Lagrange Multipliers

→ Minimization of Functional with Constraints

① Lagrange Multiplier Method
② Penalty Function Method → often used in contact problems

functional : $I(u,v) = \int_a^b F(x,u,u',v,v')\,dx$ ... and u and v are both functions of x

constraint : $G(u,u',v,v') = 0$

$\text{Area}_{deformed} - A_{initial} = 0$

$u \mapsto u + \overbrace{\epsilon\eta}^{\delta u}$ $\quad u' \mapsto u' + \overbrace{\epsilon\eta'}^{\delta u'}$

$v \mapsto v + \overbrace{\epsilon\phi}^{\delta v}$ $\quad v' \mapsto v' + \overbrace{\epsilon\phi'}^{\delta v'}$

$\mathcal{U}(u) \mapsto \delta\mathcal{U} = \frac{\partial \mathcal{U}}{\partial u}\delta u$

$\mathcal{U}(u,v) \mapsto \delta\mathcal{U} = \frac{\partial \mathcal{U}}{\partial u}\delta u + \frac{\partial \mathcal{U}}{\partial v}\delta v$

$\mathcal{U}(u,u') \mapsto \delta\mathcal{U} = \frac{\partial \mathcal{U}}{\partial u}\delta u + \frac{\partial \mathcal{U}}{\partial u'}\delta u'$ ← integration by parts here

Lagrange Multiplier

$\delta I = 0$

$\delta I = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v'}\delta v'\right)dx = 0$

u and v need to satisfy : $G(u,u',v,v') = 0$

$\delta G = 0$

$\delta G = \frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial u'}\delta u' + \frac{\partial G}{\partial v}\delta v + \frac{\partial G}{\partial v'}\delta v'$

Step ①: multiply by an arbitrary parameter λ

Step ②: integrate over the interval, e.g. (a, b)

Step ③: add that result to δI

side note:

$d(\cdot) \mapsto$ ordinary derivative
1 ind. variable

$\partial(\cdot) \mapsto$ partial derivative

$\delta(\cdot) \mapsto$ first variation of quantity,
($\delta^2(\cdot)$ ... 2nd variation, etc.)

Therefore,

$\delta I + \lambda\delta G = \int_a^b \left[\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial F}{\partial v}\delta v + \frac{\partial F}{\partial v'}\delta v' + \lambda\left(\frac{\partial G}{\partial u}\delta u + \frac{\partial G}{\partial u'}\delta u' + \frac{\partial G}{\partial v}\delta v + \frac{\partial G}{\partial v'}\delta v'\right)\right]dx = 0$

these are quantities that need to integrate by parts

Step ④: integrate by parts

Step ⑤: $\delta v(a) = \delta v(b) = \delta u(a) = \delta u(b) = 0$

integrate by parts:

$$\int_a^b \Bigg\{ \underbrace{\left[\frac{\partial F}{\partial u} + \lambda\frac{\partial G}{\partial u} - \underbrace{\frac{d}{dx}\left(\frac{\partial F}{\partial u'} + \lambda\frac{\partial G}{\partial u'}\right)}_{\text{result of integration by parts}}\right]\delta u}_{\text{one equation}} + \underbrace{\left[\frac{\partial F}{\partial v} + \lambda\frac{\partial G}{\partial v} - \underbrace{\frac{d}{dx}\left(\frac{\partial F}{\partial v'} + \lambda\frac{\partial G}{\partial v'}\right)}_{\text{result of integration by parts}}\right]\delta v}_{\text{second equation}} \Bigg\}\,dx = 0$$

these must both be zero

① $\frac{\partial}{\partial u}(F + \lambda G) - \frac{d}{dx}\left[\frac{\partial}{\partial u'}(F + \lambda G)\right] = 0$

② $\frac{\partial}{\partial v}(F + \lambda G) - \frac{d}{dx}\left[\frac{\partial}{\partial v'}(F + \lambda G)\right] = 0$

③ $G(u,u',v,v') = 0$

the constraint

now have three equations with three unknowns

Penalty Function

penalty parameter – positive, "large" number

$$P(u,v) = I(u,v) + \frac{\gamma}{2}\int_a^b \left[G(u,u',v,v')\right]^2 dx$$

add quadratic term G^2

$\delta P = 0$

$$\left.\begin{aligned} \frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) + \gamma\left[G\frac{\partial G}{\partial u} - \frac{d}{dx}\left(G\frac{\partial G}{\partial u'}\right)\right] = 0 \\ \frac{\partial F}{\partial v} - \frac{d}{dx}\left(\frac{\partial F}{\partial v'}\right) + \gamma\left[G\frac{\partial G}{\partial v} - \frac{d}{dx}\left(G\frac{\partial G}{\partial v'}\right)\right] = 0 \end{aligned}\right\} \text{ as } \gamma \to \infty \; (?)$$

→ constraint satisfied approximately

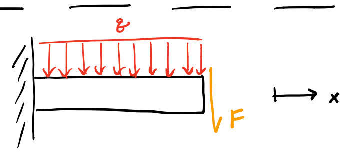
→ no additional unknowns

Example:

angle of deflection relative to x-axis

$$\Pi(w, \theta_x) = \int_0^L \left[\frac{B}{2}\left(\frac{d\theta}{dx}\right)^2 + q\,w(x)\right]dx - \underbrace{F\,w(L)}_{\text{shear force @ end of the beam}}$$

beam elasticity — deflection — B = bending rigidity — distributed load



$$G(w', \theta_x) = \frac{dw}{dx} + \theta_x = 0$$

$$w(0) = \theta_x(0) = 0$$

really productive group session!

Π total pot. energy of system, this is I

=

$$\pi(w, \theta_x, \lambda) = \pi(w, \theta_x) + \int_0^L \lambda\left(\frac{dw}{dx} + \theta_x\right)dx$$

from $\delta w \longrightarrow$ $-\frac{d\lambda}{dx} + q = 0$ $\qquad 0 < x < L$

$\lambda - F = 0$ $\qquad$ @ $x = L$ $\qquad$ tells us Lagrange multiplier is shear force

from $\delta\theta \longrightarrow$ $-\frac{d}{dx}\left(B\frac{d\theta}{dx}\right) + \lambda = 0$ $\qquad 0 < x < L$

$B\frac{d\theta}{dx} = 0$ $\qquad$ @ $x = L$

from $\delta\lambda \longrightarrow$ $\theta + \frac{dw}{dx} = 0$ $\qquad 0 < x < L$

*Problem Set coming... going to be HARD

→ may need to do research on the concepts when get stuck

→ don't worry so much about being correct

*OH via appointment/email ... advising week