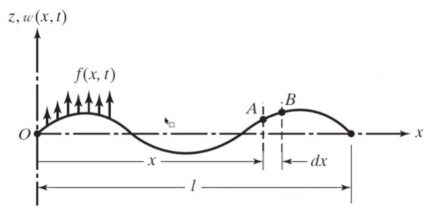
Transverse Vibration of a String or Cable

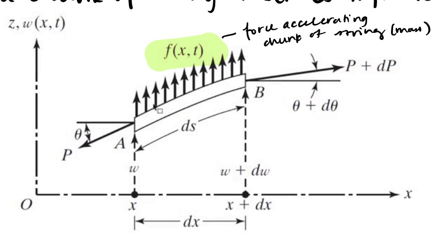
Vibrations - Lec 15
Tue., 2021-03-23

- the vibrating string is a continuous system
  - → everything we've done before is a discrete system (mass, spring, dashpot, certain # of displacements you're tracking aka DoF) ... FEA
- no more linear algebra, now treat continous system as continuous PDE



(a)

cut a chunk of string that is infinitesimally small



(b)

ds is length of string
dx is distance between end points of string

↳ if lying down on x-axis, as in string at rest and not vibrating, then ds = dx

Can this string be stretched?
→ yes, but our analysis will treat this as inextensible

What could f(x,t) be? ex. drag force like on an instrument, maybe also magnetism

This equation is Newton's Law in the vertical direction for the string:

$$\uparrow \oplus \quad (P + dP)\sin(\theta + d\theta) + f\,dx - P\sin\theta = \rho\,dx\,\frac{\partial^2 w}{\partial t^2}$$

(left side is forces in vertical direction, right side is ma)

where P is the tension, ρ is mass per unit length, θ is angle deflected string makes w/ x-axis

↳ for an elemental length dx: $dP = \frac{\partial P}{\partial x}dx$

↳ $\sin\theta \simeq \tan\theta = \frac{\partial w}{\partial x}$ ← small θ assumption

↳ $\sin(\theta + d\theta) \simeq \tan(\theta + d\theta) = \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}dx$ ← almost like a Taylor series

w = xverse displacement of string, $w(x,t)$
f = xverse force per unit length, $f(x,t)$

The Linearized Equation (Most-general): P is allowed to depend on displacement

$$\frac{\partial}{\partial x}\left[P\,\frac{\partial w(x,t)}{\partial x}\right] + f(x,t) = \rho(x)\,\frac{\partial^2 w(x,t)}{\partial t^2}$$

The Wave Equation: when P is constant

$$P\,\frac{\partial^2 w(x,t)}{\partial x^2} + f(x,t) = \rho\,\frac{\partial^2 w(x,t)}{\partial t^2}$$

* hallmarks of wave equation: 2nd derivate wrt displacement on one side, 2nd derivative wrt time on other

→ if $f(x,t) = 0$ then we obtain the free-vibration equation (no forcing)

$$P \frac{\partial^2 w(x,t)}{\partial x^2} = \rho \frac{\partial^2 w(x,t)}{\partial t^2}$$

$$\Downarrow$$

$$\boxed{c^2 \frac{\partial^2 w}{\partial x^2} = \frac{\partial^2 w}{\partial t^2}}$$

c = the speed at which your wave will travel

Let's solve this wave equation

$$c = \text{sound speed} = \sqrt{\frac{P}{\rho}} = \sqrt{\frac{F}{M/L}} = \sqrt{\frac{ML/T^2}{M/L}} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T} \checkmark$$

Initial conditions: need to know

$w(x, t=0) = w_0(x)$ ← initial displacement

$\frac{\partial w}{\partial t}(x, t=0) = \dot{w}_0(x)$ ← initial velocity

(we're assuming that the string is pinned at both ends @ $t=0$)

Boundary conditions:

$w(x=0, t) = w(x=l, t) = 0$

Separation of Variables: $w(x,t)$

make a guess: $w(x,t) = W(x)\,T(t)$ ← make two separate functions multiplied by each other that each take one variable (note that not every function can do this, such as x^t or $\log_x t$)

$$\frac{\partial^2 w}{\partial x^2} = W''T$$

$$\underbrace{\frac{\partial^2 w}{\partial t^2} = WT''}_{\text{plug these into our wave equation}}$$

$$c^2 W'' T = W T''$$

(we want dependence on x on one side, dependence on t on other side)

$$c^2 \frac{W''(x)}{W(x)} = \frac{T''(t)}{T(t)}$$

true for all x from 0 to l

true for time @ zero and on

⋮

what if we call this ratio a constant a? ... then a must also be equal to the time ratio ... a is a "separation constant"

So now we've taken a PDE and split it into 2 ODE's

$c^2 \frac{W''}{W} = a \quad \Rightarrow \quad c^2 W'' - aW = 0$ (a displacement equation)

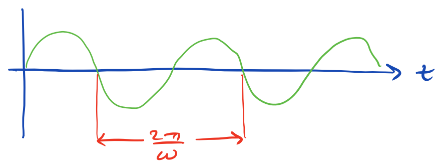
$\frac{T''}{T} = a \quad \Rightarrow \quad T'' - aT = 0$ (a time equation)

we typically see ω^2 here...
set $a = -\omega^2$ ← this is omega, as in $\omega = \text{rad/s}$

$$T'' + \omega^2 T = 0$$

so omega is the frequency of this function:

soln to the time equation → $T(t) = C\cos\omega t + D\sin\omega t$



$$W'' + \frac{\omega^2}{c^2} W = 0$$

$W(x) = A\cos\left(\frac{\omega x}{c}\right) + B\sin\left(\frac{\omega x}{c}\right)$ ← soln to the spatial equation

In Acoustics (not in book):

$k = \frac{\omega}{c}$ aka the "wave number"

↳ $W(x) = A\cos(kx) + B\sin(kx)$
$T(t) = C\cos(\omega t) + D\sin(\omega t)$

$x \longleftrightarrow t$

$\left(\frac{1}{m}\right)\ k \qquad \omega\ \left(\frac{rad}{s}\right)$

remember, can't take sine or cosine of a dimensional #

Period : $2\pi/\omega$
(T_0)

the wavelength λ is the analogous in the spatial domain

$\Rightarrow \lambda = \frac{2\pi}{k}$