Polar Moment of Inertia – property of x-sectional area

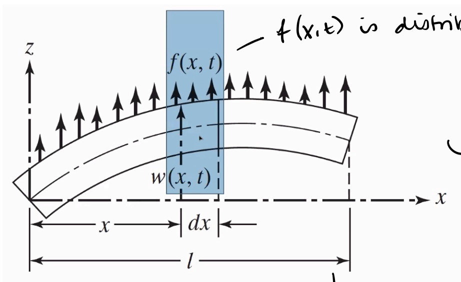
Vibrations - Lecture 19
Tue., 2021-04-06

- symbol J (analogous to I to beams)
- unit : length^4

I_0 = mass polar moment of inertia per unit length

don't really need this word, it's redundant

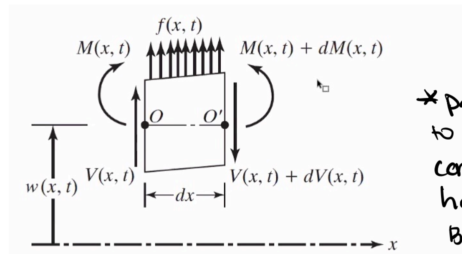
The beam theory we're about to talk about assumes that the beam is not under tension



f(x,t) is distributed load

working at entire length of beam since l

making cut



bending moment & shear force (vertical)

* pay attention to sign convention here!!! affect B.C.

ρ = mass density now

$$\underbrace{\rho A dx}_{[m]} \underbrace{\frac{\partial^2 w}{\partial t^2}(x,t)}_{\left[\frac{m\,l}{t^2}\right]} = \underbrace{-(V+dV) + f(x,t)\,dx + V}_{\text{force equation in the z-direction}}$$

aka acceleration

* Newton's Law

=> sum moments about left hand side (of right image, dx)

$$(M + dM) - (V+dV)\underbrace{dx}_{\text{moment arm}} + f(x,t)\,dx\underbrace{\left(\frac{dx}{2}\right)}_{\text{moment arm}} - M = 0 \qquad \oplus \circlearrowleft \text{ ccw}$$

↳ but if dx is vanishing ... put centroid in middle and then assume $dx/2$ bc need to concentrate the force!!!

↳ now can we relate M and V?

$$dV = \frac{\partial V}{\partial x}dx \qquad dM = \frac{\partial M}{\partial x}dx$$

→ ignore higher powers in dx

$$-\frac{\partial V}{\partial x}(x,t) + f(x,t) = \rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t)$$

$$\frac{\partial M}{\partial x}(x,t) - V(x,t) = 0$$

By using the relation $V = \partial M/\partial x$ then

$$-\frac{\partial^2 M}{\partial x^2}(x,t) + f(x,t) = \rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t)$$

from Euler-Bernoulli: relationship between bending moment and deflection can be expressed as: $M(x,t) = EI(x)\frac{\partial^2 w}{\partial x^2}(x,t)$

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial x^2}(x,t)\right] + \rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t) = f(x,t)$$

EI and ρA are often considered properties of beam

spatially independent properties argument

$$EI\frac{\partial^4 w}{\partial x^4}(x,t) + \rho A\frac{\partial^2 w}{\partial t^2}(x,t) = f(x,t)$$

for vibration, this term is $\emptyset$

$$c^2\frac{\partial^4 w}{\partial x^4}(x,t) + \frac{\partial^2 w}{\partial t^2}(x,t) = 0$$

when $c = \sqrt{\frac{EI}{\rho A}}$ ... what is c? will discuss it's significance in another class.

I.C.

$w(x, t=0) = w_0(x)$

$\frac{\partial w}{\partial t}(x, t=0) = \dot{w}_0(x)$

Equation of motion for beam:

$$c^2\frac{\partial^4 w}{\partial x^4}(x,t) + \frac{\partial^2 w}{\partial t^2}(x,t) = 0 \qquad c = \sqrt{\frac{EI}{\rho A}}$$

→ separation of variables:

$w(x,t) = W(x)T(t)$

$$\left.\begin{aligned}\frac{\partial^2 w(x,t)}{\partial t^2} &= W(x)T''(t)\\ \frac{\partial^4 w(x,t)}{\partial x^4} &= W^{IV}(x)T(t)\end{aligned}\right\}$$ substitute these into equation of motion

$$c^2 W^{IV} T + W T'' = 0 \qquad \text{divide by } WT$$

$$c^2\frac{W^{IV}}{W} = \frac{-T''}{T} = \omega^2$$

What do the separated equations look like?

$$T'' + T\omega^2 = 0 \quad (1)$$

$$W^{IV} - \frac{\omega^2}{c^2} W = 0 \quad (2)$$

... define $\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI}$

↳ β will turn out to be a wave number-like thing

(1)

$$T(t) = A\cos\omega t + B\sin\omega t$$

↳ rad/s

by inspection: $W = \exp(\beta x) \rightarrow W^{IV} = \beta^4 \exp(\beta x)$

what if $W = \exp(-\beta x)$? $\rightarrow W^{IV} = \beta^4 \exp(-\beta x)$

① $\exp(i\beta x) = \cos\beta x + i\sin\beta x$
② $\exp(-i\beta x) = \cos\beta x - i\sin\beta x$
③ $\exp(\beta x)$
④ $\exp(-\beta x)$

$$W(x) = C_1 \exp(i\beta x) + C_2 \exp(-i\beta x) + C_3 \exp(\beta x) + C_4 \exp(-\beta x)$$

$$\cosh(\beta x) = \frac{\exp(\beta x) + \exp(-\beta x)}{2}$$

$$\sinh(\beta x) = \frac{\exp(\beta x) - \exp(-\beta x)}{2}$$

$$W(x) = A\cos\beta x + B\sin\beta x + C\cosh\beta x + D\sinh\beta x$$

other notes: $W = \exp(\alpha x)$

$W^{IV} - \beta^4 W = 0$

$\alpha^4 \exp(\alpha x) - \beta^4 \exp(\alpha x) = 0$

$\alpha^4 = \beta^4$

↳ $\alpha_1 = \beta \quad \alpha_3 = i\beta$

$\alpha_2 = -\beta \quad \alpha_4 = -i\beta$

these different forms of W are helpful for finding boundary conditions

$$W(x) = C_1 \exp(\beta x) + C_2 \exp(-\beta x) + C_3 \exp(i\beta x) + C_4 \exp(-i\beta x)$$

(note that C_1, C_2, C_3, C_4 are not same as before)

remember that $\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI}$

* the natural frequencies of the beam

$$\omega = \beta^2 \sqrt{\frac{EI}{\rho A}} = (\beta l)^2 = \sqrt{\frac{EI}{\rho A l^4}}$$

(βl) is dimensionless! so can take sin, cos, sinh, cosh