ME 712 : Applied Mathematics in Mechanics
Holmes
Problem Set #1

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1. The luminosity of certain giant and supergiant stars varies in a periodic manner. It is hypothesized that the period p depends upon the star's average radius r, its mass m, and the gravitational constant G.
   - 1.1 Newton's law of gravitation asserts that the attractive force between two bodies is proportional to the product of their masses divided by the square of the distance between them, that is $F = Gm_1m_2/d^2$, where G is the gravitational constant. From this, determine the fundamental dimensions of G.
   - 1.2 Use dimensional analysis to determine the functional dependence of p on m, r, and G.
   - 1.3 Arthur Eddington used the theory for thermodynamic heat engines to show that $p = (3\pi/2\gamma G\rho)^{1/2}$, where γ is the ratio of specific heats of the stellar materials, and ρ is the star's average density, how does this differ from your result? (?)

→ luminosity varies in periodic manner
→ p (period) depends on r (ave. radius), m (mass), G (gravitational constant)

(1.1) $F = Gm_1m_2d^{-2}$ Find fundamental dimensions of G.

$G = Fd^2m_1^{-1}m_2^{-1}$ ... using MLT system: $[F = MLT^{-2}]$, $[d = L]$, $[m = M]$

$[G] = [MLT^{-2}]^1[L]^2[M]^{-1}[M]^{-1}$
$= M^{-1}L^3T^{-2}$

$$\boxed{[G] = [M^{-1}L^3T^{-2}]}$$

(1.2) $p = f(m, r, G)$ ... period is time per cycle: $[p = T]$

$[p] = [m^a, r^b, G^c]$ ⇒ $T = (M)^a(L)^b(M^{-1}L^3T^{-2})^c$

mass radius gravitational constant

$T = M^{a-c}L^{b+3c}T^{-2c}$

⇓

M: $a - c = 0$
L: $b + 3c = 0$ ⇒ $a = -1/2$, $b = 3/2$, $c = -1/2$ ⇒ use α as numerical prefactor
T: $-2c = 1$

$$\boxed{p = \alpha m^{-1/2} r^{3/2} G^{-1/2}}$$

(1.3) $p = (3\pi)^{1/2}(2\gamma G\rho)^{-1/2}$ γ = ratio of sp. heats, constant ρ = ave density, $[ML^{-3}]$

from 1.2 ⇒ $p = \alpha[G]^{-1/2}m^{-1/2}r^{3/2}$ and $p = \left(\frac{3\pi}{2\gamma}\right)^{1/2}[G]^{-1/2}\left[\frac{M}{L^3}\right]^{-1/2}$

$$\alpha\cancel{[G]^{-1/2}}\left(\frac{r^3}{m}\right)^{1/2} = \left(\frac{3\pi}{2\gamma}\right)^{1/2}\cancel{[G]^{-1/2}}\left(\frac{3}{4\pi}\frac{m}{r^3}\right)^{-1/2}$$

↑ $\frac{\text{mass}}{\text{volume}} = \frac{\text{mass}}{\left(\frac{4\pi}{3}\right)r^3}$

...relate to ... m, r, G

volume of sphere, assuming perfect sphere

$$\alpha\cancel{\left(\frac{r^3}{m}\right)^{1/2}} = \left(\frac{\cancel{3}\pi}{2\gamma}\right)^{1/2}\left(\frac{4\pi}{\cancel{3}}\right)^{1/2}\cancel{\left(\frac{r^3}{m}\right)^{1/2}}$$

$$\alpha = \left(\frac{2\pi^2}{\gamma}\right)^{1/2}$$

$$\boxed{\text{using Eddington, we conclude that } \alpha = 2^{1/2}\pi\gamma^{-1/2}}$$

2. When a drop of liquid hits a wetted surface a crown formation appears. It has been found that the number of points N on the crown depends on the speed U at which the drop hits the surface, the radius r and density ρ of the drop, and the surface tension σ of the liquid making up the drop.

2.1 • Use dimensional reduction to determine the functional dependence of N on U, r, ρ, and σ. Express your answer in terms of the Weber number, $We = \rho U^2 r/\sigma$.

2.2 • The value N has been measured as a function of the initial height h of the drop. Express your answer in terms of h by writing U in terms of h and g. Assume the drop starts with zero velocity.

2.3 • Experimental results show a linear dependence of N on h with a slope of about 1/4. Use this result to find the unknown function in the first part of the problem. If we take $r = 3.6$ mm, $\rho = 1.1014$ gm/cm^3, and $\sigma = 50.5$ dyn/cm, what must the initial height of the drop be to produce 80 points?

→ N (# of pts) depends on U (speed), r (radius), ρ (density), σ (surface tension)

(2.1) $We = \frac{\rho U^2 r}{\sigma}$ $N = f(U, r, \rho, \sigma)$ $U = LT^{-1}$ $r = L$ $\rho = ML^{-3}$ $\sigma = MT^{-2}$

↑ dimensionless

$$[N] = [U^a r^b \rho^c \sigma^d] \Rightarrow (LT^{-1})^a (L)^b (ML^{-3})^c (MT^{-2})^d \Rightarrow M^{c+d} L^{a+b-3c} T^{-a-2d}$$

M: $c + d = 0$

L: $a + b - 3c = 0$ 3 eq, 4 unknowns ⇒ take d to be the unknown

T: $-a - 2d = 0$

$a = -2d$
$b = -d$
$c = -d$

$$N = U^{-2d} r^{-d} \rho^{-d} \sigma^d = \left(\frac{\sigma}{U^2 r \rho}\right)^d = \alpha\left(\frac{1}{We}\right)^d \sim \alpha(We)^{-d} \Rightarrow \boxed{N = \alpha f(We)} \Rightarrow \boxed{N = f(We)}$$

(with π labelling $\left(\frac{\sigma}{U^2 r\rho}\right)$)

(2.2) new function for N in terms of h and g for U

$$\overset{(1)}{-g} = \frac{d^2y}{dt^2} = a \Rightarrow \overset{(2)}{-gt} + c_1 = \frac{dy}{dt} = U \Rightarrow \overset{(3)}{-gt^2}\left(\tfrac{1}{2}\right) + c_1 t + c_2 = y$$

I.C. $\begin{cases} t = 0,\ y = h \overset{(3)}{:}\ -g(0)^2\left(\tfrac{1}{2}\right) + c_1(0) + c_2 = h, & h = c_2 \\ t = 0,\ U = 0 \overset{(2)}{:}\ -g(0) + c_1 = 0, & 0 = c_1 \end{cases}$

⇒ $y = h - gt^2\left(\tfrac{1}{2}\right)$, when drop hits surface $y = 0 \therefore h = gt^2\left(\tfrac{1}{2}\right)$ and @ $t = (2h/g)^{1/2}$

plug t into (2): $-g(2h/g)^{1/2} = U = -(2gh)^{1/2}$ (4)

plug (4) into N:

$$N = \left(\frac{\sigma}{\left(-(2gh)^{1/2}\right)^2 r\rho}\right)^d = \left(\frac{\sigma}{2ghr\rho}\right)^d \Rightarrow \boxed{N = \cancel{\alpha} f\left(\frac{\sigma}{2ghr\rho}\right)}$$

don't need to write α as coefficient when equating (=) vs. (~)

(2.3) $r = 3.6(10^{-3})\,m$ $\rho = \frac{1.1014\,(10^{-3})\,kg}{(10^{-2}\,m)^3}$ $\sigma = 50.5\ \frac{dyn}{(10^{-2}\,m)} = 50.5(10^2)\ kg/s^2$ $\sigma = \frac{M}{T^2}$

linear dependence, $\alpha = 1/4$ N = 80 pts. find h. dyn = dynamic viscosity = $ML^{-1}T^{-1}$ $= \frac{kg}{m\cdot s}$

$$N = \alpha f\left(\frac{\sigma}{2ghr\rho}\right) \Rightarrow 80 = \left(\frac{1}{4}\right)\left(\frac{50.5(10^2)\ \cancel{kg}/\cancel{s^2}}{2(9.81\,m/\cancel{s^2})\,h(3.6\cdot\cancel{10^{-3}}\,m)(1.1014\cdot\cancel{10^3}\ \cancel{kg}/m^3)}\right)$$

d = -

$80\,h = 16.229$ m

$h = 0.203$ m ⇒ $\boxed{h = 203\,mm}$

→ 320 cm

$\alpha = \frac{1/4\ \sigma}{2gr\rho}$ from $N = \alpha\underbrace{\left(\frac{2gr\rho}{\sigma}\right)}_{\frac{1}{4}} h$

3. A ball is dropped from a height h_0 and it rebounds to a height h.

3.1 • Identify the parameters you suspect will dictate the rebound height, and find a dimensionally reduced form of h.

3.2 • Assume h depends linearly on h_0 (with $h = 0$ if $h_0 = 0$). How does this reduce your formula for h?

(3.1) initial height h_0

rebound height h

height should depend on density ρ $[ML^{-3}]$, radius r $[L]$ of ball, gravitational constant g $[LT^{-2}]$

... in addition to rebound height

$[h] = [h_0 \rho r g]$

$L = (\cancel{L})(M\cancel{L^{-3}})(\cancel{L})(\cancel{L}T^{-2})$

$L = MT^{-2}$ ∴ there must be at least one more parameter with dimensions $M^{-1}LT^{2}$

→ elastic modulus is $ML^{-1}T^{-2}$ ... $L = \cancel{M}\cancel{T^{-2}}\left(\frac{1}{\cancel{M}L^{-1}\cancel{T^{-2}}}\right)$ ✓

find dimensionally reduced form:

$[h] = [h_0^{\,a} \rho^b r^c g^d E^e] \Rightarrow L = (L)^a (ML^{-3})^b (L)^c (LT^{-2})^d (ML^{-1}T^{-2})^e$

$L = M^{b+e} L^{a-3b+c+d-e} T^{-2d-2e}$

$M: \; b + e = 0$ $\quad e = -b$

$L: \; a - 3b + c + d - e = 1$ $\quad e = -b = -d$ $\quad a - 3b + c + b + b = 1$

$T: \; -2d - 2e = 0$ $\quad e = -d$ $\quad a - b + c = 1$

$h = \alpha h_0^{\,a} \rho^b r^{1-a+b} g^b E^{-b} \approx \alpha r (h_0 r^{-1})^a (\rho r g E^{-1})^b$

$\approx \alpha r \, \pi_1^{\,a} \pi_2^{\,b}$ $\quad \pi_1 = \frac{h_0}{r}$ $\quad \pi_2 = \frac{\rho r g}{E}$

$$\boxed{h = \cancel{\alpha} r f(\pi_1, \pi_2) \text{ where } \pi_1 = \frac{h_0}{r} \text{ and } \pi_2 = \frac{\rho r g}{E}}$$

don't need to write alpha

(3.2) h depends linearly on h_0

→ if h depends linearly on h_0 where if $h_0 = 0$ then $h = 0$ then $a = 1$

$h = \alpha \cancel{r} (h_0 \cancel{r^{-1}}) (\rho r g E^{-1})^b$

$$\boxed{h = \cancel{\alpha} h_0 f(\pi_2) \text{ where } \pi_2 = \frac{\rho r g}{E}}$$

again don't need alpha

4. The frequency ω of waves on a deep ocean is found to depend on the wavelength λ of the wave, the surface tension σ and density ρ of the water, and gravity.

4.1 • Use dimensional analysis to determine the functional dependence of ω on λ, σ, ρ, and g.

4.2 • In fluid dynamics, it is shown that $\omega = \left(gk + \sigma k^3/\rho\right)^{1/2}$, where $k = 2\pi/\lambda$ is the wavenumber. How does this differ from your result?

→ frequency ω depends on wavelength λ, surface tension σ, density ρ, and g

(4.1) $\omega = f(\lambda, \sigma, \rho, g)$

(T^{-1}) (L) (MT^{-2}) (ML^{-3}) (LT^{-2})

$$[\omega] = [\lambda^a \sigma^b \rho^c g^d] \Rightarrow T^{-1} = (L)^a (MT^{-2})^b (ML^{-3})^c (LT^{-2})^d = M^{b+c} L^{a-3c+d} T^{-2b-2d}$$

M: $b + c = 0$ $\quad c = -b$

L: $a - 3c + d = 0$ $\quad a - 3(-b) + d = 0 \Rightarrow a + 3b + (\frac{1}{2} - b) = 0 \Rightarrow a = -\frac{1}{2} - 2b$

T: $-2b - 2d = -1$ $\quad d = \frac{1}{2} - b$

$$\omega = \alpha \lambda^{-\frac{1}{2}} \lambda^{-2b} \sigma^b \rho^{-b} g^{\frac{1}{2}} g^{-b} = \alpha \left(\frac{g}{\lambda}\right)^{1/2} \left(\frac{\lambda^2 \rho g}{\sigma}\right)^{-b} \Rightarrow \boxed{\omega = \alpha \left(\frac{g}{\lambda}\right)^{1/2} f\left(\frac{\lambda^2 \rho g}{\sigma}\right)}$$

where α is an arbitrary #.

(4.2) in fluid dynamics: $\omega = (gk + \sigma k^3/\rho)^{1/2}$ where $k = 2\pi/\lambda$ is the wavenumber

$$\omega = \left(\frac{g 2\pi}{\lambda} + \frac{8\pi^3 \sigma}{\lambda^3 \rho}\right)^{1/2} = \left[\left(\frac{2\pi}{\lambda}\right)\left(g + \frac{4\pi^2 \sigma}{\lambda^2 \rho}\right)\right]^{1/2} = \left[\left(\frac{2\pi g}{\lambda}\right)\left(1 + \frac{4\pi^2 \sigma}{\lambda^2 \rho g}\right)\right]^{1/2}$$

CHECK THIS

$$= \underbrace{(2\pi)^{1/2}}_{\alpha} \underbrace{\left(\frac{g}{\lambda}\right)^{1/2}}_{\left(\frac{g}{\lambda}\right)^{1/2}} \underbrace{\left[1 + 4\pi^2 \left(\frac{\lambda^2 \rho g}{\sigma}\right)^{-1}\right]^{1/2}}_{f()}$$

$\omega = \alpha \left(\frac{g}{\lambda}\right)^{1/2} f()$ from above, so $\boxed{\alpha = (2\pi)^{1/2}}$ as it relates to (4.1)

5. A ball, when released underwater, will rise towards the surface with velocity v. This velocity depends on the density ρ_b and radius R of the ball, on gravity g, and on the density ρ_f and kinematic viscosity ν of the fluid.

- 5.1 • Find a dimensionally reduced form for v.
- 5.2 • In fluid dynamics, using Stoke's Law, it is found that $v = 2gR^2(\rho_b - \rho_f)/9\nu\rho_f$, how does this differ from your result?

→ v (velocity) depends on ρ_b (density), R (radius), g (gravity), ρ_f (fluid dens.), ν (kin. viscosity)

(5.1) $[v] = [\rho_b^a R^b g^c \rho_f^d \nu^e] \Rightarrow LT^{-1} = (ML^{-3})^a (L)^b (LT^{-2})^c (ML^{-3})^d (L^2T^{-1})^e$

$LT^{-1} = M^{a+d} L^{-3a+b+c-3d+2e} T^{-2c-e}$

$M: \; a + d = 0 \Rightarrow a = -d$

$L: \; -3a + b + c - 3d + 2e = -1 \Rightarrow b + c + 2e = 1$

$T: \; -2c - e = -1 \Rightarrow e = 1 - 2c$

$\Rightarrow b + c + 2 - 4c = 1$

$b + 1 = 3c \Rightarrow c = \frac{b+1}{3} \Rightarrow e = \frac{1}{3}(1-2b)$

$$v = \rho_b^a R^b g^{\frac{b+1}{3}} \rho_f^{-a} \nu^{\frac{1-2b}{3}}$$

$$= \left(\frac{\rho_b}{\rho_f}\right)^a R^{\frac{3}{3}b} g^{b/3} g^{1/3} \nu^{1/3} \nu^{-2/3 b} = (g\nu)^{1/3} \underbrace{\left(\frac{\rho_b}{\rho_f}\right)^a}_{\pi_1} \underbrace{\left(\frac{R^3 g}{\nu^2}\right)^{b/3}}_{\pi_2}$$

$$\Rightarrow \boxed{\begin{array}{l} v = \alpha (g\nu)^{1/3} f(\pi_1, \pi_2) \\ \text{where } \pi_1 = \rho_b/\rho_a \text{ and } \pi_2 = R^3 g/\nu^2 \end{array}}$$

(5.2) $v = \dfrac{2gR^2(\rho_b - \rho_f)}{9\nu\rho_f}$ Stoke's Law

$$v = \frac{2gR^2(\rho_b - \rho_f)}{9\nu\rho_f} = \frac{2}{9}\frac{g^2}{g}\frac{R^3}{R}\frac{\nu}{\nu^2}\left(\frac{\rho_b - \rho_f}{\rho_f}\right) = \frac{2}{9}\underbrace{(g\nu)}_{LT^{-2}L^2T^{-1} = \frac{L^3}{T^3}}\underbrace{\left(\frac{R^3 g}{\nu^2}\right)}_{f(\pi_2)}\underbrace{\left(\frac{1}{gR}\right)}_{\frac{1}{LT^{-2}L} = \frac{T^2}{L^2}}\underbrace{\left(\frac{\rho_b}{\rho_f} - 1\right)}_{f(\pi_1)} = \frac{2}{9}(g\nu)\left(\frac{1}{gR}\right) f(\pi_1, \pi_2)$$

$\frac{\nu}{R}$ will give velocity

$\nu \Leftrightarrow L^2T^{-1}$

$$v = \frac{2}{9}(g\nu)\left(\frac{1}{gR}\right)\cancel{f(\pi_1, \pi_2)} = \alpha (g\nu)^{1/3}\cancel{f(\pi_1, \pi_2)}$$

$$\frac{2}{9}(g\nu)^{2/3}(gR)^{-3/3} = \alpha \Rightarrow \frac{2}{9} g^{-1/3} \nu^{2/3} R^{-3/3} = \frac{2}{9}\underbrace{\left(\frac{\nu^2}{R^3 g}\right)^{1/3}}_{\text{dimensionless } (\pi_2)}$$

$$\Rightarrow \boxed{\alpha = \frac{2}{9}}$$

6. From bartenders to infamous "Ice King" of Boston, Frederic Tudor, it has been long known that small ice cubes melt faster than large ones, meaning that small ice cubes will cool your drink faster than large ones. The common explanation of this effect is that small cubes has a larger surface area for the same total volume as large cubes. To test this hypothesis, we would expect the time for a drink to cool down will be proportional to L^2. Using dimensional analysis, and recognizing that the relevant parameters are length L, time T, temperature Θ, thermal conductivity $[\kappa] = LT^{-3}\Theta^{-1}M$, and volumetric heat capacity $[s] = L^{-1}T^{-2}\Theta^{-1}M$, show that cooling time is proportional to L^2.

→ small ice cubes melt faster than large, will cool faster than large

(small ice cubes: larger surface area for same total volume)

$t_c \,\frac{1}{\alpha}\, L^2$ inv. prop.

(t_c: time to cool; L^2: SA)

$$t_c = f(l, t_0, \theta_0, K, s) \Rightarrow [T] = [(L)^a (T)^b (\theta)^c (LT^{-3}\theta^{-1}M)^d (L^{-1}T^{-2}\theta^{-1}M)^e]$$
$$= M^{d+e} L^{a+d-e} T^{b-3d-2e} \theta^{c-d-e}$$

$$\left.\begin{aligned} M&: \quad d+e = 0 \\ L&: \quad a+d-e = 0 \\ T&: \quad b-3d-2e = 1 \\ \theta&: \quad c-d-e = 0 \end{aligned}\right\} \qquad \begin{aligned} e &= -d \\ a &= -2d \\ b &= 1+d \\ c &= 0 \end{aligned}$$

$$t_c = L^{-2d} t_0^{1+d} \theta_0^0 K^d s^{-d} = \alpha\, t_0 (L^{-2} t_0 K s^{-1})^d$$

$$\boxed{t_c = \alpha\, t_0 \left(\frac{t_0 K}{L^2 s}\right)^d}$$

∴ t_c, the time to cool is inversely proportional to the surface area, meaning that the larger the SA, the quicker to cool the drink

Homework #1 Review Lecture

ex. $A \sim \alpha\, xy \left(\frac{z}{B}\right)^a \longrightarrow A = xy\, \Phi\left(\frac{z}{B}\right)$

scaling

we don't know anything about this quantity

dimensions of xy must be equal to A

$\frac{A}{xy} \sim \left(\frac{z}{B}\right)^a$

... dimensionless on both sides

$\Rightarrow A \sim \alpha_1 xy \left(\frac{z}{B}\right)^{a_1} + \alpha_2 xy \left(\frac{z}{B}\right)^{a_2} \ldots$ is exactly dimensionally the same as the top line and also just as valid

why can we make this leap? because from dimensional analysis we can't rule this out as a solution ... why $A = xy\, \Phi\left(\frac{z}{B}\right)$ is the best thing for us to do

Now with two pi groups: $A = xy\, \underbrace{\Phi(\pi_1, \pi_2)}$

↳ identical to saying $f(x, y)$

$A \sim \alpha_1 xy\, (\pi_1)^{a_1} (\pi_2)^{b_1} + \alpha_2 xy\, (\pi_1)^{a_2} (\pi_2)^{b_2} + \ldots \Rightarrow$ why it's better to write $A = xy\, \Phi(\pi_1, \pi_2)$