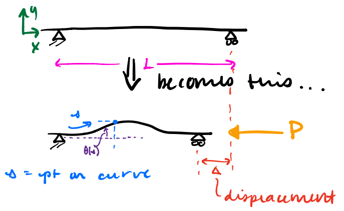
# Variational Calculus Continued ~ Elastica

## Problem: 1D - beam



(remember, work = force · displacement)

... but what does x, y mean here?

→ probably better to parametrize problem to work in a coordinate system unique to our problem

The Langrangian: $\mathcal{L} = \cancelto{0}{T} - V$ => using what's derived below => $U_b - W = \int_0^L \left[\frac{B}{2}(\theta')^2 - P(1-\cos\theta)\right] ds$

↑ this Lagrangian is function of $s, \theta$, and $\bar{s}$(?)

↑ potential energy of an elastic structure:

strain energy - work

strain energy → $\mathcal{U} = \frac{1}{2}\int_{\forall} \sigma \varepsilon \, d\forall$ ← strain

↖ stress

$\forall$ / volume of elastic material

stress = $\sigma = E\varepsilon$

$$\mathcal{U} = \frac{1}{2}\int_{\forall} \frac{\sigma^2}{E} \, d\forall$$

in bending: $\sigma = My/I$

so we'll just look at bending strain energy:

$$\mathcal{U}_b = \frac{1}{2}\int_s \frac{M^2}{EI^2} \, ds \underbrace{\iint_A y^2 dA}$$

this is the second moment of the area: $I$

↑ integral along axis

↑ integral through thickness

$$\mathcal{U}_b = \frac{1}{2}\int_s \frac{M^2}{EI} \, ds$$

$M = \kappa(s) EI$

$$\mathcal{U}_b = \frac{EI}{2}\int_s \kappa^2(s) \, ds$$

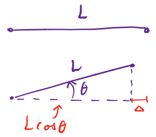
↖ kappa, the curvature

$B = EI$

B = bending rigidity of the object

$$\Rightarrow \boxed{\mathcal{U}_b = \frac{B}{2}\int_s (\theta'(s))^2 ds} \Rightarrow \boxed{\mathcal{U}_b = \frac{B}{2}\int_0^L (\theta'(s))^2 ds}$$

→ work:



$\Rightarrow \Delta = L - L\cos\theta = L(1-\cos\theta)$

$$W = \int_s P(1-\cos\theta) \, ds$$

$$W = \int_0^L P(1-\cos\theta) \, ds$$

$$U_b - W = \int_0^L \left[\frac{B}{2}(\theta')^2 - P(1-\cos\theta)\right] ds$$

$\frac{ds}{d\epsilon}$ = gateaux derivative

*crucial step: can get rid of that derivative by doing integration by parts

Integration by Parts:

- product rule backwards

$$\downarrow$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\Rightarrow \int u\,dv = uv - \int v\,du$$

Work in breakout room:

$$S = \int_0^L \mathcal{L}(s, \theta, \theta')\,ds$$

$$\tilde{S} = \int_0^L \mathcal{L}(s, \theta + \epsilon\eta, \theta' + \epsilon\eta')\,ds$$

$$\delta S = \frac{d\tilde{S}}{d\epsilon}\Big|_{\epsilon=0} = \int_0^L \left[\underbrace{\frac{\partial \mathcal{L}}{\partial s}\frac{\partial s}{\partial \epsilon}}_{\substack{0 \\ \text{(can't perturb} \\ \text{the ind. var.)}}} + \frac{\partial \mathcal{L}}{\partial \tilde{\theta}}\underbrace{\frac{\partial \tilde{\theta}}{\partial \epsilon}}_{\eta} + \frac{\partial \mathcal{L}}{\partial \tilde{\theta}'}\underbrace{\frac{\partial \tilde{\theta}'}{\partial \epsilon}}_{\eta'}\right] ds$$

$$= \int_0^L \left[\frac{\partial \mathcal{L}}{\partial \theta}\eta + \underbrace{\frac{\partial \mathcal{L}}{\partial \theta'}\eta'}_{\text{integration by parts}}\right] ds = 0$$

$$u = \frac{\partial \mathcal{L}}{\partial \theta'} \qquad dv = \eta'$$

$$du = \qquad v = \eta$$

$$\Rightarrow \underbrace{\eta\frac{\partial \mathcal{L}}{\partial \theta}\Big|_0^L}_{\emptyset \text{ bc } \eta(0)=\eta(L)=0} - \int_0^L \eta \frac{\partial}{\partial s}\frac{\partial \mathcal{L}}{\partial \theta}\,ds$$

so now we have: $$\int_0^L \overset{\uparrow}{\eta}\underbrace{\left[\frac{\partial \mathcal{L}}{\partial \theta} - \frac{\partial}{\partial s}\frac{\partial \mathcal{L}}{\partial \theta}\right]}_{\text{must be zero}} ds = 0$$

can't be zero because otherwise would be exact

$$\Downarrow$$

$$\frac{\partial}{\partial s}\left(\frac{\partial \mathcal{L}}{\partial \theta'}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$

$$U_b - W = \int_0^L \left[\frac{B}{2}(\theta')^2 - P(1-\cos\theta)\right] ds$$

← now apply this

$$\frac{d\mathcal{L}}{d\theta'} = B\theta'$$

$$\frac{d}{ds}(B\theta') = B\theta''(s)$$

$$\frac{d\mathcal{L}}{d\theta} = -P\sin(\theta(s))$$

$$\boxed{B\theta''(s) + P\sin(\theta(s)) = 0}$$ ← the first variation

↑ governing equation

… now go thru boundary conditions

→ look @ simply supported beam B.C.

$$\eta \frac{d\mathcal{L}}{d\theta'}\Big|_0^L = \eta\frac{d\mathcal{L}}{\theta'}(L) - \eta\frac{d\mathcal{L}}{\theta'}(0)$$

$$\eta(L)\cancel{B}\theta'(L) - \eta(0)\cancel{B}\theta'(0) = 0$$

$$\underbrace{\eta(L)}_{\phi}\underbrace{\theta'(L)}_{\phi} - \underbrace{\eta(0)}_{\phi}\underbrace{\theta'(0)}_{\phi} = 0$$

What does this say about the boundary equations?

- simply supported?
- cantilevered?

<u>Elastica</u>:

governing equation: $\theta''(s) + \overset{\lambda}{\boxed{\frac{P}{B}}}\sin(\theta(s)) = 0$

$$B\theta'\eta\Big|_0^L = 0$$

↓

if this is zero:

$w''(L) \approx \theta'(L) = \kappa(L) = 0$
$w''(0) \approx \theta'(0) = \kappa(0) = 0$

↑ kappa for curvature

} natural / static boundary conditions → from variational calculus

↑ equilibrium

the deflection

$w(L) = w(0) = 0$ } essential / kinematic boundary conditions ] ⇐ what we don't get from variational calculus (?)