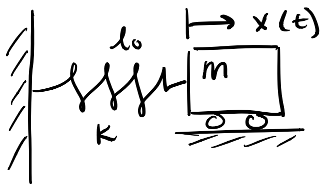
Dynamics – Lecture #23



Equation of motion

$$m\ddot{x} + kx = 0$$

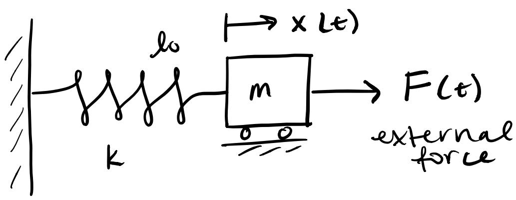
$$\ddot{x} + \omega_n^2 x = 0 \qquad \omega_n = \sqrt{k/m}$$

solution for $x(t)$ is

$$x(t) = A\sin\omega_n t + B\cos\omega_n t$$
$$= C\sin(\omega_n t + \phi)$$

A, B or C, ϕ determined from initial conditions

Forced oscillator



Equation of motion:

$$m\ddot{x} + kx = F(t) \qquad (2)$$

solution of (2):

$$x(t) = \underbrace{x_h(t)}_{\text{homogeneous soln}} + \underbrace{x_p(t)}_{\text{particular soln}}$$

$x_h(t)$ is the general solution of $m\ddot{x}_h + kx_h = 0$

$F(t) = F_0 \sin\omega_n t$ (what we assume, see "static response")

So,

$$x_h(t) = A\sin\omega_n t + B\cos\omega_n t = C\sin(\omega_n t + \phi)$$

In mechanics, $x_h(t)$ is the "free response" (free response of oscillator, no external force)

$x_p(t)$ is some solution of $m\ddot{x}_p + kx_p = F(t)$

$x_p(t)$ depends on $F(t)$

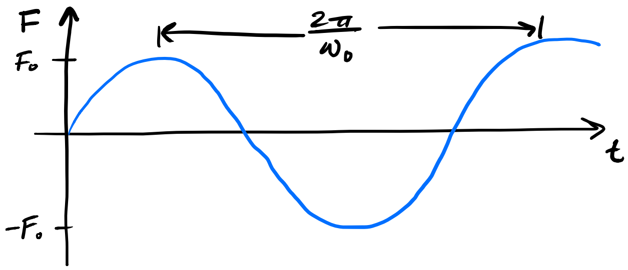
In mechanics, $x_p(t)$ is the "forced response"

Here, consider sinusoidal forcing

↳ we assume that the external force is a sine function:

$$F(t) = F_0 \sin\omega_0 t$$

F_0 = amplitude of the forcing function
ω_0 = radian frequency of the forcing function



Question: given this $\uparrow$, what is the particular soln?

For sinusoidal forcing,

$$m\ddot{x}_p + kx_p = F_0 \sin \omega_0 t \quad \underline{(3)}$$

Try $x_p = X \sin \omega_0 t$ $\underline{(4)}$ $\quad$ X = amplitude of the forced response (or particular solution)

ω_0 = radian frequency of the forced response
= radian frequency of the external force

Take linear mechanical system and force it at ω_0, if that's the frequency of the external force, then the particular soln will also have that same radian freq.

$\Rightarrow$ if force system @ ω_0 it will respond @ same freq.

$\underline{(4)} \rightarrow \underline{(3)}$

$$m(-\omega_0^2 X \cancel{\sin \omega_0 t}) + k(X \cancel{\sin \omega_0 t}) = F_0 \cancel{\sin \omega_0 t}$$

$$X(k - \omega_0^2 m) = F_0 \quad \Rightarrow \quad \underbrace{\underline{X} = \frac{F_0}{k - \omega_0^2 m}}_{\text{amplitude of part. soln}}$$

$$\underline{X} = \frac{F_0}{k - \omega_0^2 m} \cdot \frac{1/k}{1/k} = \frac{F_0/k}{1 - \omega_0^2 m/k} = \frac{F_0/k}{1 - \omega_0^2/(k/m)}$$

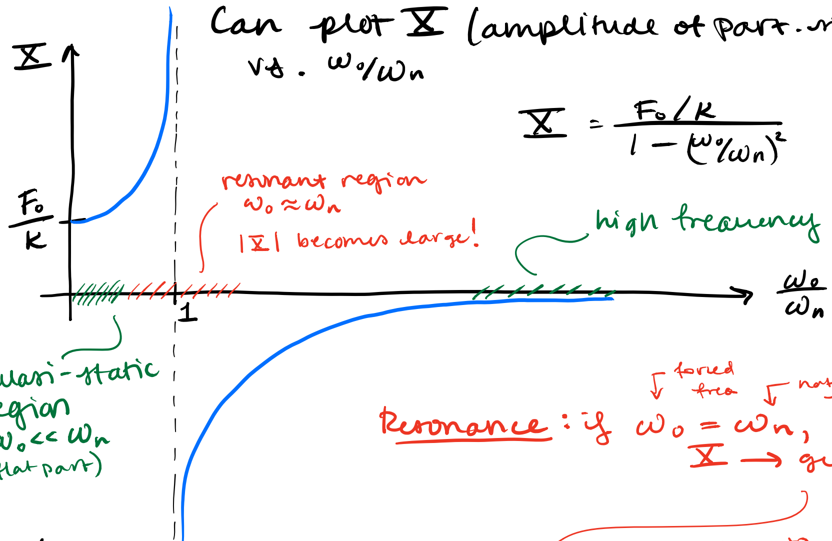
$$\left\{ \underline{X} = \frac{F_0/k}{1 - \left(\frac{\omega_0}{\omega_n}\right)^2} \right\}$$

F_0/k = "static response" of system

$\frac{\omega_0}{\omega_n}$ = "frequency ratio," ratio of forcing frequency to the natural frequency

Can plot $\bar{X}$ (amplitude of part. soln / forced response) vs. ω_0/ω_n

$$\bar{X} = \frac{F_0/k}{1-(\omega_0/\omega_n)^2}$$



resonant region $\omega_0 \approx \omega_n$ $|\bar{X}|$ becomes large!

high frequency region, $\omega_0 \gg \omega_n$ "system hardly responds at all"

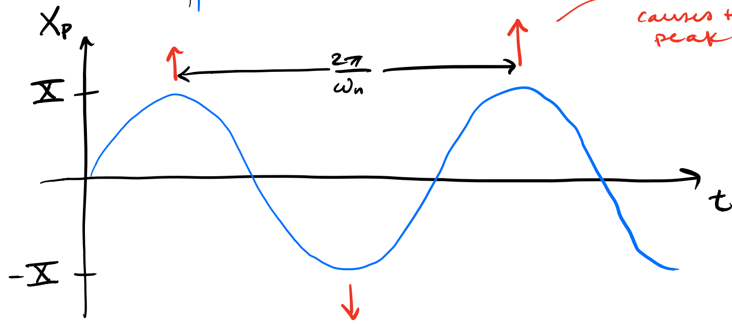
Quasi-static region $\omega_0 \ll \omega_n$ (flat part)

Resonance: if $\omega_0 = \omega_n$, (forced freq; natural freq.)

$\bar{X} \rightarrow$ gets large, ∞

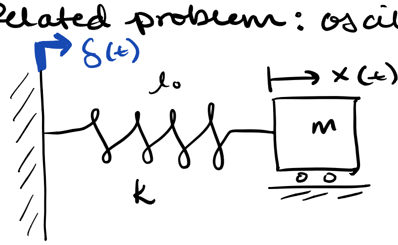
causes these to peak higher

usually a bad thing: crack, fail



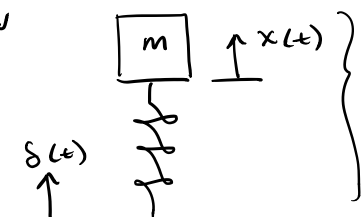
Related problem: oscillator w/ support motion

k = stiffness l_0 = unstretched length



$\delta(t)$ = displacement of support

rotate system 90°



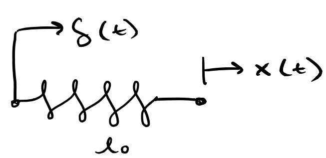
side note

building ... Earth moves, causing building to shake

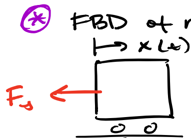
simple model of earthquake

Focus here:

Back to the horizontal case:



FBD of mass

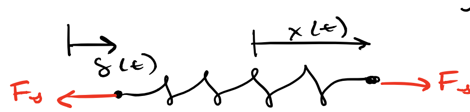


Focus here:

For mass,
$-F_s = m\ddot{x}$
$-kx + k\delta(t) = m\ddot{x}$

-or- $m\ddot{x} + kx = k\delta(t)$

new spring length:
$l_0 + x(t) - \delta(t)$



$F_s = k(x(t) - \delta(t))$

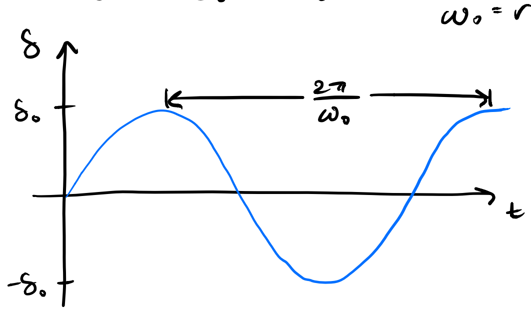
$$m\ddot{x} + kx = k\delta(t)$$

solution is $x(t) = x_h(t) + x_p(t)$

$x_h(t)$ satisfies $m\ddot{x}_h + kx_h = 0 \Rightarrow x_h = A\sin\omega_n t + B\cos\omega_n t$
$\quad = C\sin(\omega_n t + \phi)$

$x_p(t)$ is some solution of
$\quad m\ddot{x}_p + kx_p = k\delta(t)$

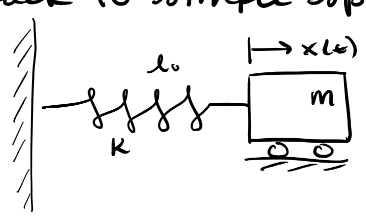
special case: sinusoidal support motion
$\quad \delta(t) = \delta_0 \sin\omega_0 t \qquad \delta_0$ = amplitude of support motion
$\qquad\qquad\qquad\qquad\quad \omega_0$ = radian frequency of support motion



Then
$\quad m\ddot{x}_p + kx_p = (k\delta_0)\sin\omega_0 t$

... previous analysis for x_p (the forcing case) applies with F_0 replaced by $k\delta_0$

Back to simple spring mass oscillator



$$E = T + V$$
$$= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}kx^2$$

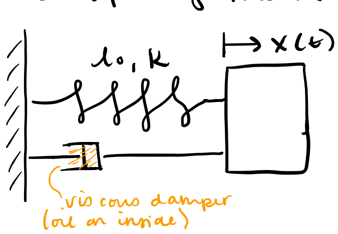
$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(\cancel{m\ddot{x} + kx})$$
$$= \phi \quad \text{or } E = \text{constant in time}$$

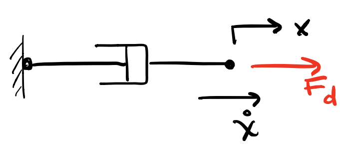
Spring-mass oscillator is a <u>conservative</u> mechanical system

Real mechanical systems are <u>not</u> conservative. Mechanical energy is <u>dissipated</u> (converted to <u>heat</u>)

↳ when mechanical energy converted to heat, can't get mechanical energy back (TD law!)

To include dissipation, can add a <u>damper</u> (or a dashpot) to the spring-mass model





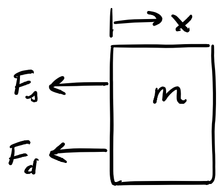
[Dashpot, shockabsorber]

object in which force applied is F_0 and it's proportional to velocity of end

$$F_d = c\dot{x}$$

c = dashpot constant

Free-Body Diagram of Mass



Newton's Law:

$$m\ddot{x} = -F_s - F_d$$
$$= -kx - c\dot{x}$$

New equation of motion is $\underline{m\ddot{x} + c\dot{x} + kx} = 0$
(now have 1st derivative)

$$E = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} kx^2$$

$$\frac{dE}{dt} = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(\underline{m\ddot{x} + kx}) = \dot{x}(-c\dot{x}) = -c\dot{x}^2 < 0$$

∴ this means that the mechanical energy E now <u>decreases</u> with time

$$m\ddot{x} + c\dot{x} + kx = 0 \quad (5)$$ ⇐ Equation of Motion of a <u>Damped</u> Oscillator

Q: what is the solution of (5)?

$$m\ddot{x} + c\dot{x} + kx = 0$$
$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$
$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \quad (6)$$

$\omega_n = \sqrt{\frac{k}{m}}$ = natural frequency, undamped radian frequency of system

$$2\zeta\omega_n = \frac{c}{m}$$
$$\zeta = \frac{c}{2\omega_n m} = \frac{c}{2\sqrt{\frac{k}{m}}\,m} = \frac{c}{2\sqrt{km}}$$
= damping ratio of the system

(6) is a linear, ordinary differential equation w/ constant coefficients (not functions of time) (linear b/c no square x)

ζ is a <u>dimensionless</u> parameter, depends directly on c (the dashpot constant)

For undamped system,
$c = 0 \Rightarrow \zeta = 0$

Solutions can be found by assuming
$$x = Ae^{st} \quad (7)$$

A, s to be determined

⇒ Linear ODE's w/ constant coefficients have exponential solutions

(7) → (6) $$s^2\cancel{Ae^{st}} + 2\zeta\omega_n s\cancel{Ae^{st}} + \omega_n^2\cancel{Ae^{st}} = 0$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$ ⇐ "characteristic equation for s"

λ = eigenvalues!!!

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$s = \begin{cases} s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} \\ s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} \end{cases}$$

Now, $x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$, $(s_1 \neq s_2)$

A_1, A_2 determined by initial conditions

Case I: $\zeta = 0$ (undamped oscillator)

$$s_1 = \omega_n\sqrt{-1} = i\omega_n$$
$$s_2 = -\omega_n\sqrt{-1} = -i\omega_n$$

s_1, s_2 are pure imaginary numbers (no real part)

So, $x = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t}$

Euler's formula:
$$e^{i\theta} = \cos\theta + i\sin\theta$$
$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$= A_1(\cos\omega_n t + i\sin\omega_n t) + A_2(\cos\omega_n t - i\sin\omega_n t)$$

$$= (iA_1 - iA_2)\sin\omega_n t + (A_1 + A_2)\cos\omega_n t$$

let $A = iA_1 - iA_2$
$B = A_1 + A_2$

then,
$$x = A\sin\omega_n t + B\cos\omega_n t$$

* undamped oscillator $\zeta = 0$ *

Next time,

Case II: $0 < \zeta < 1$ ... we'll see damping here!