Introduction to Singular Perturbations (see respective Mathematica notebook)

*only at order 1 is $\epsilon = 0$

Boundary Layers (see respective Mathematica Nb "Singular Perturbation: Boundary Layer")

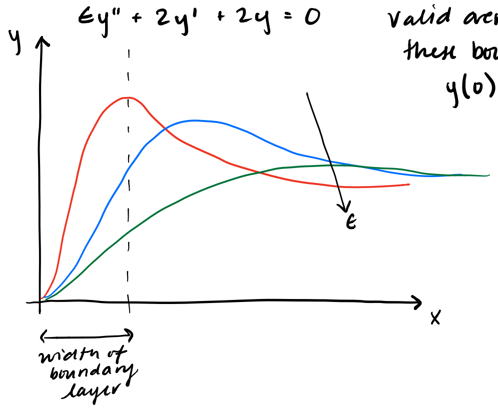
Consider the BVP:

$$\epsilon y'' + 2y' + 2y = 0$$

valid over domain: $0 < x < 1$

these boundary conditions are imposing the locations of y

$$y(0) = 0 \qquad y(1) = 1$$



⇐ what happens as we decrease ε

Step 1: Regular Perturbation (Outer Solution)

assume: $y \sim y_0(x) + \epsilon y_1(x) + \dots$

insert into ODE:

$$\epsilon\underbrace{(y_0'' + \epsilon y_1'' + \dots)}_{y''} + 2\underbrace{(y_0' + \epsilon y_1' + \dots)}_{y'} + 2\underbrace{(y_0 + \epsilon y_1 + \dots)}_{y} = 0$$

for the BC:

$$y_0(0) + \epsilon y_1(0) + \dots = 0$$
$$y_0(1) + \epsilon y_1(1) + \dots = 1$$

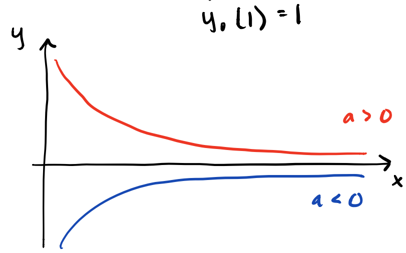
$$O(1): \cancel{\epsilon\underbrace{(y_0'' + \epsilon y_1'' + \dots)}_{y''}} + 2\underbrace{(y_0' + \cancel{\epsilon y_1' + \dots})}_{y'} + 2\underbrace{(y_0 + \cancel{\epsilon y_1 + \dots})}_{y} = 0$$

take $\epsilon \to 0$ ... we lose a very important term!

$$O(1): 2y_0' + 2y_0 = 0 \longrightarrow \boxed{y_0 = ae^{-x}} \longrightarrow \boxed{a = e}$$
$$y_0(0) = 0$$
$$y_0(1) = 1$$

$$\boxed{y_0 \sim e^{1-x}}$$

only BC we can apply is $y_0(1) = 1$



these are the only two options, which boundary condition do we apply?

- we can't apply $y_0(0) = 0$ because if we tried to solve for a, we couldn't
- we can only apply $y_0(1) = 1$, this way we can say $a = e$

$$O(\epsilon): \underbrace{\epsilon(y_0'' + \epsilon y_1'' + \ldots)}_{y''} + 2\underbrace{(y_0' + \epsilon y_1' + \ldots)}_{y'} + 2\underbrace{(y_0 + \epsilon y_1 + \ldots)}_{y} = 0$$

$$O(\epsilon): y_0'' + 2y_1' + 2y_1 = 0 \longrightarrow y_1 = \left(b - \frac{x}{2}\right)e^{1-x}$$

$$BC: \quad y_1(1) = 0$$

$$y_1 = \frac{1-x}{2}e^{1-x}$$

now combine w/ $O(1)$ ...

$$y \sim e^{1-x} + \epsilon\frac{1-x}{2}e^{1-x}$$

Step 2:

$$\bar{x} = \frac{x}{\epsilon^\gamma} \qquad \frac{d}{dx} = \frac{d\bar{x}}{dx}\frac{d}{d\bar{x}} = \frac{1}{\epsilon^\gamma}\frac{d}{d\bar{x}}$$

$$\frac{d^2}{dx^2} = \frac{1}{\epsilon^{2\gamma}}\frac{d^2}{d\bar{x}^2}$$

$$\epsilon y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y(1) = 1 \quad \rightarrow \quad \epsilon^{1-2\gamma}\bar{y}'' + 2\epsilon^{-\gamma}\bar{y}' + 2\bar{y} = 0, \quad \bar{y}(0) = 0, \quad \bar{y}(1) = 1$$

$$y(x) \to \bar{y}(\bar{x})$$

Dominant Balance: $\boxed{\gamma = 1}$

$$\bar{y}'' + 2\bar{y}' + 2\epsilon\bar{y} = 0$$

now clearly have a regular perturbation problem

$$\bar{y}(\bar{x}) \sim \bar{y}_0(\bar{x}) + \epsilon\bar{y}_1(\bar{x}) + \ldots$$

$$(\bar{y}_0'' + \epsilon\bar{y}_1'') + 2(\bar{y}_0' + \epsilon\bar{y}_1') + 2\epsilon(\bar{y}_0 + \epsilon\bar{y}_1) = 0$$

$$O(1): \left.\begin{array}{l} \bar{y}_0'' + 2\bar{y}_0' = 0 \\ \bar{y}_0(0) = 0 \end{array}\right\} \rightarrow \bar{y}_0 = A + Be^{-2\bar{x}} \longrightarrow \bar{y}_0 = A(1 - e^{-2x})$$

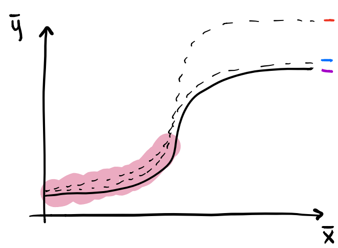
with the given BC, solve for B_1 reduces to

$$\bar{y} = \underbrace{\frac{c[1]}{2}}_{A} e^{-2\bar{x}}(e^{-2\bar{x}} - 1)$$

$$\lim_{\bar{x}\to\infty} \bar{y} = \frac{c[1]}{2} = A$$

matching:

$$\lim_{\bar{x}\to\infty} \bar{y} = \lim_{x\to 0} y$$



$\uparrow A$

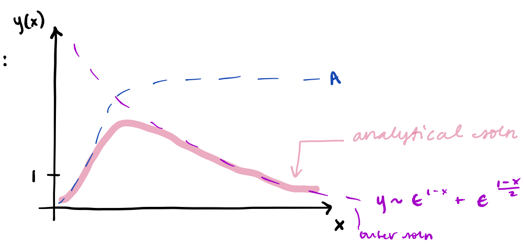
$$\bar{y}_0 \to A \text{ as } \bar{x} \to \infty$$

$$\frac{c[1]}{2} = \frac{1}{2}e(2+\epsilon)$$

$$\boxed{c[1] = e(2+\epsilon)}$$

$$\boxed{y_0(0) = \bar{y}(\infty) = e(2+\epsilon)}$$

matching:



$$\lim_{x\to 0} y = \frac{1}{2}e(2+\epsilon)$$

Construct the composite solution

$y \sim$ inner + outer − matching

have to subtract this or else we double count it

$\bar{y}_0(\bar{x})$ $y_0(x) + \epsilon y_1(x)$ $y_0(0)$

change back to y & x

$$y \sim \frac{e(2+\epsilon)}{2} e^{-2x/\epsilon}\left(2^{2x/\epsilon} - 1\right) + e^{1-x} + \epsilon \frac{1-x}{2} e^{1-x} - e(2+\epsilon)$$

*need to rewatch this lecture