Perturbation Methods

Want to reduce your problem to something you can solve, then add some small perturbation/difference to get approximations

> Diff. Eqns Tweet:
> "Knowing what is big and what is small is more important than being able to solve partial differential equations" - Stan Ulam

We have a procedure to find dimensionless groups and identify a small parameter, which refer to as ϵ

Algebraic Equation — start with a second order polynomial as an example

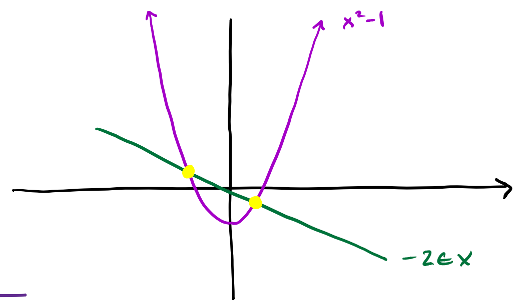
small

$$x^2 + \underbrace{2\epsilon x}_{} - 1 = 0$$

move small term to other side of the equation

$$x^2 - 1 = -2\epsilon x$$

... then plot these two lines



Regular Perturbation
- the reduced problem ($\epsilon = 0$) has the same number of solutions as the original problem

Analytical Solution: $x = -\epsilon \pm \underbrace{\sqrt{1+\epsilon^2}}_{}$

assume $\epsilon \ll 1$, use binomial expansion

$$\sqrt{1+\epsilon^2} = 1 + \tfrac{1}{2}\epsilon^2 - \tfrac{1}{8}\epsilon^4 + \dots$$

$$x = -\epsilon \pm \left(1 + \tfrac{1}{2}\epsilon^2 - \tfrac{1}{8}\epsilon^4 + \dots\right) = \pm 1 - \epsilon \pm \tfrac{1}{2}\epsilon^2 \pm \tfrac{1}{8}\epsilon^4 + \dots$$

$\Rightarrow$ at order 1: $O(1)$: $x = \pm 1$

$\Rightarrow$ at order ϵ: $O(\epsilon)$: $x = \pm 1 - \epsilon$

$\Rightarrow$ at order ϵ^2: $O(\epsilon^2)$: $x = \pm 1 - \epsilon \pm \tfrac{1}{2}\epsilon^2$

Approximations

- Taylor's theorem: our guess

$$f(x) = f(0) + x f'(0) + \tfrac{1}{2!}x^2 f''(0) + \tfrac{1}{3!}x^3 f'''(0) + \dots$$ ← technically Maclaurin since taking at zero

$\downarrow \epsilon$

$$f(\epsilon) = x(0) + \epsilon x'(0) + \tfrac{1}{2}\epsilon^2 x''(0) + \dots$$

$\downarrow$ more general form

$$x(\epsilon) \sim x_0 + \epsilon^\alpha x_1 + \epsilon^\beta x_2 + \dots$$ → "well-ordering" assumption: $0 < \alpha < \beta < \gamma \dots$

insert this into our original equation

Regular Perturbation

$$x^2 + 2\epsilon x - 1 = 0$$

$$(x_0 + \epsilon^\alpha x_1 + \epsilon^\beta x_2 + \dots)(x_0 + \epsilon^\alpha x_1 + \epsilon^\beta x_2 + \dots)$$

$$\boxed{x_0^2 + 2\epsilon^\alpha x_0 x_1} + 2\epsilon^\beta x_0 x_2 + \epsilon^{2\alpha} x_1^2 + \epsilon^{2\beta} x_2^2 + \dots$$

for simplicity, we'll just keep these 1st two terms

$$2\epsilon(x_0 + \epsilon^\alpha x_1 + \dots)$$

$$\underbrace{x_0^2 + 2\epsilon^\alpha x_0 x_1}_{x^2} + \dots + \underbrace{2\epsilon(x_0 + \epsilon^\alpha x_1 + \dots)}_{2\epsilon x} \underbrace{-1}_{-1} = 0$$

so now...

$O(1)$: let $\epsilon \to 0$

$$x_0^2 - 1 = 0 \longrightarrow x_0 = \pm 1$$

$$\pm 2\epsilon^\alpha x_1 + \dots + 2\epsilon(\pm 1 + \epsilon^\alpha x_1 + \dots) = 0$$

$$\pm 2\epsilon^\alpha x_1 + \dots \pm 2\epsilon + 2\epsilon^{1+\alpha} x_1 + \dots = 0$$

taking these terms, means $\alpha = 1$

we want to take these terms, ignore powers $> \alpha$

$O(\epsilon)$: only keep orders up to ϵ

$$\pm 2\cancel{\epsilon} x_1 \pm 2\cancel{\epsilon} = 0 \longrightarrow x_1 = -1$$

$$\pm 2\epsilon^\beta x_2 + \epsilon^2 + \dots + 2\epsilon(-\epsilon + \epsilon^\beta x_2 + \dots) = 0$$

$$\pm 2\epsilon^\beta x_2 + \epsilon^2 + \dots - 2\epsilon^2 + 2\epsilon^{1+\beta} x_2 + \dots = 0$$

$\beta = 2$

$O(\epsilon^2)$: $\pm 2\cancel{\epsilon^2} x_2 + \cancel{\epsilon^2} - 2\cancel{\epsilon^2} = 0 \longrightarrow x_2 = \pm \frac{1}{2}$

$$x(\epsilon) \sim x_0 + \epsilon^\alpha x_1 + \epsilon^\beta x_2 + \dots$$

from $O(1)$ $\rightarrow$ $x(\epsilon) \sim \pm 1 + O(\epsilon)$

from $O(\epsilon)$ $\rightarrow$ $x(\epsilon) \sim \pm 1 - \epsilon + O(\epsilon^2)$

from $O(\epsilon^2)$ $\rightarrow$ $\boxed{x(\epsilon) \sim \pm 1 - \epsilon \pm \frac{1}{2}\epsilon^2 + O(\epsilon^3)}$

↳ if we take this soln and take $\frac{dx}{d\epsilon} = -1 + \epsilon$

$$x'(0) = \frac{dx}{d\epsilon}\Big|_{\epsilon=0} = -1 \equiv x_1$$

$$\frac{1}{2}x''(0) = \frac{1}{2}\frac{d^2x}{d\epsilon^2}\Big|_{\epsilon=0} = \frac{1}{2} \equiv x_2$$

more terms we keep, more accurate our ϵ will be