Dimensional Analysis

[L] length　　[J] luminosity (candela)　　[N] mole
[M] mass　　[θ] temperature
[T] time　　[I] current

we'll mostly deal w/ these

→ radiation can be found with its radioactivity ($\sim 1/T$?)

Height of a projectile - example

X_m = maximum ht projectile goes in air
$X_m = f(g, m, v_0)$

our goal is to develop a testable hypothesis

$$X_m = f(g, m, v_0) \longrightarrow [X_m] = [m^a v_0^b g^c]$$

$$T^0 M^0 L = m^a \left(\frac{L}{T}\right)^b \left(\frac{L}{T^2}\right)^c = M^a L^{b+c} T^{-b-2c}$$

$$\left.\begin{aligned} L&: b + c = 1 \\ T&: -b - 2c = 0 \\ M&: a = 0 \end{aligned}\right\} \begin{aligned} c &= -1 \\ b &= 2 \\ a &= 0 \end{aligned} \longrightarrow \boxed{X_m \sim \frac{V_0^2}{g}}$$

single tilde means we're leaving some constant intentionally out

we came up with a prediction w/out using calculus, etc.
...now can find unknown constant

$$\boxed{X_m \sim \frac{V_0^2}{g}} \longrightarrow X_m = \alpha \frac{V_0^2}{g}$$

$$\underbrace{\ddot{x}(t)}_{\frac{d^2x}{dt^2}} = -\frac{gR^2}{(R+x)^2}$$

R = radius of Earth

$$\underbrace{x(0) = 0 \quad \dot{x}(0) = V_0}_{\text{initial conditions}}$$

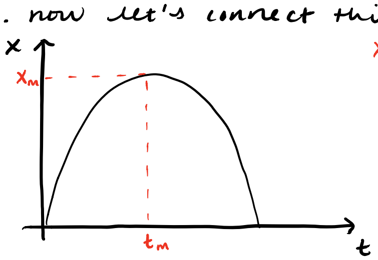
this is the annoying term ...
$R + x \approx R$ (assuming x is really small compared to R)
i.e. $\frac{x}{R} \ll 1$

$\ddot{x}(t) = -g$ ... then apply two initial conditions and integrate

↳ $\boxed{x(t) = -\frac{1}{2} g t^2 + V_0 t}$

... now let's connect this with our dimensional analysis

X_m = max ht　　t_m = max ht time



$$\dot{x} = 0 \rightarrow t = \frac{V_0}{g} \therefore \boxed{X_m = \frac{V_0^2}{2g}} \therefore \alpha = \frac{1}{2}$$

... we are assuming our gravitational field is constant for all x, which really isn't true

Drag on a sphere - example

... we usually want the drag to be as low as possible

Drag will depend on:

- the radius (R)
- the velocity (V)
- density (ρ)
- viscosity (μ)

$$D_f = f(R, V, \rho, \mu) \longrightarrow [D_f] = [R^a V^b \rho^c \mu^d]$$

$$MLT^{-2} = L^a \left(\frac{L}{T}\right)^b \left(\frac{M}{L^3}\right)^c \left(\frac{M}{LT}\right)^d$$

$$MLT^{-2} = L^{a+b-3c-d}\, T^{-b-d}\, M^{c+d}$$

$$\left.\begin{aligned} L&: a+b-3c-d = 1 \\ T&: -b-d = -2 \\ M&: c+d = 1 \end{aligned}\right\}$$

take d to be unknown ...

$b = 2-d$

$c = 1-d$

$a = 2-d$

$$\longrightarrow D_f \sim R^{2-d} V^{2-d} \rho^{1-d} \mu^d$$

$$D_f \sim R^{2-d} V^{2-d} \rho^{1-d} \mu^d \sim R^2 V^2 \rho \underbrace{\left(\frac{\mu}{RV\rho}\right)^d}_{\pi^d}$$

reciprocal of Re #

$Re = \dfrac{RV\rho}{\mu}$

π^d dimensionless product/group

numerical prefactor

$$D_f = \alpha R^2 V^2 \rho\, \pi^d$$

← these are both dimensionless

$$D_f = \alpha_1 R^2 V^2 \rho\, \pi^{d_1} + \alpha_2 R^2 V^2 \rho\, \pi^{d_2}$$

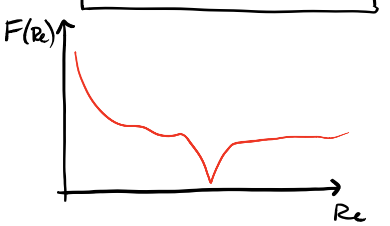
... this is just as valid as above

$$D_f = \rho R^2 V^2 \underbrace{(\alpha_1 \pi^{d_1} + \alpha_2 \pi^{d_2} + \ldots)}_{F(\pi)}$$

← we measure this experimentally

... more complete version

$$\boxed{D_f = \rho R^2 V^2 F(\pi)}$$



We can use this approach to develop scale models

- we can run an experiment at a small scale to tell us how it will be in a large scale

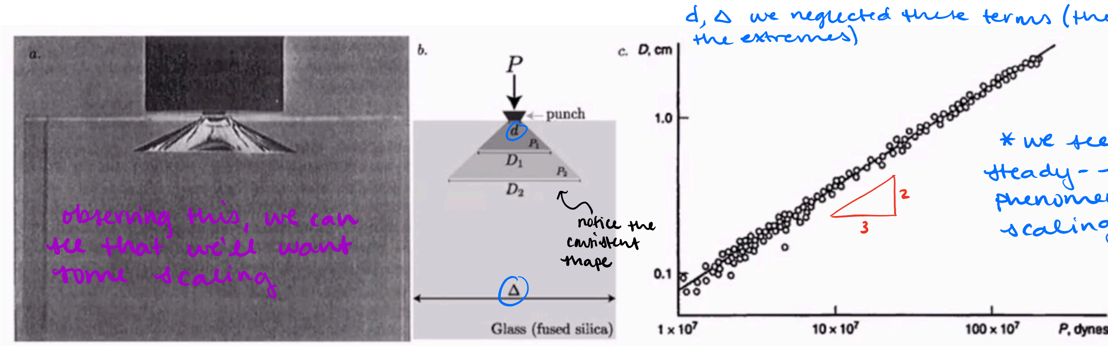
model:

as long as keep dimensionless part the same, can vary other items

$$\frac{\mu_m}{R_m V_m \rho_m} = \frac{\mu}{RV\rho} \longrightarrow V_m = \frac{\mu_m Re}{\mu R_m \rho_m} V$$

... this is implying physical similarity example problem in brittle fracture

d, Δ we neglected these terms (these are the extremes)



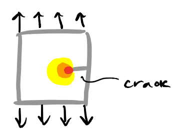
* we see this steady-state phenomenon in scaling

→ brittle fracture falls under elasticity theory
- something that breaks that can be put back together (so no plasticity)
- problem in fracture problems is that there's more boundaries than we were thinking about
  - → Strain Energy Release Rate
    - ↳ which is related to the "<u>stress intensity factor</u>" (K)

stress intensity factor : $K \sim \sigma \sqrt{s}$

ave. stress

distance from crack



how does this crack grow as increase stress?

$D = f(P, K, \nu)$

force

poisson's ratio (dimensionless)

$[P] = F = MLT^{-2}$

$[K] = \frac{F}{A}\sqrt{L} = MLT^{-2}L^{-2}L^{1/2} = \frac{MLT^{-2}}{L^{3/2}}$

unknown function of poisson's ratio

$$\boxed{D \sim \left(\frac{P}{K}\right)^{2/3}\phi(\nu)}$$

this length sets the limit of when something too small, too large

$D \sim \left(\frac{P}{K}\right)^{2/3}\phi(\nu)$

<u>valid</u>: when $\frac{d}{(P/K)^{2/3}} \ll 1$ AND $\frac{\Delta}{(P/K)^{2/3}} \gg 1$

... scaling only valid in this intermediate s-s regime

"<u>intermediate asymptotics</u>"

example of boundary layer in mechanics:

cut tennis ball in 1/2, flip dome inside-out, boundary deviates from dome at the brim

... vs in fluid mechanics where boundary layer is typically at surface from some flow

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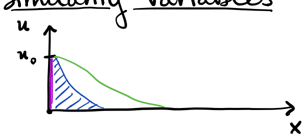
<u>Similarity Variables</u>



u = concentration

t = 0

t = t_1

t = t_2

Diffusion Equation: $D\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

$u|_{x=0} = u_0$

$u|_{x=\infty} = 0$

boundary conditions (b.c.)

$u|_{t=0} = 0$

initial condition (i.c.)

Dimensional Analysis: Derivatives / Integrals

area function of x : $A(x) = x^2 \rightarrow [A] = L^2$

$$\frac{dA}{dx} = 2x \rightarrow \left[\frac{dA}{dx}\right] = \frac{1}{L}[A] = L$$

$$\frac{d^2A}{dx^2} = 2 \rightarrow \left[\frac{d^2A}{dx^2}\right] = \frac{1}{L^2}[A] = 1 \quad \text{(dimentionless!)}$$

Integrals ↑ ↓ Derivatives

... back to diffusion equation example

$$D\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad \underbrace{u|_{x=0} = u_0 \quad u|_{x=\infty} = 0}_{B.C.} \qquad \underbrace{u|_{t=0} = 0}_{I.C.}$$

$[D\,u_{,xx}] = [u_{,t}]$

$[u] = ML^{-3}$ $\therefore [D] = L^2T^{-1}$

↑ chemical concentration has dimensions of mass per volume

• two independent variables, x & t
• D is diffusion parameter

we want to find $u = f(x, t, D, u_0)$

$$[u] = [x^a t^b D^c u_0^d] \rightarrow ML^{-3} = L^a T^b (L^2T^{-1})^c (ML^{-3})^d$$
$$ML^{-3} = L^{a+2c-3d} T^{b-c} M^d$$

$$\left\{\begin{array}{ll} L: & a + 2c - 3d = -3 \\ T: & b - c = 0 \\ D: & d = 1 \end{array}\right. \quad \therefore \begin{array}{l} d = 1 \\ b = c = -\frac{a}{2} \end{array}$$

now collect all of these terms:

$$\boxed{u \sim u_0 \left(\underbrace{\frac{x}{\sqrt{Dt}}}\right)^a}$$

$\eta = \frac{x}{\sqrt{Dt}}$ similarity variable ... these occur when have dimensionless group that includes the independent variables

... important to check that spatial parameter is in numerator

physically, what does this mean?

- we're going to have self-similar behavior (on curve)
- allows us to write in terms of η (eta), reducing from PDE to ODE so we've significantly reduced the complexity of our problem

$u \sim u_0 F(\eta)$ $\quad [u] = ML^{-3}$ $\quad \therefore D = L^3T^{-1}$

chain Rule:

$$\frac{\partial}{\partial t}[F(\eta)] = \frac{dF(\eta)}{\partial\eta}\frac{\partial\eta}{\partial t} \Rightarrow D\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$\frac{\partial u}{\partial t} \rightarrow u_0 F'(\eta)\frac{\partial\eta}{\partial t}$

$\hookrightarrow \frac{\partial\left(\frac{x}{\sqrt{Dt}}\right)}{\partial t} = \frac{-x}{2D^{1/2}t^{3/2}}$

$\rightarrow -u_0 F'(\eta)\frac{\eta}{2t}$

*point of this: reduce complicated PDE to simple ODE using a similarity transform

$$\frac{\partial^2 u}{\partial x^2} = u_0 F''(\eta)\frac{1}{Dt}$$

... now stitch this all out together...

$$\cancel{D}\left[\cancel{u_0} F''(\eta)\frac{1}{\cancel{Dt}}\right] = -\cancel{u_0} F'(\eta)\frac{\eta}{2\cancel{t}} \rightarrow \boxed{F''(\eta) = -\frac{1}{2}\eta F'(\eta)}$$

$$\boxed{F(0) = 1 \qquad F(\infty) = 0}$$

↑ normalizing our concentration by the initial conc., why $u_0 = 1$

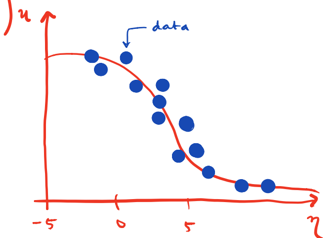
the way to solve this problem is by making a variable substitution: let $G = F'$ $\therefore G' = -\frac{1}{2}\eta G$

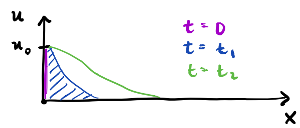
$\rightarrow G = \alpha \exp[-\eta^2/4]$

$$\therefore F(\eta) = \beta + \alpha\int_0^\eta \exp[-s^2/4]\,ds \rightarrow F(0) = 1 \therefore \beta = 1, \quad F(\infty) = 0 \therefore 1 + \alpha\underbrace{\int_0^\infty \exp\left[-\frac{s^2}{4}\right]ds}_{\sqrt{\pi}} = 0$$

$$F(\eta) = \underbrace{1 - \underbrace{\frac{1}{\sqrt{\pi}}\int_0^{\eta} \exp\left[-s^2/4\right] ds}_{\text{error function: } \mathrm{erf}(\eta)}}_{1-\mathrm{erf}(\eta) = \mathrm{erfc}(\eta) \,\ldots\, \text{complementary error function}}$$

$$\boxed{u(x,t) = u_0\, \mathrm{erfc}\left(\frac{x}{2\sqrt{Dt}}\right)}$$





When might we expect to see a similarity variable?
- when the problem we're solving lacks a characteristic space or time variable
  ex. hanging sheet of paper off of a table
  (... more hangs off, starts to bend while if hanging off only a little then can see finite length)

Buckingham π – Theorem

$$a = f(\underbrace{a_1, \ldots, a_k}, \underbrace{b_1, \ldots, b_m})$$

a: dimensional parameter we're interested in

$a_1, \ldots, a_k$: parameters w/ independent dimensions

$b_1, \ldots, b_m$: parameters that can be expressed as products of powers of $a_1, \ldots, a_k$
i.e. $b_1 = a_1^{p_1} \cdots a_k^{r_1}$

for any given problem, sometimes $m = 0$
↳ meaning that all the governing parameters have independent dimensions
sometimes k can be 0 as well
↳ meaning that all quantities/governing parameters are dimensionless
... in general, neither of these are true and $k > 0$ & $m > 0$

construct dimensionless parameters:

$$\Pi = \frac{a}{a_1 \cdots a_k} \qquad \Pi_1 = \frac{b_1}{a_1^{p_1} \cdots a_k^{r_1}} \qquad \Pi_2 = \frac{b_2}{a_1^{p_2} \cdots a_k^{r_2}} \qquad \Pi_m = \frac{b_m}{a_1^{p_m} \cdots a_k^{r_m}}$$

For instance,
Recall: Drag on a Sphere

$$\underset{a}{D_f} = \underset{a_1}{\rho}\, \underset{a_2}{R^2}\, \underset{a_3}{V^2}\, F\left(\frac{\overset{b_1}{\mu}}{\underset{a_1'}{\rho}\, \underset{a_2'}{R}\, \underset{a_3'}{V}}\right)$$

a_3' (meaning a_3 to the power of 1)

⬇

$$\underbrace{\frac{D_f}{\rho R^2 V^2}}_{\Pi} = \underset{\phi}{F}\left(\underbrace{\frac{\mu}{\rho R V}}_{(\Pi_1)}\right)$$

in general, we should be able to construct ... $\Pi = \phi(\Pi_1, \Pi_2, \ldots, \Pi_m)$

Buckingham Π–Theorem: The physical relationship between a dimensional quantity and several dimensional parameters can be rewritten as a relationship between a dimensionless parameter and several dimensionless products of the governing parameters. The number of dimensionless products is equal to the number of governing parameters minus the number of independent dimensions.

When are dimensionless products essential?
→ not too small, and not too large

Essential Dimensionless Products

e.g. $\frac{1}{10} < \underbrace{\Pi_m}_{\text{essential}} < 10$ ... might get parameter or one over parameter

→ so in general, if dimensionless parameter Π_m is small or large, can assume it to be neglected

e.g. brittle fracture problem

$$D = f(P, K, \nu, d, \Delta)$$

↑↑ if we put these in, would have gotten slightly different answer

$$D \sim \underbrace{\left(\frac{P}{K}\right)^{2/3}}_{\text{length}} \phi\left(\underset{\uparrow \Pi_1}{\nu}, \underset{\uparrow \Pi_2}{\frac{d}{(P/K)^{2/3}}}, \underset{\uparrow \Pi_3}{\frac{\Delta}{(P/K)^{2/3}}}\right)$$

we made an assumption:
$\Pi_2 \ll 1$ (very small) and $\Pi_3 \gg 1$ (very large)

$$D \sim \underbrace{\left(\frac{P}{K}\right)^{2/3}}_{\text{length}} \phi\left(\underset{\uparrow \Pi_1}{\nu}, \underset{\uparrow \Pi_2}{\cancel{\frac{d}{(P/K)^{2/3}}}}, \underset{\uparrow \Pi_3}{\cancel{\frac{\Delta}{(P/K)^{2/3}}}}\right)$$

neglect neglect

we therefore decided to neglect these

more formally:
Π_m that is <u>small</u> or <u>large</u> can be neglected if $\phi(\ldots)$ has a finite, non-zero limit as $\Pi_m \to 0$ or $\Pi_m \to \infty$

this is known as "<u>complete similarity</u>" – ability to throw out parameters that were safe to neglect ... also written as "si<u>milarity</u> of the <u>first</u> <u>kind</u>"

Why would we want to neglect things?
→ need to do an order of magnitude more of experiments which could be costly, time-killing

<u>Incomplete Similarity</u> -or- <u>Similarity of the Second Kind</u>

- if $\Pi_m \to 0$ or $\Pi_m \to \infty$ the function $\phi(\ldots)$ does <u>not</u> have a finite and non-zero limit
  - b_m can remain essential no matter how <u>small</u> or <u>large</u> its corresponding dimensionless parameter becomes
- if $\phi(\ldots)$ has a power-like asymptotic representation then you can rewrite it as

$$\phi = \Pi_m^{\alpha}\, \phi_1(\Pi_1, \ldots, \Pi_{m-1}) + \underbrace{O(\Pi_m^{\alpha})}_{\text{small compared with first term}}$$

capital 'O' meaning higher order?

allows us to write:

$$\Pi^* = \phi_1(\Pi_1, \ldots, \Pi_{m-1})$$

$$\uparrow$$

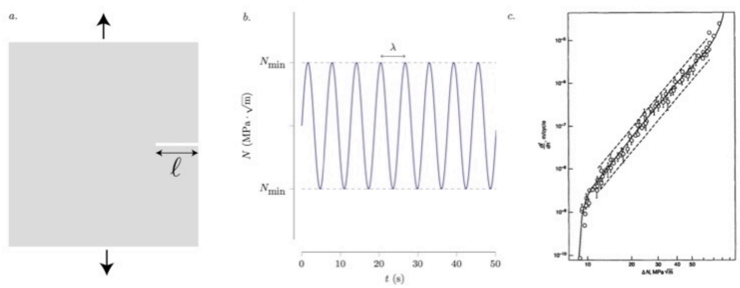
$$\frac{\Pi}{\Pi_m^{\alpha}} = \frac{a}{a_1^{p-\alpha p_m} \cdots a_k^{r-\alpha r_m} b_m}$$

↑ this quantity that want to neglect still appears

may depend on the dimensionless products in the problem

α and Π^* cannot be determined by dimensional analysis alone

Fatigue: the gradual failure of structures from cyclical loading
– an example of Incomplete Similarity



$$\frac{dl}{dn} = A(\Delta N)^m$$

material properties? geometry or specimen size?

$$\frac{dl}{dn} = f\Big(\Delta N, \underbrace{\frac{N_{max}}{N_{min}}}_{R}, f, t, h, \sigma_y, K_{Ic}\Big)$$

thickness (h); stress intensity factor (K_{Ic}); frequency (f); t_{time}; yield stress (σ_y)

ratio of stress int. factors (R)

frequency · time = dimensionless

fatigue like so propagation of a crack

dimensional analysis:

$$\frac{dl}{dn} = \left(\frac{\Delta N}{\sigma_y}\right)^2 \phi\left(\frac{\Delta N}{K_{Ic}}, R, \underbrace{\frac{\sigma_y\sqrt{h}}{K_{Ic}}}_{z}, \cancel{ft}\right)$$

ratio of ductile to brittle fracture

quantity is usually small

neglect (the π_4 term)

...assume complete similarity

$$\frac{dl}{dn} = \left(\frac{\Delta N}{\sigma_y}\right)^2 \phi(R, z)$$

... realign to make consistent with empirical formula

$$= \underbrace{\frac{\phi_1(R,z)}{\sigma_y^2}}_{A} (\Delta N)^{2} \neq A(\Delta N)^m$$

but then from experiments, find that $m \neq 2$ after comparing with our data

...so now need to assume <u>incomplete</u> <u>similarity</u>

$$\phi = \left(\frac{\Delta N}{K_{Ic}}\right)^{\alpha} \phi_1(R,z) \longrightarrow \frac{dl}{dn} = \frac{(\Delta N)^{2+\alpha}}{\sigma_y^2 K_{Ic}^{\alpha}} \phi_1(R,z)$$

α is a number but we won't know what it is, alpha could be a function of R and z, $\alpha(R,z)$

$$\frac{dl}{dn} = \underbrace{\boxed{\frac{\phi_1(R,z)}{\sigma_y^2 K_{Ic}^{\alpha}}}}_{A} (\Delta N)^{\overbrace{2+\alpha}^{m}}$$