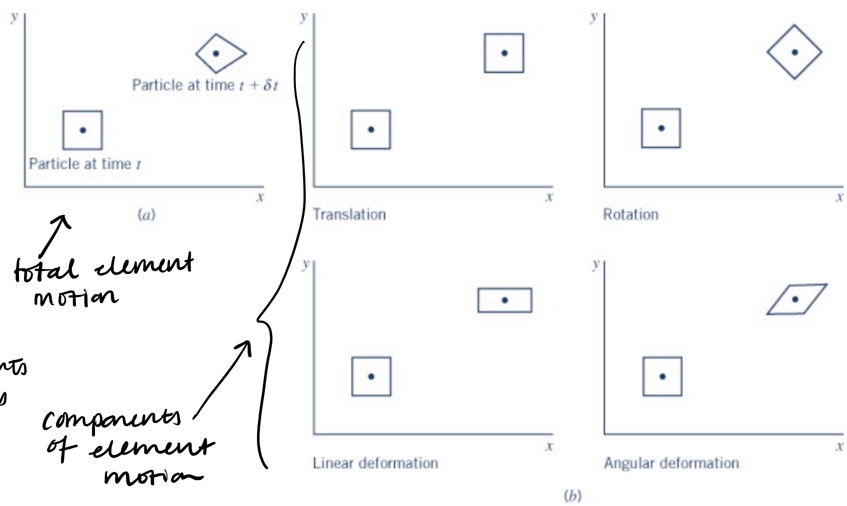
# Fluids Ch. 6 Notes

Fluid Element motion consists of
(1) translation
(2) linear deformation
(3) rotation
(4) angular deformation

*element motion and deformation are intimately related velocity and variation of velocity thru flow field

$\underset{\sim}{V}(x,y,z,t)$ = velocity of fluid particle depends on where it's located $(x,y,z)$ and time $t$
EULERIAN

$\underset{\sim}{V} = u\hat{i} + v\hat{j} + w\hat{k}$
where $u, v, w$ are the velocity components in the $x, y, z$ directions




total element motion

components of element motion

The acceleration of a fluid particle is described using the material derivative.

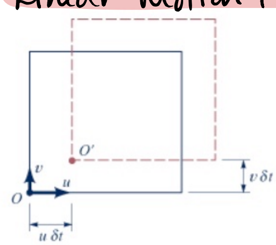
$$\underset{\sim}{a} = \frac{\partial \underset{\sim}{V}}{\partial t} + u\frac{\partial \underset{\sim}{V}}{\partial t} + v\frac{\partial \underset{\sim}{V}}{\partial t} + w\frac{\partial \underset{\sim}{V}}{\partial t}$$

$a_x = \partial u/\partial t + u\,\partial u/\partial x + v\,\partial u/\partial y + w\,\partial u/\partial z$
$a_y = \partial v/\partial t + u\,\partial v/\partial x + v\,\partial v/\partial y + w\,\partial v/\partial z$
$a_z = \partial w/\partial t + u\,\partial w/\partial x + v\,\partial w/\partial y + w\,\partial w/\partial z$

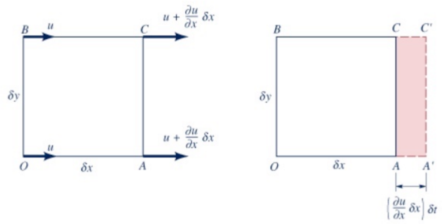
$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + u\frac{\partial(\ )}{\partial x} + v\frac{\partial(\ )}{\partial y} + w\frac{\partial(\ )}{\partial z}$$

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + (\underset{\sim}{V}\cdot\nabla)(\ )$$ where the gradient operator is $\nabla(\ ) = \frac{\partial(\ )}{\partial x}\hat{i} + \frac{\partial(\ )}{\partial y}\hat{j} + \frac{\partial(\ )}{\partial z}\hat{k}$

## Linear Motion & Deformation




← translation (simplest type of motion): in small time interval $\delta t$ a particle located at O will move to O' if all pts have the same velocity (there's no velocity gradients)




← linear deformation of fluid element, have stretching here where $\delta\forall = \left(\frac{\partial u}{\partial x}\delta x\right)\delta y\,\delta z\,(\delta t)$

and the rate at which $\forall$ changing per unit volume due to gradient $\partial u/\partial x$ is

$$\frac{1}{\delta\forall}\frac{d(\delta\forall)}{dt} = \lim_{\delta t\to 0}\left[\frac{(\partial u/\partial x)\,\delta t}{\delta t}\right] = \frac{\partial u}{\partial x}$$

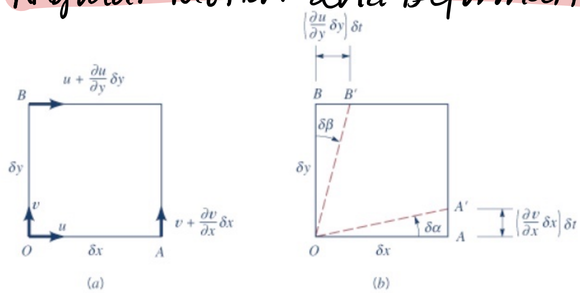
If the velocity gradients $\partial v/\partial y$ and $\partial w/\partial z$ are also present then in the general case: $\frac{1}{\delta \forall}\frac{d(\delta \forall)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \underset{\sim}{V}$

**Volume dilatation rate** = rate of change of vol per unit

- for incompressible fluid, volumetric dilatation rate must be zero since the element volume cannot change w/out a change in fluid density (element mass must be conserved

* Variations in the velocity in the direction of the velocity, as represented by the derivatives $\partial u/\partial x$, $\partial v/\partial y$, $\partial w/\partial z$ simply cause a linear deformation of the element in the sense that the shape of the element does not change

## Angular Motion and Deformation




(a) the velocity variation that causes rotation and angular deformation

(b) in a short time interval $\delta t$ the line segments OA & OB will rotate through the angles $\delta\alpha$ and $\delta\beta$ to the new positions OA' & OB'

the angular velocity of line OA, $\omega_{OA}$ is $\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t}$

and for small angles $\tan\delta\alpha \approx \delta\alpha = \frac{(\partial v/\partial x)\delta x \delta t}{\delta x} = \frac{\partial v}{\partial x}\delta t$

so that $\omega_{OA} = \lim_{\delta t \to 0}\left[\frac{(\partial v/\partial x)\delta t}{\delta t}\right] = \frac{\partial v}{\partial x}$

and if $\partial v/\partial x$ is ⊕ then ccw ... similar for OB but if $\partial u/\partial y$ ⊕ then cw

... the rotation $\omega_z$ of the element about the z-axis is defined as the average of the angular velocities $\omega_{OA}$ and $\omega_{OB}$ of the two mutually perpendicular lines OA and OB and if ccw is ⊕

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

**Rotation of fluid particles is related to certain velocity gradients in the flow field**

the rotation vector $\underset{\sim}{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} = \frac{1}{2}\text{curl}\,\underset{\sim}{V} = \frac{1}{2}\nabla \times \underset{\sim}{V}$

$$\therefore \underset{\sim}{\omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

**Vorticity $\zeta$** – twice the fluid particle rotation in a flow field

$$\zeta = 2\underset{\sim}{\omega} = \nabla \times \underset{\sim}{V}$$

**IRROTATIONAL** – $\nabla \times \underset{\sim}{V} = 0$, rotation and vorticity are zero

The change in the original right angle formed by OA and OB is the shearing strain $\delta\gamma$ where $\delta\gamma = \delta\alpha + \delta\beta$

$\dot{\gamma}$ = rate of shearing strain $= \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$

if $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ then rate of angular deformation is zero

## Conservation of Mass requires that the mass of a system remain constant

$$\frac{DM_{sys}}{Dt} = 0$$

continuity equation : $\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \vec{V} \cdot \hat{n} \, dA = 0$

## Differential Form of Continuity Equation

$$\frac{\partial}{\partial t}\int_{cv} \rho \, d\forall \simeq \frac{\partial \rho}{\partial t} \delta x \delta y \delta z$$

differential equation for conservation of mass : $\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$

... this fundamental equation is valid for
→ steady or unsteady flow
→ compressible or incompressible flow

... in vector notation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0$$

... for steady, incompressible flow then $\nabla \cdot \vec{V} = 0$ or $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$
bc $\rho$ isn't func of time for steady and $\rho$ is C throughout for incompress.

differential form of cont. equation in cylindrical coord. : $\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$

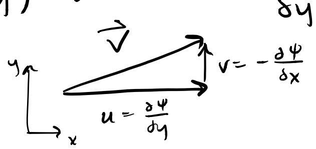
and for steady, compressible flow : $\frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial(\rho v_z)}{\partial z} = 0$

and for incompress. flow (steady or unsteady) : $\frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$

## The Stream Function

velocity components in a 2D flow field can be expressed in terms of stream function

$\psi(x,y)$ where $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$ from $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$



stream function satisfies continuity equation

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

lines along which $\psi$ is constant are streamlines

streamline line defn : $\frac{dy}{dx} = \frac{v}{u}$

the change in the value of the stream function is related to the volume rate of flow

the volume rate of flow $q$ between 2 streamlines : $q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$

the velocity components $V_r$ and $V_\theta$ can be related to stream function $\psi(r,\theta)$ through the equations

$$V_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad V_\theta = -\frac{\partial \psi}{\partial r}$$

## The Linear Momentum Equation

linear momentum eq. (Newton's 2nd Law) : $\underset{\sim}{F} = \left.\frac{D\underset{\sim}{P}}{Dt}\right|_{sys}$

where $\underset{\sim}{F}$ = resultant force acting on fluid mass

$\underset{\sim}{P}$ = linear momentum, $\underset{\sim}{P} = \int_{sys} \vec{V}\, dm$

for a finite CV: $\sum \vec{F}_{\text{contents of CV}} = \frac{\partial}{\partial t}\int_{cv} \rho \vec{V}\, d\forall + \int_{cs} \vec{V} \rho \vec{V} \cdot \hat{n}\, dA$

and for a differential system consisting of a mass $\delta m$

$$\delta \vec{F} = \frac{D(\vec{V}\delta m)}{Dt} \Rightarrow \delta \vec{F} = \delta m \frac{D\vec{V}}{Dt} \quad \text{and} \quad \frac{D\vec{V}}{Dt} = \text{acceleration } \vec{a}$$

$\therefore \delta \vec{F} = \delta m \vec{a}$

Both surface forces and body forces generally act on fluid particles

surface forces: act on differential element

body forces: distributed throughout element

only body force of interest is wt : $\delta \vec{F}_b = \delta m \vec{g}$

surface forces can be expressed in terms of shear and normal stresses