**Midterm Discussion** in beginning (rewatch for details) → thru first ~6 minutes

## Calculus of Variations

→ extremize functionals

Function: dependent variable changes with independent variable, $x(t)$

↳ extremize → Calculus: $\frac{dx(t)}{dt} = 0$

### What's a functional?

Functional: function of a function of a variable

$V(\vec{x}(t))$

potential energy ↑ ↑ some vector

→ extremize → Calculus of variations (... a little more complicated)

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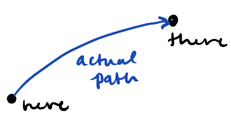
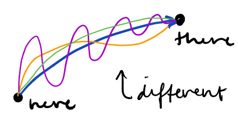
## Principle of Least Action

→ throw a ball

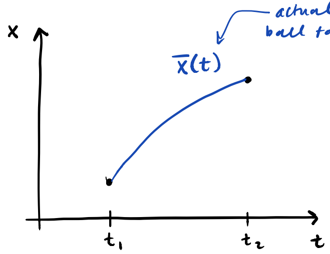



action: $S = \int_{start}^{end} (T - V)\, dt$

capital "S" — kinetic energy — potential energy




↑ different paths the ball may have taken




kinetic energy $T = \frac{1}{2} m \left(\frac{d\bar{x}}{dt}\right)^2$

potential energy $V = V(\bar{x})$

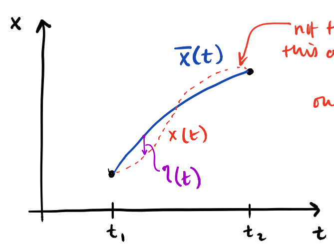
$$S = \int_{t_1}^{t_2} \left[\frac{1}{2} m \left(\frac{d\bar{x}}{dt}\right)^2 - V(x)\right] dt$$

(from calculus:)


... looking for a point that minimizes

while here, we're looking for a curve that minimizes

*going to get a function as a result of minimizing a functional*




not the actual path, imagine this as infinitely close to actual

adding small perturbation to actual solution

our perturbed answer guess:

$$x(t) = \bar{x}(t) + \epsilon\eta(t)$$

$x, \bar{x}, \eta$ are all vectors while $\epsilon$ is scalar

this is the perturbation of the actual function

*action* == difference between KE and PE integration over between two time points

*if the blue curve minimizes the action, then any other curve $\eta$ added to $\bar{x}$ is going to be larger than $\bar{x}$

* restriction on $\eta$: has to start and stop at same points as blue curve

$$S = \int_{t_1}^{t_2} \left[\frac{m}{2}\left(\frac{d\bar{x}}{dt} + \epsilon\frac{d\eta}{dt}\right)^2 - V(\bar{x} + \epsilon\eta)\right] dt$$

... now expand and simplify

$$\left(\frac{dx}{dt}\right)^2 = \underbrace{\left(\frac{d\bar{x}}{dt}\right)^2}_{O(1)} + \underbrace{2\epsilon\frac{d\bar{x}}{dt}\frac{d\eta}{dt}}_{O(\epsilon)} + \underbrace{\epsilon^2\left(\frac{d\eta}{dt}\right)^2}_{O(\epsilon^2)}$$

H.O.T. == higher order terms

$V(\bar{x}+\epsilon\eta) =$ ... take a Taylor Series Expansion around $\eta = 0$ (?)

$$= V(\bar{x}) + \underbrace{\epsilon\eta V'(\bar{x})}_{\mathcal{O}(\epsilon)} + \underbrace{\epsilon^2 \frac{\eta^2}{2} V''(\bar{x})}_{\mathcal{O}(\epsilon^2)} + \dots$$

so now ...

$$S = \int_{t_1}^{t_2} \Big[ \underbrace{\frac{m}{2}\left(\frac{d\bar{x}}{dt}\right)^2 - V(\bar{x})}_{\text{when integrated} \;=\; \bar{S}} + m\epsilon \frac{d\bar{x}}{dt}\frac{d\eta}{dt} - \epsilon\eta V'(\bar{x}) + \mathcal{O}(\epsilon^2\eta^2) \Big] dt$$

written with the $\eta$ term because typically write $x(t) = \bar{x}(t) + \delta\bar{x}(t)$

then rewrite ...

$$\delta S = \bar{S} - S = 0 \Rightarrow \text{then have found } \bar{x}$$

this is called the "first variation" since only considered $\epsilon$ terms (not $\epsilon^2$ terms for instance)

$$\delta S = 0$$

$$\delta S = \int_{t_1}^{t_2} \left[ \underline{m\frac{d\bar{x}}{dt}\epsilon\frac{d\eta}{dt}} - \epsilon\eta V'(\bar{x}) \right] dt = 0$$

if we can factor out $\eta$

$$\int_{t_1}^{t_2} A\eta \, dt = 0$$

$\eta$ not zero (otherwise it's exact) but can be anything else/arbitrary

then whatever is left on inside must be $\emptyset$

so we need to get derivative of $\eta$ out ... the underlined quantity needs to be turned into something times $\eta$

*crucial step: can get rid of that derivative by doing integration by parts

Integration by parts:

- product rule backwards

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\Rightarrow \int u\,dv = uv - \int v\,du$$

$\underline{m\frac{d\bar{x}}{dt}\epsilon\frac{d\eta}{dt}}$ $\longrightarrow$ $\displaystyle\int_{t_1}^{t_2} m\underbrace{\frac{d\bar{x}}{dt}}_{u}\epsilon\underbrace{\frac{d\eta}{dt}}_{dv}dt = \Big[\epsilon\underbrace{\eta}_{v}\underbrace{m\frac{d\bar{x}}{dt}}_{u}\Big]_{t_1}^{t_2} - \epsilon m\int_{t_1}^{t_2}\underbrace{\eta}_{v}\underbrace{\frac{d}{dt}\left(\frac{d\bar{x}}{dt}\right)}_{u}dt$

from before, what we want to integrate by parts

Back to what we're after:

$$\delta S = \epsilon\,\eta(t)\, m \frac{d\bar{x}}{dt}\Bigg|_{t_1}^{t_2} - \epsilon m \int_{t_1}^{t_2} \frac{d^2\bar{x}}{dt^2}\,\eta(t)\,dt - \epsilon \int_{t_1}^{t_2} V'(\bar{x})\,\eta\,dt$$

insert $t_1$ and $t_2$ here

* remember that $\eta$ is the difference between $x$ and $\bar{x}$

... but @ $t_1$ and $t_2$ there is no difference

$\therefore\ \eta(t_1) = \eta(t_2) = 0$

$\therefore$ can take out this entire term!

first variation of action : $\delta S = \int_{t_1}^{t_2} \left[ -m \frac{d^2\bar{x}}{dt^2} - V'(x) \right] \epsilon\,\eta(t)\,dt$

$$\downarrow$$

$$-m \frac{d^2\bar{x}}{dt^2} - V'(\bar{x}) = 0$$

acceleration

force

force $= -V'(\bar{x})$

$\Rightarrow$ we found Newton's Law governing equation!

$$\boxed{F = ma}$$