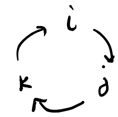
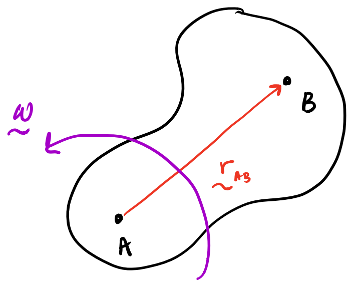
Dynamics – Lecture #13

HW #5 still due 3/18/2020, to be submitted via blackboard

Review of Rigid Body kinematics



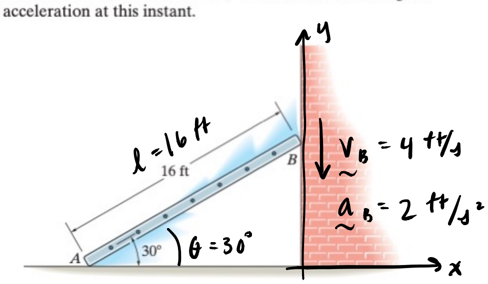


$$\underset{\sim}{V}_B = \underset{\sim}{V}_A + \underset{\sim}{\omega} \times \underset{\sim}{r}_{AB}$$

$$\underset{\sim}{a}_B = \underset{\sim}{a}_A + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}_{AB} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r})$$

$$= \underset{\sim}{a}_A + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}_{AB} - \omega^2 \underset{\sim}{r}_{AB}$$

16–105. At a given instant the top B of the ladder has an acceleration $a_B = 2\ \mathrm{ft/s^2}$ and a velocity of $v_B = 4\ \mathrm{ft/s}$, both acting downward. Determine the acceleration of the bottom A of the ladder, and the ladder's angular acceleration at this instant.



Calculate the angular velocity $\underset{\sim}{\omega}$ and the angular acceleration $\dot{\underset{\sim}{\omega}}$ of the bar.

$$\underset{\sim}{V}_B = \underset{\sim}{V}_A + \underset{\sim}{\omega} \times \underset{\sim}{r}_{AB}$$

$$-V_B \underset{\sim}{j} = V_A \underset{\sim}{i} + \omega \underset{\sim}{k} \times l(\cos\theta \underset{\sim}{i} + \sin\theta \underset{\sim}{j})$$

$$= V_A \underset{\sim}{i} + \omega l \cos\theta \underset{\sim}{j} - \omega l \sin\theta \underset{\sim}{i}$$

separate components:

$$\begin{cases} \underset{\sim}{i}: & 0 = V_A - \omega l \sin\theta \\ \underset{\sim}{j}: & -V_B = \omega l \cos\theta \end{cases}$$

$-V_B = \omega l \cos\theta \Rightarrow$ gives angular velocity of the bar

$$\omega = \frac{-V_B}{l\cos\theta}$$

... this means that it's $-\underset{\sim}{k}$ so clockwise rotation!

Find angular acceleration:

$$\underset{\sim}{a}_B = \underset{\sim}{a}_A + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}_{AB} - \omega_{AB}^2 \underset{\sim}{r}_{AB} \qquad \underset{\sim}{\alpha} = \dot{\underset{\sim}{\omega}}$$

$$-a_B \underset{\sim}{j} = a_A \underset{\sim}{i} + \underbrace{\alpha \underset{\sim}{k} \times l(\cos\theta \underset{\sim}{i} + \sin\theta \underset{\sim}{j})}_{\dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}_{AB}} - \underbrace{\omega^2 l(\cos\theta \underset{\sim}{i} + \sin\theta \underset{\sim}{j})}_{\omega_{AB}^2 \underset{\sim}{r}_{AB}}$$

$$-a_B \underset{\sim}{j} = a_A \underset{\sim}{i} + \alpha l \cos\theta \underset{\sim}{j} - \alpha l \sin\theta \underset{\sim}{i} - \omega^2 l(\cos\theta \underset{\sim}{i} + \sin\theta \underset{\sim}{j})$$

Separate components:

$$\begin{cases} \underset{\sim}{i}: & 0 = a_A - \alpha l \sin\theta - \omega^2 l \cos\theta \\ \underset{\sim}{j}: & -a_B = \alpha l \cos\theta - \omega^2 l \sin\theta \end{cases}$$

$-a_B = \alpha l \cos\theta - \omega^2 l \sin\theta \Rightarrow$ use this to get α

$$\alpha l \cos\theta = \omega^2 l \sin\theta - a_B$$

$$\alpha = \underbrace{\omega^2 \tan\theta}_{(+)\text{ term}} - \underbrace{\frac{a_B}{l\cos\theta}}_{\text{together } (-) \text{ term}} \qquad a_B = 2\ ft/s^2$$

... note that we wrote a_B above as $-a_B \underset{\sim}{j}$
just need to be consistent!

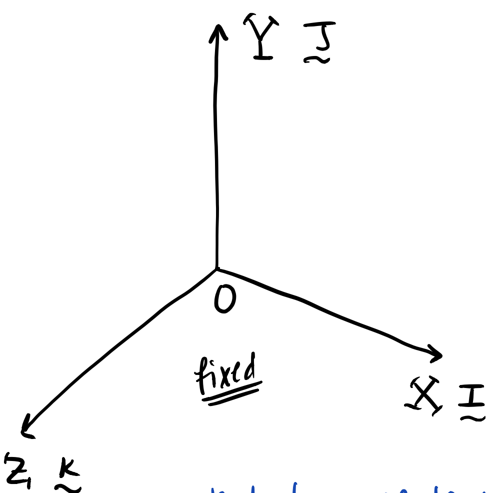
... takeaway, angular velocity is simple to visualize but angular acceleration is much more difficult

Section 16.8:

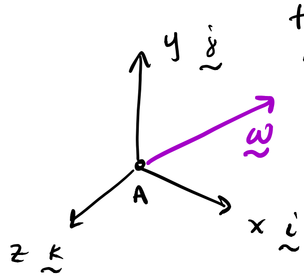
Kinematic analysis w/moving reference frame

(... previous problem had fixed reference frame)

FIXED COORDINATE SYSTEM

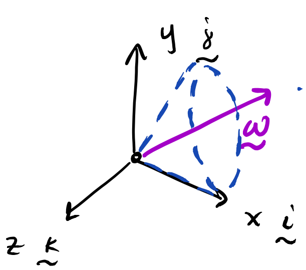


OTHER: this frame can translate and rotate



$\underset{\sim}{\omega}$ = angular velocity of the moving frame (wrt the fixed frame)

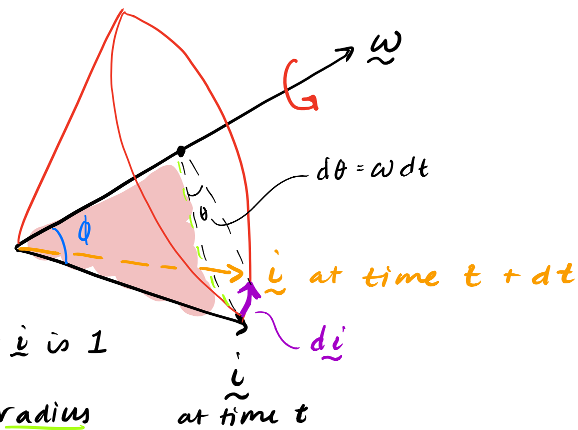
what happens to unit vector $\underset{\sim}{i}$ when frame rotates?
↳ $\underset{\sim}{i}$ sweeps out a cone!



... all three unit vectors will trace out cones

Question: $\frac{d\underset{\sim}{i}}{dt} = ?$
↑ as seen (wrt) in fixed frame

... definitely not zero bc it moves and traces out a cone same picture as before but only showing i and ω vectors



what's the angle that's swept out?

ϕ = cone angle

... length of $\underset{\sim}{i}$ is 1

... so the radius of circle of the cone is $(1) \sin\phi$

$$|d\underset{\sim}{i}| = \sin\phi\, \omega\, dt \quad \Rightarrow \quad \left|\frac{d\underset{\sim}{i}}{dt}\right| = \sin\phi\, \omega \quad (1)$$

Direction of $d\underset{\sim}{i}$ is $\perp$ to both $\underset{\sim}{i}$ itself and $\underset{\sim}{\omega}$

plane that contains $\underset{\sim}{i}$ and $\underset{\sim}{\omega}$

so, Direction of $\frac{d\underset{\sim}{i}}{dt}$ is $\perp$ to $\underset{\sim}{i}$ and $\underset{\sim}{\omega}$ (2)

(1) and (2) are equivalent to

$$\frac{d\underset{\sim}{i}}{dt} = \underset{\sim}{\omega} \times \underset{\sim}{i}$$

Similarly,

$$\frac{d\underset{\sim}{j}}{dt} = \underset{\sim}{\omega} \times \underset{\sim}{j} \quad \text{and} \quad \frac{d\underset{\sim}{k}}{dt} = \underset{\sim}{\omega} \times \underset{\sim}{k}$$

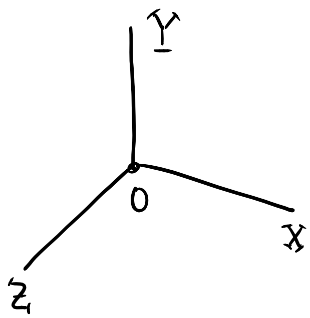
back to the above moving coordinate system:

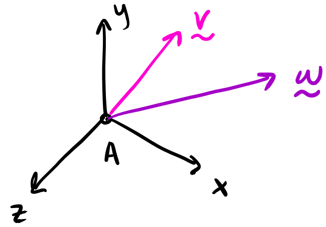
A_0

$$\frac{d\underset{\sim}{i}}{dt} = \underset{\sim}{\omega} \times \underset{\sim}{i}$$

$$\frac{d\underset{\sim}{j}}{dt} = \underset{\sim}{\omega} \times \underset{\sim}{j}$$

$$\frac{d\underset{\sim}{k}}{dt} = \underset{\sim}{\omega} \times \underset{\sim}{k}$$





$\mathbf{V}$ = vector that moves in moving frame

Q: what is $\frac{d\mathbf{V}}{dt}$ wrt <u>fixed</u> frame?

... whole thing spins and ω determines the spinning (?)

$$\mathbf{V} = V_x \mathbf{i} + V_y \mathbf{j} + V_z \mathbf{k}$$

$$\frac{d\mathbf{V}}{dt} = \dot{V}_x \mathbf{i} + V_x \frac{d\mathbf{i}}{dt} + \dot{V}_y \mathbf{j} + V_y \frac{d\mathbf{j}}{dt} + \dot{V}_z \mathbf{k} + V_z \frac{d\mathbf{k}}{dt}$$

now group the terms...

$$= \dot{V}_x \mathbf{i} + \dot{V}_y \mathbf{j} + \dot{V}_z \mathbf{k} + V_x \boldsymbol{\omega} \times \mathbf{i} + V_y \boldsymbol{\omega} \times \mathbf{j} + V_z \boldsymbol{\omega} \times \mathbf{k}$$

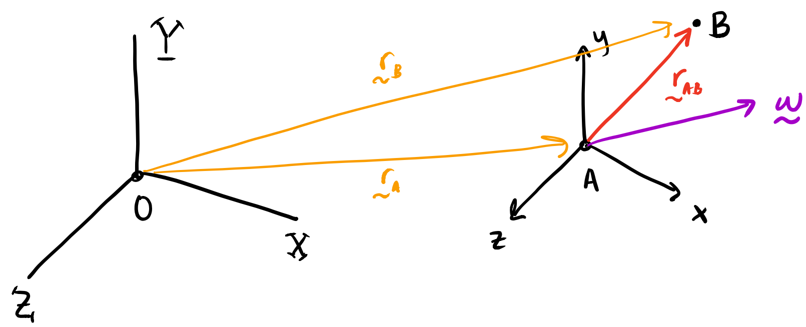
$$= \underbrace{\dot{V}_x \mathbf{i} + \dot{V}_y \mathbf{j} + \dot{V}_z \mathbf{k}}_{\text{derivative if stayed in moving frame}} + \underbrace{\boldsymbol{\omega} \times (V_x \mathbf{i} + V_y \mathbf{j} + V_z \mathbf{k})}_{\boldsymbol{\omega} \times \mathbf{V}}$$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{\text{relative}}$$

Q: what is $\frac{d\mathbf{V}}{dt}$ wrt <u>fixed</u> frame?

A:

$$\boxed{\frac{d\mathbf{V}}{dt} = \left(\frac{d\mathbf{V}}{dt}\right)_{rel} + \boldsymbol{\omega} \times \mathbf{V}}$$



B can move in the moving frame

Q: what are $\underset{\sim}{v}_B$ and $\underset{\sim}{a}_B$, wrt fixed frame? ... we'll see it's similar to what we found for rigid bodies...

... start by writing out r vectors

$$\underset{\sim}{r}_B = \underset{\sim}{r}_A + \underset{\sim}{r}_{AB}$$

$$\frac{d\underset{\sim}{r}_B}{dt} = \frac{d\underset{\sim}{r}_A}{dt} + \frac{d}{dt}\underset{\sim}{r}_{AB}$$

$$\Downarrow \qquad \Downarrow \qquad \Downarrow$$

$\underset{\sim}{v}_B = \underset{\sim}{v}_A +$ to calculate that third term, we'll need to remember that it's in moving frame and will need the earlier terms

$$\underset{\sim}{v}_B = \underset{\sim}{v}_A + \left(\frac{d\underset{\sim}{r}_{AB}}{dt}\right)_{rel} + \underset{\sim}{\omega} \times \underset{\sim}{r}_{AB}$$

A:

$$\boxed{\underset{\sim}{v}_B = \underset{\sim}{v}_A + \left(\underset{\sim}{v}_B\right)_{rel} + \underset{\sim}{\omega} \times \underset{\sim}{r}_{AB}}$$

velocity of B wrt fixed frame

velocity of B in moving frame

suppose AB is rigid body ... then there's no relative velocity! It's zero!
... the other terms are the exact same as before

$$\underset{\sim}{v}_B = \underset{\sim}{v}_A + \underset{\sim}{\omega} \times \underset{\sim}{r}_{AB} + \cancelto{0}{\left(\underset{\sim}{v}_B\right)_{rel}}$$

Remember...

$\underset{\sim}{\omega}$ tells how moving frame spins wrt fixed frame

ω = angular velocity of moving frame wrt fixed frame

$$\underset{\sim}{v}_B = \underset{\sim}{v}_A + \underset{\sim}{\omega} \times \underset{\sim}{r}_{AB} + (\underset{\sim}{v}_B)_{rel}$$

$\rightarrow$ if want acceleration, take $\frac{d}{dt}$

$$\boxed{\frac{d\underset{\sim}{V}}{dt} = \left(\frac{d\underset{\sim}{V}}{dt}\right)_{rel} + \underset{\sim}{\omega} \times \underset{\sim}{V}}$$

note that this comes from the above

$$\frac{d}{dt}\underset{\sim}{v}_B = \frac{d}{dt}\underset{\sim}{v}_A + \frac{d}{dt}(\underset{\sim}{\omega} \times \underset{\sim}{r}_{AB}) + \frac{d}{dt}\left((\underset{\sim}{v}_B)_{rel}\right)$$

$$\underset{\sim}{a}_B = \underset{\sim}{a}_A + \frac{d}{dt}\underset{\sim}{\omega} \times \underset{\sim}{r}_{AB} + \underset{\sim}{\omega} \times \frac{d}{dt}\underset{\sim}{r}_{AB} + \left(\frac{d(\underset{\sim}{v}_B)_{rel}}{dt}\right)_{rel} + \underset{\sim}{\omega} \times (\underset{\sim}{v}_B)_{rel}$$

$$\underset{\sim}{a}_B = \underset{\sim}{a}_A + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}_{AB} + \underset{\sim}{\omega} \times \left(\left(\frac{d\underset{\sim}{r}_{AB}}{dt}\right)_{rel} + \underset{\sim}{\omega} \times \underset{\sim}{r}_{AB}\right) + (\underset{\sim}{a}_B)_{rel} + \underset{\sim}{\omega} \times (\underset{\sim}{v}_B)_{rel}$$

these terms are the same!

$$\underset{\sim}{a}_B = \underset{\sim}{a}_A + \dot{\underset{\sim}{\omega}} \times \underset{\sim}{r}_{AB} + \underset{\sim}{\omega} \times (\underset{\sim}{\omega} \times \underset{\sim}{r}_{AB}) + (\underset{\sim}{a}_B)_{rel} + \underbrace{2\underset{\sim}{\omega} \times (\underset{\sim}{v}_B)_{rel}}_{\text{coriolis acceleration}}$$

we have this when have pt that moves in rotating reference frame