DERIVATION OF THE CONTINUITY EQUATION

* the amount of mass in a system does not change with time *

a system is defined as a collection of unchanging contents so the conservation of mass principle for a system is stated as the time rate of change of the system mass = 0, or

$$\frac{DM_{sys}}{Dt} = 0$$ where the system mass is generally expressed as $$M_{sys} = \int_{sys} \rho \, d\forall$$

→ basically means that the system mass is equal to the sum of all the density-volume element products for the contents of the system

REYNOLD'S TRANSPORT THEOREM allows us to state that for a system with a fixed, nondeforming control volume that

B = mass

b = 1

$$\frac{D}{Dt}\int_{sys} \rho \, d\forall = \frac{d}{dt}\int_{cv} \rho \, d\forall + \int_{cs} \rho \vec{V} \cdot \hat{n} \, dA$$

$$\begin{bmatrix}\text{time rate} \\ \text{of change} \\ \text{of mass of} \\ \text{system}\end{bmatrix} = \begin{bmatrix}\text{time rate of} \\ \text{change of} \\ \text{mass of} \\ \text{contents of CV}\end{bmatrix} + \begin{bmatrix}\text{net rate flow} \\ \text{of mass} \\ \text{through the} \\ \text{control surface}\end{bmatrix}$$

when a flow is steady, all field properties including density remain constant w/time and the time rate of change of the mass of the contents of the CV is zero,

$$\frac{d}{dt}\int_{cv} \rho \, d\forall = \emptyset$$

the integrand $\vec{V} \cdot \hat{n} \, dA$ in the mass flow rate integral represents the product of the component of velocity ⊥ to the CS and differential area dA ⇒ then $\vec{V} \cdot \hat{n} \, dA$ is the volume flowrate through dA and $\rho \vec{V} \cdot \hat{n} \, dA$ is the mass flow rate through dA

(+) $\vec{V} \cdot \hat{n} \, dA$ for flow out of the CV

(−) $\vec{V} \cdot \hat{n} \, dA$ for flow into the CV

{ $\hat{n}$ is (+) when points out of CV }

The net mass flow rate through the CS is then

$$\int_{cs} \rho \vec{V} \cdot \hat{n} \, dA = \sum \dot{m}_{out} - \sum \dot{m}_{in}$$

when all of the differential quantities $\rho \vec{V} \cdot \hat{n} \, dA$ are summed over the entire control surface

→ if the integral is (+), net flow is out of the CV

→ if the integral is (−), net flow is into the CV

The control volume expression for conservation of mass is called the **Continuity Equation** for a fixed, nondeforming control volume

$$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \vec{V} \cdot \hat{n} \, dA = 0$$

{the continuity equation is a statement that mass is conserved}

* this equation states that because mass is conserved the time rate of change of the mass of the contents of the control volume plus the net rate of mass flow through the control surface must equal zero. We get the same result if we equate the rates of mass flow into and out of the cv to the rates of accumulation and depletion of mass within the CV. We can see that Reynold's Transport Theorem works in this case *

Mass flowrate equals the product of density and volume flowrate

$$\dot{m} = \rho Q = \rho A V$$

where V is the component of fluid velocity $\perp$ to the control surface area A

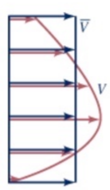
IMPORTANT:

since $\dot{m} = \int_A \rho \vec{V} \cdot \hat{n} \, dA$, the equation $\dot{m} = \rho Q = \rho A V$ involves the use of <u>representative or average</u> values of fluid density ρ & fluid velocity V.

→ for incompressible flows, ρ is uniformly distributed over area A.

→ for compressible flows, we'll assume a uniformly distributed fluid density at each section of flow and allow density changes to occur only from section to section

∴ the appropriate fluid velocity to use in $\dot{m} = \rho Q = \rho A V$ is the average value of the component of velocity normal to the section area involved. The average value $\bar{V}$ is defined as



$$\bar{V} = \frac{\int_A \rho V \cdot \hat{n} \, dA}{\rho A}$$

if the velocity $\vec{V}$ is assumed to be uniformly distributed (one-dimensional flow) over the section area A, then $\bar{V} = \frac{\int_A \rho \vec{V} \cdot \hat{n} dA}{\rho A} = V$ and the bar notation is unnecessary

*when the flow is not uniformly distributed over the flow cross-sectional area, the bar notation reminds us that an ave. velocity is being used.

For steady-flow, mass flowrate in equals mass flowrate out

$$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall = 0 \qquad \sum \dot{m}_{out} - \sum \dot{m}_{in} = 0$$

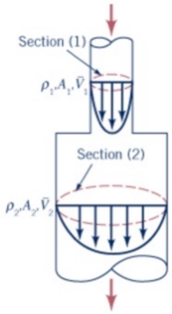
if steady flow is incompressible: $\sum Q_{out} - \sum Q_{in} = 0$

An unsteady but cyclical flow can be considered steady on a time average basis. When the flow is unsteady, the instantaneous time rate of change of the mass of the contents of the control volume is not necessarily zero. When the value of $\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall$ is

⊕ ... the mass contents of the cv is increasing

⊖ ... the mass contents of the cv is decreasing

When the flow is uniformly distributed over the opening in the CS $\dot{m} = \rho A V$ where V is the uniform value of the velocity component normal to the section area A. When the velocity is nonuniformly distributed over the opening in the CS, $\dot{m} = \rho A \bar{V}$ where $\bar{V}$ is the average value of the component of velocity normal to the section area A



For steady flow involving only one stream of a specific fluid flowing through the CV at sections (1) and (2)

$$\dot{m} = \rho_1 A_1 \bar{V}_1 = \rho_2 A_2 \bar{V}_2$$

and for incompressible flow, $Q = A_1 \bar{V}_1 = A_2 \bar{V}_2$

For steady flow involving more than one stream of a specific fluid or more than one specific fluid flowing through the control volume $\sum \dot{m}_{in} = \sum \dot{m}_{out}$

EXAMPLE 5.1 Conservation of Mass—Steady, Incompressible Flow

GIVEN

A great danger to workers in confined spaces involves the consumption of breathable air (oxygen) and its replacement with other gases such as carbon dioxide (CO_2). To prevent this from happening, confined spaces need to be ventilated. Although there is no standard for air exchange rates, a complete change of the air every 3 minutes has been accepted by industry as providing effective ventilation.

A worker is performing maintenance in a small rectangular tank with a height of 10 ft and a square base 6 ft by 6 ft. Fresh air enters through an 8-in.-diameter hose and exits through a 4-in.-diameter port on the tank wall. The flow is assumed steady and incompressible.

FIND

Determine

(a) the exchange rate needed (ft^3 / min) for this tank and

(b) the velocity of the air entering and exiting the tank at this exchange rate.

$$\forall = (10\,ft)(6\,ft)(6\,ft) = 360\,ft^3$$

$$Q = \frac{\forall}{t}$$

$$Q = \frac{360\,ft^3}{3\,min}$$

$$\boxed{Q = 120\,ft^3/min}$$

$$\cancel{\frac{\partial}{\partial t}\int_{CV} \rho\, dV} + \sum \rho_{out} V_{out} A_{out} - \sum \rho_{in} V_{in} A_{in} = 0$$

0 bc steady & incompressible

$\rho_{out} = \rho_{in}$

$$\Rightarrow V_{out} A_{out} - V_{in} A_{in} = 0$$

or $V_{out} A_{out} - V_{in} A_{in} = Q$

$$V_{out} = \frac{Q}{A_{out}} = \frac{120\,ft^3/min}{(\pi/4)(4/12\,ft)^2}\left(\frac{1\,min}{60\,s}\right) = \boxed{22.9\,ft/s = V_{out}}$$

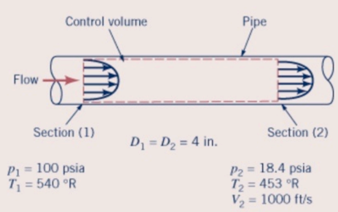
$$V_{in} = \frac{Q}{A_{in}} = \frac{120\,ft^3/min}{(\pi/4)(8/12\,ft)^2}\left(\frac{1\,min}{60\,s}\right) = \boxed{5.73\,ft/s = V_{in}}$$

EXAMPLE 5.2 Conservation of Mass—Steady, Compressible Flow

GIVEN

Air flows steadily between two sections in a long, straight portion of 4-in. inside diameter pipe as indicated in Fig. E5.2. The uniformly distributed temperature and pressure at each section are given. The average air velocity (nonuniform velocity distribution) at section (2) is 1000 ft/s.

AIR = ideal gas



■ **Figure E5.2**

∴ continuity equation is valid for compressible also

FIND Calculate the average air velocity at section (1).

$$\frac{\partial}{\partial t}\underbrace{\cancel{\int_{CV} \rho\, d\forall}}_{0\ \text{steady}} + \int_{CS} \rho \vec{V} \cdot \hat{n}\, dA = 0$$

$\int_{CS} \rho \vec{V} \cdot \hat{n}\, dA = \dot{m}_2 - \dot{m}_1 = 0$ or

$\dot{m}_1 = \dot{m}_2 \rightarrow \rho_1 A_1 \bar{V}_1 = \rho_2 A_2 \bar{V}_2$

since $A_1 = A_2$

$\rightarrow \bar{V}_1 = \frac{\rho_2}{\rho_1} \bar{V}_2$, $\rho = \frac{p}{RT}$ (IDEAL)

$$\rightarrow \bar{V}_1 = \frac{p_2 T_1 \bar{V}_2}{p_1 T_2}$$

$$\boxed{\bar{V}_1 = 219\,ft/s}$$

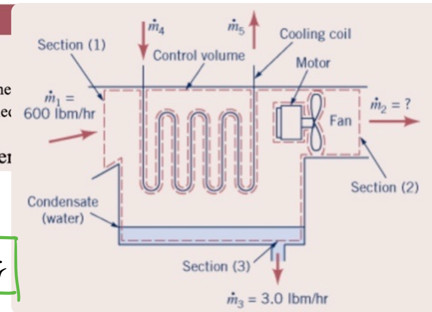
*non uniform velocity can be handled w/average velocity concept

EXAMPLE 5.3 Conservation of Mass—Two Fluids

GIVEN

The inner workings of a dehumidifier are shown in Fig. E5.3a. Moist air (a mixture of dry air and water vapor) enters the dehumidifier at the rate of 600 lbm / hr. Liquid water drains out of the dehumidifier at a rate of 3.0 lbm / hr. A simplified sketch of the process is provided in Fig. E5.3b.

FIND Determine the mass flowrate of the dry air and the water vapor leaving the dehumidifier



$$\int_{cs} \rho \vec{V}\cdot\hat{n}\,dA = -\dot m_1 + \dot m_2 + \dot m_3 = 0$$

$$\dot m_2 = \dot m_1 - \dot m_3 = 600\ \text{lbm/hr} - 3.0\ \text{lbm/hr} \qquad \boxed{\dot m = 597\ \frac{\text{lbm}}{\text{hr}}}$$

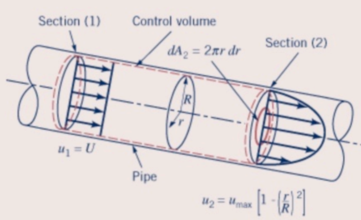
EXAMPLE 5.4 Conservation of Mass—Nonuniform Velocity Profile

GIVEN

Incompressible, laminar water flow develops in a straight pipe having radius R as indicated in Fig. E5.4a. At section (1), the velocity profile is uniform; the velocity is equal to a constant value U and is parallel to the pipe axis everywhere. At section (2), the velocity profile is axisymmetric and parabolic, with zero velocity at the pipe wall and a maximum value of u_{max} at the centerline.

$$\frac{d}{dt}\cancel{\int_{cv}}\overset{0\ (\text{steady})}{}\rho\, d\forall + \int_{cs}\rho\vec{V}\cdot\hat{n}\,dA = 0$$

$$\int_{(1)}\rho\vec{V}\cdot\hat{n}\,dA = -\rho_1 A_1 U$$



■ **Figure E5.4a**

At section (2) velocity is not uniform. But the net flow rate thru this section is sum of flows thru numerous small washer-shaped areas of size $dA_2 = 2\pi r dr$. On each of these infinitesimal areas the fluid velocity is denoted as u_2. Thus, the limit of infinitesimal area elements, the summation is replaced by an integration and the outflow thru section (2) is $\int_{(2)}\rho\vec{V}\cdot\hat{n}\,dA = \rho_2\int_0^R u_2 2\pi r\,dr$

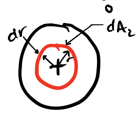
FIND

(a) How are U and u_{max} related?

(b) How are the average velocity at section (2), $\overline{V}_2$, and u_{max} related?

$$-\rho_1 A_1 U + \rho_2\int_0^R u_2 2\pi r\,dr = 0$$ and since the flow is incompressible $\rho_1 = \rho_2$

$$2\pi u_{max}\int_0^R\left[1-\left(\frac{r}{R}\right)^2\right]r\,dr - A_1 U = 0 \;\Rightarrow\; 2\pi u_{max}\left(\frac{r^2}{2} - \frac{r^4}{4R^2}\right)_0^R - \pi R^2 U = 0$$



$$\boxed{u_{max} = 2U}$$

(b) since flow is incompressible and area is constant, U is the average velocity at all sections of the CV. Thus, the ave velocity at section (2), $\overline{V}_2$ is one half the maximum velocity, u_{max}, there or $\boxed{\overline{V}_2 = \frac{u_{max}}{2}}$

EXAMPLE 5.5 Conservation of Mass—Unsteady Flow

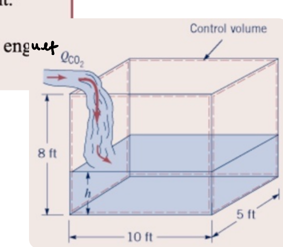
GIVEN

Construction workers in a trench of the type shown in Fig. E5.5a are installing a new waterline. The trench is 10 ft long, 5 ft wide, and 8 ft deep. As a result of being near an intersection, carbon dioxide from vehicle exhaust enters the trench at a rate of 10 ft^3 / min. Because carbon dioxide has a greater density than air, it will settle to the bottom of the trench and displace the air the workers need to breathe. Assume that there is negligible mixing between the air and carbon dioxide.

FIND

(a) Estimate the time rate of change of the depth of carbon dioxide in the trench, $\partial h/\partial t$, in feet per minute at any instant.

(b) Calculate the time, $t_{h=6}$, it would take for the level of carbon dioxide to reach 6 ft, the approximate height to fully engulf the utility workers.



(a)

$$\frac{d}{dt}\int_{\substack{\text{air}\\ \text{volume}}}\rho_{air}\,d\forall_{air} + \frac{d}{dt}\int_{\substack{CO_2\\ \text{volume}}}\rho_{CO_2}\,d\forall_{CO_2} - \dot m_{CO_2} + \dot m_{air} = 0$$

$$dm = \rho\, d\forall$$

FOR AIR:

$$\frac{d}{dt}\int_{\text{air volume}} \rho_{air}\, d\forall_{air} + \dot{m}_{air} = 0 \quad \underline{\underline{(1)}}$$

FOR CARBON DIOXIDE:

$$\frac{d}{dt}\int_{CO_2 \text{ volume}} \rho_{CO_2}\, d\forall_{CO_2} = \dot{m}_{CO_2} \quad \underline{\underline{(2)}}$$

$$\underline{\underline{(3)}} \quad \int_{CO_2 \text{ vol.}} \rho_{CO_2}\, d\forall_{CO_2} = \rho_{CO_2}[h(10\,ft)(5\,ft)]$$

$$\boxed{\frac{dh}{dt} = 0.20\ ft/min}$$

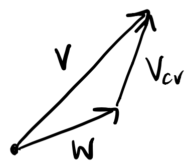
$$\underline{(2)} \ \&\ \underline{(3)} \quad \rho_{CO_2}(50\,ft^2)\frac{dh}{dt} = \dot{m}_{CO_2}$$

and since $\dot{m} = \rho Q \rightarrow \frac{dh}{dt} = \frac{Q_{CO_2}}{50\,ft^2} = \frac{10\,ft^3/min}{50\,ft^2}$

(b) $h = 6\,ft$ $\quad t_{h=6ft} = \frac{6\,ft}{0.2\,ft/min} = \boxed{30\ min}$

MOVING, NONDEFORMING CONTROL VOLUME

when a moving control volume is used, the fluid velocity relative to the moving control volume (relative velocity) is an important flow field variable



W = relative velocity (fluid velocity seen by observer moving w/CV)
V_{cv} = velocity of CV as seen from fixed coordinate system
V = absolute velocity (fluid velocity seen by stationary observer in fixed coor-system)

For a system and a moving, nondeforming control volume that are coincident at an instant of time, the Reynold's Transport Theorem for a moving CV leads to

$$\frac{DM_{sys}}{Dt} = \frac{d}{dt}\int_{cv}\rho\, d\forall + \int_{cs}\rho\vec{W}\cdot\hat{n}\, dA$$

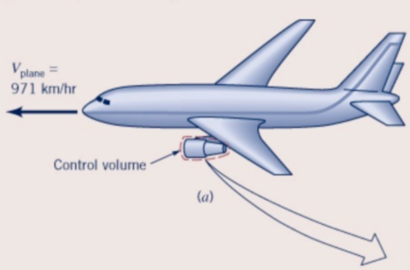
We can get the CV expression for conservation of mass (continuity equation) for a moving, nondeforming CV, namely, $\frac{d}{dt}\int_{cv}\rho\, d\forall + \int_{cs}\rho\vec{W}\cdot\hat{n}\, dA = 0$

EXAMPLE 5.6 Conservation of Mass—Compressible Flow with a Moving Control Volume

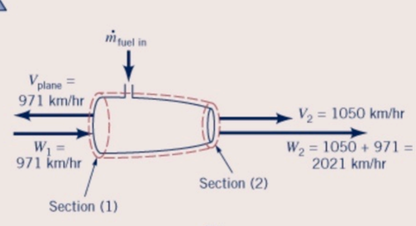
GIVEN

An airplane moves forward at a speed of 971 km/hr as shown in Fig. E5.6a. The frontal intake area of the jet engine is 0.80 m^2 and the entering air density is 0.736 kg/m^3. A stationary observer determines that relative to the Earth, the jet engine exhaust gases move away from the engine with a speed of 1050 km/hr. The engine exhaust area is 0.558 m^2, and the exhaust gas density is 0.515 kg/m^3.

FIND: Estimate the mass flowrate of fuel into the engine in kg/hr



(a)



(b)

flow relative to moving CV is considered steady on time-average basis

$$\cancel{\frac{d}{dt}\int_{cv}\rho\, d\forall}^{\,0} + \int_{cs}\rho\vec{W}\cdot\hat{n}\, dA = 0$$

Assuming 1-D flow:

$$-\dot{m}_{\substack{fuel\\in}} - \rho_1 A_1 W_1 + \rho_2 A_2 W_2 = 0$$

$$\dot{m}_{\substack{fuel\\in}} = \rho_2 A_2 W_2 - \rho_1 A_1 W_1$$

$$V_2 = W_2 + V_{plane}$$

$$W_2 = V_2 - V_{plane} = 1050\ \frac{km}{hr} - \left(-971\ \frac{km}{hr}\right) = 2021\ km/hr$$

$$\dot{m}_{\substack{fuel\\in}} = (0.515\ kg/m^3)(0.558\ m^2)(2021\ km/hr)(1000\ m/km) - (0.763\ \tfrac{kg}{m^3})(0.80\ m^2)(971\ \tfrac{km}{hr})(1000\ m/km)$$
$$= (580{,}800 - 571{,}700)\ kg/hr$$

$$\boxed{\dot{m}_{\substack{fuel\\in}} = 9100\ kg/hr}$$

EXAMPLE 5.7 **Conservation of Mass—Relative Velocity**

GIVEN

Water enters a rotating lawn sprinkler through its base at the steady rate of 1000 ml/s as sketched in Fig. E5.7. The exit area of each of the two nozzles is 30 mm^2.

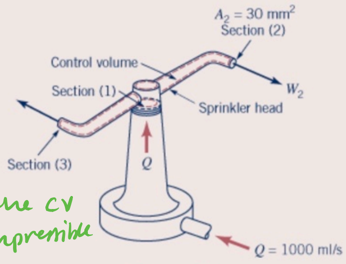


Figure E5.7

∅ flow is steady or the CV is filled w/an incompressible fluid

FIND: Determine the average speed of the water leaving the nozzle, relative to the nozzle, if
(a) rotary sprinkler head stationary
(b) sprinkler head rotates at 600 rpm
(c) sprinkler head accelerates from 0 to 600 rpm

$$\cancel{\frac{\partial}{\partial t}}\int_{cv}\rho\, d\mathcal{V} + \int_{cs}\rho\vec{W}\cdot\hat{n}\, dA = 0 \Rightarrow \sum\rho_{out}A_{out}W_{out} - \sum\rho_{in}A_{in}W_{in} = 0$$

The time rate of change of the mass of water in CV is zero because the flow is steady and the CV is filled w/water. Because there's only one inflow (at the base, section (1)), and two outflows (sections (2) and (3)) each with the same area and fluid velocity

$$\rho_2A_2W_2 + \rho_3A_3W_3 - \rho_1A_1W_1 = 0$$ and because of incompressible flow $\Rightarrow A_2W_2 + A_3W_3 - A_1W_1 = 0$

$Q = A_1W_1$, $A_2 = A_3$, $W_2 = W_3$

$$W_2 = \frac{Q}{2A_2} = \frac{(1000\ ml/s)(0.001\ m^3/liter)(10^6\ mm^2/m^2)}{(1000\ ml/liter)(2)(30\ mm^2)} = \boxed{16.7\ m/s = W_2}$$

for (b) and (c), the value of W_2 is independent of the speed of rotation of the sprinkler head and represents the average velocity of the water exiting from each nozzle wrt the nozzle for cases a, b, c.

The velocity of water discharging from each nozzle, when viewed from a stationary reference (V_2), will vary as the rotation speed of the sprinkler head varies since $V_2 = W_2 - U$
where $U = \omega R$ is the speed of nozzle (R is radius of head, ω = ang. vel)

Takeaway, if the flow within the moving CV is steady, or steady on a time-average basis, the time rate of change of the mass of the contents of the CV is zero. Velocities seen from the CV reference frame (relative velocities) must be used in the continuity equation. Relative and absolute velocities are related by a vector equation $\vec{V} = \vec{W} + \vec{V}_{cv}$ which also involves the control volume velocity.

FLOWRATE IS CALCULATED USING THE VELOCITY RELATIVE TO THE CV

<u>Deforming CV</u> – involves changing volume size and control surface movement. Thus, the RTT for a moving CV can be used for this case

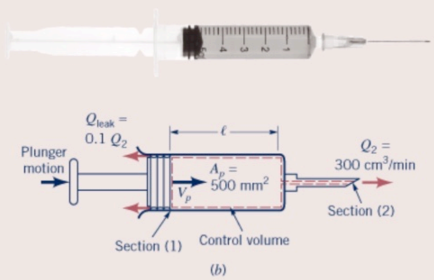
$$\frac{DM_{sys}}{Dt} = \underbrace{\frac{\partial}{\partial t}\int_{cv}\rho\, d\mathcal{V}}_{\text{this time rate of change term is usually nonzero and needs to be evaluated carefully}} + \underbrace{\int_{cs}\rho\vec{W}\cdot\hat{n}\, dA}_{\text{mass flow rate term must be determined with the relative velocity } \vec{W}\text{, the velocity referenced to the control surface}} = 0$$

Since the CV is deforming, the control surface velocity is not necessarily uniform and identical to the CV velocity V_{cv}, as was true for moving, nondeforming control volumes. The velocity of the surface of a deforming control volume is not the same at all points on the surface. For the deforming CV, $\vec{V} = \vec{W} + \vec{V}_{cs}$ where $\vec{V}_{cs}$ is the velocity of the cs as seen by a fixed observer. The relative velocity $\vec{W}$ must be ascertained with care wherever fluid crosses the control surface

EXAMPLE 5.8 Conservation of Mass—Deforming Control Volume

GIVEN
A syringe (Fig. E5.8a) is used to inoculate a cow. The plunger has a face area of 500 mm². The liquid in the syringe is to be injected steadily at a rate of 300 cm³/min. The leakage rate past the plunger is 0.10 times the volume flowrate out of the needle.

we can say $A_1 = A_p$ (even though not technically true since leakage occurs)



■ Figure E5.8

$$\frac{\partial}{\partial t}\int_{cv} \rho\, d\forall + \dot{m}_2 + \rho Q_{leak} = 0$$

*even though Q_{leak} and flow thru A_2 are steady, time rate of change of mass of liquid in shrinking CV is not zero bc the CV is getting smaller, so

$\int_{cv} \rho\, d\forall = \rho(\ell A_1 + \forall_{needle})$ where ℓ = changing length of CV

FIND With what speed should the plunger be advanced?

$\frac{\partial}{\partial t}\int_{cv} \rho\, d\forall = \rho A_1 \frac{\partial \ell}{\partial t}$ where $-\frac{\partial \ell}{\partial t} = V_p$ and V_p is what we're solving for ...

combining it all: $-\rho A_1 V_p + \dot{m}_2 + \rho Q_{leak} = 0$ and $\dot{m}_2 = \rho Q_2$

$$-\rho A_1 V_p + \rho Q_2 + \rho Q_{leak} = 0 \implies V_p = \frac{Q_2 + Q_{leak}}{A_1} \qquad Q_{leak} = 0.1\,Q_2$$

$$V_p = \frac{1.1\,Q_2}{A_1} = \frac{(1.1)(300\ \text{cm}^3/\text{min})}{500\ \text{mm}^2}\left(\frac{1000\ \text{mm}^3}{\text{cm}^3}\right) \rightarrow \boxed{V_p = 660\ \text{mm/min}}$$

EXAMPLE 5.9 Conservation of Mass—Deforming Control Volume

GIVEN
Consider Example 5.5.

FIND Solve the problem of Example 5.5 using a deforming control volume that includes only the carbon dioxide accumulating in the trench.

For this deforming CV $\rightarrow$ $\underbrace{\frac{\partial}{\partial t}\int_{CO_2\ \text{volume}} \rho\, d\forall}_{①} + \underbrace{\int_{cs} \rho\, \vec{W}\cdot\hat{n}\, dA}_{②} = 0$

① $\frac{\partial}{\partial t}\int_{CO_2\ \text{vol}} \rho\, d\forall = \frac{\partial}{\partial t}\left[\rho h (10\ \text{ft})(5\ \text{ft})\right] = \rho(50\ \text{ft}^2)\frac{\partial h}{\partial t}$

② $\int_{cs} \rho\, \vec{W}\cdot\hat{n}\, dA = -\rho\left(V_{CO_2} + \frac{\partial h}{\partial t}\right) A_{CO_2}$ the overall mass flow term is ⊖ since CO_2 is entering cv

$$\rho(50\ \text{ft}^2)\frac{\partial h}{\partial t} + -\rho\left(V_{CO_2} + \frac{\partial h}{\partial t}\right)A_{CO_2} = 0 \rightarrow (50\ \text{ft}^2)\frac{\partial h}{\partial t} - V_{CO_2}A_{CO_2} - A_{CO_2}\frac{\partial h}{\partial t} = 0$$

$$\frac{\partial h}{\partial t}(50\ \text{ft}^2 - A_{CO_2}) = V_{CO_2}A_{CO_2} \longrightarrow \frac{\partial h}{\partial t} = \frac{V_{CO_2}A_{CO_2}}{50\ \text{ft}^2 - A_{CO_2}} = \frac{Q_{CO_2}}{50\ \text{ft}^2 - A_{CO_2}}$$

taking $A_{CO_2} \ll 50\ \text{ft}^2$ $\rightarrow$ $\frac{\partial h}{\partial t} = \frac{10\ \text{ft}^3/\text{min}}{50\ \text{ft}^2} = \boxed{0.2\ \text{ft/min}}$

REMEMBER, in general where fluid flows through CS, it's advisable to make CS ⊥ to the flow

NEWTON'S SECOND LAW — THE LINEAR MOMENTUM AND MOMENT-OF-MOMENTUM EQUATIONS

time rate of change of linear momentum of sys = sum of external forces acting on the system

momentum = mass times velocity

→ the momentum of a small particle of mass $\rho d\forall$ is ∴ $\vec{V}\rho d\forall$

then the momentum of an entire system is $\int_{sys} \vec{V}\rho d\forall$ and Newton's Law becomes $\frac{D}{Dt}\int_{sys}\vec{V}\rho d\forall = \sum \vec{F}_{sys}$

any reference or coordinate system for which this statement is true is called <u>inertial</u>. A fixed coordinate system is <u>inertial</u>. A coordinate system that moves in a straight line w/ constant velocity and is thus without acceleration is also <u>inertial</u>.

Forces acting on a flowing fluid can change its velocity magnitude and/or direction.

For a system and the contents of a coincident control volume that is fixed and nondeforming, the RTT (with b = velocity or momentum per unit mass, and B_{sys} = system momentum) allows us to conclude that

$$\underbrace{\frac{D}{Dt}\int_{sys}\vec{V}\rho d\forall}_{\text{time rate of change of linear momentum of sys.}} = \underbrace{\frac{\partial}{\partial t}\int_{cv}\vec{V}\rho d\forall}_{\text{time rate of change of linear momentum of contents of CV}} + \underbrace{\int_{cs}\vec{V}\rho\vec{V}\cdot\hat{n}\,dA}_{\text{net rate of flow of linear momentum through the CS}}$$

This equation states that the time rate of change of system linear momentum is expressed as the sum of the two CV quantities:

(1) the time rate of change of the <u>linear momentum of the contents of CV</u>

(2) net rate of <u>linear momentum flow through the control surface</u>

For a CV that is fixed (thus inertial) and nondeforming, Newton's 2nd Law of Motion can be expressed as

$$\left[\frac{\partial}{\partial t}\int_{cv}\vec{V}\rho d\forall + \int_{cs}\vec{V}\rho\vec{V}\cdot\hat{n}\,dA = \sum F_{\text{contents of cv}}\right]$$

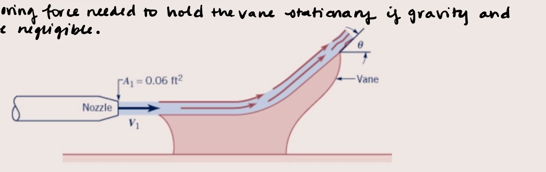
* THE LINEAR MOMENTUM EQUATION *

EXAMPLE 5.10 ***Linear Momentum—Change in Flow Direction***

GIVEN

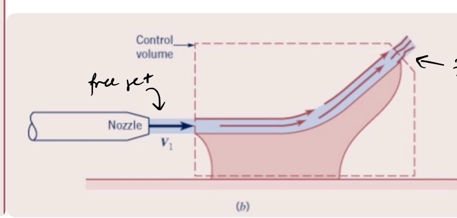
As shown in Fig. E5.10*a*, a horizontal jet of water exits a nozzle with a uniform speed of $V_1 = 10$ ft/s, strikes a vane, and is turned through an angle θ. A similar situation is also shown in Video V5.6.

Determine the anchoring force needed to hold the vane stationary if gravity and viscous effects are negligible.

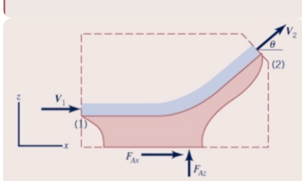


(a)

EXAMPLE w/ STEADY, INCOMPRESSIBLE FLOW



(b)



(c)

$$\cancel{\frac{\partial}{\partial t}\int_{cv} u\rho d\forall}^{\,0} + \int_{cs} u\rho\vec{V}\cdot\hat{n}\,dA = \sum F_x$$

FLOW IS STEADY

$$\cancel{\frac{\partial}{\partial t}\int_{cv} w\rho d\forall}^{\,0} + \int w\rho\vec{V}\cdot\hat{n}\,dA = \sum F_z$$

at free jets, $p_1 = p_2 = 0$

then speed of fluid remains constant (Bernoulli) so that $V_1 = V_2 = 10$ ft/s

(negligible gravity & viscous effects)

then... $u_2 \rho A_2 V_2 - u_1 \rho A_1 V_1 = \Sigma F_x$ where $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

$w_2 \rho A_2 V_2 - w_1 \rho A_1 V_1 = \Sigma F_z$

At Section (1): $u_1 = V_1,\ w_1 = 0$ At section (2): $u_2 = V_1 \cos\theta,\ w_2 = V_1 \sin\theta$

$\rightarrow \quad V_1 \cos\theta \rho A_2 V_2 - V_1 \rho A_1 V_1 = F_x$

$\rightarrow \quad V_1 \sin\theta \rho A_2 V_2 - 0 \rho A_1 V_1 = F_z$

using conservation of mass simplification, for this incompressible flow
$A_1 V_1 = A_2 V_2$ or $A_1 = A_2$ since $V_1 = V_2$

$\rightarrow F_x = \rho A_1 V_1^2 \cos\theta - \rho A_1 V_1^2 = \rho A_1 V_1^2 (\cos\theta - 1)$

$\rightarrow F_z = \rho A_1 V_1^2 \sin\theta$

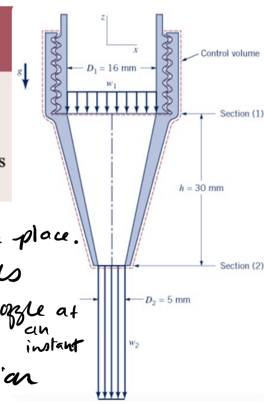
$F_x = (1.94\ \text{slugs/ft}^3)(0.06\ \text{ft}^2)(10\ \text{ft/s})^2(\cos\theta - 1) = \boxed{11.64(\cos\theta - 1)\ \text{lb} = F_x}$
aka $\dot{m} V_1 (\cos\theta - 1)$

$F_z = (1.94\ \text{slugs/ft}^3)(0.06\ \text{ft}^2)(10\ \text{ft/s})^2 \sin\theta = \boxed{11.64 \sin\theta\ \text{lb} = F_z}$
aka $\dot{m} V_1 \sin\theta$

EXAMPLE 5.11 ***Linear Momentum—Weight, Pressure, and Change in Speed***

GIVEN

As shown in Fig. E5.11a, water flows through a nozzle attached to the end of a laboratory sink faucet with a flowrate of 0.6 liters/s. The nozzle inlet and exit diameters are 16 mm and 5 mm, respectively, and the nozzle axis is vertical. The mass of the nozzle is 0.1 kg. The pressure at section (1) is 464 kPa.

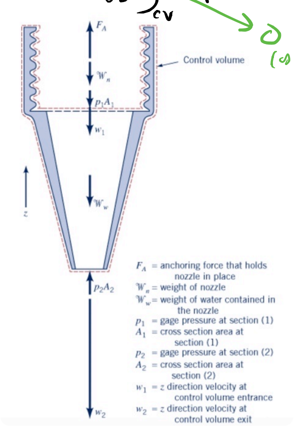


FIND: Determine the anchoring force required to hold the nozzle in place.
→ anchoring force is rxn force between faucet & nozzle threads
→ select CV that includes the entire nozzle & H_2O contained in nozzle at an instant
→ action of atmospheric pressure cancels out in every direction
→ gage pressure forces do not cancel out in the vertical direction

looking at the z-direction:

$$\cancel{\frac{\partial}{\partial t}\int_{cv} w \rho \, d\forall} + \int_{cs} w \rho \vec{V} \cdot \hat{n} \, dA = F_A - \mathcal{W}_n - p_1 A_1 - \mathcal{W}_w + p_2 A_2$$

0 (steady flow)



$\vec{V} \cdot \hat{n}$ (+) for flow out, $\vec{V} \cdot \hat{n}$ (−) for flow in

$\vec{V} \cdot \hat{n} \, dA = \pm |w| \, dA$

to evaluate the CS integral, we assume a distribution for fluid velocity w and fluid density ρ. we assume that w is uniformly distributed or constant, with magnitudes of w_1 and w_2 over cross-sectional areas A_1 and A_2. This flow is incompressible so the fluid density ρ constant throughout

$$(-\dot{m}_1)(-w_1) + (\dot{m}_2)(-w_2) = F_A - \mathcal{W}_n - p_1 A_1 - \mathcal{W}_w + p_2 A_2$$

where $\dot{m} = \rho A V$ is the mass flowrate and w_1 and w_2 are ⊖ bc both velocities are down, and $\dot{m}_1$ is ⊖ bc of the flow into the CV

$\therefore F_A = \dot{m}_1 w_1 - \dot{m}_2 w_2 + \mathcal{W}_n + p_1 A + \mathcal{W}_w - p_2 A_2$

and from the conservation of mass equation, $\dot{m}_1 = \dot{m}_2 = \dot{m}$, then we get

$$F_A = \dot{m}(w_1 - w_2) + \mathcal{W}_n + p_1 A_1 + \mathcal{W}_w - p_2 A_2$$

* note that nozzle wt, H_2O wt, and gage pressure at section (1) increase the anchoring force while the gage pressure force at (2) decreases it
* the change in vertical momentum flowrate, $\dot{m}(w_1 - w_2)$ will then decrease the anchoring force bc the change is ⊖ $(w_2 > w_1)$

$$\dot{m} = \rho w_1 A_1 = \rho Q = (999 \text{ kg/m}^3)(0.6 \text{ liter/s})(10^{-3} \text{ m}^3\text{/liter}) = 0.599 \text{ kg/s}$$

$$w_1 = \frac{Q}{A_1} = \frac{Q}{\pi (D_1^2/4)} = \frac{(0.6 \text{ liter/s})(10^{-3} \text{ m}^3\text{/liter})}{\pi (16 \text{ mm})^2 / 4 (1000^2 \text{ mm}^2\text{/m}^2)} = 2.98 \text{ m/s}$$

$$w_2 = \frac{Q}{A_2} = \frac{Q}{\pi (D_2^2/4)} = \frac{(0.6 \text{ liter/s})(10^{-3} \text{ m}^3\text{/liter})}{\pi (5 \text{ mm})^2 / 4 (1000^2 \text{ mm}^2\text{/m}^2)} = 30.6 \text{ m/s}$$

$$\mathcal{W}_n = m_n g = (0.1 \text{ kg})(9.81 \text{ m/s}^2) = 0.981 \text{ N}$$

$$\mathcal{W}_w = \rho \forall_w g \quad \text{where } \forall_w = \tfrac{1}{12}\pi h (D_1^2 + D_2^2 + D_1 D_2) = \tfrac{1}{12}\pi \frac{30 \text{ mm}}{1000 \text{ mm/m}} \left(\frac{(16 \text{ mm})^2 + (5 \text{ mm})^2 + (16 \text{ mm})(5 \text{ mm})}{(1000^2 \text{ mm}^2\text{/m}^2)} \right)$$

$$\forall_w = 2.84 \times 10^{-6} \text{ m}^3$$

$$\mathcal{W}_w = (999 \text{ kg/m}^3)(2.84 \times 10^{-6} \text{ m}^3)(9.81 \text{ m/s}^2) = 0.0278 \text{ N}$$

gage pressure at (2) = 0

$$F_A = (0.599 \tfrac{\text{kg}}{\text{s}})(2.98 \text{ m/s} - 30.6 \text{ m/s}) + 0.981 \text{ N} + 464(10^3)\text{Pa} \left(\frac{\pi (16 \text{ mm})^2}{4(10^6 \text{ mm}^2\text{/m}^2)} \right) + 0.0278 \text{ N} - 0$$

$$\boxed{F_A = 77.9 \text{ N}}$$

since the anchoring force is ⊕ it acts in upward direction (nozzle would be pushed off pipe otherwise)

Important generalities about application of linear momentum equation

1. when flow is uniform, integral is simplified
2. linear momentum is directional
3. flow into is ⊖ $\vec{V} \cdot \hat{n}$ dot product, flow out is ⊕ $\vec{V} \cdot \hat{n}$ dot product
4. if steady flow, time rate of change of linear momentum of contents of nondeforming CV $\frac{\partial}{\partial t}\int_{cv} \vec{V} \rho \, d\forall$ is $\emptyset$
5. if CS is ⊥ to flow, the surface force exerted at these locations by fluid outside CV on fluid inside will be due to pressure
6. the forces due to atmospheric pressure on CS cancel each other out and gage pressures may be used
7. only external forces acting on contents of CV are considered in linear momentum equation
8. force required to anchor an object will generally exist in response to surface pressure and/or shear forces acting on control surface, to a change in linear momentum flow thru CV containing object, and to weight of object and fluid contained in CV
   ex. from above, the nozzle anchoring force was required mainly bc of pressure forces and partly because of a change in linear momentum flow associated w/ accelerating the fluid in the nozzle ($\mathcal{W}_w$ and $\mathcal{W}_n$ only contributed slightly)