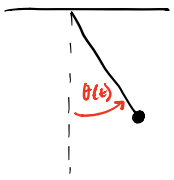
## Multiple Scales : Two-Timing

Pendulum:




$\theta'' + \sin\theta = 0$
$\theta'(0) = 0$
$\theta(0) = \epsilon$

$\sin\theta \sim \theta - \frac{1}{6}\theta$ [fill this in]

we know that in the absence of damping, this will forever swing back + forth (not realistic, but go with it)

... expect something sinusoidal

$\theta \sim \theta_0 + \epsilon^\alpha \theta_1 + \dots$

$\theta_0 = 0$ & $\alpha = 1$

$\theta \sim \epsilon(\theta_0(t) + \epsilon^\alpha \theta_1(t) + \dots)$ ← this rescaling is a little silly because $\theta_0(0) = 0$ but go with it...

$\theta(\epsilon)$: $\theta_0'' + \theta_0 = 0$, $\theta_0(0) = 1$, $\theta_0'(0) = 0$ $\Rightarrow \theta_0 \sim \cos(t)$ $\longrightarrow$ $\theta \sim \epsilon\cos t$ ... worked but didn't capture the whole picture

↙ higher order now need to try

$\theta(\epsilon^3)$: $\theta_1'' + \theta_1 = \frac{1}{6}\theta_0^3$, $\theta_1(0) = 0$, $\theta_1'(0) = 0$ $\Rightarrow \theta_1 = \frac{1}{16}t\sin t + \frac{1}{192}\cos t - \frac{1}{192}\cos(3t)$

↳ from this, captured the period a little better ... but the amplitude seems to increase still

↳ we get a "secular term" → we want to eliminate this because this is what's causing the amplitude to increase over time

Introduce two time scales

$t_1 = t$
$t_2 = \epsilon^2 t$

① change of variable
② elimination of secular term(s)

$$\theta \sim \epsilon\cos\left(t - \epsilon^2 t/16\right)$$