

Linear Program Solver

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Context

Today, optimization of processes and systems is a necessity.

Linear programming is a method to achieve the minimum or maximum outcome in a mathematical model whose constraints are represented by linear relationship.

- Max Profit
- Shortest Path

Method taught in ISyE 3833 to solve linear programming problems is the simplex method

Simplex Algorithm Overview

- Simplex algorithm is an algorithm to solve linear programming problems.
- The linear programming problems we are to solve have a special constraint on the decision variables requiring them to be non-negative.

Example

$$\text{maximize } 3x_1 + 2x_2 + 5x_3$$

$$\begin{aligned} & 2x_1 + 3x_2 + 6x_3 \geq 5 \\ \text{s.t. } & 7x_1 + 5x_2 + 2x_3 \leq 1 \\ & 3x_1 + 4x_2 = 2 \end{aligned}$$

$$\forall i \in \{1, 2, 3\}, x_i \geq 0$$

Transformation

$$\text{maximize } 3x_1 + 2x_2 + 5x_3$$

$$\begin{array}{rcl} 2x_1 + 3x_2 + 6x_3 - x_4 + a_1 & = & 5 \\ s.t. \quad 7x_1 + 5x_2 + 2x_3 + x_5 & = & 1 \\ 3x_1 + 4x_2 + a_2 & = & 2 \end{array}$$

The added x variables are called slack variables.

The added a variables are called auxiliary/artificial variables.

Notations

- \vec{c}_b : Coefficient vector of Basic Variables (Variables that have non zero value) in the Objective Function
- \vec{c}_n : Coefficient vector of Non-Basic Variables (Variables are zero) in the Objective Function

- B : Coefficient matrix of Basic Variables in the constraints
- N : Coefficient matrix of Non-Basic Variables in the constraints
- \vec{b} : Vector of the RHS values in the constraints
- \vec{x}_b : Vector of Basic Variables

- \vec{x}_n : Vector of Non-Basic Variables

Matrix Representation

$$\text{maximize } \vec{c}_b^T \cdot \vec{x}_b + \vec{c}_n^T \cdot \vec{x}_n$$

$$\text{s.t. } B\vec{x}_b + N\vec{x}_n = \vec{b}$$

The Objective function can also be written as:

$$c_b^T B^{-1} \vec{b} + (c_n^T - c_b^T B^{-1} N) \vec{x}_n$$

The goal of the simplex algorithm is to set all dimensions of the vector $c_n^T - c_b^T B^{-1} N$ to be negative by swapping Basic and Non-Basic Variables (pivot).

The objective value is $c_b^T B^{-1} \vec{b}$

Each pivot will “improve” the objective value

Two Phases

- If \exists Artificial Variable(s), “Phase One” is required
- The objective of “Phase One” is to maximize $-\sum_i a_i$
- If the optimal value of “Phase One” $\neq 0$, the linear program is infeasible; otherwise, it is feasible

Implementation

- Single phase solver – Finished
- “Phase One” builder – Work in Process

```
struct Model {  
    int is_max;  
    int stat;  
    int num_non_basic;  
    int num_basic;  
    int num_art;  
    double** b_matrix;  
    double** n_matrix;  
    double* cb_vector;  
    double* cn_vector;
```

```
int* xb_index_vector;  
int* xn_index_vector;  
double* b_vector;  
double** art_matrix;  
int* art_ind_vector;  
};
```

```
int pivot(Model* model, int in, int out);  
int check(Model* model, int* in, int* out);  
int ratio_check(Model* model, int in_ind,  
                int* ind);
```

Sample I/O

Input

3 3

1 10 3 5

2 4 -4 0 4

1 -20 -5 0 7

4 2 3 0 10

$$\textit{maximize } 10x_1 + 3x_2 + 5x_3$$

$$\begin{array}{lcl} & 2x_1 + 4x_2 - 4x_3 & \leq 5 \\ s.t. & 1x_1 - 20x_2 - 5x_3 & \leq 1 \\ & 4x_1 + 2x_2 + 3x_3 & \leq 2 \end{array}$$

Output

$x_1 = 2.363636$

$x_3 = 0.181818$

$x_2 = 0.000000$

$opt_val = 24.545455$

Output from Xpress IVE

2

Optimum found

$x_1 = 2.36364$

$x_2 = 0$

$x_3 = 0.181818$

24.5455

Input

3 3

1 10 3 5

2 4 -4 0 4

1 -20 -5 0 7

4 2 -3 0 10

$$\text{maximize } 10x_1 + 3x_2 + 5x_3$$

$$\begin{array}{lcl} & 2x_1 + 4x_2 - 4x_3 & \leq 5 \\ s.t. & 1x_1 - 20x_2 - 5x_3 & \leq 1 \\ & 4x_1 + 2x_2 - 3x_3 & \leq 2 \end{array}$$

Output

Unbounded

Output from Xpress IVE

8

Unbounded

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

0

Matrix Multiplication

Implement with the “Divide and Conquer” idea.

A is an $a \times b$ matrix, B is an $b \times c$ matrix.

- If $a \leq b$: divide A horizontally into “numofthreads” parts and multiply each part with matrix B

$$C = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} B = \begin{pmatrix} A_1 B \\ A_2 B \end{pmatrix}$$

- Else: divide B vertically into “numofthreads” parts and multiply matrix A with each submatrix of B

$$C = A (B_1 \quad B_2) = (AB_1 \quad AB_2)$$

Detailed Implementation

1, The splitting of matrix -- modifying the traditional matrix multiplication

Traditional matrix multiplication

```
for(int i=0;i<n1;i++){  
    for(int k=0;k<n3;k++){  
        c[i][k] = 0;  
        for (int j=0;j<n2;j++){  
            c[i][k] += a[i][j]*b[j][k];  
        }  
    }  
}
```

Modified matrix multiplication with splitted matrices

```
for(int i=nralower;i<nraupper;i++){  
    for(int k=ncblower;k<ncbupper;k++){  
        out[i][k] = 0;  
        for (int j=0;j<nca;j++){  
            out[i][k] += a[i][j]*b[j][k];  
        }  
    }  
}
```

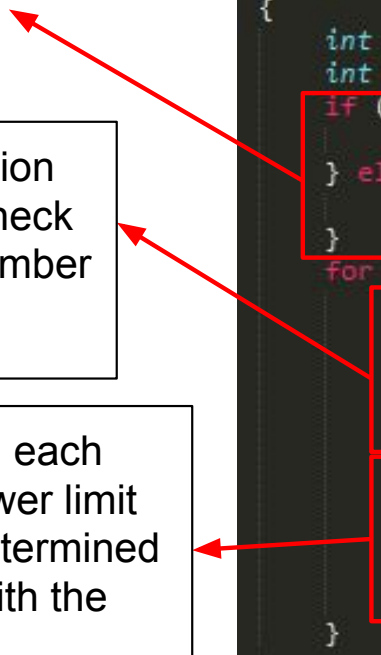
Detailed Implementation -- Real Algorithm

Determine which matrix to split

Set the upper limit of division for the current loop and check whether it exceeds the number of rows/columns

Split the selected matrix in each thread according to the lower limit (loop variable i) and predetermined upper limit, and multiply with the other complete matrix


```
#pragma omp parallel
{
    int id = omp_get_thread_num();
    int nt = omp_get_num_threads();
    if (nra > ncb) {
        int split = nra;
    } else {
        int split = ncb;
    }
    for (int i = id; i < split; i += nt) {
        if ((i + nt) <= split) {
            int uplimit = i + nt;
        } else {
            int uplimit = split;
        }
        if (split == nra) {
            basic_matrix_multiply(i, uplimit, nca, 0, ncb, a, b, output);
        } else {
            basic_matrix_multiply(0, nra, nca, i, uplimit, a, b, output);
        }
    }
}
```



Other Matrix Functions

1, Right Multiply: Multiply a vector with a matrix (vector a times matrix b)


```
#pragma omp parallel
{
    int id = omp_get_thread_num();
    int nt = omp_get_num_threads();
    for (int j = id; j < ncb; j += nt) {
        for (int i = 0; i < da; i++) {
            output[j] += a[i] * b[i][j];
        }
    }
}
```



First parse through the columns of matrix b with omp parallel

2, Left Multiply: Multiply a matrix with a vector (matrix a times vector b)

```
#pragma omp parallel
{
    int id = omp_get_thread_num();
    int nt = omp_get_num_threads();
    for (int i = id; i < nra; i += nt) {
        for (int j = 0; j < db; j++) {
            output[i] += a[i][j] * b[j];
        }
    }
}
```



First parse through the rows of matrix a with omp parallel

Other Matrix Functions

1, Dot Product (vector a times vector b)

```
#pragma omp parallel
{
    int id = omp_get_thread_num();
    int inc = omp_get_num_threads();
    double temp = 0;
    for (int i = id; i < dim; i += inc) {
        temp += a[i] * b[i];
    }
    #pragma omp critical
    {
        *output += temp;
    }
}
```

Parse through each entry of the vector with multi-threads

A critical session to update the result with the product of each pair of the entries

2, Find Max (find the entry with maximum value in a vector)

```
*max = vector[0];
*max_ind = 0;
#pragma omp parallel for
for (int i = 1; i < dim; i++) {
    #pragma omp critical
    {
        if (vector[i] > *max) {
            *max_ind = i;
            *max = vector[i];
        }
    }
}
```

Parse through each entry of the vector. Achieve the same effect as the method in class with only one layer of loop

A critical session to check whether the current processed entry is larger than the existing max. If true, update the max value and its index with the entry's.

Matrix Inverse -- Gaussian Elimination

```
int matrix_invert(int dim, double** a, double** output) {
    if (a == NULL || output == NULL) {
        puts("matrix invert NULL ptr");
        return -1;
    }
    matrix_copy(dim, dim, a, output);
    double** i_matrix = create_identity(dim);
    for (int i = 0; i < dim; i++) {
        if (output[i][i] == 0) {
            int j;
            for (j = i; j < dim && output[j][i] == 0; j++);
            if (output[j][i] == 0) {
                return -1;
            }
            swap_row(i, j, output, output);
        }
        double scale = output[i][i];
        #pragma omp parallel for
        for (int j = 0; j < dim; j++) {
            output[i][j] = output[i][j] / scale;
            i_matrix[i][j] = i_matrix[i][j] / scale;
        }
        #pragma omp parallel for
        for (int j = i + 1; j < dim; j++) {
            double factor = output[j][i];
            #pragma omp parallel for
            for (int k = 0; k < dim; k++) {
                output[j][k] = output[j][k] - factor * output[i][k];
                i_matrix[j][k] = i_matrix[j][k] - factor * i_matrix[i][k];
            }
        }
    }
}
```

Matrix Inverse -- Back Substitution

```
for (int i = dim - 1; i > 0; i--) {  
    #pragma parallel for  
    for (int j = i - 1; j > -1; j--) {  
        double factor = output[j][i];  
        #pragma omp parallel for  
        for (int k = 0; k < dim; k++) {  
            output[j][k] = output[j][k] - factor * output[i][k];  
            i_matrix[j][k] = i_matrix[j][k] - factor * i_matrix[i][k];  
        }  
    }  
}
```

Bellman-Ford Shortest Path Algorithm

Algorithm to find shortest path and distance from source node to all other nodes in the graph.

Relaxes edge distances between graphs, if new relaxation provides a shorter path to the node, the distance is updated.

In Serial

```
for (int i = 0; i <= V-1; i++){
    listNode *currentNode = graph->array[i].head;
    for(int j = 0; j<graph->array[i].size; j++){
        int u = currentNode->vertexNumber;
        int weight = currentNode->weight;

        if (StoreDistance[u][1] == -1 || StoreDistance[i][1] + weight < StoreDistance[u][1]){
            StoreDistance[u][1] = StoreDistance[i][1] + weight;
        }
        currentNode = currentNode->nextListNode;
    }
}
```

Parallelizing Bellman-Ford

Parallelize the relaxation process

Compares known distance to neighbor with the distance including the new edge for each of the edges in parallel. Each neighbor receives its own processor

In parallel

```
omp_set_num_threads(NUM_THREADS);
#pragma omp parallel
{
    int i, id, nthrds;
    id = omp_get_thread_num();
    nthrds = omp_get_num_threads();
    if(id==0) nthrds = nthrds;
    for(int j = id; j < V; j+=nthrds){
        for (listNode* current = graph->array[j].head; current != NULL; current = current->nextListNode) {
            int vNum = current->vertexNumber;

            if(StoreDistance[vNum][1] == -1 || StoreDistance[j][1] + current->weight < StoreDistance[vNum][1]){
                #pragma omp critical
                {
                    StoreDistance[vNum][1] = StoreDistance[j][1] + current->weight;
                }
            }
        }
    }
}
```

```
Vertex Distance from Source Vertex
0      0
1      420
2      175
3      570
4      670
EROSTENSTROM@Erik-PC:/mnt/c/Users/Erik/eclipse-workspace/4010_project$
```


Time Analysis of Bellman-Ford vs Simplex

Ultimately, we will compare the runtime of the Bellman-Ford algorithm and the simplex method

We will also analyze how the runtimes compare to the theoretical times

Question?

Comments?

Concerns?