DYNAMICS – FINAL REVIEW

Ch. 17 – Planar Kinematics of a Rigid Body: Force and Acceleration

Mass Moment of Inertia

Parallel Axis Theorem

$$I = I_G + md^2$$

Radius of Gyration

$$I = mk^2 \quad \text{or} \quad k = \sqrt{\frac{I}{m}}$$

Planar Kinetic Equations of Motion

Equation of Translational Motion

=> sum of all the external forces acting on the body is equal to the body's mass times acceleration of its mass center G

$$\Sigma F_x = m(a_G)_x \qquad \Sigma F_y = m(a_G)_y$$

Equation of Rotational Motion

=> sum of the moments of all the external forces about the body's mass center G is equal to the product of the moment of inertia of the body about an axis passing through G and the body's angular acceleration

$$\Sigma M_G = I_G \alpha \quad \Rightarrow \quad \Sigma M_P = \Sigma(M_k)_P \text{ (more general}$$

Equations of Motion: Translation

when rigid body undergoes a translation, all the particles of the body have the same acceleration

=> $\alpha = 0$, $\Sigma M_G = 0$

Rectilinear Translation

$$\Sigma F_x = m(a_G)_x \quad \Sigma F_y = m(a_G)_y \quad \Sigma M_G = 0 \qquad a_G = \frac{dv_G}{dt} \quad a_G\,ds_G = v_G\,dv_G$$

Curvilinear Translation

when rigid body subjected to curvilinear translation, all the particles of the body have the same accelerations as they travel along curved paths

$$\Sigma F_n = m(a_G)_n \quad \Sigma F_t = m(a_G)_t \quad \Sigma M_G = 0$$

$$v_G = (v_G)_0 + a_G t \qquad (v_G)_0^2 + 2a_G[s_G - (s_G)_0]$$

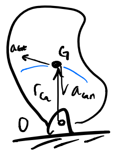
$$s_G = (s_G)_0 + (v_G)_0 t + \frac{1}{2}a_G t^2$$

$$(a_G)_n = v_G^2/\rho$$

$$(a_G)_t = dv_G/dt \qquad (a_G)_t\,ds_G = v_G\,dv_G$$

Equations of Motion: Rotation about a Fixed Axis

=> rigid body constrained to rotate in vertical plane about fixed axis ⊥ to page and passing thru pt. O.



Body's COM G moves around a circular path

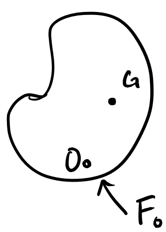
- tangential comp. of acc has magnitude:
  $(a_G)_t = \alpha r_G$
  and acts in direction consistent w/ body's angular acceleration α

- magnitude of normal comp. of acc:
  $(a_G)_n = \omega^2 r_G$
  and always directed from G to O, regardless of rotational sense ω

$\Sigma F_n = m(a_G)_n = m\omega^2 r_G \quad \Sigma F_t = m(a_G)_t = m\alpha r_G \quad \Sigma M_G = I_G \alpha$

Moment Equation About Point O

=> used when convenient to sum moments about the pin at O in order to eliminate an unknown force F_o



$\Sigma M_o = (I_G + m r_G^2)\alpha$ from P-A Th: $I_o = I_G + md^2$

$\underbrace{}$ => moment of inertia of body about fixed axis of rotation passing through O.

$\Sigma F_n = m(a_G)_n = m\omega^2 r_G \quad \Sigma F_t = m(a_G)_t = m\alpha r_G \quad \Sigma M_o = I_o \alpha$

*note that $I_o \alpha$ accounts for the "moment" of both $m(a_G)_t$ and $I_G \alpha$ about pt. O.

$\alpha = \frac{d\omega}{dt} \quad \alpha\, d\theta = \omega\, d\omega \quad \omega = \frac{d\theta}{dt}$ } variable angular acceleration

$\omega = \omega_0 + \alpha_c t \quad \theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_c t^2$
$\omega^2 = \omega_0^2 + 2\alpha_c(\theta - \theta_0)$ } angular acc is constant

Equations of Motion: General Plane Motion

$\Sigma F_x = m(a_G)_x$
$\Sigma F_y = m(a_G)_y$
$\Sigma M_G = I_G \alpha$

Moment equation About the IC

=> involves uniform disc that rolls on a rough surface w/out slipping
=> sum moments about IC of zero velocity,

$\Sigma M_{IC} = I_{IC}\alpha$

$a_G = \alpha r$

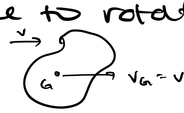
Chapter 18: Planar Kinetics of a Rigid Body: Work + Energy

Kinetic Energy

Translation

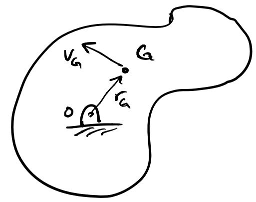
⇒ when rigid body subjected to either rectilinear or curvilinear translation, KE due to rotation = 0 since $\underset{\sim}{\omega} = 0$

$$T = \frac{1}{2} m v_G^2$$



Rotation About a Fixed Axis

⇒ when body rotates about a fixed axis passing thru pt. O body has both translational and rotational kinetic Energy



$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$$

note that $v_G = r_G \omega$ here

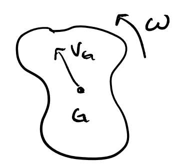
$$T = \frac{1}{2}(I_G + m r_G^2)\omega^2$$

⇑ PA Th.

$$T = \frac{1}{2} I_O \omega^2$$

General Plane Motion

⇒ when body subjected to general plane motion, it has angular velocity $\underset{\sim}{\omega}$ and its mass center has velocity $\underset{\sim}{v_G}$



$$T = \frac{1}{2} m v_G^2 + \frac{1}{2} I_G \omega^2$$

— or —

$$T = \frac{1}{2} I_{IC} \omega^2$$ where I_{IC} is moment of inertia of body about its instantaneous center

The Work of a Force

Work of a Variable Force

$$U_F = \int \underset{\sim}{F} \cdot d\underset{\sim}{r} = \int_s F\cos\theta \, ds$$

⇒ ext. force $\underset{\sim}{F}$ acts on body, work done by force when body moves on path s

Work of a Constant Force

⇒ ext. force $\underset{\sim}{F_c}$ acts on body and maintains constant magnitude F_c and constant direction θ, while the body undergoes a translation s

$$U_{F_c} = (F_c \cos\theta) s$$

Work of a Weight

⇒ weight of a body does work only when body's center of mass G undergoes a vertical displacement Δy. If this displacement is upward, work is negative since weight opposite of displ.

$$U_W = -W\Delta y$$

⇒ if disp downward, (−Δy), work is +

Work of a Spring Force

=> spring force $F_s = ks$ acting on body does work when spring either stretches or compresses from s_1 to a farther position s_2

=> in both cases, work will be ⊖ since disp of body is in opposite direction of force

$$U_s = -\left(\tfrac{1}{2}ks_2^2 - \tfrac{1}{2}ks_1^2\right) \quad \text{where } |s_2| > |s_1|$$

Forces That Do No Work

=> act at fixed pts on body: reactions @ pin supports, normal rxn as body moves along surface, wt of body when center of gravity of body moves in a horizontal plane

=> direction of force ⊥ to their disp

=> friction acting on round body as it rolls w/out slipping

Principle of Work and Energy

$$T_1 + \Sigma U_{1-2} = T_2$$

=> body's initial translational <u>and</u> rotational kinetic energy plus work done by external forces as body moves from its initial to its final position is equal to body's final xlational <u>and</u> rotational KE

Conservation of Energy

=> cons. of energy can be applied when a force system acting on a rigid body consists only of conservative forces

=> work of a conservative force is independent of path taken, depends only on initial and final positions of body

Gravitational Potential Energy

$V_g = Wy_G$ ⊕ when y_G is ⊕ upward

if G located below datum, $-y_G$, then ⊖

Elastic Potential Energy

=> spring stretched or compressed from an initial undeformed position ($s=0$) to final position s

$$V_e = +\tfrac{1}{2}ks^2$$

Translational KE:
$\frac{1}{2}mV_G^2$

Rotational KE:
$\frac{1}{2}I_G\omega^2$

Conservation of Energy

$$V = V_g + V_e$$

$$T_1 + V_1 + (\Sigma U_{1-2})_{nc} = T_2 + V_2$$

$$\text{if } (\Sigma U_{1-2})_{nc} = 0$$

$$T_1 + V_1 = T_2 + V_2$$ *cons of mech. energy*

note that friction or other drag-resistant forces, which depend on velocity or acceleration, are nonconservative

Chapter 22: Vibrations

Undamped Free Vibration

vibration ⇒ oscillating motion of a body or system of connected bodies displaced from a position of equilibrium

⇒ Two types of vibration

I. Free vibration: motion is maintained by gravitational or elastic restoring forces, such as the swinging motion of a pendulum or vibration of an elastic rod

II. Forced vibration: caused by an external periodic or intermittent force applied to the system

⇒ both of these types of vibrations can be damped or undamped

undamped vibrations exclude frictional effects

damped motion of all vibrating bodies in reality since there's both internal & external frictional forces

$\xrightarrow{+} \Sigma F_x = ma_x \quad \Rightarrow \quad -kx = m\ddot{x}$ "simple harmonic motion"

$\ddot{x} + \omega_n^2 x = 0 \qquad \omega_n = \text{natural frequency} = \sqrt{\frac{k}{m}} \quad [\text{rad/sec}]$

$+\downarrow \Sigma F_y = ma_y \quad \Rightarrow \quad \underbrace{-W - ky}_{\text{magnitude of spring force}} + \underset{\text{weight}}{W} = m\ddot{y} \qquad \Rightarrow \ddot{y} + \omega_n^2 y = 0$

⇒ homogeneous, second-order, linear, differential equation

general soln: $x = A\sin\omega_n t + B\cos\omega_n t$

$v = \dot{x} = A\omega_n \cos\omega_n t - B\omega_n \sin\omega_n t$

$a = \ddot{x} = -A\omega_n^2 \sin\omega_n t - B\omega_n^2 \cos\omega_n t$

constants A, B determined from initial conditions

⇒ expressed in simple sinusoidal motion

$A = C\cos\phi \qquad B = C\sin\phi$

$x = C\cos\phi\sin\omega_n t + C\sin\phi\cos\omega_n t \qquad [\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi]$

$x = C\sin(\omega_n t + \phi)$

↳ maximum displacement of block from its eq. position is the AMPLITUDE of vibration

C = amplitude

ϕ = phase angle, represents amount curve displaced from origin when $t = 0$

$C = \sqrt{A^2 + B^2} \qquad \phi = \tan^{-1}\left(\frac{B}{A}\right)$

the sine curve completes one cycle in time $t = \tau$ when

$\omega_n \tau = 2\pi$ or $\tau = \frac{2\pi}{\omega_n}$ "PERIOD" = time interval $= \tau = 2\pi\sqrt{\frac{m}{k}}$

frequency f = number of cycles computed per unit of time, which is the reciprocal of the period

$$f = \frac{1}{\tau} = \frac{\omega_n}{2\pi} \quad \text{or} \quad f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad \left[\text{cycle/sec} = \text{Hz} = \frac{2\pi \text{ rad}}{s}\right]$$

Energy Methods

⇒ remember: simple, harmonic motion of a body is due only to gravitational and elastic restoring forces acting on the body ... and since these forces are conservative, can use cons of energy!

$$T + V = C \quad \Rightarrow \quad \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2 = C$$

→ differentiating wrt time

$$m\dot{x}\ddot{x} + kx\dot{x} = 0$$
$$\dot{x}(m\ddot{x} + kx) = 0$$

and since velocity $\dot{x}$ not always zero in a vibrating system

$$\ddot{x} + \omega_n^2 x = 0 \qquad \omega_n = \sqrt{k/m}$$

Undamped Forced Vibration

Periodic Force

system subjected to a periodic force $F = F_0 \sin\omega_0 t$

⇒ amplitude F_0, forcing frequency ω_0

$$\overset{+}{\rightarrow} \Sigma F_x = ma_x \qquad F_0 \sin\omega_0 t - kx = m\ddot{x}$$

— or —

$$\ddot{x} + \frac{k}{m}x = \frac{F_0}{m}\sin\omega_0 t$$

→ nonhomogeneous, second-order DE with complementary soln x_c plus a particular soln x_p

⇒ complementary soln, setting term on right = 0

$$x_c = C\sin(\omega_n t + \phi) \qquad \omega_n = \sqrt{\frac{k}{m}}, \text{ natural freq.}$$

⇒ since periodic, particular soln determined using

$$x_p = X\sin\omega_0 t$$

→ take 2nd time derivative

$$-X\omega_0^2 \sin\omega_0 t + \frac{k}{m}(X\sin\omega_0 t) = \frac{F_0}{m}\sin\omega_0 t$$

→ factor out $\sin\omega_0 t$, solve for X

$$X = \frac{F_0/m}{(k/m - \omega_0^2)} = \frac{F_0/k}{1-(\omega_0/\omega_n)^2}$$

$$x_p = \frac{F_0/k}{1-(\omega_0/\omega_n)^2}\sin\omega_0 t$$

⇒ general soln:

$$x = x_c + x_p = C\sin(\omega_n t + \phi) + \frac{F_0/k}{1-(\omega_0/\omega_n)^2}\sin\omega_0 t$$

x_c = free vibration → dampen out "transient"
x_p = forced vibration "steady-state"

Periodic Support Displacement

⇒ periodic vibration caused by harmonic movement $\delta = \delta_0 \sin \omega_0 t$

$\overset{+}{\rightarrow}$ $F_x = ma_x$: $\quad -k(x - \delta_0 \sin \omega_0 t) = m\ddot{x}$

$$\ddot{x} + \frac{k}{m}x = \frac{k\delta_0}{m}\sin \omega_0 t$$

(same equation, just F_0 replaced w/ $k\delta_0$)

Viscous Damped Free Vibration

⇒ damping force $F = c\dot{x}$, $c = C$

$\overset{+}{\rightarrow}$ $\Sigma F_x = ma_x$ $\quad -kx - c\dot{x} = m\ddot{x}$

$$m\ddot{x} + c\dot{x} + kx = 0$$
$$x = e^{\lambda t}$$

$$m\lambda^2 e^{\lambda t} + c\lambda e^{\lambda t} + ke^{\lambda t} = 0$$

$$e^{\lambda t}(m\lambda^2 + c\lambda + k) = 0 \quad \Rightarrow \quad m\lambda^2 + c\lambda + k = 0$$

$$\lambda_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \qquad \lambda_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

⇒ critical damping coefficient c_c makes value c that makes radical = 0

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0 \qquad c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

Overdamped System

when $c > c_c$ $\quad \lambda_1$ & λ_2 are both real

$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

⇒ motion is nonvibrating, effect of damping so strong that when block is displaced and released, it simply creeps back to its original position w/out oscillating

Critically Damped System

when $c = c_c$ $\quad$ then $\lambda_1 = \lambda_2 = -c_c/2m = -\omega_n$

⇒ condition where c has smallest value necessary to cause system to be nonvibrating

$$x = (A + Bt)e^{-\omega_n t}$$

Underdamped System

most often $c < c_c$, roots λ_1 and λ_2 are complex