* discussion of final MATLAB project here *

* discussion of Dynamic MATLAB homework *

New topics for rest of semester:

① parabolic problems ⟶ heat transfer

② nonlinear FEM

③ beam elements

Parabolic Differential Equations

$$\frac{du}{dt} = \frac{d^2u}{dx^2} \qquad 0 < x < 1$$

Boundary Condition: $u(x=0,t) = 0$

"Flux BC" ⟵ $\frac{du}{dx}(x=1,t) = 1$

Initial Condition: $u(x, t=0) = 1$

→ Examples of parabolic equations

① Diffusion ⟶ Ficks Law: $\frac{dc}{dt} = D\frac{d^2c}{dx^2}$

c = concentration of atomic species of trying to model

D = diffusion coefficient

② Temperature (heat conduction)

$$\rho C_p \frac{dT}{dt} = K\frac{d^2T}{dx^2}$$

ρ = density K = conductivity C_p = heat capacity T = temperature

Derive weak form: multiply by weight function and integrate over the domain

$$\frac{du}{dt} = \frac{d^2u}{dx^2} \Rightarrow \int_0^L \delta u \frac{du}{dt}\,dx = \int_0^L \delta u \frac{d^2u}{dx^2}\,dx$$

$\Rightarrow$ mechanical analog: $\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2}$

FEM: $M\ddot{x} + Kx = f^{ext}$ $\Rightarrow$ conserves energy (no damping)

if $\frac{d^2u}{dt^2} + \frac{du}{dt} = \frac{d^2u}{dx^2}$

then FEM: $M\ddot{x} + \underbrace{C\dot{x}}_{\text{damping term}} + Kx = f^{ext}$ $\Rightarrow$ i.e. take energy out of system

→ Approximate weight function using FEM approximation

→ Approximate field (u) with finite elements

→ Integrate by parts

$$\int_0^L \frac{d}{dx}\left(\delta u \frac{du}{dx}\right)dx = \int_0^L \frac{d}{dx}(\delta u)\frac{du}{dx}\,dx + \int_0^L \delta u \frac{d^2u}{dx^2}\,dx$$

$$\int_0^L \delta u \frac{du}{dt}\,dx = \int_0^L \frac{d}{dx}\left(\delta u \frac{du}{dx}\right)dx - \underbrace{\int_0^L \frac{d}{dx}(\delta u)\frac{du}{dx}\,dx}_{\text{no second derivative}}$$

$$\int_0^L \delta u \frac{du}{dt}\,dx + \int_0^L \frac{d}{dx}(\delta u)\frac{du}{dx}\,dx = \int_0^L \frac{d}{dx}\left(\delta u \frac{du}{dx}\right)dx$$

→ Make FEM approximation

$\delta u = N\delta d$, $u = Nd$

$$\int_0^L N\delta d N \frac{dd}{dt}\,dx + \int_0^L \frac{d}{dx}(N\delta d)\frac{d}{dx}(Nd)\,dx = \int_0^L \frac{d}{dx}\left(N\delta d \frac{dN}{dx} d\right)dx$$

Take transposes

$$\int_0^L [N\delta d]^T N \frac{dd}{dt} dx + \int_0^L \left[\frac{d}{dx}(N\delta d)\right]^T \frac{d}{dx}(Nd)\, dx = \int_0^L \left[\frac{d}{dx}(N\delta d)\right]^T \frac{dN}{dx} d)\, dx$$

$$= \delta d^T \underbrace{\int_0^L N^T N dx}_{M} \frac{dd}{dt} + \delta d^T \underbrace{\int_0^L B^T B dx}_{K} d = \underbrace{\delta d^T(1) \frac{du}{dx}(1) - \delta d^T(0) \frac{dd}{dx}(0)}_{B.C.}$$

so have : $M\dot{d} + Kd = f^{ext}$

compare to : $M\ddot{d} + Kd = f^{ext}$ for momentum equation

Another concept:

→ similar form except $\dot{d}$ vs. $\ddot{d}$

→ so change time integration approach

→ so α-family finite difference integration

so : $(1-\alpha)\dot{u}_s + \alpha \dot{u}_{s+1} = \dfrac{u_{s+1} - u_s}{\Delta t_{s+1}}$, $\Delta t_{s+1} = t_{s+1} - t_s$

$0 \leq \alpha \leq 1$

→ weighted average of time derivative approximated by linear interpolation of variable values at 2 time steps

two limiting cases

① when $\alpha = 0$: $\dot{u}_s = \dfrac{u_{s+1} - u_s}{\Delta t_{s+1}}$ "forward difference"

② when $\alpha = 1$: $\dot{u}_{s+1} = \dfrac{u_{s+1} - u_s}{\Delta t_{s+1}}$ "backward difference"

In these notes:

→ finish parabolic equations

→ start introducing nonlinear continuum mechanics, FEM

Parabolic Equations

form: $\frac{du}{dt} = \frac{d^2u}{dx^2}$

weak form: $M\dot{u} + Ku = f$ $\qquad M = \int N^T N\,dx \qquad K = \int \left(\frac{dN}{dx}\right)^T \frac{dN}{dx}\,dx = \int B^T B\,dx$

mass ↗ ↑ velocity, stiffness ↑ ↑ displacement, ↑ force

time integration using finite difference

$$(1-\alpha)\dot{u}_s + \alpha \dot{u}_{s+1} = \underbrace{\frac{u_{s+1} - u_s}{\Delta t_{s+1}}}_{\text{finite difference}}$$

then discretize weak form in time:

at time s: $M\dot{u}_s = f_s - Ku_s \quad \Rightarrow \quad \dot{u}_s = M^{-1}(f_s - Ku_s)$

at time $s+1$: $M\dot{u}_{s+1} = f_{s+1} - Ku_{s+1} \quad \Rightarrow \quad \dot{u}_{s+1} = M^{-1}(f_{s+1} - Ku_{s+1})$

substitute these in to the above

so: $$(1-\alpha)M^{-1}(f_s - Ku_s) + \alpha M^{-1}(f_{s+1} - Ku_{s+1}) = \frac{u_{s+1} - u_s}{\Delta t_{s+1}}$$

solve for u_{s+1}, assume u_s is known, for example via initial conditions

$$u_{s+1} = (M + \alpha\Delta t_{s+1}K)^{-1}(M - \Delta t_{s+1}(1-\alpha)K)u_s + \Delta t_{s+1}(M + \alpha\Delta t_{s+1}K)^{-1}[\alpha f_{s+1} + (1-\alpha)f_s]$$

can think of it like this:

$$u_{s+1} = \underbrace{Au_s}_{} + F_{s,s+1} \qquad A = (M + \alpha\Delta t_{s+1}K)^{-1}(M - \Delta t_{s+1}(1-\alpha)K)$$

$$F_{s,s+1} = \Delta t_{s+1}(M + \alpha\Delta t_{s+1}K)^{-1}(\alpha f_{s+1} + (1-\alpha)f_s)$$

⇒ A matrix is called the "Amplification operator"

u_s is an approx. soln be truncating at Δt

→ means there's an error each time integrate forward in time

→ error $= u_{exact}(t_s) - u_s$

→ error will grow if $|A| > 1$

→ so for stability $|A| \leq 1$

→ we can see that A depends on: mass, K, α ... not in control, but we can control Δt_{s+1}

→ this tells us that places restrictions on time step Δt_{s+1} since M, K, α are fixed values

→ if $\alpha = 0$!

$M u_{s+1} = (M - \Delta t_{s+1}K)u_s + \Delta t_{s+1} f_s$

then have easy explicit update

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what we had before:

$\rho A \frac{d^2u}{dt^2} + bA\frac{d^2u}{dx^2} = 0$ ← "hyperbolic"

$\underbrace{\rho C_p \frac{dT}{dt}}_{M} = \underbrace{K\frac{d^2T}{dx^2}}_{K}$ ← parabolic