MATH 271, Homework 10, *Solutions*

Problem 1. For each description below, construct a single 3×3 transformation matrix.

(a) A rotation by $\frac{\pi}{2}$ in the yz-plane.

(b) A rotation by $\frac{\pi}{2}$ in the xy-plane, an interchange of the x and y coordinates.

(c) A matrix which undoes everything in part (b).

Solution 1. First, note that these rotation matrices are in my notes as well as readily available online. Specifically, we have

$$[\mathrm{Rot}]_{xy,\theta} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$[\mathrm{Rot}]_{yz,\theta} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Can you derive what the matrix for rotating in the xz-plane would be?

(a) Taking $[\mathrm{Rot}]_{yz,\theta}$ above with $\theta = \pi/2$, we get

$$[\mathrm{Rot}]_{yz,\pi/2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(b) We first create the rotation matrix

$$[\mathrm{Rot}]_{xy,\pi/2} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then, the matrix that exchanges x and y coordinates is

$$[S] = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the matrix for (b) is given by

$$[S][\mathrm{Rot}]_{xy,\pi/2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) Now, note that in (b) we found the matrix simply reflects across the yz-plane by outputting negative the input x-value. Thus, the inverse for this matrix is in fact itself!

Remark 1. The matrix in (c) generates a group by multiplication. Specifically, this is the group with the elements

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is called the *cyclic group of order 2* and denoted by C_2. This group is the reflection symmetry group as reflecting across the same plane twice does nothing.

Problem 2. Consider the system of linear equations

$$\begin{aligned} 2x - y - 2z &= 4 \\ -x + 3y + 4z &= -2 \\ 2x + y - 2z &= 8. \end{aligned}$$

(a) Write this system of equations as a matrix/vector equation

$$[A]\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}}$$

(b) Solve the system of equations by finding the inverse of the matrix $[A]$.

Solution 2. Let

$$\vec{\boldsymbol{x}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{and} \qquad \vec{\boldsymbol{y}} = \begin{pmatrix} 4 \\ -2 \\ 8 \end{pmatrix}$$

Then we want a matrix $[A]$ such that we have

$$[A]\vec{\boldsymbol{x}} = \vec{\boldsymbol{y}},$$

which is given by the coefficients on the system above. That is,

$$[A] = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 3 & 4 \\ 2 & 1 & -2 \end{pmatrix}$$

This gives us the augmented matrix

$$[M] = \left(\begin{array}{ccc|c} 2 & -1 & -2 & 4 \\ -1 & 3 & 4 & -2 \\ 2 & 1 & -2 & 8 \end{array}\right).$$

Then we can row reduce this matrix by first subtracting R_1 from R_3 to get

$$\left(\begin{array}{ccc|c} 2 & -1 & -2 & 4 \\ -1 & 3 & 4 & -2 \\ 0 & 2 & 0 & 4 \end{array}\right),$$

then adding $1/2 \cdot R_1$ to R_2 to get

$$\left(\begin{array}{ccc|c} 2 & -1 & -2 & 4 \\ 0 & 5/2 & 3 & 0 \\ 0 & 2 & 0 & 4 \end{array}\right),$$

then subtracting $5/4 \cdot R_3$ from R_2 to get

$$\left(\begin{array}{ccc|c} 2 & -1 & -2 & 4 \\ 0 & 0 & 3 & -5 \\ 0 & 2 & 0 & 4 \end{array}\right),$$

then adding $1/2 \cdot R_3$ to R_1 to get

$$\left(\begin{array}{ccc|c} 2 & 0 & -2 & 6 \\ 0 & 0 & 3 & -5 \\ 0 & 2 & 0 & 4 \end{array}\right),$$

then we can swap R_2 and R_3 and divide by some constants to get

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & & 2 \\ 0 & 0 & 1 & -5/3 \end{array}\right),$$

and finally add R_3 to R_1 to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4/3 \\ 0 & 1 & & 2 \\ 0 & 0 & 1 & -5/3 \end{array}\right)$$

which gives us $x = 4/3$, $y = 2$, and $z = -5/3$.

Now, we can repeat those same steps above, but with a different augmented matrix. Namely, we start with

$$[M] = \left(\begin{array}{ccc|ccc} 2 & -1 & -2 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 2 & 1 & -2 & 0 & 0 & 1 \end{array}\right).$$

Perform the exact same row operations, and we get

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5/6 & 1/3 & -1/6 \\ 0 & 1 & 0 & -1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 7/12 & 1/3 & -5/12 \end{array}\right),$$

which means the inverse matrix is

$$[A]^{-1} = \begin{pmatrix} 5/6 & 1/3 & -1/6 \\ -1/2 & 0 & 1/2 \\ 7/12 & 1/3 & -5/12 \end{pmatrix}.$$

Then to solve the equation we take

$$\vec{\boldsymbol{x}} = [A]^{-1}\vec{\boldsymbol{y}},$$

and we get that

$$\vec{\boldsymbol{x}} = \begin{pmatrix} 4/3 \\ 2 \\ -5/3 \end{pmatrix}.$$

Remark 2. This is why computing the inverse is advantageous. It takes no more work to compute, but it makes the job easier if we had to compute the solution to many more equations.

Problem 3. Consider the matrix

$$[A] = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

(a) Compute the eigenvalues and eigenvectors for the matrix $[A]$.

(b) Show that $\det([A]) = \lambda_1\lambda_2\lambda_3$ and $\mathrm{tr}([A]) = \lambda_1 + \lambda_2 + \lambda_3$ where λ_1, λ_2, and λ_3 are the eigenvalues you found in (a).

(c) Argue why the eigenvalues of $[A]^{-1}$ must be $\frac{1}{\lambda_1}$, $\frac{1}{\lambda_2}$, and $\frac{1}{\lambda_3}$.

Solution 3.

(a) We first compute the characteristic polynomial by

$$\det([A] - \lambda[I]) = \begin{vmatrix} 1-\lambda & 3 & 0 \\ 3 & 1-\lambda & 0 \\ 0 & 4 & 1-\lambda \end{vmatrix}.$$

Then, expand along the right column to get

$$\begin{vmatrix} 1-\lambda & 3 & 0 \\ 3 & 1-\lambda & 0 \\ 0 & 4 & 1-\lambda \end{vmatrix} = (1-\lambda)((1-\lambda)^2 - 9).$$

Then we find that one root is $\lambda_1 = 1$ and we have left

$$(1-\lambda)^2 - 9 = 0,$$

which has the roots $\lambda_2 = -2$ and $\lambda_3 = 4$.

Now we find the eigenvectors. First for $\lambda_1 = 1$, we have

$$[A] - 1[I] = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix}.$$

Hence, it must be that $x = y = 0$ and z is free. So the first eigenvector is

$$\vec{e}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Next, for $\lambda_2 = -2$, we have

$$[A] + 2[I] = \begin{pmatrix} 3 & 3 & 0 \\ 3 & 3 & 0 \\ 0 & 4 & 3 \end{pmatrix}.$$

Hence $-x = y$ and $z = -4/3y$. Hence we can choose $x = 3$ (just to make the vector have integer entries) to get the second eigenvector

$$\vec{\boldsymbol{e}}_2 = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix}.$$

Lastly, for $\lambda_3 = 4$, we have

$$[A] - 4[I] = \begin{pmatrix} -3 & 3 & 0 \\ 3 & -3 & 0 \\ 0 & 4 & -3 \end{pmatrix}.$$

Thus we have $x = y$ and $z = 4/3y$. Again, choose $x = 3$ to get the last eigenvector

$$\vec{\boldsymbol{e}}_3 = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}.$$

(b) Now, we have by expanding along the right column

$$\det([A]) = 1(1 \cdot 1 - 3 \cdot 3) = -8.$$

Then we also have

$$\lambda_1 \lambda_2 \lambda_3 = -8.$$

Lastly, we have

$$\operatorname{tr}([A]) = 1 + 1 + 1 = 3$$

and

$$\lambda_1 + \lambda_2 + \lambda_3 = 3.$$

(c) Now, we know that

$$\det([A]^{-1}) = \frac{1}{\det([A])},$$

which means that

$$\det([A]^{-1}) = \frac{1}{\lambda_1 \lambda_2 \lambda_3}.$$

However, this is not quite enough to realize that the eigenvalues are in fact $\frac{1}{\lambda_1}$, $\frac{1}{\lambda_2}$, and $\frac{1}{\lambda_3}$. To see this, we note that if $[A]^{-1}$ does exist, we have

$$\begin{aligned} [A]\vec{\boldsymbol{e}}_i &= \lambda_i \vec{\boldsymbol{e}}_i \\ \iff [A]^{-1}[A]\vec{\boldsymbol{e}}_i &= [A]^{-1}\lambda_i \vec{\boldsymbol{e}}_i \\ \iff \vec{\boldsymbol{e}}_i &= \lambda_i([A]^{-1}\vec{\boldsymbol{e}}_i), \end{aligned}$$

which means that we must have

$$[A]^{-1}\vec{\boldsymbol{e}}_i = \frac{1}{\lambda_i}.$$

Thus it follows that the eigenvalues of $[A]^{-1}$ are reciprocals of the eigenvalues of $[A]$.

Problem 4. Consider the matrix

$$[A] = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}.$$

(a) Find the eigenvalues and eigenvectors for this matrix.

(b) Construct the matrix $[P]$ such that

$$[\Lambda] = [P]^{-1}[A][P]$$

from the eigenvectors you found.

(c) Find $[P]^{-1}$ and compute

$$[P]^{-1}[A][P].$$

Is this $[\Lambda]$ diagonal?

Solution 4.

(a) First we find the eigenvalues by putting

$$[A] - \lambda[I] = \begin{pmatrix} 1-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix},$$

and taking the determinant to get the characteristic polynomial

$$\det([A] - \lambda[I]) = (1-\lambda)(2-\lambda).$$

Then we set the characteristic polynomial equal to zero and solve

$$(1-\lambda)(2-\lambda) = 0$$

to get $\lambda_1 = 1$ and $\lambda_2 = 2$.

Then we can find the eigenvectors. First, we will find $\mathrm{Null}([A] - \lambda_1[I])$ to get the eigenvector corresponding to λ_1. So we have

$$[A] - 1[I] = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

which means that $y = 0$ and x is free to be anything. Hence, we have that

$$\vec{\boldsymbol{e}}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

by simply choosing $x = 1$. Next, we consider the case for $\lambda_2 = 2$ and we have

$$[A] - 2[I] = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix},$$

which means that $x = y$. Thus, we can choose $x = 1$ which forces $y = 1$ as well and we have

$$\vec{\boldsymbol{e}}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b) We have that

$$[P] = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(c) Then we can compute $[P]^{-1}$ fairly easily to get

$$[P]^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Then we can compute

$$[P]^{-1}[A][P] = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

So yes, $[\Lambda]$ is diagonal.

Problem 5. Let $A\colon \mathbb{C}^n \to \mathbb{C}^n$ be hermitian. We prove the two theorems in the text.

(a) Show that all eigenvalues of A are real.

(b) Show that eigenvectors corresponding to different eigenvalues are orthogonal with the hermitian inner product.

Solution 5.

(a) Let $\vec{\boldsymbol{e}}$ and λ be an eigenvector/eigenvalue pair for the transformation A (i.e., $A\vec{\boldsymbol{e}} = \lambda\vec{\boldsymbol{e}}$). Then since A is hermitian, we have

$$\langle A\vec{\boldsymbol{e}}, \vec{\boldsymbol{e}}\rangle = \langle \vec{\boldsymbol{e}}, A\vec{\boldsymbol{e}}\rangle.$$

The left hand side yields

$$\langle A\vec{\boldsymbol{e}}, \vec{\boldsymbol{e}}\rangle = \langle \lambda\vec{\boldsymbol{e}}, \vec{\boldsymbol{e}}\rangle = \lambda\langle \vec{\boldsymbol{e}}, \vec{\boldsymbol{e}}\rangle,$$

whereas the right hand side yields

$$\langle \vec{\boldsymbol{e}}, A\vec{\boldsymbol{e}}\rangle = \langle \vec{\boldsymbol{e}}, \lambda\vec{\boldsymbol{e}}\rangle = \lambda^*\langle \vec{\boldsymbol{e}}, \vec{\boldsymbol{e}}\rangle.$$

Thus, since the left hand side and the right hand side must be equal we have

$$\lambda = \lambda^*.$$

It must be that $\lambda \in \mathbb{R}$ since if λ had a nonzero imaginary part, this would not hold. Hence, we have proven the statement.

(b) Let $\vec{\boldsymbol{e}}_1$ and λ_1 be an eigenvector/eigenvalue pair and $\vec{\boldsymbol{e}}_2$ and λ_2 be another pair with $\lambda_1 \neq \lambda_2$. Then we can note that since A is hermitian

$$\langle A\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2\rangle = \langle \vec{\boldsymbol{e}}_1, A\vec{\boldsymbol{e}}_2\rangle.$$

The left hand side yields

$$\langle A\vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2\rangle = \lambda_1\langle \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2\rangle,$$

and the right hand side

$$\langle \vec{\boldsymbol{e}}_1, A\vec{\boldsymbol{e}}_2\rangle = \lambda_2^*\langle \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2\rangle.$$

We know that $\lambda_1 \neq \lambda_2$ and we also know that $\lambda_1, \lambda_2 \in \mathbb{R}$ by part (a), so it must be that $\langle \vec{\boldsymbol{e}}_1, \vec{\boldsymbol{e}}_2\rangle = 0$ for equality to be possible since it cannot be that $\lambda_1 = \lambda_2^*$. Thus, the eigenvectors must be orthogonal.