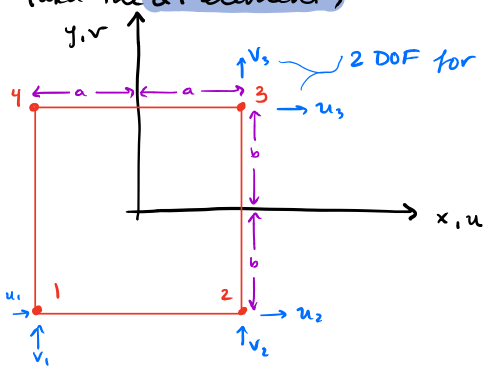
Other 2D Finite Elements

→ most commonly used 2D Finite Element: bilinear 4-node quadrilateral element (aka the "Q4 element")



2 DOF for each node (move in x, y)

4-node plane element with 8 DOFs

Displacement fields

$$u = a_1 + a_2 x + a_3 y + a_4 xy$$

constant ; linear in x ; linear in y ; bilinear term

$$v = a_5 + a_6 x + a_7 y + a_8 xy$$

Normal Strains:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = a_2 + a_4 y \quad \text{(linear as function of } y\text{)}$$

$$\varepsilon_{yy} = \frac{\partial u}{\partial y} = a_7 + a_8 x \quad \text{(linear as function of } x\text{)}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_3 + a_4 x + a_6 + a_8 y \quad \text{(constants + linear in } x \text{ + linear in } y\text{)}$$

Shape Functions: (assume rectangular, so $a \neq b$)

$$N_1(x,y) = \frac{(a-x)(b-y)}{4ab}$$ shape function for node 1

→ NOTE: $N_1(x=-a, y=-b \ldots$ coordinate for node 1$) = 1$
→ remember that shape function values must be 1 at the node, varies linearly away, is Ø at other nodes

$$N_2(x,y) = \frac{(a+x)(b-y)}{4ab}$$ shape function for node 2

$$N_3(x,y) = \frac{(a+x)(b+y)}{4ab}$$ shape function for node 3

$$N_4(x,y) = \frac{(a-x)(b+y)}{4ab}$$ shape function for node 4

Q4 element is called "BILINEAR"

NOTE: $N_1(x,y) = \frac{(a-x)(b-y)}{4ab}$

→ x direction $N_1(x) = \frac{a-x}{2a}$ — if $x=-a$, $N_1=1$; if $x=+a$, $N_1=0$

→ y direction $N_1(y) = \frac{b-y}{2b}$ — if $y=-b$, $N_1=1$; if $y=+b$, $N_1=0$

these are basically the 1D linear shape function

called BILINEAR BC IS PRODUCT OF 2 1D LINEAR POLYNOMIALS

Finite Element Approximation

$u = Nd$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$$u = \sum_{I=1}^{4} N_I u_I \qquad \varepsilon_{xx} = \frac{\partial u}{\partial x} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial x} u_I$$

$$v = \sum_{I=1}^{4} N_I v_I \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial y} v_I$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

We can take this and put this into a B matrix

$$\underbrace{K}_{8\times 8} = \int_\Omega \underbrace{B^T}_{8\times 3} \underbrace{C}_{3\times 3} \underbrace{B}_{3\times 8} \, d\Omega$$

$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}$$

more explicitly ...

strain

$$\underbrace{\varepsilon}_{3\times 1} = \underbrace{B}_{3\times 8} \underbrace{d}_{8\times 1} \qquad \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{bmatrix} = B \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

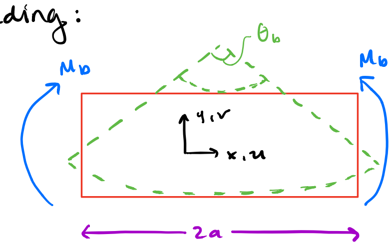
$$B = \frac{1}{4ab} \begin{bmatrix} -(b-y) & 0 & (b-y) & 0 & (b+y) & 0 & (b+y) & 0 \\ 0 & -(a-x) & 0 & -(a+x) & 0 & (a+x) & 0 & (a-x) \\ -(a-x) & -(b-y) & -(a+x) & (b-y) & (a+x) & (b+y) & (a-x) & -(b+y) \end{bmatrix}$$

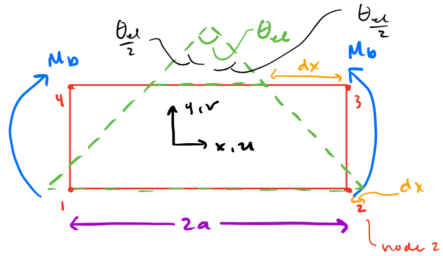
Element Defects

→ Bilinear elements (2D) or trilinear elements (3D) are perhaps most widely used elements, so good to know potential sources for error

→ Like constant strain triangle, Q4 element cannot exhibit pure bending, so it displays "parasitic" shear strain → absorbs energy in bending, so require larger bending moment than expected to produce correct value. So element shows "shear locking"

Bending:





(a) Pure/idealized bending

$$\varepsilon_{xx} = -\frac{\theta_b y}{2a}$$

$$\varepsilon_{yy} = \nu \frac{\theta_b y}{2a}$$

$$\gamma_{xy} = 0$$

(b) Q4 element under bending

$$\varepsilon_{xx} = -\frac{\theta_{el} y}{2a} \quad \text{(exact)}$$

$$\varepsilon_{yy} = 0 \quad \text{(exact if } \nu = 0\text{)}$$

$$\gamma_{xy} = -\frac{\theta_{el} x}{2a} \quad \text{(should be } \emptyset\text{)}$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial x} u_I$$

exaggerated picture above

$$= \frac{1}{4ab}\begin{bmatrix} (-b+y) & (b-y) & (b+y) & -(b+y) \end{bmatrix} \begin{bmatrix} u_1 = -dx \\ u_2 = +dx \\ u_3 = -dx \\ u_4 = +dx \end{bmatrix}$$

$$\varepsilon_{xx} = \frac{1}{4ab}(-4y\,dx) = \frac{-y\,dx}{ab} \quad \text{but } \frac{dx}{b} = \frac{\theta_{el}}{2}$$

$$\varepsilon_{xx} = -\frac{y\theta_{el}}{2a}$$

... exactly as we derived

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial y} v_I = \frac{1}{4ab}\begin{bmatrix} -(a-x) & -(a+x) & (a+x) & (a-x) \end{bmatrix} \begin{bmatrix} v_1 = 0 \\ v_2 = 0 \\ v_3 = 0 \\ v_4 = 0 \end{bmatrix}$$

$$\varepsilon_{yy} = 0$$

same as above

$$\gamma_{xy} = \sum_{I=1}^{4} \frac{\partial N_I}{\partial y} u_I + \sum_{I=1}^{4} \frac{\partial N_I}{\partial x} v_I = \frac{1}{4ab}\begin{bmatrix} -(a-x) & -(b-y) & -(a+x) & (b-y) & (a+x) & (b+y) & (a-x) & -(b+y) \end{bmatrix} \begin{bmatrix} u_1 = -dx \\ v_1 = 0 \\ u_2 = dx \\ v_2 = 0 \\ u_3 = -dx \\ v_3 = 0 \\ u_4 = dx \\ v_4 = 0 \end{bmatrix}$$

$$\gamma_{xy} = \frac{-4x\,dx}{4ab} = \frac{-x\,dx}{ab}, \quad \text{so} \quad \gamma_{xy} = \frac{-x\theta_{el}}{2ab}$$

↑ but $\frac{dx}{b} = \frac{\theta_{el}}{2}$

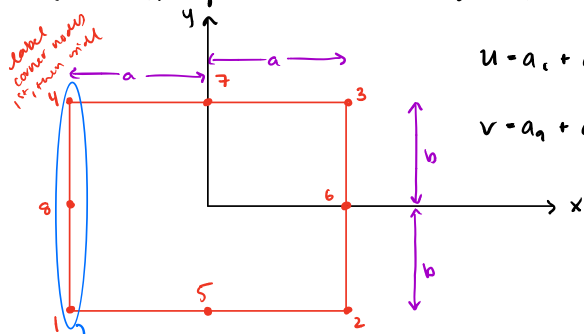
→ Spurious shear strain, comes directly from FEM shape functions, shape functions come from element type

→ Q4 element is stiff in bending

Quadratic Rectangle (Q8) Element ... 8 nodes

(for final MATLAB HW, will also study a 9-node (Q9) element)

→ what can higher order element do about shear locking?



$$u = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^2 y + a_8 xy^2$$

$$v = a_9 + a_{10} x + a_{11} y + a_{12} x^2 + a_{13} xy + a_{14} y^2 + a_{15} x^2 y + a_{16} xy^2$$

x = constant → u if $x = C$ is quadratic function of y

→ implies that element sides can curve

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = a_2 + 2a_4 x + a_5 y + 2a_7 xy + a_8 y^2$$

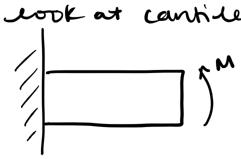
$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = a_{11} + a_{13} x + 2a_{14} y + a_{15} x^2 + 2a_{16} xy$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = (a_3 + a_{10}) + (a_5 + 2a_{12}) x + (2a_6 + a_{13}) y + a_7 x^2 + 2(a_8 + a_{15}) xy + a_{16} y^2$$

B for Q8 element is 3×16

→ How does Q8 perform under bending?

look at cantilever beam w/ moment applied at free end



Exact solution:

$$u = -Cxy$$

$$v = \frac{C}{2}(x^2 + \nu y^2)$$

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -Cy$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = \nu C y$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Recall Q8 displacement field:

$$u = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^2 y + a_8 xy^2$$

$$v = a_9 + a_{10} x + a_{11} y + a_{12} x^2 + a_{13} xy + a_{14} y^2 + a_{15} x^2 y + a_{16} xy^2$$

→ set all a's = 0 except: $a_5 = -C$, $a_{12} = C/2$, $a_{14} = \nu \frac{C}{2}$

(because a's are just constants)

$$\left.\begin{aligned} u &= -Cxy \\ v &= \frac{C}{2} x^2 + \nu \frac{C}{2} y^2 \end{aligned}\right\} \text{ now Q8 displacement field is exact}$$

$a_5 \to xy$ ← Q4 has

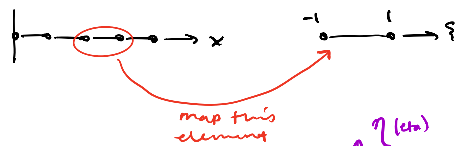
$\left.\begin{aligned} a_{12} &\to x^2 \\ a_{14} &\to y^2 \end{aligned}\right\}$ Q4 doesn't have

IMPORTANT FOR LAST MATLAB HW

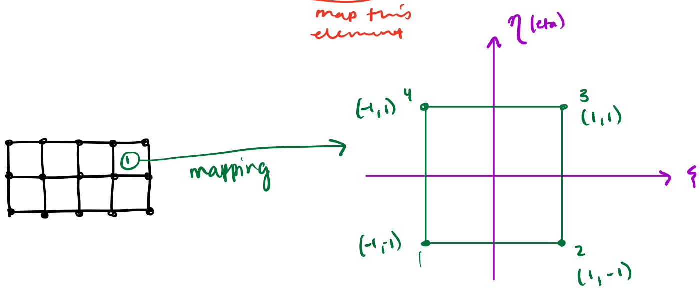
⇒ TAKEAWAY: Q8 element should perform well in bending because there's no spurious shear strain!

4-Node Isoparametric Element

1D Reminder:



2D:



$$x(\xi,\eta) = \sum_{a=1}^{4} N_a(\xi,\eta)\, x_a$$

$$y(\xi,\eta) = \sum_{a=1}^{4} N_a(\xi,\eta)\, y_a$$

Shape functions:

$$N_1(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1-\eta)$$
$$N_2(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1-\eta)$$
$$N_3(\xi,\eta) = \tfrac{1}{4}(1+\xi)(1+\eta)$$
$$N_4(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1+\eta)$$

check on shape functions:

① $N_1 + N_2 + N_3 + N_4 = 1$

② Check: shape functions = 1 at node, ∅ at other nodes, and linear in between nodes

ex.

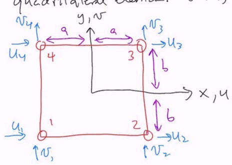
$N_1(\xi,\eta) = \tfrac{1}{4}(1-\xi)(1-\eta) \rightarrow N_1(-1,-1) = \tfrac{1}{4}(1--1)(1--1) = 1$ ✓ first node

$\rightarrow N_1(1,-1) = \tfrac{1}{4}(1-1)(1--1) = 0$ ✓ second node

$\rightarrow N_1(1,1) = \tfrac{1}{4}(1-1)(1-1) = 0$ ✓ third node

$\rightarrow N_1(-1,1) = \tfrac{1}{4}(1--1)(1-1) = 0$ ✓ fourth node other 2D finite elements

→ Most commonly used 2D element: bilinear 4-node quadrilateral element (Q4)



4-node plane element, with 8 DOF's

Displacement fields

$u = a_1 + a_2 x + a_3 y + a_4 xy$

$v = a_5 + a_6 x + a_7 y + a_8 xy$

strains = $\varepsilon_{xx} = \frac{\partial u}{\partial x} = a_2 + a_4 y$

$\varepsilon_{yy} = \frac{\partial v}{\partial y} = a_7 + a_8 x$

$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_3 + a_4 x + a_6 + a_8 y$

Shape Functions ($a \neq b$)

$N_1(x,y) = \frac{(a-x)(b-y)}{4ab}$

→ note:

$N_1(x=-a,\ y=-b) = 1$ → so shape function value is 1 at node, varies linearly away, is ϕ at other node

$N_2(x,y) = \frac{(a+x)(b-y)}{4ab}$

$N_3(x,y) = \frac{(a+x)(b+y)}{4ab}$

$N_4(x,y) = \frac{(a-x)(b+y)}{4ab}$

Q4 element is called "bilinear" — because is product of 2 1D linear polynomials

NOTE: $N_1(x,y) = \frac{(a-x)(b-y)}{4ab}$ → x_direction $N_1(x) = \frac{a-x}{2a}$ if $x=-a, N=1$ if $x=+a, N=0$

→ y-direction: $N_1(y) = \frac{b-y}{2b}$ if $y=-b, N_1=1$, if $y=+b, N_1=0$

$u = N d$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$u = \sum_{i=1}^{4} N_I u_I \qquad \varepsilon_{xx} = \frac{\partial u}{\partial x} = \sum_{i=1}^{4} \frac{\partial N_I}{\partial x} u_I$

$v = \sum_{i=1}^{4} N_I v_I \qquad \varepsilon_{yy} = \frac{\partial v}{\partial y} = \sum_{i=1}^{4} \frac{\partial N_I}{\partial y} v_I$

$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$

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$$B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 & \frac{\partial N_4}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} & 0 & \frac{\partial N_4}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial y} & \frac{\partial N_4}{\partial x} \end{bmatrix}$$

$$\underset{3\times1}{\varepsilon} = \underset{3\times8}{B}\;\underset{8\times1}{D} \qquad \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{bmatrix} = B \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

$$B = \frac{1}{4ab} \begin{bmatrix} -(b-y) & 0 & (b-y) & 0 & (b+y) & 0 & -(b+y) & 0 \\ 0 & -(a-x) & 0 & -(a+x) & 0 & (a+x) & 0 & (a-x) \\ -(a-x) & -(b-y) & -(a+x) & (b-y) & (a+x) & (b+y) & (a-x) & -(b+y) \end{bmatrix}$$

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