Minimum Potential Energy

→ this will lead to same finite element equations ie $Kd = F^{ext}$

let's define: π_p = total potential energy of system
u = internal strain energy
Ω = external forces potential energy (work done by external forces)

$$\pi_p = u + \Omega$$

the idea: write this potential energy as a function of all the nodal displacements

$$\pi_p = \pi_p(d_{1x}, d_{2x}, \ldots, d_{nx})$$

units of energy: $J = N \cdot m$

to minimize a function: take derivative of energy $\frac{d\pi_p}{dd} \Rightarrow \frac{N \cdot m}{m} = N$

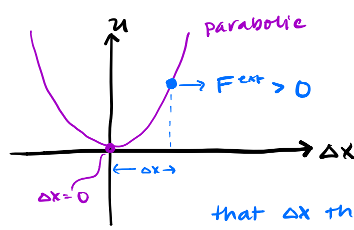
when minimize PE of system, get $Kd = F^{ext}$

<u>Principle of Minimum Potential Energy</u>: of all geometrically possible shapes a body can take, the true one, corresponding to satisfaction of stable equilibrium of the body, is identified by a minimum value of the total potential energy

* the shape it takes is the one with lowest PE it can find

<u>ex.</u>

$u = \frac{1}{2} K \Delta x^2$



that Δx that the spring takes is the lowest possible energy state for the work that you put in

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Spring PE + Finite Elements

$\pi_p = u + \Omega$

$du = F\,dx$ $\quad du$ = change in PE
assume linear spring, so $F = Kx$ $\quad \ldots \; du = Kx\,dx$

$\Rightarrow$ total PE $\quad u = \int_0^x Kx\,dx = \frac{1}{2}Kx^2$

$\Omega = -Fx$

so, $\pi_p = \frac{1}{2}Kx^2 - Fx$

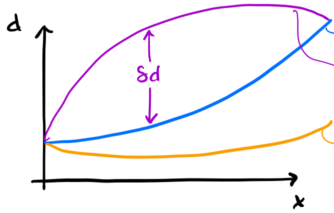
Now, we want to minimize PE, so typically $\pi_p(d_{1x}, d_{2x}, \ldots)$

Do a variation: $\delta \pi_p = \frac{\partial \pi_p}{\partial d_{1x}} \delta d_1 + \frac{\partial \pi_p}{\partial d_{2x}} \delta d_{2x} + \ldots + \frac{\partial \pi_p}{\partial d_{nx}} \delta d_{nx}$

chain rule

<u>Principle</u>: equilibrium exists when d_{ix} define a state such that $\delta \pi_p = 0$ for any arbitrary admissible variations δd_{ix} from equilibrium

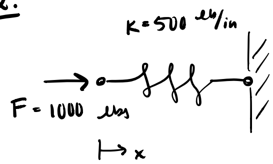
<u>Admissible Variation</u>: (not as important now, but will be) one where finite elements displacement field satisfies the boundary conditions, and is continuous



Any of $\delta d_{ix} = 0$, to satisfy $\delta \pi_p = 0$, all coefficients associated with δd_{ix} should be zero independently

Impose $\frac{\partial \pi_p}{\partial d_{ix}} = 0$ ... results in n equations to solve for n values of d_{nx} $(i = 1, 2, 3, \ldots n)$

<u>ex.</u>



$\pi_p = \mathcal{U} + \Omega$

$\mathcal{U} = \frac{1}{2} k x^2, \quad \Omega = -F_x$

$\pi_p = \frac{1}{2} k x^2 - F_x$

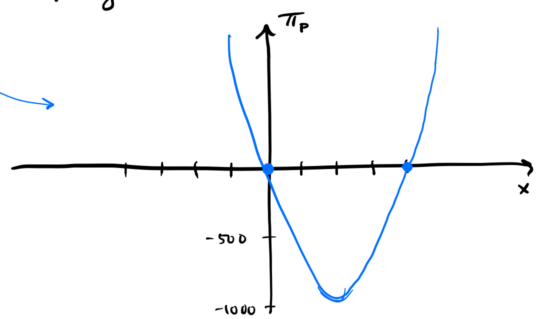
$\delta \pi_p = 0 = \frac{\partial \pi_p}{\partial x} \delta x$

δx is arbitrary may not be $\emptyset$
then ensure $\delta \pi_p = 0$ by saying $\frac{\partial \pi_p}{\partial x} = 0$

$\pi_p = 250x^2 - 1000x$

$\frac{\partial \pi_p}{\partial x} = 0 = 500x - 1000$

$\therefore x = 2$ in (the equilibrium length)

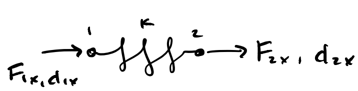


Derive spring element equations + stiffness matrix using PE

recall: $\pi_p = \frac{1}{2} k x^2 - F x$

write for this element:

$\pi_p = \frac{1}{2} k (d_{2x} - d_{1x})^2 - F_{1x} d_{1x} - F_{2x} d_{2x}$



$\frac{\partial \pi_p}{\partial d_{1x}} = 0 = \frac{1}{2} k (-2d_{2x} + 2d_{1x}) - F_{1x}$

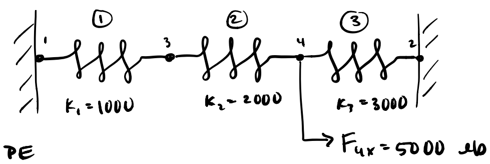
$\frac{\partial \pi_p}{\partial d_{2x}} = 0 = \frac{1}{2} k (2d_{2x} - 2d_{1x}) - F_{2x}$

Now have two equations with two unknowns, put in matrix form:

$$\underbrace{\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{2x} \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{2x} \end{bmatrix}}_{Kd = F^{ext}}$$

What is this telling us?
- one interpretation: physically meaningful that displacements are ones that minimize the PE of the structure

Example:



Find: ① total PE
② minimized value

① Total PE:

$$\pi_p = \sum_{e=1}^{3} \pi_p^{(e)} = \frac{1}{2} k_1 (d_{3x} - d_{1x})^2 - f_{1x}^{(1)} d_{1x} - f_{3x}^{(1)} d_{3x}$$
$$+ \frac{1}{2} k_2 (d_{4x} - d_{3x})^2 - f_{3x}^{(2)} d_{3x} - f_{4x}^{(2)} d_{4x}$$
$$+ \frac{1}{2} k_3 (d_{2x} - d_{4x})^2 - f_{4x}^{(3)} d_{4x} - f_{2x}^{(3)} d_{2x}$$

② minimize:

$$\frac{\partial \pi_p}{\partial d_{1x}} = 0 = -k_1 d_{3x} + k_1 d_{1x} - f_{1x}^{(1)}$$

$$\frac{\partial \pi_p}{\partial d_{2x}} = 0 = k_3 d_{2x} - k_3 d_{4x} - f_{2x}^{(3)}$$

$$\frac{\partial \pi_p}{\partial d_{3x}} = 0 = k_1 d_{3x} - k_1 d_{1x} - k_2 d_{4x} + k_2 d_{3x} - f_{3x}^{(1)} - f_{3x}^{(2)}$$

$$\frac{\partial \pi_p}{\partial d_{4x}} = 0 = k_2 d_{4x} - k_2 d_{3x} - k_3 d_{2x} + k_3 d_{4x} - f_{4x}^{(2)} - f_{4x}^{(3)}$$

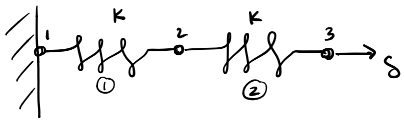
→ put in matrix form

$$\underbrace{\begin{bmatrix} k_1 & 0 & -k_1 & 0 \\ 0 & k_3 & 0 & -k_3 \\ -k_1 & 0 & k_1+k_2 & -k_2 \\ 0 & -k_3 & -k_2 & k_2+k_3 \end{bmatrix}} \begin{bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \\ d_{4x} \end{bmatrix} = \begin{bmatrix} f_{1x}^{(1)} \\ f_{2x}^{(3)} \\ f_{3x}^{(1)} + f_{3x}^{(2)} \\ f_{4x}^{(2)} + f_{4x}^{(3)} \end{bmatrix} = \begin{bmatrix} F_{1x} = 0 \\ F_{2x} = 0 \\ F_{3x} = 0 \\ F_{4x} = 5000 \end{bmatrix}$$

identical to obtained via direct stiffness
→ satisfies all conditions on K

↑ this example that we went over had prescribed forces.
... what if we had prescribed displacements instead?

ex. Prescribed Displacements



note that first node fixed in order to have invertible system

"work done on node 3 by element 2"

$$\pi_p = U + \Omega$$
$$= \tfrac{1}{2}k(d_2-d_1)^2 + \tfrac{1}{2}k(d_3-d_2)^2 - f_1^{(1)}d_1 - f_2^{(1)}d_2 - f_2^{(2)}d_2 - f_3^{(2)}d_3$$

we want to minimize the potential energy by each one individually ...

matrix form

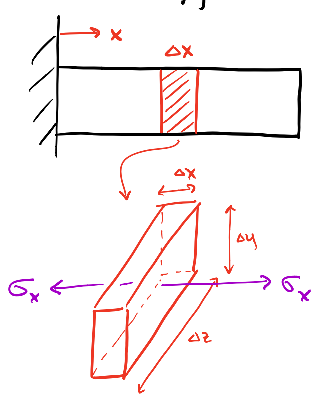
$$\begin{cases} \dfrac{\partial \pi_p}{\partial d_1} = 0 = -kd_2 + kd_1 - f_1^{(1)} \\ \dfrac{\partial \pi_p}{\partial d_2} = 0 = 2kd_2 - kd_1 - kd_3 - f_2^{(1)} - f_2^{(2)} \\ \dfrac{\partial \pi_p}{\partial d_3} = 0 = kd_3 - kd_2 - f_3^{(2)} \end{cases}$$

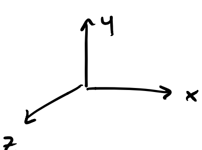
$$\rightarrow \underbrace{\begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}}_{K \text{ is independent of boundary conditions}} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} + f_2^{(2)} \\ f_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow$$

→ solving is same as before, so only need to solve for d_2:

$$-k\cancel{d_1}^{\,0} + 2kd_2 - kd_3 = 0$$
$$2kd_2 = kd_3 = k\delta \quad \ldots \quad \underline{d_2 = \delta/2}$$

Potential Energy + Connections to Finite Elements





strain

Displacement of face at x: $\Delta x\, \varepsilon_x$

Displacement of face at $x + \Delta x$: $\Delta x(\varepsilon_x + d\varepsilon_x)$

Total change in displacement: $\Delta x\, d\varepsilon_x$

$$\pi_p = U + \Omega$$
$$dU = \underbrace{\sigma_x(\Delta y)(\Delta z)}_{\text{force}} \times \underbrace{\Delta x\, d\varepsilon_x}_{\text{displacement}} \Rightarrow dU = \sigma_x\, d\varepsilon_x\, dV$$

dV - volume = $\Delta x \Delta y \Delta z$

For entire bar, integrate strain energy over the volume and the strain ...

For entire bar : $U = \int dU = \int_{\forall} \left[\int_0^{\varepsilon_x} \sigma_x \, d\varepsilon_x \right] d\forall$

assume material is linear elastic (Hooke's Law): $\sigma_x = E\varepsilon_x$

so :

$$U = \int_{\forall} \left[\int_0^{\varepsilon_x} E\varepsilon_x \, d\varepsilon_x \right] d\forall$$

$$U = \int_{\forall} \tfrac{1}{2} E \varepsilon_x^2 \, d\forall = \tfrac{1}{2} \int_{\forall} \sigma_x \varepsilon_x \, d\forall$$

internal work

→ now consider work by external forces (Ω)

$$\Omega = -\int_{\forall} \underbrace{X_b U \, d\forall}_{} - \int_{S} \underbrace{T_x U_s \, ds}_{} - \underbrace{\sum_{i=1}^{M} f_{ix} d_{ix}}_{}$$

volume integral

surface integral

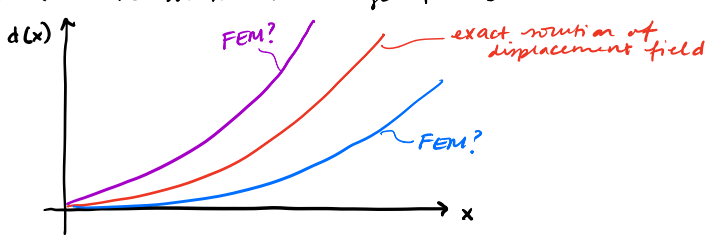
work done by applied nodal forces (what we've had all along)

X_b = "body forces" that act over entire $\forall$
ex. gravity

T_x = "surface loading"
ex. pressure, friction,
distributed loads (beam bending),
contact forces (normal)

→ most loading conditions are acting over surfaces or at specific points, not over entire volume

→ reminder: FE solution minimizes the PE



Is your FE solution going to start from above or below the exact and then approach exact from above or below? In other words, does FEM over- or under-estimate the potential energy?

→ FEM is "stiff" → small finite # of degrees of freedom

→ as you use more element/nodes FEM gets more flexible, less stiff

$$K d = F$$

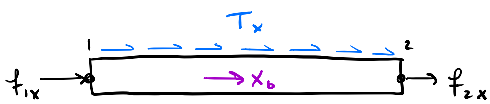
larger in FEM than reality, causing d to be smaller

... so the blue solution is correct.

more elements you have, more degrees of freedom, less constraints, more flexible, less stiffness!

Connecting Potential Energy to FEM

General forces on a 1D bar: (assume length L, cross-sectional area A)



X_b = gravity
T_x = surface load / traction

$$\pi_p = \frac{A}{2}\int_0^L \sigma_x \varepsilon_x \, dx - f_{1x} d_{1x} - f_{2x} d_{2x} - \int_S U_s T_x \, dS - \int_{\forall} X_b U \, d\forall$$

typically integrate over $\forall$ but pulling out the area A

displacements on surfaces of the structure

→ assume finite element approximation of displacement field
Do this using shape functions

$$\underbrace{U(x)}_{\substack{\text{displacement field} \\ 1\times1}} = \underbrace{N(x)}_{\substack{\text{shape function} \\ 1\times2}} \underbrace{d}_{\substack{\text{nodal displacement} \\ 2\times1}}$$

⇒ N might look like this: $N = \left[1 - \frac{x}{L} \quad \frac{x}{L}\right]$ $\quad d = \begin{bmatrix} d_{1x} \\ d_{2x} \end{bmatrix}$

product gives scalar value

$$U_s(x) = N_s(x) d$$

Strain: $\varepsilon_x = \frac{dU}{dx} = \frac{dN}{dx} d = Bd$

where B is $\frac{dN}{dx}$

$$\underbrace{\sigma_x}_{1\times1} = \underbrace{E}_{1\times1}\underbrace{\varepsilon_x}_{1\times1} = \underbrace{E}_{1\times1}\underbrace{B}_{1\times2}\underbrace{d}_{2\times1}$$

→ substitute these FEM approximations for $U, \varepsilon_x, \sigma_x$ into total PE:

$$\pi_p = \frac{A}{2}\int_0^L (EBd)^T Bd \, dx - f_{1x} d_{1x} - f_{2x} d_{2x} - \int_S (N_s d)^T T_x \, dS - \int_{\forall} (Nd)^T X_b \, d\forall$$

transpose some stuff!

$$\pi_p = \frac{A}{2}\int_0^L \underbrace{\underbrace{d^T}_{1\times2} \underbrace{B^T}_{2\times1} \underbrace{E}_{1\times1} \underbrace{B}_{1\times2} \underbrace{d}_{2\times1}}_{1\times1} \, dx - f_{1x} d_{1x} - f_{2x} d_{2x} - \int_S d^T N_s^T T_x \, dS - \int_{\forall} d^T N^T X_b \, d\forall$$

scalar so transpose has no effect

1x1 → energy in 1-D is a scalar.

→ d not a function of x, so can pull out of integral
→ B also not a function of x, because a constant → $N = \left[1 - \frac{x}{L} \quad \frac{x}{L}\right]$, $B = \frac{dN}{dx} = \left[-\frac{1}{L} \quad \frac{1}{L}\right]$

$$\pi_p = \underbrace{\overbrace{\frac{AL}{2} d^T B^T E B d}^{U}}_{\text{internal work}} - \overbrace{d^T f^{ext}}^{\Omega}$$

$$f^{ext} = \underbrace{P}_{\text{point/nodal forces}} + \underbrace{\int_S N_s^T T_x \, dS}_{\text{surface forces}} + \underbrace{\int_{\forall} N^T X_b \, d\forall}_{\text{body forces}}$$

two displacement degrees of freedom $\quad d = \begin{bmatrix} d_{1x} \\ d_{2x} \end{bmatrix}$

Minimize Π_p wrt d_{1x}, d_{2x}.

Define $U = \frac{AL}{2} d^T B^T E B d$

$$= \frac{AL}{2}\begin{bmatrix} d_{1x} & d_{2x}\end{bmatrix}\begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L}\end{bmatrix} E \begin{bmatrix} -\frac{1}{L} & \frac{1}{L}\end{bmatrix}\begin{bmatrix} d_{1x} \\ d_{2x}\end{bmatrix}$$

$$= \frac{EA}{2L}\left(d_{1x}^2 - 2d_{1x}d_{2x} + d_{2x}^2\right)$$

$$\Omega = d^T f^{ext} = d_{1x} f_{1x}^{ext} + d_{2x} f_{2x}^{ext}$$

2 equations, 2 unknowns

$$\left\{\begin{aligned} \frac{\partial \Pi_p}{\partial d_{1x}} &= 0 = \frac{AL}{2}\left[\frac{E}{L^2}(2d_{1x} - 2d_{2x})\right] - f_{1x}^{ext} \\ \frac{\partial \Pi_p}{\partial d_{2x}} &= 0 = \frac{AL}{2}\left[\frac{E}{L^2}(2d_{2x} - 2d_{1x})\right] - f_{2x}^{ext}\end{aligned}\right.$$

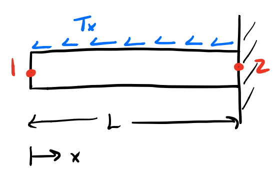
$\rightarrow$ matrix form $\Rightarrow$

$$\underbrace{\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1\end{bmatrix}\begin{bmatrix} d_{1x} \\ d_{2x}\end{bmatrix}}_{Kd} = \underbrace{\begin{bmatrix} f_{1x}^{ext} \\ f_{2x}^{ext}\end{bmatrix}}_{f^{ext}}$$

$$K = \frac{AE}{L} = \frac{(m^2)(N/m^2)}{m} = \frac{N}{m}$$ ... makes sense!

f^{ext} now has point, surface, and volume contributions

Ex. Axially Loaded Bar



1 linear element (2 nodes)
$A = 2\ in^2$
$E = 30\,(10^6)$ psi
$L = 60$ in
$T_x = -10x$ lb/in

Find:
1. Axial displacement
2. Axial stress

$$K = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1\end{bmatrix} = \frac{(2\ in^2)(30(10^6)\ lb/in^2)}{60\ in} = 10^6\begin{bmatrix} 1 & -1 \\ -1 & 1\end{bmatrix}\left(\frac{lb}{in}\right)$$

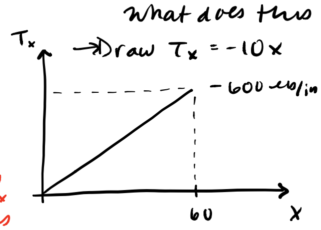
$\rightarrow$ use FEM to find distributed load (T_x)

surface force $\quad F_s = \int_0^L N^T T_x\,dx = \int_0^L \begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L}\end{bmatrix}(-10x)\,dx \qquad N = \begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L}\end{bmatrix}$

$$\begin{bmatrix} F_{1x} \\ F_{2x}\end{bmatrix} = \int_0^L \begin{bmatrix} -10x + \frac{10x^2}{L} \\ -\frac{10x^2}{L}\end{bmatrix}dx = \underbrace{\begin{bmatrix} -6000\ lbs \\ -12000\ lbs\end{bmatrix}}_{\text{what does this mean?}}$$

-6000 lbs ~ acting on node 1
-12000 lbs ~ acting on node 2

$F_s = \int_0^L N^T T_x\,dx$
N^T: choice (made by you, the user) impacts how T_x is distributed to the nodes



total load due to:
$T_x = \frac{1}{2}bh = \frac{1}{2}(60\,in)(-600\,\frac{lb}{in}) = -18\,000$ lbs

$$\begin{bmatrix} f_{1x} \\ f_{2x}\end{bmatrix} = \begin{bmatrix} -6000 \\ -12000\end{bmatrix} \rightarrow \text{sum} = -18\,000\ \text{lbs}$$

forces are conserved!

The discrete system conserves the forces that are applied

$Kd = F^{ext}$

$$K = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} d_{1x} \\ d_{2x} \end{bmatrix} = \begin{bmatrix} -6000 \\ -12000 \end{bmatrix} \qquad \frac{AE}{L} = 10^6$$

$$10^6\begin{bmatrix} 1 & \cancel{-1} \\ \cancel{-1} & \cancel{1} \end{bmatrix}\begin{bmatrix} d_{1x} \\ \cancel{d_{2x}} \end{bmatrix} = \begin{bmatrix} -6000 \\ \cancel{-12000} \end{bmatrix}$$

because $d_{2x} = 0$

$$10^6 d_{1x} = -6000, \quad \underline{d_{1x} = -.006 \text{ in} \;\; \& \;\; d_{2x} = 0}$$

so the displacement field between nodes varies linearly.

② calculate the Stress

$$\sigma_x = E\varepsilon_x \qquad u(x) = N(x)d$$

$$\varepsilon_x = \frac{du}{dx} = \underbrace{\frac{dN}{dx}}_{B} d \qquad \Rightarrow \qquad \underset{1\times1}{\sigma_x} = \underset{1\times2\;\;2\times1}{EBd}$$

$$N = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \rightarrow B = \frac{dN}{dx} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$$

$$\sigma_x = E\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{bmatrix} d_{1x} \\ d_{2x} \end{bmatrix} = \frac{E}{L}(d_{2x} - d_{1x}) = \frac{E}{L}(0 - -0.006) = \frac{E}{L}(0.006)$$

$$\underline{\sigma_x = 3000 \text{ psi}}$$

Questions:

① If displacement is linear in element, what is stress variation in element?
stress is <u>constant</u>, because depends on derivation of displacement

② Why is the stress positive, $\sigma_x > 0$?
bar is in <u>tension</u> so getting longer ∴ stress should be ⊕

③ Is this FEM solution exact?
likely solution is nonlinear but would have to work out analytically