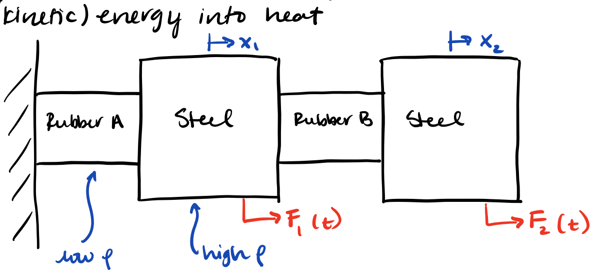
Multidegree-of-Freedom Systems will give way to lots of linear equations

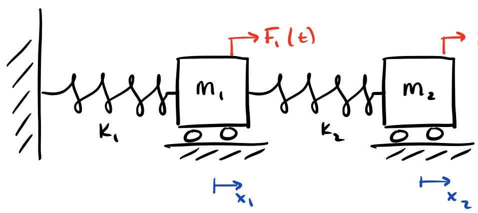
Vibrations - Lec 2
2021-01-28

Dashpots are fictitious systems that we'll frequently use, which turns vibrations (kinetic) energy into heat



→ here, we take this system and make an assumption that rubbers stretch (storing potential energy), steel displaces (K.E.)

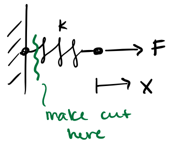
→ we model things w/potential energy as springs and those with KE as rigid bodies



This is called a lumped-parameter approximation

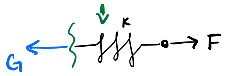
the displacements of m_1 and m_2 are responsible for giving insight into spring forces thereby k_1 and k_2, which in turn cause an acceleration on m_1 and m_2

Rules for Springs (something more flexible than massive)



$F = kx$

ut tensio sic vis



$\Sigma F = \cancel{m}a^{\,0}$

$F - G = 0$

$F = G$

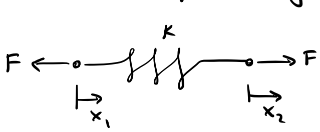
guessing the direction the force is in

$\hat{\imath} \to \oplus$

applied force to right meaning wall pulls back in other direction

→ if F ⊕ then tension, ⊖ compression

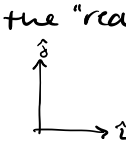
so far we've assumed that left end is always fixed, but what if it gets to move independently like the other... this leads to the "real" Hooke's Law



$F = k\delta$ $\quad \delta = x_2 - x_1$

tension

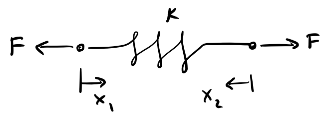
elongation



Let's play a game: $F = k\delta$ (tension, elongation)

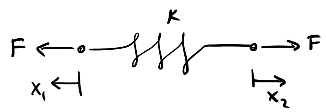
arrows pointing to inside of spring $\ominus$
arrows pointing to outside of spring $\oplus$

$F = k(x_2 - x_1)$

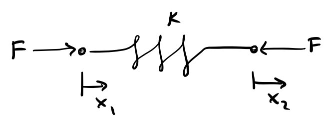


$F = k(-x_2 - x_1)$

means that F is a compressive force



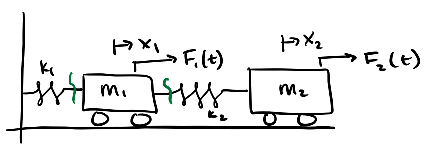
$F = k(x_2 + x_1)$



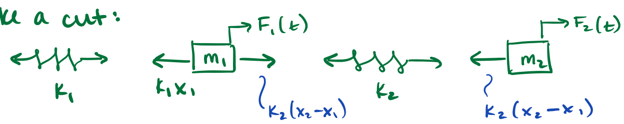
so then $-F$ $\iff$

$-F = k(x_2 - x_1)$

Derive Equations of Motion for this System



make a cut:



(make assumption $x_2 > x_1$)

$\longrightarrow \hat{\imath}$

$$\Sigma F_x = m_1 \ddot{x}_1$$

mass 1

$$-k_1 x_1 \hat{\imath} + F_1(t)\hat{\imath} + k_2(x_2 - x_1)\hat{\imath} = m_1 \ddot{x}_1 \hat{\imath}$$

$$\underline{\underline{(1)}} \quad m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = F_1(t)$$

mass 2

$$-k_2(x_2 - x_1)\hat{\imath} + F_2(t)\hat{\imath} = m_2 \ddot{x}_2 \hat{\imath}$$

$$\underline{\underline{(2)}} \quad m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 = F_2(t)$$

The displacement x_2 will influence the elongation of k_2 which in turn affects the acceleration of m_1, etc.

In order to solve, need to know initial conditions $(x_1(0), \dot{x}_1(0), x_2(0), \dot{x}_2(0))$ and equations of motion $(F_1(t)$ & $F_2(t))$ ... find $x_1(t)$ and $x_2(t)$

Putting these two equations of motion into matrix form to solve.

$$\underset{\text{x vector}}{\{x\}} = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \text{displacement vector}$$

$$\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \end{Bmatrix}$$

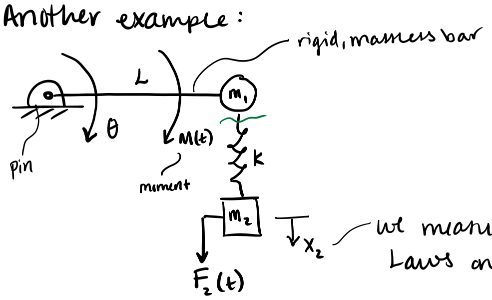
$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}}_{\substack{2\times 2 \\ \text{Mass Matrix}}} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \underbrace{\begin{bmatrix} (k_1+k_2) & -k_2 \\ -k_2 & k_1 \end{bmatrix}}_{\substack{2\times 2 \\ \text{Stiffness Matrix}}} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \underbrace{\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}}_{\text{Force Vector}}$$

① first row is eq. ($\underline{1}$)

② second row is eq. ($\underline{2}$)

2 equations of motion, 2 Degree of Freedom system

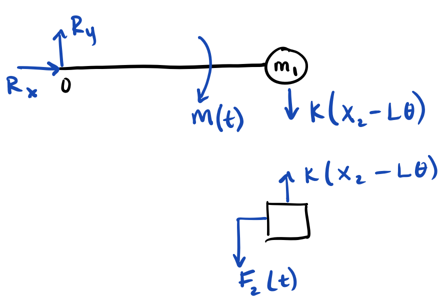
Another example:



we measure x_2 relative to its original position, Newton's Laws only valid in inertial reference system

assume θ is small, approximate
$\sin\theta \approx \theta$
$\cos\theta \approx 1$

FBD: can assume vertical displacement of m_1 to be $L\theta$ because small θ and that $x_2 > L\theta$



$$\Sigma M_0 = \underset{\text{mass moment about 0}}{J_0}\ddot{\theta}$$

take θ clockwise ⊕, ∴ $M(t)$ ⊕

$$M(t) + kL(x_2 - L\theta) = J_0\ddot{\theta}$$

$J_0 = m_1 L^2$ since m_1 pt. mass

$$\underline{(1)} \quad M(t) + kL(x_2 - L\theta) = m_1 L^2 \ddot{\theta}$$

$$\underline{(2)} \quad F_2(t) - k(x_2 - L\theta) = m_2 \ddot{x}_2$$

$$\begin{bmatrix} J_0 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} kL^2 & -kL \\ -kL & k \end{bmatrix} \begin{Bmatrix} \theta \\ x_2 \end{Bmatrix} = \begin{Bmatrix} M(t) \\ F_2(t) \end{Bmatrix}$$

Supplemental Book Notes

Procedure to derive equations of motion of multidegree-of-freedom system using Newton's 2nd Law of Motion

1. set up coordinate system, assume suitable ⊕ directions
2. determine static equilibrium configuration and measure displacements of masses + rigid bodies from respective static equilibrium positions
3. draw FBD
4. apply Newton's 2nd to each mass or rigid body shown by FBD as

$$m_i \ddot{x}_i = \sum_j F_{ij} \quad \text{(for mass } m_i\text{)}$$

—or—

$$J_i \ddot{\theta}_i = \sum_j M_{ij} \quad \text{(for rigid body of inertia } J_i\text{)}$$

where ΣF_{ij} denotes the sum of all forces acting on mass m_i and ΣM_{ij} indicates sum of moments of all forces about suitable axis acting on the rigid body of mass moment of inertia J_i

$$\overset{\text{mass}}{[m]}\ddot{\vec{x}} + \overset{\text{damping}}{[c]}\dot{\vec{x}} + \overset{\text{stiffness}}{[k]}\vec{x} = \vec{F}$$

in general form, mass, damping, and stiffness matrices given by:

mass matrix
$$[m] = \begin{bmatrix} m_{11} & m_{12} & m_{13} & \cdots & m_{1n} \\ m_{21} & m_{22} & m_{23} & \cdots & m_{2n} \\ \vdots & & & & \\ m_{n1} & m_{n2} & m_{n3} & \cdots & m_{nn} \end{bmatrix}$$

displacement: $\vec{x}$
$$\begin{Bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{Bmatrix}$$

damping matrix
$$[c] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2n} \\ \vdots & & & & \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{bmatrix}$$

velocity: $\dot{\vec{x}}$
$$\begin{Bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{Bmatrix}$$

acceleration: $\ddot{\vec{x}}$
$$\begin{Bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \vdots \\ \ddot{x}_n(t) \end{Bmatrix}$$

stiffness matrix
$$[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} & \cdots & k_{1n} \\ k_{21} & k_{22} & k_{23} & \cdots & k_{2n} \\ \vdots & & & & \\ k_{n1} & k_{n2} & k_{n3} & \cdots & k_{nn} \end{bmatrix}$$

force: $\vec{F}$
$$\begin{Bmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{Bmatrix}$$

for an undamped system, the equations of motion reduce to

$$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{F}$$