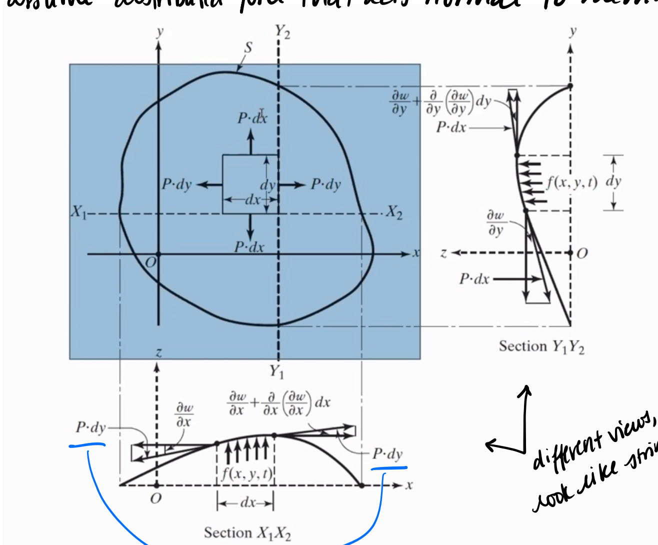
Assignment #7 Announcements
- see example problem 8.4
- mostly pen + paper

Vibrations - Lecture 21
Tue., 2021-04-13

Membranes – 2D string (uncancelled component of tension in string is the restoring force)
- under tension
- bending stiffness, rigidity basically zero
- really, really thin material ($I \to 0$, no bending stiffness)
- only 2D problem we'll work through this semester
  - mathematically 3D (2 spatial: x, y; time)

w is towards us, function of x, y, t
assume tension is everywhere in all directions (P)
assume distributed force that acts normal to membrane
} blue square



different views, look like strings!

force is same on both sides, but different angles => we'll have net vertical force that results from these different angles

8.6 $\Sigma F =$

inertia term: $\underbrace{\rho(x,y)}_{\text{mass per unit area}} \underbrace{\frac{\partial^2 w}{\partial t^2}}_{\text{acceleration}} \underbrace{dx\,dy}_{\text{area}}$ (sum of the forces equals this)

→ equation of motion of the membrane:

$$\underbrace{P}_{\text{tension } \frac{F}{L}}\underbrace{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)}_{\frac{L}{L^2} = \frac{1}{L}} + \underbrace{f}_{\frac{F}{L^2}} = \rho \frac{\partial^2 w}{\partial t^2}$$

$\frac{M}{L^2}, \frac{L}{T^2} \Rightarrow$ multiply by $\frac{FT^2}{ML} \Rightarrow \frac{F}{L^2}$

f: externally applied force in vertical direction (per unit area)
↳ will be zero in free vibration

*all of these terms are pressures

- Free vibration, $f = 0$
- Define sound speed $c = \left(\frac{P}{\rho}\right)^{1/2}$ $\quad \left[\frac{F/L}{M/L^2}\right]^{1/2} \rightarrow \left[\frac{F/L}{M/L^2} \cdot \frac{ML}{FT^2}\right]^{1/2} \rightarrow \frac{L}{T}$ (speed!)

$$\boxed{c^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{\partial^2 w}{\partial t^2}}$$

same as wave equation for string, except adding this new term: "2D"

Laplacian: $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

$$\nabla^2 w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

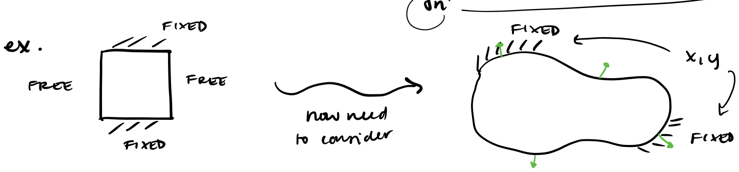
$$c^2 \nabla^2 w = \frac{\partial^2 w}{\partial t^2}$$

- Initial Conditions: $w(x, y, t=0) = w_0(x, y)$

$$\left.\frac{\partial w}{\partial t}(x, y, t)\right|_{t=0} = \dot{w}_0(x, y)$$

- Boundary conditions:
  - fixed over some region (x_1, y_1) then $w(x, y, t=0) = 0$
  - free over some (x_2, y_2) then $P\frac{\partial w}{\partial n}(x_2, y_2, t) = 0$ — n is normal to the boundary

ex.



Separation of variables on this 2D wave equation

$\rightarrow$ we assume $w(x, y, t) = W(x, y)\,T(t)$
$= X(x)\,Y(y)\,T(t)$

$$\frac{\partial^2 w}{\partial x^2} = X''(x)\,Y\,T \qquad \frac{\partial^2 w}{\partial y^2} = X\,Y''(y)\,T \qquad \frac{\partial^2 w}{\partial t^2} = X\,Y\,T''(t)$$

substitute into wave equation: $c^2\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{\partial^2 w}{\partial t^2}$

$c^2\left(X''YT + XY''T\right) = XYT \quad \rightarrow$ divide by XYT

$$c^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right) = \frac{T''}{T} = -\omega^2$$

make this equal to a constant

$$\frac{T''}{T} = -\omega^2 \quad \longrightarrow \quad T'' + \omega^2 T = 0$$

$$c^2\left(\frac{X''}{X} + \frac{Y''}{Y}\right) = -\omega^2 \quad \longrightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} - \left(\frac{\omega}{c}\right)^2 = -\alpha^2$$

$$X'' + \alpha^2 X = 0$$

now have 3 ODE's from 1 PDE with an algebraic relation

$$\frac{Y''}{Y} = -\left(\frac{\omega}{c}\right)^2 + \alpha^2 = \alpha^2 - \left(\frac{\omega}{c}\right)^2 = \beta^2$$

$$\boxed{\alpha^2 + \beta^2 = (\omega/c)^2}$$

$$Y'' + \beta^2 Y = 0$$

it will turn out that:

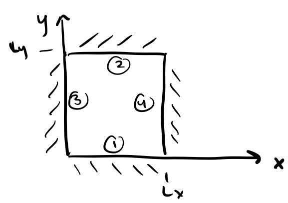
α is a wave number in x (wave number you'd see in x-direction)

β is a wave number in y

$$\boxed{\alpha^2 + \beta^2 = (\omega/c)^2} \longrightarrow \left(\frac{2\pi}{\lambda x}\right)^2 + \left(\frac{2\pi}{\lambda y}\right)^2 = \left(\frac{2\pi}{\lambda}\right)^2$$

remember, $k = \frac{2\pi}{\lambda}$

$$X = C_1 \cos\alpha x + C_2 \sin\alpha x$$
$$Y = C_3 \cos\beta y + C_4 \sin\beta y$$
$$T = A \cos\omega t + B \sin\omega t$$



① $w(x, y=0, t) = 0$

$X(x)\,Y(0)\,T(t) = 0$

$Y(0) = 0$ $\qquad Y(0) = C_3 = 0$

② $w(x, y=L_y, t) = 0$

$X(x)\,Y(L_y)\,T(t) = 0$

$Y(L_y) = 0 \quad = C_4 \sin\beta L_y = 0$

$\beta_n = \frac{n\pi}{L_y} \qquad n = 1, 2, 3, \ldots$

③ $w(x=0, y, t) = X(0)\,Y(y)\,T(t) = 0$

$X(0) = 0$

④ $w(x=L_x, y, t) = X(L_x)\,Y(y)\,T(t) = 0$

$X(L_x) = 0$

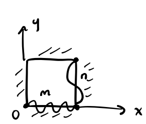
$X = C_2 \sin\alpha x \qquad \alpha_m = \frac{m\pi}{L_x} \qquad m = 1, 2, 3, \ldots$

$$w(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{m,n} \sin(\alpha_m x)\sin(\beta_n y)\,T(t)$$

if you choose any combination of m and n, that's a mode of vibration

$$\alpha_m^2 + \beta_n^2 = \left(\frac{\omega_{mn}}{c}\right)^2$$

can play w/ different mode shapes:



for instance, this one!