Stability + Bifurcation

Bifurcations — qualitative changes in behavior as a parameter is changed
- fixed points: created/destroyed
- fixed points: changing stability

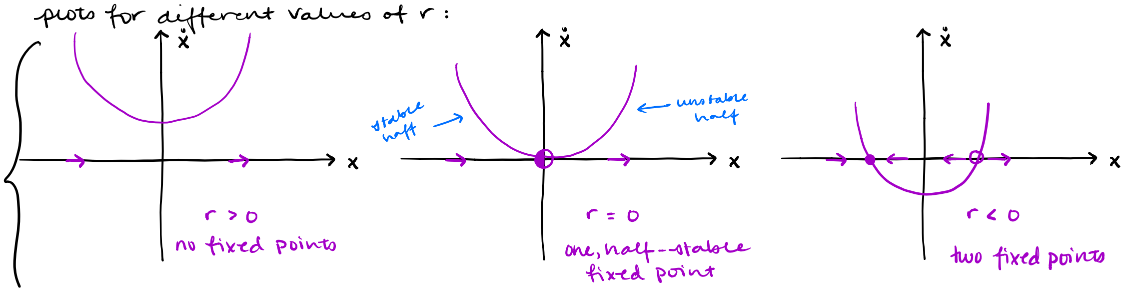
Three Canonical Bifurcations

① Saddle-Node Bifurcation (aka "Blue sky Bifurcation")
↳ creating/destroying fixed points

$f(x) = \dot{x} = r \pm x^2$

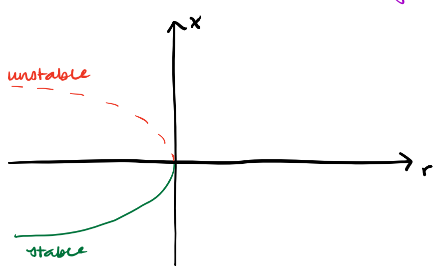
↳ r is the parameter whose value we change (can be + or −)

plots for different values of r:



$f(x) = r + x^2$

* these are qualitatively different vector fields *



also applies to $f(x) = r + x^2$

* simplest bifurcation we'll find *

ex. $\boxed{\dot{x} = r - x - e^{-x}}$

↳ Taylor Series Expansion about $x = 0$

approximate this about $x = 0$:

$\dot{x} = r - x - [1 - x + \frac{x^2}{2!} + \dots]$

→ simplify ⇒

$\dot{x} = (r-1) - \frac{x^2}{2} + \dots$

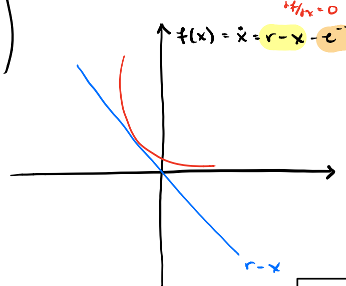
this has the same algebraic form as $r + x^2$

the normal form:

$\dot{x} = f(x, r)$ bifurcation: $x = x^*$ & $r = r_c$

$$\dot{x} = \underbrace{\underset{0}{f(x^*, r_c)} + (x - x^*)\frac{\partial f}{\partial x}\Big|_{(x^*, r_c)}}_{\text{fixed points if curves intersect tangentially } \partial f/\partial x = 0} + \underbrace{(r - r_c)\frac{\partial f}{\partial r}\Big|_{(x^*, r_c)}}_{x_0} + \underbrace{\frac{1}{2}(x - x^*)^2 \frac{\partial^2 f}{\partial x^2}\Big|_{(x^*, r_c)}}_{x_1} + \dots$$

like from perturbation methods



bifurcation will occur when these two curves touch each other

$\boxed{\dot{x} = x_0 (r - r_0) + x_1 (x - x^*)^2}$

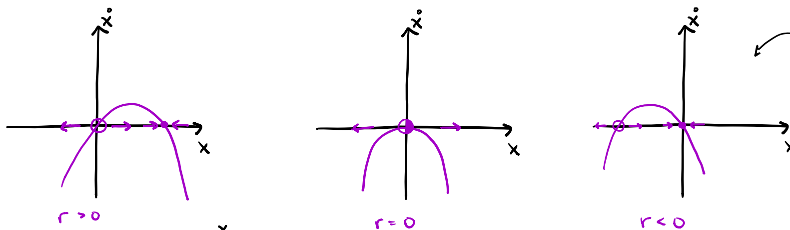
Takeaway: if some function has parabola anywhere then there's emergence of fixed points and system will change behavior when parameter adjusted (see video)

② Transcritical Bifurcation

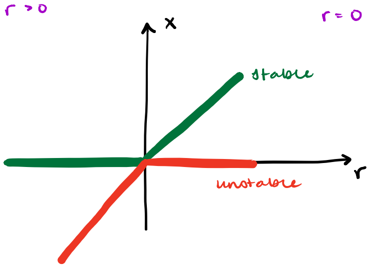
- fixed point <u>always</u> exists for all values of a parameter $\longrightarrow$ can never be destroyed
  - ↳ <u>changes stability</u>

$f(x) = \dot{x} = rx - x^2$ e.g. logistic equation

$\dot{N} = rN\left(1 - \frac{N}{K}\right) \longmapsto \dot{x} = rx\left(1 - \frac{x}{K}\right) = rx - r\frac{x^2}{K}$



we see the point at the origin going from unstable to stable



* solid state laser, ex. of transcritical bifurcation

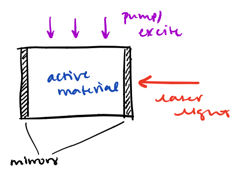
e.g. the model for a solid-state laser

# of photons in laser field $n(t)$

$\boxed{\dot{n} = \text{gain} - \text{loss} = GnN - kn}$ $k = \frac{1}{\tau}$

gain coefficient (material property)

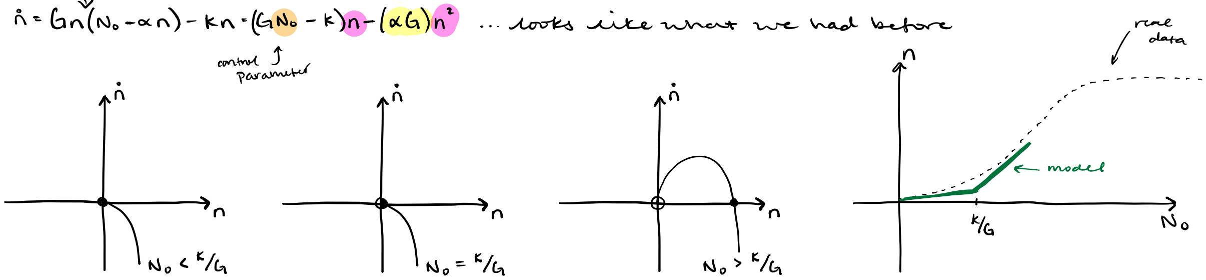
excited atoms



$N(t) = N_0 - \alpha n$

$\dot{n} = Gn(N_0 - \alpha n) - kn = (GN_0 - k)n - (\alpha G)n^2$ … looks like what we had before

control parameter



③ Pitchfork Bifurcation

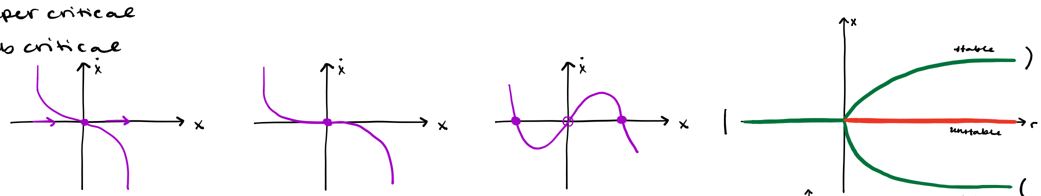
↳ fixed points appear/disappear in pairs

ⓘ super critical

ⓘⓘ sub critical

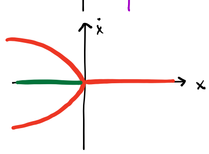
<u>supercritical</u>

$\dot{x} = rx - x^3$



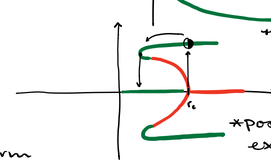
<u>subcritical</u>

$\dot{x} = rx + x^3$



$\rightsquigarrow \dot{x} = rx + x^3 - x^5$

↑ stabilized by higher order term



*post it note example