Viscous Flow In Pipes

→ assume pipe is fully filled w/ fluid so that can maintain Δp

laminar → transitional → turbulent

⇓ only one comp of velocity $\vec{V} = u\hat{i}$

↳ predominant velocity still along pipe, but is unsteady $\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$

→ pipe flow characteristics are <u>dependent</u> on value of $Re = \frac{\rho V D}{\mu}$

$Re < 2100$ LAMINAR

TRANSITIONAL (switches between the two randomly)

$Re > 4000$ TURBULENT

1. fluid enters pipe w/ uniform velocity profile [ENTRANCE]
2. as fluid moves, viscous effects cause stick to pipe wall [ENT. REGION)
   - → no slip boundary condition
   - → boundary layer produced along wall
     - causes initial velocity profile to change w/ distance, x
     - fluid outside boundary layer = inviscid core
3. once fluid reaches end of entrance length, velocity profile does not vary w/ x
   - → entrance length is func of Re

Fully developed steady flow may be driven by gravity or press. forces

→ horizontal pipe: just the press diff: $\Delta p = p_1 - p_2$

Steady, fully developed pipe flow experiences no acceleration

Fully developed horizontal pipe flow is balance between press & viscous forces

$$p_1 \pi r^2 - (p_1 - \Delta p)\pi r^2 - (\tau) 2\pi r l = 0$$ — acceleration = 0

pressure difference acting on end of cylinder of area πr^2

shear stress acting on lateral surface of cylinder of area $2\pi r l$

→ laminar flow of Newtonian Fluid, shear stress ∝ velocity gradient

$\tau = \mu \frac{du}{dy}$ ⇒ pipe flow notation: $\tau = -\mu \frac{du}{dy}$

Darcy Friction Factor: $f = \frac{64}{Re}$

$Re = \frac{\rho V \ell}{\mu} = \frac{\text{inertia force}}{\text{viscous force}}$

$Fr = \frac{V}{\sqrt{g\ell}} = \frac{\text{inertia force}}{\text{gravit. force}}$

$Eu = \frac{p}{\rho V^2} = \frac{\text{press force}}{\text{inertia force}}$

$Ca = \frac{\rho V^2}{E_v} = \frac{\text{inertia force}}{\text{compress. force}}$

$We = \frac{\rho V^2 \ell}{\sigma} = \frac{\text{inertia force}}{\text{surface tension force}}$

$Ma = \frac{V}{c} = \frac{\text{inertia force}}{\text{compressibility f.}}$

$K_L = \frac{h_L}{V^2/2g} = \frac{\text{energy loss}}{\text{kinetic energy}}$ (loss coefficient)

$C_d = \frac{D}{\frac{1}{2}\rho V^2 \ell^2} = \frac{\text{drag force}}{\text{inertia force}}$ (drag coefficient)

$C_L = \frac{L}{\frac{1}{2}\rho V^2 \ell^2} = \frac{\text{lift force}}{\text{inertia force}}$ (lift coefficient)

$C_f = \frac{\tau_w}{\frac{1}{2}\rho V^2} = \frac{\text{shear force}}{\text{inertia force}}$ (friction coefficient)

turbulent flow: τ_w func of ρ while laminar flow: τ_w independent of ρ so μ is only important fluid property

unlike laminar pipe flow characteristics, turbulent pipe flow characteristics depend on fluid density and pipe roughness

$$f = \text{friction factor} = \frac{\Delta p D}{\ell \rho V^2 \frac{1}{2}}$$

the head form of the mechanical energy equation for steady, incompressible flow

$$\frac{p_1}{\gamma} + \alpha_1 \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \alpha_2 \frac{V_2^2}{2g} + z_2 + h_L$$

where h_L = head loss between sections (1) and (2)

$\left.\begin{array}{l}\rightarrow \text{assume constant diameter, } D_1 = D_2 \text{ then } V_1 = V_2 \\ \rightarrow \text{assume horizontal } z_1 = z_2 \\ \rightarrow \text{assume fully developed flow } \alpha_1 = \alpha_2\end{array}\right\} \Delta p = p_1 - p_2 = \gamma h_L$

$$h_L = \frac{\Delta p}{\gamma} = \frac{1}{\gamma}\left(f \frac{\ell}{D} \rho V^2 \frac{1}{2}\right) \qquad \gamma = \rho g$$

$$= \frac{1}{\rho g}\left(f \frac{\ell}{D} \rho V^2 \frac{1}{2}\right)$$

$$h_L = f \frac{\ell}{D} \frac{V^2}{2g}$$ } the major head loss in pipe flow is given in terms of the friction factor

*part of the pressure change is due to the elevation change (z) and part is due to head loss associated w/ frictional effects for $V_1 = V_2$

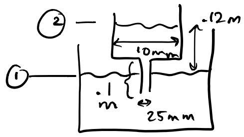
$$p_1 - p_2 = \gamma(z_2 - z_1) + \gamma h_L = \gamma(z_2 - z_1) + f \frac{\ell}{D} \frac{\rho V^2}{2}$$

1. Syringe injecting fluid into another reservoir

$SG = .96$

$\mu = 9.2(10^{-4})\ \frac{Ns}{m^2}$ } fluid drawn thru syringe

$P_v = 1.2(10^4)\ N/m^2$



$P_{atm} = 101\ kPa$

Max flow rate for no cavitation?

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2 + \left(f\frac{\ell}{D} + \Sigma K_L\right)\frac{V^2}{2g}$$

$P_1 = P_{atm} = 101\ kPa$

$z_1 = 0$

$V_1 = 0$

$z_2 = 0.12\ m$

Max flow rate occurs when P_2 is at a minimum → minimum it can be is cavitation press.

$P_2 = P_v = 1.2(10^4)\ Pa$

$$\frac{P_1}{\gamma} = \frac{P_v}{\gamma} + \frac{V_2^2}{2g} + z_2 + \left(f\frac{\ell}{D} + K_{L\,entrance} + K_{L\,exit}\right)\frac{V^2}{2g}$$

$$V_2 = \frac{VA}{A_2} = V\left(\frac{D}{D_2}\right)^2 = V\left(\frac{.25\ mm}{10\ mm}\right)^2 \Rightarrow V_2 = 0.000625V$$

∴ negligible, $V_2 \approx 0$

$$\frac{[101(10^3) - 1.2(10^4)]\ \frac{N}{m^2}}{(.96)(9.8(10^3)\ \frac{N}{m})} = .12\left(f\left(\frac{.1\ m}{.25\times10^{-3}\ m}\right) + .5 + 1\right)\frac{V^2}{2(9.81\ m/s^2)}$$

base values for loss factors

$$122 = (267f + 1)V^2$$

assumption → flow laminar, then $f = \frac{64}{Re} = \frac{64\mu}{\rho VD}$

$$F = \frac{64(9.2\times10^{-4})}{.96(999)(.25\times10^{-3})V} = \frac{.246}{V}$$

$$122 = \left(267\left(\frac{.246}{V}\right) + 1\right)V^2$$

$$V^2 + 65.7V - 122 = 0 \Rightarrow V = \underline{1.81\ m/s} \text{ or } \cancel{-65.7}\ m/s$$

$$Q = AV = \frac{\pi}{4}(.25\times10^{-3}\ m)^2(1.81\ m/s)$$

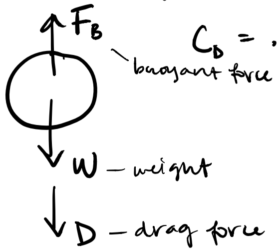
$$\Rightarrow \boxed{Q = 8.88(10^{-8})\ \frac{m}{s}}$$ maximum flow rate w/out cavitation

verify: we assumed laminar

$$Re = \frac{\rho VD}{\mu} = \frac{(.96)(999\ kg/m^3)(1.81\ m/s)(.25\times10^{-3}\ m)}{9.2\times10^{-4}\ \frac{N\cdot s}{m^2}} = 472 < 2100,$$

so ok! laminar

Tennis ball released from bottom of pool and need to find final velocity as rises to surface.



$C_D = .4$ diameter $= 38.1\text{ mm}$ $W = 0.0245\text{ N}$

steady rise at this pt. $\Sigma F = 0$

$F_B = W + D$

$D = C_D \frac{1}{2} \rho V^2 \frac{\pi}{4} D^2$ $\quad F_B = \gamma \underbrace{\pi \frac{4}{3} \left(\frac{D}{2}\right)^3}_{\forall}$

$$\gamma \pi \left(\frac{4}{3}\right)\left(\frac{D}{2}\right)^3 = W + \frac{1}{2} C_D \rho V^2 \frac{\pi}{4} D^2$$

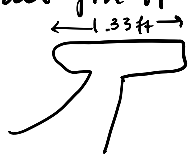
water, SI units

$$(9.8 \times 10^3)\left(\frac{4\pi}{3}\right)\left(\frac{.0381}{2}\right)^3 = 0.0245 + \frac{1}{2}(.4)(999)V^2 \frac{\pi}{4}(.0381)^2$$

$V = \pm 1.06\text{ m/s}$, $\boxed{V = 1.06\text{ m/s}}$

↑ will stop accelerating at top

Tail fin of race car



$b =$ depth $= 4\text{ ft}$ ↑ into page

$C_L = .87$ $C_D = .0195$

↑ lift ↑ drag

μ friction $= .6$ (tires)

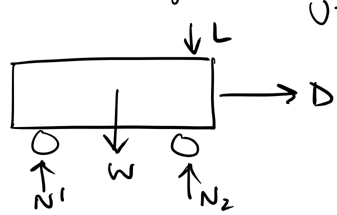
$\rho_{air} = 0.722 \frac{\text{lbm}}{\text{ft}^3}$

$V = 200\text{ mph} = 294\text{ ft/s}$

How much does fin increase max traction force between car and ground?

$F_2 = \mu N_2$

$\Delta F_2 = \mu \Delta N_2 = \mu L$



$$L = \frac{1}{2}(C_L \rho V^2 A) = \frac{1}{2}(.87)(0.0722 \frac{\text{lbm}}{\text{ft}^3})(294 \frac{\text{ft}}{\text{s}})^2 (1.33\text{ ft} \cdot 4\text{ ft})\left(\frac{1\text{ lbm} \cdot \text{s}^2}{32.2 \text{ lbm} \cdot \text{ft}}\right)$$

$L = 450$

$\Delta F_2 = (.6)(450) = \boxed{270\text{ lb}}$ how much more traction get when fin added

$D = \frac{1}{2} C_D \rho V^2 A = \frac{1}{2}(.0195)(.0722)(294)(5.33)\frac{1}{32.2} \rightarrow \boxed{D = 10.1\text{ lbf}}$

New problem: Dimensional Analysis

$D = f(V_p, \rho_p, L_p, \mu_p)$ prototype car, p   model car, m

$k - r = \underset{\text{variables}}{5} - \underset{\text{terms}}{3}$

$D = \left[\frac{ML}{T^2}\right]$ $V_p = [L/T]$ $\rho_p = [M/L^3]$ $L_p = [L]$

$\mu_p = \left[\frac{M}{LT}\right]$

Π_1 = dependent

$= D V_p^a \rho_p^b L_p^c$

$M^0 L^0 T^0 = \left(\frac{ML}{T^2}\right)\left(\frac{L}{T}\right)^a\left(\frac{M}{L^3}\right)^b L^c$

M : $1 + b = 0$ $b = -1$

L : $1 - 2 + 3 + c = 0$ $c = -2$

T : $-2 - a = 0$ $a = -2$

$$\Pi_1 = \frac{D}{\rho_p V_p^2 L_p^2}$$

$\Pi_2 = \mu V_p^a \rho_b^b L_p^c$

by inspection:

$$\Pi_2 = \frac{\rho_p V_p L_p}{\mu} = Re$$

$$\frac{D}{\rho V^2 L^2} = \phi(Re)$$

What if we had 1/5 scale model of car? Predict drag on car if going 90 km/hr. How fast wind tunnel need to be?

$\frac{L_m}{L_p} = \frac{1}{5}$ $V_p = 90 \frac{km}{hr}$

$\Pi_{2p} = \Pi_{2m}$ $\Pi_{1p} = \Pi_{1m}$

$$\frac{\cancel{\rho_p} V_p L_p}{\cancel{\mu_p}} = \frac{\cancel{\rho_m} V_m L_m}{\cancel{\mu_m}}$$ cancels bc both air

$V_m = V_p\left(\frac{L_p}{L_m}\right) = \boxed{450 \frac{km}{hr} = V_m}$

$D_m = 110 N$, what would drag on prototype be if changed $V_p = 100 \frac{km}{hr}$

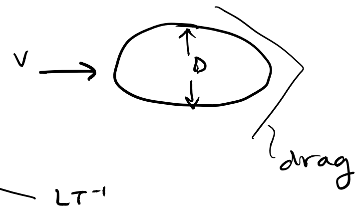
$\frac{V_m}{V_p} = 5$ $\frac{D_m}{\rho_m V_m^2 L_m^2} = \frac{D_p}{\rho_p V_p^2 L_p^2}$ $D_p = D_m\left(\frac{V_p}{V_m}\right)^2\left(\frac{L_p}{L_m}\right)^2$

$= 110\left(\frac{1}{5}\right)^2(5)^2 = D_p = 110 N$

Cope Pod, water crustacean, moving thru fresh water

diameter = 1mm

ave. vel. 2.53 cm/s



$F_D = f(\rho, \mu, L, V)$

MLT^{-2} ML^{-3} $ML^{-1}T^{-1}$ L LT^{-1}

$k = 5 - 3 = 2$ pi

$\pi_1 = F_D V^a \rho^b L^c$

$M^0 L^0 T^0 = (MLT^{-2})(LT^{-1})^a (ML^{-3})^b (L^c)$

$M: 0 = 1 + b$ $b = -1$

$L: 0 = 1 + a - 3b + c$ $c = -2$

$T: 0 = -2 - a$ $a = -2$

$\pi_1 = \dfrac{F_D}{\rho V^2 L^2}$

$\boxed{\pi_2 = Re, \dfrac{V\rho L}{\mu}}$

$$\frac{F_D}{\rho V^2 L^2} = \phi(Re)$$

Scale model that's 100x as large as cope pod and changing from water to glycerin

$\dfrac{L_m}{L_p} = 100$ $\pi_{2p} = \pi_{2m}$

$$\frac{V_p \rho_p L_p}{\mu_p} = \frac{V_m \rho_m L_m}{\mu_m}$$

$\rho_p = H_2O$

$\rho_m = gly$

$$V_m = \frac{\mu_m}{\mu_p} \frac{\rho_p}{\rho_m} \frac{L_p}{L_m} V_p$$

$$= \left(\frac{.95 \text{ Pa-s}}{.001 \text{ Pa-s}}\right)\left(\frac{1000 \text{ kg/m}^3}{1263 \text{ kg/m}^3}\right)\left(\frac{1}{100}\right)\left(253 \frac{m}{s}\right)$$

$\boxed{V_m = 19.03 \text{ cm/s}}$

actual drag

$\boxed{F_D = 1.82(10^{-6})\text{ N}}$

$= \dfrac{1000}{1263}\left(\dfrac{2.53}{19.03}\right)^2\left(\dfrac{1}{100}\right) 1.3$

$\text{Drag}_m = F_{Dm} = 1.3\text{N}$ ← given

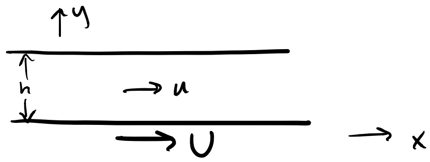
F_{Dp} (the actual)

$\pi_{1p} = \pi_{1m}$ => $\dfrac{F_{Dp}}{\rho_p V_p^2 L_p^2} = \dfrac{F_{Dm}}{\rho_m V_m^2 L_m^2}$

$F_{Dp} = \dfrac{\rho_p}{\rho_m} \dfrac{V_p^2}{V_m^2} \dfrac{L_p^2}{L_m^2} F_{Dm}$

Finding a Dimensionless Governing Eq.

- incompressible fluid between 2 plates
- upper plate fixed
- lower pulled at U suddenly at start



N-S Eq: "governing eq." 2 boundary conditions, 1 initial cond.

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2}$$ } this is our starting equation and we want to make it dimensionless

use h, U as reference parameters for y, u

use $\tau = \frac{h^2 \rho}{\mu}$ $\rightarrow t$ (ref par.)
↖ unit of time

$$\underbrace{y_1 = \frac{y}{h} \qquad t_1 = \frac{t}{\tau} \qquad u_1 = \frac{u}{U}}_{\text{nondimensionalize these}} \quad \Rightarrow \quad y = y_1 h \quad t = t_1 \tau \quad u = u_1 U$$

need to replace $\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial y^2}$

2 ways to do this

1. plug in known relations

$$\frac{du}{dt} = \frac{\partial (u_1 U)}{\partial (t_1 \tau)} = \boxed{\frac{U}{\tau} \underbrace{\frac{du_1}{dt_1}}_{\text{dimensionless}}}$$

$$\frac{du}{dy} = \frac{\partial (u_1 U)}{\partial (y_1 h)} = \frac{U}{h} \frac{du_1}{dy_1}$$

$$\frac{d^2 u}{dy^2} = \frac{\partial}{\partial y}\left(\frac{U}{h}\frac{\partial u_1}{\partial y}\right) = \frac{\partial}{\partial (y_1 h)}\left(\frac{U}{h}\frac{\partial u_1}{\partial y_1}\right) \Rightarrow \frac{\partial^2 u}{\partial y^2} = \boxed{\frac{U}{h^2}\frac{\partial^2 u_1}{\partial y_1^2}}$$

2. more general, chain rule

$$\frac{du}{dt} = \frac{du}{du_1}\frac{du_1}{dt} = \frac{du}{du_1}\frac{du_1}{dt_1}\frac{dt_1}{dt}$$

$\frac{d}{dt}\left(\frac{t}{\tau}\right) = \frac{1}{\tau}$

$\frac{d}{du_1}(u_1 U) = U$

$$= \frac{U}{\tau}\left(\frac{du_1}{dt_1}\right)$$

now plug back into starting equation:
everything w/ subscript 1 is inherently dimensionless as defined earlier

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} \qquad \rho\left(\frac{U}{\tau}\frac{\partial u_1}{\partial t_1}\right) = \mu\left(\frac{U}{h^2}\frac{\partial^2 u_1}{\partial y_1^2}\right)$$

$$\frac{\rho U}{\tau}\frac{\partial u_1}{\partial t_1} = \frac{\mu U}{h^2}\frac{\partial^2 u_1}{\partial y_1^2} \qquad \tau = \frac{h^2 \rho}{\mu}$$

$$\frac{\mu U}{h^2}\frac{\partial u_1}{\partial t_1} = \frac{\mu U}{h_2}\frac{\partial^2 u_1}{\partial y_1^2}$$

so the final dimensionless governing equation for movement of bottom plate

$$\boxed{\frac{\partial u_1}{\partial t_1} = \frac{\partial^2 u_1}{\partial y_1^2}}$$

I.C./B.C.

$u(t) : u(0) = 0$ ← velocity of fluid
$u(y) : u(0) = U$ ← bottom plate velocity
$u(y) : u(h) = 0$

Non dimensional BC/IC

$$\underbrace{\begin{aligned} u_1(t_1) &\Rightarrow u_1(0) = 0 \\ u_1(y_1) &\Rightarrow u_1(0) = 1 \\ u_1(y_1) &\Rightarrow u_1(1) = 0 \end{aligned}}_{\text{these are new B.C.}} \qquad t_1 = \frac{t}{\tau} \quad y_1 = \frac{y}{h} \quad u_1 = \frac{u}{U}$$

$$(f)\quad \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = \mu\frac{\partial v}{\partial x} = \mu\left(\frac{\rho g}{2}(2x-h)\right)$$

$$\boxed{F = \tau_{xy} b l}$$

Part 3 b (25pt)

Time available to <u>solve</u> and <u>submit</u> your answer: <u>25 minutes</u>

Exercise: Consider a steady, incompressible, laminar flow of a Newtonian fluid falling slowly down between two vertical plates at a distance *h*. Gravity acts downwards.

<u>PART_1</u>: Consider the case illustrated in <u>figure **a)**</u> where both plates are <u>fixed</u>. There is no applied (forced) pressure driving the flow, it falls by gravity alone.

$$(a)\quad \cancel{\frac{\partial \rho}{\partial t}} + \cancel{\frac{\partial u}{\partial x}} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0$$

$$\frac{\partial v}{\partial y} = 0 \quad \therefore v(x) = v$$

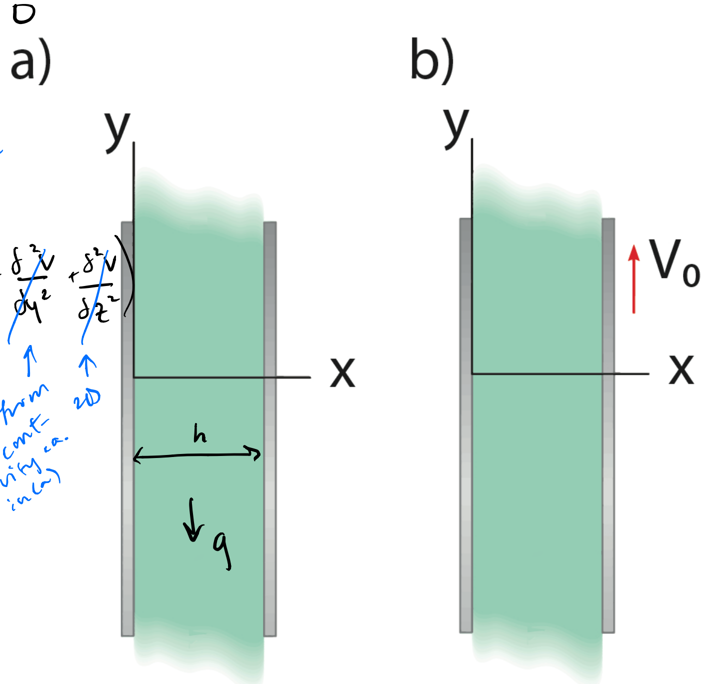
steady, no u vel, no w vel

$$(b)\quad \rho\left(\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + v\frac{\partial v}{\partial y} + \cancel{w\frac{\partial v}{\partial z}}\right) = -\cancel{\frac{\partial p}{\partial y}} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \cancel{\frac{\partial^2 v}{\partial y^2}} + \cancel{\frac{\partial^2 v}{\partial z^2}}\right)$$

no press gradient driving liq, only grav.

from cont. eq. in (a)

2D



$$\mu\frac{d^2 v}{dx^2} = \rho g$$

(c) $v(0) = 0$
$v(h) = 0$

(d) $\frac{dv}{dx} = \frac{\rho g}{\mu}x + C_1$

$v = \frac{\rho g}{2\mu}x^2 + C_1 x + C_2$

$C_2 = 0$

$\frac{\rho g}{2\mu}h^2 + C_1 h = 0$

$C_1 = -\frac{\rho g}{2\mu}h$

$$v = \frac{\rho g x}{2\mu}(x-h)$$

a) **(2pt)** Write the conservation of mass equation for this specific case. Indicate which terms are zero in this specific case, if any. Discuss the meaning of the result obtained.
b) **(4pt)** Write the Navier Stokes equations for this specific case and <u>justify</u> why some terms are zero if any.
c) **(2pt)** List the boundary conditions.
d) **(3pt)** Obtain the velocity field, write the expression for the maximum velocity
e) **(2pt)** Find the volumetric flowrate (per unit depth)
f) **(4pt)** Find the shear stress distribution across the channel and indicate the expression for the force on a section of width *b* and length *l* of the left plate.

$$\text{max:}\quad \frac{dv}{dx} = \frac{d}{dx}\left(\frac{\rho g x^2}{2\mu} - \frac{\rho g}{2\mu}x h\right) = \frac{\rho g x}{\mu} - \frac{\rho g h}{2\mu} = \frac{\rho g}{2\mu}(2x-h)$$

$$2x - h = 0 \Rightarrow x = \frac{h}{2} \qquad V = \frac{\rho g \, h/2}{2\mu}\left(-\frac{h}{2}\right) \Rightarrow \boxed{V = -\frac{\rho g h^2}{8\mu}}$$

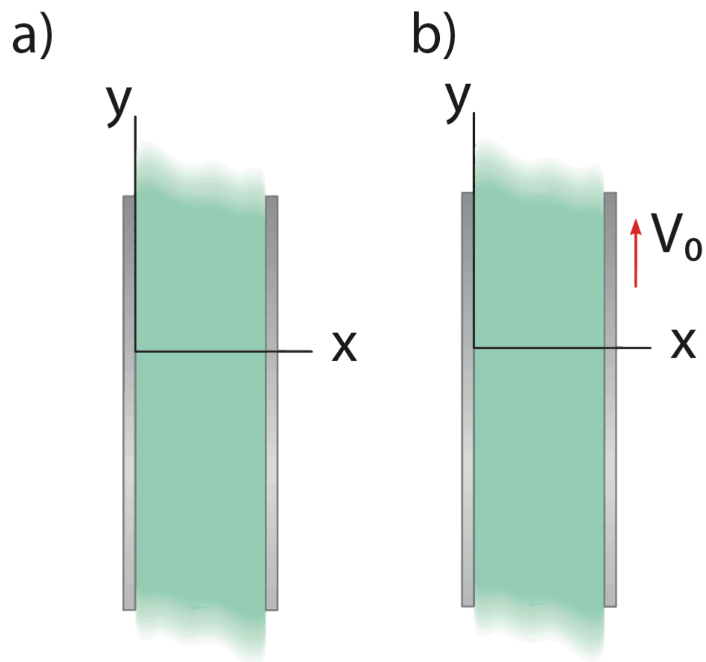
(e) $Q = AV = (d)hV$

$$\frac{Q}{d} = \int_0^h V\,dx = \int_0^h \left[\frac{\rho g x^2}{2\mu} - \frac{\rho g h}{2\mu}x\right]dx = \left[\frac{\rho g x^3}{6\mu} - \frac{\rho g h x^2}{4\mu}\right]_0^h = \frac{Q}{d} = -\frac{\rho g h^3}{12\mu}$$

PART_2: Consider the case illustrate in figure b) where the right plate is moving at a constant speed V_0.

(g) same result

(h) Navier-Stokes same



g) **(2pt)** Discuss and detail if/why the conservation of mass equation for this specific case changes

h) **(2pt)** Write the Navier Stokes equations for this specific case and compare them to the previous case.

i) **(4pt)** Find for which value of the velocity V_0, the net volumetric flowrate (per unit depth) across the gap is going to be zero.

(i) $V(h) = V_0$

$$V = \frac{\rho g x^2}{2\mu} + C_1 x + \cancel{C_2}^{\,0}$$

$$V_0 = \frac{\rho g h^2}{2\mu} + C_1 h$$

$$C_1 = \frac{V_0}{h} - \frac{\rho g h}{2\mu}$$

$$V = \frac{\rho g x^2}{2\mu} + x\left(\frac{V_0}{h} - \frac{\rho g h}{2\mu}\right)$$

$$V = \underbrace{\frac{V_0 x}{h} + \frac{\rho g x}{2\mu}(x-h)}$$

$$\frac{Q}{d} = \int_0^h (\qquad) = \left[\frac{V_0 x^2}{2h} + \frac{\rho g x^3}{6\mu} - \frac{\rho g x^2 h}{4\mu}\right]_0^h$$

$$\frac{Q}{d} = \frac{V_0 h}{2} - \frac{\rho g h^3}{12\mu} = 0$$

$$\boxed{\text{when } V_0 = \frac{\rho g h^2}{6\mu}}$$