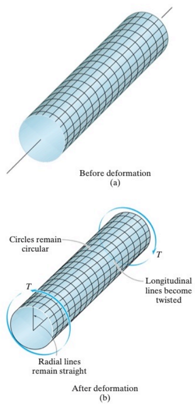
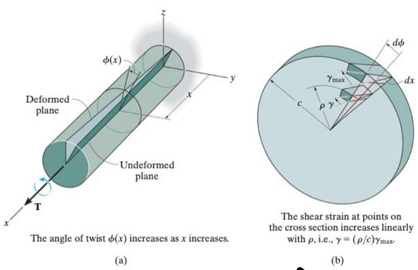
Materials — TORSION

Torsional Deformation of a circular shaft

Torque – a moment that tends to twist a member about its longitudinal axis



- when the torque is applied, the longitudinal grid lines tend to distort into a helix that intersects the circles at equal angles
- all of the cross-sections of the shaft will remain flat, so they do not wrap or bulge in or out, and radial lines remain straight and rotate during this deformation
- provided the angle of twist is small, then the length of the shaft and its radius will remain prac. unchanged



The angle of twist $\phi(x)$ increases as x increases.
(a)

The shear strain at points on the cross section increases linearly with ρ, i.e., $\gamma = (\rho/c)\gamma_{max}$.
(b)

small disc element located at x from the end of the shaft

- if shaft is fixed at one end and a torque is applied to its other end, then the dark green shaded plane will distort into a skewed form as shown

Angle of twist — the radial line located on the cross-section at a distance x from the fixed end of the shaft will rotate through an angle $\phi(x)$
→ $\phi(x)$ depends on the position x and will vary along the shaft as shown

⇓

due to deformation, front and rear faces of element will undergo rotation – the back face by $\phi(x)$ and the front face by $\phi(x) + dx$
↳ as a result, the <u>difference</u> in these rotations, $d\phi$, causes the element to be subjected to a shear strain γ.

↳ this angle, or shear strain, can be related to the angle $d\phi$ by noting that the length of the red arc is

$$\rho\, d\phi = dx\,\gamma \quad \text{or} \quad \gamma = \rho \frac{d\phi}{dx} \quad (1)$$

and since dx and $d\phi$ are the same for all elements, then $\frac{d\phi}{dx}$ is constant over the cross section

(1) states that the magnitude of the shear strain varies only with its radial distance ρ from the axis of the shaft.

since $\frac{d\phi}{dx} = \frac{\gamma}{\rho} = \frac{\gamma_{max}}{c}$ then $\gamma = \left(\frac{\rho}{c}\right)\gamma_{max}$ (2)

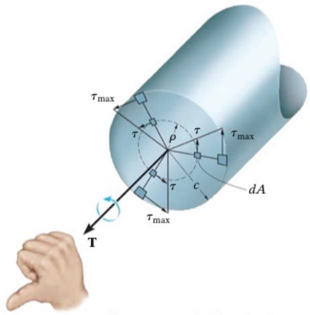
∴ the shear strain within the shaft <u>varies linearly</u> along any radial line, from $\emptyset$ at an axis of the shaft to a maximum γ_{max} at its outer boundary

The Torsion Formula

when an external torque is applied to a shaft, it creates a corresponding internal torque within the shaft

If the material is linear elastic, then Hooke's law applies, $\tau = G\gamma$, or $\tau_{max} = G\gamma_{max}$, and consequently a linear variation in shear strain leads to a corresponding linear variation in shear stress along any radial line

↳ so τ will vary from zero at the shaft's longitudinal axis to a maximum value, τ_{max}, at its outer surface.



Shear stress varies linearly along each radial line of the cross section.

$$\tau = \left(\frac{\rho}{c}\right)\tau_{max} \quad (3)$$

since each element of area dA, located at ρ, is subjected to a force of $dF = \tau dA$, then the torque produced by this force is then $dT = \rho(\tau dA)$, and for the entire cross-section we have

$$T = \int_A \rho(\tau dA) = \int_A \rho\left(\frac{\rho}{c}\right)\tau_{max}\, dA \quad (4)$$

and since τ_{max}/c is a constant, then

$$T = \frac{\tau_{max}}{c}\int_A \rho^2 dA \quad (5)$$

the integral represents the polar moment of inertia of the shaft's cross-sectional area about the shaft's longitudinal axis, J

$$\boxed{\tau_{max} = \frac{Tc}{J}} \quad (6)$$

τ_{max} = maximum shear stress in the shaft, which occurs at its outer surface

T = resultant internal torque acting at the cross section ... its value is determined from the method of sections and the equation of moment equilibrium applied about the shaft's longitudinal axis

J = polar moment of inertia of the cross-sectional area

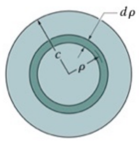
c = outer radius of the shaft

substituting (3) into (6) we can find the shear stress at the intermediate distance ρ on the cross section

$$\boxed{\tau = \frac{T\rho}{J}} \quad (7)$$

(6) & (7) referred to as the torsion formula [used only if the shaft has a circular cross-section, material is homogeneous, and behaves in linear elastic manner (since derived from Hooke's Law)]

POLAR MOMENT OF INERTIA



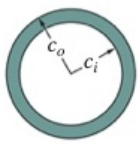
• if the shaft has a solid circular cross-section, the polar moment of inertia J can be determined using an area element in the form of a differential ring or annulus having a thickness dρ and circumference $2\pi\rho$. for this ring, $dA = 2\pi\rho\, d\rho$ so

$$J = \int_A \rho^2 dA = \int_0^c \rho^2 (2\pi\rho\, d\rho)$$

$$= 2\pi \int_0^c \rho^3 d\rho = 2\pi \left(\frac{1}{4}\right) \rho^4 \Big|_0^c \implies \boxed{J = \frac{\pi}{2} c^4} \quad (8)$$

a solid section

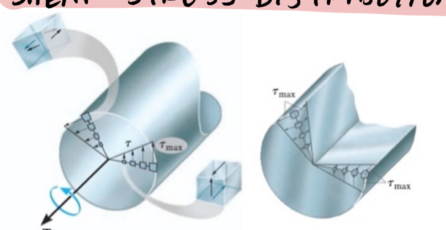
* note that J is always ⊕, common units are mm^4 or in^4



• if shaft has a tubular cross section with inner radius c_i and outer radius c_o, then from (8) can get its polar moment of inertia by subtracting J for a shaft of radius c_i from that determined for a shaft of radius c_o

$$\boxed{J = \frac{\pi}{2}(c_o^4 - c_i^4)} \quad (9)$$

SHEAR STRESS DISTRIBUTION



(a)

Shear stress varies linearly along each radial line of the cross section.

(b)

• if an element of material on the cross section of the shaft or tube is isolated, then due to the complementary property of shear, equal shear stresses must also act on four of its adjacent faces

∴ the internal torque T develops a linear distribution of shear stress along each radial line in the plane of the cross-sectional area, and also an associated shear-stress distribution is developed along an axial plane

* note : b/c of this axial distribution of shear stress, shafts made of wood tend to split along the axial plane when subjected to excessive torque

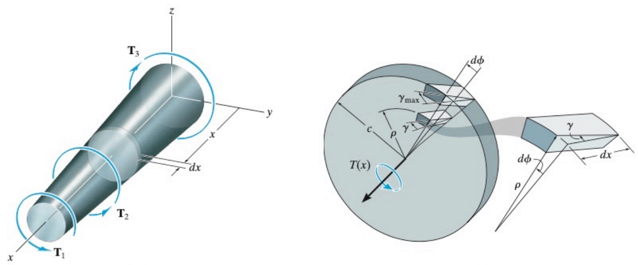
↳ bc wood is <u>anisotropic</u> so its shear resistance parallel to its grains or fibers, directed along the axis of the shaft, is much less than its resistance perpendicular to the fibers within the plane of the cross section

IMPORTANT POINTS

- when a shaft having circular cross section is subjected to a torque, the x-section remains plane while radial lines rotate → causing shear strain w/in the material that varies linearly along any radial line, from ∅ at the axis of the shaft to a maximum at its outer boundary
- for linear elastic homogeneous material, the shear stress along any radial line of the shaft also varies linearly, from ∅ at its axis to a maximum at its outer boundary → the max shear stress must <u>not</u> exceed the proportional limit

Angle of Twist

- assume shaft has circular x-section that can gradually vary along its length (a)
- material is homogeneous, behaves linear elastically when torque applied



(a) (b)

- a differential disc of thickness dx located at position x is isolated from shaft
- here, the internal torque is T(x), since the external loading may cause it to change along the shaft

→ due to T(x), the disc will twist such that the relative rotation of one of its faces wrt the other face is $d\phi$
→ then an element of material located at an arbitrary radius ρ within the disc will undergo a shear strain γ

$$d\phi = \gamma \frac{dx}{\rho} \quad (10)$$

Since Hooke's Law: $\gamma = \tau / G$, applies and the shear stress can be expressed in terms of the applied torque using the torsion formula $\tau = T(x)\rho / J(x)$ then $\gamma = T(x)\rho / J(x)G(x)$

$$d\phi = \frac{T(x)}{J(x)\,G(x)}\,dx$$

and integrating over entire length L of the shaft

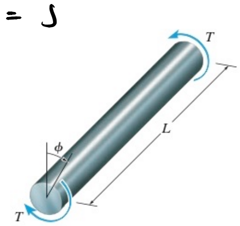
$$\boxed{\phi = \int_0^L \frac{T(x)\,dx}{J(x)\,G(x)}} \quad (11)$$

ϕ = angle of twist of one end of shaft wrt other end, measured in radians
T(x) = internal torque at arbitrary position x, found from method of sections and equation of moment equilibrium applied about the shaft's axis
J(x) = shaft's polar moment of inertia expressed as function of x
G(x) = shear modulus of elasticity for the material expressed as function of x

CONSTANT TORQUE AND CROSS SECTIONAL AREA

- when the material is homogeneous so that G = C, x-sectional area and external torque are C along length of shaft
  → internal torque T(x) = T
  → polar moment of inertia J(x) = J

$$\boxed{\phi = \frac{TL}{JG}} \quad (12)$$



Equation (12) often used to determine shear modulus of elasticity G of a material.

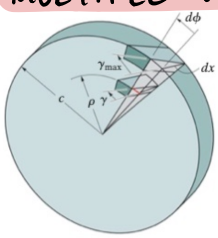
→ need to know length & diameter

→ applied torque T and angle of twist ϕ then measured along length L

→ need several tests to get a reliable value of G

$$G = \frac{TL}{J\phi}$$

MULTIPLE TORQUES



The shear strain at points on the cross section increases linearly with ρ, i.e., $\gamma = (\rho/c)\gamma_{max}$.

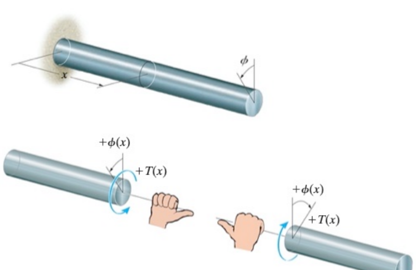
- if shaft subjected to several different torques, or cross sectional area or shear modulus changes abruptly from one region of shaft to next then (12) should be applied to each segment of the shaft where these quantities are all constant
- angle of twist of one end of shaft wrt other is found from algebraic addition of angles of twist of each segment

$$\boxed{\phi = \sum \frac{TL}{JG}} \quad (13)$$

SIGN CONVENTION

- best way to apply (13) is to use sign convention for both internal torque and angle of twist of one end of the shaft wrt to other end

RHR → both torque and angle (+) provided thumb is directed outward from shaft while fingers curl in direction of torque



Positive sign convention for T and ϕ.

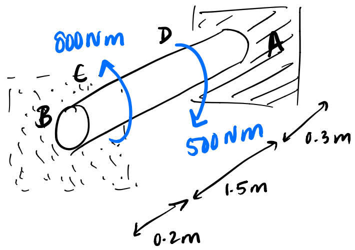
IMPORTANT POINT

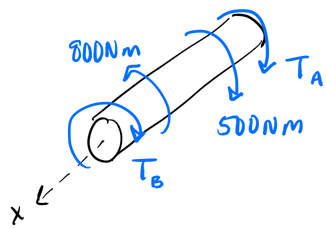
- when applying (11) to determine ϕ, it's important that the applied torques do not cause yielding of the material, and that the material is homogeneous and behaves in a linear elastic manner.

Statically Indeterminant Torque-Loaded Members

- need compatability condition
  → angle of twist of one end of the shaft wrt the other end to be equal to zero, $\phi_{A/B} = 0$

ex. solid steel shaft has diameter 20mm. If it is subjected to the two torques, determine the reactions at the fixed supports A & B.





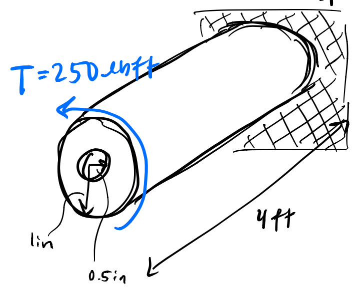
$\Sigma M_x = 0; \quad -T_B + 800Nm - 500Nm - T_A = 0 \quad (1)$

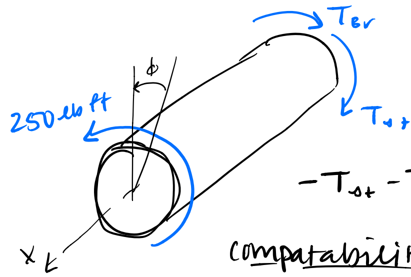
compatability condition: $\phi_{A/B} = 0$

load displacement, $\phi = TL/JG$

$$-\frac{T_B(0.2m)}{JG} + \frac{(800 - T_B)(1.5m)}{JG} + \frac{(300 - T_B)(0.3m)}{JG} = 0$$

$\boxed{T_B = 645Nm \quad T_A = -345Nm}$ (so T_A acts in opp. of dir shown) ex. The shaft is made from a steel tube, which is bonded to a brass core. If a torque of $T = 250\ lb\cdot ft$ is applied at its end, plot the shear-stress distribution along a radial line on its cross-section. $G_{st} = 11.4\,(10^3)$ ksi $\quad G_{Br} = 5.20\,(10^3)$ ksi





$$-T_{st} - T_{Br} + (250\ lb\,ft)(12\ in/ft) = 0 \quad \underline{(1)}$$

compatability : $\phi = \phi_{st} = \phi_{Br}$

load displacement: $\phi = \dfrac{TL}{JG}$

$$\frac{T_{st}\,L}{\frac{\pi}{2}\left[(1in)^4 - (0.5in)^4\right] 11.4(10^3)\frac{kip}{in^2}} = \frac{T_{Br}\,L}{\frac{\pi}{2}(0.5in)^4\, 5.20(10^3)\,kip/in^2} \Rightarrow T_{st} = 32.88\,T_{Br} \quad \underline{(2)}$$

$\underline{(1)}$ & $\underline{(2)}$ $\rightarrow$ $T_{st} = \underset{2911.5\ lb\,in}{246.6\ lb\,ft}$ & $T_{Br} = \underset{88.5\ lb\,in}{7.38\ lb\,ft}$

torsion formula

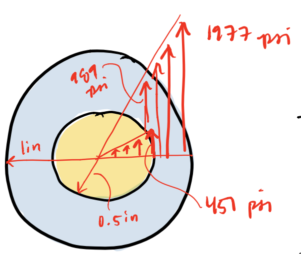
- shear stress in brass core varies from zero at its center to a max at the interface where it contacts the steel tube

$$(\tau_{Br})_{max} = \frac{(88.5\ lb\,in)(0.5\,in)}{\pi/2\,(0.5\,in)^4} = 451\ psi$$

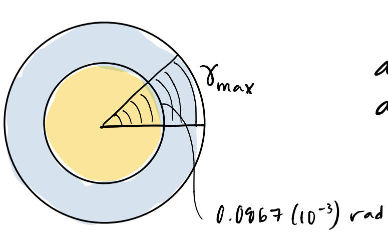
- for the steel, minimum and maximum shear stresses are

$$(\tau_{st})_{min} = \frac{(2911.5\ lb\,in)(0.5\,in)}{(\pi/2)\left[(1in)^4 - (0.5in)^4\right]} = 989\ psi$$

$$(\tau_{st})_{max} = \frac{(2911.5\ lb\,in)(1\,in)}{(\pi/2)\left[(1in)^4 - (0.5in)^4\right]} = 1977\ psi$$



← note the discontinuity of shear stress at the brass and steel interface (which is expected since materials have different moduli of rigidity; ie steel is stiffer than brass ($G_{st} > G_{Br}$) and thus it carries more shear stress at the interface

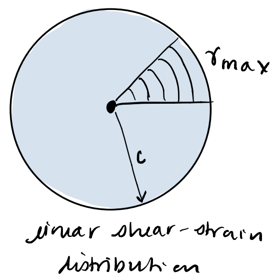


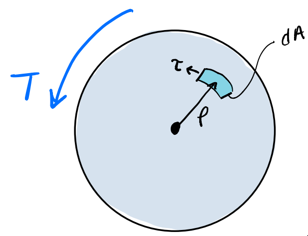
although the shear stress is discontinuous at the interface, the shear strain is not. Rather, it is the same on either side of brass-steel interface

Inelastic Torsion

we know so far that...

- regardless of material behavior, the shear strains that develop in a circular shaft will vary <u>linearly</u>, from zero at the center to a maximum at its outer boundary

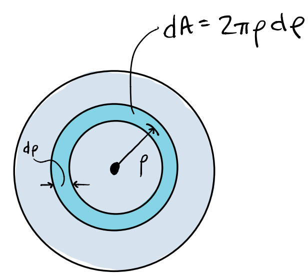




since the shear stress τ acting on an element of area dA produces a force $dF = \tau dA$, then the torque about the axis of the shaft is $dT = \rho\, dF = \rho(\tau dA)$

for the entire shaft we require: $T = \int_A \rho \tau \, dA$

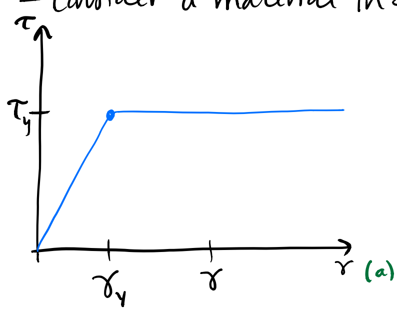
→ if the area dA over which τ acts is defined as a differential ring having an area of $dA = 2\pi\rho\, d\rho$ then the above equation can be written as:



$$\boxed{T = 2\pi \int_0^c \tau \rho^2 \, d\rho} \qquad (14)$$

Elastic - Plastic Torque

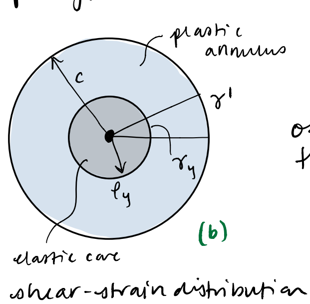
- consider a material in shaft to exhibit an elastic perfectly plastic behavior



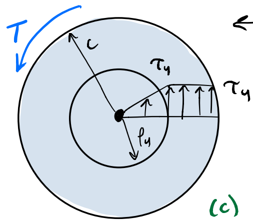
if the internal torque produces the maximum <u>elastic</u> shear strain γ_y at the outer boundary of the shaft, then the maximum elastic torque T_y that produces this strain can be found from the torsion formula $\tau_y = T_y c / [(\pi/2) c^4]$ so that

$$T_y = \frac{\pi}{2} \tau_y c^3$$

- if the applied torque ↑ in magnitude above T_y, it will begin to cause yielding, which will start at the outer boundary of the shaft, $\rho = c$.
- as the maximum shear strain increases to say γ', then if the material is elastic perfectly plastic the yielding boundary will progress inward toward the shaft's center (below, left)



← this produces an <u>elastic core</u> where by proportion the radius of the core is $\rho_y = (\gamma_y/\gamma')c$. The outer portion of the material forms a plastic annulus (or ring) since the shear strains γ within this region are greater than γ_y



(c)

shear stress distribution

← shear-stress distribution along a radial line of the shaft. This is established by taking successive points on the shear-strain distribution in the above figure and finding the corresponding value of shear stress from the $\tau - \gamma$ diagram. For example, at $\rho = c$, γ' gives τ_y at at $\rho = \rho_y$ γ_y also gives τ_y, etc.

since τ can now be expressed as function of ρ, we can determine the torque as follows

$$T = 2\pi \int_0^c \tau \rho^2 d\rho$$

$$= 2\pi \int_0^{\rho_y} \left(\tau_y \frac{\rho}{\rho_y}\right) \rho^2 d\rho + 2\pi \int_{\rho_y}^{c} \rho^2 d\rho$$

$$= \frac{2\pi}{\rho_y} \tau_y \int_0^{\rho_y} \rho^3 d\rho + 2\pi \tau_y \int_{\rho_y}^{c} \rho^2 d\rho$$

$$= \frac{\pi}{2\rho_y} \tau_y \rho_y^4 + \frac{2\pi}{3} \tau_y (c^3 - \rho_y^3) \Rightarrow T = \frac{\pi \tau_y}{6}(4c^3 - \rho_y^3)$$

if the applied torque causes the material to exceed the elastic limit, then the stress distribution won't be proportional to the radial distance from the center line of the shaft. instead, the internal torque is related to the stress distrib. using the shear stress-shear strain diagram + equilibrium

Plastic Torque

– further increases in T tend to shrink the radius of the elastic core until all the material yields i.e. $\rho_y \to 0$ (b)

– the material of the shaft will then be subjected to perfectly plastic behavior and the shear-stress distribution becomes uniform so that $\tau = \tau_y$

– we can apply equation (14) to determine the plastic torque T_p, which represents the largest possible torque the shaft will support

$$T_p = 2\pi \int_0^c \tau_y \rho^2 d\rho$$

$$= \frac{2\pi}{3} \tau_y c^3$$

... compared w/ the maximum elastic torque T_y it can be seen that $T_p = \frac{4}{3} T_y$

(d)

fully plastic torque

← essentially, this means that the plastic torque is 33% greater than the maximum elastic torque

... unfortunately, the angle of twist ϕ for the shear-stress distribution in (d) cannot be uniquely defined. This is b/c $\tau = \tau_y$ does not correspond to any unique value of shear strain $\gamma \geq \gamma_y$.

→ as a result, once T_p is applied, the shaft will continue to deform or twist with no corresponding increase in shear stress.