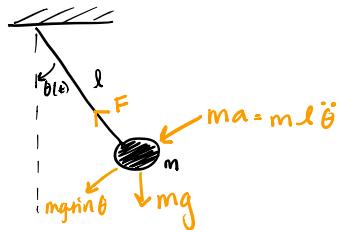


Continuum Mechanics

- study of fluids or solids in a continuum space (rather than discrete)

Pendulum, point mass



$$a = \ddot{\theta}l$$

$$ml\ddot{\theta} + mg\sin\theta = 0 \quad (\text{the governing eq.})$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0 \quad (2^{\text{nd}} \text{ order DE})$$

suppose θ is small, in radians $\rightarrow \sin\theta \approx \theta$

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\theta = A\cos(\sqrt{\frac{g}{l}}t) + B\sin(\sqrt{\frac{g}{l}}t)$$

$$\omega = \sqrt{\frac{g}{l}} \text{ - natural frequency, determines period}$$

$$\theta = A\cos(\sqrt{\frac{g}{l}}t) + B\sin(\sqrt{\frac{g}{l}}t)$$

initial conditions

$$\dot{\theta}(0) = 0$$

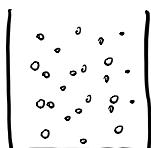
$$\theta(0) = \theta_0$$

$$\left. \begin{array}{l} \theta = \theta_0 \cos(\sqrt{\frac{g}{l}}t) \\ \theta_0, g, l \text{ all known} \end{array} \right\}$$

$$T = \text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

$$F = mg\cos\theta = mg\cos[\theta_0 \cos(\sqrt{\frac{g}{l}}t)]$$

Now suppose we want to find pressure from 1L of air in a dish ...
@ 25°C



$$N = 2.5 \times 10^{22}$$

... computer, 10^{16} operations per second

$$t = \frac{2.5 \times 10^{22}}{10^{16} \text{ ops}} = 2.5 \times 10^6 \text{ s} \approx 30 \text{ days}!!!$$

... so we will solve using continuum model

l_M = macroscopic length scale

ex. rubber band, amount that is stretched

l_μ = microscopic length scale

ex. at molecular level

} this ratio $\frac{l_\mu}{l_M} \ll 1$
 $\Rightarrow \underline{\text{CONTINUUM}}$

Question: Can we use continuum mechanics to measure volume change of lungs? (alveoli, microscopic)

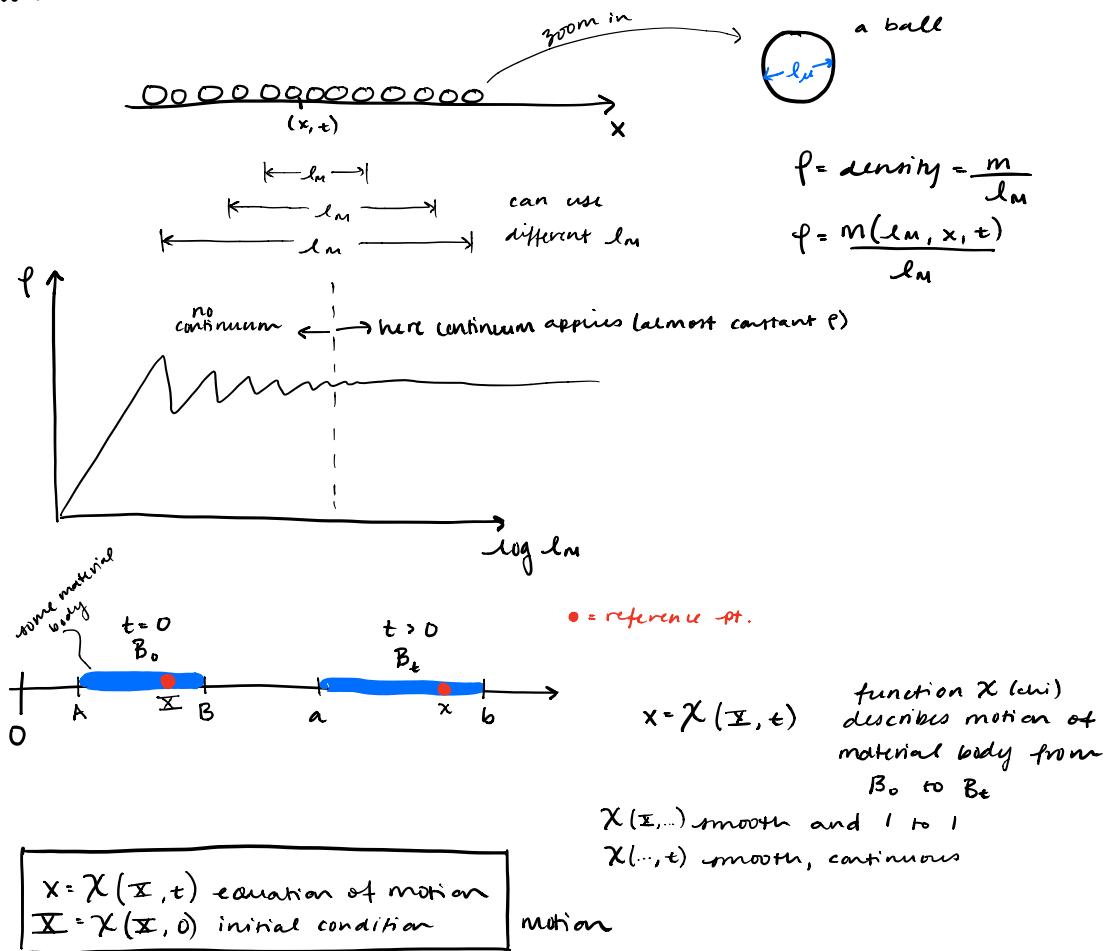
ex. $l_m = 10^2 \text{ mm}$ (diameter of alveolus)
 $\Delta V_m = 10^8 \text{ cm}^3$ (volume of lungs during normal breathing)

$$\frac{l_m}{l_M} \ll 1 ? \quad \text{What is } l_M? \dots \text{take cube root of volume}$$

$$l_M = \sqrt[3]{\Delta V_m} \approx 5 \text{ cm}$$

$$\frac{l_m}{l_M} \ll 1 \quad \left\{ \right. = 2 \times 10^{-3} \ll 1 \quad \text{so in this case we can use continuum mechanics}$$

1-Dimensional Continuum Mechanics: where it works and where it breaks down:



$$v = v(x, t) = \frac{dx}{dt} \quad \text{velocity}$$

$$a = a(x, t) = \frac{dv}{dt} = \frac{d^2x}{dt^2} \quad \text{acceleration}$$

(since in 1D, v and a are scalars ... in 3D they are vectors)

① Material Description, or Lagrangian Description
 material coordinates \underline{x} } solids

② Spatial, or Eulerian description
 Spatial coordinates x } fluids

$$x = \chi(\underline{x}, t) \quad \left\{ \begin{array}{l} \text{1-to-1 in } \underline{x} \\ \Rightarrow \underline{x} = \chi^{-1}(x, t) \end{array} \right.$$

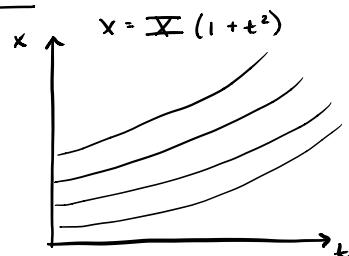
$$\begin{array}{ll} v = v(\underline{x}, t) = v(\chi^{-1}(x, t), t) = \tilde{v}(x, t) & \\ \text{Lagrangian} & \text{Eulerian} \\ (\text{material description}) & (\text{spatial description}) \end{array}$$

$$a = a(\underline{x}, t) = \tilde{a}(x, t)$$

Lagrangian Eulerian

$$\Phi = \Phi(\underline{x}, t) - \Phi(\chi^{-1}(x, t), t) = \tilde{\Phi}(x, t)$$

Example:



Does this satisfy equation of motion?
 yes.

$$\text{Lagrangian: } v = v(\underline{x}, t) = \underbrace{\frac{\partial x}{\partial t}}_{\text{green wavy line}} = 2\underbrace{\underline{x}}_{\text{green wavy line}} t \quad a = a(\underline{x}, t) = \underbrace{\frac{\partial v}{\partial t}}_{\text{purple wavy line}} = 2\underbrace{\underline{x}}_{\text{purple wavy line}}$$

Eulerian: need to invert equation of motion

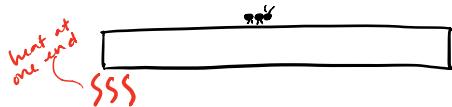
$$\underline{x} = \frac{x}{1+t^2} = \chi^{-1}(x, t)$$

$$v = \tilde{v}(x, t) = \underbrace{\frac{2xt}{1+t^2}}_{\text{green wavy line}} \quad a = \tilde{a}(x, t) = \underbrace{\frac{2x}{1+t^2}}_{\text{purple wavy line}}$$

In Lagrangian form, velocity increases linearly w/time. But Eulerian is non-linear ... same idea with Eulerian

→ they describe the same quantities in different ways

Example: metal rod w/ant on top



$$\varphi = \varphi(\mathbf{x}, t) \rightarrow \dot{\varphi} = \frac{\partial \varphi(\mathbf{x}, t)}{\partial t} \quad \text{material time derivative}$$

Recap: continuum mechanics models bodies as a continuum
... in reality we have a real continuum body composed of discrete particles

Real (discrete)
body

... so within boundaries we re-evaluate these discrete bodies as single continuum



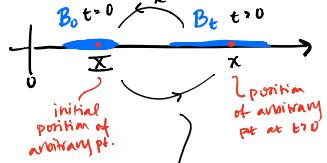
continuum

'particle', a mathematical point w/ certain physical properties (temp, force, energy, momentum, etc.)

$\varphi = \varphi(x, t)$ particle function φ as function of space and time (can be any physical thing, such as temp, energy, whatever)

continuum mechanics is a theoretical principle, applied mathematics

Along x-axis, follow movement of single body



think train only moving forward + backward, not up or down ... assume 1-d continuum

equation of motion: $\chi(\mathbf{x}, t) = x$

- chi is smooth, aka always differentiable
- chi is 1-to-1, $\chi(\mathbf{x}, \cdot)$... this therefore becomes invertible then $\chi^{-1}(x, t) = \mathbf{x}$
- $\chi(\mathbf{x}, 0) = \mathbf{x}$ initial condition

$$v \equiv \frac{\partial \chi(\mathbf{x}, t)}{\partial t} \quad \text{velocity}$$

$$a \equiv \frac{\partial v(\mathbf{x}, t)}{\partial t} = \frac{\partial^2 \chi(\mathbf{x}, t)}{\partial t^2} \quad \text{acceleration}$$

motion

Lagrangian - describes in terms of material properties

$$\varphi = \varphi(\mathbf{x}, t)$$

... this is most natural for solids

Eulerian - describes in terms of spatial

$$\varphi = \bar{\varphi}(x, t)$$

ex. of pollution in a river: can measure change in concentration over time (Eulerian), or you can follow single pollution particle (Lagrangian)

→ they'll each give the same result but the equations will be different

Now let's describe mechanical properties in terms of Eulerian & Lagrangian

How φ changes w/time t ?

$$\varphi = \varphi(\mathbf{x}, t) \quad \text{or}$$

Lagrangian
 \mathbf{x} not dependent
 on time t

$$\varphi = \bar{\varphi}(x, t)$$

Eulerian
 x is dependent on
 time t through $\Rightarrow x = \chi(\mathbf{x}, t)$
 equation of motion

Let's consider Lagrangian description first

$$\dot{\varphi} = \frac{\partial \varphi(\mathbf{x}, t)}{\partial t} \quad \begin{matrix} \text{material time derivative} \\ \text{or Lagrangian} \end{matrix}$$

Now Eulerian - depends on t and t through x (chain rule)

$$\dot{\varphi} = \frac{\partial \bar{\varphi}(x, t)}{\partial t} + \frac{\partial \bar{\varphi}(x, t)}{\partial x} \frac{\partial x}{\partial t}$$

$$\dot{\varphi} = \underbrace{\frac{\partial \bar{\varphi}(x, t)}{\partial t}}_{\text{spatial time derivative}} + \underbrace{\frac{\partial \bar{\varphi}(x, t)}{\partial x} \bar{v}(x, t)}_{\text{convective time derivative}}$$

- how φ changes w/time at given pt.
- how φ changes w/time ...

in the future $\therefore \dot{\varphi} = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x} v$
 will drop bar

$$\dot{\varphi} = \frac{D\varphi}{Dt} \quad (\text{includes spatial + time derivative})$$

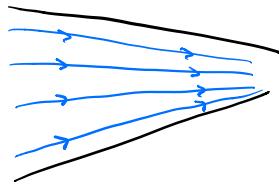
For Lagrangian description

$$\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t} \quad \text{because there is no convective term}$$

For Eulerian description

$$\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x} v$$

ex of fluid moving from large x -sectional area to small



at steady-state but there's changes in velocity so there's a convective acceleration term
 $v = \bar{v}(x) \rightarrow a = \ddot{v} = \frac{\partial \bar{v}(x, t)}{\partial x} \bar{v}(x, t)$

Example 2: find Eulerian Acceleration

$$v = \frac{2xt}{1+t^2} = \bar{v}(x, t) \quad a = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} v$$

find spatial time derivative, aka spatial acceleration

$$\frac{\partial v}{\partial t} = \frac{2x(1+t^2) - 4xt^2}{(1+t^2)^2} = \frac{2x(1-t^2)}{(1+t^2)^2}$$

find convective time derivative:

$$\frac{\partial v}{\partial x} = \frac{2t}{1+t^2} \quad \frac{\partial v}{\partial x} v = \frac{2t}{1+t^2} \cdot \frac{2xt}{1+t^2} = \frac{4xt^2}{(1+t^2)^2}$$

Now add these together to get a:

$$a = \frac{2x(1-t^2)}{(1+t^2)^2} + \frac{4xt^2}{(1+t^2)^2} = \frac{2x}{1+t^2} \checkmark$$

Example 3: Eulerian

given $v = \bar{v}(x, t) = \frac{2xt}{1+t^2}$, find x ?

No velocity field is given, now want to find trajectory

$$v = \frac{dx}{dt} = \frac{2xt}{1+t^2} \quad \dots \text{use separation of variables + integrate}$$

$$\int \frac{dx}{x} = \int \frac{2t dt}{1+t^2} = \int \frac{dt}{1+t^2} \Rightarrow x = C(1+t^2) \Rightarrow \ln \frac{x}{C} = \ln(1+t^2)$$

$$\frac{x}{C} = (1+t^2)$$

$$x = C(1+t^2)$$

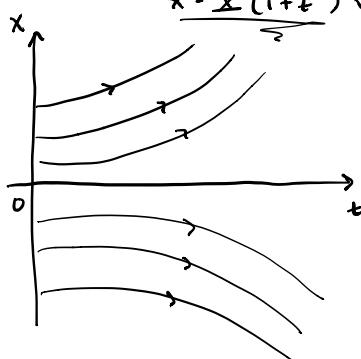
$$x = X(\underline{x}, t)$$

$$\underline{x} = X(\underline{x}, 0)$$

find C using initial condition:

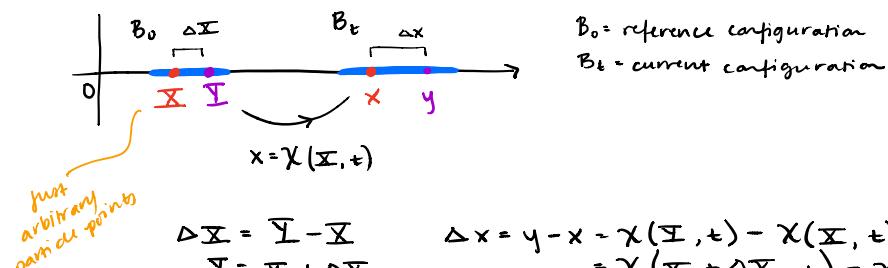
$$x \text{ at } 0 = \underline{x} \Rightarrow \underline{x} = C(1+0) = C \checkmark$$

$$x = \underline{x}(1+t^2) \checkmark$$



for different \underline{x} 's will have different trajectories (divergent field)

Local kinematics



B_0 = reference configuration

B_t = current configuration

$$\begin{aligned} \Delta \underline{x} &= \underline{x} - \underline{x} \\ \underline{x} &= \underline{x} + \Delta \underline{x} \end{aligned} \quad \begin{aligned} \Delta x &= y - x = X(\underline{x}, t) - X(\underline{x}, t) \\ &= X(\underline{x} + \Delta \underline{x}, t) - X(\underline{x}, t) \end{aligned}$$

$$\frac{\Delta x}{\Delta \underline{x}} = \lim_{\Delta \underline{x} \rightarrow 0} \frac{\Delta x}{\Delta \underline{x}} = \frac{y - x}{\underline{x} - \underline{x}} = \lim_{\Delta \underline{x} \rightarrow 0} \frac{X(\underline{x} + \Delta \underline{x}, t) - X(\underline{x}, t)}{\Delta \underline{x}} = \frac{\partial X(\underline{x}, t)}{\partial \underline{x}} = F(\underline{x}, t)$$

$$\boxed{dx = F(\underline{x}, t) d\underline{x}}$$

F = deformation gradient

λ = local stretch (the stretch around some arbitrary pt.)

$$\boxed{\lambda \equiv \frac{dx}{d\bar{x}} = F(\bar{x}, t)}$$

because we assume that the equation of motion is 1-to-1

because $X(\bar{x}, \cdot)$ is 1 to 1, $F(\bar{x}, t) \neq 0$

$$\begin{aligned} \bar{x} = X(\bar{x}, 0) \text{ initial condition} &\Rightarrow F(\bar{x}, 0) = 1 \\ \therefore \boxed{F(\bar{x}, t) > 0} &\quad \left. \begin{aligned} &\text{cannot be negative since 1 to 1 and} \\ &\text{would have to go thru 0 to be negative} \end{aligned} \right\} \\ &\quad t \left(\frac{dx(X, 0)}{d\bar{x}} = \frac{d\bar{x}}{d\bar{x}} = 1 \right) \\ &\quad dx \neq 0, d\bar{x} \neq 0 \end{aligned}$$

Lagrangian

... this means that if we start with a finite amount of matter, we preserve the conservation of matter

This is a mathematical constraint

so because F is strictly positive, then λ is also strictly \oplus

$$\boxed{\lambda > 0}$$

$\lambda > 1$ elongation

$0 < \lambda < 1$ compression (because can never compress to zero, all defects have to be finite)

$$dx = F(\bar{x}, t)d\bar{x} \Rightarrow d\bar{x} = F^{-1}dx$$

$$F^{-1} = \frac{d\bar{x}}{dx} = \frac{dx^{-1}(x, t)}{dx} = \bar{F}^{-1}(x, t)$$

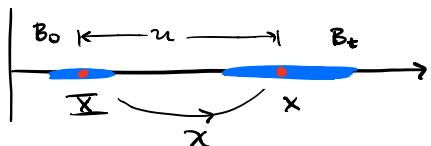
Eulerian

... take inverse of F to get the Eulerian description of deformation (bar just means Eulerian)

$$\dot{F} = \frac{DF}{Dt} = \frac{D}{Dt} \left(\frac{\partial X(\bar{x}, t)}{\partial \bar{x}} \right) = \underbrace{\frac{\partial}{\partial \bar{x}} \left(\frac{\partial X(\bar{x}, t)}{\partial t} \right)}_{v = \frac{\partial X(\bar{x}, t)}{\partial t}} = \frac{\partial v}{\partial \bar{x}} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial \bar{x}} = \frac{\partial v}{\partial x} \underbrace{\frac{\partial X(\bar{x}, t)}{\partial \bar{x}}}_{F} = \frac{\partial v}{\partial x} F$$

$$\boxed{\dot{F} = \frac{DF}{Dt} = \frac{\partial v}{\partial x} F}$$

Displacement



Lagrangian Displacement:

$$u = u(\bar{x}, t) \equiv x - \bar{x} = X(\bar{x}, t) - \bar{x}$$

\equiv means "by definition"

Eulerian Displacement

$$u = \bar{u}(x, t) \equiv x - \bar{x} = x - X^{-1}(x, t)$$

Everything in terms of x

Lagrangian Displacement Gradient:

$$\frac{\partial u}{\partial \bar{x}} = \frac{\partial X(\bar{x}, t)}{\partial \bar{x}} - 1 = F - 1$$

Eulerian Displacement Gradient:

$$\frac{\partial u}{\partial x} = 1 - \frac{\partial X^{-1}(x, t)}{\partial x} = 1 - F^{-1}$$

Strain - change of length wrt reference (true for small deformation)
 → for large deformation, strain is not linear

Lagrangian strain, E :

$$E \equiv \lim_{\mathbf{y} \rightarrow \mathbf{x}} \frac{\Delta \mathbf{x}^2 - \Delta \mathbf{X}^2}{2 \Delta \mathbf{X}^2} = \frac{1}{2} \left[\lim_{\mathbf{y} \rightarrow \mathbf{x}} \left(\frac{\Delta \mathbf{x}}{\Delta \mathbf{X}} \right)^2 - 1 \right] = \frac{1}{2} [\lambda^2 - 1] = \frac{1}{2} (F^2 - 1)$$

remember: (for Lagrangian) $\frac{\partial u}{\partial X} = \frac{\partial X}{\partial x}(x, t) - 1 = F - 1 \Rightarrow F = 1 + \frac{\partial u}{\partial x}$

$$E = \frac{\partial u}{\partial X} + \frac{1}{2} \left(\frac{\partial u}{\partial X} \right)^2$$

Linear strain Nonlinear

Eulerian strain, e :

$$e \equiv \lim_{\mathbf{y} \rightarrow \mathbf{x}} \frac{\Delta \mathbf{x}^2 - \Delta \mathbf{X}^2}{2 \Delta \mathbf{x}^2} = \frac{1}{2} \left[1 - \lim_{\mathbf{y} \rightarrow \mathbf{x}} \left(\frac{\Delta \mathbf{x}}{\Delta \mathbf{X}} \right)^2 \right] = \frac{1}{2} [1 - \lambda^2] = \frac{1}{2} (1 - F^{-2})$$

remember: (for Eulerian) $\frac{\partial u}{\partial x} = 1 - \frac{\partial X^{-1}}{\partial x}(x, t) = 1 - F^{-1} \Rightarrow F^{-1} = 1 - \frac{\partial u}{\partial x}$

$$e = \frac{\partial u}{\partial x} - \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2$$

Eulerian strain

Linear (engineering) strain ε :

$$\varepsilon = \lim_{\mathbf{Y} \rightarrow \mathbf{x}} \frac{\Delta \mathbf{x} - \Delta \mathbf{X}}{\Delta \mathbf{X}} = \lim_{\mathbf{Y} \rightarrow \mathbf{x}} \left(\frac{\Delta \mathbf{x}}{\Delta \mathbf{X}} \right) - 1 = \lambda - 1 = F - 1$$

$$F = 1 + \frac{\partial u}{\partial x} \Rightarrow \boxed{\varepsilon = \frac{\partial u}{\partial x}}$$

... so if deformation is small, $\frac{\partial u}{\partial x}$ is small, $(\frac{\partial u}{\partial x})^2 \approx 0$
 $\therefore \varepsilon \approx E$

... furthermore, small deformation $\Rightarrow x \approx \mathbf{X}$
 $\therefore \varepsilon \approx e$

... then $\varepsilon \approx E \approx e$... but this is not the case for large deformations,
 such as in soft tissue injury (biomechanics)

Stretching - the fractional rate of length of change, K (kappa)

ex. see change in length of streamline in flow

$$K \equiv \lim_{\mathbf{y} \rightarrow \mathbf{x}} \frac{(\Delta \mathbf{x})^\circ}{\Delta \mathbf{x}} = \lim_{\mathbf{y} \rightarrow \mathbf{x}} \frac{(y - x)}{y - x} = \lim_{\mathbf{y} \rightarrow \mathbf{x}} \frac{\bar{v}(y, t) - \bar{v}(x, t)}{y - x} = \frac{\partial \bar{v}}{\partial x} = \frac{\partial v}{\partial x}$$

$$\boxed{K = \frac{\partial v}{\partial x} \text{ velocity gradient}}$$

$$K \neq \dot{\varepsilon} \text{ b/c } \dot{\varepsilon} = \frac{D\varepsilon}{Dt} = \frac{D}{Dt} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial \dot{u}}{\partial x} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial x} \frac{\partial x}{\partial X} = \frac{\partial v}{\partial x} F = K(1 + \varepsilon)$$

$$\therefore K = \frac{\dot{\varepsilon}}{1 + \varepsilon}$$

... however, for $\varepsilon \ll 1$, then $K \approx \dot{\varepsilon}$

Example 4:

$$x = \underline{\underline{X}}(1+t^2) \quad \lambda, E, \epsilon, \dot{\epsilon}, K, \dot{\dot{\epsilon}} ?$$

$$\lambda = F = \frac{\partial \underline{\underline{X}}}{\partial \underline{\underline{x}}} = \underbrace{(1+t^2)}_{?} = \lambda$$

$$\frac{\partial u}{\partial \underline{\underline{x}}} = \lambda - 1 = t^2 \Rightarrow E = \frac{\partial u}{\partial \underline{\underline{x}}} + \frac{1}{2} \left(\frac{\partial u}{\partial \underline{\underline{x}}} \right)^2 = \underbrace{t^2 (1 + \frac{1}{2} t^2)}_{?} = E$$

$$\epsilon \text{ (Eulerian strain)}: \frac{\partial u}{\partial x} = ? \rightarrow u = x - \underline{\underline{X}} = \underline{\underline{X}}(1+t^2) - \underline{\underline{X}} = \underline{\underline{X}} + t^2 = \frac{x+t^2}{1+t^2}$$

$$\frac{\partial u}{\partial x} = \frac{t^2}{1+t^2} \Rightarrow \epsilon = \frac{\partial u}{\partial x} - \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 = \underbrace{\frac{t^2}{1+t^2} \left(1 + \frac{1}{2} t^2 \right)}_{?} = \epsilon$$

$$\epsilon: \epsilon = \frac{\partial u}{\partial \underline{\underline{x}}} = \underbrace{t^2}_{?} = \epsilon$$

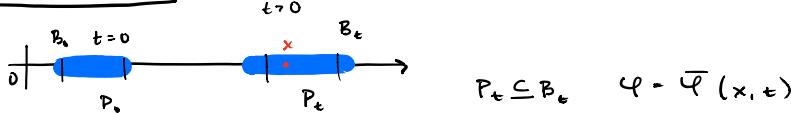
see example 2

$$K: \frac{\partial v}{\partial x} \rightarrow v = \frac{2xt}{1+t^2} \Rightarrow \frac{\partial v}{\partial x} = \frac{2t}{1+t^2} = K$$

$$\dot{\epsilon}: K(1+\epsilon) = \frac{\partial v}{\partial x} (1 + \frac{\partial u}{\partial x}) = \frac{2t}{1+t^2} (1+t^2) = \underbrace{2t}_{?} = \dot{\epsilon}$$

$$\text{also: } \dot{\dot{\epsilon}} = \frac{D\epsilon}{Dt} = \frac{2t}{?} \text{ (same as above)}$$

Field Equations



$$P_t \subseteq B_t \quad \varphi = \bar{\varphi}(x, t)$$

$\int_{P_t} \bar{\varphi}(x, t) dx \dots \text{want to find } \frac{D}{Dt} \int_{P_t} \varphi dx = ? \quad (\dots \text{aka Reynolds Transport Theorem})$

↑ this integral describes
physical property of P_t
at same time

$\frac{D}{Dt} \int_{P_t} \varphi dx \dots \text{can't bring } \frac{D}{Dt} \text{ into integral since } P_t \text{ depends on time}$

... remember $dx = F d\underline{\underline{x}}$

$$\frac{D}{Dt} \int_{P_t} \varphi dx = \frac{D}{Dt} \int_{P_0} \varphi F d\underline{\underline{x}} = \int_{P_0} \frac{D(\varphi F)}{Dt} d\underline{\underline{x}} = \int_{P_0} (\dot{\varphi} F + \varphi \dot{F}) d\underline{\underline{x}} = \int_{P_0} (\dot{\varphi} F + \varphi \frac{\partial v}{\partial x} F) d\underline{\underline{x}}$$

↑ bc P_0 not time dependent,
can bring derivative inside \int

$$\Rightarrow \text{in Lagrangian form} \Rightarrow \int_{P_0} (\dot{\varphi} F + \varphi \frac{\partial v}{\partial x} F) d\underline{\underline{x}} = \int_{P_0} (\dot{\varphi} + \varphi \frac{\partial v}{\partial x}) F d\underline{\underline{x}} = \int_{P_t} \underbrace{\left(\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x} v + \varphi \frac{\partial v}{\partial x} \right)}_{\frac{\delta(\varphi v)}{\delta x}} dx$$

$$\Rightarrow \int_{P_t} \left[\frac{\partial \varphi}{\partial t} + \frac{\delta(\varphi v)}{\delta x} \right] dx$$

and now we can define Reynold's Transport Theorem

$$\boxed{\frac{D}{Dt} \int_{P_t} \varphi dx = \int_{P_t} \left[\frac{\partial \varphi}{\partial t} + \frac{\partial (\varphi v)}{\partial x} \right] dx}$$

we will use RTT to derive other items, such as balance of mass

Balance of Mass

$\varphi = \bar{\rho}(x, t)$ Eulerian mass density at x at t

$$M_{P_t} = \int_{P_t} \varphi dx \quad \text{from physics} \Rightarrow \frac{D M_{P_t}}{Dt} = 0 \quad \text{because mass is constant}$$

$$\Rightarrow \therefore \int_{P_t} \left[\frac{\partial \varphi}{\partial t} + \frac{\partial (\varphi v)}{\partial x} \right] dx = 0 \quad \forall P_t \subseteq B_t$$

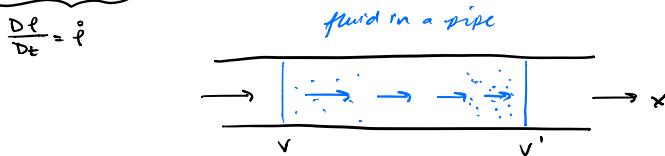
↑ for every

$$\therefore \boxed{\frac{\partial \varphi}{\partial t} + \frac{\partial (\varphi v)}{\partial x} = 0} \quad \text{the continuity equation}$$

↑ represents a local balance of mass, true for every $(\forall) x \in B_t$ and $t > 0$

physical interpretation:

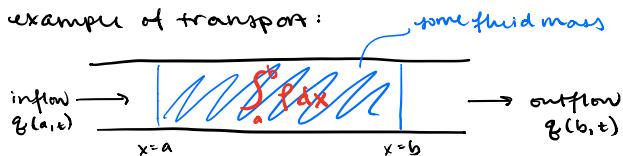
$$\underbrace{\frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x} v}_{\frac{D \varphi}{Dt} = \dot{\varphi}} + \varphi \frac{\partial v}{\partial x} > 0 \Rightarrow \boxed{\frac{1}{\varphi} \frac{D \varphi}{Dt} = -\frac{\partial v}{\partial x}} \quad \text{also continuity, but in more physically apparent way}$$



since $\frac{\partial v}{\partial x} < 0$ then $v > v'$

... as the velocity slows down, particles will begin accumulating at the right end

example of transport:



how does total mass change with time?

$$q_b(x, t) = \bar{\rho}(x, t) \bar{v}(x, t)$$

$$\frac{\partial}{\partial t} \int_a^b \bar{\rho}(x, t) dx = \int_a^b \frac{\partial \bar{\rho}(x, t)}{\partial t} dx = \bar{\rho}(a, t) \bar{v}(a, t) - \bar{\rho}(b, t) \bar{v}(b, t) = - \int_a^b d(\varphi v) = - \int_a^b \frac{\partial (\varphi v)}{\partial x} dx$$

note that we are partial since fixed integral

$$\int_a^b \left[\frac{\partial \bar{\rho}(x, t)}{\partial t} + \frac{\partial (\varphi v)}{\partial x} \right] dx = 0 \quad \forall a, b \quad \therefore \boxed{\frac{\partial \varphi}{\partial t} + \frac{\partial (\varphi v)}{\partial x} = 0}$$