Differential Geometry + Tensor Analysis

→ DG always had a home in physics, now making way in Mechanical Engineering
- always us to think about deformation, changes in shape, for example
  - bubbles, droplets, curved structures
  - thin structures: plates, shells, etc.

→ going to be a very conceptual introduction

Invariance ⇒ we're often interested in how physical quantities change as change coordinate systems

① transformations ↦ tensors
② measurement
③ curved spaces & calculus

} the three topics we'll cover

Cartesian to **Polar** coordinates

$(x, y) \longmapsto (r, \theta)$ $\quad x = r\cos\theta \mapsto dx = dr\cos\theta - d\theta r\sin\theta$
$\quad y = r\sin\theta \mapsto dy = dr\sin\theta + d\theta r\cos\theta$

] $ds^2 = dx^2 + dy^2$ (Eucledian Distance)

$$ds^2 = dx^2 + dy^2$$
$$= (dr\cos\theta - r\sin\theta d\theta)^2 + (dr\sin\theta + r\cos\theta d\theta)^2$$
$$= dr^2 + r^2 d\theta^2$$

(in 2D so two components: the differential/infinitesimal & what multiplies this)

↓ in general

call our coordinates: $x^\mu = (x^1, x^2, \dots x^D)$
$\quad x^1 \to r$, $x^2 \to \theta$

$$ds^2 = \sum_{\mu=1}^{D}\sum_{\nu=1}^{D} g_{\mu\nu}\, dx^\mu dx^\nu$$

← the μ and ν will "contract"... like cancel out but not technically mathematically correct

ds^2: squared quantity, not superscript (length of line)

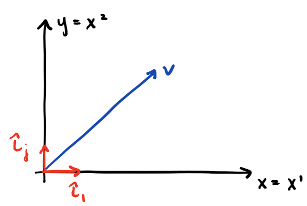
$dx^\mu dx^\nu$: coordinates

Einstein's Summation Notation:

$$ds^2 = \sum_{\mu=1}^{D}\sum_{\nu=1}^{D} g_{\mu\nu}\, dx^\mu dx^\nu = g_{\mu\nu}\, dx^\mu dx^\nu$$

THE FIRST FUNDAMENTAL FORM

$g_{\mu\nu}$ → the metric tensor



→ the metric tensor is constructed from the basis vectors in your coordinate system

- unit vectors have length 1
- basis vectors are set of linearly independent vectors that can be used to make up any vector in a vector space

Metric tensor: $g_\mu \cdot g_\nu = g_{\mu\nu}$

dotting 2 vectors → $\hat{\imath}_1$, $r\hat{\imath}_2$; tensor

metric tensor is composed of dotting two basis vectors (in 2D)

in polar coordinates
↓
$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$$

} $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

$\quad g_{\mu\nu} = \delta_\mu^\nu = \mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

in flat space: $ds^2 = dx^2 + dy^2$

$$ds^2 = \underset{1}{g_{11}}\, \underset{dx}{dx^1}\, \underset{dx}{dx^1} + \underset{0}{\cancel{g_{12}}}\, dx^1 dx^2 + \underset{0}{\cancel{g_{21}}}\, dx^2 dx^1 + \underset{1}{g_{22}}\, \underset{dy}{dx^2}\, \underset{dy}{dx^2}$$

$$ds^2 = dx^2 + dy^2$$

$$ds^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} dx \\ dy \end{bmatrix} = dx^2 + dy^2$$

$$ds^2 = dr^2 + r^2 d\theta^2$$

Conceptually, what is the metric tensor doing?

→ allows us to determine lengths and angles between tangent vectors

→ the metric tensor doesn't change when change coordinate systems, but their components inside the matrix do.

- how do these quantities transform then? do they change in a way that is similar? do they covary? contravary?

* upper indices: contravariant → in general, contravariant vectors typically have length in the numerator: length, displacement, velocity, acceleration, etc., and typically column vector (tangent vectors)

...if change in opposite way of basis vectors then contravariant (think velocity m/s → cm/s example), while if same way of basis vectors then covariant

* lower indices: covariant → gradient (row vector) (covectors)

When you take a vector that is contravariant with a vector that is covariant... very familiar

→ take arbitrary vector a_λ and multiply by vector b^λ

→ when have something "downstairs" and "upstairs" and multiply, then we contract and they "go away"

$$\underbrace{a_\lambda}_{\substack{\text{vector}\\(1^{st}\text{ order tensor})}} \overset{\text{vector}}{b^\lambda} = \underbrace{c}_{\text{scalar with angle relevant to those vectors}}$$

A detour to multivariate calculus:

$\forall = \int_\forall dx\,dy\,dz = \iiint dx\,dy\,dz$ ... can't just switch dx w/ dr, dy w/ dθ, dz w/ dφ

↳ but in general: $\forall = \iiint \sqrt{|g|}\, dx^1 dx^2 dx^3$

sqrt of det(g): scalar density

metric tensor

$$g_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$ — metric tensor in spherical coordinates

$$|g| = \det(g) = r^4\sin^2\theta \mapsto \sqrt{|g|} = r^2\sin\theta$$

$$\forall = \iiint r^2\sin\theta\, dr\, d\theta\, d\phi$$

Tensor Notation:

a	scalar	[0th order tensor]
b_i	vector	[1st order tensor]
c_{ij}	tensor	[2nd order tensor] ↳ stress, metric
d_{ijk}	tensor	[3rd order tensor] ↳ $\epsilon_{ijk} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}$ (permutation tensor)
f_{ijkl}	→ elastic modulus tensor $\sigma_{ij} = A^{ij}_{kl}\,\varepsilon^{kl}$	

$\left.\begin{matrix} c_{ij} \\ d_{ijk} \end{matrix}\right\}$ $a_{ij}\, b_k = d_{ijk}$

Valid Tensor Equations

$r_{il}\, s^{lmn}\, t_n\, u_m{}^k = V_i{}^k$

then contract, component doesn't appear on other side of equation

raising and lowering indices

you have the vector V^λ but want covariant V_λ ... what do you do?

$V^\lambda g_{\mu\lambda} = V_\mu$ ... subscript is different but it's really a "dummy" subscript anyway

$a_\gamma g^{\gamma\lambda} = a^\lambda$

t remember that g is the metric tensor (it's incredibly powerful!)

contravariant metric tensor $g^{\gamma\lambda}$ = covariant metric tensor inverse, $(g_{\gamma\lambda})^{-1}$

↳ what is it in polar coordinates?

$$g_{\gamma\lambda} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix} \quad \therefore \quad g^{\gamma\lambda} = \begin{bmatrix} 1 & 0 \\ 0 & 1/r^2 \end{bmatrix}$$

we dotted the covariant basis vectors earlier, can also do this with contravariant

$g_\mu \cdot g_\nu = g_{\mu\nu}$ covariant

$g^\mu \cdot g^\nu = g^{\mu\nu}$ contravariant

$g_\mu \cdot g^\nu = \delta_\mu{}^\nu$ kronecker's delta = $\mathbb{1}$ (identity matrix)

unpacking more... let's take 2nd order tensor

$r_{il}\, g^{lk} = r_i{}^k$ ← contravariant component; ↑ covariant component

⇒ this is a mixed tensor ... $r_1{}^1$ or $r_1{}^2$ or $r_1{}^3$

↳ tell me what coordinate system want to be in and we'll unpack from there... just a confusing way to not include the metric tensor

$r_{il}\, g^{lk} = r_i{}^k$ *can use to get from mixed to covariant, or mixed to contravariant for instance