Nonlinear Continuum + Finite Elements

① Strains

② Stress

③ Nonlinear FEM weak form

Nonlinear Strains

Why? Strains must vanish for rigid body motion + rotation + translation. This is why linear strains are abandoned in nonlinear theory.

→ main nonlinear strain: E = Green Strain (not Young's modulus)

F = deformation gradient $= \frac{dx}{dX} = 1$ if no strain

$ds^2 - dS^2 = 2dX \cdot E \cdot dX$

ds = line length in current configuration

dS = line length in reference configuration

Key idea: green strain changes with square of lengths (makes it nonlinear)

$$ds^2 - dS^2 = dx \cdot dx - dX dX = 2dX \cdot E \cdot dX$$

$dx = F \cdot dX$, so

$$(FdX)^T \cdot (FdX) - dX dX = 2dX \cdot E \cdot dX$$

$$= dX^T F^T F dX - dX \cdot dX = 2dX \cdot E \cdot dX$$

$$= dX \cdot F^T \cdot F \cdot dX - dX \cdot I \cdot dX = 2dX \cdot E \cdot dX$$

$$= dX \cdot (F^T F - I - 2E) \cdot dX = 0$$

$$2E = F^T F - I \qquad \boxed{E = \tfrac{1}{2}(F^T F - I)}$$

→ connection to linear strain

$$F^T F = \frac{dx_k}{dX_i} \frac{dx_k}{dX_j} = F_{ik}^T F_{jk} \qquad u = x - X \quad \text{so} \quad x = u + X$$

$$= \left(\frac{du_k}{dX_i} + \frac{dx_k}{dX_i}\right)\left(\frac{du_k}{dX_j} + \frac{dx_k}{dX_j}\right)$$

$$= \left(\frac{du_k}{dX_i} + \delta_{ij}\right)\left(\frac{du_k}{dX_j} + \delta_{kj}\right)$$

$$= \frac{du_k}{dX_i}\frac{du_k}{dX_j} + \delta_{ij}\frac{du_k}{dX_j} + \frac{du_k}{dX_i}\delta_{kj} + \delta_{ij}\delta_{kj}$$

$$= \frac{du_k}{dX_i}\frac{du_k}{dX_j} + \frac{du_k}{dX_i} + \frac{du_j}{dX_i} + \delta_{ij}$$

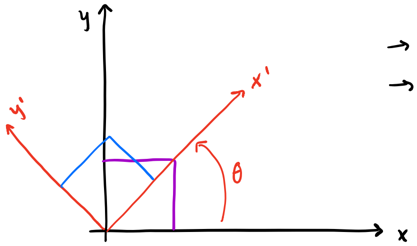
if $i, j, k = 1$

$$= (\varepsilon_{xx}\varepsilon_{xx}) + (\varepsilon_{xx}) + (\varepsilon_{xx}) + 1$$

$$E = \tfrac{1}{2}(F^T F - I) = \tfrac{1}{2}(\varepsilon_{xx}^2 + 2\varepsilon_{xx} + 1 - 1)$$

$$E = \underbrace{\varepsilon_{xx}}_{\text{standard linear strain}} + \underbrace{\tfrac{1}{2}\varepsilon_{xx}^2}_{\text{nonlinear strain}} \quad \text{in 1D}$$

Why does linear strain fail?



→ idea: rigid body rotation of block through angle θ

→ can define rotation through rotation matrix:

$$x = R\underline{X}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix}$$

$$u = x - \underline{X} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{bmatrix} \begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix} \leftarrow \text{now calculate strains}$$

$$u_x = (\cos\theta - 1)\underline{X} - \sin\theta\, \underline{Y}$$
$$u_y = \sin\theta\, \underline{X} + (\cos\theta - 1)\underline{Y}$$

$$\underbrace{\varepsilon_{xx} = \frac{\partial u_x}{\partial \underline{X}} = \cos\theta - 1 \qquad \varepsilon_{yy} = \frac{\partial u_y}{\partial \underline{Y}} = \cos\theta - 1}_{} \qquad \gamma_{xy} = \frac{\partial u_x}{\partial \underline{Y}} + \frac{\partial u_y}{\partial \underline{X}} = -\sin\theta + \sin\theta$$

$$\gamma_{xy} = 0$$

What does this tell you?

→ only $\emptyset$ if $\theta = 0$ ...when there's no rotation at all

→ $\varepsilon_{xx}, \varepsilon_{yy}$ nonzero for <u>any</u> nonzero value of θ

→ so if $\theta \neq \emptyset$, extensional strains do not vanish in linear theory

→ called "geometric nonlinearity"

→ this result <u>does</u> <u>not</u> depend on the material properties!

→ How about nonlinear (green strain)?

$$E = \tfrac{1}{2}(F^T F - I)$$

$$x = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix}$$

$$F = \begin{bmatrix} \partial x/\partial \underline{X} & \partial x/\partial \underline{Y} \\ \partial y/\partial \underline{X} & \partial y/\partial \underline{Y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$F^T F = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

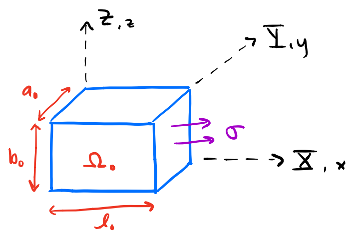
$$\tfrac{1}{2}(F^T F - I) = 0 \leftarrow \text{so Green strain does not cause spurious strains for any } \theta.$$

Nonlinear Stresses → 3 main stress values

① Cauchy Stress : σ
② Nominal Stress : P (1st Piola Kirchoff Stress)
③ 2nd Piola-Kirchoff Stress : S
→ difference between these is whether stress is a reference or current configuration value
→ can convert between all stress values

$$\left.\begin{aligned} P &= SF^{T} \\ \sigma &= J^{-1}FP \\ S &= JF^{-1}\sigma F^{-T} \end{aligned}\right\} \text{ point is, if know one stress, can convert to others}$$

Ex. uniaxial stress



under axial loading:

$a_0 \to a$
$b_0 \to b$ so
$\ell_0 \to \ell$

$$x = \frac{\ell}{\ell_0}X \qquad y = \frac{a}{a_0}Y \qquad z = \frac{b}{b_0}Z$$

→ goal: relate P and S to σ

deformation gradient

$$F = \begin{bmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix} = \begin{bmatrix} \frac{\ell}{\ell_0} & 0 & 0 \\ 0 & \frac{a}{a_0} & 0 \\ 0 & 0 & \frac{b}{b_0} \end{bmatrix}$$

$$J = \mathrm{Det}(F) = \frac{ab\ell}{a_0 b_0 \ell_0}$$

$$F^{-1} = \begin{bmatrix} \ell_0/\ell & 0 & 0 \\ 0 & a_0/a & 0 \\ 0 & 0 & b_0/b \end{bmatrix} \qquad \sigma = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ uniaxial stress}$$

$$P = F^{-1}J\sigma = \begin{bmatrix} \ell_0/\ell & 0 & 0 \\ 0 & a_0/a & 0 \\ 0 & 0 & b_0/b \end{bmatrix} \frac{ab\ell}{a_0 b_0 \ell_0} \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{ab\sigma_x}{a_0 b_0} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to P_{xx} = \frac{ab}{a_0 b_0}\sigma_x = \frac{\overset{\text{area } ab}{A}}{\underset{\text{reference area}}{A_0}}\sigma_x$$

... so stress · area is force $\quad \therefore \boxed{P_{xx} = \frac{F}{A_0}}$ ← engineering stress

$$S = F^{-T}P = \frac{\ell_0}{\ell}\frac{A}{A_0}\sigma_x = \boxed{\frac{\ell_0}{\ell}\frac{F}{A_0}}$$ ← physical meaning ?

Strains for uniaxial stress:

$$E = \frac{1}{2}(F^T F - I) : \; E_{11} = \frac{l^2 - l_0^2}{2 l_0^2}, \quad E_{22} = \frac{a^2 - a_0^2}{2 a_0^2}, \quad E_{33} = \frac{b^2 - b_0^2}{2 b_0^2}$$

strains are differences of length squared rather than just lengths $\Rightarrow$ for comparison: linear strain $\varepsilon_{xx} = \frac{l - l_0}{l_0}$