# Dynamical Systems + Stability

## Linearity

re-examine harmonic oscillator midterm task:

$$m\ddot{x} + a\dot{x} + kx = 0$$

general framework

$$\left.\begin{aligned}\dot{x}_1 &= f_1(x_1, \dots, x_n)\\ &\vdots\\ \dot{x}_n &= f_n(x_1, \dots, x_n)\end{aligned}\right\} \quad \begin{aligned}x_1 &= x\\ x_2 &= \dot{x}\end{aligned} \quad \mapsto \quad \begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= \ddot{x} = -\frac{a}{m}\dot{x} - \frac{k}{m}x = \underbrace{-\frac{a}{m}x_2 - \frac{k}{m}x_1}\end{aligned}$$

this system of equations is linear because the $x_i$ on the RHS here apply only to the first power

$\Leftarrow$ it would be NONLINEAR if:

$$\left\{\begin{aligned}\text{RHS} - \ &x_1 x_2\\ &(x_1)^3\\ &\cos x_2\\ &\dots\end{aligned}\right.$$

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\lambda \sin x_1\end{aligned} \quad \longleftarrow \quad \ddot{x} + \lambda \sin x = 0$$

$\lambda$: $g/L$ (pendulum), $\frac{P}{B}$ (elastica)

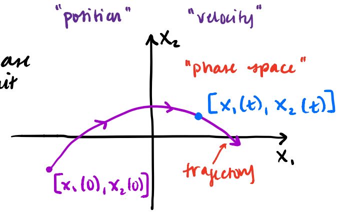
most things we model in the world are nonlinear
→ pickles and ice cream example

## Geometric Methods

solution: $x_1(t)$ & $x_2(t)$

$x_1(t) \downarrow$ "position"; $x_2(t) \downarrow$ "velocity"

the phase portrait

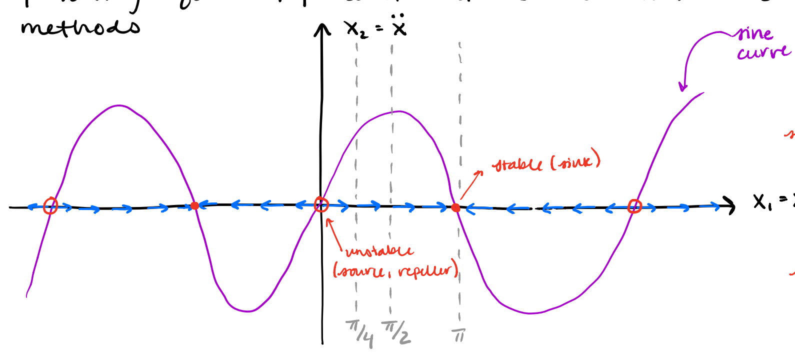



we want to be able to work backwards using geometric methods:

**Differential Equations as a Vector Field**

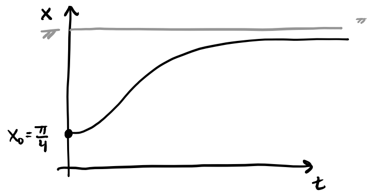
consider: $\dot{x} = \sin x$, $x(0) = x_0$ ] → N.L. solve by separation of variables: $dt = \frac{dx}{\sin x}$ → $t = \int \csc x \, dx = \ln\left|\frac{\csc x_0 + \cot x_0}{\csc x + \cot x}\right|$

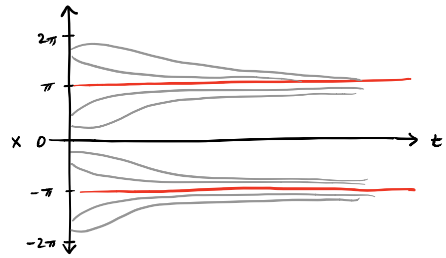
so instead of solving equation, pretend that we can't solve and implement geometric methods




source: o when arrows pt away from each other, unstable

sink: • when arrows point toward a fixed point, we say that fixed pt is stable




## Population Growth

simplest model : $\dot{N} = rN$

population change — growth rate — $N(t)$ # of organisms at time $t$

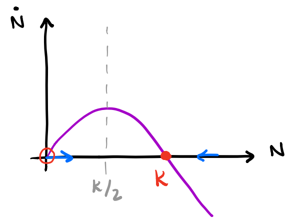
add to this model the carrying capacity $\rightarrow$ $\frac{\dot{N}}{N}$ = will decrease when N exceeds $K$ $\longrightarrow$ $r = \frac{\dot{N}}{N}$ decreases linearly with N

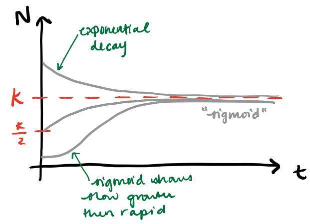
carrying capacity

### Logistic Equation

$$\dot{N} = \underbrace{rN\left(1 - \frac{N}{K}\right)}$$

when rate of change (+), move to right
when negative, move to left




we've now obtained qualitative solns without doing any math

## Linear Stability Analysis — determine the quantitative takeaways

- linearize about fixed point $x^*$

perturb about $x^*$ : $\eta(t) = x(t) - x^*$ ... we want to perturb our system and determine whether this perturbation grows or decays in time

how does our perturbation change in time?

$$\dot{\eta}(t) = \frac{d}{dt}(x - x^*) = \dot{x}(t) = f(x) = f(x^* + \eta)$$

$$\dot{\eta}(t) = f(x^* + \eta)$$

$\downarrow$ linearize by expanding in a Taylor Series

$$f(x^* + \eta) = \cancel{f(x^*)} + \eta f'(x^*) + \cancel{O(\eta^2)} \implies \boxed{\dot{\eta} = \eta f'(x^*)}$$

$\emptyset$ By definition this is zero

$\emptyset$ we can ignore as long as $f'(x^*) = \emptyset$

tells us that our perturbation will
$\rightarrow$ grow exponentially if the derivative at the fixed pt, $f'(x^*)$ is (+)
$\Rightarrow$ $f'(x^*) > 0$ : UNSTABLE
$\rightarrow$ will decay exponentially if
$\Rightarrow$ $f'(x^*) < 0$ : STABLE

... [ insert what was said here ]
the magnitude of the slope determines how stable/unstable it is

characteristic time scale : $\tau \sim \frac{1}{|f'(x^*)|}$

Read This When Have Time ↴