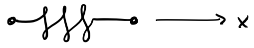


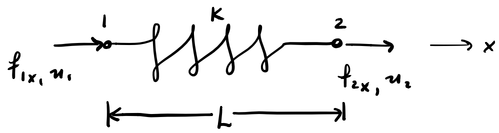
A single spring element

For an **element**, a stiffness matrix $[k]$ is a matrix such that

$$\{f\} = [k]\{d\}$$

where $[k]$ relates nodal displacements $\{d\}$ to nodal forces $\{f\}$ of a single element

Derivation of the Stiffness Matrix for a Spring Element



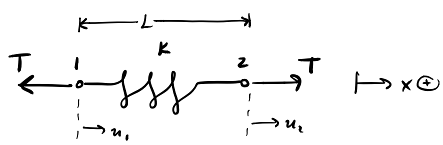
a spring element that obeys Hooke's Law

1 & 2 are reference pts, or nodes
f_{1x} & f_{2x} are local nodal forces associated w/the local x-axis
u_1 & u_2 are local nodal displacements, called the degrees of freedom
k is the spring constant

we want to develop a relationship between nodal forces and nodal displacements and this relationship is what we call the stiffness matrix

$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

k_{ij} represent the force F_i in the i^{th} degree of freedom due to a unit displacement d_j in the j^{th} degree of freedom while all other displacements are zero
→ meaning, when we let $d_j = 1$ and $d_k = 0$ for $k \neq j$, force $F_i = k_{ij}$



Define strain/displacement and stress/strain relationships
- tensile forces T produce an elongation δ
  total deformation, $\delta = u_2 - u_1$
- no strain/displacement relationship necessary to consider since force related to deformation
- stress/strain relationship expressed in terms of force/deformation relationship instead as
  $T = k\delta$

$$\therefore \; T = k(u_2 - u_1)$$

Derive Element Stiffness Matrix and Equations
- $f_{1x} = -T$ and $f_{2x} = T$

$$\therefore \; T = -f_{1x} = k(u_2 - u_1) \quad \text{and} \quad T = f_{2x} = k(u_2 - u_1)$$

- or -

$$f_{1x} = k(u_1 - u_2)$$
$$f_{2x} = k(u_2 - u_1)$$
$$\Rightarrow \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Here, $[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$ is the local stiffness matrix for the element

Note that [k] has the following properties:

1. symmetric matrix, so $k_{ij} = k_{ji}$ for $i \neq j$
2. square matrix as it relates the same # of nodal forces to nodal displacements
3. singular, determinant of [k] is zero and [k] cannot be inverted

Assemble Element Equations to Obtain Global Equations and Introduce B.C.

$$[K] = \sum_{e=1}^{N} [k^{(e)}] \qquad \{F\} = \sum_{e=1}^{N} \{f^{(e)}\}$$

where $[k^{(e)}]$ and $\{f^{(e)}\}$ are now element stiffness and force matrices expressed in a global reference frame

Solve for Nodal Displacements

$$\{F\} = [K]\{d\}$$

Solve for Element Forces

- done using back substitution

Potential Energy Approach to Derive Spring Element Equations

Principle of Minimum Potential Energy

- of all the geometrically possible shapes that a body can assume, the true one, corresponding to the satisfaction of stable equilibrium of the body, is identified by a minimum value of the potential energy
- the total potential energy π_p of a structure is expressed in terms of displacements. in the finite element formulation, these will generally be nodal displacements such that

$$\pi_p = \pi_p(d_1, d_2, \ldots, d_n)$$

and when π_p is minimized wrt these displacements, equilibrium equations result

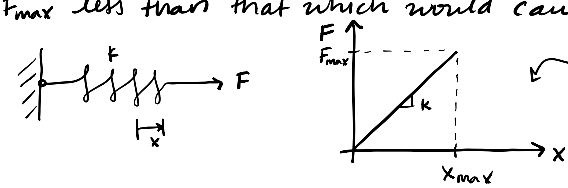
Total potential Energy – sum of the internal strain energy U and the potential energy of the external forces Ω,

$$\pi_p = U + \Omega$$

strain energy – capacity of internal forces (or stresses) to do work through deformations (strains) in the structure

Ω – capacity of forces such as body forces, surface traction forces, and applied nodal forces to do work through deformation of the structure

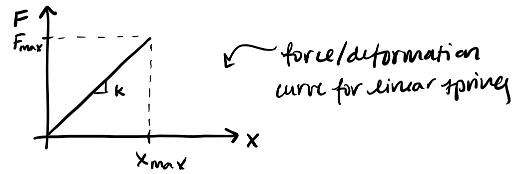
External work is done on a linear-elastic behaving member by applying a gradually increasing magnitude force F to the end of the spring up to some maximum value F_{max} less than that which would cause permant deformation in the spring



force/deformation curve for linear spring

the maximum deformation X_{max} occurs when the maximum force occurs as shown

The external work is given by the area under the force/deformation curve, where the slope of the straight line is equal to the spring constant k



The external work is then given as

$$W_e = \int F \cdot dx = \int_0^{x_{max}} F_{max}\left(\frac{x}{x_{max}}\right) dx = \frac{F_{max} x_{max}}{2} \qquad \text{where } F \text{ is given by } F = F_{max}(x/x_{max})$$

From the conservation of mechanical energy, the ext. work due to applied force F is xformed into the internal strain energy U of the spring

$$U = W_e = \frac{F_{max} X_{max}}{2}$$

Upon gradual reduction of the force to zero, the spring returns to its original undeformed state and this returned energy that is stored is called the internal strain energy

$$F_{max} = k X_{max} \quad \Rightarrow \quad U = k x_{max}^2 \left(\tfrac{1}{2}\right)$$

The potential energy of F^{ext}, being opposite in sign from the W_e expression because the potential energy of F^{ext} is lost when the work is done by F^{ext}

$$\Omega = -F_{max} X_{max} \quad \Rightarrow \quad \pi_p = \tfrac{1}{2} k x_{max}^2 - F_{max} X_{max}$$

In general, we replace X_{max} w/ x and F_{max} w/ F and express U and Ω as

$$U(x) = \tfrac{1}{2} k x^2 \qquad \Omega(x) = -F_x$$

We then express the total potential energy as

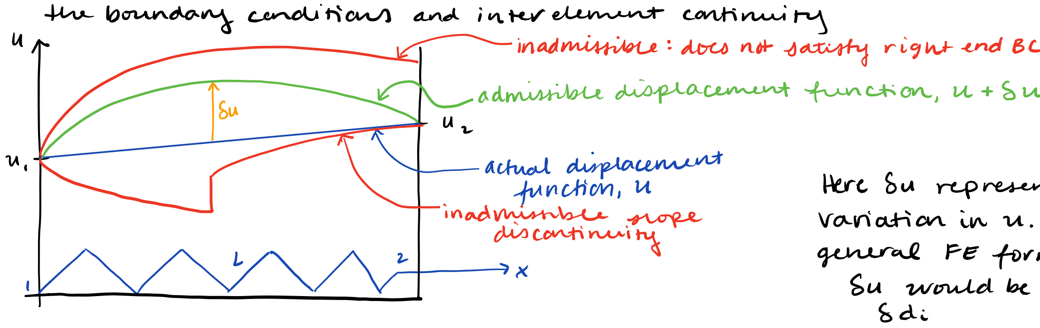
$$\pi_p(x) = \tfrac{1}{2} k x^2 - F x$$

Now, to apply the principle of minimum potential energy, so to <u>minimize</u> π_p, we take the variation (δ) of π_p, which is a function of nodal displacements d_i defined in general as

$$\delta \pi_p = \frac{\delta \pi_p}{\delta d_1}\delta d_1 + \frac{\delta \pi_p}{\delta d_2}\delta d_2 + \cdots + \frac{\delta \pi_p}{\delta d_n} d_n$$

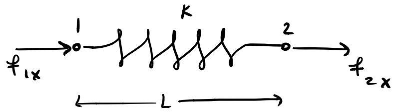
Equilibrium exists when the d_i define a structure state such that $\delta \pi_p = 0$ (change in potential energy = 0) for arbitrary admissible variations in displacement δd_i from the equilibrium state

→ an admissable is one in which the displacement field still satisfies the boundary conditions and interelement continuity



Here δu represents the variation in u. In the general FE formulation, δu would be replaced by δd_i

Derivation of Spring Element Equations and Stiffness Matrix Using the Principle of Minimum Potential Energy



linear spring subjected to nodal forces

total PE: $\pi_p = \frac{1}{2}k(u_2 - u_1)^2 - f_{1x}u_1 - f_{2x}u_2$

→ where $u_2 - u_1$ = deformation of the spring

→ first term on right = strain energy

$$\pi_p = \frac{1}{2}k(u_2^2 - 2u_2u_1 + u_1^2) - f_{1x}u_1 - f_{2x}u_2$$

the minimization of π_p wrt each nodal displacement requires taking partial derivatives of π_p wrt each nodal displacement such that

$$\frac{\partial \pi_p}{\partial u_1} = \frac{1}{2}k(-2u_2 + 2u_1) - f_{1x} = 0$$

$$\frac{\partial \pi_p}{\partial u_2} = \frac{1}{2}k(2u_2 - 2u_1) - f_{2x} = 0$$

simplifying ...

$$k(-u_2 + u_1) = f_{1x}$$
$$k(u_2 - u_1) = f_{2x}$$

in matrix form ...

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix}$$

$\Rightarrow$ because $\{f\} = [k]\{d\}$ we have the stiffness matrix for the spring element obtained from $[k] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}$

identical to what we found before

for a system of springs:

$$U = \frac{1}{2}\{d\}^T[k]\{d\}$$