[0, 1], \text{pointwise topology}) \subset (C[0, 1], d), d(f, g) = \sup_x |f(x) - g(x)| = \|f - g\|_\infty$

Definition 1.1.5. (X, τ) is separable if there is a countable dense subset. ie $A = \{x_n\}_{n \in \mathbb{N}}, \bar{A} = X$.
(X, τ) is first countable if for all $x \in X$, there is a countable neighbourhood base at x. ie $\exists B = \{U_n : x \in U_n \in \tau\}$ such that if $x \in U \in \tau$, then $\exists U_n \in B$ such that $U_n \subset U$.
(X, τ) is second countable if there is a countable base.

Example 1.1.4.
- Suppose (X, d) is a metric space, then X is first countable. Then $\left\{b_{1/n}(x) : n \in \mathbb{N}\right\}$ is a countable local base.
- For $(X, \text{discrete})$, a base must contain (be) $\{\{x\} : x \in X\}$. So X is second countable iff X is countable. Also X is separable iff X is countable.
- For a metric space, X is compact iff X is separable. Why?

Proposition 1.1.3. *If X is a metric space, then being separable implies being second countable. (Is converse true?)*

Proof. ($\Rightarrow$) Let $\{x_n\}_{n \in \mathbb{N}}$ be dense. Let $S = \left\{b_{1/k}(x_n) : k, n \in \mathbb{N}\right\}$, which is countable. Let U be an open neighbourhood of $x \in X$. Then there is $r > 0$ such that $b_r(x) \subset U$. Pick $1/n < r/2$. Then $b_{2/n}(x) \subset b_r(x) \subset U$. Pick $x_k \in b_{1/n}(x)$, which is allowed because the x_k's are dense. Then $x \in b_{1/n}(x_k) \subseteq b_{2/n}(x) \subset U$. Therefore $U = \cup \left\{b_{1/n}(x_k)\right\}$ implies S is a countable base. $\square$

Definition 1.1.6. $f : (X, \tau) \to (Y, \sigma)$ is continuous if for all $V \in \sigma$, $f^{-1}(V) \in \tau$.
f is a homeomorphism if f is a bijection and f, f^{-1} are continuous.

Example 1.1.5.
- Let $f : (\mathbb{R}, \text{discrete}) \to (\mathbb{R}, d), g : (\mathbb{R}, d) \to (\mathbb{R}, \text{trivial})$ be identity functions. Then f, g are both bijective and continuous, but f^{-1}, g^{-1} are not continuous.
- $f : (X, \text{discrete}) \to (Y, \sigma)$ is always continuous.
- $f : (X, \text{trivial}) \to (Y, d)$ is continuous iff f is constant.

Definition 1.1.7. (X, τ) is Hausdorff if $\forall x \neq y \in X$, there is an open neighbourhood of x, U, and an open neighbourhood of y, V, such that $U \cap V = \emptyset$.

Example 1.1.6.
1. Metric spaces are Hausdorff.
2. Let $X = \{0, 1\}, \tau = \{\emptyset, \{0\}, X\}$. Cannot separate $0, 1$, so not Hausdorff.

3. Let $X = [0,1) \cup \{a,b\}$. If $U \subseteq [0,1)$ is open in (X,d), then $U \in \tau$. Also $(r,1) \cup \{a\}, (r,1) \cup \{b\}$ are open. But

$$((r,1) \cup \{a\}) \cap ((s,1) \cup \{b\}) = (\max(r,s),1) \neq \emptyset$$

means it is not Hausdorff.