One final assignment on FE stuff will go out Thursday and we'll have one week to complete it

Vibrations - Lecture 23
Tue., 2021-04-20

from last time: mass matrix of a chunk of a bar l long

$$[m] = \frac{\rho A l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

shape functions: $u(x,t) = N_1(x)u_1(t) + N_2(x)u_2(t)$

u_1 = left axial displacement
u_2 = right axial displacement

$$N_1(x) = 1 - \frac{x}{l} \qquad N_2(x) = \frac{x}{l}$$

what assumptions did we make for the mass matrix?
- axial displacement is linear (if not linear, then mass matrix will not give us the right answer) ... must make elements small enough so that linear

stiffness matrix:

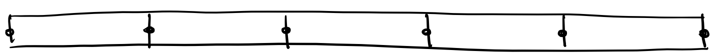
$$[k] = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

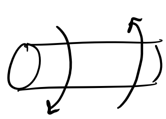
E, A ↑, more stiff
l ↑, less stiff

force vector: $\vec{f} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$

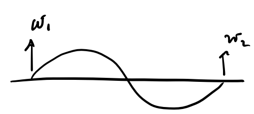
$[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{F}(t) \rightarrow$ 

turn this into global model eventually



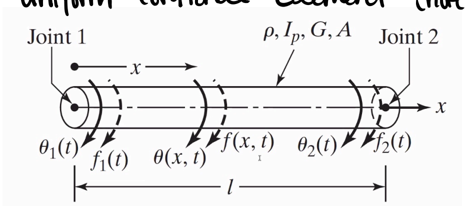


torsion
2^{nd} order PDE
so 2x2 matrices



beam bending is 4^{th} order PDE ... we'll therefore have 4x4 matrices

Uniform Torsional Element (note that l is length of element, not whole bar)



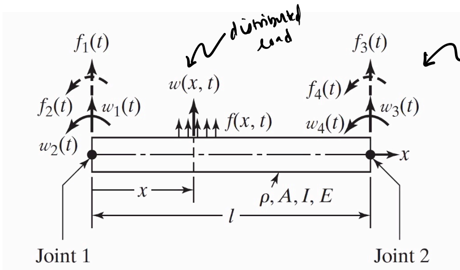
f_1 and f_2 are torsions or moments

pages 1020-1021

... but the beam element is probably most used in FEA
→ in 3D for one element, 12 DoF : 4 flexure, 4 flexure, 2 torsion, 2 compression
↳ so then get 12×12 matrices

Beam Element — every beam element has 4 DoF
- because 4^{th} order, must be modeled as cubic

$$w(x,t) = a(t) + b(t)\,x + c(t)\,x^2 + d(t)\,x^3$$



a uniform beam element

the unknown joint displacements must satisfy the conditions:

$$w(0,t) = w_1(t) \qquad dw/dx\,(0,t) = w_2(t)$$
$$w(l,t) = w_3(t) \qquad dw/dx\,(l,t) = w_4(t)$$

then:

$$a(t) = w_1(t)$$
$$b(t) = w_2(t)$$
$$c(t) = \frac{1}{l^2}\left[-3w_1(t) - 2w_2(t) + 3w_3(t) - w_4(t)\,l\right]$$
$$d(t) = \frac{1}{l^3}\left[2w_1(t) + w_2(t)\,l - 2w_3(t) + w_4(t)\,l\right]$$

substitute these into $w(x,t) = a(t) + b(t)\,x + c(t)\,x^2 + d(t)\,x^3$

$$w(x,t) = \left(1 - 3\frac{x^2}{l^2} + 2\frac{x^3}{l^3}\right)w_1(t) + \left(\frac{x}{l} - 2\frac{x^2}{l^2} + \frac{x^3}{l^3}\right)l\,w_2(t)$$
$$+ \left(3\frac{x^2}{l^2} - 2\frac{x^3}{l^3}\right)w_3(t) + \left(-\frac{x^2}{l^2} + \frac{x^3}{l^3}\right)l\,w_4(t)$$

rewritten as: $w(x,t) = \sum_{i=1}^{4} N_i(x)\,w_i(t)$ where $N_i(x)$ are the shape functions

$$N_1(x) = 1 - 3\left(\frac{x}{l}\right)^2 + 2\left(\frac{x}{l}\right)^3$$

$$N_2(x) = x - 2l\left(\frac{x}{l}\right)^2 + l\left(\frac{x}{l}\right)^3$$

$$N_3(x) = 3\left(\frac{x}{l}\right)^2 - 2\left(\frac{x}{l}\right)^3$$

$$N_4(x) = -l\left(\frac{x}{l}\right)^2 + l\left(\frac{x}{l}\right)^3$$

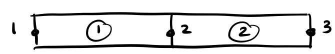
Eventually we get the 4x4 mass and stiffness matrices

$$[m] = \frac{\rho A l}{420}\begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

ρ = density of beam
E = YM
I = moment of inertia of cross section
A = cross section

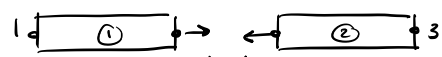
$$[k] = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

example: assembling global matrices



$[m_1]\ddot{\vec{x}}_1 + [k_1]\vec{x}_1 = F_1$

$[m_2]\ddot{\vec{x}}_2 + [k_2]\vec{x}_2 = F_2$



these displacements must be equal

$$\underset{3\times 3}{\begin{bmatrix} \; m \; \end{bmatrix}} \ddot{\vec{x}} + \underset{3\times 3}{\begin{bmatrix} \; k \; \end{bmatrix}} \vec{x} = \vec{F}$$

Equations of Motion of the Complete System of Finite Elements : now extend the eq of motion obtained for single finite elements in the global system to the complete structure

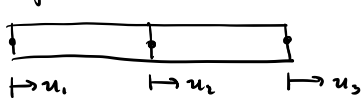
→ denote the joint displacements of the complete structure in the global coordinate system

$$\begin{Bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_M(t) \end{Bmatrix}$$

$\vec{U}^{(e)}(t) = [A^{(e)}]\,\vec{\underline{U}}(t)$

where $[A^{(e)}]$ is a rectangular matrix composed of zeros and ones (the unitary matrix) ... mostly zeros

Making connectivity matrices:



$U^e = A^e U$

$$U = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$U^e = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

MATLAB codes:

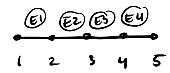
barmatrices.m

for these models, will always have one more node than elements

- computes GLOBAL stiffness and mass matrices of bar
- uses bar element
- rho is a vector (density of each element)
- l is vector of lengths of elements
- E is a vector
- nodes is a matrix
  - 2 rows
  - every column is an element

$$\left[\begin{array}{c|c|c|c} 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 & 5 \end{array}\right]$$

E1 E2 E3 E4

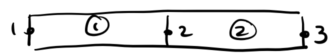


- [m], [k] are same matrices that we made last class

↳ so ρ, A, E will be 4 x 1

- create connectivity matrix A_e

↳ let's take deeper look:



$$\text{nodes} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

element 2 : $\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$ $A^e = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$

bar_simulation.m

- 500 element mesh
- fixed on left, free on right
  - pinned all of the nodes except last one, give that displacement of 1
- use integrate_viscous.m (runge kutta)

Watch end of this lecture for simulations & mode-shapes