2D Finite Elements

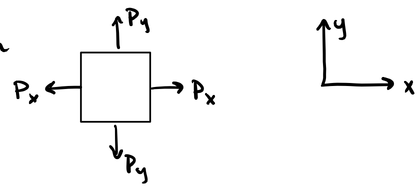
→ start with 2D elasticity
→ 2D is a specific approximation of 3D

LECTURE VIDEO
2020-11-02

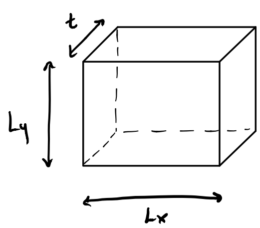
2 Types of 2D approximations

① Plane stress: all stresses perpendicular to plane = 0
- if loading in x, y - plane

all z-direction stresses = 0

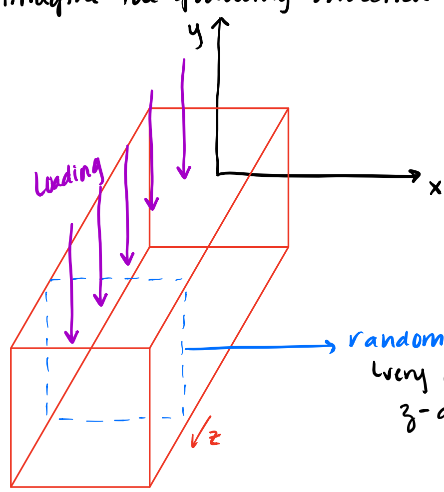


- generally: thin members loaded only in x,y-plane (plates, shells, kind of thing)
(and other thin structures)



for plane stress, $t \ll L_x$ or L_y

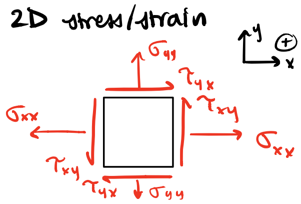
② Plane strain: situation where all strains normal to x,y-plane are 0.
→ Realistic for long bodies in the z-direction with constant cross sectional area subject to loads that act only in x or y directions, don't vary in z
→ imagine the following situation:



random cross section
very constrained in z-direction by lots of mass, so z-direction strains = 0

(dams, tunnels, etc.)

2D stress/strain



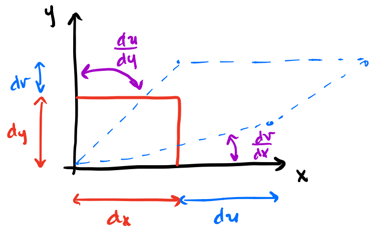
Nomenclature:
τ_{xy} → normal to plane points in x-direction
→ stress in y-direction

$$[\sigma] = \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix}$$

complementary shear $\tau_{xy} = \tau_{yx}$

→ What's this conserving? Angular momentum
(shear stresses have to be equal)

2D Strains



$$\varepsilon_{xx} = \frac{du}{dx}$$
$$\varepsilon_{yy} = \frac{dv}{dy}$$
$$\gamma_{xy} = \left(\frac{du}{dy} + \frac{dv}{dx}\right)\frac{1}{2}$$

so $[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$

Constitutive Relationships (stress-strain relationships)

- Plane Stress

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \underbrace{\frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}}_{\text{also called } C} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

Young's modulus

poisson's ratio

=> $\sigma = C\varepsilon$

Remember:

1D: $\sigma_{xx} = E\varepsilon_{xx}$ (Hooke's Law)

2D: $\sigma_{xx} = \frac{E}{1-\nu^2}\left(\varepsilon_{xx} + \nu\varepsilon_{yy}\right)$

x-stress now depends on y-strain

Plane Strain:

typically called 'C'

$$\begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{bmatrix} = \overbrace{\frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}}^{C} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

=> $\boxed{\sigma = C\varepsilon}$

→ What is valid range for ν?

Range: 0 to 1/2

but... if $\nu \to \frac{1}{2}$, then for plane strain $\frac{E}{(1+\nu)\underbrace{(1-2\nu)}_{\to 0}}$ so $\sigma \to \infty$

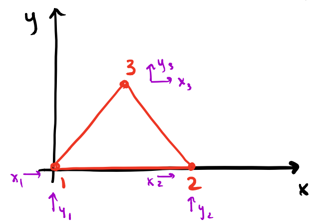
$\nu = \frac{1}{2}$ are rubbery materials/polymers typically have $\nu \sim \frac{1}{2}$

→ called "incompressible"

→ much easier to change their shape rather than change their volume or area

→ if not careful in FE simulation, results in artificially stiff FEM simulations (will discuss briefly later)

2D Elements: Constant Strain Triangle - three nodes each have 2-degrees of freedom now



Historical Note: In industry, most meshing done with tetrahedral elements, 2D equivalent is triangle

Note that $x_1 = y_1 = y_2 = 0$

Define scalar field value:

$$\phi = [1 \quad x \quad y]\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

ϕ = field quantity (i.e. displacement)
a_i = constants
$[1 \quad x \quad y] \rightarrow$ linear displacement element in x and y

$$\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} = A \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & x_2 & 0 \\ 1 & x_3 & y_3 \end{bmatrix}}_{\text{A matrix has coordinates of nodes}} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \rightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = A^{-1} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

$$\phi = \underbrace{[1 \quad x \quad y]A^{-1}}_{\substack{\text{2D shape function} \\ = N(x,y)}} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix} \qquad A^{-1} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{x_2} & \frac{1}{x_2} & 0 \\ \frac{x_3 - x_2}{x_2 y_3} & \frac{-x_3}{x_2 y_3} & \frac{1}{y_3} \end{bmatrix}$$

gradients: $$\underbrace{\begin{bmatrix} \phi_{1x} \\ \phi_{1y} \end{bmatrix}}_{2\times 1} = \underbrace{B}_{2\times 3} \underbrace{\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{bmatrix}}_{3\times 1} \qquad B = \begin{bmatrix} \frac{\partial N}{\partial x} \\ \frac{\partial N}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \end{bmatrix}$$

$$[1 \quad x \quad y]A^{-1} = \begin{bmatrix} 1 - \frac{x}{x_2} + \frac{y(x_3 - x_2)}{x_2 y_3} \\ 0 + \frac{x}{x_2} - y\frac{x_3}{y_2 x_3} \\ 0 + 0 + \frac{y}{y_3} \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{1}{x_2} & \frac{1}{x_2} & 0 \\ \frac{x_3 - x_2}{x_2 y_3} & \frac{-x_3}{x_2 y_3} & \frac{1}{y_3} \end{bmatrix} \rightarrow$$ rows of shape function derivative sum up to 0

u = x-displacement $= [1 \quad x \quad y]\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

v = y-displacement $= [1 \quad x \quad y]\begin{bmatrix} a_4 \\ a_5 \\ a_6 \end{bmatrix}$

$$\left.\begin{aligned} \varepsilon_{xx} &= \frac{du}{dx} = a_2 \\ \varepsilon_{yy} &= \frac{dv}{dy} = a_6 \\ \gamma_{xy} &= \frac{1}{2}\left(\frac{du}{dy} + \frac{dv}{dx}\right) \\ &= \frac{1}{2}(a_3 + a_5) \end{aligned}\right\} \text{strains are constant}$$

$$\underbrace{\varepsilon}_{3\times 1} = \underbrace{B}_{3\times 6}\underbrace{d}_{\substack{6\times 1 \\ \text{6 degrees of freedom} \\ \text{between 3 nodes}}} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix}$$

1st row tells us: $\varepsilon_{xx} = \frac{\partial N_1}{\partial x} u_1 + \frac{\partial N_2}{\partial x} u_2 + \frac{\partial N_3}{\partial x} u_3$

last row tells us: shear strain

So, $K = \int_\Omega \underset{6\times3}{B^T} \underset{3\times3}{C} \underset{3\times6}{B} \, d\Omega$ = 6x6 which makes sense since 2 DoF for each of 3 nodes

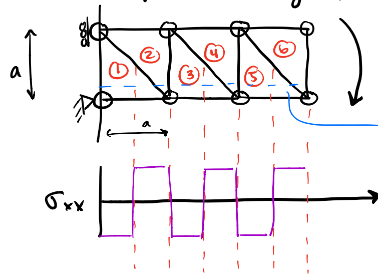
$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{x_2} & 0 & \frac{1}{x_2} & 0 & 0 & 0 \\ 0 & \frac{x_3-x_2}{x_2 y_3} & 0 & \frac{-x_3}{x_2 y_3} & 0 & \frac{1}{y_3} \\ \frac{x_3-x_2}{x_2 y_3} & -\frac{1}{x_2} & \frac{-x_3}{x_2 y_3} & \frac{1}{x_2} & \frac{1}{y_3} & 0 \end{bmatrix}}_{B} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

Constant Strain Triangle: Element Defects

→ element may perform particularly poorly for certain types of loading

→ consider pure bending of a beam

$\sigma = \frac{My}{I}$

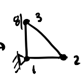


① σ_{xx} not along beam centerline should be linear with y

→ but FEM stress is constant

→ wrong

② consider shear stress in element 1 →



$u_1 = u_3 = 0$, $v_1 = 0$

$v_3 = 0$ (if truly pure bending)

$$\gamma_{xy} = \left(\frac{x_3 - x_2}{x_2 y_3}\right)(u_1) - \frac{1}{x_2}(v_1) - \frac{x_3}{x_2 y_3} u_2 + \frac{1}{x_2} v_2 + \frac{1}{y_3} u_3 + 0 (v_3)$$

bc $x_2 = x_3$ $v_1 = 0$ $x_3 = 0$ $u_3 = 0$ 0

$\gamma_{xy} = \frac{1}{x_2} v_2 \Rightarrow \boxed{\gamma_{xy} = \frac{v_2}{x_2}}$

So for constant strain triangle element, $\gamma_{xy} \neq 0$

→ What should γ_{xy} be for bending? $\gamma_{xy} = 0$

→ so constant strain triangle absorbs energy in shear when it should not.

called "spurious shear strain"

→ triangle diverts some work to deforming in shear, so beam doesn't bend as much as anticipated, so FE mesh is "stiff" in bending

→ flaw due to constant strain nature of element