Numerical Integration

Here's where we are so far:

① Weak form:

$$f^{body} = \int_0^L N^T bA\,dx$$

② Isoparametric elements/mapping:

$$f^{body} = \int_{-1}^{1} \underbrace{N^T(\xi)\, bA \overbrace{\frac{dx}{d\xi}}^{\text{Jacobian}}}\, d\xi$$

↑ "Parent" domain (ξ) for all element integrals

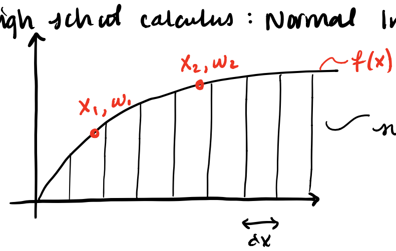
① All integrals have same limits of integration

② Integrand is a polynomial of known order ⇒ order is known because as the user, you choose $N(\xi)$

So, this enables automated, numerical evaluation of element integrals for all weak form terms, i.e. computer does it for you

→ Numerical integration <=> "quadrature" ... use these terms interchangeably

→ high school calculus : Normal Integral : $\int_a^b f(x)\,dx$



~ sum up all of the pieces under the curve to get the area

... but numerical integration is different :

Numerical Integration:
- → don't sum up all dx's
- → have fixed # of $\underbrace{\text{sampling points}}_{x_1,\ x_2}$ + $\underbrace{\text{weights}}_{w_1,\ w_2}$

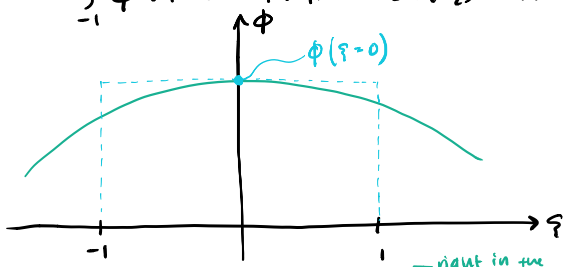
$$\int_a^b f(x)\,dx = \sum_{i=1}^{NP} f(x_i)\,w_i \qquad \text{where } NP = \#\text{ of sampling points}$$

⇒ Numerical integration is discrete, we will use "Gaussian Quadrature"
⇒ Use sampling points + weights to minimize integration error
⇒ Error = 0 for polynomial integrands assuming use sufficient # of points/weights

1-D Example

$$I = \int_{-1}^{1} \phi \, d\xi \approx w_1 \phi(\xi_1) + w_2 \phi(\xi_2) + \ldots + w_n \phi(\xi_n)$$

with n sampling pts, the numerical integration is the sum of all of the functions at each sampling pt., multiplied by the wt.



right in the middle

it's 2 bc the weight should sample the entire length (so −1 to 1, which is the length of the ξ domain (=2))

↳ Q: why weights not ranging from 0 to 1?

A: ex. $\int_{-1}^{1} c \, d\xi = 2c$

• use 1 quadratic point: $\xi_1 = 0$, $w_1 = 2$

$$\int_{-1}^{1} \phi \, d\xi \approx \phi(0) \cdot 2 = \boxed{2\phi(\xi=0)}$$

⇒ more specific example:

Integrate a 3rd order polynomial: $\phi = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$

$$\int_{-1}^{1} \phi \, d\xi = \int_{-1}^{1} [a_0 + \cancel{a_1\xi}^{0} + a_2\xi^2 + \cancel{a_3\xi^3}^{0}] \, d\xi$$

$$= (a_0\xi + a_2(1/3)\xi^3 \Big|_{-1}^{1} = \boxed{2a_0 + (2/3)a_2} \leftarrow \text{exact!}$$

⇒ **1-Point Quadrature:**

$$\int_{-1}^{1} (a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3) \, d\xi \approx [a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3]\underbrace{(2)}_{\text{weight}}$$

$= 2a_0$ ← not exact, but can integrate constant correctly with 1-pt quadrature

... tells us we'll need more sampling pts

⇒ **2-Point Quadrature:**

$\xi_1 = \frac{1}{\sqrt{3}}$, $\xi_2 = -\frac{1}{\sqrt{3}}$

$w_1 = 1$, $w_2 = 1$ → $w_1 + w_2 = 2$, good because length of ξ domain $= 2$

$$\int_{-1}^{1} (a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3) \, d\xi \approx [a_0 + \cancel{a_1\underbrace{\left(\tfrac{1}{\sqrt{3}}\right)}_{\xi_1}} + a_2\left(\tfrac{1}{\sqrt{3}}\right)^2 + \cancel{a_3\left(\tfrac{1}{\sqrt{3}}\right)^3}]\underbrace{(1)}_{w_1}$$

$$\int_{-1}^{1} (a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3) \, d\xi \approx [a_0 + \cancel{a_1\underbrace{\left(\tfrac{-1}{\sqrt{3}}\right)}_{\xi_2}} + a_2\left(\tfrac{-1}{\sqrt{3}}\right)^2 + \cancel{a_3\left(\tfrac{-1}{\sqrt{3}}\right)^3}]\underbrace{(1)}_{w_2}$$

... we can see immediately that the odd terms will cancel

$$= \boxed{2a_0 + (2/3)a_2} \leftarrow \text{exact! same as before}$$

What's the point of this?

1. If have correct # of points, can integrate polynomials of any order exactly
2. If not right # of points, can't integrate exactly
3. What if use more points than necessary?
   ↳ we'll come back to this...

⇒ Thinking ahead: (in an actual FEM code)

① loop over each element (for $i = 1 : NE$)

② for each element, loop over each integration point (for $j = 1 : QP's$)

③ for each quadrature point, calculate k or f^{body}

i.e.

$$f_i^{body} = f_i^{body} + \sum_{j=1}^{QP's} N^T(\xi_j) bA \frac{dx}{d\xi} w_j$$

NE : # of elements

QP : # of integration parts/element

Deriving Gaussian Quadrature (use 2 points for example)

$$I = \int_a^b f(x)\,dx \approx C_1 f(x_1) + C_2 f(x_2)$$

all are unknown $\begin{cases} C_1, C_2 = \text{weights} \\ x_1, x_2 = \text{quadrature pt. locations} \end{cases}$

Let's consider 3rd order polynomial

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$$

$$\int_a^b f(x)\,dx = \int_a^b (a_0 + a_1 x + a_2 x^2 + a_3 x^3)\,dx$$

$$\boxed{= a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right)}$$ exact

Now do numerical integration with 2 points:

$$\int_a^b f(x)\,dx \approx C_1 f(x_1) + C_2 f(x_2)$$

$$\approx C_1(a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3) + C_2(a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^3)$$

regroup

$$\approx a_0(C_1 + C_2) + a_1(C_1 x_1 + C_2 x_2) + a_2(C_1 x_1^2 + C_2 x_2^2) + a_3(C_1 x_1^3 + C_2 x_2^3)$$

compare a_0, a_1, a_2, a_3 terms to above...

so we get:

$$\left.\begin{aligned} a_0 &: b-a = C_1 + C_2 \\ a_1 &: (b^2-a^2)/2 = C_1 x_1 + C_2 x_2 \\ a_2 &: (b^3-a^3)/3 = C_1 x_1^2 + C_2 x_2^2 \\ a_3 &: (b^4-a^4)/4 = C_1 x_1^3 + C_2 x_2^3 \end{aligned}\right\}$$ 4 nonlinear equations, 4 unknowns ... theoretically solvable

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solving:

$$\left.\begin{aligned} C_1 &= C_2 = (b-a)/2 \\ x_1 &= [(b-a)/2][-1/\sqrt{3}] + (b+a)/2 \\ x_2 &= [(b-a)/2][1/\sqrt{3}] + (b+a)/2 \end{aligned}\right\} \Rightarrow$$

in our case, need to map to isoparametric domain: $a = -1$, $b = -1$

... substitute this in:

$C_1 = C_2 = 1$, $C_1 + C_2 = 2$ ✓

$x_1 = -\frac{1}{\sqrt{3}}$, $x_2 = +\frac{1}{\sqrt{3}}$

⇒ shows that need only 2 points to integrate a 3rd order polynomial exactly

⇒ what order polynomial can you integrate exactly, with

① 1 point? → $\int_a^b f(x)\,dx \approx C_1 f(x_1)$ → 1st order polynomial (2 unknowns, need 2 equations for a_0 & a_1)

② 3 points? → 5th order polynomial

⇒ how many points do you need to integrate a quadratic polynomial exactly? → 2 points

Numerical Integration Example

$$f^{body} = \int_0^L N^T(x)\, bA\, dx$$

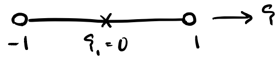
$$= \int_{-1}^{1} N^T(\xi)\, bA \frac{dx}{d\xi} d\xi$$

→ Assume $N(\xi)$ is linear, how many integration points needed to evaluate element integral exactly?

$$\approx \sum_{i=1}^{QP's} \underbrace{N^T(\xi_i)\, bA \frac{dx}{d\xi}}_{} w_i$$

polynomial order = 1st order → 1 pt integration is sufficient

$\xi_1 = 0, \quad w_1 = 2$



$$N(\xi) = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix}$$

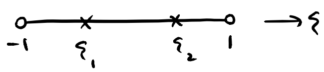
→ evaluate this:

$$\approx \underbrace{\begin{bmatrix} \frac{1-0}{2} \\ \frac{1+0}{2} \end{bmatrix}}_{N^T(\xi_1 = 0)} bA \underbrace{\frac{L}{2}}_{\frac{dx}{d\xi}} \underbrace{(2)}_{w_1} = f^{body} = \frac{bAL}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

← same as before!

⇒ What if we had used 2-point integration? (which is overkill)

$\xi_1 = -\frac{1}{\sqrt{3}}, \quad \xi_2 = +\frac{1}{\sqrt{3}} \qquad w_1 = w_2 = 1$



$$f^{body} = \frac{bAL}{2}\begin{bmatrix} \frac{1-(-1/\sqrt{3})}{2} \\ \frac{1+(-1/\sqrt{3})}{2} \end{bmatrix}(1) + \begin{bmatrix} \frac{1-(1/\sqrt{3})}{2} \\ \frac{1+(1/\sqrt{3})}{2} \end{bmatrix}(1) = \frac{bAL}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

{... same as before, just made computer work harder...}

Info for first ANSYS lab:

- * recommend downloading student version vs. CITRIX
- * don't worry too much about formatting / rubric