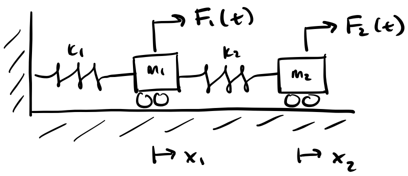
Vibrations - Lecture 10
2021-03-02

→ covering section 6.14
- orthogonality + eigenvalue problem more extensively

What is a Forced-Undamped Multidegree of Freedom System?



Known: $m_1, m_2, k_1, k_2, F_1^{(t)}, F_2^{(t)}$
$\vec{x}_0$, $\dot{\vec{x}}_0$ initial conditions

Find: $x_1(t)$ and $x_2(t)$. There is a unique solution. The shape of vibration is the eigenvector for the free vibration state... what about forced?

What's the relationship between the vibration caused by the force and the vibration caused by the initial conditions?

If put force as such:

$$F_1 = 2\sin \underbrace{3\pi}_{} t \qquad \hookrightarrow \omega = 2\pi f$$
$$F_2 = 0 \qquad f = 3/2 \text{ Hz}$$

It will vibrate @ initial condition natural frequencies and at 1.5 Hz (aka the forcing frequency)
→ What if we're unlucky and this f (3/2 Hz) lines up with one of the natural frequencies?
- bad things happen if force a system at one of its natural frequencies
- mathematically speaking, the vibration builds up until the displacements become very large until we're exceeding the linear limit of the spring and now becomes a nonlinear spring and then keeps increasing until spring breaks (wine glass shatters @ natural frequencies, etc)

→ if you excite it at natural frequencies, our equations will tell us infinity but reality is breakage

Particular soln means particular to the force
Homogeneous soln doesn't change if force changes, RHS always equal to zero
↳ then add these two to get full solution
→ some people say "transient solution" which is same as homogeneous

Ex. $m\ddot{x} + kx = F_0 \cos\omega t$
↑ this omega is <u>not</u> the natural frequency, it's just a frequency
→ when omega does not have a subscript, it's typically a forcing frequency

the homogeneous soln:
$x_h(t) = c_1 \cos\omega_n t + c_2 \sin\omega_n t$ where natural frequency is $\omega_n = (k/m)^{1/2}$

The homogeneous soln is synonymous w/ free vibration, such as in assign. 3

In the force problem though, we "postpone the use of initial conditions"

$X_p(t) = \bar{X}\cos\omega t$ ... MUCK = method of undetermined coefficients

where X is a constant that denotes the maximum amplitude of $x_p(t)$ ... solving for X:

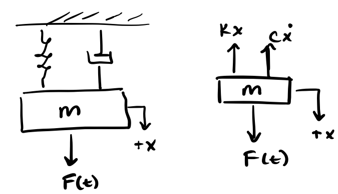
$$\bar{X} = \frac{F_0}{k - m\omega^2} = \frac{\delta_{st}}{1-(\omega/\omega_n)^2}$$ ← nondimensional form

when ω large: mass-controlled

when ω small: k dominates

where $\delta_{st} = F_0/k$ denotes the deflection of the mass under a Force F_0 and is sometimes called static deflection because F_0 is a constant (static) force. Thus the total solution becomes

$$x(t) = c_1\cos\omega_n t + c_2\sin\omega_n t + \frac{F_0}{k-m\omega^2}\cos\omega t$$



then using initial conditions x_0 and $\dot{x}_0$ we find that

$$c_1 = x_0 - \frac{F_0}{k-m\omega^2} \qquad c_2 = \frac{\dot{x}_0}{\omega_n}$$

Now Multidegree of Freedom System: 6.14

$$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{F}(t)$$

$$\vec{x} = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} \qquad \vec{F}(t) = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

• consider Eigenvalue Problem

$$\omega_i^2[m]\vec{\bar{X}}^{(i)} = [k]\vec{\bar{X}}^{(i)} \quad \text{for } i = 1, 2, \dots, n \quad (\#\text{ DoF})$$

eigenvector

→ use eigenpairs as tools:

the eigenvectors are linearly independent ... expansion theorem

For our problem, at any particular time t:

$$\vec{x}(t) = q_1(t)\vec{\bar{X}}^{(1)} + q_2(t)\vec{\bar{X}}^{(2)} + \dots\ q_n(t)\vec{\bar{X}}^{(n)}$$

↳ how are we going to find q?

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{Bmatrix} = \vec{x}(t)$$ ... there are n time-dependent functions

then we can have $\begin{Bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_n(t) \end{Bmatrix}$ ... knowing q is to know x because we know the eigenvector $\vec{\bar{X}}$

... we're going to trade currency, because finding q's will be easier than finding x's (especially when [M] and [K] are not diagonal

$$\vec{x}(t) = [X]\,\vec{q}(t)$$

$$\uparrow [\vec{X}^{(1)} \;\; \vec{X}^{(2)} \; \ldots \; \vec{X}^{(n)}]$$

... when we go to find the q's we'll have mathematically uncoupled ODE's

How do we find q?

$$\ddot{\vec{x}}(t) = [X]\ddot{\vec{q}}(t) \qquad [X]\vec{q}(t)$$

$$[m]\ddot{\vec{x}} + k\vec{x} = \vec{F}(t)$$

$$\underbrace{[m][X]}_{?}\ddot{\vec{q}}(t) + \underbrace{[k][X]}_{?}\vec{q}(t) = \vec{F}(t)$$ ... we get the equations of motion in terms of q instead of x

How can orthogonality help us? "Dog rowing boat"

$$[X]^T[m][X] = [I]$$

$$[X]^T[k][X] = [\Omega^2]$$

$$= \begin{bmatrix} \omega_1^2 & 0 & 0 & \ldots \\ 0 & \omega_2^2 & 0 & \ldots \\ 0 & 0 & \ldots & \ldots \end{bmatrix} = \operatorname{diag}(\omega_n^2)$$

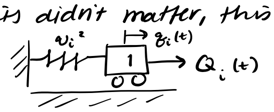
so we premultiply by $[X]^T$

$$[X]^T[m][X]\ddot{\vec{q}}(t) + [X]^T[k][X]\vec{q}(t) = \underbrace{[X]^T\vec{F}(t)}_{\vec{Q}(t)}$$ ... call this capital Q now

$$[I]\ddot{\vec{q}}(t) + [\Omega^2]\vec{q}(t) = \vec{Q}(t)$$

row i^{th}: $\ddot{q}_i(t) + \omega_i^2 q_i(t) = Q_i(t)$ ← this is the form of the single DoF system

↳ if the units didn't matter, this would be the equation of motion for



for $i = 1, 2, \ldots, n$

→ x is very difficult to find, but q is very easy ... just a normal ODE

↑ q's are the uncoupled equations of motion

→ remember, q_1 corresponds to ω_1

→ $q_i(t) \sim i^{th}$ modal coordinate

→ $Q_i(t) \sim i^{th}$ modal force

$$\begin{cases} x_1(t) = \mathrm{X}_{11}\, q_1(t) + \mathrm{X}_{12}\, q_2(t) \\ x_2(t) = \mathrm{X}_{21}\, q_2(t) + \mathrm{X}_{22}\, q_2(t) \end{cases}$$

$[\mathrm{X}] = [\mathrm{X}^{(1)} \;\; \mathrm{X}^{(2)}]$

this way if we find the q's instead, then we can find the x's

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \vec{\mathrm{X}}^{(1)} q_1(t) + \vec{\mathrm{X}}^{(2)} q_2(t)$$

... the first modal coordinate multiplies the first mode shape (eigenvector), etc.

the displacement vector

now let's pretend:

$$[\mathrm{X}] = [\mathrm{X}^{(1)} \;\; \mathrm{X}^{(2)}] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

then,

$$x_1(t) = 1 \cdot q_1(t) + 2 q_2(t)$$
$$x_2(t) = 3 \cdot q_1(t) + 4 q_2(t)$$

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 3 \end{Bmatrix} q_1(t) + \begin{Bmatrix} 2 \\ 4 \end{Bmatrix} q_2(t)$$

a comment on initial conditions: we have $\vec{x}_0$ and $\dot{\vec{x}}_0$

→ what is $q_1(0)$ and $\dot{q}_1(0)$?

... go back to our initial relationship

$$\vec{x}(t) = [\mathrm{X}]\, \vec{q}(t)$$

evaluate at time $t = 0$

$$\vec{x}(0) = \vec{x}_0 = [\mathrm{X}]\, \vec{q}_0$$

since $[\mathrm{X}]$ has linearly independent columns, could do inverse $[\mathrm{X}]^{-1} \vec{x}_0 = \vec{q}_0$ but not so great w/huge DOF systems because the inverse could look crazy

... instead, premultiply by $[\mathrm{X}]^T [m]$

$$[\mathrm{X}]^T [m]\, \vec{x}(t) = \vec{q}(t)$$

$$\vec{q}(0) = \vec{q}_0 = [\mathrm{X}]^T [m]\, \vec{x}(0)$$

$$\dot{\vec{q}}(0) = \dot{\vec{q}}_0 = [\mathrm{X}]^T [m]\, \dot{\vec{x}}(0)$$