# Differential Geometry & Curved Spaces

Book: *Einstein's Gravity in a Nutshell*

## First Fundamental Form

$ds^2 = g_{ij}\, dx^i dx^j = \bar{g}_{kl}\, d\bar{x}^i d\bar{x}^j$

$ds^2 = \bar{g}_{kl}\, d\bar{x}^i d\bar{x}^j$

Notes:

$g_{ij} \rightarrow$ metric tensor in 3D or 4D

$a \rightarrow$ metric tensor in 2D

$g_{\lambda\mu} \rightleftarrows$ 4D, 2D

$g_{ij} \rightarrow$ 2D (latin)

ex. plate : $a_{\alpha\beta}$

blob of jello: $g_{ij}$

from ordinary calculus:

differentials

$$\underbrace{d\bar{x}^3}_{dz} = \frac{\partial \bar{x}^3}{\partial x^1}\underbrace{dx^1}_{\text{e.g. } dr} + \frac{\partial \bar{x}^3}{\partial x^2}\underbrace{dx^2}_{d\theta} + \frac{\partial \bar{x}^3}{\partial x^3}\underbrace{dx^3}_{d\psi} + \dots + \frac{\partial \bar{x}^3}{\partial x^D} dx^D = \sum_{\nu=1}^{D} \frac{\partial \bar{x}^3}{\partial x^\nu} dx^\nu$$

$d\bar{x}^2 = \dots$

$d\bar{x}^1 = \dots$

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$$\boxed{d\bar{x}^i = \frac{\partial \bar{x}^i}{\partial x^j} dx^j} \longleftrightarrow dx^i = \frac{dx^i}{d\bar{x}^j} d\bar{x}^j$$

$S^i{}_j$ (rows: $i$, columns: $j$) if linear, reduces to something we've seen before $\longrightarrow \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ in 2D

$\boxed{\frac{d}{dx^j} = \partial_j} \longrightarrow$ "gradient is covariant"

contravariant metric tensor $\longrightarrow$ covariant metric tensor

$g_{ij} \longrightarrow g^{ij} = (g_{ij})^{-1}$

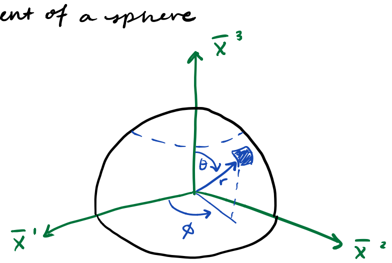
## changing coordinate system

$ds^2 = g_{ij}\, dx^i dx^j$

$\therefore\ \bar{g}_{kl}\, d\bar{x}^k d\bar{x}^l = g_{ij} \frac{dx^i}{d\bar{x}^k} \frac{dx^i}{d\bar{x}^l} d\bar{x}^k d\bar{x}^l$

$\bar{g}_{kl} = g_{ij} \frac{dx^i}{d\bar{x}^k} \frac{dx^i}{d\bar{x}^l}$

Ex. segment of a sphere




$$\bar{r} = \begin{bmatrix} r\sin\theta\cos\phi \\ r\sin\theta\sin\phi \\ r\cos\theta \end{bmatrix}$$

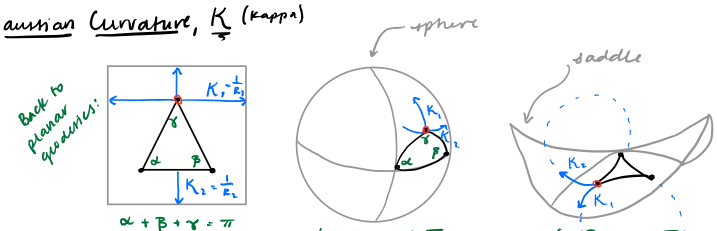
$$\frac{\partial \bar{r}}{\partial r} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} \quad \frac{\partial \bar{r}}{\partial \theta} = \begin{bmatrix} r\cos\theta\cos\phi \\ r\cos\theta\sin\phi \\ -r\sin\theta \end{bmatrix} \quad \frac{\partial \bar{r}}{\partial \phi} = \begin{bmatrix} -r\cos\theta\sin\phi \\ r\sin\theta\cos\phi \\ 0 \end{bmatrix}$$

## THE METRIC TENSOR

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

$\frac{\partial \bar{r}}{\partial r} \cdot \frac{\partial \bar{r}}{\partial r} = 1 \qquad \frac{\partial \bar{r}}{\partial \theta} \cdot \frac{\partial \bar{r}}{\partial \theta} = r^2 \qquad \frac{\partial \bar{r}}{\partial \phi} \cdot \frac{\partial \bar{r}}{\partial \phi} = r^2\sin^2\theta$

## Gaussian Curvature, $\underset{\kappa}{K}$ (kappa)




$\alpha + \beta + \gamma = \pi$ $\quad$ $\alpha + \beta + \gamma > \pi$ $\quad$ $\alpha + \beta + \gamma < \pi$

$K = K_1 K_2 = 0$ $\quad$ $K = K_1 K_2 > 0$ $\quad$ $K = K_1 K_2 < 0$

## Gauss's Theorem Egregium

$K$ is invariant under local isometry ... gaussian curvature will not change if the metric does not change

deformation that does not ~~stretch~~ a surface

... impossible to wrap a soccer ball without crumpling the wrapping paper... the lines that result are where the stretch are

... if you change gaussian curvature, you are always stretching

Let's say we live on surface of a sphere:

$$g_{ij} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{bmatrix}}$$

$$ds^2 = g_{ij}\,dx^i x^j = \cancel{dr^2} + r^2 d\theta + r^2 \sin^2\theta\, d\phi^2$$

$dr^2 \to 0$: let's say we don't care about changes in $r$

$r \to 1$: let's pretend $r=1$

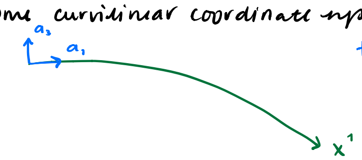
$$\Rightarrow ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2 \quad \text{(on surface)}$$

$$a_{\alpha\beta} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$$

$$\gamma_{\alpha\beta} = \frac{1}{2}\left(\overline{a}_{\alpha\beta} - a_{\alpha\beta}\right)$$

$\gamma_{\alpha\beta}$: green-st. venant strain tensor; $\overline{a}_{\alpha\beta}$: deformed; $a_{\alpha\beta}$: original

have some curvilinear coordinate system:




the normal (take cross product): $a_1 \times a_2 = \dfrac{\partial r}{\partial x^1} \times \dfrac{\partial r}{\partial x^2}$

take unit normal: $a_3 = \dfrac{a_1 \times a_2}{|a_1 \times a_2|}$

change in unit normal, $a_3$

$$\partial_\alpha a_3 = \frac{\partial a_3}{\partial x^\alpha}$$

projection onto $a_1$ and $a_2$

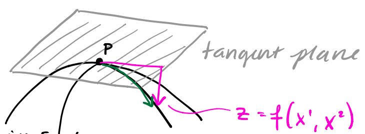
$$\partial_\alpha a_3\, a^\beta = -b_\alpha^\beta$$

* $b$ is a matrix (tensor, 2x2), $2^{nd}$ order *

$\hookrightarrow$ Second Fundamental Form

$$\mathrm{II}: \frac{1}{2} b_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$b_{\alpha\beta} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$$




Taylor Series Exp (but ignoring first order terms):

$$f(x^1,x^2) \approx \underbrace{\tfrac{1}{2}(x^1)^2 \frac{\partial^2 f(0,0)}{\partial (x^1)^2}}_{L} + \underbrace{x^1x^2 \frac{\partial f(0,0)}{\partial x^1}\frac{\partial f(0,0)}{\partial x^2}}_{M} + \underbrace{\tfrac{1}{2}(x^2)^2 \frac{\partial^2 f(0,0)}{\partial (x^2)^2}}_{N} + \dots$$

Hessian Matrix, $b_{\alpha\beta} = \begin{bmatrix} L & M \\ M & N \end{bmatrix}$

*principal curvatures are eigenvalues of matrix... two x two, two eigenvalues*

$\mathrm{II}: \; L\,dx^{1^2} + 2M\,dx^1dx^2 + N\,dx^{2^2}$

$f(0,0)$ ; $\frac{\partial f(0,0)}{\partial x^1}$ ; $\frac{\partial f(0,0)}{\partial x^2}$

$$\boxed{\begin{aligned} H &= \tfrac{1}{2} a^{\alpha\beta} b_{\alpha\beta} = \tfrac{1}{2} b^{\beta}_{\alpha} \\ K &= \frac{\det(b)}{\det(a)} = b_1^1 b_2^2 - b_2^1 b_1^2 \end{aligned}}$$

eigenvalue 1:

mean curvature (extrinsic) $H = \dots$ trace of matrix in curved space $\tfrac{1}{2} a^{\alpha\beta} b_{\alpha\beta} = \tfrac{1}{2} b^{\alpha}_{\alpha}$

eigenvalue 2:

gaussian curvature (intrinsic) $K = \dots$ determinant $\frac{\det(b)}{\det(a)} = b_1^1 b_2^2 - b_2^1 b_1^2$

## Calculus on Curved Manifolds

- gradients ⟶ increases your tensor order by 1
  - scalar ⟶ vector
  - vector ⟶ tensor
- Divergence ⟶ decreases tensor order by 1
  - vector ⟶ scalar
  - tensor ⟶ vector
- Laplacian ⟶ divergence of gradient of object ⟶ maintains tensor order!
  - vector ⟶ vector
  - tensor ⟶ tensor

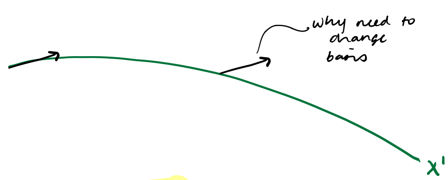
$\nabla_\mu \nabla^\mu V^\lambda = b^\lambda$ vector

$\nabla_\mu \nabla^\mu \phi = \phi$ scalar

since both depend on x, take product rule

The covariant derivative: $\nabla_\mu \vec{V} \;\leadsto\; \partial_\mu \vec{V} = \partial_\mu (V^\lambda_{(x)} e_\lambda^{(x)}) = \partial_\mu V^\lambda e_\lambda + V^\lambda \partial_\mu e_\lambda$

↳ ordinary derivative + change in reference frame
(how much vector changing) + (how much basis vector changing)




$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\nu$

↳ derivatives of the metric

$\Gamma^\rho_{\mu\nu} = \tfrac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$