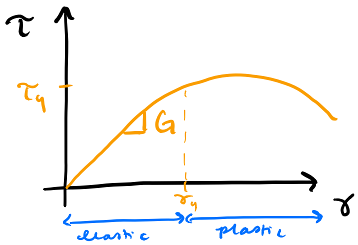
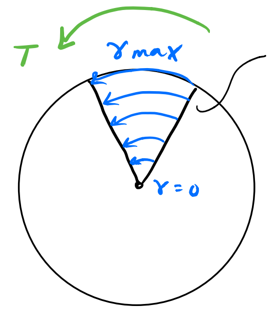
Materials – Lecture #12

HW due Tuesday after Spring Break!

Inelastic Torsion

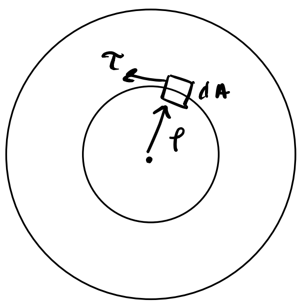
→ $\tau = G\gamma$: linear elastic torque, where stress & strain are linearly proportional





shear strain distribution

shear strain varies linearly from center to outer boundary



$dF = \tau dA$ = force
$dT = \rho dF = \rho \tau dA$ = torque

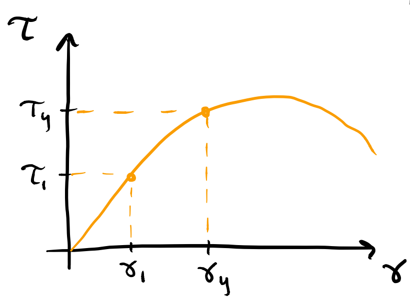
For entire cross section, integrate over the area

$$T = \int \rho \tau dA \rightarrow A = \pi \rho^2$$
$$dA = 2\pi\rho d\rho$$

then

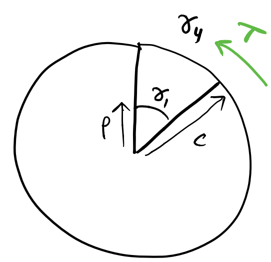
$$T = \int \rho \tau 2\pi \rho d\rho$$

$$\boxed{T = 2\pi \int \tau \rho^2 d\rho}$$



outer surface yields first

if apply enough torque to get to yield, the surface is going to yield first



there will be point on interior w/ smaller strain γ_1

Write shear stress distribution as such:

$\tau = \tau_y \left(\frac{\rho}{c}\right)$: if considering $\rho = c$, then $\tau = \tau_y$

: if consider $\rho = 0$ (at the center), then $\tau = 0$

: the rest, linearly varies between these two points

We can calculate yield torque as follows:

$$T = T_y = 2\pi \int_0^c \tau_y \left(\frac{\rho}{c}\right) \rho^2 \, d\rho$$

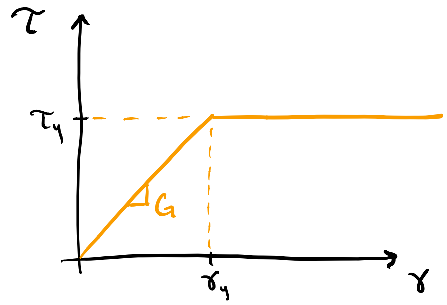
then,

$$\boxed{T_y = \frac{\pi}{2} \tau_y c^3}$$

yield torque ↑ ↑ yield stress (material property)

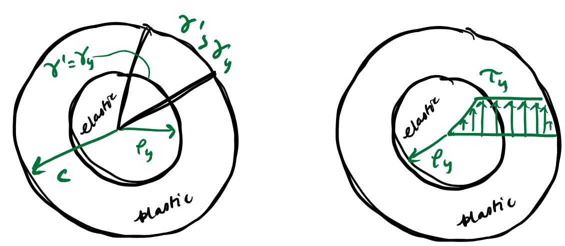
A specific kind of plastic stress...

ELASTIC-PLASTIC TORQUE (or elastic/perfectly plastic)



most materials don't behave like this but allows for more simple math

Now, what if load > T_y?



$$\rho_y = c\left(\frac{\gamma_y}{\gamma'}\right) \rightarrow \tau = \tau_y \left(\frac{\rho}{\rho_y}\right)$$

→ $\rho = \rho_y \ldots \tau = \tau_y$

→ $\rho = 0 \ldots \tau = 0$

We can calculate how much torque by breaking into two parts: plastic & elastic

→ Elastic-Plastic Torque

$$T = 2\pi \int_0^c \tau \rho^2 \, d\rho$$

$$T = \underbrace{2\pi \int_0^{\rho_y} (\tau_y)\left(\frac{\rho}{\rho_y}\right) \rho^2 \, d\rho}_{\text{elastic}} + \underbrace{2\pi \int_{\rho_y}^{c} \tau_y \rho^2 \, d\rho}_{\text{plastic}}$$

$$T = T_{elastic} + T_{plastic}$$

$$= 2\pi \int_0^{\rho_y} (\tau_y)\left(\frac{\rho}{\rho_y}\right) \rho^2 \, d\rho + 2\pi \int_{\rho_y}^{c} \tau_y \rho^2 \, d\rho$$

$$= \frac{\pi}{2\rho_y} \tau_y \rho_y^4 + \frac{2\pi}{3} \tau_y (c^3 - \rho_y^3)$$

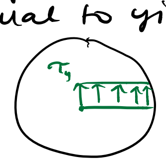
$$\boxed{T = \frac{\pi \tau_y}{6} (4c^3 - \rho_y^3)}$$

c = radius
τ_y = yield stress
ρ_y = radius of elastic core

Now what if the entire cross-section is yielded?
→ anywhere in cross-section, stress is equal to yield stress

PLASTIC TORQUE: whole cross-section yields

T_p = plastic torque $= 2\pi \int_0^c \tau_y \rho^2 \, dp$



$$\boxed{T_p = \frac{2\pi}{3}\tau_y c^3}$$

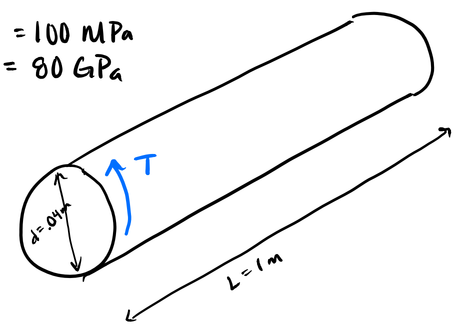
Ratio of T_p to T_y : $\frac{T_p}{T_y} = \frac{\frac{2\pi}{3}(\tau_y c^3)}{\frac{\pi}{2}(\tau_y c^3)} = \frac{4}{3}$

basically saying that if apply 33% more torque, can go from just outer surface yielding to whole cross-section yielding

33% more torque than T_y to make entire cross-section yield

In-class Example

$\tau_y = 100$ MPa
$G = 80$ GPa



Find:
① Max elastic torque T_y
(so structure doesn't yield)
② corresponding angle of twist
③ what is angle of twist when $T = 1.2T_y$?

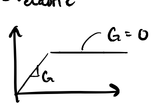
①
$$T_y = \frac{\pi}{2}\tau_y c^3$$
$$= \frac{\pi}{2}(100)(10^6)\left[\frac{N}{m^2}\right](0.2\,m)^3 \Rightarrow T_y = 1256.637061\ N\cdot m$$
$$\boxed{= 1.257\ kN\cdot m}$$

② $\phi = \frac{T_y L}{JG}$ $\quad J = \frac{\pi}{2}c^4 \Rightarrow J = \frac{\pi}{2}(0.02\,m)^4$

$$\phi = \frac{(1256.637061\ N\cdot m)(1\,m)}{(\frac{\pi}{2})(0.02m)^4(80)(10^9)\left[\frac{N}{m^2}\right]} = \boxed{0.0625\ \text{radians}} = 3.58°$$

③ $T = 1.2T_y$

$G_{elastic} = 80$ GPa



is there a general way to calculate changes in angle of a shaft w/out any material assumptions?

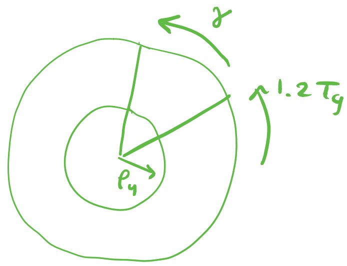
$\rho_y = 0.014723$ m from $1.2T = \frac{\pi \tau_y}{6}(4c^3 - \rho_y^3)$

$\rho_y = c\left(\frac{\gamma_y}{\gamma'}\right)$ $\quad \gamma' = \frac{c\gamma_y}{\rho_y} = \frac{(0.02m)(0.0625\ rad)}{(0.014723m)} = \boxed{0.0849\ \text{radians}}$

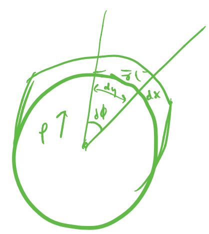
how he found answer:

$\tau_y = 1256$ $\quad \tau_y(1.2) =$

$\gamma_y = \frac{\tau_y}{G} = .00125$ rad



$\rho_y = 0.0147$ m



$dy = \gamma dx = \rho d\phi$

$d\phi = \frac{\gamma dx}{\rho}$

$\phi = \frac{\gamma_y L}{\rho_y} = \frac{(0.00125)(1)}{.0147} = .085$

if we had used

$\phi = \frac{TL}{JG} = \frac{1.2T_y L}{JG} = 0.0748$ which is less. why?

0.0748 (not the answer) < 0.085 . . . why does this make sense?

→ after yielding, the material gets softer and 0.0748 would only account for it being still stiff.