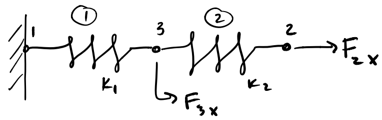
Assembly of a global stiffness matrix from element stiffnesses



→ assembled finite element equations:

$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1+k_2 \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{2x} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} 0 \\ F_{2x} \\ F_{3x} \end{bmatrix}$$

⇒ represents $Kd = F^{ext}$

"global stiffness matrix"
↳ what we get when all elements assembled together

→ properties of global stiffness matrix

① K is 3x3 → 3 nodes, 3 degrees of freedom
- more generally, always a square matrix: #DOF x #DOF

② K is symmetric → amount of computer memory storage needed decreases, can also use efficient solvers to invert symmetric matrices

③ diagonal entries all > 0 → why?
- fix one node: $k_2 d_{2x} = F_{2x}$
- physically meaningful, otherwise nodes could have negative displacements for positive forces

④ sum of rows/columns: adds to zero → why?
- because internal forces should = 0 if no external force

⑤ is global K (or element stiffness) invertible? why or why not?
- linear dependence, rows + columns can be combined
- one eigenvalue is zero, because a singular matrix
- physical meaning:
  → in 1-dimension: $U = \frac{1}{2} E \varepsilon^2$
  ↑ strain energy
  → zero eigenvalues of K represent "zero energy" or rigid body motions

*we don't want these zero-energy nodes
  → we remove rigid body nodes by applying constraints, more specifically: apply boundary conditions
  → if don't apply boundary conditions, K is singular and can't invert it

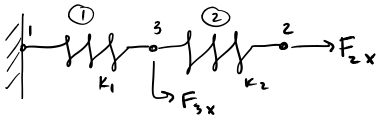
2 Basic Types of Boundary Conditions for solid mechanics problems

① prescribed displacements (i.e. fixed nodes)
② prescribed force

→ general rule: need to apply as many fixed node constraints as # of rigid body modes

1D: one rigid body node (only translation)
2D: three rigid body nodes (x, y translation; 1 rotation)

How to Impose Boundary Conditions?



$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1+k_2 \end{bmatrix} \begin{bmatrix} d_{1x}^{=0} \\ d_{2x} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} 0 \\ F_{2x} \\ F_{3x} \end{bmatrix}$$

1^{st} node fixed to wall, $d_{1x} = 0$ ... Expand these equations:

→ ~~first row: $k_1(0) + 0(d_{2x}) - k_1(d_{3x}) = F_{1x}$~~ ← reaction force — goes away

→ second row: $\cancel{(0)(0)} + k_2(d_{2x}) - k_2(d_{3x}) = F_{2x}$

→ third row: $\cancel{-k_1(0)} - k_2(d_{2x}) + (k_1+k_2)d_{3x} = F_{3x}$

goes away

so we're left with: removing row + column of fixed node

... can write in 2 ways:

① Reduced 2x2 system: $\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1+k_2 \end{bmatrix} \begin{bmatrix} d_{2x} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} F_{2x} \\ F_{3x} \end{bmatrix}$

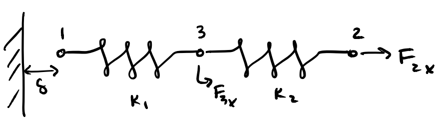
→ note that rows/columns are not linearly dependent

② Full 3x3 system: zero out rows/columns of fixed node, put 1 on diagonal

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_1+k_2 \end{bmatrix} \begin{bmatrix} d_{1x}=0 \\ d_{2x} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} 0 \\ F_{2x} \\ F_{3x} \end{bmatrix}$$

→ first equation: $1(d_{1x}) + 0(d_{2x}) + 0(d_{3x}) = 0$ ... $d_{1x} = 0$

Boundary Condition Example: prescribed non-zero displacement



i.e. $d_{1x} = \delta$

→ because k's are same for elements, global K doesn't change

$$\begin{bmatrix} k_1 & 0 & -k_1 \\ 0 & k_2 & -k_2 \\ -k_1 & -k_2 & k_1+k_2 \end{bmatrix} \begin{bmatrix} d_{1x}=\delta \\ d_{2x} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} 0 \\ F_{2x} \\ F_{3x} \end{bmatrix}$$

first row: $k_1\delta + 0(d_{2x}) - k_1(d_{3x}) = 0$

second row: $0\delta + k_2(d_{2x}) - k_2(d_{3x}) = F_{2x}$

third row: $-k_1\delta + -k_2(d_{2x}) + (k_1+k_2)(d_{3x}) = F_{3x}$

doesn't necessarily drop out because $\delta \neq 0$

ignore first row because $d_{1x} = \delta$, write 2^{nd}, 3^{rd} rows again:

2^{nd} row: ① $k_2 d_{2x} - k_2 d_{3x} = F_{2x}$ ← same as before

3^{rd} row: $-k_1\delta - k_2 d_{2x} + (k_1+k_2)d_{3x} = F_{3x}$

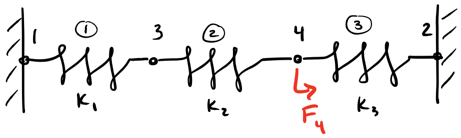
② $-k d_{2x} + (k_1+k_2)d_{3x} = F_{3x} + k_1\delta$

put in matrix form ① and ②:

same as zero displacement case! → $\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_1+k_2 \end{bmatrix} \begin{bmatrix} d_{2x} \\ d_{3x} \end{bmatrix} = \begin{bmatrix} F_{2x} \\ F_{3x} + k_1\delta \end{bmatrix}$ — due to non-zero node 1 displacement - if $\delta = 0$, same as 0 displacement case!

Example:

4 nodes and 3 elements



$k_1 = 1000$ lbs/in $\quad k_2 = 2000$ lbs/in $\quad k_3 = 3000$ lbs/in

$F_4 = 5000$ lbs (external force)

Find:

① global K (should be 4x4 matrix)

② d_{3x}, d_{4x}

③ reaction forces at nodes 1 + 2

④ forces in each spring (element)

Element k's:

connectivity to nodes 1 and 3

$$k^{(1)} = \begin{array}{cc} 1 & 3 \\ \end{array} \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{array}{c} 1 \\ 3 \end{array} \qquad k^{(2)} = \begin{array}{cc} 3 & 4 \\ \end{array} \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix} \begin{array}{c} 3 \\ 4 \end{array} \qquad k^{(3)} = \begin{array}{cc} 4 & 2 \\ \end{array} \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix} \begin{array}{c} 4 \\ 2 \end{array}$$

①

global K:

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ & \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 1000 + 2000 & -2000 \\ 0 & -3000 & -2000 & 2000 + 3000 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} \Rightarrow K = \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix}$$

multiple connections @ node

②

→ now apply boundary conditions to K. Nodes 1, 2 = fixed, so zero out or remove these rows/columns

$$K = \begin{bmatrix} \cancel{1000} & \cancel{0} & \cancel{-1000} & \cancel{0} \\ \cancel{0} & \cancel{3000} & \cancel{0} & \cancel{-3000} \\ \cancel{-1000} & \cancel{0} & 3000 & -2000 \\ \cancel{0} & \cancel{-3000} & -2000 & 5000 \end{bmatrix}$$

so we get: $\begin{bmatrix} 3000 & -2000 \\ -2000 & 5000 \end{bmatrix} \begin{bmatrix} d_{3x} \\ d_{4x} \end{bmatrix} = \begin{bmatrix} 0 \\ 5000 \end{bmatrix}$

$d_{3x} = \frac{10}{11}$ in, $d_{4x} = \frac{15}{11}$ in

③

Find rxn Forces

Properties of K: what discussed prior

$$K = \begin{bmatrix} 1000 & 0 & -1000 & 0 \\ 0 & 3000 & 0 & -3000 \\ -1000 & 0 & 3000 & -2000 \\ 0 & -3000 & -2000 & 5000 \end{bmatrix} \begin{bmatrix} d_{1x} = 0 \\ d_{2x} = 0 \\ d_{3x} = \frac{10}{11} \\ d_{4x} = \frac{15}{11} \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{2x} \\ F_{3x} \\ F_{4x} \end{bmatrix}$$

rxn forces

$F_{1x} = 1000(0) + 0(0) - 1000(10/11) + 0(15/11) = -\frac{10000}{11}$ lbs

$F_{2x} = 0(0) + 3000(0) + 0(10/11) - 3000(15/11) = -\frac{45000}{11}$ lbs

$F_{3x} = 3000(10/11) - 2000(15/11) = 0$

$F_{4x} = -2000(10/11) + 5000(15/11) = \frac{55000}{11}$ lbs $\quad F^{ext}$ on node 4

$\Sigma F_{ix} = 0: -\frac{10000}{11} - \frac{45000}{11} + \frac{55000}{11} = 0$ ✓

Are these rxn forces reasonable?
→ should be negative since external force is (+)
→ F_{2x} is significantly larger than F_{1x}
- larger based on geometry as well as spring stiffness!

Why is the force at node 3 zero?
- $Kd \rightarrow F^{int} = F^{ext}$

④
Element Forces

element 1 : $\begin{bmatrix} f_{1x}^{(1)} \\ f_{3x}^{(1)} \end{bmatrix} = \begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{bmatrix} 0 \\ 10/11 \end{bmatrix} \rightarrow \underbrace{f_{1x}^{(1)} = -\frac{10000}{11}}_{\text{reaction force}} \qquad f_{3x}^{(1)} = +\frac{10000}{11}$

$\rightarrow f_{1x}^{(1)} + f_{3x}^{(1)} = 0$ (adding up nodal forces for el. 1)

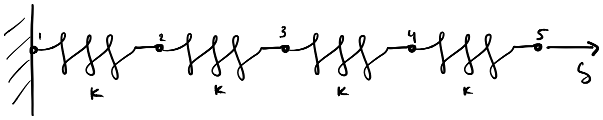
element 2 : $\begin{bmatrix} f_{3x}^{(2)} \\ f_{4x}^{(2)} \end{bmatrix} = \begin{bmatrix} 2000 & -2000 \\ -2000 & 2000 \end{bmatrix} \begin{bmatrix} 10/11 \\ 15/11 \end{bmatrix} \rightarrow f_{3x}^{(2)} = -\frac{10000}{11} \qquad f_{4x}^{(2)} = +\frac{10000}{11}$

$\rightarrow f_{3x}^{(1)} + f_{3x}^{(2)} = 0$ ... because there's no force applied to node 3!
$\rightarrow f_{3x}^{(2)} + f_{4x}^{(2)} = 0$

element 3 : $\begin{bmatrix} f_{4x}^{(3)} \\ f_{2x}^{(3)} \end{bmatrix} = \begin{bmatrix} 3000 & -3000 \\ -3000 & 3000 \end{bmatrix} \begin{bmatrix} 15/11 \\ 0 \end{bmatrix} \rightarrow f_{4x}^{(3)} = +\frac{45000}{11} \qquad f_{2x}^{(3)} = -\frac{45000}{11}$

$\rightarrow f_{4x}^{(2)} + f_{4x}^{(3)} = \frac{55000}{11} = \underbrace{5000 \text{ lbs}}_{\text{same as applied } F^{ext} \text{ on node 4}}$

Direct Stiffness Method Example with prescribed displacement



$\delta = 20$ mm $k = 200$ kN/m

Find:
① global K
② d_{2x}, d_{3x}, d_{4x}
③ Resulting forces on nodes

$$k^{(1)} = k^{(2)} = k^{(3)} = k^{(4)} = \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix}$$

① global K:

$$\begin{Bmatrix} 200 & -200 & 0 & 0 & 0 \\ -200 & 400 & -200 & 0 & 0 \\ 0 & -200 & 400 & -200 & 0 \\ 0 & 0 & -200 & 400 & -200 \\ 0 & 0 & 0 & -200 & 200 \end{Bmatrix}$$

② d_{2x}, d_{3x}, d_{4x}:

$$\begin{array}{c} F_{2x} \\ F_{3x} \\ F_{4x} \end{array} \underset{3\times 1}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} -200 & 400 & -200 & 0 & 0 \\ 0 & -200 & 400 & -200 & 0 \\ 0 & 0 & -200 & 400 & -200 \end{bmatrix} \begin{bmatrix} d_{1x} = 0 \\ d_{2x} \\ d_{3x} \\ d_{4x} \\ d_{5x} = \delta = .02\text{m} \end{bmatrix}$$

displacement vector

$F_{4x} = 0 = 0(d_{1x}) + 0(d_{2x}) - 200(d_{3x}) + 400(d_{4x}) - \boxed{200(d_{5x})}$

add to force because $d_{5x} \neq 0$

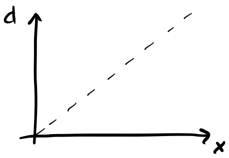
$d_{5x} \neq 0$ so cannot drop out of equation

$$\begin{bmatrix} 0 \\ 0 \\ 200\,^{kN}/_{m}(.02m) \end{bmatrix} = \begin{bmatrix} 400 & -200 & 0 \\ -200 & 400 & -200 \\ 0 & -200 & 400 \end{bmatrix} \begin{bmatrix} d_{2x} \\ d_{3x} \\ d_{4x} \end{bmatrix}$$

solve:

$d_{1x} = 0, \; d_{2x} = .005\,m, \; d_{3x} = .01\,m, \; d_{4x} = .015\,m, \; d_{5x} = .02\,m$

... notice that the displacement field is linear!



<u>Question</u>: how many linear finite elements are needed to get this exact solution?

only one.

③ resulting nodal forces: $k \cdot d$

$F_{1x} = (-200)(.005) = -1\,kN$ ← rxn force on node 1, force is to left to keep bar non-moving

$F_{2x} = (400)(.005) - 200(.01) = 0$ ← no external force applied here so makes sense

$F_{3x} = -200(.005) + 400(.01) - 200(.015) = 0$

$F_{4x} = -200(.01) + 400(.015) - 200(.02) = 0$

$F_{5x} = -200(.015) + 200(.02) = 1\,kN$

$\Sigma F = F_{1x} + F_{2x} + F_{3x} + F_{4x} + F_{5x} = 0$ ✓

↳ what does 1 kN force on node 5 mean?

- it's the amount of force needed to apply to displace node 5 by $\delta = .02m$
- in mechanics, can prescribe displacement or force, but not both