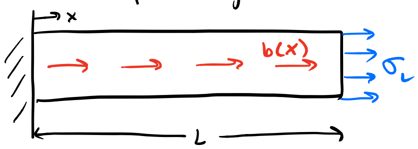
Motivation for Weighted Residual (Variational) Methods



Ignore surface forces

ε_x = strain = $\frac{du}{dx}$ = derivative of displacement

$b(x)$ = body force

σ_L = stress

Potential Energy: $\Pi_p = \int_0^L \frac{1}{2} E \varepsilon_x^2 A\,dx - \int_0^L u\,b(x)\,A\,dx - u_L A \sigma_L$

because body force is force per volume

assume A is constant:

$$\Pi_p = \underbrace{\frac{AE}{2}\int_0^L \varepsilon_x^2\,dx}_{\text{internal work done by strains}} - \underbrace{A\int_0^L b\,u\,dx - (A\sigma_L)u_L}_{\text{external work done by body forces (applied loads)}}$$

displacement field; u is function of x

assume that $u = u(x)$ is "admissible"

① u is continuous

② satisfies displacement boundary condition $u=0$ at $x=0$

Now perturb u by a small amount δu

define $\delta u = e\eta$ $\quad$ e = small #, $\eta = \eta(x)$ is also admissible, satisfies BC and continuous elsewhere

Perturbed field $u + e\eta$ is also admissible, satisfies same BC as u

Perturbed strain: $\varepsilon_x = \frac{du}{dx} + e\frac{d\eta}{dx}$

Boundary condition: $u_L = u_L + e\eta_L$ $\quad$ (η_L is η at $x=L$)

Define new energy: $\Pi_p^{new} = \Pi_p + \delta\Pi_p$

Write Π_p^{new}!

Recall: $\Pi_p = \frac{AE}{2}\int_0^L \varepsilon_x^2\,dx - A\int_0^L b\,u\,dx - (A\sigma_L)u_L$

$$\Pi_p^{new} = \frac{AE}{2}\int_0^L \left(\frac{du}{dx} + e\frac{d\eta}{dx}\right)^2 dx - A\int_0^L b(u + e\eta)\,dx - (A\sigma_L)(u_L + e\eta_L)$$

$$\Pi_p = \frac{AE}{2}\int_0^L \left(\frac{du}{dx}\right)^2 dx - A\int_0^L b\,u\,dx - (A\sigma_L)u_L$$

$$\delta\Pi_p = \Pi_p^{new} - \Pi_p = \frac{AE}{2}\int_0^L \left(2e\frac{du}{dx}\frac{d\eta}{dx} + e^2\left(\frac{d\eta}{dx}\right)^2\right)dx - \int_0^L A\,b\,e\,\eta\,dx - A\sigma_L e\eta_L$$

$$= \boxed{e\left[AE\int_0^L \frac{du}{dx}\frac{d\eta}{dx}\,dx - \int_0^L A\,b\,\eta\,dx - (A\sigma_L)\eta_L\right] + e^2\frac{AE}{2}\int_0^L\left(\frac{d\eta}{dx}\right)^2 dx}$$

For stable equilibrium, Π_p is at relative minimum, so $\delta\Pi_p > 0$ (must increase from relative minimum) for any admissible η

How to guarantee that $\Pi_p > 0$?

look @ second term : $\epsilon^2 \frac{AE}{2}\int_0^L \left(\frac{d\eta}{dx}\right)^2 dx \longrightarrow$ must be ≥ 0

so if $\delta \pi_p > 0$ for all ϵ: $\boxed{AE\int_0^L \underbrace{\frac{du}{dx}\frac{d\eta}{dx}}_{\text{integrate by parts}} dx - \int_0^L Ab\eta\, dx - (A\sigma_L)\eta_L = 0}$

the "weak form" of governing differential equation

$$AE\int_0^L \frac{d}{dx}\left(\frac{du}{dx}\eta\right)dx = AE\int_0^L \frac{d^2u}{dx^2}\eta\, dx + AE\int_0^L \frac{du}{dx}\frac{d\eta}{dx}dx$$

chain rule derivative

$$AE\int_0^L \frac{du}{dx}\frac{d\eta}{dx}dx = \underbrace{AE\int_0^L \frac{d}{dx}\left(\frac{du}{dx}\eta\right) - AE\int_0^L \frac{d^2u}{dx^2}\eta\, dx}_{\text{substitute this in to above}}$$

so we get:

$$\underbrace{AE\int_0^L \frac{d}{dx}\left(\frac{du}{dx}\eta\right)dx}_{\text{simplify this term:}} - AE\int_0^L \frac{d^2u}{dx^2}\eta\, dx - \int_0^L Ab\eta\, dx - (A\sigma_L)\eta_L = 0$$

$$AE\left[\frac{du}{dx}(x=L)\,\eta(x=L) - \frac{du}{dx}(x=0)\,\cancelto{0}{\eta(x=0)}\right]$$

$$AE\,\frac{du}{dx}(x=L)\,\eta(x=L) - \underbrace{AE\int_0^L \frac{d^2u}{dx^2}\eta\, dx - \int_0^L Ab\eta\, dx}_{\text{combine}} - (A\sigma_L)\eta_L = 0$$

combine

$$0 = -\int_0^L \left(AE\frac{d^2u}{dx^2} + Ab\right)\eta\, dx + \underbrace{AE\frac{du}{dx}(x=L)}_{E\frac{du}{dx}(x=L)=\sigma_L}\underbrace{\eta(x=L)}_{\eta(x=L)=\eta_L} - (A\sigma_L)\eta_L$$

$$0 = -\int_0^L \left(AE\frac{d^2u}{dx^2} + Ab\right)\eta\, dx + A\eta_L\left(E\frac{du}{dx}(x=L) - \sigma_L\right)$$

so: $\underbrace{AE\frac{d^2u}{dx^2} + Ab = 0}_{\text{governing DE for solids "equilibrium equation"}}$ for $0 < x < L$ AND $\underbrace{E\frac{du}{dx} - \sigma_L = 0}_{\text{"natural" boundary condition}}$ at $x = L$

Weak form:

$$\underbrace{AE\int_0^L \frac{du}{dx}\frac{d\eta}{dx}dx}_{\text{internal work}} - \underbrace{\int_0^L Ab\eta\, dx - A\sigma_L\eta_L}_{\text{external work}} = 0$$

<u>Meaning</u>: total internal and external work should balance for any admissible infinitesimal displacement η from an equilibrium configuration.

Connection to Finite Elements:

Say we want approximate solution $\tilde{u} = \tilde{u}(x)$ = FEM soln

say $\tilde{u}(x) = a_1 x + a_2 x^2 + a_3 x^3 + \ldots$ a's are constants

$\rightarrow$ $\tilde{u}(x)$ does not (in general) satisfy governing equation $AE \frac{d^2u}{dx^2} + Ab = 0$

so there is a "residual" or error, i.e. $R(x) = \text{residual} = AE\frac{d^2\tilde{u}(x)}{dx^2} + Ab \neq 0$

Idea of FEM: don't satisfy differential equation at every point!

$\rightarrow$ instead: try to satisfy DE in a weighted integral sense

$$\int_0^L \underbrace{\left(AE\frac{d^2\tilde{u}}{dx^2} + Ab\right)}_{\text{residual}} \eta \, dx = 0$$

η ← weight by some function η

$\int_0^L$ ← integrate over domain

$\rightarrow$ integrate by parts

leads to weak form!

Flowchart for FEM Simulation

① What's governing differential equation ("strong form")

$\rightarrow$ defining / picking problem type

(ex. fluid problem? electromagnetic? solid-fluid?)

② Create weak/variational form (equivalent to strong form)

~ "weighted residual" or "Galerkin FEM Approximation"

$\rightarrow$ FEA softwares have these pre-defined

$\rightarrow$ Not all softwares have same weak forms (mainly for nonlinear problems)

③ Approximate weak form using finite elements

$\rightarrow$ select element type

$\rightarrow$ select material type

④ Element-level assembly of stiffness matrix

$\rightarrow$ convert elements from real space to <u>isoparametric</u> <u>domain</u>

$\rightarrow$ Numerical integration

⑤ Assemble global K from element stiffnesses

⑥ Impose boundary conditions

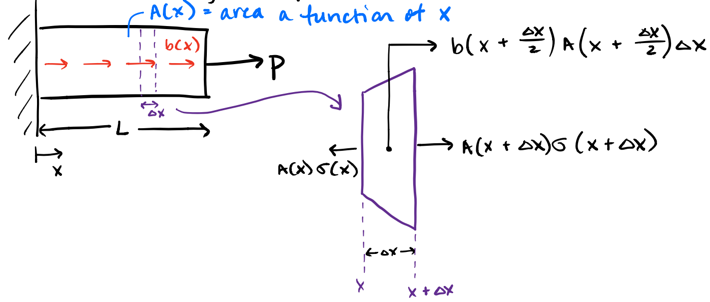
⑦ Solve $Kd = F^{ext}$

⑧ Postprocess/visualize results

} same as before

From the flowchart...

Step ①: governing DE of 1-D bar



(drawing body force as going through centroid)

$\sum f_x = 0 = A(x+\Delta x)\sigma(x+\Delta x) + b(x+\frac{\Delta x}{2})A(x+\frac{\Delta x}{2})\Delta x - A(x)\sigma(x)$

$= 0 = A(x+\Delta x)\sigma(x+\Delta x) - A(x)\sigma(x) + b(x+\frac{\Delta x}{2})A(x+\frac{\Delta x}{2})\Delta x$

→ divide by Δx, take limit $\Delta x \to 0$

$$\lim_{\Delta x \to 0}\left[\underbrace{\frac{A(x+\Delta x)\sigma(x+\Delta x) - A(x)\sigma(x)}{\Delta x}}_{\text{definition of a derivative}} + b(x+\tfrac{\Delta x}{2})A(x+\tfrac{\Delta x}{2}) = 0\right]$$

$$\boxed{\frac{d}{dx}(A\sigma) + bA = 0}$$

→ "strong form" OR (more generally) governing DE

"Equilibrium" Equation

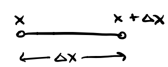
→ governing equation for solid mechanics

Rewriting this more generally,

$$\frac{d}{dx}\left[\underline{A(x)\sigma(x)}\right] + b(x)A(x) = 0$$ → units? force/length

If constant area, equation simplifies a lot: $\frac{d\sigma}{dx} + b = 0$

→ Define strain? $\varepsilon = \frac{\Delta L}{L_0} = \frac{L - L_0}{L}$ (change in length over initial length)



elongation of an element: $u(x+\Delta x) - u(x)$

$$\lim_{\Delta x \to 0}\left[\frac{u(x+\Delta x) - u(x)}{\Delta x}\right] = \frac{du(x)}{dx} = \varepsilon$$

→ Stress, assume Hooke's Law: $\underline{\sigma = E\,\varepsilon(x) = E\frac{du}{dx}}$ combine with above underlined term

Back to strong form:

$$\frac{d}{dx}\left[A(x)E\frac{du(x)}{dx}\right] + b(x)A(x) = 0$$

Assume $A(x)$, area, is constant $\Rightarrow$ $\boxed{AE\frac{d^2u}{dx^2} + bA = 0}$ strong form in terms of displacements ... requires 2nd derivative of displacement

Problem Definition:

strong form: $\frac{d}{dx}(A\sigma) + bA = 0$ OR $AE\frac{d^2u}{dx^2} + bA = 0$

boundary conditions: $u(x=0) = 0$ $\quad A\sigma\big|_{x=L} = P$ or $AE\frac{du}{dx}\big|_{x=L} = P$

⇒ these BC's specific to bar fixed at left end, loaded at right

→ For these problems, strong form remains same, but boundary conditions change

→ back to strong form ...

$AE\frac{d^2u}{dx^2} + bA = 0$

... why do we need to modify this equation?

let's u is linear → then $\frac{d^2u}{dx^2} = 0$

↳ 2^{nd} derivative in u requires "higher order" continuity in approximation for u → so u must be at least quadratic

↳ then need to "weaken" continuity requirements on u, i.e. can we get equation that is mathematically equivalent to strong form, but has weaker continuity requirements on u, i.e. is valid for linear u.

STRONG FORM TO WEAK FORM

$$\frac{d}{dx}(A\sigma) + bA = 0$$

define "virtual" displacement (weight function) : δu

δu is admissible, i.e. $\delta u(x=0) = 0$ ← $\delta u(x=0)$ is specific for this problem

How to get weak form?

Multiply strong form by weight function, integrate over the domain

$$\underbrace{0 = \int_0^L \Big[\underbrace{\tfrac{d}{dx}(A\sigma)}_{F/L} + bA\Big]\underbrace{\delta u}_{L}\,\underbrace{dx}_{L}}$$ $= F \cdot L =$ work (energy)

weighted residual satisfy in weighted, volume-averaged sense instead of at all points x

δu physical meaning = displacement

$$0 = \underbrace{\int_0^L \delta u \frac{d}{dx}(A\sigma)\,dx}_{\delta W^{int}\ ①} + \underbrace{\int_0^L \delta u bA\,dx}_{\delta W^{ext}\ ②}$$

stress is internal work

body force is external work

$\Rightarrow \delta W^{ext} = \delta W^{int}$

work on term ①: integrate by parts

$$\int_0^L \frac{d}{dx}(\delta u A\sigma)\,dx = \int_0^L (\sigma A)\frac{d}{dx}(\delta u)\,dx + \underbrace{\int_0^L \delta u \frac{d}{dx}(A\sigma)\,dx}_{①}$$

$$\underbrace{\int_0^L \frac{d}{dx}(\delta u A\sigma)\,dx}_{①} = \underbrace{\int_0^L \frac{d}{dx}(\delta u A\sigma)\,dx - \int_0^L (\sigma A)\frac{d}{dx}(\delta u)\,dx}_{\text{substitute for } ①}$$

$$0 = \underbrace{\int_0^L \frac{d}{dx}(\delta u A\sigma)\,dx}_{\delta u A\sigma|_0^L} - \int_0^L (\sigma A)\frac{d}{dx}(\delta u)\,dx + \int_0^L \delta u bA\,dx$$

$$\delta u A\sigma\Big|_0^L - \int_0^L (\sigma A)\frac{d}{dx}(\delta u)\,dx + \int_0^L \delta u bA\,dx = 0$$

$$\delta u(L)A(L)\sigma(L) - \delta u\cancel{(0)}A(0)\sigma(0) - \int_0^L (\sigma A)\frac{d}{dx}\delta u\,dx + \int_0^L \delta u bA\,dx = 0$$

0 because $\delta u(0) \to 0$

$$\delta u(L)\underbrace{A(L)\sigma(L)}_{=P} - \int_0^L (\sigma A)\frac{d}{dx}\delta u\,dx + \int_0^L \delta u\, bA\,dx = 0$$

$$\boxed{P\,\delta u(L) - \int_0^L (\sigma A)\frac{d}{dx}(\delta u) + \int_0^L \delta u\, bA\,dx = 0}$$ WEAK FORM

term depends on applied forces, so this changes depending on B.C.

do not change so long as solving equilibrium equation

→ weak form is mathematically equivalent (equal to) strong form

→ no approximation (yet)

→ approximation (and error) comes when choosing FEM approximation for $u(x)$

$$0 = P\,\delta u(L) - \boxed{\int_0^L (\sigma A)\frac{d}{dx}(\delta u)\,dx} + \int_0^L \delta u\, bA\,dx$$

What have we gained by deriving weak form?

→ compare strong form: $\underbrace{AE\frac{d^2u}{dx^2}}_{\text{2nd order derivative wrt } u}$ vs. $-\int_0^L \underbrace{(\sigma A)}_{\sigma = E\frac{du}{dx}} \underbrace{\frac{d}{dx}(\delta u)}_{\frac{d}{dx}(\delta u)}\,dx$

2nd derivative replaced by product of 1st derivatives

→ have "weakened" continuity requirements on u

now can use linear displacement field for u and δu ←

How is u related δu?

... learn this soon

Short recap:

① Strong form governing DE:

$$\frac{d}{dx}(A\sigma) + bA = 0 \quad \text{or equivalently} \quad AE\frac{d^2u}{dx^2} + bA = 0$$

② Obtain mathematically equivalent weak form

$$F(L)\delta u(L) - \int_0^L \underbrace{\frac{d}{dx}(\delta u)A\sigma}\,dx + \int_0^L \delta u\, bA = 0$$

→ contains no second derivatives of displacement u

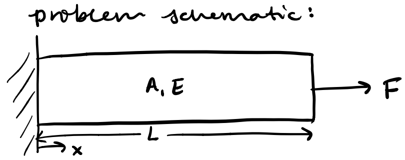
→ product of 2 first derivatives of u

Next Step: approximate weak form with FE, meaning assume a displacement field u (linear, quadratic, etc) and also for weight function δu

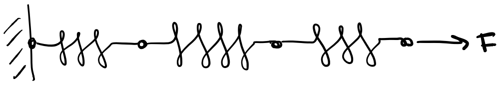
How do we do this conceptually?

$$\underbrace{-\int_0^L \frac{d}{dx}(\delta u)A\sigma - dx}_{\text{integral over entire bar}} \rightarrow$$ will eventually become our stiffness term

problem schematic:



we now turn the problem schematic into a discrete structure of elements



... but need to calculate for each element (change the integral)

$$-\int_0^L \frac{d}{dx}(\delta u) A\sigma - dx = \sum_{e=1}^{\#\,elements} \underbrace{\int_{L^e} \frac{d}{dx}(\delta u) A\sigma dx}_{\text{evaluate for each element}}$$

body force term: $\int_0^L \delta u b A dx = \sum_{e=1}^{\#\,elements} \underbrace{\int_{L^e} \delta u b A dx}_{\text{evaluate for each element}}$

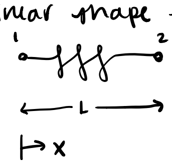
force term: don't need to integrate since @ point $\underbrace{F(L)\delta u(L)}_{\text{only evaluate at node } x=2\text{, no integral since pt. force}}$

What properties do we want FEM approximation to have?

$$u^{FEM}(x) = \sum_{I=1}^{\#\,nodes/elements} \underbrace{N_I(x)}_{\text{shape function}} \underbrace{d_I}_{\text{nodal displacement}}$$

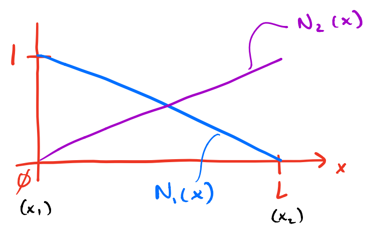
shape function → user-chosen → defines polynomial order of the approximation

Ex. linear shape function



$$N(x) = \left[\underbrace{1 - \frac{x}{L}}_{N_1(x)} \quad \underbrace{\frac{x}{L}}_{N_2(x)}\right]$$

→ Plot.



let's say we want displacement u at the point x_1: $u(x_1) = N_1(x_1)d_1 + N_2(x_1)d_2$

... say $x_1 = 0$ and $x_2 = L$, can we simplify this?

$u(x_1) = \left(1 - \frac{0}{L}\right)d_1 + \left(\frac{0}{L}\right)d_2$

$u(x_1) = d_1$ ... which is reasonable, makes sense

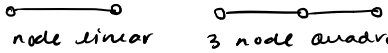
let's evaluate displacement at x_2:

$u(x_2) = N_1(x_2)d_1 + N_2(x_2)d_2$

$u(x_2) = \left(1 - \frac{L}{L}\right)d_1 + \left(\frac{L}{L}\right)d_2$

$u(x_2) = d_2$ ... which also makes sense

→ # of nodes for an element depends on order of $N(x)$



2 node linear element

3 node quadratic element

$$u^{FEM}(x) = \sum_{I=1}^{\#\,nodes/element} N_I(x) d_I \qquad \underline{\text{Strain}}: \frac{du^{FEM}}{dx} = \sum_{I=1}^{\#\,nodes/element} \frac{d N_I(x)}{dx} d_I = \sum_{I=1}^{\#\,nodes/elem} B_I d_I \qquad B = \frac{dN}{dx}$$

Weight Functions → also need FEM approximation

↳ $\delta u(x)$: assume that use same functions as for $u(x)$

↳ "Galerkin" approximation → very important for FEM

What this means is, we approximate the same way

$$\delta u(x) = \sum_{I=1}^{\#\text{nodes/element}} N_I(x)\,\delta d_I$$

$$\frac{d}{dx}(\delta u(x)) = \delta \varepsilon(x) = \sum_{I=1}^{\#\text{nodes/element}} \frac{dN_I}{dx}\delta d_I = \sum_{I=1}^{\#\text{nodes/element}} B_I(x)\,\delta d_I$$

...

Back to weak form:

$$\underbrace{F(L)\delta u(L) - \int_0^L \frac{d}{dx}(\delta u) A\sigma\, dx + \int_0^L \delta u\, b A\, dx = 0}$$

substitute FEM approximations for u and δu into weak form

→ this is where we introduce error into solution of DE

Start with $-\int_0^L \frac{d}{dx}(\delta u) A\sigma\, dx = -\int_0^L \delta\varepsilon A \sigma\, dx = -\int_0^L \delta \varepsilon A E \frac{du}{dx} dx$

weighted strain, area, stress

$$= -\int_0^L \underset{1\times2}{B}\underset{2\times1}{\delta d} A E \underset{1\times2}{B}\underset{2\times1}{d}\, dx$$

now take transposes

$= -\int_0^L (B\delta d)^T A E B d\, dx = -\int_0^L \delta d^T B^T A E B d\, dx$ → δd^T, d are nodal values, not functions of x, therefore constants

$$= -\delta d^T \int_0^L B^T A E B\, dx\; d$$

↳ sum over all elements: $\displaystyle\sum_{e=1}^{\#\text{elements}} \underset{1\times2}{\delta d^T} \underbrace{\left[\int_{L^e} \underset{2\times1}{B^T} \underset{1\times1}{A} \underset{1\times1}{E} \underset{1\times2}{B}\, dx\right]}_{} \underset{2\times1}{d}$ ... going to be scalar, units of energy

K^e = element stiffness matrix (2x2 matrix for linear element)

*side note: why can we transpose? *

$\underset{1\times2}{B}\ \underset{2\times1}{\delta d} = 1\times1$ $\quad (B\delta d)^T = \underset{1\times2}{\delta d^T}\ \underset{2\times1}{B^T} = 1\times1$

Body force: $\int_0^L \delta u\, b A\, dx = \int_0^L N\delta d\, b A\, dx \Rightarrow \int_0^L (N\delta d)^T b A\, dx = \int_0^L \underbrace{\delta d^T} N^T b A\, dx$

not function of x, pull out of integral

$$= \delta d^T \int_0^L N^T b A\, dx$$

↳ sum over elements: $\displaystyle\sum_{e=1}^{\#\text{elements}} \delta d^T \underbrace{\left[\int_{L^e} \underset{2\times1}{N^T} \underset{1\times1}{b} \underset{1\times1}{A}\, dx\right]}_{}$ $= 2\times1$

element body force

External force: $F(L)\delta u(L) = F(L)N(L)\delta d \Rightarrow F(L)[N(L)\delta d]^T = \delta d^T \underbrace{N^T(L)}F(L)$

① only acts at $x = L$

② because shape function values = 1 at node, ϕ elsewhere, external force only appears at $x = L$

So FEM approximation of weak form looks like:

$$\delta d^T\left[\int_0^L B^T AE B\,dx\, d\right] = \delta d^T\left[N^T(L)F(L) + \int_0^L N^T bA\,dx\right]$$

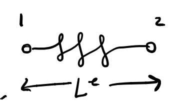
or: $Kd = f^{ext} + f^{body}$

$$K^e = \int_{L^e} B^T AE B\,dx$$

$$f^{ext} = N^T(L)F(L)$$

$$f^{body} = \int_{L^e} N^T bA\,dx$$

Evaluate stiffness matrix for one element



$$K^e = \int_0^{L^e} B^T AE B\,dx \qquad N(x) = \begin{bmatrix} 1-\frac{x}{L^e} & \frac{x}{L^e} \end{bmatrix} \leftarrow \text{linear and add to 1}$$

$$B(x) = \begin{bmatrix} -\frac{1}{L^e} & \frac{1}{L^e} \end{bmatrix} \leftarrow \text{the derivatives}$$

$$= \begin{bmatrix} B_1 & B_2 \end{bmatrix}$$

$$K^e = \int_0^{L^e} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} AE \begin{bmatrix} B_1 & B_2 \end{bmatrix} dx$$ ← assume area and young's modulus are constant for simplicity

$$= AE\int_0^{L^e} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\begin{bmatrix} B_1 & B_2 \end{bmatrix} dx$$

$$= AE\int_0^{L^e} \underbrace{\begin{bmatrix} B_1^2 & B_1B_2 \\ B_1B_2 & B_2 \end{bmatrix}}_{\substack{\text{① symmetric} \\ \text{② diagonal entries} > \emptyset}} dx = AE\int_0^{L^e}\begin{bmatrix} \left(-\frac{1}{L^e}\right)^2 & \left(-\frac{1}{L^e}\right)\left(\frac{1}{L^e}\right) \\ \left(-\frac{1}{L^e}\right)\left(\frac{1}{L^e}\right) & \left(\frac{1}{L^e}\right)^2 \end{bmatrix} dx$$

$$= AE\int_0^{L^e}\begin{bmatrix} \frac{1}{L^{e2}} & -\frac{1}{L^{e2}} \\ -\frac{1}{L^{e2}} & \frac{1}{L^{e2}} \end{bmatrix} dx$$

$$\underbrace{K^e = \frac{AE}{L^e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{\text{③ same element } k \text{ as before}}$$

→ Back to weak form:
- k comes from $-\int_0^L \underline{\frac{d}{dx}(\delta u)} A\sigma\,dx = -\int_0^L \underline{\frac{d}{dx}(\delta u)} AE \underline{\frac{du}{dx}}\,dx$
  - becomes: $K^e = \int_0^{L^e} B^T AE B\,dx$

→ Galerkin approximation (ie approximate δu and u the same way) leads to symmetric k, one that is same as before
 * not necessarily in fluids, but in solids this is the most beneficial form of the approximation

Body force:

$$f^{body} = \int_0^{L^e} N^T bA\,dx \qquad N = \left[1-\frac{x}{L^e} \quad \frac{x}{L^e}\right]$$

$$= \int_0^{L^e} \begin{bmatrix} 1-\frac{x}{L^e} \\ \frac{x}{L^e} \end{bmatrix} bA\,dx = \int_0^{L^e} \begin{bmatrix} x - \frac{x^2}{2L^e} \\ \frac{x^2}{2L^e} \end{bmatrix} bA \Bigg|_0^{L^e}$$

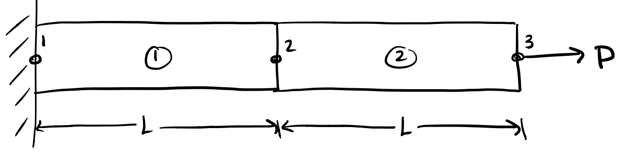
$$f^{body} = \underbrace{bAL^e \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}}$$ for 1 linear element

so each node is essentially getting 1/2 of total body force → linear element splits force equally.

total body force in element: $\underbrace{b}_{\text{force/vol}}\underbrace{AL^e}_{\text{vol}}$

Example: Element Assembly

– how does this weak form based approach work for more than 1 element?



Find:
① K-matrix / body forces for each element
② assembley of equations

A, E, b

Element ①: $K^{(1)} = \int_0^L B^T AEB\,dx = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$f^{body\,(1)} = \int_0^L N^T bA\,dx = \frac{bAL}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

shape function: $N^{(1)} = \left[1-\frac{x}{L} \quad \frac{x}{L}\right]$

Element ②: $K^{(2)} = \int_L^{2L} B^T AEB\,dx =$

$f^{body\,(2)} = \int_L^{2L} N^T bA\,dx =$

Question: $N^{(2)} = \left[1-\frac{x}{L} \quad \frac{x}{L}\right]$?
Can we use same shape function?
◦ shape function should have value = 1 at node, ∅ @ other node

$x_2 = L$
$x_3 = 2L$

$N(x_2) = \left[1-\frac{L}{L} \quad \frac{L}{L}\right] = [0 \quad 1]$ ... so zero @ node, 1 at the other
so backwards of what we want
... should be $[1 \quad 0]$

$N(x_3) = \left[1-\frac{2L}{L} \quad \frac{2L}{L}\right] = [-1 \quad 2]$
... but should have $[0 \quad 1]$

→ so need new shape function for each element!
◦ easy for simple geometries, but not in general
* ◦ would be best if had same shape function for each element *

define a new shape function: $N(x) = \left[2-\frac{x}{L} \quad \frac{x}{L}-1\right]$ ← valid for element 2

$\therefore B = \left[-\frac{1}{L} \quad \frac{1}{L}\right] \Rightarrow K^{(2)} = \int_L^{2L}\begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} AE \left[-\frac{1}{L} \quad \frac{1}{L}\right] dx = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$f^{body\,(2)} = \int_2^{2L} \begin{bmatrix} 2-\frac{x}{L} \\ \frac{x}{L}-1 \end{bmatrix} bA\,dx = \frac{bAL}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Ex. cont'd

Assembly is same as before

$$\left(\underbrace{\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{element 1}} + \underbrace{\frac{AE}{L}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}}_{\text{element 2}} \right) d = \underbrace{\begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix}}_{\text{external force}} + \underbrace{\frac{bAL}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\text{element 1}} + \underbrace{\frac{bAL}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\text{element 2}}$$

$$\frac{AE}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} d = \begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix} + \frac{bAL}{2}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

node 2 coupling terms are bigger

we've now gotten our FE equations 3 different ways:

1. direct stiffness
2. potential energy
3. start w/ DE, come up w/ equivalent weak form, approximate weak form w/ FE, and eventually getting the finite element equations