FLUIDS EXAM II STUDY GUIDE

$$\frac{1}{\delta \forall}\frac{d(\delta \forall)}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \underbrace{\underset{\uparrow\,\text{GRADIENT}}{\vec{\nabla}} \cdot \vec{V}}_{\text{DIVERGENCE}}$$

Rate of change of volume per unit volume => **VOLUMETRIC DILATATION RATE**

→ for an incompressible fluid, volumetric dilatation rate = ∅ since the element volume can't change without a change in fluid density (element mass must be conserved

→ velocity variations $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial w}{\partial z}$ cause linear deformation

→ cross derivatives such as $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ cause rotation and generally angular deformation

Angular motion & deformation

Rotation of fluid element about:

z-axis => $\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

y-axis => $\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$

x-axis => $\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$

$$\frac{1}{2}\vec{\nabla} \times \vec{V} = \frac{1}{2}\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$\Uparrow$ ω $\Uparrow$

rotation vector $\omega = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

vorticity: twice fluid particle rotation

$$\zeta = 2\omega = \nabla \times V$$

→ when $\partial u/\partial y = -\partial v/\partial x$ rotation about z-axis = ∅

⇒ if $\vec{\nabla} \times \vec{V} = 0$ then the rotation (and vorticity ζ) are zero
IRROTATIONAL

* flow is irrotational if angular velocity = ∅
* incompressible flow field satisfies COM if dilatation = ∅
  ↳ $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \emptyset$ for conservation of mass

Infinitesimal strain tensor

$$\begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{bmatrix}$$

→ diagonal: linear deformation, normal stresses

→ off-diagonal: shear stresses

→ trace: volumetric dilatation rate (trace is sum of diagonal elements) and for incompressible fluids, the trace and therefore volumetric dilation rate is zero.

Conservation of Mass

$$\frac{DM_{sys}}{Dt} = 0 = \frac{\partial}{\partial t}\int_{cv} \rho \, d\forall + \int_{cs} \rho \vec{V}\cdot\hat{n}\, dA$$

↑ rate of mass outflow − inflow

↑ rate at which mass w/in cv is changing

$$\frac{\partial}{\partial t}\int_{cv} \rho\, d\forall \approx \frac{\partial \rho}{\partial t}\,\delta x\,\delta y\,\delta z \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho v)}{\partial z} = 0}$$

* aka continuity equation *

→ can be written as $\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{V} = 0$

→ valid for steady or unsteady flow, and compressible or incompressible flow

→ steady compressible flow
$\vec{\nabla}\cdot\rho\vec{V} = 0$

→ for incompressible flow
$\vec{\nabla}\cdot\vec{V} = 0$

Cylindrical Polar Coordinates

$$\vec{V} = V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{e}_z$$

continuity equation in differential form: $\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial(r\rho V_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho V_\theta)}{\partial \theta} + \frac{\partial(\rho V_z)}{\partial z} = 0$

→ for steady, compressible flow

$$\frac{1}{r}\frac{\partial(r\rho V_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho V_\theta)}{\partial \theta} + \frac{\partial(\rho V_z)}{\partial z} = 0$$

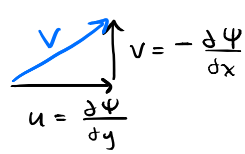
→ for incompressible flow, steady or unsteady

$$\frac{1}{r}\frac{\partial(r V_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\partial V_z}{\partial z} = 0$$

Stream function

for steady, incompressible, plane, 2D flow, the continuity equation reduces to $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$\psi(x,y) \Rightarrow \boxed{u = \frac{\partial \psi}{\partial y} \quad v = -\frac{\partial \psi}{\partial x}}$



→ lines along which ψ is constant are <u>streamlines</u> $\Rightarrow \frac{dy}{dx} = \frac{v}{u}$

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = -v\,dx + u\,dy$$

and if constant ψ, then $d\psi = 0 \Rightarrow -v\,dx + u\,dy = 0$

→ change in value of the stream function is related to the volume rate of flow

$$dq = u\,dy - v\,dx$$

$$dq = \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = d\psi \quad \therefore \quad q = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1$$

→ in cylindrical coordinates the continuity equation for incompressible, plane, 2D flow reduces to

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} = 0$$

and the velocity components v_r and v_θ can be related to $\psi(r,\theta)$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = -\frac{\partial \psi}{\partial r}$$

* stream function ψ *
- constant along streamlines
- used to compute volumetric flow rate by integrating between values of ψ
- 2D, inviscid, incompressible, steady
- ψ satisfies conservation of mass
- ψ can be used to derive the velocity field

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x} \quad \Rightarrow \quad \vec{V} = u\hat{i} + v\hat{j}$$

*in 2D potential flow stream func satisfies Laplace *

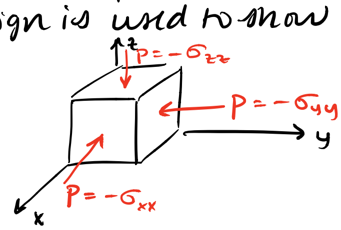
Inviscid flow

→ inviscid = flow fields in which shearing stresses are negligible

→ normal stresses on fluid element $= \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$

and negative sign is used to show that compressive normal stress

$$\begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$



Euler's Equations of Motion

→ apply to an inviscid flow field

$$\rho\vec{g} - \nabla p = \rho\left[\underset{\text{local}}{\frac{\partial \vec{V}}{\partial t}} + \underset{\text{convective}}{(\vec{V}\cdot\vec{\nabla})\vec{V}}\right]$$

non linear velocity terms

*represent differential form of the momentum equation for ideal fluids

* if steady, results in Bernoulli (streamline).

* if steady & irrotational, result in Bernoulli and can be used anywhere

→ steady flow: $\rho\vec{g} - \nabla p = \rho(\vec{V}\cdot\vec{\nabla})\vec{V}$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = C \quad \text{along streamline}$$

$$\frac{p_1}{\gamma} + \frac{V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{V_2^2}{2g} + z_2$$

Bernoulli } restricted to: inviscid, steady, incompressible, along a streamline

→ irrotational flow; $u = U,\ v = 0,\ w = 0$ in x-direction

for incompressible, irrotational flow, Bernoulli applies w/ same considerations as before

The Velocity Potential, ϕ

for an irrotational flow the velocity components can be expressed in terms of scalar function $\phi(x,y,z,t)$ as

→ $u = \frac{\partial\phi}{\partial x}$ $\quad v = \frac{\partial\phi}{\partial y}$ $\quad w = \frac{\partial\phi}{\partial z}$

→ $\vec{V} = \vec{\nabla}\phi$ so that for irrotational flow the velocity is the gradient of the ϕ

* velocity potential is a consequence of the irrotationality of the flow field *
* stream function is a consequence of conservation of mass *
* velocity potential can be defined for general 3D flow while stream function is restricted to 2D flow *

→ for incompressible, irrotational flow ($\vec{V} = \vec{\nabla}\phi$)

$$\vec{\nabla}^2\phi = 0 \quad \text{where } \vec{\nabla}^2 = \nabla\cdot\nabla(\ ) = \text{Laplacian operator}$$

→ inviscid, incompressible, irrotational flow fields are governed by Laplace's equation and are called potential flows

... vorticity = 0 throughout

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

→ if vorticity present, flow can't be described by Laplace's Eq.

$$\nabla(\ ) = \frac{\partial(\ )}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial(\ )}{\partial\theta}\hat{e}_\theta + \frac{\partial(\ )}{\partial z}\hat{e}_z$$

$$\nabla\phi = \frac{\partial\phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{e}_\theta + \frac{\partial\phi}{\partial z}\hat{e}_z$$

$$\phi = \phi(r,\theta,z) \Rightarrow \vec{V} = V_r\hat{e}_r + V_\theta\hat{e}_\theta + V_z\hat{e}_z$$

$$\vec{V} = \vec{\nabla}\phi \Rightarrow V_r = \frac{\partial\phi}{\partial r} \quad V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} \quad V_z = \frac{\partial\phi}{\partial z}$$

Laplace's Eq. cylindrical coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

* velocity potential is constant <u>across</u> streamlines *
* in 2D potential flow, like stream function, satisfies Laplace *

→ lines of constant ϕ = equipotential lines, orthogonal to lines of constant ψ (streamlines) at all points where they intersect

Uniform Flow

$\frac{\partial \phi}{\partial x} = U \quad \frac{\partial \phi}{\partial y} = 0 \longrightarrow$ integrate $\longrightarrow \phi = Ux + C_1$ if $C_1 = \emptyset$ then

$\phi = Ux$

$\frac{\partial \psi}{\partial y} = U \quad \frac{\partial \psi}{\partial x} = 0 \longrightarrow$ integrate $\longrightarrow \psi = Uy$

for the general case:

any angle α w/ x-axis

$$\phi = U(x\cos\alpha + y\sin\alpha)$$

$$\psi = U(y\cos\alpha - x\sin\alpha)$$

Source & Sink

$\phi = \frac{m}{2\pi} \ln r$

if $m \oplus$ ... source
if $m \ominus$ ... sink

m = strength of source or sink

$\phi = C_1 \Rightarrow$ equipotential lines are concentric circles

$\psi = \frac{m}{2\pi}\theta \qquad \psi = C_1 \Rightarrow$ streamlines are radial lines

Vortex

streamlines are concentric circles

$\phi = k\theta \qquad \psi = -k\ln r$

tangential velocity varies inversely w/ distance from origin

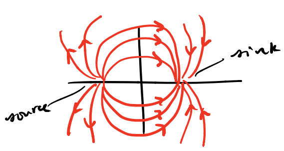
circulation: $\Gamma \Rightarrow$ line integral of the tangential component of the velocity taken around a closed curve in the flow field

$k = \frac{\Gamma}{2\pi}$

$\Rightarrow$ vortex : $\psi = -\frac{\Gamma}{2\pi}\ln r$ or $\phi = \frac{\Gamma}{2\pi}\theta$

Doublet

source-sink pair



$\psi = -\frac{K\sin\theta}{r}$

$K = \frac{ma}{\pi}$ (strength of doublet)

$\phi = \frac{K\cos\theta}{r}$

Flow around a half-body

$$\psi = \psi_{uniform} + \psi_{source} = Ur\sin\theta + \frac{m}{2\pi}\theta$$

$$\phi = Ur\cos\theta + \frac{m}{2\pi}\ln r$$

Flow around a circular cylinder

$$\psi = \psi_{uniform} + \psi_{doublet} = Ur\left(1-\frac{a^2}{r^2}\right)\sin\theta$$

$$\phi = Ur\left(1+\frac{a^2}{r^2}\right)\cos\theta$$

→ pressure distribution on cylinder surface is obtained from Bernoulli equation

P_0 = pt far from cylinder pressure

$$P_0 + \frac{1}{2}\rho U^2 = P_s + \frac{1}{2}\rho V_{\theta s}^2$$

P_s = surface pressure

since $V_{\theta s} = -2U\sin\theta$

then $P_s = P_0 + \frac{1}{2}\rho U^2(1-4\sin^2\theta)$

Navier-Stokes Equations

these three equations, combined w/ conservation of mass equation, provide a complete mathematical description of the flow of incompressible newtonian Fluids

* conservation of momentum *

$$\rho\left(\frac{\partial\vec{V}}{\partial t} + \vec{V}\cdot\vec{\nabla}\vec{V}\right) = -\vec{\nabla}p + \rho\vec{g} + \mu\vec{\nabla}^2\vec{V}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

$$\underset{\text{INERTIA}}{\rho\left(\frac{\partial\vec{V}}{\partial t} + \vec{V}\cdot\vec{\nabla}\vec{V}\right)} = \underset{\text{EXTERNAL PRESSURE}}{-\vec{\nabla}p} + \underset{\text{GRAVITATIONAL FORCES}}{\rho\vec{g}} + \underset{\text{VISCOUS FORCES}}{\mu\vec{\nabla}^2\vec{V}}$$

$$= \left[\frac{kg}{m^2 s^2}\right] = \frac{\text{Force}}{\forall}$$