An Application of Coulomb Torsion Theory & the Twisting of Uniform Shafts

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1 Pre-lab

Materials Tortion Pre-lab

1.

Material	Modulus of Rigidity (G)	Yield Strength (σ)
2011-T3 Aluminum	3770000 psi	43000 psi
360 Brass	5370000 psi	52200 psi
304 stainless steel	12500000 psi	31200 psi

Matweb.com

2. Find torque for 18 in long circular shaft w/ 1/8 in diameter. $\tau_y = \frac{1}{2}\sigma_y$

$$J = \frac{\pi}{2}\left(\frac{1}{16}\right)^4 \qquad \frac{\tau J}{\rho} = T \quad \rightarrow \quad \frac{\frac{1}{2}\sigma_y\left(\frac{\pi}{2}\right)\left(\frac{1}{16}\right)^4}{\left(\frac{1}{16}\right)} = T$$

$$\boxed{T = \frac{\pi\sigma_y}{16384}}$$

3. Compute total rotation ϕ

$$\phi = \frac{TL}{JG}$$

$$\phi_{Al} = \left(\frac{\cancel{\pi}(\sigma_y)}{16384} in^3\right)(18\ in)\left(\frac{2(16)^4}{\cancel{\pi}\ in^4}\right)\left(\frac{1}{G}\right) = \left(\frac{43000}{16384}\right)(18)\left(2(16)^4\right)\left(\frac{1}{3770000}\right)\left(\frac{180^\circ}{\pi\ rad}\right) = 94.10$$

$$\phi_{Br} = \left(\frac{52200}{16384}\right)(18)\left(2(16^4)\right)\left(\frac{1}{5370000}\right)\left(\frac{180^\circ}{\pi\ rad}\right) = 80.20^\circ$$

$$\phi_{St} = \left(\frac{31200}{16384}\right)(18)\left(2(16)^4\right)\left(\frac{1}{12500000}\right)\left(\frac{180^\circ}{\pi\ rad}\right) = 20.59^\circ$$

4. Find force P if applied on lever of 6 in gives yield torques.

Torque = distance $\times$ P

$$P_{Al} = \frac{\pi(43000)}{16384}\left(\frac{1}{6}\right) = 1.37\ lb \qquad P_{St} = \frac{\pi(31200)}{16384}\left(\frac{1}{6}\right) = 0.997\ lb$$

$$P_{Br} = \frac{\pi(52200)}{16384}\left(\frac{1}{6}\right) = 1.67\ lb$$

5. Find mass in grams whose weight on Earth gives loads P computed.

$$\underline{1\ lb}\left(\frac{453.59\ g}{1\ lb}\right) \quad \Rightarrow \quad m_{Al} = 621.4\ g \quad m_{St} = 452.2\ g$$

$$m_{Br} = 757.5\ g$$

2 Introduction

Engineers interact with a variety of materials every day in their work and research and it is therefore vital to have a substantial understanding of a material's inherent properties before commencing a new project or experiment. Often, engineers want to know how strong a material is, which types of loads can be applied, among many other things. Of significant importance is the understanding of a material's response to a applied torque.

Detailed in this report is the study of circular shafts made of three different materials subjected to various torques, for the ultimate goal of gaining a broader understanding of the behavior of these materials and why they may be chosen for their everyday commercial uses. Each of these shafts were comprised of uniform, homogeneous material and different torques were applied to test Coulomb torsion theory. Coulomb torsion theory tells us that one end of the shaft will twist and deform with an elastic behavior that will be linearly related to the torque applied. This was accomplished by conducting torsion testing upon shafts made of 2011-T3 Aluminum, 360 Copper Brass, and 304 Stainless Steel using a Megazord apparatus and a digital angle gauge (DLAG) at Boston University. The angle of twist was documented for various applied torques. Torsion testing is of particular importance as it gives insight into a material's behavior when subjected to a torsional load. This is then useful to understand how and when such material will deform so that its strength can be evaluated and compared to other materials. As such, the data obtained from the testing was then used to investigate the relationship between the angle of twist and the torque applied.

3 Theory

When studying the torsional deformation of a uniform circular shaft made of linear elastic material that is fixed at one end, Coulomb's torsion theory tells us that an applied torque on the shaft's other end will cause a twisting distortion that is linearly related to the magnitude of the torque. The deformed end of the shaft will rotate through an angle ϕ(x) that depends on the position x, which will vary along the shaft, and cause a shear strain. The magnitude of this shear strain will vary linearly along a radial distance from the shaft's axis to its outermost surface and the shear modulus G will be constant throughout for a homogeneous shaft. Since Hooke's Law tells us that shear stress is equal to the torque divided by the shear modulus $\gamma = \tau_{\overline{G}}$ then the shear stress can be expressed in terms of torque. (Hibbeler, 2017)

Torque is a measure of force x distance, while G is a measure of force per unit area. In order to relate an angle of twist ϕ to a material's shear modulus and an applied torque, one must factor in the polar moment of inertia J for a shaft of solid circular cross-section with diameter d, described by Equation 1.

$$J = \frac{\pi d^4}{32} \tag{1}$$

These previously described terms together give us a torsion equation described by Equation 2 below. Since the shaft is uniform and homogenous, and doesn't vary in cross-sectional area throughout, J and G are constant and x is the position along the length of the shaft. Applying boundary conditions to this equation such that the angle of twist ϕ is equal to zero when x = 0, and that x_{max} = L, the total length of the shaft, we derive a linear equation for the angle of twist in terms of the applied torque T, length L, polar moment of inertia J, and shear modulus G (Equation 3).

$$T(x) = JG\phi(x) \tag{2}$$

$$\phi = \frac{TL}{JG} \tag{3}$$

Equation 3 can also be described in terms of a torsional spring constant k (Equation 4) that highlights the linear relationship between torque and angle of twist (Equation 5). Further, the shear modulus G can also be described in terms of this torsional spring constant in Equation 6.

$$k_{torsion} = \frac{JG}{L} \tag{4}$$

$$T = k_{torsion}\phi \tag{5}$$

$$G = \frac{TL}{J\phi} = \frac{k_{torsion}L}{J} \tag{6}$$

In this experiment, we aimed to discover the linear relationship between applied torque and angles of twist by experimentally finding the torsional spring constant for a material as well as compare measured and calculated shear moduli with those that are published.

4 Measurement Methods

The experimental setup consisted of the Megazord apparatus, a digital angle gauge (DLAG), and three different uniform shafts consisting of 360 Copper Brass, 2011-T3 Aluminum, and 304 Stainless Steel. First the dimensions of the circular shafts and the torque wheel were measured so that these values could be used for later calculations and analysis. The Megazord was then delicately balanced along the edge of the experiment table as it is only comprised of three feet of support. The samples were then inserted and a 10g mass was loaded to the wire along the torque wheel to remove any potential slack. It is important to note that this 10g mass was not factored into the total loads analyzed for these materials, though this clearly highlights a potential source of error which will be touched on in the discussion section of this report. The DLAG was then calibrated to zero and 20g of mass were added and the new angle reading recorded. This was repeated until 120g (130g including the initial 10g mass) were reached and the procedure was reported for the remaining shafts. Observations were taken for each material and only one trial was conducted for each material. Figure 1 shows the apparatus with the DLAG during the experimental testing of aluminum.

Figure 1: The Megazord apparatus and digital angle gauge (DLAG).

5 Results & Analysis

Before conducting the experimentation, measurements of the apparatus and the shafts of each material were measured. These are tabulated below in Table 1. While the torques were applied with various masses that are in units of grams, these were converted into pound-force (lb) and then multiplied by the torque wheel radius to give the applied torque in lb·in. The measured angles of twist (in degrees) of each material for the different applied torques are tabulated in Table 2.

Table 1: Measurements of Samples & Apparatus.

Material	**Diameter (in.)**	**Polar Moment of Inertia, J (in^4)**	**Length (in.)**	**Torque Wheel Diameter (in.)**
360 Copper Brass	0.125	$2.3968 \cdot 10^{-5}$	23.75	11.875
2011-T3 Aluminum	0.125	$2.3968 \cdot 10^{-5}$	23.75	11.875
304 Stainless Steel	0.125	$2.3968 \cdot 10^{-5}$	23.75	11.875

After converting the angles from degrees into radians, the torques vs. angle of twist were plotted for each material. Figure 2 displays this relationship for all three materials to allow for comparison. An analysis of their differences will be discussed further in the following section of this report. Figures 3 – 5 display the torque and angle of twist relationship for each material with linear fit lines that best describe the raw data. These plots also show the error in angle measurements of $\pm$ 0.05 degrees, though it may be difficult to see. The equations for the linear best fit lines of each material are displayed on the individual plots themselves.

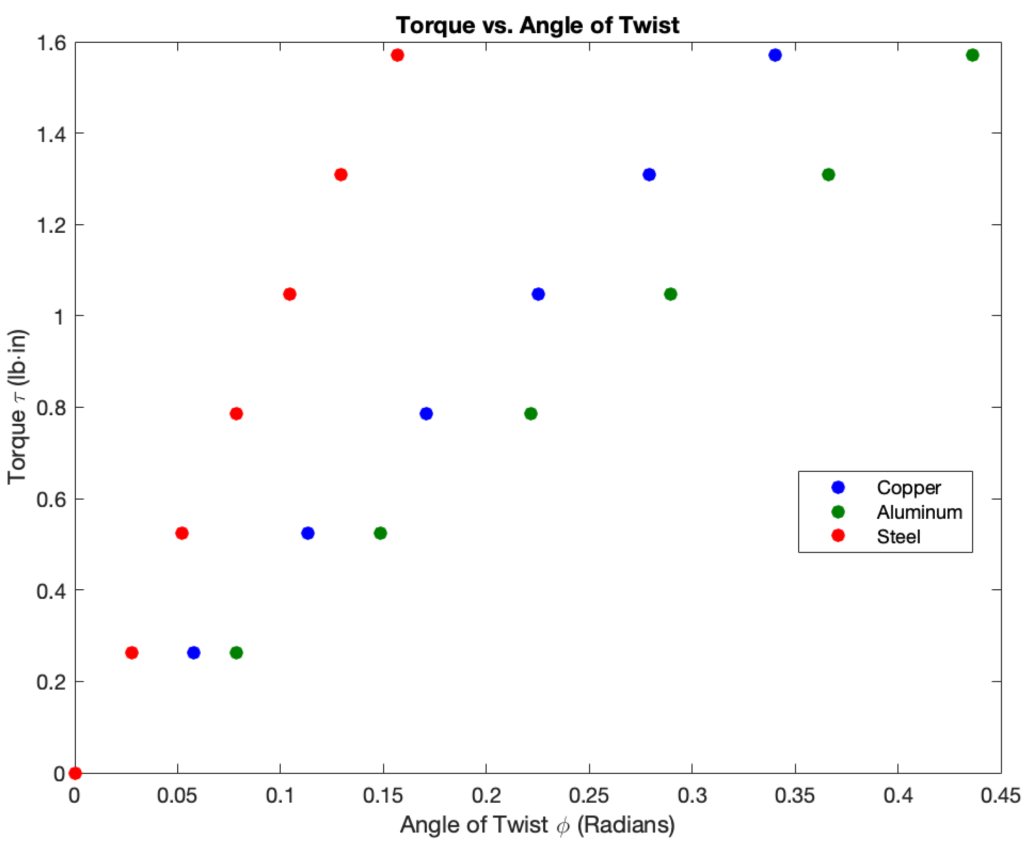

Figure 2: The torque and angle of twist of all three materials.

Table 2: Applied Torque & Angle of Twist Data.

Applied Torque τ **(lb·in)**	**Angle of Twist** ϕ **(degrees)**	**Angle of Twist** ϕ **(degrees)**	**Angle of Twist** ϕ **(degrees)**
	360 Cop. Brass	2011-T3 Alum.	304 St. Steel
0.0000	0.0	0.0	0.0
0.2619	3.3	4.5	1.6
0.5238	6.5	8.5	3.0
0.7857	9.8	12.7	4.5
1.0476	12.9	16.6	6.0
1.3094	16.0	21.0	7.4
1.5713	19.5	25.0	9.0

Table 3 displays the calculated results of each material, including the torsional spring constant, experimentally derived shear modulus, published shear modulus value, and percent error. "MATWEB2020" (n.d.)

Finally, all calculations, data analysis, and plots were generated in MATLAB.

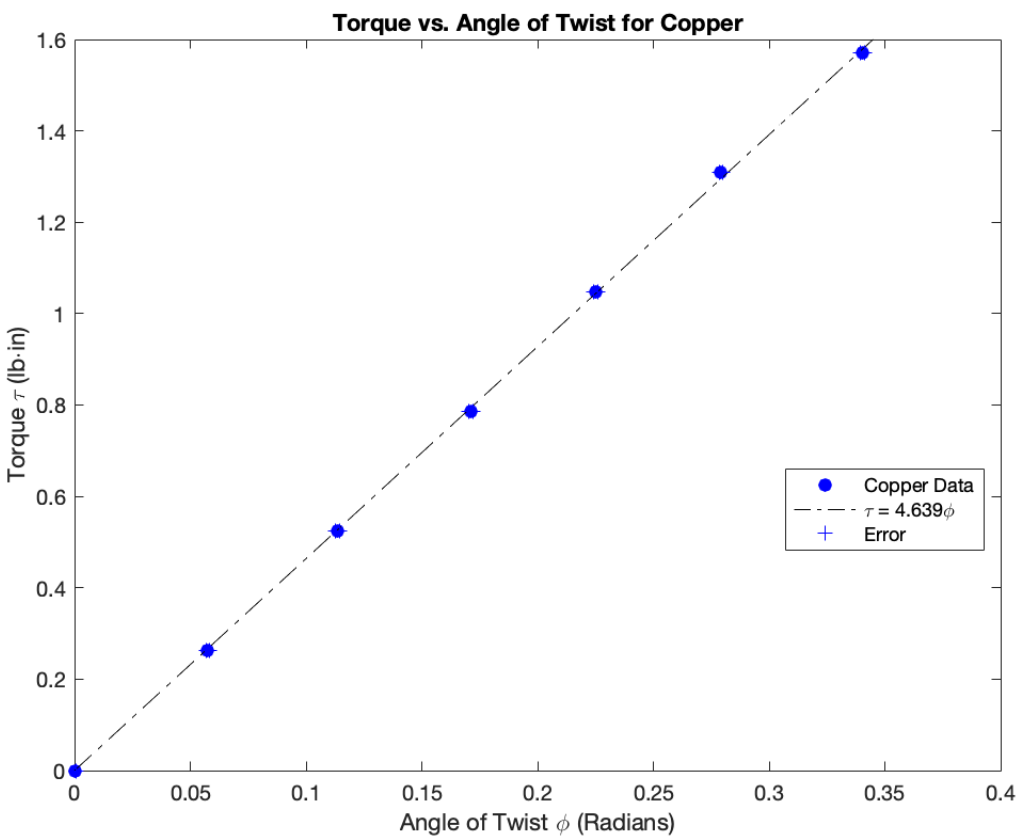

Figure 3: The torque and angle of twist for 360 Copper Brass with linear fit equation $\tau = 4.639\phi$.

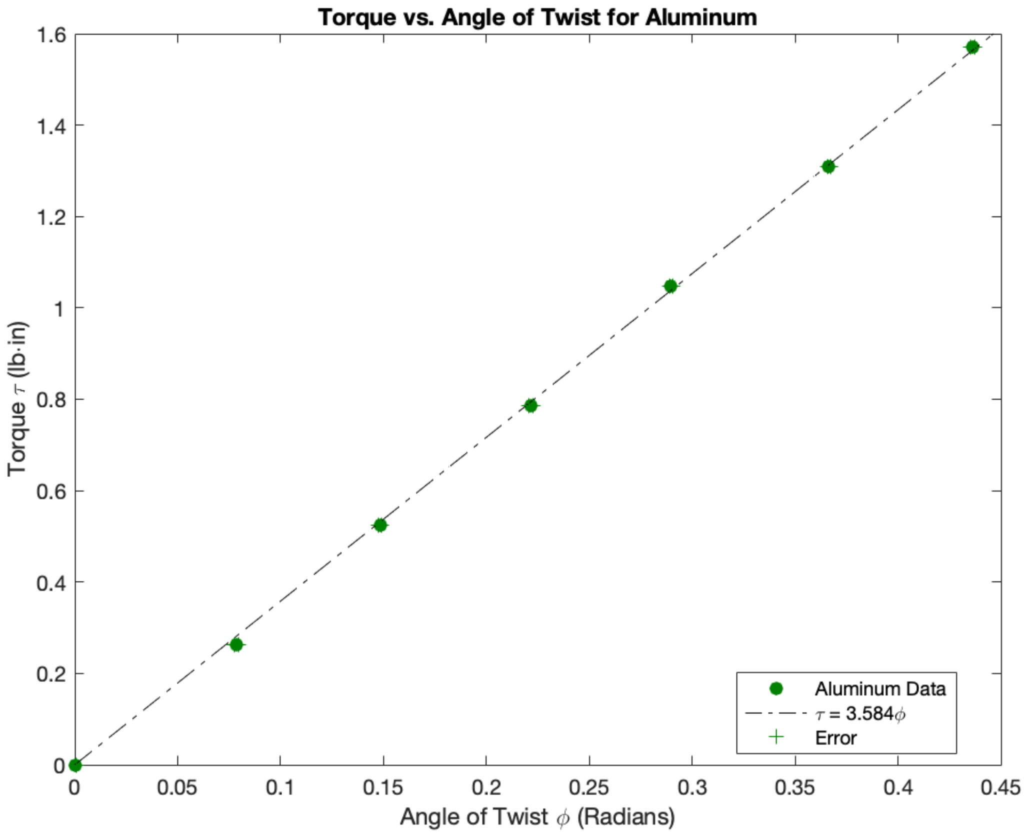

Figure 4: The torque and angle of twist for 2011-T3 Aluminum with linear fit equation $\tau = 3.584\phi$.

Table 3: Experimental & Published Shear Modulus Data.

Material	**Torsional Spring Constant, k ($\frac{lb*in}{radian}$)**	**Experimental Shear Modulus, G (psi)**	**Published Shear Modulus, G (psi)**	**Difference (psi)**	**Error**
360 Copper Brass	4.639	4596719.9	5370000	773280.1	14.40%
2011-T3 Aluminum	3.584	3551335.2	3770000	218664.8	5.80%
304 Stainless Steel	10.03	9938586.0	12500000	2561414.0	20.49%

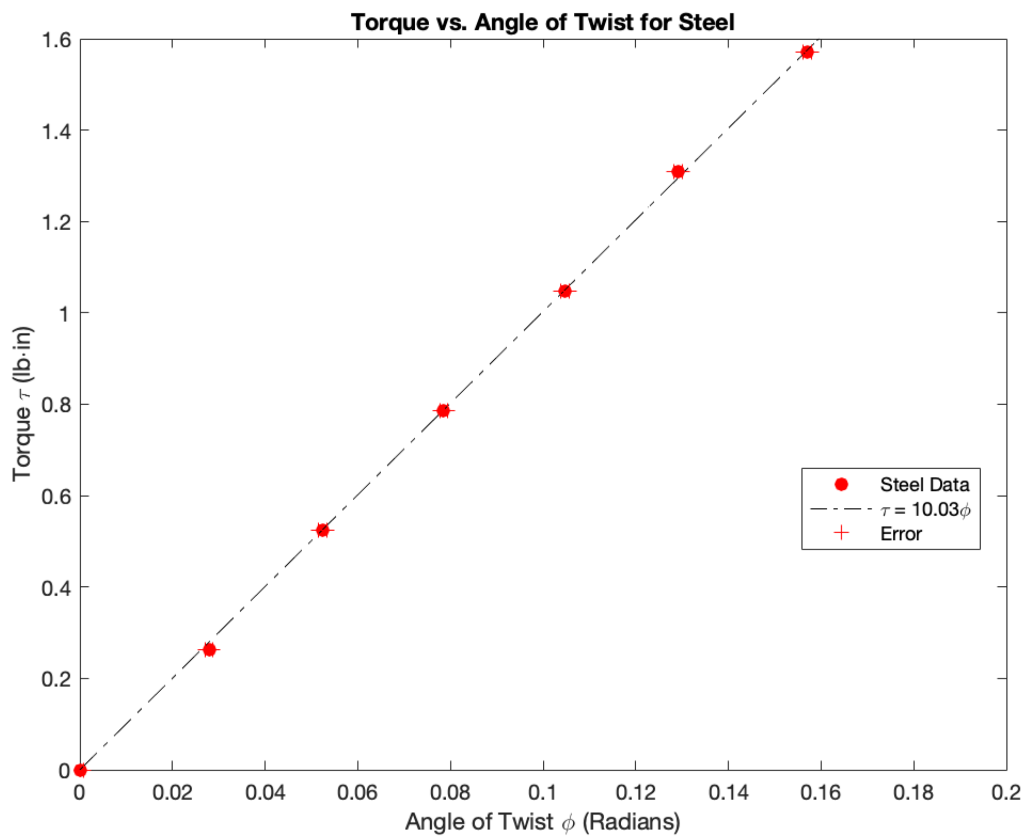

Figure 5: The torque and angle of twist for 304 Stainless Steel with linear fit equation $\tau = 10.03\phi$.

6 Discussion

As displayed by the plots for each material, there is an elastic, linear relationship between the applied torque and angle of twist, in which the slopes represent the torsional spring constant k. The raw data itself for each material is practically on the best fit line and the errors are barely visible. This supports what Coulomb torsion theory says, that there exists a linear relationship between torque and angle of twist for a circular solid shaft and that these materials behave in an elastic manner. As the data suggests, steel is most difficult to deform when torque is applied, followed by the copper brass, and then the aluminum, which is consistent with what has been studied previously and in the literature. In Table 3 from the prior section, one can see that the shear moduli are also greatest for steel, then copper brass, then aluminum which means that more force (or torque) is required to cause significant deformation for steel than for the other materials.

From this experimentation, one can see that there are some discrepancies between the measured and calculated values of shear moduli and those that have been published. For the most part, the shear modulus found for aluminum was very close to the expected published value (percent error of 5.80%) suggesting that aluminum responded to the applied torque in a way that is consistent with the research and literature. However, the copper brass and steel had moduli more deviant from the expected with percent errors of 14.40% and 20.49%, respectively.

These discrepancies may result from a few potential factors. For example, one mustn't forget that there was 10g that were never factored into these calculations, and this extra mass certainly does have some non-negligible effect. Factoring this 10g mass in may result in a smaller percent error from the published values, for example. In addition, the slopes for both the copper brass and steel, or torsional spring constant k, were not as steep as expected which affects the shear moduli value G. This suggests that the materials may have been comprised in some way from the inherent nature of doing all experimentation by hand, or by potentially being a blend of materials and not necessarily pure and uniform. (The prior equations assume homogeneity and uniformity at all times.)

There are also other potential sources of error besides those just mentioned. For example, only one trial was conducted for each material and it would behoove future researchers to run multiple (several trials) to obtain a better understanding of the materials' behaviors. Many of the published values that are in the literature today have come from running many experiments with multiple trials. In addition, another source of error may come from the testing equipment and calibration. It's possible that the apparatus was not properly balanced and that the torque wheel and wire have been compromised from consistent use and unintentional human force. It was also quite difficult to calibrate the DLAG which may have also contributed as a source of error.

Finally, given all of these potential sources of error, the experiment can be improved in a variety of ways for the future. Most importantly, there should be more trials for each material being tested. In addition, better care must be taken in the handling of the materials as well as the calibration of the measuring apparatus and devices. In addition, it may be best to test each shaft only once, rather than continually adding more and more mass after angle readings have been recorded.

7 Conclusion

In conclusion, the goal of this laboratory exercise was to discover a relationship between torque and angle of twist, as well as finding the shear moduli for three different materials. This was done successfully and inferences were made about the strengths and behaviors of materials in response to an applied torque. Steel showed that it was more resistant to deformation when a torque is applied, followed by copper brass and then aluminum. This gave great perspective on why engineers may choose to work with some materials over others, especially when considering the safety factor during construction, among other goals. Even from this experimentation, steel clearly displayed why this material is more commonly used in building and construction as its shear moduli is much greater and less likely to deform compared to copper brass and aluminum. Though the calculated values of shear moduli were different from what is published in engineering literature, the overall behavior of the materials was consistent with what was expected from Coulomb torsion theory for uniform, homogeneous circular solid shafts. However, despite the deviations for the shear moduli, given the time and material constraints of the experiment it was really interesting to witness how the materials twisted and deformed and to be able to conduct a torsion test.

References

(n.d.). *http://www.matweb.com*.

Hibbeler, R. C. (2017). Chapter 5: Torsion. *Mechanics of Materials*, *10*th*Edition*, *p*.182 − 262.