

# Chapter 2

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## 1 Continuous Compounding

### 1.1 Notation

The following notation will be used:

- $f(t)$ : Balance at time  $t$
- $r$ : Continuous compounding interest rate

### 1.2 Derivation of $f(t)$

At any given time  $t$ , accrued interest is  $rf(t)$ . Thus:

$$\frac{df(t)}{dt} = rf(t)$$

Therefore:

$$\begin{aligned}\frac{1}{f(t)} \frac{df(t)}{dt} &= r \\ \int \frac{1}{f(t)} \frac{df(t)}{dt} dt &= \int r dt \\ \int \frac{1}{f(t)} df(t) &= rt \\ \ln f(t) &= rt \\ f(t) &= e^{rt}\end{aligned}$$

## 2 Geometric Sequence Sum

### 2.1 Statement

Prove that

$$\sum_{t=a}^b z^t = \frac{z^a - z^{b+1}}{1 - z}$$

### 2.2 Proof

*Proof.*

$$\begin{aligned}\sum_{t=a}^b z^t &= z^a + z^{a+1} + \dots + z^b \\ z \sum_{t=a}^b z^t &= z^{a+1} + z^{a+2} + \dots + z^{b+1}\end{aligned}$$

∴

$$\begin{aligned}
& z \sum_{t=a}^b z^t - \sum_{t=a}^b z^t \\
&= z^{b+1} + (z^b - z^b) + \dots + (z^{a+1} - z^{a+1}) - z^a \\
&= z^{b+1} - z^a
\end{aligned}$$

∴

$$\begin{aligned}
(z-1) \sum_{t=a}^b z^t &= z^{b+1} - z^a \\
\sum_{t=a}^b z^t &= \frac{z^{b+1} - z^a}{z-1} \\
&= \frac{z^a - z^{b+1}}{1-z}
\end{aligned}$$

□

### 3 Flat Term Structure

#### 3.1 Statement

Prove that if the term structure of spot rates is flat, the term structure of par rates is also flat at the same rate.

#### 3.2 Notation

The following notation will be used:

- $T$ : Maturity
- $C(T)$ : Par rate for maturity  $T$
- $\hat{r}$ : Spot rate for term  $t$
- $r$ : Flat spot rate value

#### 3.3 Proof

*Proof.*

$$\frac{C(T)}{2} \sum_{t=1}^{2T} \left(1 + \frac{\hat{r}(t)}{2}\right)^{-t} + \left(1 + \frac{\hat{r}(t)}{2}\right)^{-2T} = 1$$

∴

$$\hat{r}(t) = \hat{r}$$

∴

$$\frac{C(T)}{2} \sum_{t=1}^{2T} \left(1 + \frac{\hat{r}}{2}\right)^{-t} + \left(1 + \frac{\hat{r}}{2}\right)^{-2T} = 1$$

∴

$$\frac{C(T)}{2} \cdot \frac{((1+\frac{\hat{r}}{2})^{-1}) - (1+\frac{\hat{r}}{2})^{-2T-1}}{1 - (1+\frac{\hat{r}}{2})^{-1}} + \left(1 + \frac{\hat{r}}{2}\right)^{-2T} = 1$$

Let

$$u = (1 + \hat{r})^{-1}$$

∴

$$\begin{aligned}
\frac{C(T)}{2} \cdot \frac{u - u^{2T+1}}{1-u} + u^{2T} &= 1 \\
\frac{C(T)}{2} &= (1 - u^{2T}) \cdot \frac{1-u}{u(1-u^{2T})} \\
C(T) &= \frac{2-2u}{u}
\end{aligned}$$

$\therefore$

$$\begin{aligned}C(T) &= \frac{2 - \frac{2}{1 + \frac{\hat{r}}{2}}}{\left(1 + \frac{\hat{r}}{2}\right)^{-1}} \\&= \frac{\hat{r}}{1 + \frac{\hat{r}}{2}} \cdot \left(1 + \frac{\hat{r}}{2}\right) \\&= \hat{r}\end{aligned}$$

□