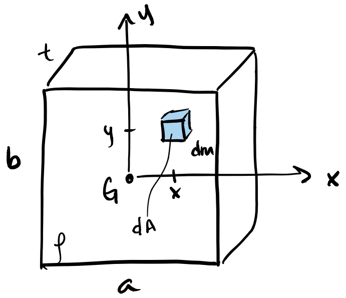
Dynamics – Lecture #15

Test will cover Ch. 14, 15, 16

Homework #6 due Wed. 03/25/2020
Test #2 : Wed. 04/01/2020 ... during normal class time ... unless the situation changes



$$I_G = \int (x^2+y^2)\,dm = \iint (x^2+y^2)\,\rho t\,dA$$

$$= \rho t \int_{y=-b/2}^{y=b/2} \int_{x=-a/2}^{x=a/2} (x^2+y^2)\,dx\,dy$$

$$= \rho t \int_{-b/2}^{b/2} \left[\frac{1}{3}\frac{a^3}{8} + \frac{1}{3}\frac{a^3}{8} + y^2\left(\frac{a}{2}+\frac{a}{2}\right)\right] dy$$

$$= \rho t \int_{-b/2}^{b/2} \left[\frac{a^3}{12} + y^2 a\right] dy$$

$\forall = abt$
$m = \rho abt$

$$= \rho t \left[\frac{a^3}{12}(b) + a\left(\frac{1}{3}\frac{b^3}{8} + \frac{1}{3}\frac{b^3}{8}\right)\right]$$

$$= \rho t\left[\frac{a^3 b}{12} + \frac{a b^3}{12}\right]$$

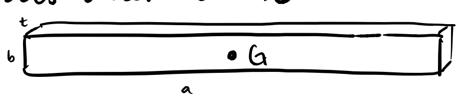
↑ moment of inertia

$$\Rightarrow I_G = \underbrace{(\rho a b t)}_{\rho(\forall) = m}\left[\frac{a^2}{12} + \frac{b^2}{12}\right]$$

$$I_G = m\left[\frac{a^2}{12} + \frac{b^2}{12}\right]$$

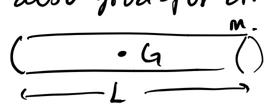
Special Case:

long rods when $a \gg b$



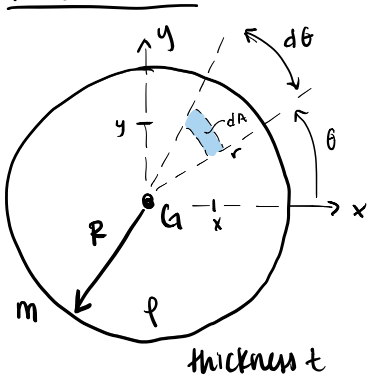
I_G, can neglect b^2 term
$I_G = \frac{1}{12} m a^2$

... this is also good for circular bars that are long and thin



... as long as its long compared to its cross-sections

Circular Disc



using polar coordinates:

$$I_G = \int (x^2+y^2)\,dm = \int\int r^2 \rho t\,dA$$

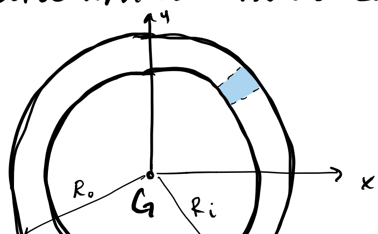
$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{R} r^2 \rho\, r\,dr\,d\theta = \rho t\, 2\pi\left(\frac{1}{4}\right)R^4$$

$$= \rho t \pi \frac{1}{2} R^4$$

$$I_G = \underbrace{(\rho t \pi R^2)}_{\text{mass } (\forall \cdot \rho)} \frac{R^2}{2} \qquad I_G = \frac{1}{2} m R^2$$

* this is a solid disc

→ disc w/ hole in middle, <u>Annular Disc</u>



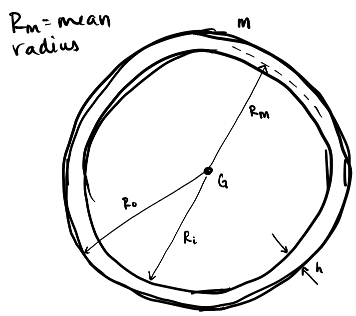
$$I_G = \int_{\theta=0}^{2\pi} \int_{r=R_i}^{R_o} r^2 \rho t \, r \, dr \, d\theta$$

$$= \rho t (2\pi) \frac{1}{4} (R_o^4 - R_i^4)$$

$$= \underbrace{\underbrace{\rho \, t \pi (R_o^2 - R_i^2)}_{\text{volume of annulus}}}_{\text{mass of annulus}} \frac{1}{2} (R_o^2 + R_i^2)$$

$$I_G = \frac{1}{2} m (R_o^2 + R_i^2)$$

→ thin hoop, annulus but the wall is very thin



$$I_G = \frac{1}{2} m (R_o^2 + R_i^2) \qquad R_o = R_m + \frac{h}{2} \qquad R_i = R_m - \frac{h}{2}$$

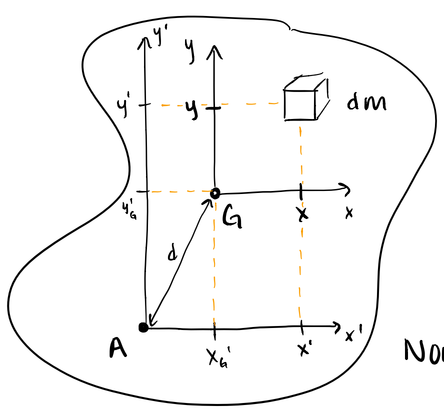
$$I_G = \frac{1}{2} m \left[\left(R_m^2 + \cancel{R_m h} + \underbrace{\frac{h^2}{4}}_{\text{small}} \right) + \left(R_m^2 - \cancel{R_m h} + \underbrace{\frac{h^2}{4}}_{\text{small}} \right) \right]$$

since thin-walled, $h \ll R_m$

$$I_G = \frac{1}{2} m 2 R_m^2 \Rightarrow I_G = m R_m^2$$

remember... solid $I_G = \frac{1}{2} m R^2$

<u>Parallel Axis Theorem</u>



d = distance between A & G

$$I_G = \int (x^2 + y^2) \, dm$$

$$I_A = \int (x'^2 + y'^2) \, dm$$ ... how we define moment of inertia w/ respect to not center of gravity

Q: How is I_A related to I_G?
A. The parallel axis theorem!

$$x' = x'_G + x \qquad y' = y'_G + y$$

Now,
$$I_A = \int \left[(x_G'^2 + 2x'_G x + x^2) + (y_G'^2 + 2y'_G y + y^2) \right] dm$$

$$= \int (x_G'^2 + y_G'^2) \, dm + \int 2x'_G x \, dm + \int 2y'_G y \, dm + \int (x^2 + y^2) \, dm$$

$$= (x_G'^2 + y_G'^2) \int dm + 2x'_G \cancelto{0}{\int x \, dm} + 2y'_G \cancelto{0}{\int y \, dm} + I_G$$

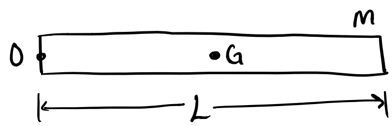
$$I_A = m d^2 + I_G$$

Why?

I_G for many bodies can be found in tables

→ use tables + parallel axis theorem to find I_A

Ex. thin rod, parallel axis theorem



$$I_G = \frac{1}{12} mL^2$$

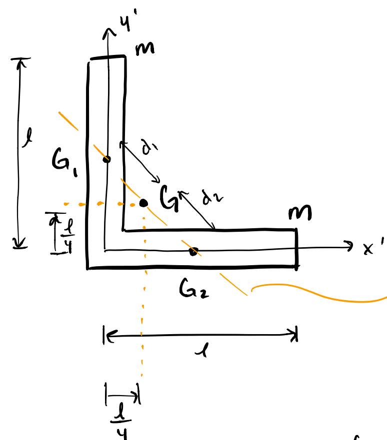
$$I_O = I_G + md^2$$
$$= \frac{1}{12} mL^2 + m\left(\frac{L}{2}\right)^2 = mL^2\left(\frac{1}{12} + \frac{1}{4}\right) \Rightarrow I_O = \frac{1}{3} mL^2$$

moment of inertia of the long rod about the end O

note that $I_O > I_G$

Composite Bodies

Find I_G.



$$x'_G = \frac{m}{2m} x'_{G_1} + \frac{m}{2m} x'_{G_2} = \frac{1}{2}(0) + \frac{1}{2}\frac{l}{2} = \frac{l}{4}$$

$$y'_G = \frac{m}{2m} y'_{G_1} + \frac{m}{2m} y'_{G_2} = \frac{1}{2}\frac{l}{2} + \frac{1}{2}(0) = \frac{l}{4}$$

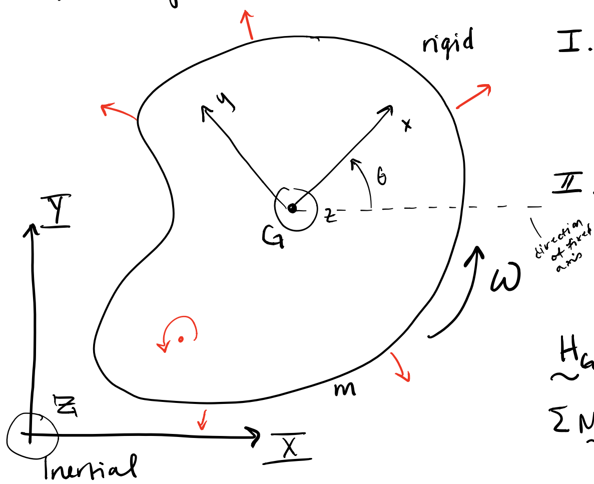
note that the center of mass is on the straight line connecting the individual centers of mass (and exactly ½ way since masses are the same)

$$d_1^2 = \left(\frac{l}{4}\right)^2 + \left(\frac{l}{4}\right)^2 = 2\left(\frac{l^2}{16}\right)$$

$$d_2^2 = \left(\frac{l}{4}\right)^2 + \left(\frac{l}{4}\right)^2 = 2\left(\frac{l^2}{16}\right)$$

$$I_G = (I_G)_{body\,1} + (I_G)_{body\,2}$$
$$= \left(I_{G_1} + md_1^2\right) + \left(I_{G_2} + md_2^2\right)$$
$$= \left[\frac{1}{12} ml^2 + m\left(\frac{l^2}{8}\right)\right] + \left[\frac{1}{12} ml^2 + m\left(\frac{l^2}{8}\right)\right]$$
$$= \frac{2}{12} ml^2 + \frac{2}{8} ml^2 \Rightarrow I_G = \frac{5}{12} ml^2$$
$$= \frac{4}{24} ml^2 + \frac{6}{24} ml^2$$
$$= \frac{10}{24} ml^2$$
$$= \frac{5}{12} ml^2$$

Rigid Body Kinetics



I. linear momentum principle

$$\sum \underset{\sim}{F} = m\underset{\sim}{a}_G \quad \left(\begin{array}{c}\text{how com moves}\\ \text{translational part}\end{array}\right)$$

II. angular momentum principle

$$\sum \underset{\sim}{M}_G = \frac{d}{dt}\left(\underset{\sim}{H}_G\right) \quad \left(\begin{array}{c}\text{tracking rotation}\\ \text{of body as it moves}\end{array}\right)$$

$$\underset{\sim}{H}_G = I_G \,\omega\, \underset{\sim}{k} \qquad \omega = \frac{d\theta}{dt} \text{ or } \dot{\theta} \quad \left(\begin{array}{c}\text{spin}\\ \text{rate}\end{array}\right)$$

$$\sum \underset{\sim}{M}_G = \frac{d}{dt}\left(I_G\,\omega\,\underset{\sim}{k}\right)$$

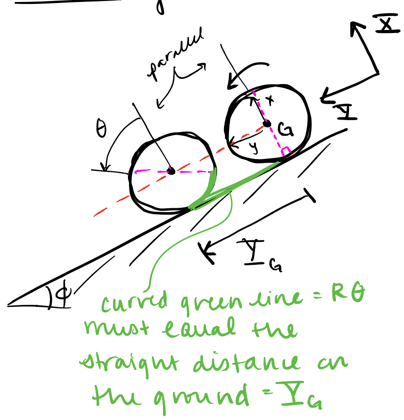
↑ function of time

$$= I_G \frac{d\omega}{dt}\underset{\sim}{k} = I_G\frac{d^2\theta}{dt^2}\underset{\sim}{k} \text{ or } I_G\ddot{\theta}\,\underset{\sim}{k}$$

$\ddot{\theta} = \alpha$ = angular acceleration

So, $\sum \underset{\sim}{M}_G = I_G\,\alpha\,\underset{\sim}{k}$

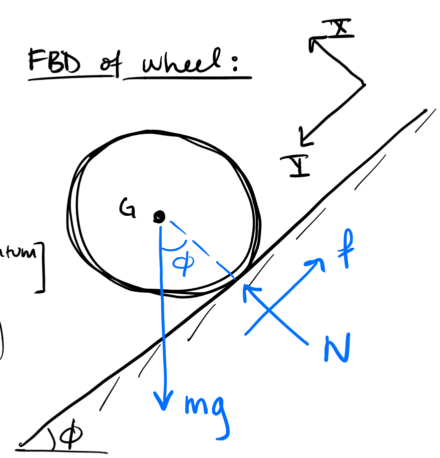
Ex. Rolling Wheel



→ wheel rolls without slip (assumption)
→ G moves in a straight line

Rolling Constraint: $R\theta = Y_G$ (1)

FBD of wheel:



I. $\sum \underset{\sim}{F} = m\underset{\sim}{a}_G$ [linear momentum equations]

(2) $-f + mg\sin\phi = m a_{GY}$ (Y_{dir})

(3) $N + mg\cos\phi = m a_{GX}$ (X_{dir})

$= 0$ (bc G only moves parallel to the road)

II. $\sum \underset{\sim}{M}_G = I_G\,\alpha\,\underset{\sim}{k}$ [angular momentum] (4)

With (1), (2), (3), and (4) can find acceleration a_{GY} downhill