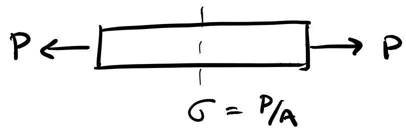
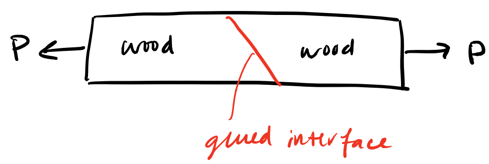
Materials – Lecture #22

4/21/20

Hibbeler Ch. 9: Stress Transformations



2 pieces of wood glued together



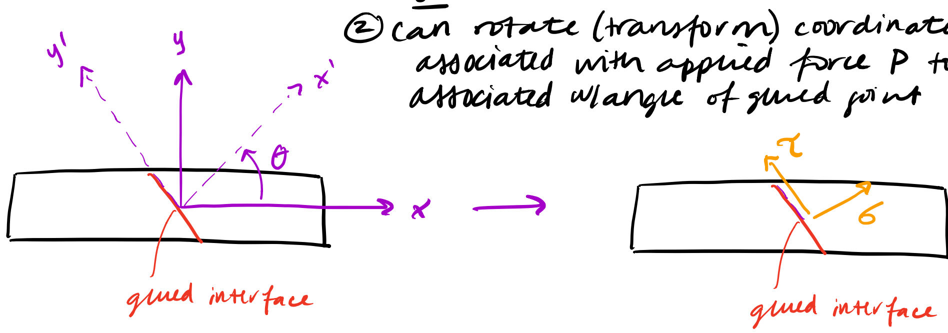
Questions to consider:

① can the glued joint sustain stress that load produces?

→ can determine normal/shear stresses normal to and parallel to joint (done previously)

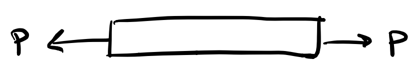
or:

② can rotate (transform) coordinates associated with applied force P to direction associated w/ angle of glued joint



→ can determine maximum or minimum normal + shear stresses, as well as orientations of those stresses

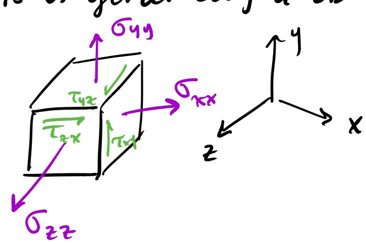
IDEA:



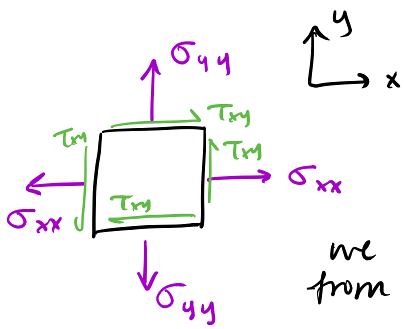
→ loading applied axially

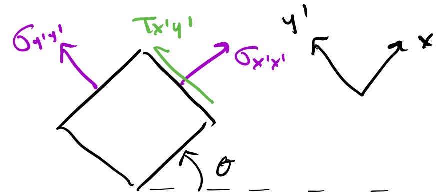
→ from this loading, can determine max/min normal/shear stresses, and their orientations, using stress transformations

– stress is generally a 3D state



But engineers often assume that stress produced can be analyzed in a single (2D) plane – "plane stress"



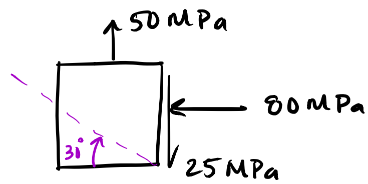


we will determine how to transform stresses from one orientation to another

IMPORTANT: both drawings above represent the <u>same</u> state of stress

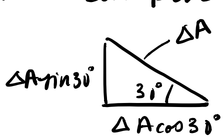
- transforming stress similar to transforming forces: orientation of area along w/ magnitude + direction

Ex.

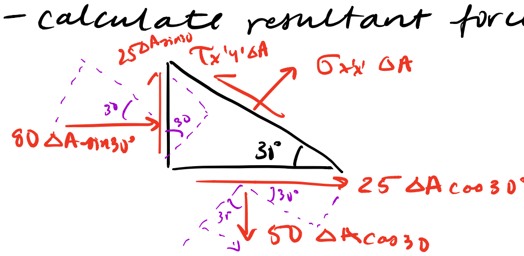


Find: state of stress if element rotated 30° clockwise

- Area components



- calculate resultant forces



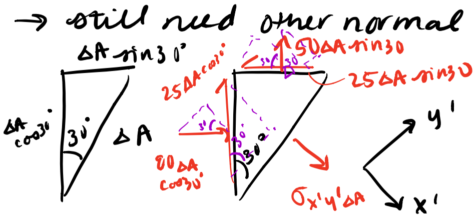
(+)↗ $\Sigma F_{x'} = 0 = \sigma_{x'x'}\Delta A - (50\Delta A\cos 30°)(\cos 30°) + (25\Delta A\cos 30°)\sin 30° + (80\Delta A\sin 30°)\sin 30° + (25\Delta A\sin 30°)\cos 30°$

$\sigma_{x'x'} = -4.15\text{ MPa}$ ✓

(+)↖ $\Sigma F_{y'} = 0 = \tau_{x'y'}\Delta A - (50\Delta A\cos 30°)\sin 30° - (25\Delta A\cos 30°)\cos 30 - (80\Delta A\sin 30°)\cos 30° + (25\Delta A\sin 30°)\sin 30°$

$\tau_{x'y'} = 68.8\text{ MPa}$ ✓

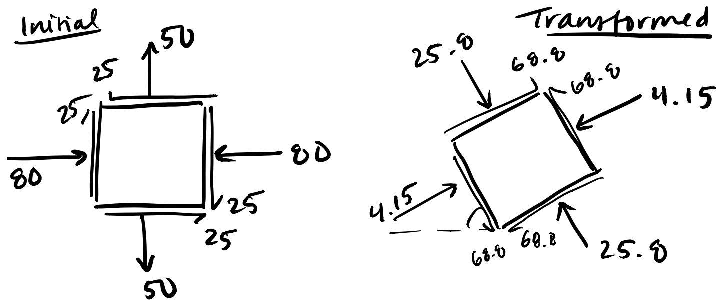
→ still need other normal stress



(+)↘ $\Sigma F_{x'} = 0 = \sigma_{x'x'}\Delta A - (25\Delta A\cos 30)\sin 30 + (80\Delta A\cos 30°)\cos 30° + (25\Delta A\sin 30°)\cos 30° - (50\Delta A\sin 30°)\sin 30°$

$\sigma_{x'x'} = -25.8\text{ MPA}$

→ stress xformation is complete (2 normal stresses, 1 shear stress in 2D)



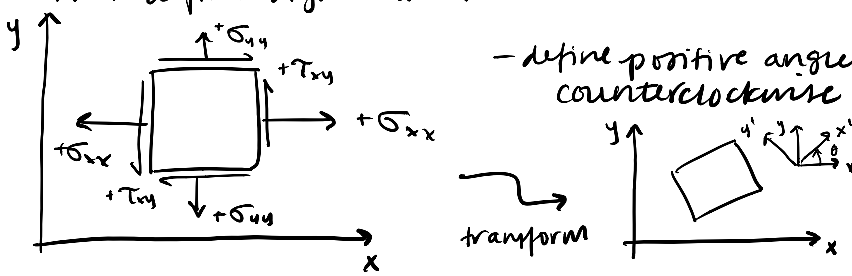
*note that don't need to ever calculate ΔA bc ΔA will cancel

these are equivalent! represent same state of stress. One just rotated 30° clockwise with respect to the other

→ THAT WAS PAINFUL

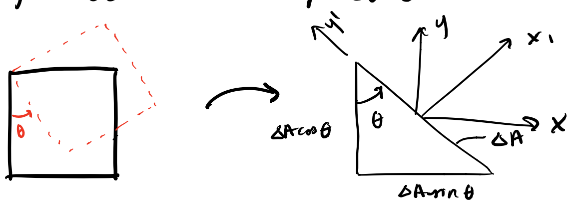
Try to do this in more efficient way ...

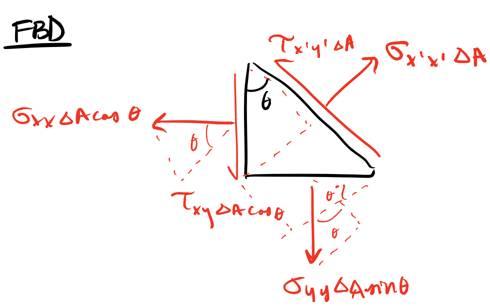
Now find general equations of plane stress transformation
– First define sign convention:



– define positive angles of rotation as counterclockwise

→ goal: find stress for this new rotated/transformed x', y' – coordinate system





$$\overset{+}{\nearrow} \Sigma F_{x'} = 0 = \sigma_{x'x'}\Delta A - (\tau_{xy}\Delta A \sin\theta)\cos\theta - (\sigma_{yy}\Delta A\sin\theta)\sin\theta - (\tau_{xy}\Delta A\cos\theta)\sin\theta - (\sigma_{xx}\Delta A\cos\theta)\cos\theta$$

$$\boxed{\sigma_{x'x'} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + \tau_{xy}(2\sin\theta\cos\theta)}$$

$\uparrow^{+} \Sigma F_{y'} = 0 = \tau_{x'y'}\Delta A + (\tau_{xy}\Delta A \sin\theta)\sin\theta - (\sigma_{yy}\Delta A \sin\theta)\cos\theta$
$- (\tau_{xy}\Delta A \cos\theta)\cos\theta + (\sigma_{xx}\Delta A \cos\theta)\sin\theta$

$$\boxed{\tau_{x'y'} = (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)}$$

Trig Identities:

$\sin 2\theta = 2\sin\theta\cos\theta \qquad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta) \qquad \sin^2\theta = 1 - \frac{1}{2}(1+\cos 2\theta)$

$\rightarrow$ SO,

$$\sigma_{x'x'} = \sigma_{xx}\left(\frac{1}{2}(1+\cos 2\theta)\right) + \sigma_{yy}\left(1 - \frac{1}{2}(1+\cos 2\theta)\right) + \tau_{xy}\sin 2\theta$$

$$* \ \boxed{\sigma_{x'x'} = \frac{1}{2}(\sigma_{xx} + \sigma_{yy}) + \frac{\cos 2\theta}{2}(\sigma_{xx} - \sigma_{yy}) + \tau_{xy}\sin 2\theta}$$

$$* \ \boxed{\tau_{x'y'} = -\frac{1}{2}\sin 2\theta(\sigma_{xx} - \sigma_{yy}) + \tau_{xy}\cos 2\theta}$$

* general expressions *

$$\sigma_{y'y'} = \sigma_{x'x'}(\theta + 90^\circ)$$

$$* \ \boxed{\sigma_{y'y'} = \left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right) - \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta}$$

Stress Transformation $\rightarrow$ can do using equations above

$\rightarrow$ can also do using matrix multiplication

$$\begin{bmatrix} \sigma_{x'x'} & \tau_{x'y'} \\ \tau_{x'y'} & \sigma_{y'y'} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$Q = \text{rotation matrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\boxed{[\sigma'] = Q\sigma Q'}$$

$\{$ see his notes since no recording here $\}$

$$Q = \text{rotation matrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\boxed{[\sigma'] = Q \cdot \sigma \cdot Q^T}$$

- Principal Stresses / Maximum In-plane shear stress

→ Engineer: find orientation that causes normal / shear stresses to be maximum or minimum.

Because likely to cause failure

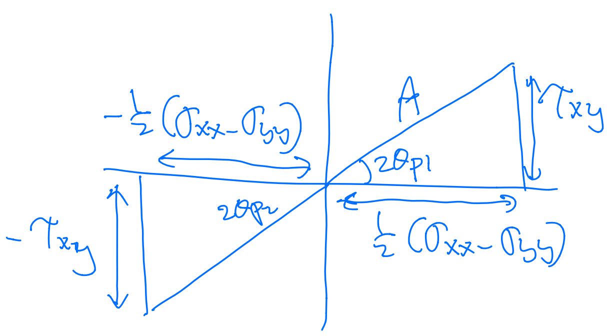
(1) $$\sigma_{x'x'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\frac{d\sigma_{x'x'}}{d\theta} = 0 = -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + 2\tau_{xy}\cos 2\theta$$

$$\boxed{\tan 2\theta_p = \frac{\tau_{xy}}{\frac{1}{2}(\sigma_{xx}-\sigma_{yy})}}$$

θ_p → angle that gives orientation of maximum or minimum normal stresses (principal stresses)

→ Solution has 2 roots, θ_{p1}, θ_{p2}



- $2\theta_{p1}$ and $2\theta_{p2}$ are 180° apart

$$A = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$\sin 2\theta_{p1} = \frac{\tau_{xy}}{A}$$

$$\cos 2\theta_{p1} = \frac{\frac{1}{2}(\sigma_{xx}-\sigma_{yy})}{A}$$

substitute into (1)

$$\sigma_{x'x'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2}{A} + \frac{\tau_{xy}^2}{A}$$

$$= \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}{A}$$

$$= \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{A^2}{A}$$

why? $\theta_{p1} \rightarrow +$

$\theta_{p2} \rightarrow -$

$$\sigma_{x'x'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

OR: $\sigma_{1,2} \rightarrow$ principal stresses $= \dfrac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\dfrac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$

→ Principal stresses represent maximum and minimum in-plane normal stresses at a point; $\sigma_1 > \sigma_2$

→ $\sigma_{1,2}$ = principal stresses, corresponding planes (angles) called "principal planes".

→ What are shear stresses on principal planes?

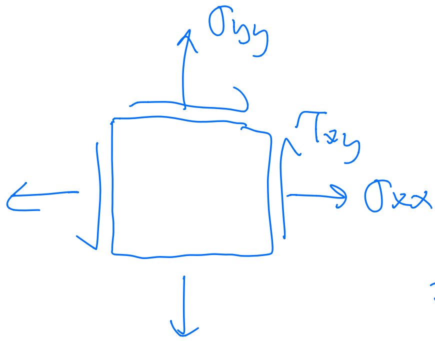
$$\tau_{x'y'} = -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

substitute in $\sin 2\theta_p$ and $\cos 2\theta_p$

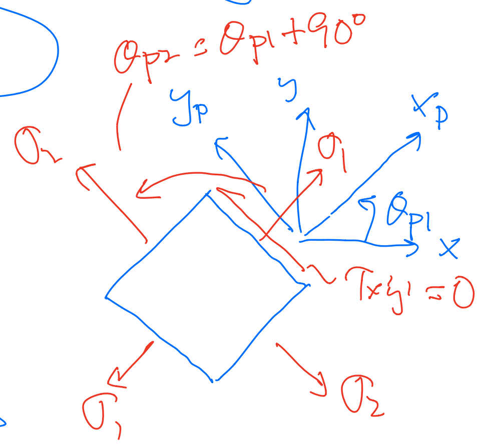
$$\sin 2\theta_{p1} = \frac{\tau_{xy}}{A}, \quad \cos 2\theta_{p1} = \frac{\frac{1}{2}(\sigma_{xx}-\sigma_{yy})}{A}$$

$$\tau_{x'y'} = -\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)\frac{\tau_{xy}}{A} + \frac{\tau_{xy}}{A}\left[\frac{1}{2}(\sigma_{xx}-\sigma_{yy})\right]$$

$$\boxed{\tau_{x'y'} = 0 \text{ on principal planes}}$$



Transform to principal planes



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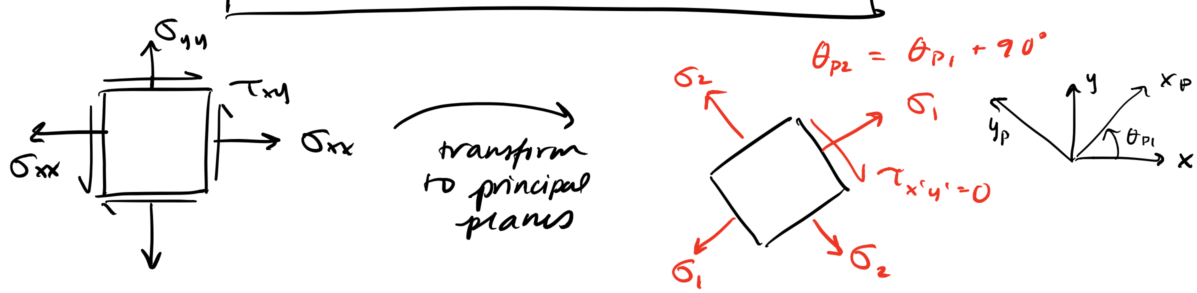

Maximum in plane shear stress

$$\tau_{x'y'} = -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\frac{d\tau_{x'y'}}{d\theta} = 0 = -(\sigma_{xx}-\sigma_{yy})\cos 2\theta - 2\tau_{xy}\sin 2\theta$$

$$\tan 2\theta_s = \frac{-\frac{1}{2}(\sigma_{xx}-\sigma_{yy})}{\tau_{xy}}$$

↑ maximum in plane shear

θ_s = angle where shear stress is a maximum

↳ this angle is negative reciprocal of the principal angles $\tan 2\theta_p$