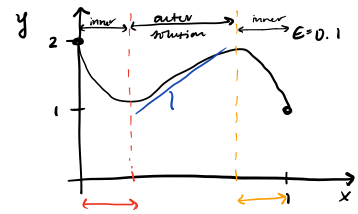
Boundary Value Problem (BVP)

$$\epsilon^2 y'' + \epsilon x y' - y = -e^x \qquad 0 < x < 1$$

$y(0) = 2$
$y(1) = 1$



① outer solution: take $\epsilon \to 0$

$$\cancel{\epsilon^2 y''} + \cancel{\epsilon x y'} - y = -e^x$$

- can't apply BC because DE to solve, no done

If we did $y(x) \sim y_0 + \epsilon y_1(x) + \epsilon^2 y_2(x) + \ldots$ (take outer solution to higher order) we wouldn't be able to apply BC

② Inner Solution: first rescale variables, do method of dominant balance, regular perturbation

1. Rescale (arbitrarily) ⟶ in general: $\boxed{\bar{x} = \frac{x - x_{bl}}{\epsilon^\gamma}}$ where bl = boundary layer
2. Method of Dominant Balance (find the rescaling)
3. Regular Perturbation on Rescaled Equation
4. Match to outer solution ⟶ composite solution

let's move to boundary layer, stretch out that approximation

$$\bar{x} = \frac{x - x_{bl}}{\epsilon^\gamma}$$

$$\bar{x} = \frac{x}{\epsilon^\gamma} \qquad (x_{bl} = 0) \quad y(0) = 2$$

$$\epsilon^2 y'' + \epsilon x y' - y = -e^x$$

insert rescaling

$$\epsilon^{2-2\gamma} \bar{y}'' + \epsilon \bar{x} \bar{y}' - \bar{y} = -e^{\epsilon^\gamma \bar{x}}$$

valid for us to say "let's do Taylor Series Approx of this"

$$e^{\epsilon^\gamma \bar{x}} \sim 1 + \epsilon^\gamma \bar{x} + \text{H.O.T}$$

$\gamma = 1$

$$\bar{y}'' + \epsilon \bar{x} \bar{y}' - \bar{y} = -e^{\epsilon \bar{x}}$$

assume: $\bar{y} \sim \bar{y}_0 + \epsilon \bar{y}_1 + \ldots$

regular perturbation

$$\bar{Y}_0 = 1 + Ae^{\bar{x}} + Be^{-\bar{x}}$$

$\bar{y}(0) = 2$ ... write B in terms of A

$$\bar{Y}_0 = 1 + Ae^{\bar{x}} + (1-A)e^{-\bar{x}}$$

physically this means...

$$\lim_{\bar{x} \to \infty} \bar{Y}_0 = \lim_{x \to 0} Y_0$$

(b.l. soln) (outer soln)

$$\lim_{\bar{x} \to \infty} e^{\bar{x}} = \infty \qquad \therefore A = 0$$

$$\boxed{\bar{Y}_0(\bar{x}) \sim 1 + e^{-\bar{x}}}$$ ~ the first term approximation in this boundary layer

BL @ x = 1

$$\tilde{x} = \frac{x - x_{bl}}{\epsilon^{\lambda}} = \frac{x-1}{\epsilon^{\lambda}}$$

now rescaling:

$$\epsilon^{2-2\lambda}\tilde{y}'' + \epsilon^{1-\lambda}(1+\epsilon^{\lambda}\tilde{x})\tilde{y}' - \tilde{y} = -e^{1+\epsilon^{\lambda}\tilde{x}}$$

$$\frac{d}{dx} = \frac{d\tilde{x}}{dx}\frac{d}{d\tilde{x}} = \epsilon^{-\lambda}\frac{d}{d\tilde{x}}$$

$$\Big\downarrow \lambda = 1$$

$$\tilde{y}'' + (1+\epsilon\tilde{x})\tilde{y}' - \tilde{y} = -e^{1+\epsilon\tilde{x}}$$

assume: $\tilde{y} \sim \tilde{y}_0 + \epsilon\tilde{y}_1 + \dots$

$\downarrow$ insert into boundary, do regular perturbation

since we reshifted x-axis/origin, need $\tilde{y}(0) = 1$

BL @ $x = 1$

$$\tilde{y}_0 \sim e + Ce^{\tilde{x} r_+} + (1-e-C)e^{r_-\tilde{x}}$$

$r_\pm = \frac{-1 \pm \sqrt{5}}{2}$ ... just using cleaner notation

$$\lim_{\tilde{x}\to-\infty}\tilde{Y}_0 = \lim_{x\to 1} Y_0$$

$$\lim_{\tilde{x}\to-\infty} e^{r_-\tilde{x}} = \infty \quad \& \quad \lim_{\tilde{x}\to-\infty} e^{r_+\tilde{x}} = 0 \longrightarrow \boxed{\tilde{Y}_0(\tilde{x}) = e + \underbrace{(1-e)}_{C}e^{r_+\tilde{x}}}$$

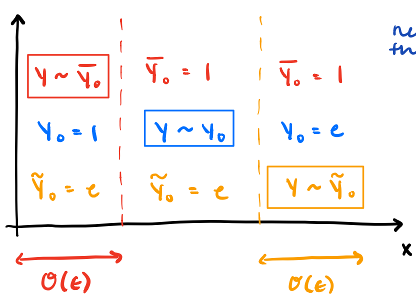
from this, we find: $1 - e - C = 0$

composite:

$$y \sim y_0(x) + \bar{y}_0(\bar{x}) + \tilde{y}_0(\tilde{x}) - \overbrace{y_0(0) - y_0(1)}^{1+e} \longrightarrow \boxed{y \sim e^{x} + e^{-x/\epsilon} + (1-e)e^{r_+(x-1)/\epsilon}}$$

*pretty good curve, see mathematica NB

need to subtract these off



$\longrightarrow \bar{Y}_0(\bar{x}) = 1 + e^{-\bar{x}} \therefore \bar{Y}_0(1) = 1 + e^{-1/\epsilon}$ ← goes to zero

$\longrightarrow Y_0(x) = e^{x} \therefore Y_0(1) = e^{1} = e$