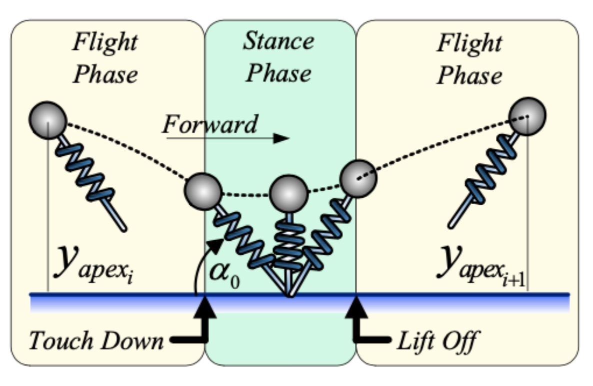
Approximating the Stance Map of the SLIP Runner Based on Perturbation Approach

Haitao Yu, Mantian Li, and Hegao Cai



Presented by Leah Gaeta for ME 712

Tuesday, December 8, 2020

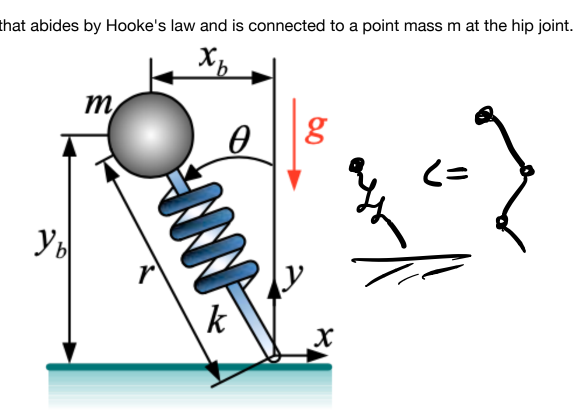
Introduction

- Engineers want to study legged dynamic walking to develop analagous machines (robots that can run smoothly)
- The Spring Loaded Inverted Pendulum (SLIP), or monopedal runner, is widely used to depict running and hopping in mammalian and human locomotion
  - Nonlinear model that has nonintegrable terms due to gravity in the stance phase
  - Most previous approximations proposed ignore gravity
  - Effects of gravitational forces cannot be neglected in locomotion analysis
- Goal: produce an accurate approximation of stance map that works in both symmetric and asymmetric cases, and includes gravitational effects
  - Perturbation methods are used to obtain an approximate solution

The Spring Loaded Inverted Pendulum (SLIP) Model

The 2-Degrees of Freedom Model

The SLIP model is comprised of a massless leg (spring) that abides by Hooke's law and is connected to a point mass m at the hip joint.

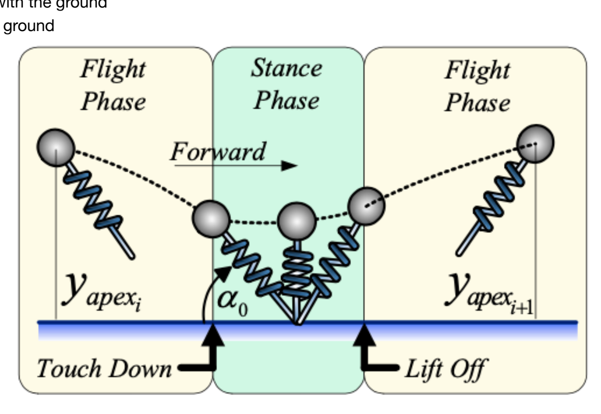


The Spring Loaded Inverted Pendulum (SLIP) Model

Complete Hopping Gait Period

In a complete gait period, the alternating touch down and lift off events contribute to the characteristic locomotion switching between the stance and flight phases

- Touch down: toe makes contact with the ground
- Lift off: toe loses contact with the ground



Assumptions:

- No energy dissipation during locomotion
- No torque applied at the hip in stance phase, meaning the toe of the planting foot does not slip and is stationary

The Spring Loaded Inverted Pendulum (SLIP) Model

The Dynamics

Symbol	Variable
m	Body mass
k	Leg spring stiffness
r	Leg length
r_0	Leg rest length
θ	Leg angle
α_0	Touchdown angle
g	Gravitation Acceleration
x_b	Body horizontal position
y_b	Body vertical position
s	Time scale
P_r	Radial momentum
P_θ	Angular momentum
Es	Total system mechanical energy
TD	Touchdown mark
LO	Liftoff mark
$apex$	Apex mark

The Spring Loaded Inverted Pendulum (SLIP) Model

The Dynamics

The analytical solution of the SLIP model in flight phase can be found but an accurate solution for the stance phase is more difficult.

The Lagrange function of stance phase in polar coordinates (r, θ) is given by

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{2}(r_0 - r)^2 - mgrcos\theta \tag{1}$$

The equations of motion of the stance phase are derived as

$$m\ddot{r} + k(r - r_0) + mgcos\theta - mr\dot{\theta}^2 = 0 \tag{2}$$
$$\frac{d}{dt}(mr^2\dot{\theta}) - mgrsin\theta = 0$$

Note that (2) is a set of nonlinear differential equations. However, exact analytical solutions for the stance phase are difficult to obtain due to the nonintegrable terms from gravity in stance dynamics.

The Spring Loaded Inverted Pendulum (SLIP) Model

Dimensional Analysis

s is the time scale and differentiation $()' \equiv \frac{d}{d\bar{t}}$ is with respect to nondimensional $\bar{t}$

(3)

Variable	Dimension	Change of Variables	First Derivative	Second Derivative
Time	$[T]$	$\bar{t} = \frac{t}{s}$		
Leg Length	$[L]$	$\bar{r} = \frac{r}{r_0}$	$\bar{r}' = s\frac{\dot{r}}{r_0}$	$\bar{r}'' = s^2\frac{\ddot{r}}{r_0}$
Horizontal Position	$[L]$	$\bar{x} = \frac{x}{r_0}$	$\bar{x}' = s\frac{\dot{x}}{r_0}$	$\bar{x}'' = s^2\frac{\ddot{x}}{r_0}$
Vertical Position	$[L]$	$\bar{y} = \frac{y}{r_0}$	$\bar{y}' = s\frac{\dot{y}}{r_0}$	$\bar{y}'' = s^2\frac{\ddot{y}}{r_0}$
Leg Angle		$\bar{\theta} = \theta$	$\bar{\theta}' = s\dot{\theta}$	$\bar{\theta}'' = s^2\ddot{\theta}$

The natural frequency of the horizontal oscillation motion is

$$\frac{s^2 g}{r_0} = 1 \qquad \Rightarrow \qquad s = \sqrt{\frac{r_0}{g}} \tag{4}$$

The relative leg stiffness (5), radial momentum (6), and angular momentum (7) are described below:

$$\bar{k} = \frac{kr_0}{mg} \tag{5}$$
$$\bar{P}_r = \frac{P_r s}{mr_0} \tag{6}$$
$$\bar{P}_\theta = \frac{P_\theta mr_0^2}{s} \tag{7}$$

The Spring Loaded Inverted Pendulum (SLIP) Model

Dimensional Analysis

Substituting (3), (4), and (5) into (2), we get the equations of stance dynamics in dimensionless form.

$$\bar{r}'' + \bar{k}(\bar{r} - 1) + cos\bar{\theta} - \bar{r}\bar{\theta}'^2 = 0 \tag{8}$$
$$(\bar{r}^2\bar{\theta}')' - \bar{r}sin\bar{\theta} = 0 \tag{9}$$

Assuming a small angular span during stance phase $\Delta\theta$, $cos\theta \approx 1$ and $sin\theta \approx \theta$ and substituting this into (8) and (9) and get

$$\bar{r}'' + \bar{k}\bar{r} = \bar{k} - 1 + \bar{r}\bar{\theta}'^2 \tag{10}$$
$$(\bar{r}^2\bar{\theta}')' = \bar{r}\bar{\theta} \tag{11}$$

The Perturbation Solutions for a Stance Map

Conservation of Angular Momentum

It's assumed that the angular momentum is conserved at the initial value P_θ (touch down (TD)) for solving (10) and then this constraint is relaxed to solve (11). The initial angular momentum C_r at TD is defined as

$$C_r = \bar{r}^2\bar{\theta}' = r_{\bar{TD}}{}^2\theta_{\bar{TD}}{}' \tag{12}$$

The conservation of angular momentum can be written as

$$\bar{r}^2\bar{\theta}' = C_r \tag{13}$$

Substituting (13) into (10) then yields

$$\bar{r}'' + \bar{k}\bar{r} = \bar{k} - 1 + \frac{C_r^2}{\bar{r}^3} \tag{14}$$

The Pertubation Solutions for a Stance Map

Relative Stiffness & Radial Motion

From assuming of sufficiently rough relative stiffness, we have $0 < 1 - \bar{r} << 1$ and thus the term $\frac{1}{\bar{r}^3}$ can be expanded in a Taylor Series (i.e. $f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \ldots$) around the *initial spring compression* $1 - \bar{r}_0 = 0$.

$$\frac{1}{\bar{r}^3} = \frac{1}{1 - (1 - \bar{r}))^3} = 1 + 3(1 - \bar{r}) + O((1 - \bar{r})^2) + \ldots \tag{15}$$

A small value of $1 - \bar{r}$ allows for truncation of the square and higher terms in (15). Therefore, (14) turns into:

$$\bar{r}'' + \bar{r}(\bar{k} + 3C_r^2) = \bar{k} - 1 + 4C_r^2 \tag{16}$$

This is solved to obtain the radial motion:

$$r_b = \frac{\bar{k} + 4C_r^2 - 1}{\bar{k} + 3C_r^2} \qquad \omega_0 = \sqrt{\bar{k} + 3C_r^2} \qquad \bar{r} = r_b + \alpha cos(\omega_0\bar{t} + \beta) \tag{17}$$

Given initial conditions $\bar{r}_{TD} = 1$ and $\bar{r}'_{TD}$, constants α and β are determined to be:

$$\alpha = (1 - r_b)\sqrt{(1 - r_b)^2 + \bar{r}'^2_{TD}/\omega_0^2} \qquad \beta = tan^{-1}(\bar{r}'_{TD}/(\omega_0(r_b - 1))) \tag{18}$$

The Perturbation Solutions for a Stance Map

Here, perturbation techniques are showcased to solve θ with non-conservation of angular momentum, with the radial motion solved in (17), turning equation (11) into:

$$\bar{\theta}'' + \frac{2\bar{r}'}{\bar{r}}\bar{\theta}' - \frac{1}{\bar{r}}\bar{\theta} = 0 \tag{19}$$

Then (19) can be transformed into one without the first derivative by making the substitution

$$\theta(\bar{t}) = p(\bar{t})u(\bar{t}) \tag{20}$$

while noting that $p(\bar{t})$ is an undetermined function for simplifying (19). Substituting (20) into (19)

$$pu'' + \left(2p' + \frac{2r'p}{r}\right)u' + \left(\frac{p'' + 2r'p'}{r} - \frac{1}{r}\right)u = 0 \tag{21}$$

Setting the coefficient of u' equal to zero, we then have $2p' + 2r'p/r = 0$. After separating variables and integrating, we get $p = 1/r$. Substituting this into (19) yields:

$$u'' - \frac{1 + r''}{r}u = 0 \tag{22}$$

Substituting (17) into (22) then gives:

$$u'' - \left[-\omega_0^2 + \left(\omega_0^2 + \frac{1}{r_b}\right)\frac{1}{1 + \epsilon cos\psi}\right]u = 0 \tag{23}$$

where $\epsilon = \alpha/r_b$ and $\psi = \omega_0\bar{t} + \beta$

The Pertubation Solutions for a Stance Map

Using a small leg compression assumption, we assume that ϵ is small allowing the fractional term in (23) to be expanded as:

$$\frac{1}{1 + \epsilon cos\psi} = 1 - \epsilon cos\psi + \epsilon^2 cos^2\psi + .. \tag{24}$$

Neglecting the higher terms of ϵ in (24) gives:

$$u'' - (\lambda^2 - \epsilon\delta cos\psi)u = 0 \tag{25}$$

where $\lambda = \frac{1}{\sqrt{r_b}} > 0$ and $\delta = \omega_0^2 + \frac{1}{r_b} > 0$. A regular perturbation solution to (25) is then created in terms of a power series in ϵ:

$$u(\bar{t}, \epsilon) = u_0(\bar{t}) + \epsilon u_1(\bar{t}) + \epsilon^2 u_2(\bar{t}) + \ldots \tag{26}$$

Substituting (26) into (25) and balancing the coefficients of ϵ allows the first order expansions to be derived

$$O(1) : u_0'' - \lambda^2 u_0 = 0$$
$$O(\epsilon) : u_1'' - \lambda^2 u_1 = -\delta u_0 cos\psi$$

The **general solution** of $O(1)$ is then written as $u_0 = C_1 e^{\lambda\bar{t}} + C_2 e^{-\lambda\bar{t}}$ where C_1 and C_2 are arbitrary constants based on the initial conditions. Substituting this into $O(\epsilon)$ gives an inhomogeneous equation of which the particular solution can be expressed as:

$$u_1 = \frac{\delta C_1}{\omega_0^2 + 4\lambda^2}e^{\lambda\bar{t}}(cos\psi - \frac{2\lambda}{\omega_0}sin\psi) + \frac{\delta C_2}{\omega_0^2 + 4\lambda^2}e^{-\lambda\bar{t}}(cos\psi + \frac{2\lambda}{\omega_0}sin\psi) \tag{27}$$

The approximate solution is restricted to the first order expansion. Applying the initial conditions for $u(0)$ and $u'(0)$, we can determine the constants C_1 and C_2. An approximate solution of θ can then be expressed as:

$$\theta(\bar{t}) = \frac{u_0(\bar{t}) + \epsilon u_1(\bar{t})}{\bar{r}(\bar{t})} \tag{28}$$

The above approximations predict stance trajectory of SLIP in for symmetric and assymetric cases.

Previous Existing Approximation Tools for Stance Map

Schwind et al.

- Use mean value theorem and Picard's iterations to acquire a gravity constrained iterative solution
- Based on Hamiltonian Dynamics

Geyer et al.

- Assumes conservation of angular momentum; focused on symmetric stride length about a neutral point
- Assumes small relative spring compression (therefore leg stiffness high)
- Works well in symmetric locomotion about a vertical line

Arslan et al.

- Extension of Geyer but with gravity correction scheme in asymmetric locomotion
- Introduce an angular momentum correction during stance phase, improving accuracy in symmetric and asymmetric case

Model Comparison

Simulations are operated in 2-D of initial states including height y_{apex} and horizontal velocity $\dot{x}_{apex}$ at the apex.

Considering biological data mentioned in literature:

- human with body mass 80 kg
- leg length r_0 of 1 m
- running speed range of 2 - 6 m/s
- relative leg stiffness $\bar{k}$ remains constant at 10

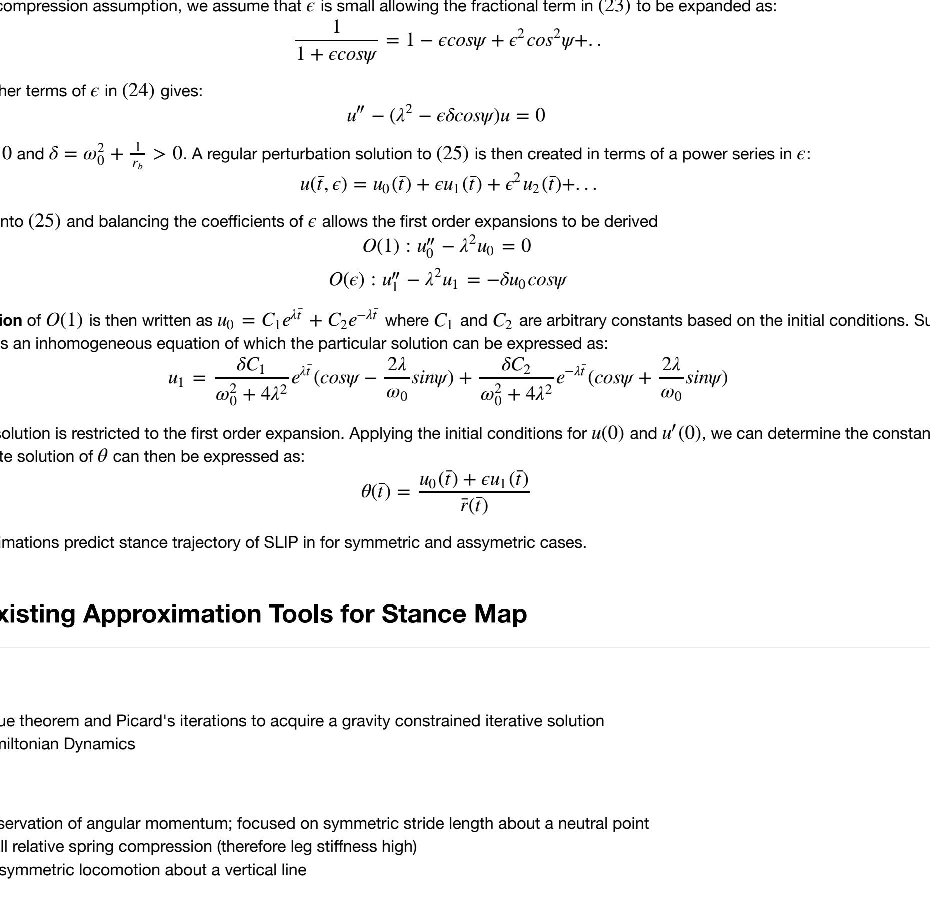
Chose dimensionless initial conditions as:

- $\bar{k} = 10$
- y_{apex} in the range of 0.95 to 1.05
- $\dot{x}_{apex}$ in the range of 2 to 6

Ran 6500 simulations for different initial conditions in MATLAB, applying variable time steps and fourth order Runge-kutta method to solve SLIP system dynamics.

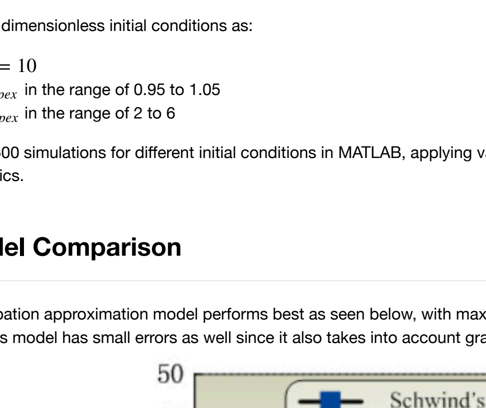
Model Comparison

Perturbation approximation model performs best as seen below, with max percentage errors less than 15% and mean value percentage errors less than 5%. Arslan's model has small errors as well since it also takes into account gravitational effects.

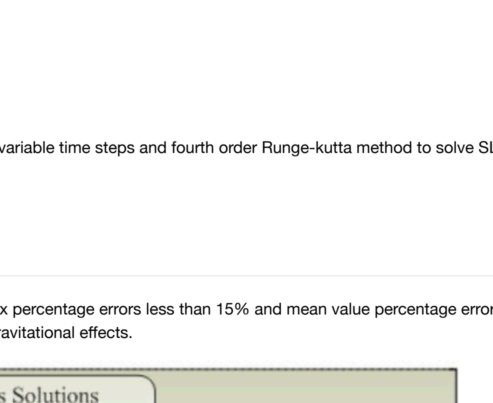


Model Comparison

The perturbation approximation performs better than the other 3 models in both symmetric and asymmetric cases. Since Arslan's model also outperforms the other two, this strongly suggests that gravitational forces cannot be neglected in asymmetric stance locomotion.



PE_{yapex}



$PE_{x'_{apex}}$

Conclusion

The perturbation model is quite precise in predicting the apex variables (height and horizontal velocity) as seen from the percentage error comparisons with the other existing approximations. It also performs better in both symmetric and asymmetric motion.

Author Suggested Future work:

- Extend the perturbation of the SLIP model to include damping
- Focus on more robust robot behavior in tough terrain

My thoughts on future work:

- Modify the SLIP model to account for absolute thigh & shank angles and relative hip, knee, and ankle angles
- Apply similar perturbation approach
- Run simulations for other populations
  - Disabled
  - Elite athletes
  - etc.

Questions and thoughts?

2012 IEEE International Conference on Robotics and Automation
RiverCentre, Saint Paul, Minnesota, USA
May 14-18, 2012

Approximating the Stance Map of the SLIP Runner Based on Perturbation Approach

Haitao Yu, Mantian Li and Hegao Cai

Abstract— **The Spring-Loaded Inverted Pendulum (SLIP), or monopedal runner, is widely used to depict running and hopping in mammalian and human locomotion, which is also serving as a template for running robot design. This classic model describes quite a simple mechanical system. Nevertheless issue of seeking the accurate analytic solution revealing the characteristics of the motion during stance remains unsettled due to the nonintegrable terms contained in the system equations. Moreover, several existing analytic approximations by simply ignoring or linearizing the gravitational force can not reveal the entire dynamical behavior of nonlinear system as well as can be breakdown rapidly when applied to a non-symmetric motion case. In this paper, a novel method with perturbation technique is proposed to obtain analytic approximate solutions to the SLIP dynamics in stance phase with considering the effect of gravity. The perturbation solution achieves higher accuracy in predicting the apex trajectory and stance locomotion by comparing with typical existing analytical approximations. Particularly, our solution is validated for non-symmetric case in a large angle range. Additionally, the prediction for stance trajectory is also verified through numerical evaluation.**

I. INTRODUCTION

THE dexterous and robust behavior displayed by legged animals when negotiating rough terrain outclasses any man-made mobile robots at present. This incomparable superiority of legged dynamic walking has been attracting the interest of scientists and engineers to develop analogous machines with stunning performance. Despite a succession of legged robots emerges such as Railbert's runners [1], the ARL-Monopod [2] and Rhex [3], some of which exhibit astonishing behaviors, this work is still in its infancy. In the past two decades, throughout the biomechanical and zoology research, the Spring-loaded Inverted Pendulum (SLIP) model (as illustrated in Fig. 1) has been considered as an archetypal model for analyzing running, especially in high speed case in mammalians [4, 5, 6]. The SLIP model not only has a better performance in fitting biological data, but also abstracts the appropriate features of running behavior for robotics research and design [7, 8], based on which passive running dynamics has been studied with minimizing the energy of torque cost function [2].

To gain further understanding of SLIP dynamics, researchers intend to seek accurate analytic approximations from the nonlinear model which contains nonintegrable terms generated by gravity in stance [9]. Several approximations have been proposed by simply ignoring the gravity [10, 11]. However, it is demonstrated that the effects of gravitational forces can not be neglected in the locomotion analysis [12].

An idealized case of SLIP dynamics had been studied in absence of gravity by Ghigliazza et al in [7]. Schwind and Koditschek used the mean value theorem and Picard's iterations to acquire a gravity-contained iterative solution [12]. The performance of the approximation related closely to the iteration steps as the accuracy would enhance with step increasing. Another approximation presented by Geyer et al. with assumption of the conservation of angular momentum focused on symmetric stride about the neutral point [10]. Arslan et al presented a gravity correction scheme in [13]. The approximate stance map was extended from Geyer's solutions by introducing a simple angular momentum correction during stance phase and provides comparable accuracy both in symmetric and non-symmetric case. Similar to Schwind's solution form, the approximations exhibit a two-step iterative character which mathematically attenuates the tractability of parametric dependency analysis.

The analytical task on approximating the stance map of SLIP model is, in general, a trade-off between the accuracy of approximations and mathematical complexity of explicit form. The aim of this work is to pursue an accurate approximation of stance map valid in both symmetric and non-symmetric case including gravitational effects. The approximation proposed in this paper is derived by elementary functions so as to study the SLIP dynamics and benefits the motion controller design of monopod robot.

This remainder of the paper is structured as follows: Section II introduces the general SLIP. In section III, three typical analytical approximate solutions are briefly reviewed. In section IV a novel perturbation method is proposed to obtain an analytic approximate solution considering gravity effects during stance phase. The performance of the perturbation solution is analyzed in section V by comparing with other existing approximations and the performance analysis is also realized. Finally, section VI presents the conclusion and future work.

This work is supported by National Hi-tech Research and Development Program of China (863 Program, Grant No.2008AA04Z211), National Natural Science Foundation of China (No. 61005076) and Self-Planned Task (NO. SKLR2008011B) of State Key Lab. of Robotics and System (HIT)

Haitao Yu is with the Dept. of Mechanical Eng., Harbin Institute of Technology, 150001 Harbin, China (E_mail:arcoyu@gmail.com)

Mantian Li and Hegao Cai are with the Dept. of Mechanical Eng., Harbin Institute of Technology, 150001 Harbin, China (E_mail:{limt, Hgcai}@hit.edu.cn).

I. The SLIP Model

A. The 2-DOF Model

The SLIP template, depicted in Fig. 1(a), consists of a massless leg with Hooke's spring law connected a point mass m at hip. During a complete hopping gait period as illustrated in Fig. 2(b), the alternative touchdown (the toe makes contact with ground) and the liftoff events (the toe loses contact with ground) contribute to the characteristic locomotion switching between the stance and flight phase. It is assumed that there is no energy dissipation during locomotion. With no torque applied at hip in stance phase, the toe of foot keeps stationary without slipping. In the flight phase, the center of mass trajectory is ballistic due to the gravitation force. The system reaches the apex as the body reaches the maximum height, where the vertical velocity decreases to zero. Noting that the leg has no moment of inertia, it is reasonable to set the touchdown angle α_0 fixed at the beginning of each stance phase [7]. Moreover, the apex return map could be established from the current apex variable y_{apexi} to the next apex variable $y_{apexi+1}$.

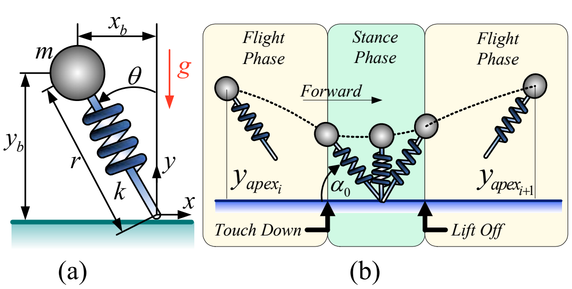


Fig. 1. The 2-DOF SLIP Model. (a) The physical template with coordinates and model parameters. (b) Hopping Gait period including flight and stance phase with phase transition events.

TABLE I
NOTATION AND TERMINOLOGY USED THROUGHOUT THE PAPER

Symbol	Variable
m	Body mass
k	Leg spring stiffness
r	Leg length
r_0	Leg rest length
θ	Leg angle
α_0	Touchdown angle
g	Gravitation Acceleration
x_b	Body horizontal position
y_b	Body vertical position
s	Time scale
P_r	Radial momentum
P_θ	Angular momentum
Es	Total system mechanical energy
TD	Touchdown mark
LO	Liftoff mark
$apex$	Apex mark

B. The SLIP Dynamics

The analytic solution of the SLIP model in flight phase can be easily obtained while the accurate analytic solution for stance phase still remains unsettled. As Fig. 1(b) shows, the Lagrange function of stance phase using polar coordinates (r,θ) is given by

$$L=\frac{m}{2}(\dot{r}^2+r^2\dot{\theta}^2)-\frac{k}{2}(r_0-r)^2-mgr\cos\theta. \tag{1}$$

The motion equations of stance phase can be derived as

$$\begin{aligned} &m\ddot{r}+k(r-r_0)+mg\cos\theta-mr\dot{\theta}^2=0\\ &\frac{d}{dt}(mr^2\dot{\theta})-mgr\sin\theta=0. \end{aligned} \tag{2}$$

Note that (2) are coupled nonlinear differential equations and a closed form solution with elementary functions cannot be found. Actually, the exact analytical solutions for stance phase are inaccessible caused by the non-integrable terms due to the gravity in stance dynamics [9].

C. Dimensional Analysis

The parameters intrinsic to the SLIP model can be reduced to a minimal set of nondimensional parameters necessary to characterize the physical model. Aiming at simplify the stance dynamics, the dimensionless variables are as follows

$$\begin{aligned} &\bar{t}=t/s\\ &\bar{r}=r/r_0,\ \bar{r}'=s\dot{r}/r_0,\ \bar{r}''=s^2\ddot{r}/r_0\\ &\bar{x}=x/r_0,\ \bar{x}'=s\dot{x}/r_0,\ \bar{x}''=s^2\ddot{x}/r_0\\ &\bar{y}=y/r_0,\ \bar{y}'=s\dot{y}/r_0,\ \bar{y}''=s^2\ddot{y}/r_0\\ &\bar{\theta}=\theta,\quad \bar{\theta}'=s\dot{\theta},\quad \bar{\theta}''=s^2\ddot{\theta} \end{aligned} \tag{3}$$

where s is the time scale and differentiation $()'\equiv d/d\,\bar{t}$ is with respect to the nondimensional time $\bar{t}$. By substituting (3) into (2), one acquires a dimensionless version of the system. Choose the natural frequency of the horizontal oscillation motion by setting

$$s^2g/r_0=1\Rightarrow s=\sqrt{r_0/g}. \tag{4}$$

Note that the relationships among parameters remain invariant even if one selects other time scales. Accordingly the relative leg stiffness, radial momentum and angular momentum can be normalized as

$$\bar{k}=kr_0/mg. \tag{5}$$

$$\bar{P}_r=P_rs/(mr_0) \tag{6}$$

$$\bar{P}_\theta=P_\theta mr_0^2/s \tag{7}$$

Ultimately substituting (3), (4), and (6) into (2), we get the dimensionless version of stance dynamics as followed

$$\bar{r}''+\bar{k}(\bar{r}-1)+\cos\theta-\bar{r}\theta'^2=0 \tag{8}$$

$$(\bar{r}^2\theta')'-\bar{r}\sin\theta=0. \tag{9}$$

II. Existing Analytical Tools for Stance Map

As mentioned above, the stance phase dynamics of the SLIP model are non-integrable. The approximate solutions are critical when shedding light on the dynamic properties of the hopping behavior. In the following sections, we briefly review three typical analytical approximate solutions among which the gravitational effect during stance has been considered elaborately.

A. Iterative Solutions for Stance Map by Schwind et al.

Schwind and Koditschek proposed closed-form analytical approximations for stance map containing nonintegrable Hamiltonian systems in [12]. The mean value theorem has been introduced for integration simplification, based on which the final solutions were attained by means of applying iteration similar to Picard Method. The analytical tools are based on a Hamiltonian Dynamics, the dimensionless version for which can be written as

$$H := \frac{1}{2}(\bar{P}_r^2 + \frac{\bar{P}_\theta^2}{\bar{r}^2}) + \frac{1}{2}k(1-\bar{r})^2 + \bar{r}\cos\theta \tag{10}$$

Given a specify level of energy E, the radial momentum $\bar{P}_r$ can be solved in form of $\bar{P}_r = H^*(\bar{r}, \theta, \bar{P}_\theta, E)$. Consequently the approximations for decompression phase could be acquired through

$$\begin{aligned}
&{}^{n+1}\tilde{t}(\bar{r}) = t_b + (\bar{r} - \bar{r}_b)/H_n^* \\
&{}^{n+1}\tilde{\theta}(\bar{r}) = \theta_b + {}^{n}\tilde{P}_\theta(\tilde{\xi})(\bar{r} - \bar{r}_b)/(\tilde{\xi}^2 H_n^*) \\
&{}^{n+1}\tilde{P}_\theta(\bar{r}) = P_{\theta_b} + \tilde{\xi}\sin({}^{n}\tilde{\theta}(\tilde{\xi}))(\bar{r} - \bar{r}_b)/H_n^* \\
&{}^{n+1}\tilde{P}_r(\bar{r}) = H_{n+1}^*
\end{aligned} \tag{11}$$

Where n represents the iteration number, $\tilde{\xi} := 3\bar{r}_b/4 + \bar{r}/4$ is generated by integral operator with mean value theorem and b is a state marker for system at bottom. Similarly the solutions for compression phase could be derived as below

$$\begin{aligned}
&{}^{n+1}\tilde{t}(\bar{r}) = t_{TD} + (\bar{r} - \bar{r}_{TD})/H_n^* \\
&{}^{n+1}\tilde{\theta}(\bar{r}) = \theta_{TD} - {}^{n}\tilde{P}_\theta(\tilde{\xi})(\bar{r} - \bar{r}_b)/(\tilde{\xi}^2 H_n^*) \\
&{}^{n+1}\tilde{P}_\theta(\bar{r}) = P_{\theta_{TD}} - \tilde{\xi}\sin({}^{n}\tilde{\theta}(\tilde{\xi}))(\bar{r} - \bar{r}_{TD})/H_n^* \\
&{}^{n+1}\tilde{P}_r(\bar{r}) = -H_{n+1}^*.
\end{aligned} \tag{12}$$

The accuracy of the approximations depends on the steps of iterations with an increasing number of which the performance will be improved. It is also important to note that the iterative form reduces the tractability of the solutions.

B. Simple Approximate Solutions by Geyer et al.

In [10], Geyer and Seyfarth et al derived analytical solutions to the stance map of an energy conservation SLIP model. It was assumed in their paper that the angular swept during stance is rather small that $\sin\theta \approx 0$. From (9) the conservation of P_θ can be derived easily. Another assumption is that the relative spring compression is sufficiently small (the effective leg stiffness is tough enough accordingly). These two remarkable assumptions above guarantee the success of approximating process.

Using the conservation of P_θ and several approximations in [**10**], the dimensionless version of analytic solutions in polar coordinates can be acquired as

$$\bar{r}(t) = 1 + a + b\cos(\hat{\omega}_0 t + \beta) \tag{13}$$

$$\begin{aligned}
\bar{\theta}(t) &= \theta_{TD} + \omega(1-2a)t \\
&\quad + 2\omega/\hat{\omega}_0[a\sin\hat{\omega}_0 t + \sqrt{b^2 - a^2}(1 - \cos\hat{\omega}_0 t)]
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\omega &= \bar{P}_\theta \\
\hat{\omega}_0 &= \sqrt{\omega_0^2 + 3\omega^2} \\
a &= (\omega^2 - 1)/\hat{\omega}_0^2 \\
b &= \sqrt{a^2 + (2\bar{E}_s - \bar{P}_\theta^2 - 2)/\hat{\omega}_0^2}.
\end{aligned} \tag{15}$$

Based on the approximations described above, the overall apex return map can be established with the ballistic flight dynamics additionally. The solutions perform well in symmetry locomotion about the vertical line. When it comes to asymmetry case, the non-zero effect of gravity leads to non-conversation of the angular momentum during the entire stance phase. Accordingly, the simple solutions proposed in [10] would become inaccurate due to the torque generated by gravity. In short these approximations have its limitation in predicting the asymmetry moment of SLIP model.

C. Approximations with Gravity Correction by Arslan et al.

On purpose of compensating the effect of gravity in non-symmetric locomotion, Arslan and Saranli et al in [13] proposed an explicit gravity correction algorithm on the base of Geyer's approximations. On stance phase, the angular momentum can be written as

$$\bar{P}(t) = \bar{P}_{TD} + \int_{t_{TD}}^{t} \bar{r}(\eta)\sin\theta(\eta)d\eta \tag{16}$$

Choosing arbitrary two moment t_i and t_f within the interval $[t_{TD}, t_{LO}]$ in stance phase, the average of $\bar{r}$ can be evaluated through

$$\bar{r}_{av}(t) = \frac{1}{t_f - t_i}\int_{t_i}^{t_f}[1 + a + b\cos(\hat{\omega}_0 t + \beta)]dt \tag{17}$$

where a, b and $\hat{\omega}_0$ can be calculated by (15) in Geyer's method. Hence the total effect of gravity can be calculated by

$$\bar{P}_C = (t_f - t_i)\bar{r}_{av}(t_i, t_f)\left(\sin\theta(t_i) + \sin\theta(t_f)\right)/2 \tag{18}$$

Finally the angular momentum with gravity correction term is calculated as

$$\hat{P}_\theta = \bar{P}_{TD} + \bar{P}_C \tag{19}$$

III. Perturbation Solutions for Stance Map

In this section we reuse the Geyer's assumptions to access approximate solutions. With a sufficiently small angular span $\Delta\theta$, we have $\cos\theta \approx 1$ and $\sin\theta \approx \theta$. Subsequently (8) and (9) have simple forms as

$$\bar{r}'' + \bar{k}\cdot\bar{r} = \bar{k} - 1 + \bar{r}\theta'^2 \tag{20}$$

$$(\bar{r}^2\theta')' = \bar{r}\theta\ . \tag{21}$$

It is extremely important to note that on this circumstance of equation (21), the angular momentum is no longer conservative. Different from Geyer's and Arslan's method, we tackle with the non-integrable terms directly without any other compensations of angular momentum. Firstly we assume that the angular momentum is conserved at the initial value $P_\theta|_{TD}$ for solving (20).We then relax this constraint and use the solution identified for r(t) to solve (21). The initial angular momentum C_r is defined as

$$C_r = \bar{r}^2\theta'\Big|_{\bar{t}=\bar{t}_{TD}} = \bar{r}_{TD}^2\theta'_{TD}\big|_{\bar{r}=1} = \theta'_{TD} \tag{22}$$

The conservation of angular momentum can be written as

$$\bar{r}^2\theta' = C_r \tag{23}$$

Substituting (23) into (20), yields

$$\bar{r}'' + \bar{k}\cdot\bar{r} = (\bar{k}-1) + \frac{C_r^2}{\bar{r}^3}. \tag{24}$$

With the assumption of sufficiently rough relative stiffness, we have $0 < 1-\bar{r} \ll 1$. Thus the term $1/\bar{r}^3$ can be expanded in Taylor series around the initial spring compression $1-\bar{r}_0 = 0$ (hereafter, the subscript 0 denotes the touchdown state)

$$\frac{1}{\bar{r}^3} = \frac{1}{\left[1-(1-\bar{r})\right]^3}\Bigg|_{\bar{r}=1} = 1 + 3(1-\bar{r}) + o\left((1-\bar{r})^2\right). \tag{25}$$

As noted above, a small value of $1-\bar{r}$ permits truncation of square and higher terms in (25). Hence (24) rewrites into

$$\bar{r}'' + (\bar{k} + 3C_r^2)\bar{r} = 4C_r^2 + \bar{k} - 1 \tag{26}$$

which can be solved easily and yields the close form of radial motion as followed

$$\bar{r} = r_b + a\cos(\omega_0\bar{t} + \beta) \tag{27}$$

where

$$r_b = \frac{\bar{k} + 4C_r^2 - 1}{\bar{k} + 3C_r^2}, \quad \omega_0 = \sqrt{\bar{k} + 3C_r^2}.$$

Given initial conditions $\bar{r}_{TD} = 1$ and $\bar{r}'_{TD}$, constants α and β can be determined as

$$\alpha = (1-r_b)\sqrt{(1-r_b)^2 + \bar{r}_{TD}'^2/\omega_0^2}, \; \beta = \tan^{-1}\left(\bar{r}'_{TD}/(\omega_0(r_b-1))\right) \tag{28}$$

A traditional method to solve angle θ is to simply ignore the terms caused by the gravity as mentioned in [7]. However, it is also indicated that the effects of gravity not only plays an irreplaceable role in monopod locomotion and analyzing biological data but also can hardly be neglected [12]. For this reason, we propose a novel method with perturbation techniques to solve θ with non-conservation of angular momentum. With the radial motion solved in (27), equation (21) has the form

$$\theta'' + \frac{2\bar{r}'}{\bar{r}}\theta' - \frac{1}{\bar{r}}\theta = 0 \tag{29}$$

which can be transformed into one without the first derivative by making the substitution

$$\theta(\bar{t}) = p(\bar{t})u(\bar{t}). \tag{30}$$

Noting that $p(\bar{t})$ works as an undetermined function for simplifying (29) in the following. Substituting (30) into (29) yields

$$pu'' + (2p' + 2r'p/r)u' + (p'' + 2r'p'/r - 1/r)u = 0 \tag{31}$$

Setting the coefficient of u' equal to zero, we have

$$2p' + 2r'p/r = 0. \tag{32}$$

Separating variables and integrating gives

$$p = 1/r \tag{33}$$

Then (29) becomes

$$u'' - \frac{1+r''}{r}u = 0. \tag{34}$$

Substituting (27) into (34) yields

$$u'' - \left[-\omega_0^2 + (\omega_0^2 + 1/r_b)\frac{1}{(1+\varepsilon\cos\psi)}\right]u = 0 \tag{35}$$

where $\quad \varepsilon = a/r_b, \quad \psi = \omega_0\bar{t} + \beta$

If it is assumed that ε is rather small (according to the small leg compression assumption), then the fractional term can be expanded as

$$\frac{1}{1+\varepsilon\cos\psi} = 1 - \varepsilon\cos\psi + \varepsilon^2\cos^2\psi + \cdots \tag{36}$$

Neglecting the higher term of ε in (36), we have

$$u'' - (\lambda^2 - \varepsilon\delta\cos\psi)u = 0 \tag{37}$$

where $\lambda = 1/\sqrt{r_b} > 0, \quad \delta = \omega_0^2 + 1/r_b > 0$.

A regular perturbation solution of Mathieu equation (37) can be developed in terms of power series in ε [14] as follow

$$u(\bar{t},\varepsilon) = u_1(\bar{t}) + \varepsilon u_1(\bar{t}) + \varepsilon^2 u_2(\bar{t}) + \cdots. \tag{38}$$

Substituting (38) into (37) and balancing the power of the coefficient of ε, the first-order expansions are derived

Order ε^0 $$u_0'' - \lambda^2 u_0 = 0 \tag{39}$$

Order ε^1 $$u_1'' - \lambda^2 u_1 = -\delta u_0\cos\psi. \tag{30}$$

The general solution of (29) can be written as

$$u_0 = c_1 e^{\lambda\bar{t}} + c_2 e^{-\lambda\bar{t}} \tag{41}$$

where c_1 and c_2 are arbitrary constants determined by initial conditions. Substituting u_0 into (40) yields an inhomogeneous equation, of which the particular solution can be expressed as

$$\begin{aligned} u_1 = &\frac{\delta c_1}{\omega_0^2 + 4\lambda^2}e^{\lambda\bar{t}}\left(\cos\psi - \frac{2\lambda}{\omega_0}\sin\psi\right) \\ &+ \frac{\delta c_2}{\omega_0^2 + 4\lambda^2}e^{-\lambda\bar{t}}\left(\cos\psi + \frac{2\lambda}{\omega_0}\sin\psi\right) \end{aligned} \tag{42}$$

Higher order terms require more expense of algebra computation. Here, we restrict the approximate solution within the first-order expansion. Given initial conditions for $u(0)$ and $u'(0)$, constants c_1 and c_2 can be determined rapidly. Eventually we obtain a close form of approximate solution of θ, which is expressed as (note that the solution is so tedious that we don't display the details here)

$$\theta(\bar{t}) = \left(u_0(\bar{t}) + \varepsilon u_1(\bar{t})\right)/\bar{r}(\bar{t}). \tag{43}$$

Up to now, the complete approximate solutions have been achieved with the close form (27) and (43). Compared with other approximations described above, these two analytical solutions not only could predict the stance trajectory of SLIP in both symmetric and asymmetric case, but also have strong tractability due to their straight forward form with elementary functions.

IV. Performance Analysis

A. Performance Criteria

Exact solutions of SLIP model in stance phase are proved to be unreachable as mentioned above. Perturbation techniques direct another way to approach the inaccessible solutions as close as possible. The quality of approximation is crucial to analyzing SLIP dynamics, especially the apex return mapping which plays a key role in controller design of monopod running. In this section, the approximate quality will be evaluated in two ways. Firstly, the percentage error (PE) proposed in [12] is employed to compute the relative errors of apex variables (apex height y_{apex} and horizontal velocity x'_{apex}) which is defined as

$$PE = 100\frac{\left\| X_{true} - X_{approximate} \right\|_2}{\left\| X_{true} \right\|_2}. \tag{44}$$

Secondly, in order to evaluate the performance of approximate solutions in both symmetric and asymmetric motion, A scalar term, sum of angles at touchdown and liftoff state $\theta_{TD}+\theta_{LO}$, is employed to describe on what extend the system displays a symmetric locomotion, noting that zero value means symmetric trajectory, namely, fixed point of apex return map of energy-conservation SLIP model. All the approximations including we proposed in this paper will undergo the simulation test.

Additionally, aiming at evaluating the full state of SLIP locomotion in stance phase, we intend to calculate the stance trajectories of approximations presented above. It should be noted that the Schwind's solution could not reveal the stance trajectory because it is the touchdown, bottom and liftoff state rather than entire stance process that the method focuses on. Among Geyer's, Arslan's and perturbation solution presented, full state evaluation of stance phase could be carried out. It's convenient to figure out that the perturbation and Geyer's solution have the same version of radial motion $\bar{r}(t)$. Consequently, we just calculate the angular velocity $\theta'(t)$ to evaluate the performance of the methods in stance trajectory prediction. The accumulative error we choose as criteria index is given below

$$Err = \sqrt{\frac{1}{(T_{LO}-T_{TD})}\int_{T_{TD}}^{T_{LO}} \left| \theta'_{true}(t) - \theta'_{approximate}(t) \right|^2 dt} \tag{45}$$

B. Simulation Conditions Settings

For comparing between existing approximations and the perturbation solution, the simulations are operated under two dimensions of initial states including the height y_{apex} and horizontal velocity $\dot{x}_{apex}$ at apex. Considering biological data mentioned in the associated literatures [15], a human (with a body mass $m = 80$kg and a leg length $r_0 = 1$m on average) runs within the speed range of 2-6 m/s, of which the relative leg stiffness $\bar{k}$ almost remains constant at 10. Motivated by these experimental data, we choose the dimensionless initial conditions as $\bar{k}$ =10, $y_{apex} \in [0.95,\ 1.05]$, $\dot{x}_{apex} \in [2,\ 6]$. All simulations with 6500 different initial states are run in Matlab environment, which applied a variable time step, fourth order Runge-kutta method to numerical solving SLIP system dynamics. The X_{true} and θ'_{true} are computed through numerical integration.

C. Simulation Results

The simulation results are shown in Fig. 2, in which the mean, maximum and standard deviation value for the apex height and horizontal velocity percentage errors, PE_{yapex} and $PE_{x'apex}$ are presented for existing approximations and perturbation solutions throughout the total valid conditions. Compared with apex index of existing method, our solution performs an significant improvement in the apex variables calculation with maximum percentage errors less than 15% and mean value percentage errors less than 5%. Note that the mean value differences between Arslan's and ours are rather small since both of methods take gravitational effect into considerations.

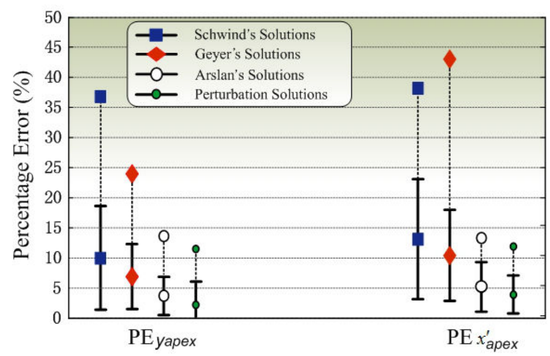
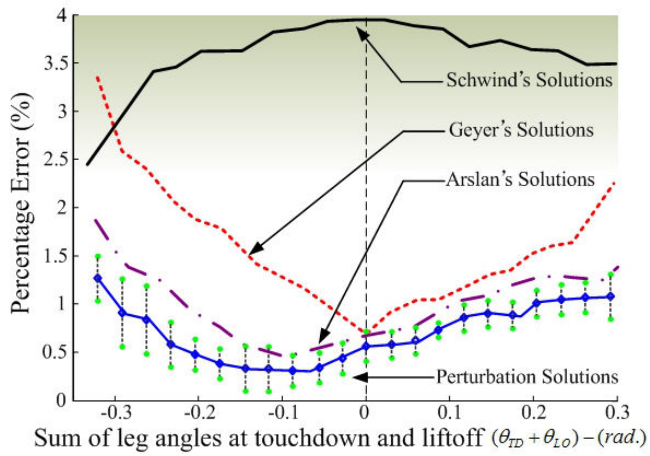


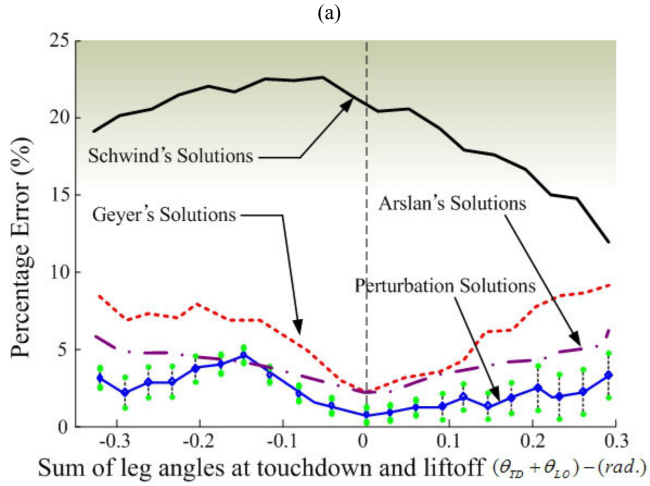
Fig. 2. Performance comparison among existing approximations and perturbation solutions. The evaluation index PE_{yapex} (left) and $PE_{x'apex}$ (right) are the apex height and apex horizontal velocity percentage errors. The upper solid markers, lower solid markers and colored vertical bars are respectively maximum, mean and standard deviations of approximate solutions.

Furthermore, the comparison of different approximations in asymmetric locomotion cases has been achieved as shown in Fig. 3. The perturbation solutions perform better than the other three methods in both symmetric and asymmetric case. Among four approximations, the performance of Arslan's and perturbation solutions outclasses than the other two. This indicates that gravitational forces play key roles in asymmetric stance locomotion and could not be roughly ignored. Schwind's solutions perform relatively smoothly throughout the entire simulations. Geyer's solutions retain high predictive accuracy around symmetric case in which the integration of angular momentum generated by gravity is nearly zero. Therefore Geyer's assumption for angular momentum conservation at touchdown and liftoff state could be achieved roughly. Also note that Geyer's and Arslan's solution perform identically in symmetric case as angular momentum correction term is zero in Arslan's solution. Overall, the perturbation solutions show sufficient accuracy in approximation of apex variables and exhibit validity for non-symmetric motion in a large angle range.

The last experiment tests the trajectory accuracy during stance phase with considering biological factors. The initial state $\bar{r}'_{TD}$ varies within [-2,-0.5], $\bar{\theta}'_{TD}$ within [-0.5,-0.1] and touchdown angle among 65º, 75ºand 85º. The results are depicted in Fig.4. All the three approximations lose accuracy with angular velocity at touchdown and radial spring compression rate (the absolute value) increasing. Note that the three methods are under the identical assumption that relative spring compression rate is sufficiently small. Thus the SLIP model with a "soft" spring might bring intrinsic barrier to approximating. Specifically, under the same touchdown angle, the perturbation solutions perform better than Geyer's and Arslan's with smaller angular velocity at touchdown and radial spring compression rate. Unfortunately, the *Err* of perturbation solutions rapidly arises as the two initial states increasing. Considering the elementary function of perturbation solutions, the existence of exponent terms accelerates the divergence of the solutions while Geyer's and Arslan's solution of angular only contain harmonic and linear terms. Generally speaking, the perturbation solutions present a higher accuracy of stance trajectory prediction in larger regions than Geyer's and Arslan's approximations.



(a)



(b)

Fig. 3. Mean value of apex index percentage errors versus symmetric motion evaluation index $\theta_{TD}+\theta_{LO}$ (a) PE_{yapex} versus $\theta_{TD}+\theta_{LO}$. (b) $PE_{x'apex}$ versus $\theta_{TD}+\theta_{LO}$. The vertical bars represent standard derivations of perturbation solutions presented in this paper.

V. CONCLUSION AND FUTURE WORK

In this paper, we proposed a novel method with perturbation technique to obtain analytic approximate solutions to the SLIP dynamics in stance phase. By means of the regular perturbation technique and taking the effects of gravity into consideration, we derived the straightforward

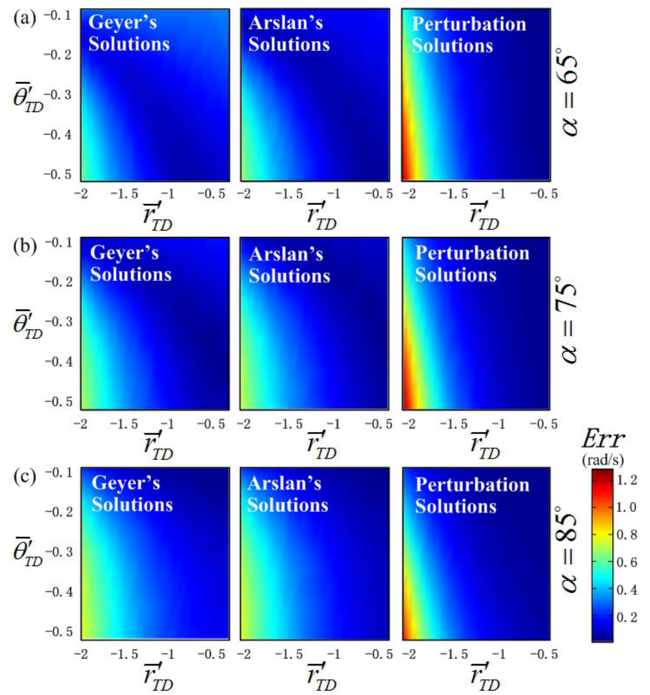


Fig. 4. The trajectory error Err of stance phase among Geyer's, Arslan's and perturbation solutions. Errs are presented through color bar. (a) α_0 =65º.(b) α_0=75º.(c) α_0=85º.

expansion solutions in elementary functions. We also demonstrated that the perturbation solutions reveal high precision in predicting the apex variables (apex height and horizontal velocity) by comparing percentage errors with existing typical analytical approximations. Additionally, our solutions were corroborated to have a better performance in both symmetric and asymmetric motion within a comparatively larger angle range than existing models. Furthermore, the evaluation for stance trajectory prediction had been carried out. The proposed method in this paper performed better in relatively wider regions than existing methods.

The future work will aim towards two extensions. Foremost, we intend to establish apex return mapping based on perturbation solutions, based on which the existence and stability of fixed point on Poincare Map should be studied so as to restrict initial conditions and parameters. The perturbation solution will extend to SLIP model with damping. Subsequently we focus on the design of running controller which provides the robot robust behavior when negotiating rough terrain.

REFERENCES

[1] M. H. Raibert, *Legged Robots that Balance*. MIT Press, Cambridge, MA, 1986.

[2] P. Gregorio, M. Ahmadi, M. Buehler, "Design, Control, and Energetics of an Electrically Actuated Legged Robot," *IEEE Trans. Systems, Man and Cybernetics*, vol. 27, pp. 626-634, August, 1997.

[3] R. Altendorfer, R. W. Longman, M. Buehler, "Rhex: A biologically inspired hexapod runner," Autonomous Robots, vol.11, pp. 207-213, 2001.

[4] R. M. Alexander, "Three uses for springs in legged locomotion," Int. J. Robotics Res, vol.9, pp. 53-61, 1990.

[5] R. Blickhan, "The spring-mass model for running and hopping," J. Biomechanics, vol. 185, pp. 27-40, 1989.

[6] R. Alexander, "Optimization and gaits in the locomotion of vertebrates," Physiol. Rev, vol. 69, pp. 1199-1227, 1989.

[7] R. M. Ghigliazza, R. Altendorfer, P Holmes, "A simply stabilized running model," SIAM J. Applied Dynamical Systems, vol. 2, pp, 187-218, 2003.

[8] A. Seyfarth, H. Geyer, M. Gunther, "A movement criterion for running," J. Biomechanics, vol. 35, pp. 649-655.

[9] P. Holmes, "Poincaŕe, celestial mechanics, dynamical-systems theory and 'chaos'," Physics Reports, vol. 193, pp. 137-163, Sept. 1990.

[10] H. Geyer, A. Seyfarth, R. Blickhan, "Spring-mass running: simple approximate solution and application to gait stability," J. Theoretical Biology, vol. 232, pp. 315-328, Sept. 2005.

[11] R.T. M'Closkey, J. W. Burdick, " Periodic Motions of a Hopping Robot With Vertical and Forward Motion," Int. J. Robotics Res, vol. 12, pp. 197-218, June, 1993.

[12] W.J. Schwind, D.E. Koditschek, "Approximating the Stance Map of a 2-DOF Monoped Runner," J. Nonlinear Sci., Vol. 10, pp. 533-568, 2000.

[13] Ömür Arslan, Uluc Saranli, Ömer Morgül, "An Approximate Stance Map of The Spring Mass Hopper with Gravity Correction For Nonsymmetric Locomotion," In Proc. Of the Int. Conf. On Robotics and Automation, pages 2388-2393, Kobe, Japan, May,2009

[14] A.H. Nayfeh, Introduction to perturbation techniques.CA: John Wiley & Sons, 1981.

[15] C.T. Farley, O. Gonzalez, "Leg stiffness and stride frequency in human running," J. Biomechanics, Vol. 29, pp. 181-186, 1996.