Nonlinear FEM strong + weak form

→ VERY similar to linear FEM result

→ there's 2 nonlinear FEM formulations (different coordinate systems)

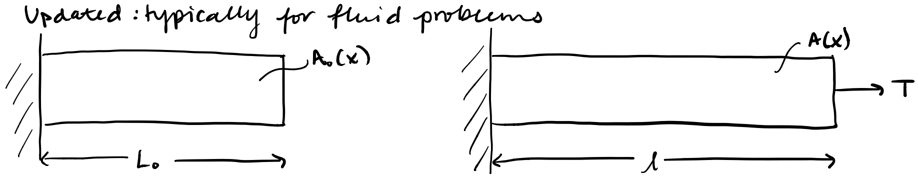
① total lagrangian: derivative wrt reference coordinate (X)

② updated lagrangian: derivative wrt current coordinates (x)

→ total + updated Lagrangian are equivalent!

Total: typically for solid problems

Updated: typically for fluid problems



initial/reference   current/deformed

- displacement: $u = x - X$
- deformation gradient: $F = \frac{dx}{dX}$
- because $F=1$ when no deformation, so define strain $\varepsilon(X,t) = F(X,t) - 1 = \frac{dx}{dX} - 1 = \frac{du}{dX}$

- stress measure:

① Cauchy stress (σ): $\frac{T}{A}$ (true stress: updated Lagrangian)

② Nominal stress (P): $\frac{T}{A}$ (nominal stress = engineering stress: total Lagrangian)

Strong Form (governing DE):

Conservation of Momentum

$x = X$ from here on out for these notes

$$\frac{d}{dX}(A_0 P) + \rho_0 A_0 b_0 = \rho_0 A_0 \frac{d^2u}{dt^2} \leftarrow \text{nonlinear}$$

$$\underbrace{\frac{d}{dX}\left(A_0 E \frac{du}{dX}\right)}_{\text{slight difference here}} + \underbrace{\rho_0 A_0 b_0}_{f^{\text{body}}} = \underbrace{\rho_0 A_0 \frac{d^2u}{dt^2}}_{\text{same: mass}\cdot\text{acceleration}} \leftarrow \text{linear}$$

slight difference here:

- nonlinear stress (P); no assumption that $P = E\varepsilon$

Just like before: need boundary and initial conditions

ICs: $u(x, t=0) = u_0(x)$

$v(x, t=0) = v_0(x)$

BCs: - displacement BC

- traction (force) BC

Strong form → derive into weak form: use weight functions $\delta u(x)$, assume $\delta u(x)$ is smooth, satisfies displacent BCs (same as before)

$$\int_{X_a}^{X_b} \delta u \left[\underbrace{\frac{d}{dX}(A_0 P)}_{\text{integrate stress term by parts}} + \underbrace{\rho_0 A_0 b_0 - \rho_0 A_0 \ddot{u}}_{\text{don't modify}}\right] dX = 0$$ (exact same as what we did before)

both same as for linear

integrate by parts:

$$\int_{X_a}^{X_b} \frac{d}{dX}\left[\delta u A_o P\right] dX = \int_{X_a}^{X_b} \frac{d}{dX}(\delta u) A_o P \, dX + \int_{X_a}^{X_b} \delta u \frac{d}{dX}(A_o P) \, dX$$

so:

$$\int_{X_a}^{X_b} \delta u \frac{d}{dX}(A_o P) \, dX = \underbrace{\int_{X_a}^{X_b} \frac{d}{dX}\left[A_o P \delta u\right] dX}_{} - \int_{X_a}^{X_b} \frac{d}{dX}(\delta u) A_o P \, dX$$

E_x = applied force/traction $= (\delta u A_o E_x)\Big|_{X_a}^{X_b} - \int_{X_a}^{X_b} \frac{d}{dX}(\delta u) A_o P \, dX$

so we end up with:

$$\int_{X_a}^{X_b} \left[\frac{d}{dX}(\delta u) A_o P - \delta u(\rho_o A_o b_o - \rho_o A_o \ddot{u})\right] dX = (\delta u A_o E_x)\Big|_{X_a}^{X_b}$$

→ What did we accomplish? Assume stress is linear elastic, ie $P = C\varepsilon$

Then $\int_{X_a}^{X_b} \underline{\frac{d}{dX}(\delta u)} A_o C \underline{\frac{du}{dX}} \, dX$

product of first derivatives, same as linear FEM

– so allow usage of linear shape functions

So end up with Principle of Virtual Work (work = w)

$$\delta W^{ext} = \int_{X_a}^{X_b} \delta u \rho_o A_o b_o \, dX + (\delta u A_o E_x)\Big|_{X_a}^{X_b}$$

$$\delta W^{kin} = \int_{X_a}^{X_b} \delta u \rho_o A_o \frac{d^2 u}{dt^2} dX \quad \text{(inertia)}$$

} same as linear

$$\delta W^{int} = \int_{X_a}^{X_b} \frac{d}{dX}(\delta u) P A_o \, dX$$

... simplify: $\delta u = \delta x - \delta X$

so: $\frac{d}{dX}(\delta x - \cancel{\delta X})^{\to 0}$

so: $\frac{d}{dX}(\delta u) = \frac{d}{dX}(\delta x) = \delta F$

$$= \int_{X_a}^{X_b} \delta F P A_o \, dX$$

- doesn't require an assumed model for stress P
- linear $f^{int} = Kd$, ie K is independent of d

$$\delta W^{kin} = \delta W^{ext} - \delta W^{int}$$

→ δW^{int} is a function of d

- through stress P
- implies that will require a nonlinear solution methodology

→ Now make FEM approximation to weak form:

→ same as before: divide domain into elements, elements governed by shape functions

$u(X,t) = \sum_{I=1}^{\#\text{nodes}} N_I(X) u_I(t)$ *same shape functions/elements as for linear FEM

$$\delta W^{int} = \int_{X_a}^{X_b} \left[\frac{d}{dX}(\delta u)\right]^T P A_o \, dX$$

$$= \delta u^T \int_{X_a}^{X_b} \left(\frac{dN}{dX}\right)^T P A_o \, dX$$

↑ weight function

so $f^{int} = \int_{X_a}^{X_b} B_o^T P A_o \, dX$ where $B_o = \frac{dN}{dX}$

Linear $f^{int} = \int_{X_a}^{X_b} B^T A E B \, dX \, d$

$$\delta W^{ext} = \int_{X_a}^{X_b} \delta u \rho_0 A_0 b_0 \, dX + (\delta u A_0 E_x)\Big|_{X_a}^{X_b}$$

$$= \int_{X_a}^{X_b} \delta u^T \rho_0 A_0 b_0 \, dX + (\delta u^T A_0 E_x)\Big|_{X_a}^{X_b}$$

$$= \delta u^T \int_{X_a}^{X_b} N^T \rho_0 A_0 b_0 \, dX + \delta u^T (N^T A_0 E_x)\Big|_{X_a}^{X_b} \leftarrow \text{same as linear}$$

$$\delta W^{kin} = \int_{X_a}^{X_b} \delta u \rho_0 A_0 \frac{d^2 u}{dt^2} \, dX$$

$$= \int_{X_a}^{X_b} \delta u^T \rho_0 A_0 \frac{d^2 u}{dt^2} \, dX$$

$$= \delta u^T \int_{X_a}^{X_b} N^T \rho_0 A_0 N \, dX \frac{d^2 u}{dt^2} \leftarrow \text{same mass matrix as linear FEM}$$

combine: $Ma = f^{ext} - f^{int}$

→ integrate in time using explicit central difference, same as before

Other notes on nonlinear FEM:

① mapping to isoparametric domain is same

② Numerical integration/gaussian quadrature is exactly the same

→ Example: 1D linear element for f^{int}

$$f^{int} = \int_{X_a}^{X_b} B_0^T P A_0 \, dX \rightarrow \int_{-1}^{1} \left[\frac{dN}{d\xi} \frac{d\xi}{dX} \right]^T P A_0 \frac{dX}{d\xi} \, d\xi$$

$$= \int_{-1}^{1} \left(\frac{dN}{d\xi} \right)^T P A_0 \, d\xi$$

Reminder: $N(\xi) = \left[\frac{1-\xi}{2} \quad \frac{1+\xi}{2} \right]$

$$\frac{dN}{d\xi} = \left[-\frac{1}{2} \quad \frac{1}{2} \right]$$

so $f^{int} = \int_{-1}^{1} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} P A_0 \, d\xi \approx \sum_{i=1}^{QP} B^T(\xi_i) P A_0 w_i$

$$= \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} P A_0 (2)$$

so $f^{int} = \begin{bmatrix} -PA_0 \\ PA_0 \end{bmatrix}$, sum of $f^{int} = 0$

→ only depends on what P is

Could essentially redo 1D-MATLAB dynamic HW, just replace stress!

P is usually a (nonlinear) function of displacement u

① dynamic problems: if explicit, no change for previous because only need previous time step values

② static + dynamic problems if not explicit (i.e. implicit):

→ stress depends on current time value of displacement

→ requires an iteration solution methodology/linearized system

Equivalence of total and updated Lagrangian formulations

→ Do this for internal force

Total Lagrangian: $f^{int} = \int_{X_a}^{X_b} B_0^T P A_0 \, dX = \int \left(\frac{dN}{dX} \right)^T P A_0 \, dX$

Updated Lagrangian: $f^{int} = \int_{x_a}^{x_b} \left(\frac{dN}{dx} \right)^T \sigma A \, dx$ $\quad \sigma$ = cauchy stress

How can we turn updated Lagrangian into total Lagrangian?

① Change shape function: $\frac{dN}{dx} = \frac{dN}{dX}\frac{dX}{dx}$

$$\int_{x_1}^{x_2} \left(\frac{dN}{dx}\right)^T \sigma A \, dx = \int_{X_1}^{X_2} \left(\frac{dN}{dX}\right)^T \sigma A \frac{dX}{dx} \, dx$$

$$= \int_{X_1}^{X_2} \left(\frac{dN}{dX}\right)^T \sigma A \, dX$$

Recall: $\sigma = \frac{T}{A}$, cauchy stress is load over current area while engineering stress is $P = T/A_0$, or load over reference area

so: $T = \sigma A = P A_0$

so $f^{int} = \int_{X_1}^{X_2} \left(\frac{dN}{dX}\right)^T P A_0 \, dX$ → so total Lagrangian intergral force is equivalent to updated Lagrangian integral force

→ can show same for mass and body force

ME 538: Nov. 30, 2020

Nonlinear FEM strong + weak forms

→ VERY similar to linear FEM result

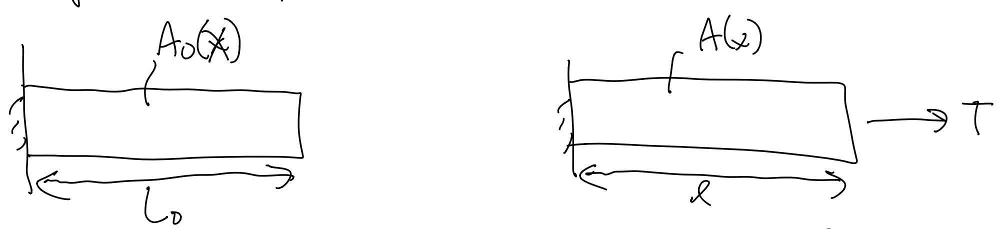
→ 2 nonlinear FEM formulations

① Total Lagrangian: derivatives with respect to reference coordinate (X)

② Updated Lagrangian: derivatives with respect to current coordinates (x)

→ Total + updated Lagrangian are equivalent!

(b) Total → typically for solid problems
Updated → typically for fluid problems



Initial / reference

Current / Deformed

- Displacement $(u) = x - X$
- Deformation gradient $(F) = \dfrac{dx}{dX}$

→ Because $F = 1$ when no deformation,
so define strain $\varepsilon(X,t) = F(X,t) - 1 = \dfrac{dx}{dX} - 1 = \dfrac{du}{dX}$

→ Stress measure:

① Cauchy stress $(\sigma) = \dfrac{T}{A}$ (true stress: updated Lagrangian)

② Nominal stress $(P) = \dfrac{T}{A_0}$ (nominal stress = engineering stress
↳ total Lagrangian)

Strong Form: conservation of momentum

$$\boxed{\frac{d}{dX}(A_0 P)} + \boxed{\rho_0 A_0 b_0} = \boxed{\rho_0 A_0 \frac{d^2u}{dt^2}} \quad \leftarrow \text{Nonlinear}$$

$$\boxed{\frac{d}{dx}\left(A_0 E \frac{du}{dx}\right)} + \boxed{\rho_0 A_0 b_0} = \boxed{\rho_0 A_0 \frac{d^2u}{dt^2}} \quad \leftarrow \text{Linear}$$

Only difference: - Nonlinear stress (P); no assumption that $P = E\varepsilon$

f^{body}

same → Ma

Need boundary conditions + initial conditions

ICs: $u(X, t=0) = u_0(X)$

$v(X, t=0) = v_0(X)$

BCs: - displacement BC

- traction (force) BC

Strong form to weak form: use weight functions $\delta u(X)$, assume $\delta u(X)$ is smooth, satisfies displacement BCs (same as before)

$$\int_{X_a}^{X_b} \delta u \left[\underbrace{\frac{d}{dX}(A_0 P)}_{\text{integrate by parts}} + \underbrace{\rho_0 A_0 b_0 - \rho_0 A_0 \ddot{u}}_{\text{Don't modify}}\right] dX = 0$$

→ both same as for linear

$$\int_{X_a}^{X_b} \frac{d}{dX}\left[\delta u A_0 P\right] dX = \int_{X_a}^{X_b} \frac{d}{dX}(\delta u) A_0 P \, dX + \int_{X_a}^{X_b} \delta u \frac{d}{dX}(A_0 P) dX$$

So: $$\int_{X_a}^{X_b} \delta u \frac{d}{dX}(A_0 P) dX = \underbrace{\int_{X_a}^{X_b} \frac{d}{dX}\left[A_0 P \delta u\right] dX}_{} - \int_{X_a}^{X_b} \frac{d}{dX}(\delta u) A_0 P \, dX$$

$\bar{t}_x$ = applied force/traction

$$= \left(\delta u A_0 \bar{t}_x\right)\Big|_{X_a}^{X_b} - \int_{X_a}^{X_b} \frac{d}{dX}(\delta u) A_0 P \, dX$$

So we end up with:

$$\int_{X_a}^{X_b} \left[\frac{d}{dX}(\delta u) A_0 P - \delta u\left(\rho_0 A_0 b_0 - \rho_0 A_0 \ddot{u}\right)\right] dX = \left(\delta u A_0 \bar{t}_x\right)\Big|_{X_a}^{X_b}$$

→ What did we accomplish? Assume stress is linear elastic, ie $P = C\varepsilon$

Then $\int_{X_a}^{X_b} \underbrace{\frac{d}{dX}(\delta u)}\, A_0 \, C \underbrace{\frac{du}{dX}}\, dX$

- Product of first derivatives, same as linear FEM
- So allow usage of linear shape functions

So end up with principle of virtual work

$$\left.\begin{aligned} \delta W^{ext} &= \int_{X_a}^{X_b} \delta u \, \rho_0 A_0 b_0 \, dX + (\delta u A_0 \bar{t}_x)\Big|_{X_a}^{X_b} \\ \delta W^{kin} &= \int_{X_a}^{X_b} \delta u \, \rho_0 A_0 \frac{d^2u}{dt^2} \, dX \end{aligned}\right] \text{ same as linear}$$

$$\delta W^{int} = \int_{X_a}^{X_b} \frac{d}{dX}(\delta u) \, P A_0 \, dX$$

$$= \int_{X_a}^{X_b} \delta F \, P A_0 \, dX$$

$\delta u = \delta x - \delta X$

so $\frac{d}{dX}(\delta x - \cancel{\delta X}^{\,0})$

so $\frac{d}{dX}(\delta u) = \frac{d}{dX}(\delta x) = \delta F$

$$\delta W^{kin} = \delta W^{ext} - \delta W^{int}$$

- Doesn't require an assumed model for stress P.
- Linear $f^{int} = Kd$, ie K is independent of d

→ δW^{int} is a function of d
- through stress P
- Implies that will require a nonlinear solution methodology

→ Now make FEM approximation to weak form:

→ Same as before! divide domain into elements, elements governed by shape functions

$$u(X,t) = \sum_{I=1}^{\#nodes} N_I(X) \, u_I(t)$$

* same shape functions/elements as for linear FEM

$$\delta W^{int} = \int_{X_a}^{X_b} \left[\frac{d}{dX}(\delta u)\right]^T P A_0 \, dX$$

$$= \delta u^T \int_{X_a}^{X_b} \left(\frac{dN}{dX}\right)^T P A_0 \, dX$$

$$\text{So } f^{int} = \int_{X_a}^{X_b} B_0^T P A_0 \, dX \quad \text{where } B_0 = \frac{dN}{dX}$$

Linear $f^{int} = \int_{X_a}^{X_b} B^T A E B \, dX \, d$

$$\delta W^{ext} = \int_{X_a}^{X_b} \delta u \rho_0 A_0 b_0 \, dX + (\delta u A_0 \bar{t}_x)\Big|_{X_a}^{X_b}$$

$$= \int_{X_a}^{X_b} \delta u^T \rho_0 A_0 b_0 \, dX + (\delta u^T A_0 \bar{t}_x)\Big|_{X_a}^{X_b}$$

$$= \delta u^T \int_{X_a}^{X_b} N^T \rho_0 A_0 b_0 \, dX + \delta u^T (N^T A_0 \bar{t}_x)\Big|_{X_a}^{X_b} \quad \leftarrow \text{same as linear}$$

$$\delta W^{kin} = \int_{X_a}^{X_b} \delta u \rho_0 A_0 \frac{d^2 u}{dt^2} \, dX$$

$$= \int_{X_a}^{X_b} \delta u^T \rho_0 A_0 \frac{d^2 u}{dt^2} \, dX$$

$$= \delta u^T \int_{X_a}^{X_b} N^T \rho_0 A_0 N \, dX \, \frac{d^2 u}{dt^2} \quad \leftarrow \text{same mass matrix as linear FEM}$$

Combine: $Ma = f^{ext} - f^{int}$

→ Integrate in time using explicit central difference, same as before.

→ Other notes on nonlinear FEM:

1. Mapping to isoparametric domain is same
2. Numerical integration (gaussian quadrature is exactly the same

→ Example: 1D linear element for f^{int}

$$f^{int} = \int_{X_a}^{X_b} B_0^T P A_0 \, dX \rightarrow \int_{-1}^{1} \left[\frac{dN}{d\xi} \cancel{\frac{d\xi}{dX}}\right]^T P A_0 \cancel{\frac{dX}{d\xi}} \, d\xi$$

$$= \int_{-1}^{1} \left(\frac{dN}{d\xi}\right)^T P A_0 \, d\xi$$

$$N(\xi) = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix}$$

$$\frac{dN}{d\xi} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\text{So } f^{int} = \int_{-1}^{1} \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} PA_0 \, d\xi \approx \sum_{i=1}^{QP} B^T(\xi_i) PA_0 w_i$$

$$= \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} PA_0 (2)$$

$$\text{So } f^{int} = \begin{bmatrix} -PA_0 \\ PA_0 \end{bmatrix}, \text{ sum of } f^{int} = 0$$

→ only depends on what P is.

→ P is usually a (nonlinear) function of displacement u

① Dynamic problems: if explicit, no change from previous because only need previous time step values

② Static + dynamic problems if not explicit (ie implicit):

→ Stress depends on current time value of displacement

→ Requires an iterative solution methodology / linearized system

→ Equivalence of total + updated Lagrangian formulations

Do this for internal force

$$\text{Total Lagrangian: } f^{int} = \int_{X_a}^{X_b} B_0^T PA_0 \, dX = \int \left(\frac{dN}{dX}\right)^T PA_0 \, dX$$

$$\text{Updated Lagrangian } f^{int} = \int_{x_a}^{x_b} \left(\frac{dN}{dx}\right)^T \sigma A \, dx$$

→ Turn updated Lagrangian into total Lagrangian

① Change shape function: $\frac{dN}{dx} = \frac{dN}{dX}\frac{dX}{dx}$

$$\int_{x_a}^{x_b} \left(\frac{dN}{dx}\right)^T \sigma A \, dx = \int_{X_1}^{X_2} \left(\frac{dN}{dX}\right)^T \sigma A \underbrace{\frac{dX}{dx}}_{} dx$$

$$= \int_{X_1}^{X_2} \left(\frac{dN}{dX}\right)^T \sigma A \underbrace{dX}_{}$$

Recall: $\sigma = \frac{T}{A}$, $P = \frac{T}{A_0}$

so $T = \sigma A = P A_0$

so $f^{int} = \int_{X_1}^{X_2} \left(\frac{dN}{dX}\right)^T P A_0 \, dX$ → so total Lagrangian internal force is equivalent to updated Lagrangian internal force.

→ Can show same for mass and body force.