Solution Procedures for Nonlinear FEM

① Explicit time integration (dynamic problems)

→ same as for linear, dynamic problems

② Implicit methods (equilibrium solutions)

→ focus on these

Linear FEM: $Kd = F^{ext}$

→ easy to solve because K is constant

$K = \int B^T AEB\,dx$ ... young's modulus E is constant for linear problems

Nonlinear FEM: $Ma = f^{ext} - f^{int}$

if static, $Ma = 0$, so $f^{int} = f^{ext}$

$f^{int} = \int B_0^T P\,dx$ ← displacement d not explicitly represented

$f^{ext} = \int N^T bA\,dx$ + applied force

↳ $f^{int} = \int B_0^T \underbrace{P(d)}\,dx$

P is a nonlinear function of d

→ so need to find a linearized model to solve for d incrementally

residual = "error" = $r(d^{n+1}, t^{n+1}) = f^{int}(d^{n+1}, t^{n+1}) - f^{ext}(d^{n+1}, t^{n+1})$

→ Goal: find d^{n+1} such that $r = 0$ which means $f^{int}(d^{n+1}, t^{n+1}) = f^{ext}(d^{n+1}, t^{n+1})$

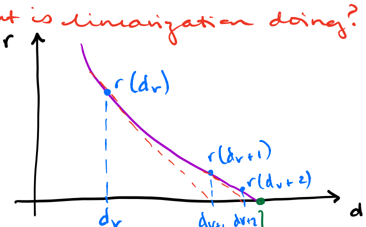
→ Take Taylor Series Expansion, to linearize, of the residual about current value of d so assume d^n from previous time step is known

$$r(d_{r+1}, t^{n+1}) = r(d_r) + \underbrace{\frac{\partial r(d_r)}{\partial d}}\Delta d + O(\Delta d^2)$$

$\Delta d = d_{r+1} - d_r$

↳ $\frac{\partial r}{\partial d} = A =$ "Jacobian Matrix"

→ What is linearization doing?



r: iteration count

solution is when residual crosses axis, in which r = 0

↳ $r + A\Delta d = 0$ ... this is the linearized model of nonlinear equation

$A\Delta d = -r$, so $\Delta d = A^{-1}[-r(d_r, t^{n+1})]$

then $d_{r+1} = d_r + \Delta d$

→ then check for convergence, if criteria isn't met

$$r(d_{r+1}) + \frac{\partial r(d_{r+1})}{\partial d}\Delta d + O(d^2) = 0 = r(d_{r+2}, t^{n+1})$$

$r(d_{r+1}) + A\Delta d = 0$

$\Delta d = A^{-1}[-r(d_{r+1}, t^{n+1})]$

$d_{r+2} = d_{r+1} + \Delta d$

→ typically iterate until $\Delta d \approx 0$, within some tolerance i.e. $\|\Delta d\| < 1e\text{-}6$

(done by computer so can never get fully to zero)

$$A = \frac{\partial r}{\partial d} = \underbrace{\frac{\partial f^{int}}{\partial d}}_{\text{"tangent stiffness"}} - \underbrace{\frac{\partial f^{ext}}{\partial d}}_{\text{"load stiffness"}}$$

↳ typically = 0 unless have non conservative external load

$$A \Delta d = -r_v$$
$$(K^{int}_v - K^{ext}_v)\Delta d = (f^{int}_v - f^{ext}_v)$$

Flowchart for equilibrium solution

① initial conditions, $d^0 = 0$, σ^0, $t = 0$

② Newton iterations for each load increment

→ calculate $f^{int} - f^{ext}$

→ compute $A = K^{int} - K^{ext}$

→ modify A for boundary conditions

→ solve $\Delta d = A^{-1} r$

→ $d_{new} = d_{new} + \Delta d$

→ check convergence, repeat if needed

→ only open question is how to calculate K^{int}

Linearization of internal forces f^{int} to set $K^{int} = \frac{\partial f^{int}}{\partial d}$

→ do this by relating small changes in force df^{int} to small changes in displacement dd

→ equivalently, $dr^{(?)}$ by relating $\dot{f}^{int}$ to $\dot{d}$

Total Lagrangian

$$f^{int} = \int_{\Omega_0} B^T P \, d\Omega_0 \qquad B_0 = \frac{dN}{dX}$$

Take time derivative: $\dot{f}^{int} = \int_{\Omega_0} B_0^T \dot{P} \, dX$

stress relationship: $P = SF^T$

so: $\dot{P} = \dot{S}F^T + S\dot{F}^T$

so: $\dot{f}^{int} = \int_{\Omega_0} B_0^T (\dot{S}F^T + S\dot{F}^T) \, dX$

→ rate (increment) of internal force has 2 parts:

① $\dot{f}^{int} = \int_{\Omega_0} B_0^T (\dot{S}F^T) \, dX$ ← depends on rate of stress ($\dot{S}$), so depends on material response

← leads to K^{mat} = material tangent stiffness

$$\dot{f}^{mat} = \int_{\Omega_0} B_0^T \dot{S} F^T \, dX$$

② $\dot{f}^{int} = \int_{\Omega_0} B_0^T (S\dot{F}^T) \, dX$ ← contains current stress (S), but also rate of change of F

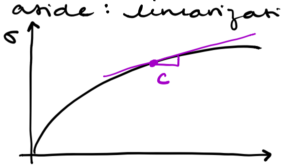
← F measures changes in stretching and rotation, so changes in geometry

← so leads to K^{geo} = geometric tangent stiffness

$$\dot{f}^{geo} = \int_{\Omega_0} B_0^T S \dot{F}^T \, dX$$

overall: $\dot{f}^{int} = \dot{f}^{geo} + \dot{f}^{mat}$

Brief aside: linearization of stress



$S = CE$

green strain

2nd P-K stress

tangent modulus

$\Rightarrow$ so: $\dot{S} = C\dot{E}$

also: $E = B_0 d$

so: $\dot{S} = C B_0 \dot{d}$

so $\dot{f}^{mat} = \int_{\Omega_0} B_0^T \dot{S} F^T d\Omega_0$

so $K^{mat} = \underbrace{\int_{\Omega_0} B_0^T C B_0 F^T d\Omega_0}_{\text{same as linear FEM if:}}$, $\dot{f}^{mat} = \int_{\Omega_0} B_0^T C B_0 F^T d\Omega_0 \dot{d}$

① tangent stiffness C is a constant (Young's Modulus)

② small changes in geometry, so $F^T \approx 1$

$\dot{f}^{geo} = \int_{\Omega_0} B_0^T S \dot{F}$

$\dot{f}^{geo} = \int_{\Omega_0} B_0^T S B_0 \dot{d}$

$$\boxed{K^{geo} = \int_{\Omega_0} B_0^T S B_0 d\Omega}$$

this term not used in linear analysis

displacement, u, is difference between current and reference pt.

$u = x - \mathbf{X}$ so $x = \mathbf{X} + u$

$$F = \frac{dx}{d\mathbf{X}} = \frac{d(u+\mathbf{X})}{d\mathbf{X}} = \frac{du}{d\mathbf{X}} + 1$$

$$\dot{F} = \frac{d}{dt}\left(\frac{du}{d\mathbf{X}} + 1\right)$$, where $u = Nd$

$$= \frac{d}{dt}(B_0 d + 1)$$ $\frac{du}{d\mathbf{X}} = B_0 d$

$\dot{F} = B_0 \dot{d}$

... back to linearized equilibrium solution

$A\Delta d = -r$ ... assume $K^{ext} = 0$

$K^{int}\Delta d = f^{ext} - f^{int}$

$(K^{mat} + \underbrace{K^{geo}}_{}) \Delta d = f^{ext} - f^{int}$

new term due to nonlinear strains $\rightarrow$ otherwise identical to linear FEM