Stress-Strain Mechanisms

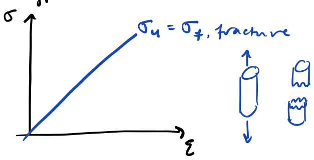
$\sigma = \frac{P}{A_i}$ engineering stress, initial, undeformed A

$\varepsilon = \frac{\Delta L}{L_i}$ engineering strain, initial, undeformed A

sometimes given as percentage where $\varepsilon\% = 100\varepsilon$

(microstrain $\varepsilon_\mu = 10^6 \varepsilon$)

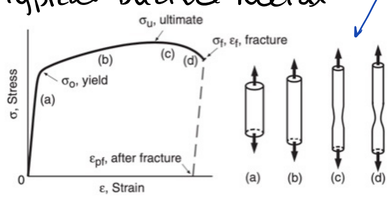
Typical Ceramic



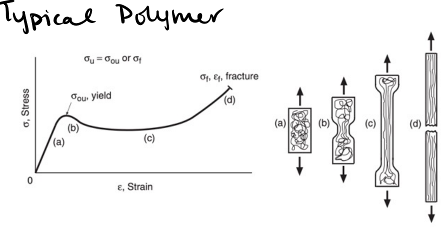
Young's Modulus, $E = \frac{\sigma_B - \sigma_A}{\varepsilon_B - \varepsilon_B}$ {tangent modulus may be used if no well-defined linear region}

necking begins at ultimate stress point

Typical Ductile Metal



Typical Polymer



σ_u = ultimate tensile strength, or just tensile strength $\sigma_u = \frac{P_{max}}{A_i}$

- highest engineering stress reached prior to fracture

σ_f = engineering fracture strength $\sigma_f = \frac{P_f}{A_i}$

- obtained at point of fracture, even if not highest force reached

Brittle materials: $\sigma_u = \sigma_f$ Ductile materials: $\sigma_u > \sigma_f$

σ_o = offset yield strength

- departure from linear-elastic behavior

Measures of Ductility – ability of a material to accomodate inelastic deformation w/out breaking

ε_f = engineering strain at fracture, usually expressed as percentage and termed:
$100\varepsilon_f$ = percent elongation at fracture

$100\varepsilon_{pf}$ = percent elongation after fracture

$\varepsilon_{pf} = \frac{L_f - L_i}{L_i}$ where L_f = fracture length, L_i = distance between tick marks before testing

Elastic strain is lost when the stress drops to zero after fracture, so this quantity is a plastic strain

→ difference between $100\varepsilon_f$ and $100\varepsilon_{pf}$ is small for ductile metals

→ see p.61 for other cases

%RA = percent reduction in area, another measure of ductility

- obtained by comparing the cross-sectional area after fracture, A_f, with the original area

$\%RA = 100\,\frac{A_i - A_f}{A_i}$

$\%RA = 100\,\frac{d_i^2 - d_f^2}{d_i^2}$

u_f = tensile toughness, the area under the entire engineering stress-strain curve up to fracture

- measure of the ability of the material to absorb energy w/out fracture
- tensile toughness should not be confused w/ fracture toughness

$$u_f \approx \varepsilon_f \left(\frac{\sigma_o + \sigma_u}{2} \right)$$ if curve relatively flat after yielding

Strain Hardening - rise in the stress-strain curve following yielding, as the material is increasing its resistance w/ increasing strain

$$\text{strain hardening ratio} = \frac{\sigma_u}{\sigma_o}$$ >1.4 == high, <1.2 == low (for metals)

$\tilde{\sigma}$ = true stress, the axial force divided by current x-sectional area

$$\tilde{\sigma} = \frac{P}{A} \qquad \tilde{\sigma} = \sigma \frac{A_i}{A}$$

$\tilde{\varepsilon}$ = true strain $\quad \tilde{\varepsilon} = \ln(1+\varepsilon)$

THE CONSTANT VOLUME ASSUMPTION

$$\tilde{\sigma} = \sigma(1+\varepsilon)$$

$$\tilde{\varepsilon} = \ln(A_i/A)$$

$$\tilde{\varepsilon} = 2\ln\left(\frac{d_i}{d}\right)$$

Corrections for Triaxial Stress due to Necking - p. 81

→ Bridgman

→ Mirone

True Stress-Strain Curves

→ for metals in region beyond yielding, stress vs. plastic strain behavior fits relationship

$$\tilde{\sigma} = H\tilde{\varepsilon}_p^n$$

↑ true plastic strain $\tilde{\varepsilon}_p = \tilde{\varepsilon} - \tilde{\sigma}/E$

→ when plotted on log-log coordinates: get a straight line w/ slope n (aka the strain hardening exponent)

→ H = strength coefficient, intercept at $\tilde{\varepsilon}_p = 1$

↑ at large strains during advanced stages of necking, need to use true stresses that have been corrected w/ Mirone (p. 81-83) $\quad \tilde{\sigma}_M = H\tilde{\varepsilon}_p^n$

$\tilde{\sigma}_f$ = true fracture strength

- obtained from force @ fracture + final area

$$\tilde{\sigma}_f = \frac{P_f}{A_f} = \sigma_f \left(\frac{A_i}{A_f}\right)$$ ... correction usually needed

$\tilde{\varepsilon}_f$ = true fracture strain

- obtained from final area or percent reduction in area

$$\tilde{\varepsilon}_f = \ln\left(\frac{A_i}{A_f}\right) = \ln\left(\frac{100}{100 - \%RA}\right)$$

$\tilde{u}_f$ = true toughness

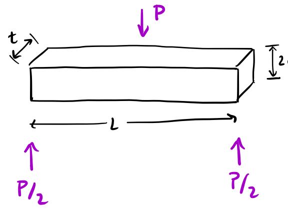
- area under true stress-strain curve up to fracture

Bending (Flexure) Tests

— assume linear elastic behavior

$\sigma = \frac{Mc}{I}$ M = bending moment

rectangular cross-section of depth $2c$ and width t, the area moment of inertia about neutral axis is $I = 2tc^3/3$



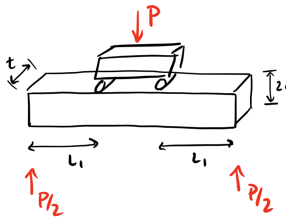
3-pt Bending

3-pt bending: highest bending moment occurs at midspan $M = PL/4$

$$\therefore \sigma_{fb} = \frac{3L}{8tc^2} P_f$$

flexural strength

fracture force

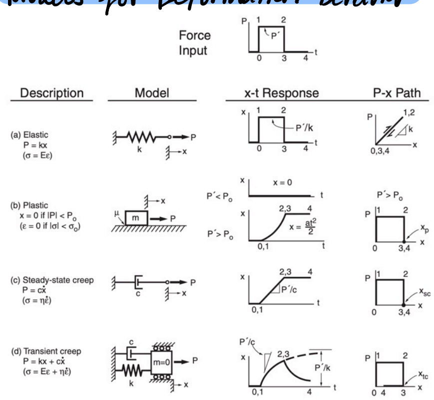


4-pt Bending

max deflection: $v = \frac{PL^3}{48EI}$

then $E = \frac{L^3}{48I}\left(\frac{dP}{dv}\right) = \frac{L^3}{32tc^3}\left(\frac{dP}{dv}\right)$

Models for Deformation Behavior



$E = \frac{\sigma}{\varepsilon}$ $\sigma = E\varepsilon$ $\varepsilon = \frac{\sigma}{E}$

let $P = kx \rightarrow E = \frac{kL}{A}$

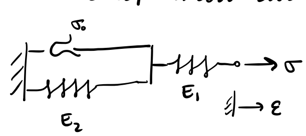
stress: $\sigma = P/A$
strain: $\varepsilon = x/L$
$d\varepsilon/dt$: $\dot{\varepsilon} = \dot{x}/L$

for plastic deformation: $\sigma_0 = P_0/A$

← steady-state creep: proceeds @ constant rate under constant force => $\eta = \frac{\sigma}{\dot{\varepsilon}}$ coefficient of tensile viscosity

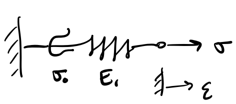
← transient creep: slows down as time passes

Plastic Deformation: $\varepsilon = \frac{\sigma}{E_1} + \frac{\sigma - \sigma_0}{E_2}$

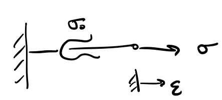


$E_t = \frac{E_1 E_2}{E_1 + E_2}$

elastic, linear hardening

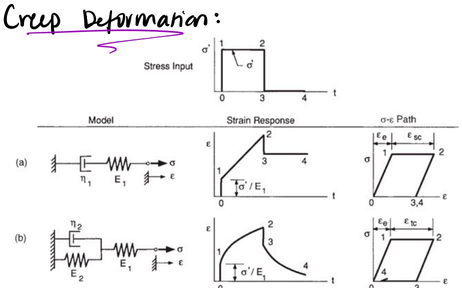


elastic, perfectly plastic



rigid, perfectly plastic

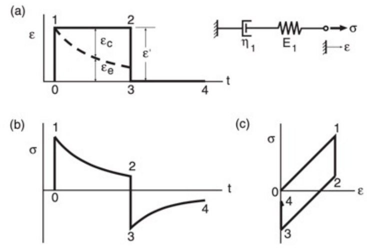
Creep Deformation:



steady-state $\varepsilon = \frac{\sigma'}{E_1} + \frac{\sigma' t}{\eta_1}$

← transient creep $\varepsilon = \frac{\sigma'}{E_1} + \frac{\sigma'}{E_2}\left(1 - e^{-E_2 t/\eta_2}\right)$

Relaxation – same phenomenon as creep, differing only in that it is observed under constant strain rather than constant stress



$$\sigma = E_1 \varepsilon' e^{-E_1 t/\eta_1}$$

↑ corresponds to a stress-time response that decays – that is, relaxes – with time (as illustrated by b.)

Elastic Deformation

$$\nu = -\frac{\text{transverse strain}}{\text{longitudinal strain}} = -\frac{\varepsilon_y}{\varepsilon_x} \quad \ldots \quad \varepsilon_y = \frac{-\nu}{E}\sigma_x$$

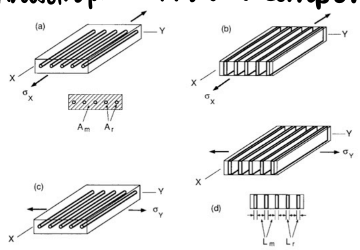
$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] + \alpha(\Delta T) \\ \varepsilon_y &= \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] + \alpha(\Delta T) \\ \varepsilon_z &= \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right] + \underbrace{\alpha(\Delta T)}_{\text{thermal term}} \end{aligned}\right\} \text{Generalized Hooke's Law} \left\{\begin{aligned} \gamma_{xy} &= \frac{\tau_{xy}}{G} \\ \gamma_{yz} &= \frac{\tau_{yz}}{G} \\ \gamma_{zx} &= \frac{\tau_{zx}}{G} \end{aligned}\right.$$

*only 2 independent elastic constants are needed for an isotropic material, so that one of E, G, and ν can be considered redundant

$$G = \frac{E}{2(1+\nu)}$$

Bulk Modulus $\quad B = \frac{\sigma_h}{\varepsilon_v} = \frac{E}{3(1-2\nu)}$

Anisotropic – Fibrous Composites – E parallel to Fibers



$$A = A_r + A_m \qquad E_X = \frac{E_r A_r + E_m A_m}{A}$$

$$V_r = \frac{A_r}{A} \qquad V_m = 1 - V_r = \frac{A_m}{A}$$

$$\boxed{E_X = V_r E_r + V_m E_m}$$

Anisotropic – Fibrous Composites – E transverse to Fibers p.166

$$\boxed{E_Y = \frac{E_r E_m}{V_r E_m + V_m E_r}}$$

Other elastic constants

major Poisson's Ratio: $\nu_{xy} = V_r \nu_r + V_m \nu_m$

shear modulus estimate: $G_{xy} = \dfrac{G_r G_m}{V_r G_m + V_m G_r}$

Plane Stress $\sigma_z = \tau_{yz} = \tau_{zx} = \emptyset$

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_n = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$ ... coordinate axes rotations for maximum and minimum values of σ

$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$ ... coordinate axes rotation for maximum shear stress

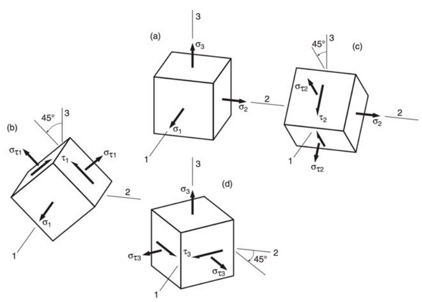
$$\tau_3 = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$ ... principal shear stress, aka maximum shear stress in x,y-plane

↑ the 2 orthogonal planes where this shear stress occurs are found to have the same normal stress of

$$\sigma_{\tau_3} = \frac{\sigma_x + \sigma_y}{2}$$

τ_3 and the accompanying normal stress can be expressed in terms of the principal normal stresses σ_1 and σ_2:

$$\tau_3 = \frac{|\sigma_1 - \sigma_2|}{2} \qquad \sigma_{\tau_3} = \frac{\sigma_1 + \sigma_2}{2}$$



(a) prin. normal stresses + prin. axes

(b) prin. shear stresses, rotation of cube 45° about axis of σ_1 gives plane where τ_1 acts

(c) similar rotation: σ_2, τ_2

(d) similar rotation: σ_3, τ_3

the principal shear stresses are τ_1, τ_2, τ_3, each accompanied by normal stresses that are the same on the two shear planes $\sigma_{\tau 1}, \sigma_{\tau 2}, \sigma_{\tau 3}$

$$\tau_1 = \frac{|\sigma_2 - \sigma_3|}{2} = \frac{|\sigma_2|}{2} \qquad \tau_2 = \frac{|\sigma_1 - \sigma_3|}{2} = \frac{|\sigma_1|}{2} \qquad \tau_3 = \frac{|\sigma_1 - \sigma_2|}{2} = \frac{|\sigma_1 - \sigma_2|}{2}$$

$$\sigma_{\tau 1} = \frac{\sigma_2 + \sigma_3}{2} \qquad \sigma_{\tau 2} = \frac{\sigma_1 + \sigma_3}{2} \qquad \sigma_{\tau 3} = \frac{\sigma_1 + \sigma_2}{2}$$

when $\sigma_z = \tau_{yz} = \tau_{zx} = 0$ then $\sigma_3 = 0$

$$\tau_{max} = \text{MAX}(\tau_1, \tau_2, \tau_3)$$

σ_n = octahedral normal stress aka hydrostatic stress

$$\sigma_n = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \Rightarrow \sigma_n = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

τ_n = octahedral shear stress

$$\tau_n = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \Rightarrow \tau_n = \frac{1}{3}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

principal normal and shear strains occur in similar manner as for stresses

→ plane strain: $\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0$

$$\tan 2\theta_n = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} \qquad \varepsilon_1, \varepsilon_2 = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\tan 2\theta_1 = -\frac{\varepsilon_x - \varepsilon_y}{\gamma_{xy}} \qquad \gamma_3 = \sqrt{(\varepsilon_x - \varepsilon_y)^2 + (\gamma_{xy})^2} \qquad \varepsilon_{\gamma 3} = \frac{\varepsilon_x + \varepsilon_y}{2}$$

3D states of strain:

$$\begin{vmatrix} (\varepsilon_x - \varepsilon) & \frac{\gamma_{xy}}{2} & \frac{\gamma_{zx}}{2} \\ \frac{\gamma_{xy}}{2} & (\varepsilon_y - \varepsilon) & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{yz}}{2} & (\varepsilon_z - \varepsilon) \end{vmatrix} = 0$$

$$\gamma_1 = |\varepsilon_2 - \varepsilon_3|, \quad \gamma_2 = |\varepsilon_1 - \varepsilon_3|, \quad \gamma_3 = |\varepsilon_1 - \varepsilon_2|$$

For cases of plane stress, $\sigma_z = \tau_{yz} = \tau_{zx} = 0$, the Poisson effect results in normal strains ε_z occuring in the out-of-plane direction, so that the state of strain is 3D

→ if isotropic, no shear γ_{yz} or γ_{zx} occur then $\varepsilon_z = \varepsilon_3$

ε_z can be obtained from Hooke's Law applying $\sigma_z = 0$:

$$\sigma_x + \sigma_y = \frac{E}{1-\nu}(\varepsilon_x + \varepsilon_y)$$

$$\varepsilon_z = \frac{-\nu}{1-\nu}(\varepsilon_x + \varepsilon_y)$$

Failure Criteria

general form: $f(\sigma_1, \sigma_2, \sigma_3) = \sigma_c$ @ failure

↑ either yield strength σ_o or ultimate strength σ_{ut} or σ_{uc} depending on whether yield or fracture of interest

$f(\sigma_1, \sigma_2, \sigma_3) = \bar{\sigma}$

↑ effective stress

$\bar{\sigma} = \sigma_c$ @ failure

$\bar{\sigma} < \sigma_c$ no failure

safety factor

$$X = \frac{\sigma_c}{\bar{\sigma}}$$

max normal stress fracture criterion → brittle materials (graphically rep. by square)

$$\bar{\sigma}_N = \mathrm{MAX}(|\sigma_1|, |\sigma_2|, |\sigma_3|)$$

fracture expected when $\bar{\sigma}_N = \sigma_u$ (either tensile or compressive)

$$X = \frac{\sigma_u}{\bar{\sigma}_N}$$

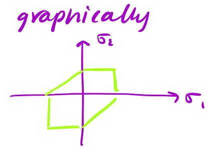
max shear stress yield criterion → ductile materials

in uniaxial: $\sigma_1 = \sigma_o \quad \sigma_2 = \sigma_3 = 0 \quad \tau_o = \sigma_o/2$

$\therefore \sigma_o = \mathrm{MAX}(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|)$ @ yielding

$$\bar{\sigma}_S = \mathrm{MAX}(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|) \qquad X = \frac{\sigma_o}{\bar{\sigma}_S}$$

graphically



octahedral shear stress yield criterion, von Mises, distortion energy criterion

*predicts yielding occurs when shear stress on octahedral planes reaches crit. value

$$\bar{\sigma}_H = \frac{1}{\sqrt{2}}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)}$$

graphically rep. by an ellipse