Continuation of vibrating String Problems

Vibrations, Lec 16
Thur., 2021-03-25

8.2 String Vibrations

$$c^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial w}{\partial t^2}$$

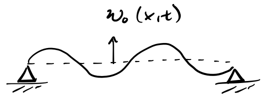
the wave equation

$$c = \left(\frac{P}{\rho}\right)^{1/2}$$

P = tension
ρ = mass/length

mass per unit length (so density * area)

What causes to vibrate?
The initial conditions



$w(x, t=0) = w_0(x)$

$\frac{\partial w}{\partial t}(x, t=0) = \dot{w}_0(x)$

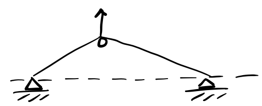
these initial conditions should not violate these B.C.

Boundary conditions because pinned:

$w(x=0, t) = 0$ $\quad$ $w(x=L, t) = 0$

$\dot{w}(x=0, t) = 0$ $\quad$ $\dot{w}(x=L, t) = 0$

when you pluck a string:
get this displacement pattern



$W(x) = A\cos\left(\frac{\omega x}{c}\right) + B\sin\left(\frac{\omega x}{c}\right)$

$T(t) = C\cos(\omega t) + D\sin(\omega t)$

from separation of variables:

$$\frac{c^2 W''}{W} = \frac{T''}{T} = a = -\omega^2$$

for any ω value

actual displacement, $w = W(x)T(t)$

Boundary Conditions:

$w(x,t) = W(x)\,T(t)$

$w(x=0, t) = 0$ (pinned)

$W(x=0)\,T(t) = 0$

get trivial solution

$W(x=0) = 0$ ①

$w(x=L, t) = 0$

$W(x=L)(T(t)) = 0$

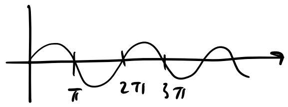
$W(x=L) = 0$ ②

$W(x) = A\cos\left(\frac{\omega x}{c}\right) + B\sin\left(\frac{\omega x}{c}\right)$ ③

apply ①, ② to ③

$W(0) = A = 0$ $\quad$ $W(L) = B\sin\left(\frac{\omega L}{c}\right) = 0$

but maybe this sine term could be zero



then $\overline{W}(L) = B\sin\left(\frac{\omega L}{c}\right)$ is zero when $\frac{\omega L}{c} = n\pi$

$n = 1, 2, \dots$

$$\frac{\omega_n L}{c} = n\pi$$

need this subscript to know which n is being used

n	ω	
1	$\pi c/L$	$= \omega_1$
2	$2\pi c/L$	$= \omega_2$
3	$3\pi c/L$	$= \omega_3$
⋮	⋮	⋮

$$\omega_n = \frac{n\pi c}{L} \quad n = 1, 2, 3$$

natural frequencies of the string

$$c = \sqrt{\frac{P}{\rho}}$$

$$w_\lambda(x,t) = W_n(x)\, T_n(t)$$

$\cancel{A_n\cos\left(\frac{\omega_n x}{c}\right)}$ $+ B_n \sin\left(\frac{\omega_n x}{c}\right)$

$C_n\cos(\omega_n t) + D_n \sin(\omega_n t)$

spatial dependence still related to way things depend on time via ω_n

$$\frac{\omega_n x}{c} = k_n x$$

$$k_n = \frac{\omega_n}{c} \qquad \lambda_n = \frac{2\pi}{k_n}$$

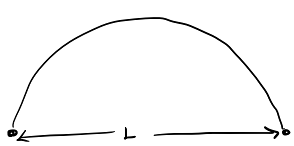
$$\lambda_n = \frac{2\pi}{\omega_n/c} = \frac{2\pi c}{\omega_n} = \frac{c}{f_n} = c\,T_n$$

period (?)

$\omega_n = \frac{n\pi c}{L}$

$$\lambda_n = \frac{2\pi c}{n\pi c/L} = \frac{2L}{n}$$

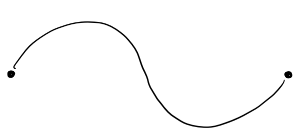
frequency going up, wavelength going down



$n = 1$

$\lambda_n = 2L = \lambda_1$

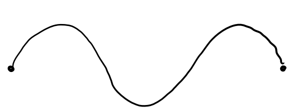
$\omega_1 = \frac{\pi c}{L}$



$n = 2$

$\lambda_n = L = \lambda_2$

$\omega_2 = \frac{2\pi c}{L}$

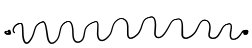


$n = 3$

$\lambda_n = \frac{2L}{3} = \lambda_3$

$\omega_3 = \frac{3\pi c}{L}$

then high n:



$$w_n(x,t) = \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{W_n(x)} \underbrace{\left(C_n \cos \frac{n\pi c t}{L} + D_n \sin \frac{n\pi c t}{L}\right)}_{T_n(t)}$$

$n = 1, 2, 3, \ldots$

initial conditions:

$$w(x, t=0) = w_0(x) = C_n \sin\left(\frac{n\pi x}{L}\right)$$

given initial disp

most general solution:

$$w(x,t) = \sum_{n=1}^{N} w_n(t)$$

$$w(x, t=0) = \boxed{w_0(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)}$$

$n=1$ … $* C_1$

$+$

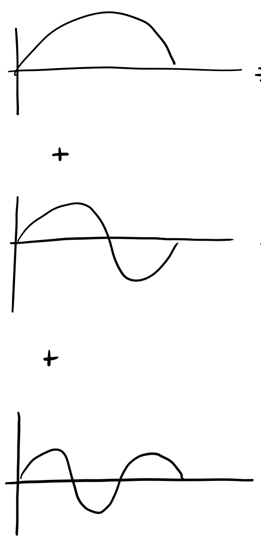
$n=2$ … $* C_2$

$+$

$n=3$ … $* C_3$

$+$

$\vdots$



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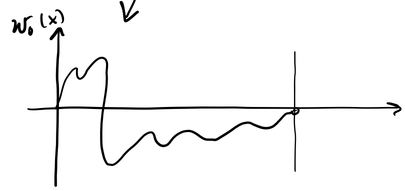

FOURIER SINE SERIES

$$C_n = \frac{2}{L}\int_0^L w_0(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

→ sine function changes w/ n
→ w_0 remains same
→ integrate over length of string

$$\dot{w}(x,t) = \sum_{n=1}^{N} \sin\left(\frac{n\pi x}{L}\right)\left(\frac{n\pi c}{L}\right)\left[-C_n \sin\left(\frac{n\pi c t}{L}\right) + D_n \cos\left(\frac{n\pi c t}{L}\right)\right]$$

$$\boxed{\dot{w}(x, t=0) = \sum_{n=1}^{\infty} \underbrace{\frac{n\pi c}{L} D_n}_{\text{call this } \hat{D}_n} \sin\left(\frac{n\pi x}{L}\right) = \dot{w}_0}$$

then solve for D_n

$$D_n = \frac{2}{n\pi c}\int_0^L \dot{w}_0(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

string_vibration.m

N = # of terms in the series, aka # of degrees of freedom
t = time, usually a vector
L = length of string
X = where on string we want to find vibration
w_0, w_dot_0 = initial conditions
c = ?

randn()

gaussian distribution of random numbers: -1 to 1, mean is 0 st.dev is 1.