Forced Dynamical Systems in Paleoclimate Modeling

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Abstract

For the last few million years, Earth's climate has experienced cycles of ice ages and warm interglacial periods. About 1 Myr ago, the length of these cycles shifted from ~40kyr to ~100kyr, a phenomenon called the Mid-Pleistocene Transition (MPT). The cause of this change remains unknown.

Conceptual climate models can be used to test hypotheses about the dynamics of these glacial cycles. Previous work has demonstrated synchronization across a wide range of external forcing, resulting in Arnold tongue structures in simple models such as the Van der Pol Oscillator. We extend this work to more complex models and types of forcing. Our results show robust Arnold tongue structures across a variety of forcing parameters in each model, but new behavior arises in some cases.

Background

Climate Background

- Leading hypothesis for cause of ice age cycles: internal free oscillations in ice sheet volume, CO_2 concentration, ocean temperature, etc. interact with external astronomical forcing.
- External forcing consists of changes in solar insolation, which is dominated by two main time scales:
  - Obliquity - Earth's tilt (~41 kyr cycles)
  - Precession - wobble of Earth's axis (~23 kyr cycles)

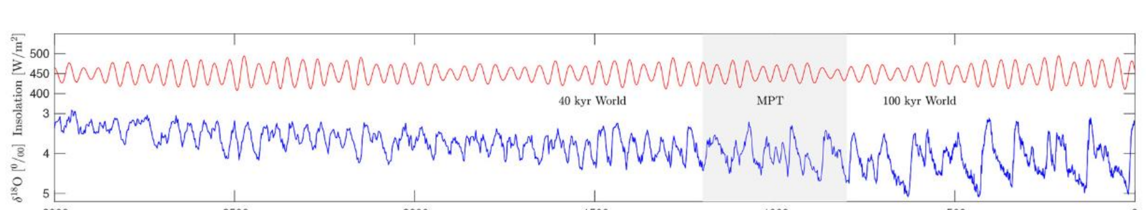


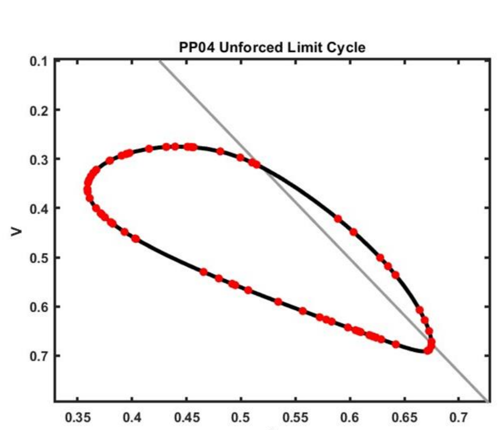
Image: von der Heydt et al. 2021

Mathematical Background

- We focus on nonlinear models with free oscillations (attractive limit cycles) modeling the late Pleistocene
- Oscillatory external forcing can induce phase-locking: synchronization, clustering, change in pacing
- Character of synchronization often (but not always) predictable based on parameters

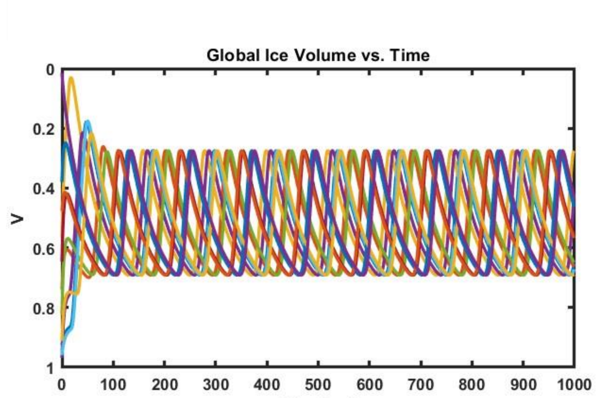
Limit Cycles and Forcing

Paillard and Perrenin 2004
Non-smooth model with asymmetric limit cycle



- Unforced attractive limit cycle
- Unforced period: ~130kyr
- 70 initial conditions on the phase space
- Unsynchronized: ending state depends on initial condition

- 10 of the 70 initial conditions shown at right
- No synchronization

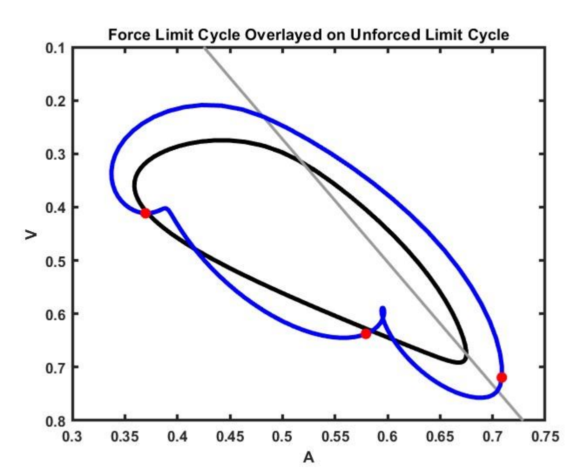


Motivating Questions

- Under what parameters do we see synchronization in climate models more complex than the Van der Pol oscillator? How is synchronization observed in these climate models?
- How do the systems respond to quasi-periodic forcing compared to periodic forcing?
- What are the differences in how different types of models respond to periodic forcing? What is the relationship between clustering and pacing?

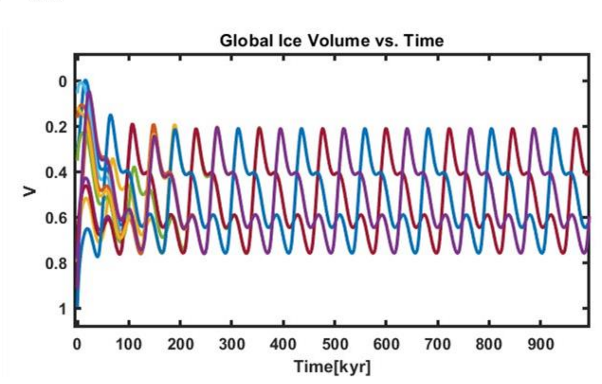
Synchronization and Pacing

- Imposing insolation as forcing term can induce synchronization
- Under some parameters, trajectories lock to the same phase & frequency, paced by the oscillation of the forcing
  - Pacing: observed oscillation of the forced system is either an integer or a rational multiple of the forcing period, not the period of the unforced limit cycle

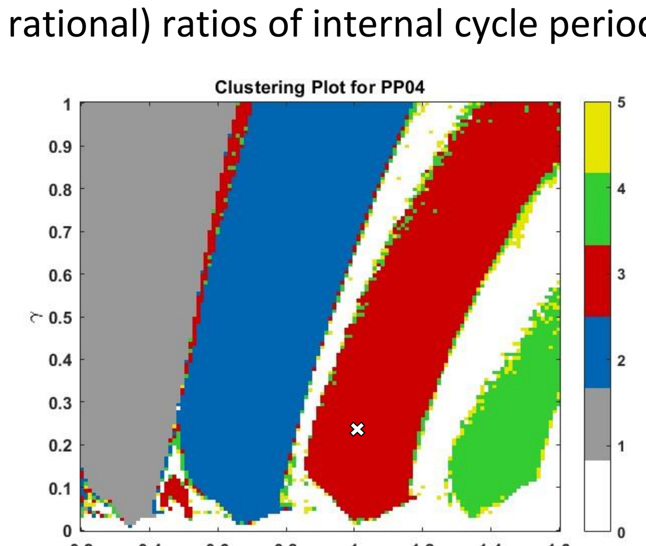


- 70 initial conditions synchronize to a new attractive limit cycle
- 3 distinct clusters
- Forced Period: ~123kyr
- Pacing: 3:1

- 70 initial conditions synchronize to 3 distinct clusters equally spaced in time



- Phase-locking to a period that is an integer multiple of the forcing period is expected if the internal oscillation's period is near that integer multiple.
- The weaker the forcing, the closer these periods must be.
- As we vary these parameters, we see regions of synchronization and unsynchronized behavior.
  - τ: the ratio of internal to external frequency
  - γ: the strength of the external forcing
- Arnold tongue structures: triangular shapes at integer (or rational) ratios of internal cycle period to forcing period



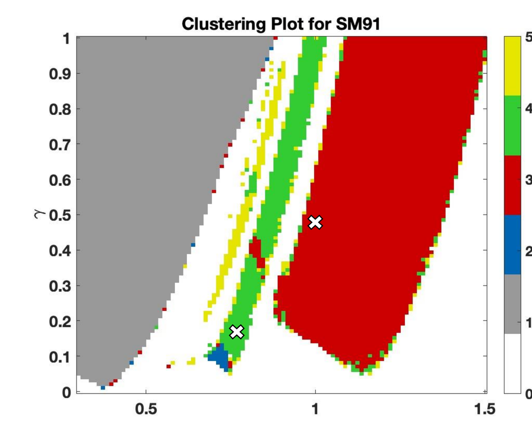
- Colors correspond to number of clusters observed
- Clustering at integer (or rational) ratios of internal to external oscillation
- Example shown above is one parameter choice (τ = 1, γ = 0.25, marked by 'x' on the left plot)

- These results are consistent with the behavior of the simple Van der Pol oscillator.

Periodic Forcing

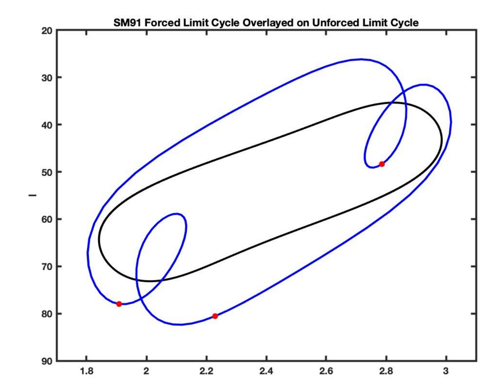
We run the same experiment as in PP04 with other models of similar complexity.

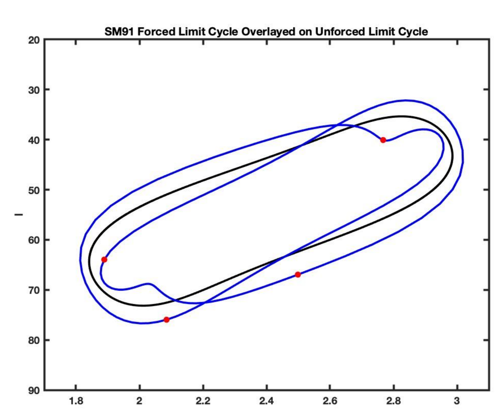
Saltzman and Maasch 1991 Model (SM91)
Smooth model with symmetric limit cycle



- Unforced Period: ~100kyr
- Arnold tongue structures remain largely intact
- Absence of expected 2 cluster region at 2:1 ratio, instead reads as 4 clusters

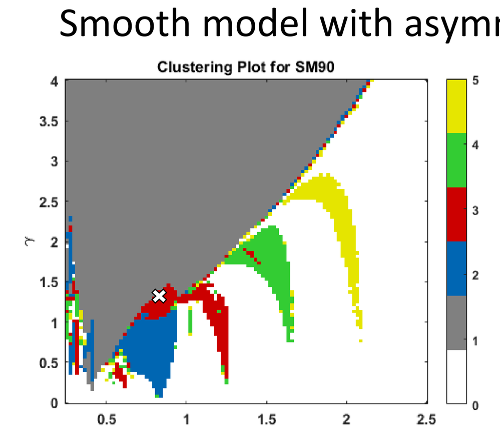
- τ = 1.0, γ = 0.50
- 3 Clusters
- Pacing: 3:1
- 1 Limit Cycle
- Forced Period: ~123kyr
- Consistent with prior results



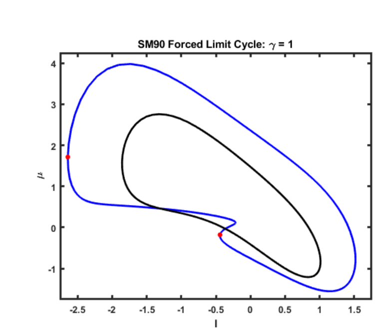


- τ = 0.75, γ = 0.15
- 4 Clusters, Pacing: 2:1
- 2 limit cycles
- Forced Period: ~82kyr per limit cycle
- New behavior: Individual limit cycles break symmetry, but overall symmetry is preserved

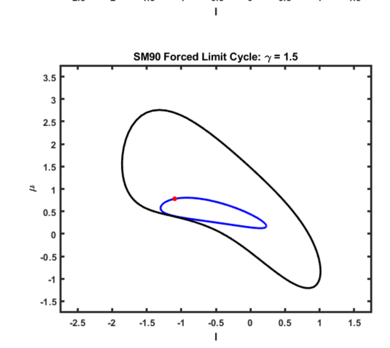
Saltzman and Maasch 1990 Model (SM90)
Smooth model with asymmetric limit cycle

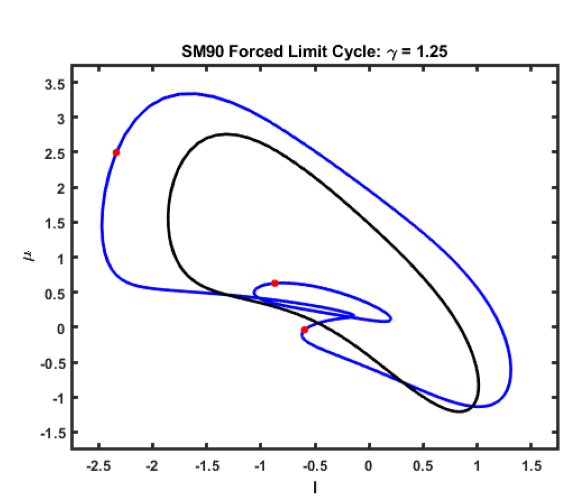


- Unforced Period: ~100kyr
- Low amplitude: Arnold Tongues
- High amplitude: tongues fall off
- New behavior: transitional region of 3 clusters between 1:1 and 2:1 tongues



- Top Plot: τ = 0.82, γ = 1; exhibits only 2:1 outer cycle
- Bottom Plot: τ = 0.82, γ = 1.5; exhibits only inner 1:1 cycle
- Both cases fit standard theory

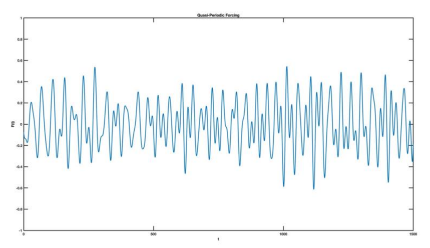




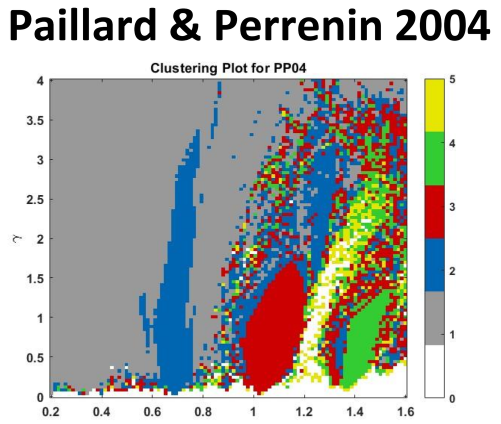
- Left: Limit cycle behavior at τ = 0.82, γ = 1.25 (indicated on cluster plot)
- Two separate limit cycles
- Inner cycle has 1 cluster, 1:1 pacing (~41kyr cycles)
- Outer cycle has 2 clusters and 2:1 pacing (~82kyr cycles)
- 'Overlap' of 1:1 and 2:1 regions

Quasi-Periodic Forcing

- Using more realistic insolation forcing, based on precession and obliquity



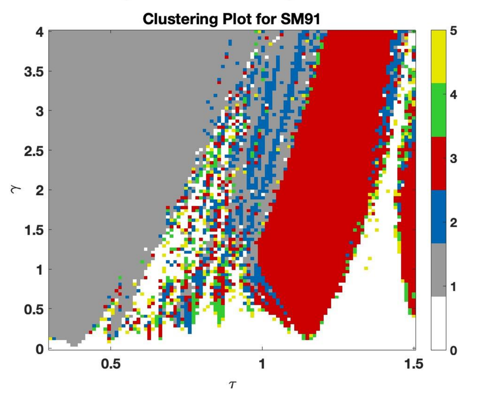
Paillard & Perrenin 2004



- Less robust Arnold tongue structures
- Retains areas of strong clustering near integer ratios
- Breakdown of structures at higher values of τ with stronger forcing

Saltzman & Maasch 1991

- Less robust Arnold tongue structures
- Keeps a large section of clustered trajectories in the 3:1 tongue
- Loses the 2:1 tongue



Conclusions

- Robust Arnold tongue structures appear across a variety of models for periodic forcing
- In some cases, we see subtle complications.
  - Breakdown in the match between clustering and pacing can occur. To preserve symmetry, two distinct limit cycles appear. More work needs to be done to understand this phenomenon and whether it can be found in non-smooth models.
  - Some models show trajectories with transitional pacing compared to the inherent system. More work is needed to explore under what conditions this occurs.
- Quasi-periodic forcing changes the system's response. There are remnants of the Arnold tongue structures but they are less robust, and synchronizing into one cluster is more common.

Future Work

- Explore features of smooth vs. non-smooth models
- Explore differences in models with symmetric vs. asymmetric limit cycles
- Explore symmetry breaking & pacing changes for periodic forcing
- Introduce stochastic forcing
- Explore how changes in clustering algorithm parameters affect number of clusters recorded
- Examine additional models

Acknowledgements

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References and Model Equations