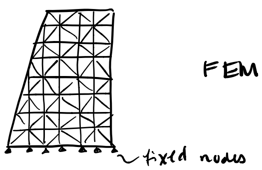
Typical problem areas of interest:

- structural analysis
- heat transfer
- fluid flow
- mass transport
- electromagnetic potential

Analytical solutions to complicated geometries and material properties are sometimes not attainable — we therefore use **numerical methods** such as the finite element method (FEM) to approximate the solution to these equations

Discretization: process of modeling a body by dividing it into an equivalent system of smaller bodies of units (finite elements) interconnected at points common to two or more elements (nodal points or nodes) and/or boundary lines and/or surfaces



FEM: instead of solving problem for entire body in one operation, formulate equations for each finite element, then combine them to obtain the soln for the whole body

ex. in structural problems, typically determine displacements at each node and stresses within each element making up the structure that is subjected to applied loads

Matrix Notation for FEM

$$\{F\} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{1z} \\ \vdots \\ F_{nx} \\ F_{ny} \\ F_{nz} \end{Bmatrix} \qquad \{d\} = \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix}$$

- F_{1x} denotes force at node 1 applied in x-direction
- x, y, z-displacements at node denoted by u, v, w
  - u_1, v_1, w_1 == displacement components in x, y, z directions at node 1

{ } for column matrices

$$[K] = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix}$$

== global structure stiffness matrix in which k_{ij} == stiffness influence coefficients

Global Stiffness Equation: represents a set of simultaneous equations

$$\{F\} = [K]\{d\}$$

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{1z} \\ \vdots \\ F_{nx} \\ F_{ny} \\ F_{nz} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix}$$

3 Primary Methods used to derive the FE equations of a physical system

(1) direct method or direct equilibrium method for structural analysis problems
(2) variational methods consisting of among the subsets energy methods & princ. of virt work
(3) weighted residual methods

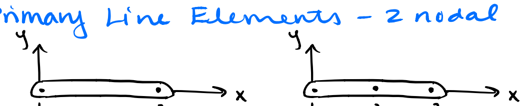
Direct Methods
- simplest, yields clear physical insight into FEM
- limited in application to deriving element stiffness matrices for 1-dimensional elements involving springs, uniaxial bars, trusses, beams
- two general direct approaches
  - (1) force/flexibility method – uses internal forces as unknowns of problem
    - → use equilibrium equations, compatibility equations
  - (2) displacement/stiffness method – assumes displacement of nodes as unknowns of problem (what we'll mostly use)
    - → compatibility condition of elements connected at common node

Variational Methods
- easier to use for deriving FE equations for 2- and 3-dimensional elements than the direct method
- uses number of principles
  - (1) theorem of minimum potential energy that applies to materials behaving in a linear-elastic manner
  - (2) principle of virtual work, applies to materials that behave in linear elastic manner

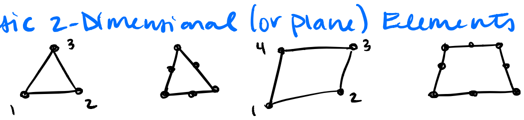
Weighted Residuals Method
- allows FEM to be applied directly to any differential equation
- modeling structure using small interconnected elements (finite elements), displacement function associated w/each finite element, every interconnected element linked through common interfaces, use stress/strain properties for material to determine behavior of given node → represent in matrix notation

FEM Formulation & Solution
→ Step 1: Discretize and Select the Element Types
- divide body into equivalent system of finite elements
  - elements small enough to give usable results, yet large enough to reduce computational effort
  - small elements used where results are changing a lot
  - large elements used where results are relatively constants
- often created w/mesh-generation programs or preprocessor programs
- choice of elements used in FEM

Primary Line Elements – 2 nodal (bar, truss, beam)

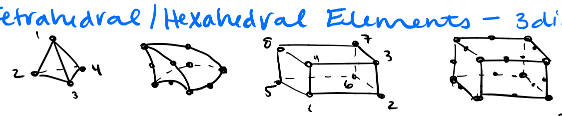


Basic 2-Dimensional (or plane) Elements – corner nodes & intermediate nodes

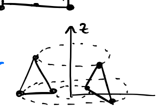


triangular, quadrilaterals

Tetrahedral/Hexahedral Elements – 3dimensional



Axisymmetric – rotating triangle or quadrilateral about fixed axis



→ Step 2: Select a Displacement Function

- typically linear, quadratic, and cubic polynomials
- functions expressed in terms of nodal unknowns
- FEM – continuous quantity, such as displacement throughout a body, is approxim. by a discrete model composed of a set piecewise-continuous functions defined within each finite domain/element

→ Step 3: Define the Strain/Displacement and Stress/Strain Relationships

- necessary for deriving equations for each finite element
- strain in the x-direction related by displacement u

$$\varepsilon_x = \frac{du}{dx}$$

- stresses related to strains (i.e. Hooke's Law)

$$\sigma_x = E\varepsilon_x \qquad \sigma_x = \text{stress in x-direction} \quad E = \text{modulus of elasticity}$$

→ Step 4: Derive the Element Stiffness Matrix and Equations

- direct equilibrium method – spring, bar, beam elements
- work or energy methods – use principle of virtual work, principle of minimum potential energy, Castigliano's Theorem
- weighted residuals method – Galerkin's method, can be applied to any diff eq.

→ Step 5: Assemble the Element Equations to obtain the Global or Total Equations and Introduce Boundary Conditions

- superposition aka direct stiffness method; implicit is concept of continuity or compatibility that requires structure remain together and no tears anywhere
- $\{F\} = [K]\{d\}$

  F = vector of global nodal forces

  K = structure global or total stiffness matrix

  d = vector of known and unknown structure nodal degrees of freedom or generalized displacements

→ note that K is singular since determinant = ∅ and to remove singularity must invoke boundary conditions, constraints, or supports so that the structure remains in place instead of moving as a rigid body

→ Step 6: Solve for the Unknown Degrees of Freedom (or Generalized Displacements)

- write $\{F\} = [K]\{d\}$ as

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{1z} \\ \vdots \\ F_{nx} \\ F_{ny} \\ F_{nz} \end{Bmatrix} = \begin{bmatrix} K_{11} & K_{12} & \cdots & K_{1n} \\ K_{21} & K_{22} & \cdots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \cdots & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_n \\ v_n \\ w_n \end{Bmatrix}$$

where now n is the structure total number of unknown nodal degrees of freedom

– can solve for ds, the primary unknowns, using Gauss elimination

→ Step 7: Solve for the Element Strains and Stresses

- secondary quantities of strain and stress (or moment and shear force) can be obtained

→ Step 8: Interpret the Results

- determination of locations in the structure where large deformations and large stresses occur is important in making design/analysis decisions

Applications of the FEM

structural problems

nonstructural problems

- stress analysis – truss, frame, holes, fillets
- buckling in columns, frames, vessels
- vibration analysis
- impact problems – crash analysis, projectile impact, bodies falling + impacting
- heat xfer – electronic chips, engines
- fluid flow – airflow, porous media
- electric & magnetic potential distribution – antennas, transistors

→ biomechanical engineering problems

→ computational fluid dynamics aka CFD to study air flow around objects

Advantages of the FEM

- model irregular shaped bodies
- handle general load conditions
- model bodies composed of different materials
- handle unlimited numbers and kinds of boundary conditions
- vary size of elements to make it possible to use small elements where necessary
- alter FEM easily (and cheaply)
- include dynamic effects
- handle nonlinear behavior

→ can do structural analysis before a prototype and can reduce number of prototypes

* all computer programs for FEM include at least the bar, beam, plane stress, plate-bending, 3-dimensional solid elements, heat xfer analysis