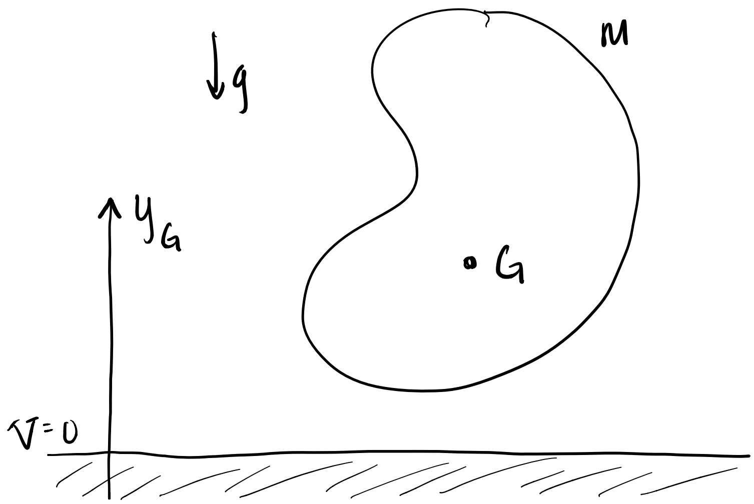
Dynamics – Lecture #19

Potential Energy of Rigid Body

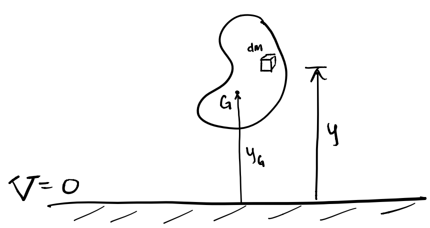


$$V = Mg\,y_G$$

CH.18 Energy Methods for Rigid Body Dynamics

→ need potential energy V and kinetic energy T for a rigid body.

Potential Energy for Rigid Body (no elastic energy since not deformable)



For element dm,

$$dV = dm\,g\,y$$

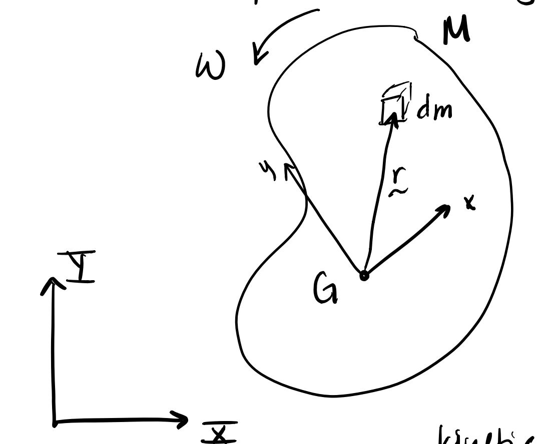
For entire body,

$$V = \int_{body} g\,y\,dm = g\int_{body} y\,dm = g(My_G)$$

therefore,

$$\boxed{V = Mg\,y_G}$$ (potential energy for a rigid body of total mass M)

Kinetic Energy of Rigid Body



velocity of element dm is

$$\mathbf{v} = \mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}$$

kinetic energy of element dm is

$$dT = \tfrac{1}{2}\, dm\, v^2 = \tfrac{1}{2}\, dm\, \mathbf{v} \cdot \mathbf{v}$$

$$= \tfrac{1}{2}\, dm\, (\mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}) \cdot (\mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r})$$

kinetic energy for body:

$$T = \int_{\text{body}} \tfrac{1}{2} (\mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r}) \cdot (\mathbf{v}_G + \boldsymbol{\omega} \times \mathbf{r})\, dm$$

$$T = \int \tfrac{1}{2} \left(\mathbf{v}_G \cdot \mathbf{v}_G + 2\, \mathbf{v}_G \cdot (\boldsymbol{\omega} \times \mathbf{r}) + (\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r}) \right) dm$$

$$= \int \tfrac{1}{2} \mathbf{v}_G \cdot \mathbf{v}_G\, dm + \int \mathbf{v}_G \cdot (\boldsymbol{\omega} \times \mathbf{r})\, dm + \int \tfrac{1}{2} (\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r})\, dm$$

$$= \tfrac{1}{2} V_G^2 \int dm + \mathbf{v} \cdot \left[\boldsymbol{\omega} \times \left(\int \mathbf{r}\, dm \right) \right] + \int \tfrac{1}{2} (\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r})\, dm$$

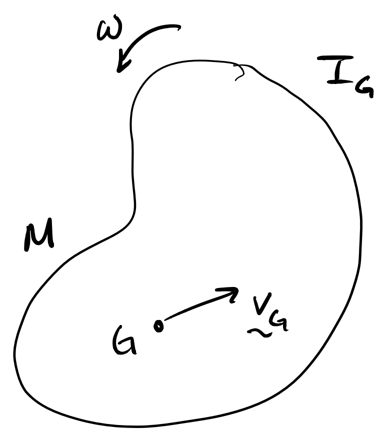
$$T = \underset{①}{\tfrac{1}{2} M V_G^2} + \int \tfrac{1}{2} \underset{②}{(\boldsymbol{\omega} \times \mathbf{r})} \cdot (\boldsymbol{\omega} \times \mathbf{r})$$

work on ②: $\boldsymbol{\omega} \times \mathbf{r} = \omega \mathbf{k} \times (x\mathbf{i} + y\mathbf{j}) = \omega x \mathbf{j} - \omega y \mathbf{i}$

$$(\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r}) = \omega^2 x^2 + \omega^2 y^2$$

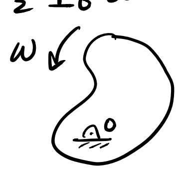
so, $T = \tfrac{1}{2} M V_G^2 + \tfrac{1}{2} \int \omega^2 (x^2 + y^2)\, dm = \tfrac{1}{2} M V_G^2 + \tfrac{1}{2} \omega^2 \int (x^2 + y^2)\, dm$

$$T = \underbrace{\tfrac{1}{2} M V_G^2}_{\text{translational KE}} + \underbrace{\tfrac{1}{2} I_G \omega^2}_{\text{rotational KE}}$$



If body has fixed pt. O, then

$$T = \tfrac{1}{2} I_O \omega^2$$



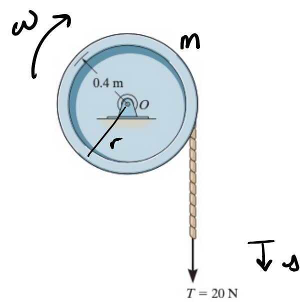
Work-Energy Principle

For a system that moves from state 1 to a state 2,

$$T_1 + V_1 + U^{nc}_{1\to2} = T_2 + V_2$$

non conservative work can be forces and moments

18–11. The force of $T = 20$ N is applied to the cord of negligible mass. Determine the angular velocity of the 20-kg wheel when it has rotated 4 revolutions starting from rest. The wheel has a radius of gyration of $k_O = 0.3$ m.



state 1: initial state of rest
state 2: pulley has made 4 rev.

$$\cancel{T_1} + \cancel{V_1} + U^{nc}_{1\to2} = T_2 + \cancel{V_2}$$

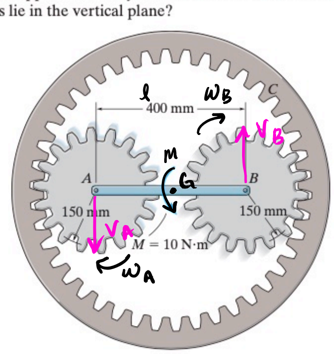
After 4 revolutions, displacement s of rope $= 4\cdot 2\pi r = (4\cdot 2\pi) r$

So, $$T\cdot 4\cdot 2\pi r = \frac{1}{2} I_o \omega_2^2$$

$$\omega_2^2 = \frac{2\cdot T\cdot 4\cdot 2\pi r}{I_o}$$

the moment of I_o and the radius of gyration k_o are related by
$$I_o = m k_o^2$$

18–33. The two 2-kg gears A and B are attached to the ends of a 3-kg slender bar. The gears roll within the fixed ring gear C, which lies in the horizontal plane. If a 10-N·m torque is applied to the center of the bar as shown, determine the number of revolutions the bar must rotate starting from rest in order for it to have an angular velocity of $\omega_{AB} = 20$ rad/s. For the calculation, assume the gears can be approximated by thin disks. What is the result if the gears lie in the vertical plane?



Prob. 18–33

$M = 10$ Nm

gears A, B roll on inside of C w/out slipping

$m_A = m_B = 2$ kg

State 1: gears at rest

State 2: $\omega_{AB} = 20$ rad/s

What is the angle of rotation θ of bar between state 1 and state 2?

Work-energy

$$\cancel{T_1} + \cancel{V_1} + U^{nc}_{1\to2} = T_2 + \cancel{V_2}$$

work of moment = M · angle of rotation

✓ = knowns

$$M\theta = \underbrace{\frac{1}{2} I_G \omega_{AB}^2}_{\text{KE of bar at state 2}} + \underbrace{\frac{1}{2} m_A v_A^2 + \frac{1}{2} I_A \omega_A^2}_{\text{KE of gear A at state 2}} + \underbrace{\frac{1}{2} m_B v_B^2 + \frac{1}{2} I_B \omega_B^2}_{\text{KE of gear B at state 2}}$$

Kinematics

$$V_A = V_B = l\,\omega_{AB}$$

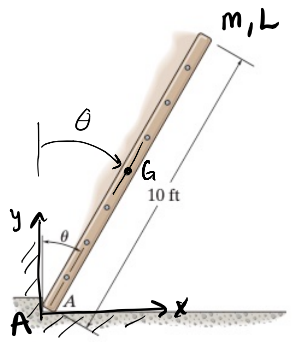
$$\omega_B = \omega_A = \frac{V_B}{r}\quad \frac{l\,\omega_{AB}}{r}$$

Eqn (1) contains only unknown θ

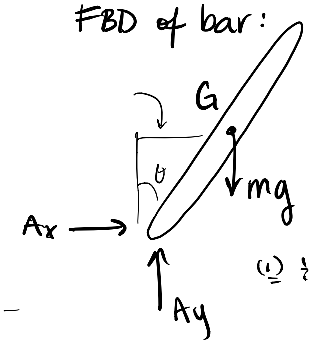
18–43. A uniform ladder having a weight of 30 lb is released from rest when it is in the vertical position. If it is allowed to fall freely, determine the angle θ at which the bottom end A starts to slide to the right of A. For the calculation, assume the ladder to be a slender rod and neglect friction at A.

State 1: $\theta = 0$, bar at rest
State 2: nonzero angle θ
} bar rotates about A

Calculate reaction forces at A as functions of θ



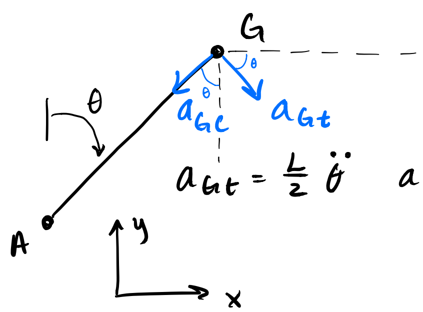
FBD of bar:



$$\Sigma \underset{\sim}{F} = m\,\underset{\sim}{a}_G \qquad (1)\ \&\ (2)$$

$$\Sigma M_A = I_A\,\alpha \qquad (3)$$

(1) & (2)
$$\begin{cases} A_x = m a_{Gx} \\ A_y - mg = m a_{Gy} \end{cases}$$



$$a_{Gt} = \frac{L}{2}\ddot{\theta} \qquad a_{Gc} = \frac{L}{2}\dot{\theta}^2$$

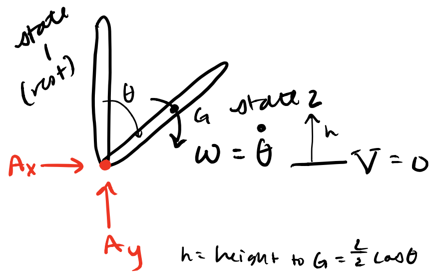
$$\begin{cases} (1)\quad A_x = m\left(\frac{L}{2}\ddot{\theta}\cos\theta - \frac{L}{2}\dot{\theta}^2\sin\theta\right) \\ (2)\quad A_y - mg = m\left(-\frac{L}{2}\ddot{\theta}\sin\theta - \frac{L}{2}\dot{\theta}^2\cos\theta\right) \end{cases}$$

(3) $$mg\frac{L}{2}\sin\theta = I_A\ddot{\theta}$$

(3) → (1), (2)
$$A_x = m\left(\frac{L}{2}\cos\theta\,\frac{mg\frac{L}{2}\sin\theta}{I_A} - \frac{L}{2}\dot{\theta}^2\sin\theta\right) \qquad (4)$$

$$A_y = mg + m\left(-\frac{L}{2}\sin\theta\,\frac{mg\frac{L}{2}\sin\theta}{I_A} - \frac{L}{2}\dot{\theta}^2\cos\theta\right) \qquad (5)$$

still need $\dot{\theta}^2$ in terms of θ; use <u>energy</u>



Energy? $\cancel{T_1} + V_1 + \cancel{U^{nc}_{1\to 2}} = T_2 + V_2$

$$mg\frac{L}{2} = \frac{1}{2} I_A \dot{\theta}^2 + mg\frac{L}{2}\cos\theta$$

$$I_A \dot{\theta}^2 = 2mg\frac{L}{2} - 2mg\frac{L}{2}\cos\theta$$

$$I_A \dot{\theta}^2 = mgL(1-\cos\theta)$$

$$\dot{\theta}^2 = \frac{1}{I_A} mgL(1-\cos\theta)$$

$$= \frac{1}{\frac{1}{3}mL^2} mgL(1-\cos\theta)$$

$$\dot{\theta}^2 = 3\frac{g}{L}(1-\cos\theta) \quad \underline{\underline{(6)}}$$

Next time, substitute $\underline{\underline{(6)}}$ into $\underline{\underline{(4)}}$ and $\underline{\underline{(5)}}$. Have A_x and A_y as functions of θ.

$\underline{\underline{(6)}}$ tells us that as $\theta \uparrow$, so as the bar falls, $\cos\theta \downarrow$

→ at $\theta = 0$, $\cos\theta = 1 \Rightarrow \dot{\theta} = 0$ at $\theta = 0$

→ at top, there is no angular velocity

→ at $\theta = \frac{\pi}{2}, 90°$, just about to hit the ground ...

$$\dot{\theta}^2 = 3\frac{g}{L}(1-0) = 3\frac{g}{L}$$