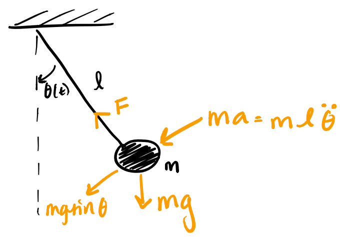
Continuum Mechanics
— study of fluids or solids in a continuum space (rather than discrete)

Pendulum, point mass




$a = \ddot{\theta} l$

$\cancel{m} l \ddot{\theta} + \cancel{m} g \sin\theta = 0$ (the governing eq.)

$\ddot{\theta} + \frac{g}{l} \sin\theta = 0$ ($2^{nd}$ order DE)

suppose $\theta$ is small, in radians $\rightarrow \sin\theta \approx \theta$

$\ddot{\theta} + \frac{g}{l}\theta = 0$

$\theta = A\cos\left(\sqrt{\frac{g}{l}}t\right) + B\sin\left(\sqrt{\frac{g}{l}}t\right)$

$\omega = \sqrt{\frac{g}{l}}$ - natural frequency, determines period

$\theta = A\cos\left(\sqrt{\frac{g}{l}}t\right) + B\sin\left(\sqrt{\frac{g}{l}}t\right)$

initial conditions

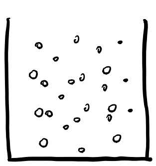
$\dot{\theta}(0) = 0$

$\theta(0) = \theta_0$

$\boxed{\theta = \theta_0 \cos\left(\sqrt{\frac{g}{l}}t\right)}$ $\theta_0, g, l$ all known

$$\boxed{\begin{array}{l} T = \text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}} \\ F = mg\cos\theta = mg\cos\left[\theta_0 \cos\left(\sqrt{\frac{g}{l}}t\right)\right] \end{array}}$$

Now suppose we want to find pressure from 1L of air in a dish ...
@ 25°C



$N = 2.5 \times 10^{22}$

... computer, $10^{16}$ operations per second

$t = \frac{2.5 \times 10^{22}}{10^{16} \, /s} = 2.5 \times 10^6 \, s \approx 30$ days!!!

... so we will solve using continuum model

$l_M$ = macroscopic length scale
ex. rubber band, amount that is stretched

$l_\mu$ = microscopic length scale
ex. at molecular level

this ratio $\frac{l_\mu}{l_M} \ll 1$

$\Rightarrow$ CONTINUUM

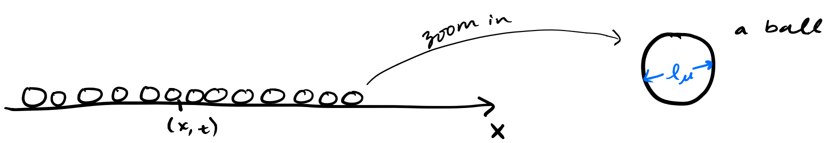
Question: can we use continuum mechanics to measure volume change of lungs? (alveoli, microscopic)

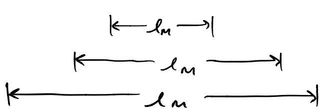
ex. $l_\mu = 10^2 \mu m$ (diameter of alveolus)
$\Delta V_M = 10^2 cm^3$ (volume of lungs during normal breathing)

$\frac{l_\mu}{l_M} \ll 1$ ? What is $l_M$? ... take cube root of volume
$l_M = \sqrt[3]{\Delta V_M} \approx 5 cm$

$\left.\frac{l_\mu}{l_M} \ll 1 \right\} = 2 \times 10^{-3} \ll 1$ so in this case we can use continuum mechanics

1-Dimensional Continuum Mechanics: where it works and where it breaks down:

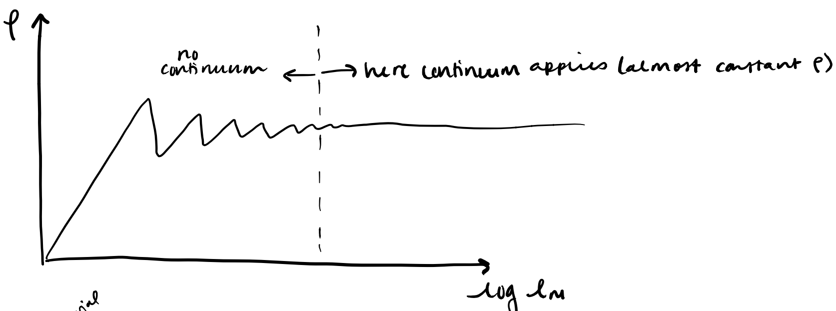


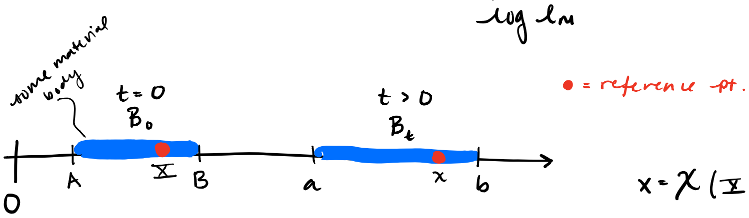




can use different $l_M$

$\rho$ = density = $\frac{m}{l_M}$

$\rho = \frac{m(l_M, x, t)}{l_M}$




$x = \chi(X, t)$ function $\chi$ (chi) describes motion of material body from $B_0$ to $B_t$

$\chi(X, \ldots)$ smooth and 1 to 1
$\chi(\ldots, t)$ smooth, continuous

$$\boxed{\begin{array}{ll} x = \chi(X, t) & \text{equation of motion} \\ X = \chi(X, 0) & \text{initial condition} \end{array}}$$ motion

$$\boxed{V = V(X, t) = \frac{\partial \chi(X, t)}{\partial t}}$$ velocity

$$\boxed{a = a(X, t) = \frac{\partial V(X, t)}{\partial t} = \frac{\partial^2 \chi(X, t)}{\partial t^2}}$$ acceleration

(since in 1D, v and a are scalar ... in 3D they are vectors)

① Material Description, or Lagrangian Description $\Big\}$ solids
material coordinates $\mathbf{X}$

② Spatial, or Eulerian description $\Big\}$ fluids
Spatial coordinates $x$

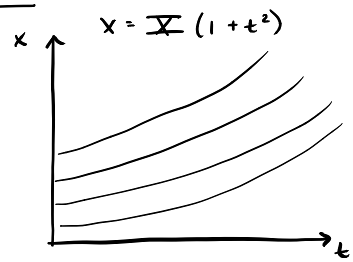
$$\left.\begin{array}{l} x = \chi(\mathbf{X}, t) \\ \text{1-to-1 in } \mathbf{X} \end{array}\right\} \Rightarrow \mathbf{X} = \chi^{-1}(x, t)$$

$$\underbrace{v = v(\mathbf{X}, t)}_{\substack{\text{Lagrangian} \\ \text{(material description)}}} = v(\chi^{-1}(x, t), t) = \underbrace{\tilde{v}(x, t)}_{\substack{\text{Eulerian} \\ \text{(spatial description)}}}$$

$$a = \underbrace{a(\mathbf{X}, t)}_{\text{Lagrangian}} = \underbrace{\tilde{a}(x, t)}_{\text{Eulerian}}$$

$$\boxed{\varphi = \varphi(\mathbf{X}, t) = \varphi(\chi^{-1}(x, t), t) = \tilde{\varphi}(x, t)}$$

Example:




Does this satisfy equation of motion?
yes.

lagrangian: $v = v(\mathbf{X}, t) = \frac{dx}{dt} = 2\mathbf{X}t$ $\qquad a = a(\mathbf{X}, t) = \frac{dv}{dt} = 2\mathbf{X}$

eulerian: need to invert equation of motion

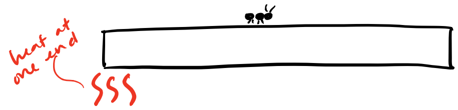
$$\mathbf{X} = \frac{x}{1 + t^2} = \chi^{-1}(x, t)$$

$$v = \tilde{v}(x, t) = \frac{2xt}{1 + t^2} \qquad a = \tilde{a}(x, t) = \frac{2x}{1 + t^2}$$

in lagrangian form, velocity increases linearly w/ time. But eulerian is nonlinear ... same idea with eulerian

→ they describe the same quantities in different ways

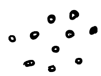
Example: metal rod w/ ant on top



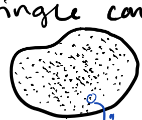

$$\varphi = \varphi(\mathbf{X}, t) \rightarrow \dot{\varphi} = \frac{d\,\varphi(\mathbf{X}, t)}{\partial t}$$ material time derivative

Recap: continuum mechanics models bodies as a continuum

... in reality we have a real continuum body composed of discrete particles




... so within boundaries we re-evaluate these discrete bodies as single continuum

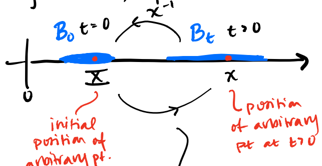



"particle": a mathematical point w/ certain physical properties (temp, force, energy, momentum, etc.)

$\varphi = \varphi(x, t)$ particle function $\varphi$ as function of space and time (can be any physical thing, such as temp, energy, whatever)

continuum mechanics is a theoretical principle, applied mathematics

Along x-axis, follow movement of single body




think train only moving forward + backward, not up or down ... assume 1-d continuum

equation of motion: $\chi(\mathbf{X}, t) = x$

- chi is smooth, aka always differentiable
- chi is 1-to-1, $\chi(\mathbf{X}, \cdot)$ ... this therefore becomes invertible then $\chi^{-1}(x, t) = \mathbf{X}$
- $\chi(\mathbf{X}, 0) = \mathbf{X}$ initial condition

} motion

$$v \equiv \frac{\partial \chi(\mathbf{X}, t)}{\partial t}$$ velocity

$$a \equiv \frac{\partial v(\mathbf{X}, t)}{\partial t} = \frac{\partial^2 \chi(\mathbf{X}, t)}{\partial t^2}$$ acceleration

Lagrangian – describes in terms of material properties

$\varphi = \varphi(\mathbf{X}, t)$

... this is most natural for solids

Eulerian – describes in terms of spatial

$\varphi = \overline{\varphi}(x, t)$

ex. of pollution in a river: can measure change in concentration over time (Eulerian), or you can follow single pollution particle (Lagrangian)

→ they'll each give the same result but the equations will be different

Now let's describe mechanical properties in terms of Eulerian & Lagrangian

How $\varphi$ changes w/time $t$?

$\varphi = \varphi(\mathbb{X}, t)$ or $\varphi = \bar{\varphi}(x, t)$

Lagrangian
- $\mathbb{X}$ not dependent on time $t$

Eulerian
- $x$ is dependent on time $t$ through equation of motion $\Rightarrow x = \chi(\mathbb{X}, t)$

Let's consider Lagrangian description first

$$\dot{\varphi} = \frac{\partial \varphi(\mathbb{X}, t)}{\partial t}$$

material time derivative or Lagrangian

Now Eulerian - depends on $t$ and $t$ through $x$ (chain rule)

$$\dot{\varphi} = \frac{\partial \bar{\varphi}(x,t)}{\partial t} + \frac{\partial \bar{\varphi}(x,t)}{\partial x} \underbrace{\frac{\partial x}{\partial t}}$$

velocity: $\frac{\partial \chi(\mathbb{X}, t)}{\partial t} = v(\mathbb{X}, t) = \bar{v}(x,t)$

$$\dot{\varphi} = \underbrace{\frac{\partial \bar{\varphi}(x,t)}{\partial t}}_{\text{spatial time derivative}} + \underbrace{\frac{\partial \bar{\varphi}(x,t)}{\partial x} \bar{v}(x,t)}_{\text{convective time derivative}}$$

spatial time derivative
- how $\varphi$ changes w/time at given pt.

convective time derivative
- how $\varphi$ changes w/time ...

in the future will drop bar: $\dot{\varphi} = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x} v$

$\dot{\varphi} = \frac{D\varphi}{Dt}$ (includes spatial & time derivative)

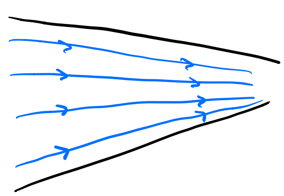
For Lagrangian description

$\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t}$ because there is no convective term

For Eulerian description

$$\frac{D\varphi}{Dt} = \frac{\partial \varphi}{\partial t} + \frac{\partial \varphi}{\partial x} v$$

ex of fluid moving from large x-sectional area to small



at steady-state but there's changes in velocity so there's a convective acceleration term

$$v = \bar{v}(x) \rightarrow a = \dot{v} = \frac{\partial \bar{v}(x,t)}{\partial x} \bar{v}(x,t)$$

**Example 2: find Eulerian Acceleration**

$v = \frac{2xt}{1+t^2} = \bar{v}(x,t)$ $\quad a = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} v$

find spatial time derivative, aka spatial acceleration

$$\frac{\partial v}{\partial t} = \frac{2x(1+t^2) - 4xt^2}{(1+t^2)^2} = \frac{2x(1-t^2)}{(1+t^2)^2}$$

find convective time derivative:

$$\frac{\partial v}{\partial x} = \frac{2t}{1+t^2} \qquad \frac{\partial v}{\partial x} v = \frac{2t}{1+t^2} \cdot \frac{2xt}{1+t^2} = \frac{4xt^2}{(1+t^2)^2}$$

Now add these together to get a:

$$a = \frac{2x(1-t^2)}{(1+t^2)^2} + \frac{4xt^2}{(1+t^2)^2} = \frac{2x}{1+t^2} \checkmark$$

## Example 3: Eulerian

given $v = \bar{v}(x,t) = \frac{2xt}{1+t^2}$, find $\chi$?

so velocity field is given, now want to find trajectory

$$v = \frac{Dx}{Dt} = \frac{2xt}{1+t^2}$$

... use separation of variables & integrate

$$\int \frac{dx}{x} = \int \frac{2t\,dt}{1+t^2} = \int \frac{d(t^2)}{1+t^2} \Rightarrow x = c(1+t^2)$$

$\Rightarrow \ln\frac{x}{c} = \ln(1+t^2)$

$\frac{x}{c} = (1+t^2)$

$x = c(1+t^2)$

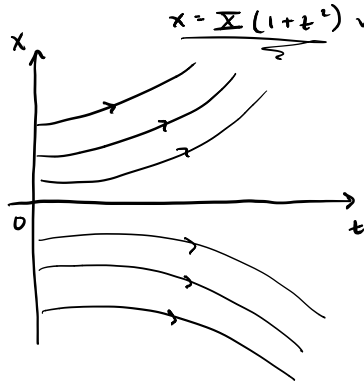
$x = \chi(X,t)$

$X = \chi(X,0)$

find c using initial condition:

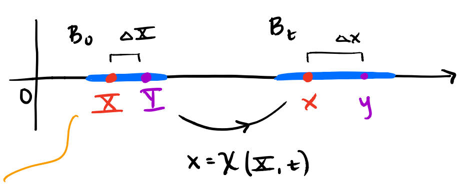
$x$ at $0 = X \Rightarrow X = c(1+0) = c \checkmark$

$$x = X(1+t^2) \checkmark$$




for different x's will have different trajectories (divergent field)

## Local kinematics




$B_0$ = reference configuration

$B_t$ = current configuration

just arbitrary particle points

$$\Delta X = Y - X \qquad Y = X + \Delta X$$

$$\Delta x = y - x = \chi(Y,t) - \chi(X,t) = \chi(X+\Delta X, t) - \chi(X,t)$$

$$\frac{dx}{dX} = \lim_{\Delta X \to 0} \frac{\Delta x}{\Delta X} = \frac{y-x}{Y-X} = \lim_{\Delta X \to 0} \frac{\chi(X+\Delta X,t) - \chi(X,t)}{\Delta X} = \frac{\partial \chi(X,t)}{\partial X} = F(X,t)$$

$$\boxed{dx = F(X,t)\,dX}$$

F = deformation gradient

$\lambda$ = local stretch (the stretch around some arbitrary pt.)

$$\boxed{\lambda \equiv \frac{dx}{d\mathbf{X}} = F(\mathbf{X},t)}$$

because we assume that the equation of motion is 1-to-1

because $\chi(\mathbf{X},\cdot)$ is 1 to 1, $F(\mathbf{X},t) \neq 0$

$\mathbf{X} = \chi(\mathbf{X},0)$ initial condition $\Rightarrow F(\mathbf{X},0) = 1$

$\left(\frac{\partial \chi(\mathbf{X},0)}{\partial \mathbf{X}} = \frac{\partial \mathbf{X}}{\partial \mathbf{X}} = 1\right)$

cannot be negative since 1 to 1 and would have to go thru 0 to be negative

$\therefore \boxed{F(\mathbf{X},t) > 0}$ $\qquad dx \neq 0,\ d\mathbf{X} \neq 0$

Lagrangian

... this means that if we start with a finite amount of matter, we preserve the conservation of matter

This is a mathematical constraint

so because $F$ is strictly positive, then $\lambda$ is also strictly ⊕

$$\boxed{\lambda > 0}$$

$\lambda > 1$ elongation

$0 < \lambda < 1$ compression (because can never compress to zero, all defects have to be finite)

Eulerian

$dx = F(\mathbf{X},t)\,d\mathbf{X} \Rightarrow d\mathbf{X} = F^{-1}dx$

$F^{-1} = \frac{d\mathbf{X}}{dx} = \frac{d\chi^{-1}(x,t)}{dx} = \bar{F}^{-1}(x,t)$ ... take inverse of $F$ to get the Eulerian description of deformation (bar just means Eulerian)

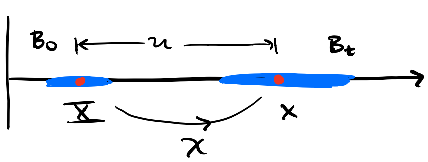
material derivative

$$\dot{F} = \frac{DF}{Dt} = \frac{D}{Dt}\left(\frac{\partial \chi(\mathbf{X},t)}{\partial \mathbf{X}}\right) = \frac{\partial}{\partial \mathbf{X}}\Bigg(\underbrace{\frac{D\chi(\mathbf{X},t)}{Dt}}_{v = \frac{\partial \chi(\mathbf{X},t)}{\partial t}}\Bigg) = \frac{\partial v}{\partial \mathbf{X}} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \mathbf{X}} = \frac{\partial v}{\partial x}\underbrace{\frac{\partial \chi(\mathbf{X},t)}{\partial \mathbf{X}}}_{F} = \frac{\partial v}{\partial x}F$$

$$\boxed{\dot{F} = \frac{DF}{Dt} = \frac{\partial v}{\partial x}F}$$

## Displacement

$\equiv$ means "by definition"




Lagrangian Displacement:

$u = u(\mathbf{X},t) \equiv x - \mathbf{X} = \chi(\mathbf{X},t) - \mathbf{X}$

everything in terms of $\mathbf{X}$

Eulerian Displacement

$u = \bar{u}(x,t) \equiv x - \mathbf{X} = x - \chi^{-1}(x,t)$

everything in terms of $x$

Lagrangian Displacement Gradient:

$$\frac{\partial u}{\partial \mathbf{X}} = \frac{\partial \chi(\mathbf{X},t)}{\partial \mathbf{X}} - 1 = F - 1$$

Eulerian Discription gradient:

$$\frac{\partial u}{\partial x} = 1 - \frac{\partial \chi^{-1}(x,t)}{\partial x} = 1 - F^{-1}$$

**Strain** – change of length wrt reference (true for small deformation)
→ for large deformation, strain is not linear

Lagrangian strain, $E$:

$$E \equiv \lim_{Y \to X} \frac{\Delta x^2 - \Delta X^2}{2\Delta X^2} = \frac{1}{2}\left[\lim \left(\frac{\Delta x}{\Delta X}\right)^2 - 1\right] = \frac{1}{2}\left[\lambda^2 - 1\right] = \frac{1}{2}(F^2 - 1)$$

remember: (for Lagrangian) $\frac{\partial u}{\partial X} = \frac{\partial \chi(X,t)}{\partial X} - 1 = F - 1 \Rightarrow F = 1 + \frac{\partial u}{\partial X}$

$$\boxed{E = \underbrace{\frac{\partial u}{\partial X}}_{\text{linear strain}} + \underbrace{\frac{1}{2}\left(\frac{\partial u}{\partial X}\right)^2}_{\text{nonlinear}}}$$ Lagrangian strain

Eulerian strain, $e$:

$$e \equiv \lim_{y \to x} \frac{\Delta x^2 - \Delta X^2}{2\Delta x^2} = \frac{1}{2}\left[1 - \lim_{y \to x}\left(\frac{\Delta X}{\Delta x}\right)^2\right] = \frac{1}{2}\left[1 - \lambda^2\right] = \frac{1}{2}(1 - F^{-2})$$

remember: (for Eulerian) $\frac{\partial u}{\partial x} = 1 - \frac{\partial \chi^{-1}(x,t)}{\partial x} = 1 - F^{-1} \Rightarrow F^{-1} = 1 - \frac{\partial u}{\partial x}$

$$\boxed{e = \frac{\partial u}{\partial x} - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2}$$ Eulerian strain

Linear (engineering) strain $\varepsilon$:

$$\varepsilon = \lim_{Y \to X} \frac{\Delta x - \Delta X}{\Delta X} = \lim_{Y \to X}\left(\frac{\Delta x}{\Delta X}\right) - 1 = \lambda - 1 = F - 1$$

$$F = 1 + \frac{\partial u}{\partial X} \Rightarrow \boxed{\varepsilon = \frac{\partial u}{\partial X}}$$

... so if deformation is small, $\frac{\partial u}{\partial X}$ is small, $\left(\frac{\partial u}{\partial X}\right)^2 \approx 0$
$\therefore \varepsilon \approx E$

... furthermore, small deformation $\Rightarrow x \approx X$
$\therefore \varepsilon \approx e$

... then $\varepsilon \approx E \approx e$ ... but this is not the case for large deformations, such as in soft tissue injury (biomechanics)

**Stretching** – the fractional rate of length of change, $\kappa$ (kappa)
ex. see change in length of streamline in flow

$$\kappa \equiv \lim_{y \to x} \frac{(\Delta x)^\circ}{\Delta x} = \lim_{y \to x} \frac{(\dot{y} - \dot{x})}{y - x} = \lim_{y \to x} \frac{\bar{v}(y,t) - \bar{v}(x,t)}{y - x} = \frac{\partial \bar{v}(x,t)}{\partial x} = \frac{\partial v}{\partial x}$$

$$\boxed{\kappa = \frac{\partial v}{\partial x} \quad \text{velocity gradient}}$$

$$\kappa \neq \mathring{\varepsilon} \quad \text{bc} \quad \mathring{\varepsilon} = \frac{D\varepsilon}{Dt} = \frac{D}{Dt}\left(\frac{\partial u}{\partial X}\right) = \frac{\partial \mathring{u}}{\partial X} = \frac{\partial v}{\partial X} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial X} = \frac{\partial v}{\partial x} F = \kappa(1 + \varepsilon)$$

$$\therefore \kappa = \frac{\mathring{\varepsilon}}{1 + \varepsilon}$$

... however, for $\varepsilon \ll 1$, then $\kappa \approx \mathring{\varepsilon}$

## Example 4:

$x = X(1+t^2) \qquad \lambda, E, e, \mathcal{E}, K, \dot{\mathcal{E}}\,?$

$$\lambda = F = \frac{\partial x}{\partial X} = (1+t^2) = \lambda$$

$$\frac{\partial u}{\partial X} = \lambda - 1 = t^2 \Rightarrow E = \frac{\partial u}{\partial X} + \frac{1}{2}\left(\frac{\partial u}{\partial X}\right)^2 = t^2\left(1 + \tfrac{1}{2}t^2\right) = E$$

$e$ (Eulerian strain): $\frac{\partial u}{\partial x} = ? \rightarrow u = x - X = X(1+t^2) - X = X t^2 = \frac{x t^2}{1+t^2}$

$$\frac{\partial u}{\partial x} = \frac{t^2}{1+t^2} \Rightarrow e = \frac{\partial u}{\partial x} - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{t^2}{1+t^2}\left(1 + \tfrac{1}{2}t^2\right) = e$$

$$\mathcal{E}: \ \mathcal{E} = \frac{\partial u}{\partial X} = t^2 = \mathcal{E}$$

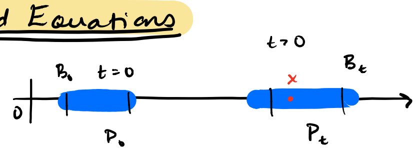
see example 2

$$K: \ \frac{\partial v}{\partial x} \rightarrow v = \frac{2xt}{1+t^2} \Rightarrow \frac{\partial v}{\partial x} = \frac{2t}{1+t^2} = K$$

$$\dot{\mathcal{E}}: \ K(1+\mathcal{E}) = \frac{\partial v}{\partial x}\left(1 + \frac{\partial u}{\partial x}\right) = \frac{2t}{1+t^2}(1+t^2) = 2t = \dot{\mathcal{E}}$$

also: $\dot{\mathcal{E}} = \frac{D\mathcal{E}}{Dt} = 2t$ (same as above)

## Field Equations




$P_t \subseteq B_t \qquad \varphi = \bar{\varphi}(x,t)$

$\int_{P_t} \bar{\varphi}(x,t)\,dx$ ... want to find $\frac{D}{Dt}\int_{P_t} \varphi\,dx = ?$ (... aka Reynold's Transport Theorem)

↑ this integral describes physical property of $P_t$ at some time

$\frac{D}{Dt}\int_{P_t} \varphi\,dx$ ... can't bring $\frac{D}{Dt}$ into integral since $P_t$ depends on time

... remember $dx = F\,dX$

$$\frac{D}{Dt}\int_{P_t} \varphi\,dx = \frac{D}{Dt}\int_{P_0} \varphi F\,dX = \int_{P_0} \frac{D}{Dt}(\varphi F)\,dX = \int_{P_0} (\dot{\varphi}F + \varphi \dot{F})\,dX = \int_{P_0} \left(\dot{\varphi}F + \varphi \frac{\partial v}{\partial x} F\right) dX$$

↑ bc $P_0$ not time dependent, can bring derivative inside $\int$

$\Rightarrow$ in Lagrangian form $\Rightarrow \int_{P_0} \left(\dot{\varphi}F + \varphi\frac{\partial v}{\partial x}F\right) dX = \int_{P_0} \left(\dot{\varphi} + \varphi \frac{\partial v}{\partial x}\right) \underbrace{F\,dX}_{dx} = \int_{P_t} \left(\frac{\partial \varphi}{\partial t} + \underbrace{\frac{\partial \varphi}{\partial x} v + \varphi \frac{\partial v}{\partial x}}_{\frac{\partial(\varphi v)}{\partial x}}\right) dx$

$$\Rightarrow \int_{P_t} \left[\frac{\partial \varphi}{\partial t} + \frac{\partial(\varphi v)}{\partial x}\right] dx$$

and now we can define Reynold's Transport Theorem

$$\boxed{\frac{D}{Dt}\int_{P_t} \varphi \, dx = \int_{P_t} \left[\frac{\partial \varphi}{\partial t} + \frac{\partial(\varphi v)}{\partial x}\right] dx}$$

we will use RTT to derive other items, such as balance of mass

## Balance of Mass

$\rho = \bar{\rho}(x,t)$ Eulerian mass density at $x$ at $t$

$M_{P_t} = \int_{P_t} \rho \, dx$ from physics $\Rightarrow \frac{DM_{P_t}}{Dt} = 0$ because mass is constant

$$\Rightarrow \therefore \int_{P_t}\left[\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x}\right] dx = 0 \quad \forall \; P_t \subseteq B_t$$

$\uparrow$ for every

$$\therefore \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0}$$

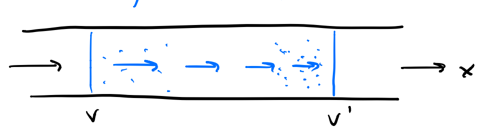
the continuity equation

$\uparrow$ represents a local balance of mass, true for every ($\forall$) $x \in B_t$ and $t > 0$

physical interpretation:

$$\underbrace{\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} v}_{\frac{D\rho}{Dt} = \dot{\rho}} + \rho \frac{\partial v}{\partial x} = 0 \Rightarrow \boxed{\frac{1}{\rho}\frac{D\rho}{Dt} = -\frac{\partial v}{\partial x}}$$

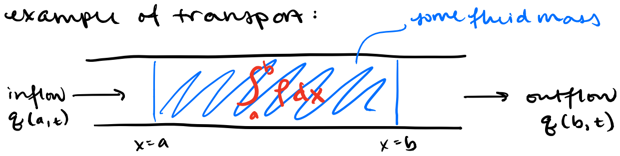
also continuity, but in more physically apparent way




since $\frac{\partial v}{\partial x} < 0$ then $v > v'$

... as the velocity slows down, particles will begin accumulating at the right end

example of transport:




how does total mass change with time?

$$q(x,t) = \bar{\rho}(x,t)\bar{v}(x,t)$$

$$\frac{d}{dt}\int_a^b \bar{\rho}(x,t)\,dx = \int_a^b \frac{\partial \bar{\rho}(x,t)}{\partial t}\,dx = \bar{\rho}(a,t)\bar{v}(a,t) - \bar{\rho}(b,t)\bar{v}(b,t) = -\int_a^b d(\rho v) = -\int_a^b \frac{\partial(\rho v)}{\partial x}\,dx$$

note that we use partial since fixed integral

$$\int_a^b \left[\frac{\partial \bar{\rho}(x,t)}{\partial t} + \frac{\partial(\rho v)}{\partial x}\right] dx = 0 \quad \forall \; a, b \qquad \therefore \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0}$$