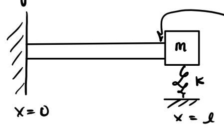
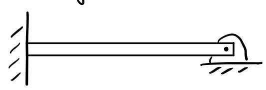
Assignment #7 last minute notes:

Vibrations - Lecture 22
Thur. 2021-04-15

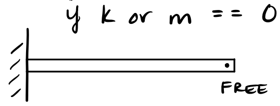


beam is pinned to this vibrational element

if k or m are infinite



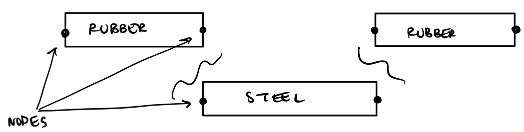
if k or m == 0



~ Rao Ch. 12 ~

FINITE ELEMENT METHOD

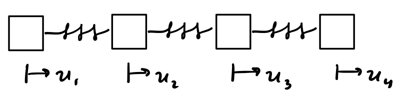
what are the boundary conditions?



← break up into three pieces to find boundary conditions ... this is essentially FEM

NODE in FEM: pt where you node something

↳ not unlike:



$[m]\ddot{\vec{u}} + [k]\vec{u} = \vec{0}$

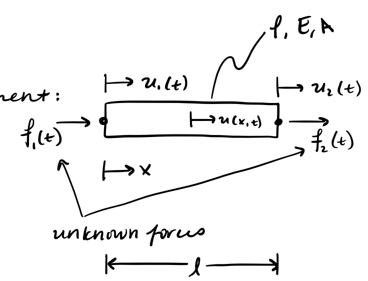
smaller element size leads to more accurate model → MESHING

h-p adaptive

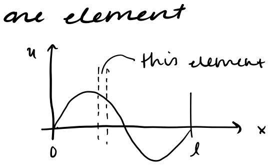
- p: polynomial order
- h: element size

~ Section 12.3 ~

uniform bar element:



... we'll make mass + stiffness matrices for one element



this element is basically linear

taylor approximation will help show us that linear element will work

Approximation $u(x,t) = a(t) + b(t)x$

linear element

x = distance from left side

a, b = coefficients that are time-dependent

$u(x=0, t) = u_1(t) = a(t)$

$u(x=l, t) = u_2(t) = a(t) + b(t)l$

$\uparrow$ $u_1(t)$

solve for $b(t)$:

$$b(t) = (u_2(t) - u_1(t))/l$$

substitute $a(t)$ & $b(t)$ into $u(x,t)$:

$$u(x,t) = \left(1 - \frac{x}{l}\right) u_1(t) + \left(\frac{x}{l}\right) u_2(t)$$

$$u(x,t) = N_1(x)u_1(t) + N_2(x)u_2(t)$$

$N_1(x) = 1 - x/l$ $\quad$ $N_2(x) = x/l$ $\quad$ N_1 + N_2 are shape functions

What would be the kinetic energy, T, of this bar?

$$T = \frac{1}{2}\int_0^l \rho A \left(\frac{\partial u(x,t)}{\partial t}\right)^2 dx$$

velocity

like $\frac{1}{2}mv^2$

substitute u for the expression we have above:

$$T = \frac{1}{2}\int_0^l \rho A\left[\left(1-\frac{x}{l}\right)\dot{u}_1 + \left(\frac{x}{l}\right)\dot{u}_2\right]^2 dx$$

$\rightarrow$ side note: $\displaystyle\int_0^l \left(1-\frac{x}{l}\right)^2 dx = \int_0^l \left(1 - 2\frac{x}{l} + \frac{x^2}{l^2}\right)dx = \left(x - \frac{x^2}{l} + \frac{1}{3}\frac{x^3}{l^2}\right)\Big|_0^l = l - l + \frac{1}{3}l = \frac{1}{3}l$

$$\int_0^l 2\left(1-\frac{x}{l}\right)\left(\frac{x}{l}\right)dx = 2\int_0^l \left(\frac{x}{l} - \frac{x^2}{l^2}\right)dx = 2\left(\frac{x^2}{2l} - \frac{x^3}{3l^2}\right)\Big|_0^l = l - \frac{2}{3}l = \frac{1}{3}l$$

$$\int_0^l \left(\frac{x}{l}\right)^2 dx = \int_0^l \frac{x^2}{l^2}dx = \frac{x^3}{3l^2}\Big|_0^l = \frac{1}{3}l$$

$\rightarrow$ back to T:

$$T = \frac{1}{2}\rho A \frac{l}{3}\left(\dot{u}_1^2 + \dot{u}_1\dot{u}_2 + \dot{u}_2^2\right) \qquad \vec{\dot{u}}(t) = \begin{Bmatrix}\dot{u}_1 \\ \dot{u}_2\end{Bmatrix}$$

find [m] based on this

$$T = \frac{1}{2}\vec{\dot{u}}^T[m]\vec{\dot{u}}$$

$$= \frac{1}{2}\begin{bmatrix}\dot{u}_1 & \dot{u}_2\end{bmatrix}\begin{bmatrix}m_{11} & m_{12} \\ m_{21} & m_{22}\end{bmatrix}\begin{Bmatrix}\dot{u}_1 \\ \dot{u}_2\end{Bmatrix} = \frac{1}{2}\begin{bmatrix}\dot{u}_1 & \dot{u}_2\end{bmatrix}\begin{Bmatrix}m_{11}\dot{u}_1 + m_{12}\dot{u}_2 \\ m_{21}\dot{u}_1 + m_{22}\dot{u}_2\end{Bmatrix}$$

$$= \frac{1}{2}\left(m_{11}\dot{u}_1^2 + m_{12}\dot{u}_1\dot{u}_2 + m_{21}\dot{u}_1\dot{u}_2 + m_{22}\dot{u}_2^2\right)$$

now match the coefficients w/ $T = \left(\frac{1}{2}\right) \rho A \left(\ell/3\right)\left(\dot{u}_1^2 + \dot{u}_1\dot{u}_2 + \dot{u}_2^2\right)$

$$T = \frac{1}{2}\left(m_{11}\dot{u}_1^2 + m_{12}\dot{u}_1\dot{u}_2 + m_{21}\dot{u}_1\dot{u}_2 + m_{22}\dot{u}_2^2\right)$$

$\therefore\ m_{11} = \rho A\left(\frac{\ell}{3}\right)$ $\quad m_{22} = \rho A\left(\frac{\ell}{3}\right)$ $\quad$ and $\quad \underbrace{m_{12} + m_{21}}$ $= \frac{\rho A \ell}{3}$

so to make the mass matrix symmetric, we say m_{12} and m_{21} are equal

$\therefore$ the mass matrix is:

$$\boxed{[m] = \frac{\rho A \ell}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}$$