Assignment 1 Notes:
- problem #2 changed

Vibrations - L5
2021-02-09

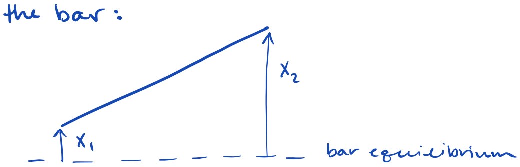
If you have 2 equations of motion:

① $a + b = F_1 + F_2$
② $c + d = F_1 - F_2$

$\left(\frac{a+b+c+d}{2}\right) = F_1$

$\left(\frac{a+b-c-d}{2}\right) = F_2$

the bar:



$\left.\begin{array}{l} \Sigma \underset{\sim}{F} = m \underset{\sim}{a}_{cg} \\ \Sigma M_o = J_o \ddot{\theta} \end{array}\right\}$ these only work for CG... need to do some geometry

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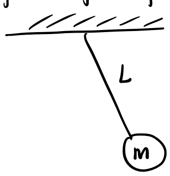
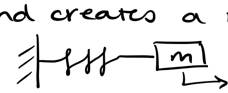
Only time gravity does something crazy to us... our friend the pendulum



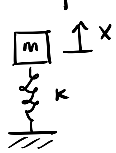
$\omega_n = \sqrt{\frac{g}{l}}$ ... in this case, gravity is our "spring"

if we deflect the pendulum, gravity acts down and creates a moment, helping return back to equilibrium
→ oscillations, mass takes time to slow down

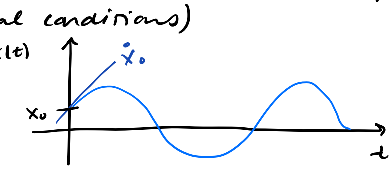


THE EIGENVALUE PROBLEM

a single degree of freedom problem, no force, free vibration (the only way to make it vibrate is to give initial conditions)



$x_0,\ \dot{x}_0$ @ $t = 0$



@ positive time, F is zero

Eq. of motion: $m\ddot{x} + kx = 0$
"the force accelerating the mass is the spring force which is the double integral of the $\ddot{x}$ and k" ... x and $\ddot{x}$ have to be in "harmony"

solve the equation: make a guess

$x = A\cos\omega_n t$
$\ddot{x} = (-\omega_n^2) A\cos\omega_n t$
$\quad = -\omega_n^2 x$

plug this in...
$[-\omega_n^2 m + k]\, \underbrace{A\cos\omega_n t}$ $= 0$

but this doesn't work for all time ... $\Rightarrow \omega_n^2 = \frac{k}{m} \rightarrow \omega_n = \sqrt{k/m}$
say A=0 that's the trivial soln.

The actual soln is: $x(t) = A\cos\omega_n t + B\sin\omega_n t$

this equation is a 1x1 eigenvalue problem: $\underset{1\times1}{k} - \omega_n^2 \underset{1\times1}{m} = 0$

$$\underset{N\times N}{k} - \omega_n^2 \underset{N\times N}{m} = 0$$ where N is the size of the system

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Let's start w/ a general N×N system:

$$\underset{N\times N}{[M]}\{\ddot{x}\} + [K]\underset{N\times 1}{\{x\}} = \{F(t)\}$$ (no damping for now)

Free vibration – how could one of those drawings in assignment 1 vibrate if there was no force? $\{F(t)\} = \{0\}$

→ initial conditions

N×1

$\{x_0\}$ initial displacement

$\{\dot{x}_0\}$ initial velocity

→ substitute:

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

assumption: $x_i(t) = X_i T(t)$ (multidegree of freedom)

i^{th} element of $\{x\}$

$i = 1, 2, \ldots, N$

there's no i subscript on T (which depends on time) but not on X which does not depend on time

⇒ so the assumption is that all of these x's have consistent time scaling

$x_1(t) = X_1 T(t)$

$x_2(t) = X_2 T(t)$

so then we write displacement vector $\{x(t)\} = \{X\}T(t)$

X tells us amplitude of element 1 relative to 2, etc.

these represent a scaling up or down

$$\begin{Bmatrix} X_1 \\ X_2 \\ \vdots \end{Bmatrix}$$

so then equation of motion:

$$[M]\{X\}\ddot{T}(t) + [K]\{X\}T(t) = \{0\}$$

Take a look at the i^{th} row of this matrix equation

$$[M]\{\Phi\}\ddot{T}(t) + [K]\{\Phi\}T(t) = \{0\}$$

then in summation form: $\left(\sum_{j=1}^{n} M_{i,j}\Phi_j\right)\overset{\text{scalar}}{\ddot{T}(t)} + \left(\sum_{j=1}^{n} K_{i,j}\Phi_j\right)\overset{\text{scalar}}{T(t)} = 0$

move items that depend on time over...

$$\left(\sum_{j=1}^{n} M_{i,j}\Phi_j\right)\frac{\ddot{T}(t)}{T(t)} + \left(\sum_{j=1}^{n} K_{i,j}\Phi_j\right) = 0$$

$$\frac{\ddot{T}(t)}{T(t)} + \frac{\left(\sum_{j=1}^{n} K_{i,j}\Phi_j\right)}{\left(\sum_{j=1}^{n} M_{i,j}\Phi_j\right)} = 0$$

$$\underbrace{\frac{\left(\sum_{j=1}^{n} K_{i,j}\Phi_j\right)}{\left(\sum_{j=1}^{n} M_{i,j}\Phi_j\right)}}_{\text{this doesn't depend on time}} = \underbrace{\frac{-\ddot{T}(t)}{T(t)}}_{\text{this side of the equation depends on time}}$$

$$\frac{-\ddot{T}(t)}{T(t)} = \frac{\left(\sum_{j=1}^{n} K_{i,j}\Phi_j\right)}{\left(\sum_{j=1}^{n} M_{i,j}\Phi_j\right)} = \omega^2$$

} 2 equations here

$$\left(\frac{-\ddot{T}(t)}{T(t)} = \omega^2\right) T(t)$$ multiply by $T(t)$

$\hookrightarrow$ $\ddot{T}(t) + \omega^2 + T(t) = 0$

one soln: $T(t) = A\cos\omega t + B\sin\omega t$

another soln: $T(t) = C_1\cos(\omega t + \phi)$ ... a shifted cosine

} these are called sinusoid ... can shift a sine or cosine, doesn't really matter

now the other eq.

$$-\omega^2\sum_{j=1}^{n} M_{i,j}\Phi_j + \sum_{j=1}^{n} K_{i,j}\Phi_j = 0$$ (reminder i^{th} row)

→ writing for all rows:

$-\omega^2[M]\{\Phi\} + [K]\{\Phi\} = \{0\}$ ⇒ associative: $\left(-\omega^2[M] + [K]\right)\{\Phi\} = \{0\}$

Side note on Linear Algebra:

$$[A]\{x\} = \{b\}$$

$|[A]| = 0$, determinant is zero, we say unique soln does not exist

if have homogeneous equation:

$$[A]\{x\} = \{0\} \qquad |[A]| = 0$$

then get zero or more than one solution

back to vibrations:

$$\underbrace{\left(-\omega^2[M] + [K]\right)}_{A(\omega)}\{X\} = \{0\}$$

$$|[A(\omega)]| = 0$$

$$|[K] - \omega^2[M]| = 0$$

how does this work if A and x are full of numbers?

$$[A]\{x\} = \{0\}$$

go back to LA:

$$\begin{cases} 2x_1 + 3x_2 = 0 \\ 4x_1 + 6x_2 = 0 \end{cases}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$x_1 = -\frac{3}{2} x_2 \qquad \frac{x_1}{x_2} = \frac{-3}{2}$ ... so as long as this ratio is satisfied it works
... there's no unique answer
... there's >1 soln to this homogeneous system of equations

$\begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$ ... infinite # of solns

↑ rank deficient matrix

... system of homogeneous solns (= 0) with many solns

back to vibrations: $|[K] - \omega^2[M]| = 0$

looking at 2 DoF system: $N=2$

$$\left| \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} - \omega^2 \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} k_{11} - \omega^2 M_{11} & k_{12} - \omega^2 M_{12} \\ k_{21} - \omega^2 M_{21} & k_{22} - \omega^2 M_{22} \end{vmatrix} = 0$$

$$(k_{11} - \omega^2 M_{11})(k_{22} - \omega^2 M_{22}) - (k_{12} - \omega^2 M_{12})(k_{21} - \omega^2 M_{21}) = 0$$

→ when expand this get: constants that don't depend on ω^2, things times ω^2, things times ω^4

$$(\quad\quad)(\omega^2)^2 + (\quad\quad)(\omega^2) + (\quad\quad) = 0$$

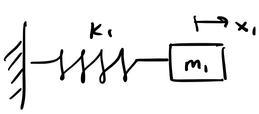
ex.

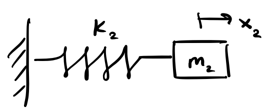
$$5(\omega^2)^2 + 4(\omega^2) + 3 = 0$$

$$\omega^2_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4(5)(3)}}{2(5)}$$

turns out that $\omega_{1,2}$ are the two natural frequencies of a 2 DoF system

in general: N natural frequencies for N DoF system



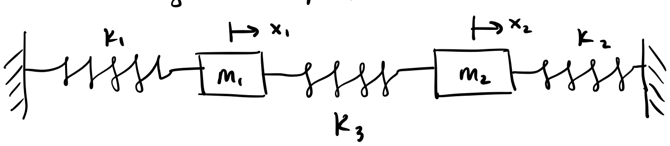


this is a 2 DoF system (can also say 2 1 DoF systems)

$$\omega_1 = \sqrt{\frac{k_1}{m_1}} \qquad \omega_2 = \sqrt{\frac{k_2}{m_2}}$$

$$[K] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \qquad [M] = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$$

same thing but flip around w/ little spring:



- still 2 natural frequencies ω_1, ω_2
  - → when ω_1 both masses move
  - → when ω_2 both masses move

  because k_3 is a coupling spring

  ... note that ω_1 and ω_2 are not supposed to correspond to $m_1 / k_1 / x_1$ or $m_2 / k_2 / x_2$ ... the ω subscript is completely different