Recap:

$$\varphi = \bar{\varphi}(x,t) \qquad \boxed{\frac{D}{Dt}\int_{P_t} \varphi \, dx = \int_{P_t} \left[\frac{\partial \varphi}{\partial t} + \frac{\partial (\varphi v)}{\partial x} \right] dx} \quad \text{RTT} \qquad P_t \subseteq B_t$$

$$\rho = \bar{\rho}(x,t) \Rightarrow m(P_t) = \text{const.} = \int_{P_t} \rho \, dx \qquad \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0}$$

Eulerian continuity equation, holds for every little x in P_t

$\forall x \in B_t$

$$\frac{D m(P_t)}{D_t} = 0$$ global balance of mass

$m(P_t) = \text{const.}$

$m(P_{t_1}) = m(P_{t_2}) = m(P_0)$

$$m(P_0) = m(P_t)$$

$$\int_{P_0} \rho(X, 0)\, dX = \int_{P_t} \bar{\rho}(x,t)\, dx \qquad \rho(X,0) \equiv \rho_0(X)$$

$\uparrow$ Lagrangian $\qquad \uparrow$ Eulerian

$$\int_{P_0} \rho_0(X)\, dX = \int_{P_0} \rho(X,t) F \, dX$$

$$\int_{P_0} (\rho_0 - \rho F)\, dX = 0 \quad \forall \; P_0 \subseteq B_0$$

$$\therefore \boxed{\rho_0 = \rho F \quad \forall X \in B_0} \longleftarrow$$ local Lagrangian balance of mass $\frac{1}{2}$ continuity equation

...will need to know how to start from this (Lagrangian) to Eulerian

Another case: thermodynamic properties always defined as specific

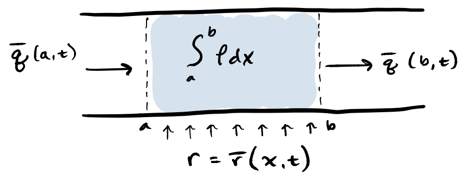
$$\psi = \frac{\varphi}{\rho} \Rightarrow \frac{D}{Dt}\int_{P_t} \rho \psi \, dx = \int_{P_t} \left[\frac{\partial (\rho\psi)}{\partial t} + \frac{\partial (\rho \psi v)}{dx} \right] dx$$

$$= \int_{P_t} \left[\frac{\partial \rho}{\partial t}\psi + \rho \frac{\partial \psi}{\partial t} + \frac{\partial (\rho v)}{\partial x}\psi + \rho v \frac{\partial \psi}{\partial x} \right] dx$$

$$= \underbrace{\int_{P_t} \left[\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} \right] \psi \, dx}_{=0 \text{ because of mass conservation}} + \underbrace{\int_{P_t} \left[\frac{\partial \psi}{\partial t} + v \frac{\partial \psi}{\partial x} \right]}_{\frac{D\psi}{Dt}} \rho \, dx$$

$$\boxed{\frac{D}{Dt}\int_{P_t} \rho \psi \, dx = \int_{P_t} \rho \frac{D\psi}{Dt} \, dx} \quad \text{RTT 2} \quad \forall \; P_t \subseteq B_t$$

Example #5: jet engine – mass not conserved, but mass added



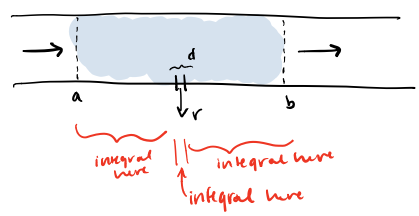
$$\frac{\partial}{\partial t}\int_a^b \bar{\rho}(x,t)\,dx = \int_a^b \frac{\partial \bar{\rho}(x,t)}{\partial t}\,dx = \bar{\rho}(a,t)\,\bar{v}(a,t) - \bar{\rho}(b,t)\,\bar{v}(b,t) + \int_a^b \bar{r}(x,t)\,dx =$$

$$= -\int_a^b d(\rho v) + \int_a^b r\,dx = -\int_a^b \frac{\partial(\rho v)}{\partial x}\,dx + \int_a^b r\,dx$$

$$= \int_a^b \left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} - r\right] dx = 0 \quad \forall a, b$$

$$\therefore \quad \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} - r = 0 \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = r}$$

now imagine there's single hole



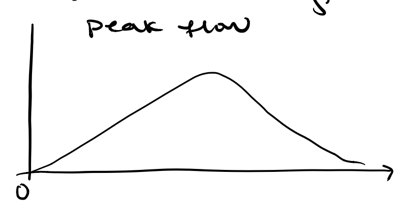
Application: Car Traffic

ρ = car density $\equiv \frac{\#\text{ cars}}{\text{length}}$

assume same amount of cars in/out: conservation

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial t} = 0} \qquad \begin{matrix}\rho = ?\\ v = ?\end{matrix}$$

$q = \rho v$, at some point between 0 and max density, will have peak flow



describes this curve

$Q(\rho) = \rho\,\bar{V}(\rho)$ ... experimental data
"state equation"

$$\therefore \frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{\partial Q}{\partial \rho}\frac{\partial \rho}{\partial x} = 0$$

$$\frac{dQ}{d\rho} = c(\rho) \quad \therefore \quad \boxed{\frac{\partial \rho}{\partial t} + c(\rho)\frac{\partial \rho}{\partial x} = 0}$$

we call this the wave equation in PDE

$\rho(x,t) = \rho^* = \text{const.}$

$\rho = \rho^* + \hat{\rho}(x,t)$ ... $\hat{\rho}$ = small disturbance

$c(\rho) = c(\rho^* + \hat{\rho}) = c(\rho^*) + \frac{dc(\rho^*)}{d\rho}\hat{\rho} + \cancel{H.O.T}^{\,0}$

$$\boxed{c(\rho)} = c(\rho^*) + \frac{dc(\rho^*)}{d\rho}\hat{\rho} = \boxed{c^* + \frac{dc(\rho^*)}{d\rho}\hat{\rho}}$$

$$\boxed{\frac{\partial \hat{\rho}}{\partial t} + c^* \frac{\partial \hat{\rho}}{\partial x} = 0}$$

unidirectional wave equation, c^* wave propagation good ← describes wave propagation in only one direction

D'Alambert's Solution

$x, t \to \xi, \eta \qquad \xi = x + c^* t \qquad \eta = x - c^* t$

$$\frac{\partial \hat{\rho}}{\partial t} = \frac{\partial \hat{\rho}}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial \hat{\rho}}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial \hat{\rho}}{\partial \xi}c^* - \frac{\partial \hat{\rho}}{\partial \eta}c^*$$

$$\frac{\partial \hat{\rho}}{\partial x} = \frac{\partial \hat{\rho}}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial \hat{\rho}}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial \hat{\rho}}{\partial \xi} + \frac{\partial \hat{\rho}}{\partial \eta}$$

$$c^*\left(\frac{\partial \hat{\rho}}{\partial \xi} - \cancel{\frac{\partial \hat{\rho}}{\partial \eta}}\right) + c^*\left(\frac{\partial \hat{\rho}}{\partial \xi} + \cancel{\frac{\partial \hat{\rho}}{\partial \eta}}\right) = 0 \Rightarrow 2c^*\frac{\partial \hat{\rho}}{\partial \xi} = 0$$

$$\therefore \frac{\partial \hat{\rho}}{\partial \xi} = 0 \Rightarrow \underline{\hat{\rho} = f(\eta) = f(x - c^* t)}$$

$f(\cdot) = ?$ Determine it from IC

suppose at $t = 0 \Rightarrow \rho(X, 0) \equiv \rho_0(X) = \rho^* + f(X)$

$\therefore f(X) = \rho_0(X) - \rho^*$ true for any X $(\forall X)$

$\hat{\rho} = f(x - c^* t) = \rho_0(x - c^* t) - \rho^*$

$\rho = \rho^* + \underbrace{\rho_0(x - c^* t) - \rho^*}_{\hat{\rho}} = \rho_0(x - c^* t)$

$$\boxed{\rho(x,t) = \rho_0(x - c^* t)}$$

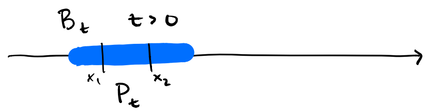
$c(\rho) \neq V(\rho)$

$Q(\rho) = \rho V(\rho) \Rightarrow \underline{\underline{c(\rho)}} = \frac{dQ}{d\rho} = V(\rho) + \rho\frac{dV}{d\rho}$

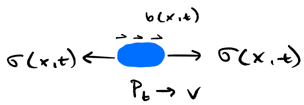
$\frac{dV}{d\rho} < 0 \qquad c(\rho) < V(\rho)$

for bumper-to-bumper $V(\rho) \approx 0 \Rightarrow c(\rho) < 0$ ... disturbance will be in opposite direction

Balance of Momentum



body forces and contact forces act on piece of material P_t
like gravity, centripetal
between bodies



$$F(P_t) \equiv \sigma(x_2,t) - \sigma(x_1,t) + \int_{P_t} \rho b(x,t)\,dx =$$

$$= \int_{P_t} d\sigma(x,t) + \int_{P_t} \rho b\,dx =$$

$$= \int_{P_t} \frac{\partial \sigma}{\partial x}\,dx + \int_{P_t} \rho b\,dx = \boxed{\int_{P_t} \left(\frac{\partial \sigma}{\partial x} + \rho b\right) dx}$$

resultant force $\forall P_t \subseteq B_t$

$$\mathcal{P}(\rho_t) \equiv \int_{P_t} \rho v\,dx \quad \forall P_t \subseteq B_t$$

1st Euler's Axiom: $\dfrac{D\mathcal{P}(P_t)}{Dt} = F(P_t)$

$$\frac{D}{Dt}\int_{P_t} \rho v\,dx = \int_{P_t}\left(\frac{\partial \sigma}{\partial x} + \rho b\right) dx$$

$\downarrow$ (by RTT2)

$$\int_{P_t} \rho \frac{Dv}{Dt}\,dx = \int_{P_t}\left(\frac{\partial \sigma}{\partial x} + \rho b\right) dx$$

$$\int_{P_t}\left(\rho \frac{Dv}{Dt} - \frac{\partial \sigma}{\partial x} - \rho b\right) dx = 0 \quad \forall P_t \subseteq B_t$$

$$\boxed{\rho \frac{Dv}{Dt} = \frac{\partial \sigma}{\partial x} + \rho b}$$

Local Eulerian balance of momentum $\forall x \in B_t,\ t$

$$\boxed{\rho \hat{v} = \frac{\partial \sigma}{\partial x} + \rho b}$$

Static conditions $(v=0)$

$$\boxed{\frac{\partial \sigma}{\partial x} + \rho b = 0}$$

Local Equilibrium

$$\rho \frac{Dv}{Dt} = \frac{\partial \sigma}{\partial x} + \rho b \Big/ F \quad \Rightarrow \quad \underbrace{\rho F}_{\rho_0} \frac{Dv}{Dt} = \underbrace{\frac{\partial \sigma}{\partial x} F}_{\frac{\partial \sigma}{\partial x}\frac{\partial x}{\partial X} = \frac{\partial \sigma}{\partial X}} + \underbrace{\rho F}_{\rho_0} b$$

$$\boxed{\rho_0 \frac{Dv}{Dt} = \frac{\partial \sigma}{\partial X} + \rho_0 b}$$

Local Lagrangian balance of momentum $\forall X \in B_0$

Lagrangian Balance of Mass can be written as

$\rho_0 = \rho F$

$\rho_0 \equiv \rho(\mathbb{X}, 0)$ true for every $\mathbb{X}$ in B_0 (aka $\forall\, \mathbb{X} \in B_0$) * local balance of mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$$ — Eulerian local balance of mass $\forall\, x \in B_t$

langrangian continuity equation is algebraic while Eulerian is a first order PDE

go from Lagrangian to Eulerian

$$\rho_0 = \rho F \longrightarrow \underbrace{\frac{D\rho_0}{Dt}}_{0} = \frac{D(\rho F)}{Dt} = \frac{D(\rho)}{Dt} F + \rho \frac{DF}{Dt} = \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} v\right) F + \rho \frac{\partial v}{\partial x} F$$

Lagrangian

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} v + \rho \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0$$

Eulerian

Material time derivative of momentum : $\frac{D\mathcal{J}(P_t)}{Dt} = F(P_t) \quad \forall\, P_t \subseteq B_t$

$$F(P_t) = \underbrace{\int_{P_t} \rho b \, dx}_{\text{body forces (forces that act at distances)}} + \underbrace{\sigma(x_2, t) - \sigma(x_1, t)}_{\text{contact forces (represent physical interactions between two bodies)}}$$

$\mathcal{J}$ = linear momentum

$$\mathcal{J}(P_t) \equiv \int_{P_t} \rho v \, dx$$

$$\boxed{\frac{\partial \sigma}{\partial x} + \rho b = \rho \frac{Dv}{Dt}} \quad \forall\, x \in B_t, t$$

Local eulerian balance of momentum

For $v = 0$ or $v = \text{const.}$ $\Rightarrow$ $\boxed{\frac{\partial \sigma}{\partial x} + \rho b = 0}$ local eulerian equilibrium

$$\left.\begin{array}{l} \underbrace{\frac{\partial \sigma}{\partial x} F}_{\frac{\partial \sigma}{\partial x}\frac{\partial x}{\partial \mathbb{X}}} + \underbrace{\rho F}_{\rho_0} b = \underbrace{\rho F}_{\rho_0} \frac{Dv}{Dt} \end{array}\right\} \Rightarrow \boxed{\frac{\partial \sigma}{\partial \mathbb{X}} + \rho_0 b = \rho_0 \frac{\partial v}{\partial t}} \quad \forall\, \mathbb{X} \in B_0, t$$

local lagrangian momentum

lagrangian time deriv.

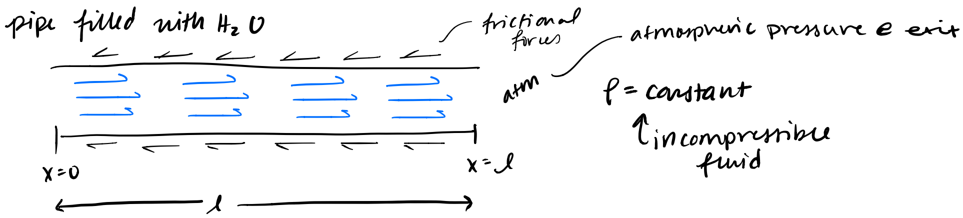
$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} \quad v = v(\mathbb{X}, t)$

Ex. 6: Simple Fluid Flow in a Pipe

For Fluids, $\sigma(x,t) = -p(x,t)$

pressure

$\ominus$ by convention bc $\uparrow$ press means $\downarrow$ volume of body



ρ = constant

incompressible fluid

a. Use continuity equation to show velocity just function of time (as in doesn't change w/x)

$v = v(t)$

$$\cancel{\frac{\partial \rho}{\partial t}} + \frac{\partial(\rho v)}{\partial x} = 0 \Rightarrow \rho\frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial v}{\partial x} = 0 \Rightarrow \underline{v = v(t)}$$

0 because incompressible fluid

b. $\rho b = -Cv$, effective body force from frictional resistance

C = const.

momentum eq: $\frac{\partial \sigma}{\partial x} + \rho b = \rho\frac{Dv}{Dt}$ $\quad \sigma$ = pressure $= -p$

$$-\frac{\partial p}{\partial x} - Cv = \rho\frac{dv}{dt}$$

why go from material to ordinary derivative? bc v just function of time

$$\therefore \frac{\partial P}{\partial x} = \underbrace{-\left(cv + \rho\frac{dv}{dt}\right)}_{\text{function of time}} \Rightarrow \underset{\text{integrate}}{\int} \Rightarrow \boxed{P = -\left(cv + \rho\frac{dv}{dt}\right)x + f(t)}$$

c. suppose at $x=0$ then $p(0,t) \equiv P_0(t)$ "main's pressure" — pressure of faucet

$x = \ell$ $\quad p(\ell,t) \equiv P_\ell(t)$ atmospheric pressure

to have good flow, always need $P_0 > P_\ell$ mains must be greater than atmospheric

momentum equation becomes $\quad \rho\frac{dv}{dt} + Cv = \frac{P_0 - P_\ell}{\ell}$ show how this is so.

use B.C. of $x=0$ and $x=\ell$

$P_0(t) = f(t)$

$p(x,t) = -\left(cv + \rho\frac{dv}{dt}\right)x + P_0(t)$

$P_\ell(t) = -\left(cv + \rho\frac{dv}{dt}\right)\ell + P_0(t) \Rightarrow \underline{\rho\frac{dv}{dt} + Cv = \frac{P_0 - P_\ell}{\ell}}$ ✓

d. obtain velocity distribution, $v(0) = 0$ find $v(t)$

$\div$ by ρ since constant and nonzero

constant of integration

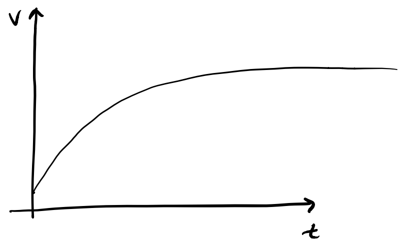
$$\frac{dv}{dt} + \frac{c}{\rho}v = \frac{P_0 - P_\ell}{\rho\ell} \Rightarrow v(t) = Ke^{-\frac{c}{\rho}t} + \frac{P_0 - P_\ell}{c\ell}$$

general solution of the DE, now need to find K

and then at $t \to \infty$ $\quad v = \frac{P_0 - P_\ell}{c\ell}$

$$0 = v(0) = K + \frac{P_0 - P_\ell}{c\ell} \Rightarrow K = \frac{P_\ell - P_0}{c\ell} \Rightarrow \boxed{v(t) = \frac{P_0 - P_\ell}{c\ell}\left(1 - e^{-\frac{c}{\rho}t}\right)}$$

as $t\uparrow$, exponential smaller, $v\uparrow$



Balance of Energy

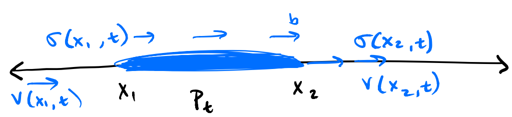
→ we'll start with mechanical

Power Th. – rate of mechanical working that is done on part or whole of material body is equal to change in KE on body

$W(P_t)$ mechanical working, rate of external forces acting

$$\boxed{W(P_t) = \frac{DK(P_t)}{Dt} + \int_{P_t} \sigma \frac{\partial v}{\partial x}\, dx \quad \forall P_t \subseteq B_t \qquad \text{Power Th.}}$$

kinetic energy $K(P_t) \equiv \int_{P_t} \frac{\rho v^2}{2} dx$ kinetic energy of $P_t \subseteq B_t$



$$W(P_t) \equiv \int_{P_t} \rho b v\, dx + \underbrace{\sigma(x_2,t)v(x_2,t) - \sigma(x_1,t)v(x_1,t)}_{\int_{P_t} d(\sigma v) = \int_{P_t} \frac{\partial(\sigma v)}{\partial x} dx} = \int_{P_t} \left[\rho b v + \frac{\partial(\sigma v)}{\partial x}\right] dx =$$

force · velocity = rate of work = power

$$\int_{P_t} (\underbrace{\rho b v + \frac{\partial \sigma}{\partial x} v}_{(\rho b + \frac{\partial \sigma}{\partial x}) v = \rho \frac{Dv}{Dt}} + \sigma \frac{\partial v}{\partial x}) dx$$

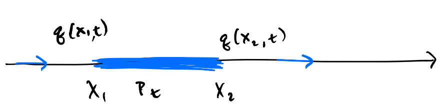
by momentum equation

$$= \int_{P_t} \rho \frac{Dv}{Dt} v\, dx + \int_{P_t} \sigma \frac{\partial v}{\partial x} dx = \underbrace{\int_{P_t} \frac{1}{2} \rho \frac{Dv^2}{Dt} dx}_{\text{from RTT 2} \quad \frac{D}{Dt} \underbrace{\int_{P_t} \frac{1}{2} \rho v^2 dx}_{K(P_t)}} + \int_{P_t} \sigma \frac{\partial v}{\partial x} dx$$

$$= \frac{DK(P_t)}{Dt} + \int_{P_t} \sigma \frac{\partial v}{\partial x} dx$$

$$\boxed{\begin{array}{l} 1^{st} \text{ Law of Thermodynamics} \\ W(P_t) = \frac{DK(P_t)}{Dt} + \frac{D\overset{\text{internal}}{U}}{Dt} - Q(P_t) \end{array}}$$

$U(P_t) \equiv \int_{P_t} \rho u\, dx$ u = specific internal energy (energy/mass)



$Q(P_t) \equiv \int_{P_t} \rho r\, dx + q(x_1,t) - q(x_2,t) \quad \forall P_t \subseteq B_t$

r = specific internal heat (heat/mass)

$$\underbrace{\cancel{\frac{DK(P_t)}{Dt}} + \int_{P_t} \sigma \frac{dv}{dx}}_{W(P_t)} = \cancel{\frac{DK(P_t)}{Dt}} + \underbrace{\frac{D}{Dt}\int_{P_t} \rho u \, dx}_{\text{use RTT2}} - \int_{P_t} \rho r \, dx + q(x_2,t) - q(x_1,t)$$

$$\int_{P_t} \sigma \frac{dv}{dx} = \int_{P_t} \rho \frac{Du}{Dt}\, dx - \int_{P_t} \rho r \, dx + \int_{P_t} \frac{dq}{dx}\, dx$$

$$\int_{P_t} \left(\rho \frac{Du}{Dt} - \sigma \frac{dv}{dx} - \rho r + \frac{dq}{dx} \right) dx = 0 \quad \forall \; P_t \subseteq B_t$$

$\therefore$ the integrand must be zero ... $\rho \frac{Du}{Dt} - \sigma \frac{dv}{dx} - \rho r + \frac{dq}{dx} = 0$

$\therefore$ $\boxed{\rho \frac{Du}{Dt} = \sigma \frac{dv}{dx} + \rho r + \frac{dq}{dx} \qquad \text{local Eulerian } 1^{st} \text{ law } \forall \; x \in B_t, t}$

multiply everything by F to get Lagrangian

$$\underbrace{\rho F}_{\rho_0} \frac{Du}{Dt} = \sigma \frac{dv}{dx} F + \underbrace{\rho F}_{\rho_0} r - \frac{dq}{dx} F$$

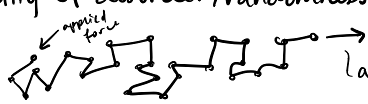
$$\rho_0 \frac{Du}{Dt} = \sigma \frac{dv}{dx}\frac{dx}{dX} + \rho_0 r - \frac{dq}{dx}\frac{dx}{dX}$$

$$\boxed{\rho_0 \frac{Du}{Dt} = \sigma \frac{dv}{X} + \rho_0 r - \frac{dq}{dX} \qquad \text{Local Lagrangian } 1^{st} \text{ law } \forall \; X \in B_0, t}$$

what we don't know from 1st law is the direction of heat xfer

<u>Entropy</u> – quantity of disorder/randomness ... ↑ disorder, ↑ entropy

take a long chain polymer



... if try to straighten/stretch, equivalent to order, entropy ↑

... if crumble, more disorder, entropy ↑

elasticity here vs. metals which are crystalline

Recap:

$$W(P_t) \equiv \int_{P_t} \rho b v \, dx + \sigma(x_2,t)\, v(x_2,t) - \sigma(x_1,t)\, v(\sigma x_1, t) =$$

$$\underbrace{\frac{DK(P_t)}{Dt}}_{\text{change of KE}} + \underbrace{\int_{P_t} \sigma \frac{dv}{dx}\, dx}_{\text{stress power}}$$

due to motion (first term) — due to deformation (second term)

$$\underbrace{W(P_t)}_{\text{work/power}} + \underbrace{Q(P_t)}_{\text{heat}} = \frac{DE_{tot}(P_t)}{Dt}$$

where $E_{tot}(P_t) = \underbrace{K(P_t)}_{\text{kinetic energy}} + \underbrace{\Phi(P_t)}_{\text{potential energy}} + \underbrace{U(P_t)}_{\text{internal energy}}$ in classical thermodynamics

Φ in continuum mechanics is part of $\mathcal{W}(P_t)$ even though not classically true

So in continuum mechanics we can write the 1st law as

$$\boxed{\mathcal{W}(P_t) + Q(P_t) = \frac{DE_{tot}(P_t)}{Dt} = \frac{DK(P_t)}{Dt} + \frac{DU(P_t)}{Dt}}$$

something DS will ask:
- where is the potential energy?
- can you quantify?
- * know concepts + math

$$\boxed{\mathcal{W}(P_t) = \frac{DK(P_t)}{Dt} + \frac{DU(P_t)}{Dt} - Q(P_t) \quad \forall P_t \subseteq B_t}$$

$$\rho \frac{Du}{Dt} = \sigma \frac{\partial v}{\partial x} + \rho r - \frac{\partial q}{\partial x} \quad \forall x \in B_t, t \qquad \text{local Eulerian 1st law}$$

$$\rho_0 \frac{Du}{Dt} = \sigma \frac{\partial v}{\partial X} + \rho_0 r - \frac{\partial q}{\partial X} \quad \forall X\, B_0, t \qquad \text{local Langrangian 1st law}$$

r = specific heat (internal heat source)

2nd Law of Thermodynamics

$$\rho \frac{Du}{Dt} = \sigma \frac{\partial v}{\partial x} + \rho r - \frac{\partial q}{\partial x}$$

$\mathcal{S}(P_t)$ = internal entropy of the body

$$\mathcal{S}(P_t) \equiv \int_{P_t} \rho \eta \, dx \quad \forall P_t \subseteq B_t$$

entropy possessed by the system

and η = specific entropy

$\dot{H}(P_t)$ = rate of entropy input into body

$$\dot{H}(P_t) \equiv \int_{P_t} \rho \frac{r}{T} dx + \frac{q(x_1,t)}{T} - \frac{q(x_2,t)}{T}$$

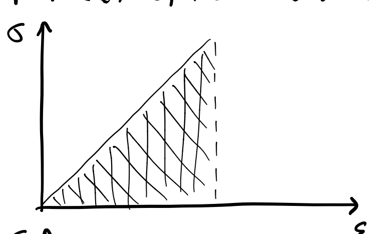
T = absolute temperature

$\Gamma(P_t)$ = total entropy production of the body

$$\boxed{\Gamma(P_t) \equiv \frac{D\,\mathcal{S}(P_t)}{Dt} - \dot{H}(P_t) \quad \forall P_t \subseteq B_t}$$

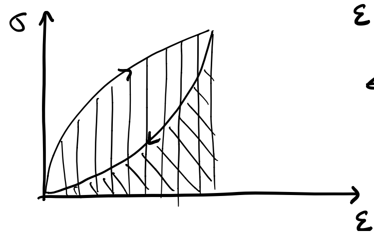
$$\boxed{\Gamma(P_t) \geq 0 \quad \forall P_t \subseteq B_t}$$
2nd Law of Thermodynamics

If $\Gamma(P_t) = 0$, then we have reversible processes
If $\Gamma(P_t) > 0$, then we have irreversible processes



Stress-Strain curve
- reversible processes

In reality, nothing is reversible, unless quasi-static, because there is internal friction. ata zero



← in reality

$$\mathring{H}(P_t) \equiv \int_{P_t} \rho \frac{r}{T} dx + \frac{q(x_1,t)}{T} - \frac{q(x_2,t)}{T} = \int_{P_t} \rho \frac{r}{T} dx - \int \frac{\partial}{\partial x}\left(\frac{q}{T}\right) dx$$

using RTT2:

$$\frac{D}{Dt}\int_{P_t} \rho \eta \, dx - \int_{P_t} \rho \frac{r}{T} dx + \int_{P_t} \frac{\partial}{\partial x}\left(\frac{q}{T}\right) dx \geq 0$$

$$\int_{P_t} \left[\rho \frac{D\eta}{Dt} + \frac{\partial}{\partial x}\left(\frac{q}{T}\right) - \rho \frac{r}{T}\right] dx \geq 0 \quad \forall P_t \subseteq B_t$$

$$\rho \frac{D\eta}{Dt} + \frac{\partial}{\partial x}\left(\frac{q}{T}\right) - \rho \frac{r}{T} \geq 0$$

local Eulerian form of the 2nd Law of TD

$$\boxed{\rho \frac{D\eta}{Dt} - \rho \frac{r}{T} + \frac{1}{T}\frac{\partial q}{\partial x} - \frac{q}{T_2}\frac{\partial T}{\partial x} \geq 0} \quad \forall x \subseteq B_t, t$$

"Clausius – Duhem Inequality"

sometimes useful to replace $\frac{\partial q}{\partial x}$ term with what was found earlier

$$\rho \frac{Du}{Dt} = \sigma \frac{\partial v}{\partial x} + \rho r - \frac{\partial q}{\partial x} \Rightarrow \frac{\partial q}{\partial x} = \sigma \frac{\partial v}{\partial x} + \rho r - \rho \frac{Du}{Dt}$$

$$\boxed{\underbrace{\rho T \frac{D\eta}{Dt} + \sigma \frac{\partial v}{\partial x} - \rho \frac{Du}{Dt}}_{\text{internal (mechanical) dissipation}} \underbrace{- \frac{q}{T}\frac{\partial T}{\partial x}}_{\text{thermal (conductive) dissipation}} \geq 0}$$

← an alternative form of the "Clausius – Duhem Inequality"

$\frac{\partial T}{\partial x} \leq 0$ because xferring heat from warm to cold body $\Rightarrow$ $\Downarrow$ ∴ this must always be ≥ 0

$$\boxed{\rho T \frac{D\eta}{Dt} + \sigma \frac{\partial v}{\partial x} - \rho \frac{Du}{Dt} \geq 0}$$

also Eulerian form

"Clausius – Planck Inequality"
A more stringent form of 2nd Law of TD postulates that mechanical dissipation is non-negative, so entropy production of system is non-negative

<u>Lagrangian</u> <u>Forms</u>:

$$\left.\begin{aligned} &\rho_0 \frac{\partial \eta}{\partial t} - \rho_0 \frac{r}{T} + \frac{1}{T}\frac{\partial q}{\partial X} - \frac{q}{T^2}\frac{\partial T}{\partial X} \geq 0 \quad \forall X \in B_0, t \\ &\rho_0 T \frac{\partial \eta}{\partial t} + \sigma \frac{\partial v}{\partial X} - \rho_0 \frac{\partial u}{\partial t} - \frac{q}{T}\frac{\partial T}{\partial X} \geq 0 \quad \forall X \in B_0, t \end{aligned}\right\} \text{Clausius – Duhem}$$

$$\rho_0 T \frac{\partial \eta}{\partial t} + \sigma \frac{\partial v}{\partial X} - \rho_0 \frac{\partial u}{\partial t} \geq 0 \quad \forall X \in B_0, t \quad \Big\} \text{Clausius-Plank}$$

Idea of Introducing 1D-CM:
+ introduce Lagrangian & Eulerian methods of finding solutions
+ introduce Field Equations