Materials – Lecture #23

continue Ch. 9 : Stress transformations

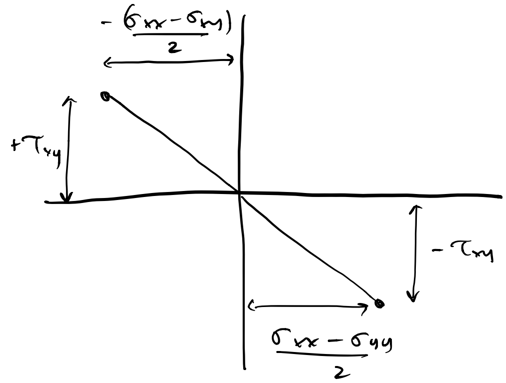
transformed normal stress: $\sigma_{x'x'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$

transformed shear stress: $\tau_{x'y'} = -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$

maximum in-plane shear stress

$$\frac{d\tau_{x'y'}}{d\theta} = 0 = -(\sigma_{xx}-\sigma_{yy})\cos 2\theta - 2\tau_{xy}\sin 2\theta$$

$$\tan 2\theta_s = \frac{-(\sigma_{xx}-\sigma_{yy})/2}{\tau_{xy}}$$ ⟸ angle θ_s at which $\tau_{x'y'}$ is a maximum



each root of $2\theta_s$ is 90° from $2\theta_p$, so this means θ_s and θ_p are 45° apart

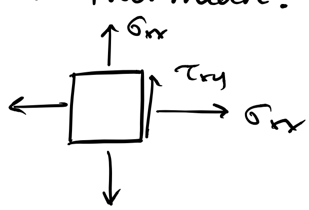
– take $\sin 2\theta_s$, $\cos 2\theta_s$, substitute into (2) for $\tau_{x'y'}$

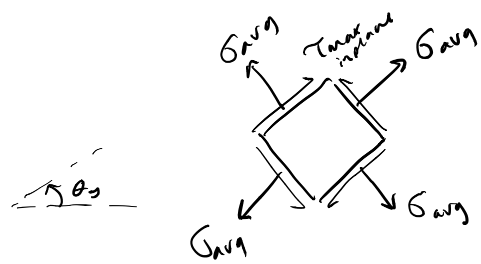
$$\tau^{max}_{inplane} = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$ ← value for largest shear stress

→ plug into (1) for $\sigma_{x'x'}$

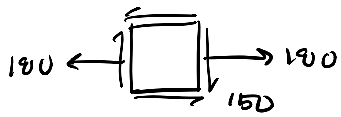
$$\sigma_{x'x'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} = \sigma_{avg}$$

What does this mean?





Example: $\sigma_{xx} = 180\,MPa$ $\quad \tau_{xy} = -150\,MPa$ (arrow on rt. hand face is down)
$\sigma_{yy} = 0\,MPa$



Find:
1. Principal stresses
2. $\tau_{\substack{max \\ inplane}}$
3. σ_{avg}

① Principal Stresses

$$\sigma_{1,2} = \frac{\sigma_{xx} + \sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

$$= \frac{180 + 0}{2} \pm \sqrt{\left(\frac{180 - 0}{2}\right)^2 + (-150)^2}$$

$$= 90 \pm \sqrt{90^2 + (-150)^2}$$

$\sigma_1 = 265\,MPa$ $\quad \sigma_2 = -85\,MPa$ ... these are the largest & smallest stresses we can have

now find orientation:

$$\tan 2\theta_p = \frac{\tau_{xy}}{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)} = \frac{(-150)}{\left(\frac{180 - 0}{2}\right)^2}$$

$\tan 2\theta_p = -1.67 \quad \rightarrow \quad \theta_{p_1} = -29.5°, \ \theta_{p2} = \theta_{p_1} + 90°$

$\theta_{p_1} = -29.5°$ $\quad \theta_{p_2} = 60.5°$

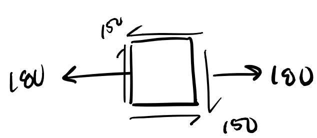
Which principal stress goes with which orientation?

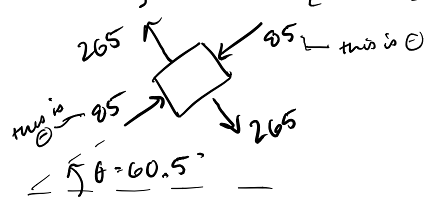
$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

substitute $\theta = 60.5°$

$$\sigma_{x'x'} = \frac{180 + 0}{2} + \frac{180 - 0}{2}\cos[2(60.5)] + (-150)\sin[(2)(60.5)]$$

$\sigma_{x'x'} = -85\,MPa$





$\tau_{x'y'} = 0$ on principal planes

② $\tau_{\substack{max \\ inplane}} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{180 - 0}{2}\right)^2 + (-150)^2} = 175\,MPa$

③ $\sigma_{avg} = \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{180 + 0}{2} = 90\,MPa$

orientation of $\tau^{max}_{in\,plane}$

$$\tan 2\theta_s = -\frac{(\sigma_{xx} - \sigma_{yy})/2}{\tau_{xy}} = -\frac{(180-0)/2}{-150} = 0.6$$

$$\theta_{s1} = 15.5^\circ, \quad \theta_{s2} = \theta_{s1} + 90^\circ = 105.5^\circ$$

Note: $\theta_{p1} = \theta_{s1} + 45^\circ = 60.5^\circ$

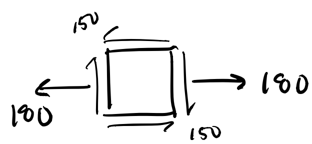
→ Formula for $\tau^{max}_{in\,plane} > 0$

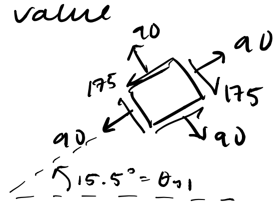
$$\tau_{x'y'} = -\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

↳ plug in $\theta = 15.5^\circ$ (bc this is angle we found)

$$\tau_{x'y'} = -90\sin 31^\circ + -150\cos 31^\circ = -175\text{ MPa}$$

so the maximum shear stress is actually a negative value





Mohr's Circle: Hibbeler 9.4

$$(1)\quad \sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{x'x'} - \frac{\sigma_{xx} + \sigma_{yy}}{2} = \frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \quad \underline{(1)}$$

$$\tau_{x'y'} = -\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta \quad \underline{(2)}$$

⇒ eliminate θ by squaring each equation and adding together

left hand side: $\left[\sigma_{x'x'} - \left(\frac{\sigma_{xx} + \sigma_{yy}}{2}\right)\right]^2 + \tau_{x'y'}^2$

right hand side: $\underline{(1)}\ \left[\frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right]\left[\frac{\sigma_{xx} - \sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta\right]$

$\underline{(2)}\ \left[-\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\right]\left[-\frac{\sigma_{xx} - \sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta\right]$

right hand side of $\underline{(1)} + \underline{(2)}$: $\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2(\sin^2 2\theta + \cos^2 2\theta) + \tau_{xy}^2(\sin^2 2\theta + \cos^2 2\theta)$

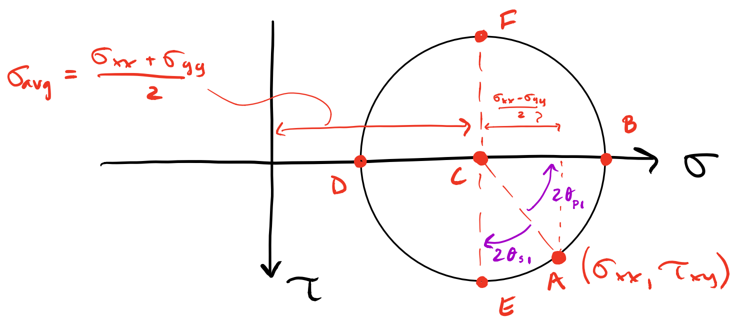
$$= \left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2$$

→ combine LHS & RHS

$$\left[\sigma_{x'x'} - \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right)\right]^2 + \tau_{x'y'}^2 = \left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2$$

or: $\left[\sigma_{x'x'} - \sigma_{avg}\right]^2 + \tau_{x'y'}^2 = R^2$, $\quad R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$

equation of circle w/ radius r



- σ is ⊕ to right
- τ is ⊕ downwards
- center of circle is at C: $(\sigma_{avg}, 0)$
- circle is called "Mohr's Circle"
- can use Mohr's circle to find the principal stresses & principal planes, avg normal stress, max in-plane shear stress, & stress on any arbitrary plane

B, D represent principal stresses σ_1, σ_2
E, F represent $\tau_{max\ in\ plane}$

$$\tan 2\theta_{p_1} = \frac{\tau_{xy}}{\frac{1}{2}(\sigma_{xx}-\sigma_{yy})}$$

$\sigma_1 = \sigma_{avg} + R$ (B) ← largest Principal Stress

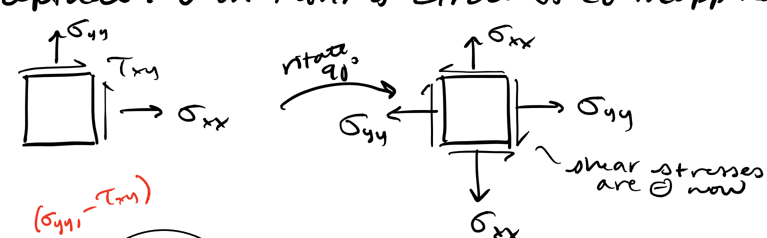
$\sigma_2 = \sigma_{avg} - R$ (D) ← smallest Principal Stress

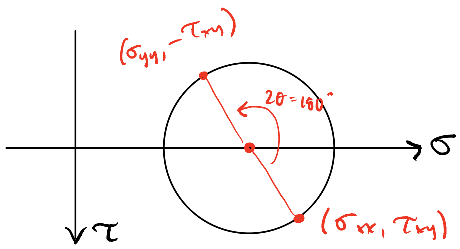
$\tau_{max\ in\ plane} = R$, $\quad R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$

→ on Mohr's circle:
rotation of angle θ on x' axis is going to be 2θ in the same direction on Mohr's circle

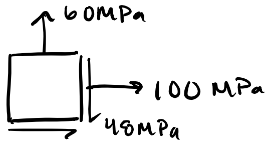
→ why is τ ⊕ downwards? Such that rotations of x' are in same direction on Mohr's circle

→ conceptual: θ on Mohr's circle is 2θ mapping





Mohr's Circle Example #1:



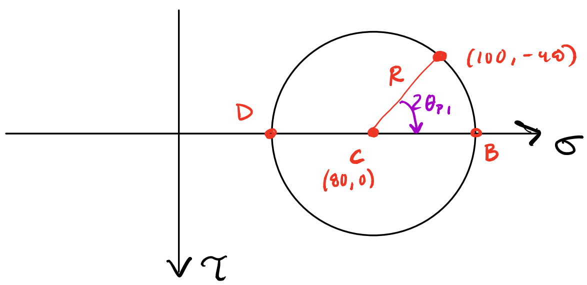
Find: ① Principal planes + principal stresses
② stress components via rotation 30° ccw

know: $\sigma_{xx} = 100$, $\sigma_{yy} = 60$, $\tau_{xy} = -48$

① construct Mohr's circle

→ $\sigma_{avg} = \dfrac{\sigma_{xx} + \sigma_{yy}}{2} = \dfrac{100+60}{2} = 80$

→ plot pt. $(\sigma_{xx}, \tau_{xy}) = (100, -48)$



$R = \sqrt{(100-80)^2 + (-48)^2}$

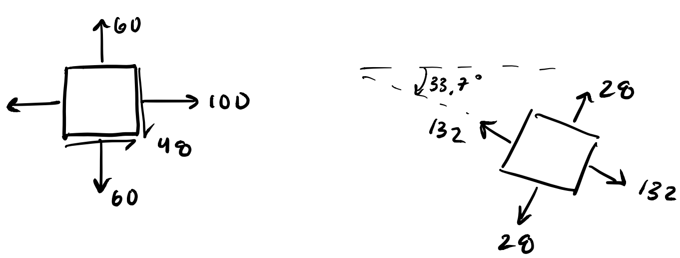
$R = 52$ MPa

$\sigma_1 = B = 80 + R = 132$ MPa (largest possible normal stress)

$\sigma_2 = D = 80 - R = 28$ MPa (smallest possible normal stress)

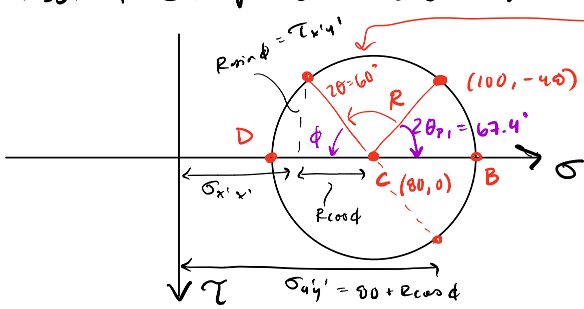
Rotate through $2\theta_{p_1}$ to get to B; at B, $\tau = 0$

$\tan 2\theta_{p_1} = \dfrac{48}{20} = 2.4$ $\quad 2\theta_{p_1} = 67.4° \rightarrow \theta_{p_1} = 33.7°$ cw



principal stresses, principal planes

→ stress components on element rotated 30° ccw



$\phi = 180 - 2\theta - 2\theta_{p_1} = 180° - 60° - 67.4°$

$\phi = 52.6°$

$\sigma_{x'x'} = 80 - R\cos\phi = 80 - 52\cos 52.6°$

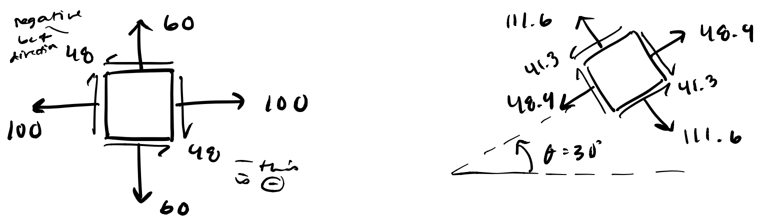
$\underline{\sigma_{x'x'} = 48.4 \text{ MPa}}$

$\tau_{x'y'} = R\sin\phi = -52\sin 52.6°$

$\underline{\tau_{x'y'} = -41.3 \text{ MPa}}$

$\sigma_{y'y'} = 80 + R\cos\phi = 80 + 52\cos 52.6°$

$\underline{\sigma_{y'y'} = 111.6 \text{ MPa}}$



check: $\tan 2\theta_p = \dfrac{\tau_{xy}}{\frac{1}{2}(\sigma_{xx} - \sigma_{yy})} = \dfrac{-48}{\frac{1}{2}(100-60)} = \dfrac{-48}{20} = -2.4$

$2\theta_p = -67.4°$, $\theta_p = -33.7°$

→ tells us that the shear stress being in ⊖ direction, gives rotation on Mohr's circle in the right direction

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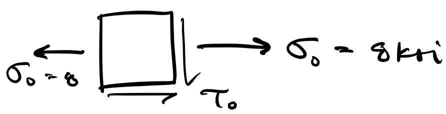
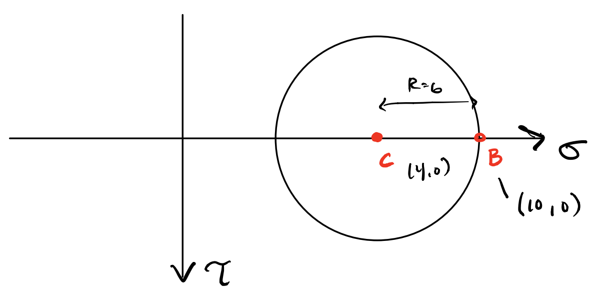
In class Mohr's Circle Example



Find:
1. value of τ_0 for which $\sigma_1 = 10$ ksi?
2. corresponding $\tau_{\substack{max \\ inplane}}$?

suggestions:
1. construct Mohr's circle
2. What is R?
3. What is τ_0?
4. θ_{p_1}?
5. τ_{max}?



→ $\sigma_{avg} = \dfrac{\sigma_{xx} + \sigma_{yy}}{2} = \dfrac{8+0}{2} = 4$ → center @ (4,0)

→ can't plot pt. (σ_{xx}, τ_{xy}) bc τ_{xy} is unknown