Beam Equations & Beam Elements

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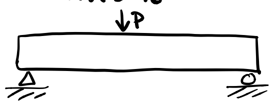
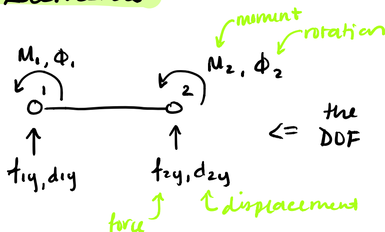
beams are usually long & slender members and the loading usually transverse



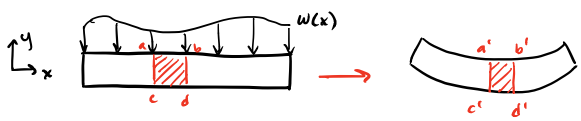


<= the DOF =>

sign convention:
1. positive moments counter cw
2. positive rotations counter cw
3. positive forces in (+) y-direction
4. positive displacement in (+) y

* The beam element has 4 DOF's:
- 2 deflections (d_{1y}, d_{2y})
- 2 rotations (ϕ_1, ϕ_2)

We assume Bernoulli-Euler beam theory:
→ assumes plane cross sections remain planar after bending



assumes a'c' and b'd' are still orthogonal to bent x' axis after bending
→ implies no change in angle across cross-sectional element, so this implies no shear

*why we use solid element for final project to show shear

Implications:
1. beam elements good for pure bending, but doesn't capture any other deformation

Reminder of beam equations:

2nd order DE : $M = EI \frac{d^2v}{dx^2}$ v = vertical deflection

3rd order DE : $V = EI \frac{d^3v}{dx^3}$ V = shear force

4th order DE : $w = EI \frac{d^4v}{dx^4}$ w = distributed load

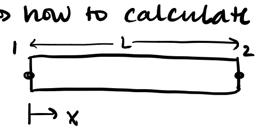
↳ assume no distributed loads, so $w = 0$, and $EI \frac{d^4v}{dx^4} = 0$ ← approximate this

→ select displacement function

$v(x) = a_1x^3 + a_2x^2 + a_3x + a_4$ ... chose cubic function bc gives 4 DOF (a_1, a_2, a_3, a_4) and matches beam element 4 DOF $(d_1, d_2, \phi_1, \phi_2)$

→ also, take four derivatives, satisfies $d^4v/dx^4 = 0$

→ how to calculate a_1, a_2, a_3, a_4:



- $v(x=0) = d_{1y} = a_4$
- $\frac{dv}{dx}(x=0) = \phi_1$ (slope at node 1) $= 3a_1x^2 + 2a_2x + a_3 \Rightarrow \phi_1 = a_3$
- $v(x=L) = d_{2y} = a_1L^3 + a_2L^2 + \phi_1 L + d_{1y}$
- $\frac{dv}{dx}(x=L) = \phi_2 = 3a_1L^2 + 2a_2L + \phi_1$

} 2 equations, 2 unknowns (a_1 and a_2)

↳ solve this system of equations:

$$a_2 = \frac{3}{L^2}(d_{2y} - d_{1y}) - \frac{1}{L}(2\phi_1 + \phi_2)$$

$$a_1 = \frac{1}{L^2}(\phi_1 + \phi_2) + \frac{2}{L^3}(d_{1y} - d_{2y})$$

now substitute $a_1 \rightarrow a_4$ into $v(x) = a_1 x^3 + a_2 x^2 + a_3 x + a_4 \ldots$

$$v(x) = \underbrace{\left[\frac{2}{L^3}(d_{1y} - d_{2y}) + \frac{1}{L^2}(\phi_1 + \phi_2)\right]}_{a_1} x^3 + \underbrace{\left[-\frac{3}{L^2}(d_{1y} - d_{2y}) - \frac{1}{L}(2\phi_1 + d_2)\right]}_{a_2} x^2 + \underbrace{\phi_1}_{a_3} x + \underbrace{d_{1y}}_{a_4}$$

put in FEM form: $v = Nd$

$N = [N_1 \; N_2 \; N_3 \; N_4]$
$$d = \begin{bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{bmatrix}$$

$$N_1 = \frac{1}{L^3}(2x^3 - 3x^2 L + L^3)$$

$$N_2 = \frac{1}{L^3}(x^3 L - 2x^2 L^2 + x L^3)$$

$$N_3 = \frac{1}{L^3}(-2x^3 + 3x^2 L)$$

$$N_4 = \frac{1}{L^3}(x^3 L - x^2 L^2)$$

Notes on shape functions here:

1. called "cubic hermite shape functions"
   - $\Sigma N_I = 1$
   - $N_1(x=0) = 1$, $N_1(x=L) = 0$
2. N_1, N_3 are dimensionless i.e. L^3/L^3
   - but N_2, N_4 have dimensions L^4/L^3, i.e. L
   - because N_2, N_4 associated with ϕ_1, ϕ_2 and rotations are derivatives of d_{1y}, d_{2y} so N_2, N_4 need higher dimension that N_1, N_3

Derive element stiffness matrix and equations:

moments + forces in opp directions here so they balance out

$$f_{1y} = V = EI \frac{d^3 v}{dx^3}(x=0) \qquad v = a_1 x^3 + a_2 x^2 + a_3 x + a_4$$

$$\rightarrow f_{1y} = V = EI(6a_1)$$

$$f_{2y} = -V = -EI \frac{d^3 v}{dx^3}(x=L)$$

$$\rightarrow f_{2y} = -V = -EI(6a_1)$$

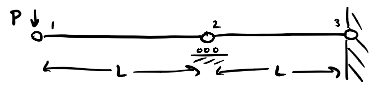
$$m_1 = -M = -EI \frac{d^2 v}{dx^2}(x=0) \quad \ldots \rightarrow \frac{d^2 v}{dx^2}(x=0) = 2a_2$$

$$m_2 = M = EI \frac{d^2 v}{dx^2}(x=L) \quad \ldots \rightarrow \frac{d^2 v}{dx^2}(x=L) = 6a_1 L + 2a_2$$

assemble these together:

$$\underbrace{\begin{bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{bmatrix}}_{f^{ext}} = \underbrace{\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}}_{K} \underbrace{\begin{bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{bmatrix}}_{d}$$

Example:



cantilever beam with 3 nodes, 2 elements, node 2 on roller, load P on node 1, element length = L

boundary conditions:
- $d_{2y} = d_{3y} = 0$, $\phi_3 = 0$

stiffness matrix assembly done in the same way!
- 6 unknown DOF

$$\begin{bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ & 4L^2 & -6L^2 & 2L & 0 & 0 \\ & & 12+12 & -6L+6L & -12 & 6L \\ & & & 4L^2+4L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ & & & & & 4L^2 \end{bmatrix} \begin{bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{bmatrix}$$

symmetric

coupling at 2x2 block due to node 2 being shared and having 2 DOFs

apply boundary conditions the same way (zero out rows/cols for fixed nodes)

$$\begin{bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 \\ & 4L^2 & -6L^2 & 2L & 0 & 0 \\ & & 12+12 & -6L+6L & -12 & 6L \\ & & & 4L^2+4L^2 & -6L & 2L^2 \\ & & & & 12 & -6L \\ & & & & & 4L^2 \end{bmatrix} \begin{bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{bmatrix}$$

$$\begin{bmatrix} F_{1y} = -P \\ M_1 = 0 \\ M_2 = 0 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & 6L \\ 6L & 4L^2 & 2L^2 \\ 6L & 2L^2 & 8L^2 \end{bmatrix} \begin{bmatrix} d_{1y} \\ \phi_1 \\ \phi_2 \end{bmatrix}$$

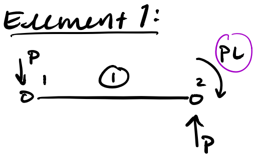
$$\Rightarrow \quad d_{1y} = \frac{-7PL^3}{12EI}, \quad \underbrace{\phi_1 = \frac{3PL^2}{4EI}, \quad \phi_2 = \frac{PL^2}{4EI}}_{\text{positive so counter clockwise rotation}}$$

calculate reaction forces (element by element)

Element 1:

$$\begin{bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} K \end{bmatrix} \begin{bmatrix} -7PL^3/12EI = d_{1y} \\ 3PL^2/4EI = \phi_1 \\ 0 = d_{2y} \\ PL^2/4EI = \phi_2 \end{bmatrix} \Rightarrow \quad f_{1y} = -P, \quad f_{2y} = P, \quad m_1 = 0, \quad m_2 = -PL$$

FBD of Element 1:



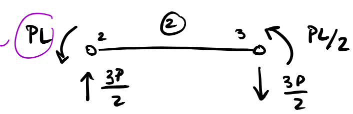
→ element 1, node 2 no contribution to global reaction force at node 2

Element 2:

$$\begin{bmatrix} f_{2y} \\ m_2 \\ f_{3y} \\ m_3 \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} & & & \\ & K & & \\ & & & \end{bmatrix}\begin{bmatrix} 0 = d_{2y} \\ PL^2/4EI = \phi_2 \\ 0 = d_{3y} \\ 0 = \phi_3 \end{bmatrix} \Rightarrow \begin{array}{ll} f_{2y} = 3P/2, & f_{3y} = 3P/L \\ m_2 = PL, & m_3 = PL/2 \end{array}$$

<u>FBD of element 2</u>:

moment at node 2 sum = 0



global reaction force at node 2:

$P + 3P/2 = 5P/2$

Shear/Moment diagram: