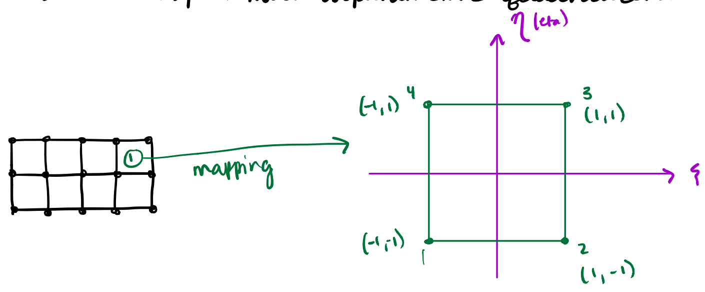
→ Finish discussion of 4-node isoparametric quadrilateral element

2D :



→ 2D analog of 1D isoparametric element

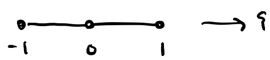
in 1D: $N(\xi) = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix}$

$N(\eta) = \begin{bmatrix} \frac{1-\eta}{2} & \frac{1+\eta}{2} \end{bmatrix}$

→ multiply various combinations of $N(\xi)$ and $N(\eta)$ together

$$\left.\begin{aligned} N_1(\xi,\eta) &= \tfrac{1}{4}(1-\xi)(1-\eta) \\ N_2(\xi,\eta) &= \tfrac{1}{4}(1+\xi)(1-\eta) \\ N_3(\xi,\eta) &= \tfrac{1}{4}(1+\xi)(1+\eta) \\ N_4(\xi,\eta) &= \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned}\right\} \text{4 shape functions corresponding to 4 nodes, respectively}$$

→ remember 1D quadratic elements in ξ?



→ can form a 2D, 9-node isoparametric element combining different combinations of $N(\xi)$ and $N(\eta)$ ... we'll use 9-node element for final MATLAB HW

(side note: Ω for integrating & Γ for integrating area)

2D Principle of Virtual Work/Weak Form

$$\underbrace{\int_\Omega \delta\varepsilon^T \sigma \, d\Omega}_{W^{int}} = \underbrace{\int_\Omega \delta u^T b \, d\Omega}_{W^{body}} + \underbrace{\int_\Gamma \delta u^T \bar{h} \, d\Gamma}_{W^{\text{traction forces}}} + \underbrace{\text{point loads}}_{\text{applied point forces}}$$

(distributed forces that act over boundary)

(doesn't really exist in 1D since a point)

δu = weight = $\begin{bmatrix} \delta u_x \\ \delta u_y \end{bmatrix}$ $\quad b$ = body force $\overset{?}{=} \begin{bmatrix} b_x \\ b_y \end{bmatrix}$ $\quad \sigma$ = stress = $\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$ $\quad \varepsilon$ = strain = $\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$

normal — shear

normal — shear

Strain-displacement Relationships: $\varepsilon_{xx} = \frac{\partial u_x}{\partial x}$, $\varepsilon_{yy} = \frac{\partial u_y}{\partial y}$, $\gamma_{xy} = 2\varepsilon_{xy} = \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$

Constitutive ($\sigma - \varepsilon$) relationship

poisson's ratio or v? double check!

→ plane stress

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

OR : $\sigma = C\varepsilon$

FEM Approximation in 2D

NEN = # of nodes per element

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \sum_{a=1}^{NEN} N_a(x,y) \begin{bmatrix} u_{xa} \\ u_{ya} \end{bmatrix}$$

→ FEM approximation for both x and y displacements, ie same as 1D except now have 2 displacement components (x+y) instead of 1 (x)

$$\varepsilon^e = \frac{\partial N}{\partial \underline{x}} d^e \rightarrow \underbrace{\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}}_{3\times 1} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} d_x \\ d_y \end{bmatrix}}_{2\times 1}$$

use FEM approximation = Nd

this is just saying:

$$\varepsilon_{xx} = \frac{\partial}{\partial x}(d_x), \quad \varepsilon_{yy} = \frac{\partial}{\partial y}(d_y), \quad \gamma_{xy} = \frac{\partial}{\partial y}(d_x) + \frac{\partial}{\partial x}(d_y)$$

x displacement y displacement

→ Now use FEM approximation for this:

$$\underbrace{\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}}_{3\times 1} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{3\times 2} \underbrace{\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}}_{2\times 6} \underbrace{\begin{bmatrix} dx_1 \\ dy_1 \\ dx_2 \\ dy_2 \\ dx_3 \\ dy_3 \end{bmatrix}}_{6\times 1}$$

3x1 (fixed size)

3x2 (fixed size)

2x6 (shape function matrix bc assumes 3-node element)
(2x8 → if Q4 element)
(2x18 → if Q9 element)

6x1
→ assumes 3 node element
(8x1 → if Q4 element)
(18x1 → if Q9 element)

these 2 matrices together give B-Matrix:

$$B = \underbrace{\begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix}}$$

3x6 if 3-nodes
(3x8 if Q4)
(3x18 if Q9)

→ back to weak form: $\delta W^{int} = \int_\Omega \delta\varepsilon^T \sigma \, d\Omega = \int_\Omega \underbrace{\delta\varepsilon^T}_{1\times 3} \underbrace{C}_{3\times 3} \underbrace{\varepsilon}_{3\times 1} \, d\Omega \;\; = 1\times 1$

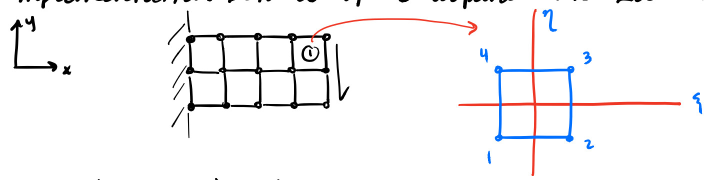
$$= \int_\Omega \delta d^T B^T C B d \, d\Omega$$

$$= \delta d^T \underbrace{\int_\Omega \underbrace{B^T}_{8\times 3} \underbrace{C}_{3\times 3} \underbrace{B}_{3\times 8} d\Omega}_{K} \, d \quad \text{(for Q4)} = 8\times 8$$

⇒ so $\delta W^{int} = \delta d^T K d$

... and for reference, in 1D:
$\delta W^{int} = \int \delta\varepsilon^T \sigma A \, dx$

Implementation Details of 2D Isoparametric Elements (FOR FINAL MATLAB PROJECT)



$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta)$

$N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$

$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta)$

$N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta)$

$$x(\xi,\eta) = \sum_{a=1}^{4} N_a(\xi,\eta)x_a, \quad y(\xi,\eta) = \sum_{a=1}^{4} N_a(\xi,\eta)y_a$$

→ we have $\frac{dN}{d\xi}$ and $\frac{dN}{d\eta}$, but need $\frac{dN}{dx}$ and $\frac{dN}{dy}$

→ recall 1D:

$$K = \int_0^L B^T AEB\,dx = \int_0^L \left(\frac{dN}{dx}\right)^T AE\,\frac{dN}{dx}\,dx$$

$$= \int_{-1}^{1} \underbrace{\left[\frac{dN}{d\xi}\frac{d\xi}{dx}\right]^T}_{=\frac{dN}{dx}} AE \underbrace{\left[\frac{dN}{d\xi}\frac{d\xi}{dx}\right]}_{=\frac{dN}{dx}} \underbrace{\left[\frac{dx}{d\xi}\right]}_{J} d\xi$$

→ do this in 2D:

→ do for each integration point!

$$\boxed{\frac{\partial N_a}{\partial \xi}} = \frac{\partial N_a}{\partial x}\boxed{\frac{\partial x}{\partial \xi}} + \frac{\partial N_a}{\partial y}\boxed{\frac{\partial y}{\partial \xi}}$$

$$\boxed{\frac{\partial N_a}{\partial \eta}} = \frac{\partial N_a}{\partial x}\boxed{\frac{\partial x}{\partial \eta}} + \frac{\partial N_a}{\partial y}\boxed{\frac{\partial y}{\partial \eta}}$$

known known known

physical coordinate

$$\frac{\partial x}{\partial \xi} = \sum_{a=1}^{4}\frac{\partial N_a}{\partial \xi}x_a \qquad \frac{\partial x}{\partial \eta} = \sum_{a=1}^{4}\frac{\partial N_a}{\partial \eta}x_a$$

$$\frac{\partial y}{\partial \xi} = \sum_{a=1}^{4}\frac{\partial N_a}{\partial \xi}y_a \qquad \frac{\partial y}{\partial \eta} = \sum_{a=1}^{4}\frac{\partial N_a}{\partial \eta}y_a$$

→ write this in matrix form:

assume Q4 element

$$\underbrace{\begin{bmatrix}\frac{\partial N_a}{\partial \xi}\\ \frac{\partial N_a}{\partial \eta}\end{bmatrix}}_{2\times 4} = \underbrace{\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}}_{\text{Jacobian }(2\times 2)} \underbrace{\begin{bmatrix}\frac{\partial N_a}{\partial x}\\ \frac{\partial N_a}{\partial y}\end{bmatrix}}_{2\times 4}$$

So,

$$\underbrace{\begin{bmatrix}\frac{\partial N_a}{\partial x}\\ \frac{\partial N_a}{\partial y}\end{bmatrix}} = \left[\text{Jacobian}\right]^{-1}\begin{bmatrix}\frac{\partial N_a}{\partial \xi}\\ \frac{\partial N_a}{\partial \eta}\end{bmatrix}$$

→ have a 2×4 matrix of shape function derivatives (2×9 for Q9)

→ need to assemble into B matrix, which is 3×8 for Q4, 3×18 for Q9

end up with: $K = \int_\Omega B^T C B\,d\Omega \Rightarrow \int_{-1}^{1}\int_{-1}^{1} B^T C B\,\mathrm{Det}(J)\,d\xi\,d\eta$

How to numerically integrate in 2D:

In 1D: $\int_0^L N^T b A\,dx = \int_{-1}^{1} N^T(\xi)\, bAJ\,d\xi \approx \sum_{i=1}^{NP} N^T(\xi_i)\, bAJ\, w_i$

In 2D: do integration by successive application of 1D quadrature rules

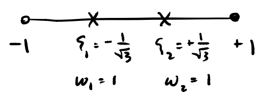
$$\int_{-1}^{1}\int_{-1}^{1} \phi(\xi,\eta)\,d\xi\,d\eta = \int_{-1}^{1}\left[\sum_i \phi(\xi_i,\eta)\,w_i\right]d\eta$$

$$= \sum_j \left[\sum_i \phi(\xi_j,\eta_j)\,w_j\right] w_j$$

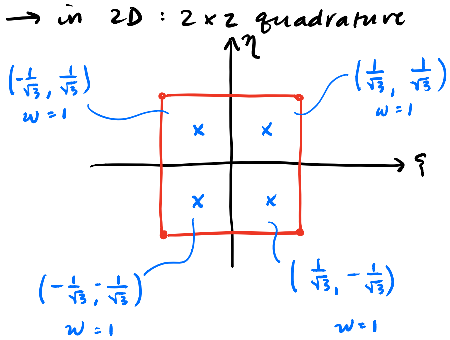
$$= \sum_i \sum_j \phi(\xi_i,\eta_j)\,w_i w_j$$

2D is, like shape functions, combination of 1D integration points

→ for example, in 1D if 2-pt. Quadrature:



→ in 2D: 2×2 quadrature



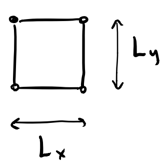
sum of weights = 4 (because the area in isoparametric domain is 4)

→ determinant of Jacobian = det(J) → used to map integrals from (x,y) to (ξ,η)

$$= \underbrace{\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta}}_{\text{rectangular/square}} - \underbrace{\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}}_{\leftarrow \text{ if mesh is distorted}}$$

Jacobian has 2 terms

→ assume in (x,y)



→ what is det(J)? $= \frac{L_x L_y}{4}$ → area ratio for $\frac{x}{\xi}$