Chapter 2

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1 Continuous Compounding

1.1 Notation

The following notation will be used:

- $f(t)$: Balance at time t
- r: Continuous compounding interest rate

1.2 Derivation of $f(t)$

At any given time t, accrued interest is $rf(t)$. Thus:

$$\frac{df(t)}{dt} = rf(t)$$

Therefore:

$$\begin{array}{lcl}
\frac{1}{f(t)}\frac{df(t)}{dt} & = & r \\
\int \frac{1}{f(t)}\frac{df(t)}{dt}dt & = & \int rdt \\
\int \frac{1}{f(t)}df(t) & = & rt \\
\ln f(t) & = & rt \\
f(t) & = & e^{rt}
\end{array}$$

2 Geometric Sequence Sum

2.1 Statement

Prove that

$$\sum_{t=a}^{b} z^t = \frac{z^a - z^{b+1}}{1-z}$$

2.2 Proof

Proof.

$$\sum_{t=a}^{b} z^t = z^a + z^{a+1} + \cdots + z^b$$

$$z\sum_{t=a}^{b} z^t = z^{a+1} + z^{a+2} + \cdots + z^{b+1}$$

$\therefore$

$$\begin{aligned}
& z \sum_{t=a}^{b} z^t - \sum_{t=a}^{b} z^t \\
= \; & z^{b+1} + \left(z^b - z^b\right) + \cdots + \left(z^{a+1} - z^{a+1}\right) - z^a \\
= \; & z^{b+1} - z^a
\end{aligned}$$

$\therefore$

$$\begin{aligned}
(z-1) \sum_{t=a}^{b} z^t &= z^{b+1} - z^a \\
\sum_{t=a}^{b} z^t &= \frac{z^{b+1} - z^a}{z-1} \\
&= \frac{z^a - z^{b+1}}{1-z}
\end{aligned}$$

$\square$

3 Flat Term Structure

3.1 Statement

Prove that if the term structure of spot rates is flat, the term structure of par rates is also flat at the same rate.

3.2 Notaion

The following notation will be used:

- T
- r: Continuous compounding interest rate

3.3 Proof

Proof.

$$\sum_{t=a}^{b} z^t = z^a + z^{a+1} + \cdots + z^b$$

$$z \sum_{t=a}^{b} z^t = z^{a+1} + z^{a+2} + \cdots + z^{b+1}$$

$\therefore$

$$\begin{aligned}
& z \sum_{t=a}^{b} z^t - \sum_{t=a}^{b} z^t \\
= \; & z^{b+1} + \left(z^b - z^b\right) + \cdots + \left(z^{a+1} - z^{a+1}\right) - z^a \\
= \; & z^{b+1} - z^a
\end{aligned}$$

$\therefore$

$$\begin{aligned}
(z-1) \sum_{t=a}^{b} z^t &= z^{b+1} - z^a \\
\sum_{t=a}^{b} z^t &= \frac{z^{b+1} - z^a}{z-1} \\
&= \frac{z^a - z^{b+1}}{1-z}
\end{aligned}$$

$\square$