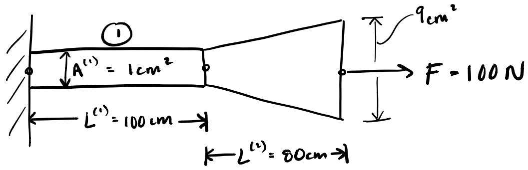
→ Varying Area Example

*Isoparametric elements



$E = 100\text{MPa} = 10^8 \text{ Pa}$

$A^{(2)} = \left(1 + \frac{x}{40}\right)^2$

assume that $x = 0$ at 100cm

* to keep simple, breaking into two areas

$$N_1(x) = \left[1 - \frac{x}{100} \quad \frac{x}{100}\right] \qquad B_1(x) = \frac{dN_1}{dx} = \left[-\frac{1}{100} \quad \frac{1}{100}\right]$$

$$K^{(1)} = \int_0^{100} B^T A E B\, dx = \int_0^{100} \begin{bmatrix} -\frac{1}{100} \\ \frac{1}{100} \end{bmatrix} (1)(10^8) \left[-\frac{1}{100} \quad \frac{1}{100}\right] dx$$

$$K^{(1)} = 10^6 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{E}{100} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

→ Now, for not constant area element #2

$$N_2(x) = \left[1 - \frac{x}{80} \quad \frac{x}{80}\right] \qquad B_2(x) = \frac{dN_2}{dx} = \left[-\frac{1}{80} \quad \frac{1}{80}\right]$$

$$K^{(2)} = \int_0^{80} B^T A E B\, dx = \int_0^{80} \begin{bmatrix} -\frac{1}{80} \\ \frac{1}{80} \end{bmatrix} \left(1 + \frac{x}{40}\right)^2 10^8 \left[-\frac{1}{80} \quad \frac{1}{80}\right] dx$$

$$= \frac{10^8}{6400} \int_0^{80} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \left(1 + \frac{2x}{40} + \frac{x^2}{1600}\right) dx$$

$$K^{(2)} = \frac{13E}{240} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

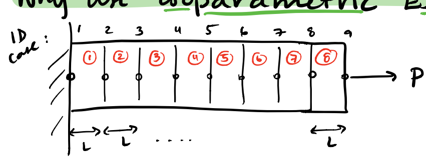
in terms of E

note that $K^{(2)} > K^{(1)}$ because of increasing area (stiffness becomes much larger)

assemble $K^{(1)}$ and $K^{(2)}$ together...

$$K^{(total)} = K^{(1)} + K^{(2)} = \frac{E}{240} \begin{bmatrix} 2.4 & 2.4 & 0 \\ -2.4 & 15.4 & -13 \\ 0 & -13 & 13 \end{bmatrix}$$

Why use Isoparametric Elements?



9 nodes, all elements length L

What do element integrals look like?

$$K^{(1)} = \int_0^{L} B^T A E B\, dx \qquad K^{(2)} = \int_L^{2L} B^T A E B\, dx \qquad K^{(3)} = \int_{2L}^{3L} B^T A E B\, dx \quad \ldots \text{ and so forth}$$

① limits of integration changing
② shape functions changing (see last example)

→ **GOAL**: ① integrate all element integrals over same domain
② constant shape functions

→ So create local coordinate system (ξ) ← psi

$$\underset{x_1}{\circ}\overset{L^e}{\text{———}}\underset{x_2}{\circ} \xrightarrow{\text{map}} \underset{-1}{\circ}\text{———}\underset{1}{\circ} \quad (\xi)$$

How to do this mapping?

$$\underset{x_A}{\circ}\overset{L^e}{\text{———}}\underset{x_{A+1}}{\circ} \xrightarrow{\text{map}} \underset{-1}{\circ}\text{———}\underset{1}{\circ} \quad (\xi)$$

Enforce linear field in ξ

$$\xi(x) = c_1 + c_2 x \qquad \begin{cases} [x_A, x_{A+1}] & x \\ [-1, 1] & \xi \end{cases}$$

$$-1 = c_1 + c_2 x_A$$

$$+1 = c_1 + c_2 x_{A+1}$$

solve for two constants

$$c_2 = -\frac{2}{(x_A - x_{A+1})} \qquad c_1 = -1 + \frac{2x_A}{(x_A - x_{A+1})} \qquad L^e = x_{A+1} - x_A$$

$$\underbrace{c_2 = \frac{2}{L_e} \qquad c_1 = -1 - \frac{2x_A}{L^e}}$$

substitute these back into ξ

$$\xi = \frac{2x - L^e - 2x_A}{L^e} = \frac{2x - (x_{A+1} - x_A) - 2x_A}{L^e} = \boxed{\frac{2x - x_A - x_{A+1}}{L^e} = \xi}$$

⇒ solve for x: $x = \dfrac{L^e\xi + x_A + x_{A+1}}{2} = \dfrac{(x_{A+1} - x_A)\xi + x_A + x_{A+1}}{2}$

$$x(\xi) = \tfrac{1}{2}x_A(1-\xi) + \tfrac{1}{2}x_{A+1}(1+\xi)$$

$$= N_1(\xi)x_A + N_2(\xi)x_{A+1} \quad \text{(i.e. } N_1 \text{ as function of psi times x sub A...)}$$

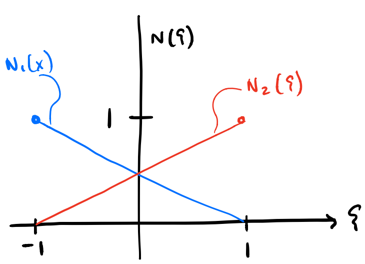
$$\boxed{x(\xi) = \sum_{A=1}^{2} N_A(\xi)x_A}$$

$N(\xi)$ are shape functions in isoparametric domain

$$N(\xi) = \left[\frac{1-\xi}{2} \quad \frac{1+\xi}{2}\right]$$

→ $N_1(\xi) + N_2(\xi) = \frac{1-\xi}{2} + \frac{1+\xi}{2} = \frac{1}{2} + \frac{1}{2} = 1$

→ linear functions of ξ



$$N_1(\xi) = \frac{1-\xi}{2}$$

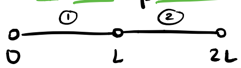
$$N_2(\xi) = \frac{1+\xi}{2}$$

⇒ takeaway: isoparametric shape functions have same properties as before!

$$x(\xi) = \left(\frac{1-\xi}{2}\right)x_A + \left(\frac{1+\xi}{2}\right)x_{A+1}$$

$$\frac{dx}{d\xi} = \tfrac{1}{2}x_{A+1} - \tfrac{1}{2}x_A = \boxed{\frac{L^e}{2} = \frac{dx}{d\xi}}$$ *important relationship

What is the practical impact of this?



Elem 1: $f^{body} = \int_0^L N_1^T bA\,dx$ ← as before, business
Elem 2: $f^{body} = \int_L^{2L} N_2^T bA\,dx$ as usual

If isoparametric:

Elem 1: $f^{body} = \int_{-1}^{1} N^T(\xi)\, bA \boxed{\frac{dx}{d\xi}}\, d\xi$ → "Jacobian" = $\frac{L^{e1}}{2}$

Elem 2: $f^{body} = \int_{-1}^{1} N^T(\xi)\, bA \boxed{\frac{dx}{d\xi}}\, d\xi$

same limits of integration — same shape function — Jacobian = $\frac{L^{e2}}{2}$

→ so only need nodal coordinates in x to do mapping

Integral mapping: $f^{body} = \int_0^L N^T(x)\, bA\,dx$ ← integral over L

$f^{body} = \int_{-1}^{1} N^T(\xi)\, bA \frac{dx}{d\xi}\, d\xi$

- limits of integration change
- still linear

integral over 2 but this is scaled by $2 \times \frac{L}{2}$
↳ becomes integral over L

both linear functions

Example: body force

already know: $\int_0^L N^T bA\,dx = \frac{bAL}{2}\begin{bmatrix}1\\1\end{bmatrix}$

now in parametric domain:

$$\int_0^L N^T(x) bA\,dx = \int_{-1}^{1} N^T(\xi)\, bA \frac{dx}{d\xi}\, d\xi = \int_{-1}^{1}\begin{bmatrix}\frac{1-\xi}{2}\\ \frac{1+\xi}{2}\end{bmatrix} bA \frac{L}{2}\, d\xi = \frac{bAL}{4}\int_{-1}^{1}\begin{bmatrix}1-\xi\\1+\xi\end{bmatrix} d\xi$$

$$= \frac{bAL}{4}\begin{bmatrix}\xi - \frac{\xi^2}{2}\\ \xi + \frac{\xi^2}{2}\end{bmatrix}\Bigg|_{-1}^{1} = \frac{bAL}{4}\begin{bmatrix}\frac{1}{2} - (-1-\frac{1}{2})\\ \frac{3}{2} - (-1+\frac{1}{2})\end{bmatrix} = \frac{bAL}{4}\begin{bmatrix}2\\2\end{bmatrix}$$

$$f^{body} = \frac{bAL}{2}\begin{bmatrix}1\\1\end{bmatrix}$$ → same as before

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Stiffness Matrix:

$$K = \int_0^L B^T AEB\,dx = \int_0^L \left(\frac{dN}{dx}\right)^T AE \left(\frac{dN}{dx}\right) dx = \int_{-1}^{1} \underbrace{\begin{bmatrix} \quad \end{bmatrix}}_{\text{what does this look like?}} d\xi$$

↳ hint: need to get dN/dx but have $N(\xi)$

my guess:

$$\int_{-1}^{1} \underbrace{\begin{bmatrix}-1/2\\1/2\end{bmatrix}}_{B^T} AE \underbrace{\begin{bmatrix}-1/2 & 1/2\end{bmatrix}}_{B=\frac{dN}{dx}} \underbrace{\frac{L}{2}}_{\frac{d\xi}{dx}}\, d\xi$$

$$\frac{dN}{dx} = \frac{d}{dx}\begin{bmatrix}\frac{1-\xi}{2} & \frac{1+\xi}{2}\end{bmatrix}\frac{dx}{d\xi}$$

$$\frac{dN}{dx} = \frac{dN}{dx}\frac{dx}{d\xi}$$

here's where we are: $K = \int_0^L \underbrace{\left(\frac{dN}{dx}\right)^T AE \left(\frac{dN}{dx}\right)} dx$

$$\int_{-1}^{1} \left[\qquad \right] \frac{dx}{d\xi} d\xi \qquad \Rightarrow \frac{dN}{dx} = \frac{dN}{d\xi}\frac{d\xi}{dx} \qquad N = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix}$$

$$\int_{-1}^{1} \left[\left(\frac{dN}{d\xi}\frac{d\xi}{dx}\right)^T AE \left(\frac{dN}{d\xi}\frac{d\xi}{dx}\right) \right] \frac{dx}{d\xi} dx$$

$$\left\{ \begin{array}{l} \frac{dN}{d\xi} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\ \frac{d\xi}{dx} = \frac{L}{2} \end{array} \right.$$

$$\frac{dN}{d\xi}\frac{d\xi}{dx} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$$

$$\Rightarrow \int_{-1}^{1} \left(\begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} AE \begin{bmatrix} \frac{1}{L} & \frac{1}{L} \end{bmatrix} \right) \frac{L}{2} \, d\xi$$

$$= \frac{AEL}{2} \int_{-1}^{1} \begin{bmatrix} 1/L^2 & -1/L^2 \\ -1/L^2 & 1/L^2 \end{bmatrix} d\xi$$

$$= \frac{AEL}{\cancel{2}} \int_{-1}^{1} \begin{bmatrix} 1/L^2 & -1/L^2 \\ -1/L^2 & 1/L^2 \end{bmatrix} (\cancel{2})$$

$$\boxed{K = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \longleftarrow \text{same as before!}$$

$$N_1(x) = \begin{bmatrix} 1 - \frac{x}{100} & \frac{x}{100} \end{bmatrix} \qquad B_1(x) = \frac{dN_1}{dx} = \begin{bmatrix} -\frac{1}{100} & \frac{1}{100} \end{bmatrix}$$

$$K^{(1)} = \int_0^{100} B^T AEB \, dx = \int_0^{100} \begin{bmatrix} -\frac{1}{100} \\ \frac{1}{100} \end{bmatrix} (1)(10^8) \begin{bmatrix} -\frac{1}{100} & \frac{1}{100} \end{bmatrix} dx$$