How to Nondimensionalize an Equation

Projectiles

- scale the independent variables using characteristic values

$$\ddot{x} = -\frac{gR^2}{(R+x)^2}$$

R = radius of Earth
x = height of ball/projectile

↳ x_c = characteristic height, t_c = characteristic time } constants

let: $x = x_c u$, $t = t_c s$

(x & x_c are both measures of length ∴ u is dimensionless)
(s is also dimensionless)

u represents dimensionless height, s dimensionless time

Recall the Chain Rule: $\frac{d}{dt} = \frac{ds}{dt}\frac{d}{ds} = \frac{1}{t_c}\frac{d}{ds}$

$$\frac{d^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}\right) = \frac{1}{t_c^2}\frac{d^2}{ds^2}$$

because s is new time variable

$s = \frac{t}{t_c}$ ∴ $\frac{ds}{dt} = \frac{1}{t_c}$

$$\frac{1}{t_c^2}\frac{d^2}{ds^2}(x_c u) = \frac{-gR^2}{(R+x_c u)^2}$$

$(R + x_c u)^2 \rightarrow \underbrace{R^2\left(1 + x_c \frac{u}{R}\right)^2}$ or $x_c^2\left(\frac{R}{x_c} + u\right)^2$

this is a more useful factorization for us

remember: our goal is to create dimensionless groups

$$\boxed{\frac{x_c}{g t_c^2}\frac{d^2u}{ds^2} = -\frac{1}{(1 + \frac{x_c}{R}u)^2}}$$

$\frac{x_c}{g t_c^2} \rightarrow \pi_1$, $\frac{x_c}{R} \rightarrow \pi_2$

I.C.'s: $u(0) = 0$, $\frac{du(0)}{ds} = \frac{t_c}{x_c}V_0$

$\frac{t_c}{x_c}V_0 \rightarrow \pi_3$

Dimensionless Groups:

1. do not involve variables (u & s) only the parameters (x_c, t_c, g, R, V_0)
2. dimensionless: accomplished by rearranging our equations
3. they are independent: not possible to π_i in terms of π_j, π_k

Characteristic Values

Rule 1: Set π_i in I.C./B.C. equal to 1 → $\pi_3 = \frac{t_c}{x_c}V_0 = 1$ ∴ $x_c = V_0 t_c$ → $\boxed{t_c = \frac{V_0}{g}}$

Rule 2: Set π_j that appear in the reduced problem equal to 1

- the reduced problem is the equation that results when throw out very small or very large dimensionless group

$$\pi_1 \frac{d^2u}{ds^2} = -\frac{1}{(1+\underbrace{\pi_2 u}_{0})^2} = -1 \rightarrow \pi_1 = \frac{x_c}{g t_c^2} = 1 \quad ∴ \quad \boxed{x_c = \frac{V_0^2}{g}}$$

reduced problem

note that throwing out π_2 means throwing out the nonlinear terms, making it easier to solve

by identifying what is essential to our problem, we can determine the characteristic length & time scale

π_2 is a small parameter — we'll refer to small par as ϵ

Dimensionless Equation

$$\frac{d^2u}{ds^2} = -\frac{1}{\left(1 + \left(\frac{V_0^2}{gR}\right)u\right)^2}, \quad \frac{V_0^2}{gR} \rightarrow \pi_2$$

$$\left.\begin{array}{l} u(0) = 0 \\ \frac{du}{ds} = 1 \end{array}\right] \rightarrow \frac{d^2u}{ds^2} = -\frac{1}{(1+\epsilon u)^2}$$

Is ϵ small?

$R = 6.4 \times 10^6$ m
$g = 9.81$ m/s²
$\Big] \rightarrow \epsilon \approx 1.6 \times 10^{-8} V_0^2$

this tells us that unless initial velocity is very large then ϵ is very small

Reaction – Diffusion (KPP equation)

$$D \frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t} - \lambda(\underbrace{\gamma - c}_{[\gamma]=[c]})\underbrace{c}_{\text{nonlinear}}$$

B.C. $c(0,t) = c(\ell, t) = 0$

I.C. $c(x,0) = c_0 \sin(5\pi \frac{x}{\ell})$

concentration $= [c] = [c_0] = ML^{-3}$
diffusion $= [D] = L^2 T$
$[\lambda(\gamma - c)c] = [{}^{dc}/_{dt}] \therefore [\lambda] = L^3 M^{-1} T^{-1}$

Change of Variables

$x = x_c u$
$t = t_c s$
$c = c_c v$

Chain Rule:

$$\frac{D c_c}{x_c^2} \frac{\partial v^2}{\partial u^2} = \frac{c_c}{t_c} \frac{\partial v}{\partial s} - \lambda c_c v (\gamma - c_c v)$$

$$\underbrace{\frac{D t_c}{x_c^2}}_{\pi_1} \frac{\partial^2 v}{\partial u^2} = \frac{\partial v}{\partial s} - \underbrace{\lambda t_c c_c}_{\pi_2} v \left(\underbrace{\frac{\gamma}{c_c}}_{\pi_3} - v\right)$$

applying boundary conditions

B.C. $v(0,s) = v\left(\underbrace{\frac{\ell}{x_c}}_{\pi_4}, s\right) = 0$

I.C. $v(u,0) = \underbrace{\frac{c_0}{c_c}}_{\pi_5} \sin\left(5\pi \frac{x_c u}{\ell}\right)$

Rule 1: $\pi_4 = \frac{\ell}{x_c} = 1$
$\pi_5 = \frac{c_0}{c_c} = 1$
$\therefore x_c = \ell$
$c_c = c_0$

$$\boxed{\pi_1} \frac{\partial^2 v}{\partial u^2} = \frac{\partial v}{\partial s} - \boxed{\pi_2} v(\pi_3 - v)$$

$\pi_2 \rightarrow$ 0 weak nonlinearity

the only way to get the reduced form (Rule 2) is to throw out the nonlinear term π_2

$\pi_1 = \frac{D t_c}{x_c^2} = 1 \rightarrow \boxed{t_c = \frac{\ell^2}{D}}$

$$\boxed{\begin{gathered}\frac{\partial^2 v}{\partial u^2} = \frac{\partial v}{\partial s} - \epsilon\left(\frac{\gamma}{c_0} - v\right) v \\ v(0,s) = v(1,s) = 0 \\ v(u,0) = \sin(5\pi u)\end{gathered}}$$

for the weakly nonlinear reaction equation

this problem is much more straightforward

$\pi_1 \rightarrow$ 0, weak diffusion

$\pi_2 = \lambda t_c c_c = 1 \therefore \boxed{t_c = \frac{1}{c_0 \lambda}}$

$$\boxed{\epsilon \frac{\partial^2 v}{\partial u^2} = \frac{\partial v}{\partial s} - \left(\frac{x}{c_0} - v\right) v}$$

ϵ multiplies highest order derivative in equation, much harder problem to solve

we looked at 2 cases here: weak diffusion & weak nonlinear