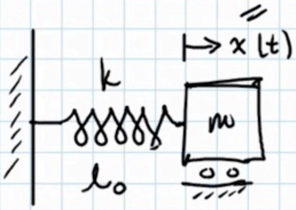
Dynamics — Lecture #22

Homework #10: 22-18, 22-23, 22-31, 22-47, 22-50, 22-56, 22-60 (4/29)

*no lab this semester. Test weights ↑ by 5% each

Final exam: Friday, 5/8/2020 6-8pm US EDT (same format as second exam, 4 problems w/30min to do and upload)



Equation of motion: $m\ddot{x} + kx = 0$

or $\ddot{x} + \frac{k}{m}x = 0$

Let $\omega_n = \sqrt{\frac{k}{m}}$ ω_n = natural frequency of system

radian frequency

Then $\ddot{x} + \omega_n^2 x = 0$ (1)

Solution of (1) is

$x(t) = A\sin\omega_n t + B\cos\omega_n t$

2nd order DE

+

→ constants A, B determined by initial conditions

Assume $x(0) = x_1$ x_1 = initial position of mass

$v(0) = \dot{x}(0) = v_1$ v_1 = initial velocity of mass

Can determine A, B in terms of x_1, v_1

$x = A\sin\omega_n t + B\cos\omega_n t$

$\dot{x} = A\omega_n\cos\omega_n t - B\omega_n\sin\omega_n t$

$x(0) = x_1 \Rightarrow x_1 = B$

$\dot{x}(0) = v_1 \Rightarrow v_1 = A\omega_n$, or $A = v_1/\omega_n$

so, $x(t) = \frac{v_1}{\omega_n}\sin\omega_n t + x_1\cos\omega_n t$ (1)

$\dot{x}(t) = v_1\cos\omega_n t - x_1\omega_n\sin\omega_n t$ (2)

Write (1) & (2) in matrix form:

$$\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} = \begin{bmatrix} \cos\omega_n t & \frac{1}{\omega_n}\sin\omega_n t \\ -\omega_n\sin\omega_n t & \cos\omega_n t \end{bmatrix} \begin{Bmatrix} x_1 \\ v_1 \end{Bmatrix} \quad (3)$$

(3) says that $\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$ at t are determined by $\begin{Bmatrix} x_1 \\ v_1 \end{Bmatrix} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$ at time $t = \phi$

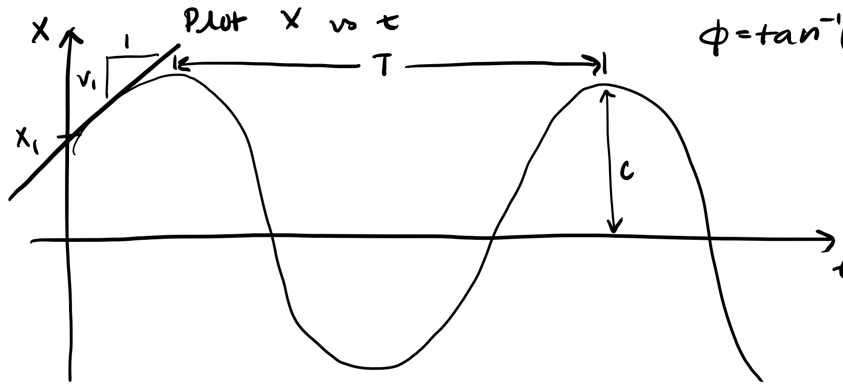
↑ state at all times (Newtonian Mechanics)

$$x = \frac{v_1}{\omega_n}\sin\omega_n t + x_1\cos\omega_n t \quad \underline{(1)}$$

can also write $\underline{(1)}$ using a trig identity as

$$x(t) = C_1\sin(\omega_n t + \phi), \text{ where } C_1 = \sqrt{\left(\frac{v_1}{\omega_n}\right)^2 + x_1^2},$$

$$\phi = \tan^{-1}\left(\frac{x_1}{v_1/\omega_n}\right) = \tan^{-1}\left(\frac{x_1\omega_n}{v_1}\right)$$



$$T = \frac{2\pi}{\omega_n}$$

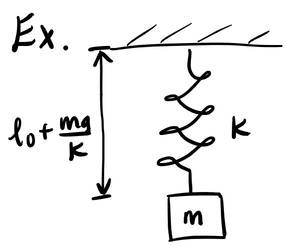
$$T = \frac{1}{f_n}$$

f_n = cycle frequency ($\frac{cyc}{s}$ or Hz)

$$\frac{2\pi}{\omega} = \frac{1}{f_n} \qquad \underline{f_n 2\pi = \omega_n}$$

ω_n = radian frequency (rad/s)

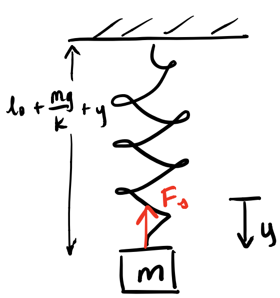
Ex.

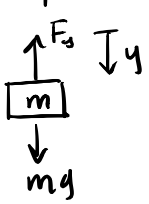


Equilibrium

when you hang the weight, the spring is __not__ unstretched!
↳ so the length is $l_0 + \frac{mg}{k}$ where $\frac{mg}{k}$ = static deflection of spring (isn't moving)

Now spring is pulled down. (spring is a "restoring force")





$$\text{Newton: } m\ddot{y} = mg - F_s$$
$$= \cancel{mg} - k\left(\cancel{\frac{mg}{k}} + y\right)$$
$$= -ky$$

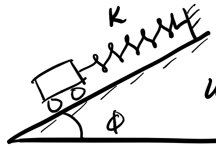
so the equation of motion is: $m\ddot{y} + ky = 0$

$$\ddot{y} + \frac{k}{m}y = 0$$

solution for $y(t)$ is same as previous solution for $x(t)$

ω_n = natural freq. of system $= \sqrt{\frac{k}{m}}$ ... just depends on spring constant & mass!

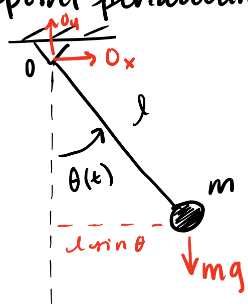
Now imagine spring-mass system now on an inclined plane



$\omega_n = ??$

Plane Pendulum

→ point pendulum w/ massless string



angular momentum principle: $\Sigma M_0 = I_0 \ddot{\theta} = ml^2\ddot{\theta}$

reaction forces have no moment about 0.

$$-\cancel{m}gl\sin\theta = \cancel{m}l^2\ddot{\theta}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$ ⇐ Equation of motion for pointmass pendulum system

... this is <u>not</u> the same form as $\ddot{x} + \omega_n^2 x = 0$

↳ but can make approximation to be in same form as springmass system ...

→ If θ is always small (don't swing pendulum too much), $\sin\theta \approx \theta$

so then, the equation of motion becomes

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$ ← which is the same form as the spring mass system

For pendulum, $\omega_n^2 = \frac{g}{l}$ so the natural frequency is $\omega_n = \sqrt{\frac{g}{l}}$

then $T = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{g}{l}}$ ... longer pendulums have longer period (will swing more slowly than a shorter pendulum)

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

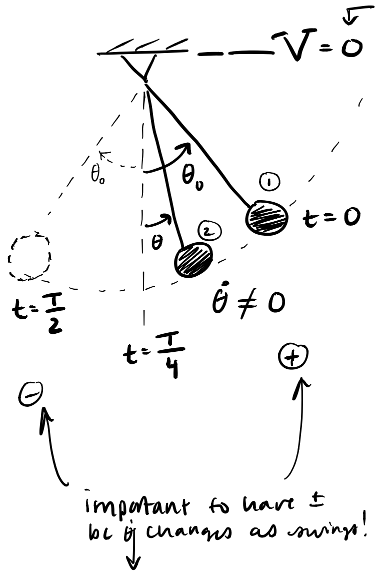
solution is $\theta = A\sin\sqrt{\frac{g}{l}}\,t + B\cos\sqrt{\frac{g}{l}}\,t$

$$\theta = \frac{\dot{\theta}_1}{\omega_n}\sin\sqrt{\frac{g}{l}}\,t + \theta_1\cos\sqrt{\frac{g}{l}}\,t$$

in the small angle approximation, the period $2\pi/\omega_n = 2\pi\sqrt{l/g}$ does not depend on initial conditions or amplitude of amplitude ... only true for approximation!!!

Can compute the period of pendulum without making small angle approximation by using an energy equation

NOT IN TEXTBOOK: Energy Equation w/ pointmass pendulum



initial state, $\dot{\theta} = 0$

θ_0 = amplitude of pendulum oscillation

pendulum is a conservative mechanical system (no air resistance) $\Rightarrow T_1 + V_1 = T_2 + V_2$

$$T_1 + V_1 = T_2 + V_2$$

$$0 + -\cancel{m}g l\cos\theta_0 = \frac{1}{2}\cancel{m}(l\dot{\theta})^2 + -\cancel{m}gl\cos\theta$$

$$\dot{\theta}^2 = 2\frac{g}{l}(\cos\theta - \cos\theta_0)$$

$$\left(\frac{d\theta}{dt}\right)^2 = 2\frac{g}{l}(\cos\theta - \cos\theta_0)$$

$$\frac{d\theta}{dt} = \pm\sqrt{2}\sqrt{\frac{g}{l}}\sqrt{\cos\theta - \cos\theta_0} \Rightarrow \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \pm\sqrt{2}\sqrt{\frac{g}{l}}\,dt$$

$$\int_{\theta_0}^{0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \pm\sqrt{2}\sqrt{\frac{g}{l}}\int_0^{T/4}dt$$

... integrate over one-quarter of the period.

↳ since going from 1 to 2 and looking at direction of swing, we want to take the negative $\ominus$!

$$\int_{\theta_0}^{0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = -\sqrt{2}\sqrt{\frac{g}{l}}\frac{T}{4}$$

... now this is not using small angle approximation

$$\frac{\sqrt{2}}{4}\sqrt{\frac{g}{l}}\,T = \int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

trig identity $\Rightarrow$ $\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$, $\cos\theta_0 = 1 - 2\sin^2\frac{\theta_0}{2}$

$$\frac{\sqrt{2}}{4}\sqrt{\frac{g}{l}}\,T = \int_0^{\theta_0}\frac{d\theta}{\sqrt{2\sin^2\frac{\theta_0}{2} - 2\sin^2\frac{\theta}{2}}}$$

change variable of integration from θ to ϕ where

$$\sin\frac{\theta}{2} = \sin\frac{\theta_0}{2}\sin\phi$$

$$\cos\theta\,\frac{1}{2}\,d\theta = \sin\frac{\theta_0}{2}\cos\phi\,d\phi$$

$$d\theta = \frac{2\sin\frac{\theta_0}{2}\cos\phi\,d\phi}{\cos\theta}$$

when $\theta = 0$, $\phi = 0$

when $\theta = \theta_0$, $\phi = \frac{\pi}{2}$

$$\frac{\sqrt{2}}{4}\sqrt{\frac{g}{l}}\,T = \int_0^{\pi/2} \frac{2\sin\frac{\theta_0}{2}\cos\phi\,d\phi}{\cos\theta\sqrt{2\sin^2\frac{\theta_0}{2} - 2\sin^2\frac{\theta_0}{2}\sin^2\phi}}$$

$$= \frac{2}{\sqrt{2}}\int_0^{\pi/2} \frac{\cancel{\sin\frac{\theta_0}{2}}\,\cancel{\cos\phi}\,d\phi}{\cos\theta\,\cancel{\sin\frac{\theta_0}{2}}\,\cancel{\sqrt{1-\sin^2\phi}}} \qquad \Rightarrow \sqrt{1-\sin^2\phi} = \cos\phi$$

$$\frac{\sqrt{2}}{4}\sqrt{\frac{g}{l}}\,T = \frac{2}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{d\phi}{\cos\theta/2} = \frac{2}{\sqrt{2}}\int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2\phi}}$$

$$\frac{2}{8}\sqrt{\frac{g}{l}}\,T = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2\phi}}$$

... approximation : TAYLOR EXP.

$$(1-\varepsilon)^{-1/2} = \underline{1+\tfrac{1}{2}\varepsilon} + \cancel{\tfrac{1}{8}}\varepsilon^2 + \cancel{\dots}$$

$$\frac{1}{4}\sqrt{\frac{g}{l}}\,T = \int_0^{\pi/2}\left(1+\frac{1}{2}\sin^2\frac{\theta_0}{2}\sin^2\phi\right)d\phi$$

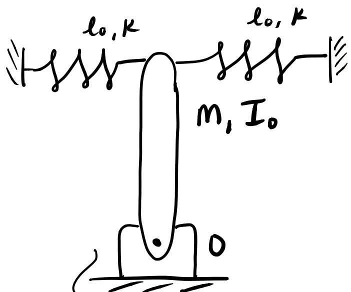
$$\frac{1}{4}\sqrt{\frac{g}{l}}\,T = \frac{\pi}{2} + \frac{1}{2}\sin^2\frac{\theta_0}{2}\,\frac{\pi}{4}$$

$$T = 4\sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + \frac{1}{2}\sin^2\frac{\theta_0}{2}\,\frac{\pi}{4}\right)$$

$$= 4\sqrt{\frac{l}{g}}\left(\frac{\pi}{2} + \frac{\pi}{8}\sin^2\frac{\theta_0}{2}\right)$$

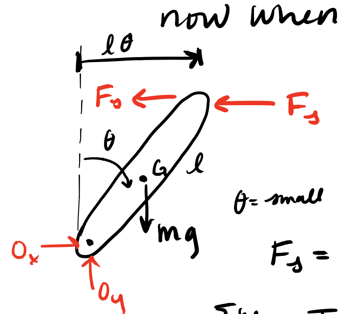
$$\boxed{T = \underbrace{2\pi\sqrt{\frac{l}{g}}}_{\text{small angle period}} + \underbrace{2\pi\left(\frac{1}{4}\right)\sqrt{\frac{l}{g}}\sin^2\frac{\theta_0}{2}}_{\text{first correction not small angle approximation}} = 2\pi\sqrt{\frac{l}{g}}\left(1 + \underbrace{\frac{1}{4}\sin^2\frac{\theta_0}{2}}_{\text{first correction}}\right)}$$

EX. Rigid bar that pivots at the bottom
... this is NOT an oscillatory system when inverted pendulum



inverted pendulum
⇕
vertical position is unstable w/out spring

... when add the springs, is it stable now when given a small push?



gravity tends to destabilize pendulum but the springs tend to stabilize pendulum

θ = small

$F_s = kl\theta$ ← for both spring forces

$$\Sigma M_0 = I_0 \ddot{\theta}$$

$$+mg\frac{l}{2}\theta - 2kl\theta \cdot l = I_0\ddot{\theta}$$

so the equation of motion for this system is

$$I_0\ddot{\theta} + \left(2kl^2 - mg\frac{l}{2}\right)\theta = 0$$

solution of DE depends on sign of $\left(2kl^2 - mg\frac{l}{2}\right)$

↳ if $\left(2kl^2 - mg\frac{l}{2}\right) > 0$, solution is

$$\theta(t) = A\sin\omega_n t + B\cos\omega_n t$$

where $\omega_n = \sqrt{\dfrac{2kl^2 - mg\,l/2}{I_0}}$

θ is oscillatory in this case

↳ if $\left(2kl^2 - mg\frac{l}{2}\right) < 0$ (negative), then

$$\theta(t) = Ae^{\lambda t} + Be^{-\lambda t}, \quad \lambda = \sqrt{\frac{mg\frac{l}{2} - 2kl^2}{I_0}}$$

and θ is not oscillatory ... it has an exponential and the $Ae^{\lambda t}$ grows with time while B term is decaying so get exponential growth & decay

... for our purposes, we'll be looking at the oscillatory cases