# First Variations & Geodesics

(Introduced Calculus of Variations last class)

**Lagrangian**: $\mathcal{L} = T - V$ (kinetic energy minus potential energy)

Hamiltonian: $H = T + V$ (total energy of the system: kinetic + potential)

*we'll focus mainly on Lagrangian: end up with fewer equations but with higher order derivatives potentially

To calculate the first variation in general:
- → $y$ and $\eta$ are functions (vectors, as well)
- → $\epsilon$ is a scalar (small)

perturbed action

true action

The first variation in $S$: $\delta \bar{S}(y, \eta) = \lim_{\epsilon \to 0} \dfrac{\bar{S}(y + \epsilon\eta) - \bar{S}(y)}{\epsilon} = \dfrac{d}{d\epsilon} \bar{S}(y + \epsilon\eta)\Big|_{\epsilon=0}$

THE GÂTEAUX DERIVATIVE $= \left(\dfrac{d\bar{S}}{d\epsilon}\right)_{\epsilon=0} = 0$
(the directional?)

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$

Back to the projectile problem:

$f(x) = \ddot{x}_0 + \epsilon \ddot{x}_1 + \epsilon^2 \ddot{x}_2 + \ldots = -1 + 2\epsilon x_0 - 3\epsilon^2 x_1^2 + \ldots$

$\dfrac{df(x)}{d\epsilon} = \ddot{x}_1 + 2\epsilon \ddot{x}_2 + \ldots = 2x_0 - 6\epsilon x_1^2 + \ldots$

→ using gâteaux derivative: evaluate at $\epsilon = 0$

$\dfrac{df(x)}{d\epsilon}\Big|_{\epsilon=0} = \ddot{x}_1 = 2x_0$ ($O(\epsilon)$ solution)

## Consider the General Functional

$S = \int_{x_1}^{x_2} \mathcal{L}(x, y, y')$

two arbitrary points in space

$\tilde{y}(x) = y(x) + \epsilon \underbrace{\eta(x)}$ ↳ $\eta(x_1) = \eta(x_2) = 0$

$\delta y = \tilde{y}(x) - y(x) = \epsilon\eta(x)$

$\dfrac{d\tilde{S}}{d\epsilon} = \dfrac{d}{d\epsilon}\int \mathcal{L}(x, y, y')dx$

perturbed function $\tilde{S} = \int_{x_1}^{x_2} \mathcal{L}(x, \tilde{y}, \tilde{y}')dx = \int_{x_1}^{x_2} \mathcal{L}(x, y + \epsilon\eta, y' + \epsilon\eta')dx$

...take derivative of functional wrt to function, take derivative of function wrt variable: $\dfrac{d\mathcal{L}}{dx}\dfrac{dx}{d\epsilon}$ etc. etc.

expand in Taylor Series Expansion: $\tilde{S} = \tilde{S}\Big|_{\epsilon=0} + \epsilon \dfrac{d\tilde{S}}{d\epsilon}\Big|_{\epsilon=0} + \dfrac{\epsilon^2}{2!}\dfrac{d^2\tilde{S}}{d\epsilon^2}\Big|_{\epsilon=0} + \ldots$

$\delta S = \tilde{S} - S = \underbrace{\epsilon \dfrac{d\tilde{S}}{d\epsilon}\Big|_{\epsilon=0}}_{\text{first variation } \delta S} + \underbrace{\dfrac{\epsilon^2}{2}\dfrac{d^2\tilde{S}}{d\epsilon^2}\Big|_{\epsilon=0}}_{\text{2nd variation } \delta^2 S} + \ldots$ (tells us minima for instance?)

first variation always $O(\epsilon)$, 2nd always $O(\epsilon^2)$, etc. etc.

Gâteaux Derivative:

$$\left(\dfrac{d\tilde{S}}{d\epsilon}\right)_{\epsilon=0} = \int_{x_1}^{x_2} \overbrace{\left(\dfrac{d\mathcal{L}}{dx}\underbrace{\boxed{\dfrac{dx}{d\epsilon}}}_{0} + \dfrac{d\mathcal{L}}{d\tilde{y}}\underbrace{\boxed{\dfrac{d\tilde{y}}{d\epsilon}}}_{\eta} + \dfrac{d\mathcal{L}}{d\tilde{y}'}\underbrace{\boxed{\dfrac{d\tilde{y}'}{d\epsilon}}}_{\eta'}\right)}^{\text{chain rule}} dx = 0$$

0 bc don't vary $x$

first variation = 0 gives equilibrium function

*don't perturb the independent variable

$$\delta S = \int_{x_1}^{x_2} \left(\dfrac{d\mathcal{L}}{d\tilde{y}}\eta + \dfrac{d\mathcal{L}}{d\tilde{y}'}\eta'\right)dx = 0$$

when get to this point, need to integrate by parts

we don't know anything about $\eta'$ so need to get rid of it by integrating by parts

→ evaluate at $\epsilon = 0$: $\tilde{y}|_{\epsilon=0} = y$ and $\tilde{y}'|_{\epsilon=0} = y'$

$$\delta S = \int_{x_1}^{x_2} \left( \frac{\partial \mathcal{L}}{\partial y} \eta + \underbrace{\frac{\partial \mathcal{L}}{\partial y'} \eta'}_{} \right) dx = 0$$

↑ integrate by parts

$$\Rightarrow \int_{x_1}^{x_2} \frac{\partial \mathcal{L}}{\partial y'} \eta' \, dx = \frac{\partial \mathcal{L}}{\partial y'} \eta \Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \left[ \frac{d}{dx}\left( \frac{\partial \mathcal{L}}{\partial y'} \right) \right] \eta \, dx$$

$$\frac{\partial \mathcal{L}}{\partial y'} \eta(x) \Big|_{x_1}^{x_2} = \frac{\partial \mathcal{L}}{\partial y'} \eta(x_2) - \frac{\partial \mathcal{L}}{\partial y'} \eta(x_1)$$

leave this in just because have to

↑ this term will go away here, but it doesn't always go away... this term is useful for deriving boundary conditions

$$\delta S = \int_{x_1}^{x_2} \left[ \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dx}\left( \frac{\partial \mathcal{L}}{\partial y'} \right) \right] \eta \, dx$$

... since $\eta$ is arbitrary, the entire term in brackets must be equal to zero

$$\therefore \boxed{\frac{d}{dx}\left( \frac{\partial \mathcal{L}}{\partial y'} \right) - \frac{\partial \mathcal{L}}{\partial y} = 0}$$

Euler-Lagrange Equations

$\left( \mathcal{L}(x, y, y') \right) \to EL \to$ 2nd ODE for $y(x)$

**Bonus Fun:**
$\mathcal{L}(x, y, y', y'')$

→ such as for deflection of a beam
→ vibration of spinning thin plates

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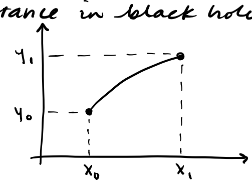
**Planar Geodesics** = "very complicated way of saying straight lines"
→ shortest distance between two points
→ the simplest problem in variational calculus
→ basis for formulating distance in black holes

$\bar{a} = (a, \alpha)$ $\quad \bar{b} = (b, \beta)$

x locations

y locations




$\Delta x = x_1 - x_0$
$\Delta y = y_1 - y_0$ $\Big\}$ $s^2 = (\Delta x)^2 + (\Delta y)^2$

$$s = \sqrt{(\Delta x)^2 + (\Delta y)^2 \left(\frac{\Delta x}{\Delta x}\right)^2}$$
$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \, \Delta x$$

$$s = \underbrace{\int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx}_{\text{arc length!}}$$

$\Downarrow$

$$S = \int_a^b \underbrace{\sqrt{1 + (y')^2}}_{} \, dx$$

this is the Lagrangian!

$\mathcal{L}(x, y, y') \to \mathcal{L}(x, y')$

$$\frac{d}{dx}\left( \frac{\partial \mathcal{L}}{\partial y'} \right) - \cancel{\frac{\partial \mathcal{L}}{\partial y}} = 0$$

bc doesn't depend on y
so zero (only y' and x)

$$\frac{d}{dx}\left( \frac{y'}{\sqrt{1 + (y')^2}} \right) = \boxed{\frac{y''}{(1 + (y')^2)^{3/2}}} = 0$$

↑ this is the curvature!

... so shortest distance is when curvature equals zero

⇒ notice that no matter value of $y'$, the denominator is always nonzero...
$\therefore y'' = 0$!

$y'' = 0 \rightsquigarrow$ integrate twice $\rightsquigarrow y = cx + d$

$x = x - a$

$\hookrightarrow \alpha$

$$c = \frac{\beta - \alpha}{b - a} \quad \ldots \quad \boxed{y = \frac{\beta - \alpha}{b - a}(x - a) + \alpha}$$

means straight, $\delta^2 s$ for minimum

* proved the extrema between two points
* in order to find minimum, need 2nd variation!