MATH 271, Homework 11

Due December 7th

Problem 1. Groups appear naturally as symmetries of physical systems. Each one of these symmetries manifests a *conserved quantity* of the system. This is the extremely famous *Noether's theorem.* For example, consider the harmonic oscillator equation

$$mx'' + kx = 0.$$

We note that this equation is a second order, linear, homogeneous, and, most importantly, autonomous equation.

(a) Noting that the force $F = ma = mx''$, argue that the force itself is independent of time t.

(b) Write down the general solution to this equation.

(c) Show that energy is conserved for any general solution. That is, show

$$E = \frac{1}{2}mx'^2 + \frac{1}{2}kx^2,$$

is constant in time.

Problem 2. Recall the hermitian inner product for vectors $\vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \in \mathbb{C}^n$ given by

$$\langle \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle = \sum_{j=1}^{n} u_j v_j^*.$$

For certain linear transformations $U\colon \mathbb{C}^n \to \mathbb{C}^n$ we have that the inner product is invariant under this transformation, i.e.,

$$\langle U\vec{\boldsymbol{u}}, U\vec{\boldsymbol{v}} \rangle = \langle \vec{\boldsymbol{u}}, \vec{\boldsymbol{v}} \rangle.$$

We refer to transformations of this kind as *unitary.*

(a) Let $n = 1$. Argue that all unitary transformations are of the form $e^{i\theta}$ for $\theta \in \mathbb{R}$. Elements of this form reside in the *unitary group* $\mathrm{U}(1)$.

(b) For a transformation $U\colon \mathbb{C}^n \to \mathbb{C}^n$ show that we must have $UU^\dagger = I$, where I is the identity transformation.

(c) Show that the set of unitary transformations form a group.

Problem 3. When two groups G_1 and G_2 behave exactly the same as one another we say these groups are *isomorphic.* That is,

i. If the groups have a correspondence between each element. (E.g., For every element $g_1 \in G_1$ there is a corresponding element in $g_2 \in G_2$ and vice-versa).

ii. The product of corresponding elements is the same. (E.g., If we have $g_1h_1 \in G_1$ then we have the corresponding element is $g_2h_2 \in G_2$).

This can be said more succinctly as there is a one-to-one and onto function $\varphi\colon G_1 \to G_2$ such that $\varphi(g_1h_1) = \varphi(g_1)\varphi(h_1)$.

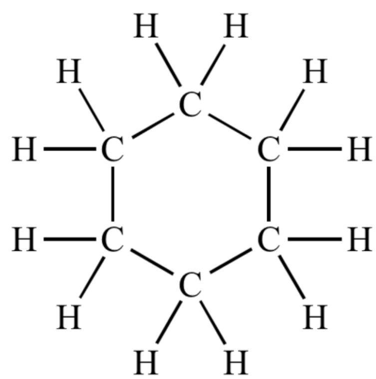
In the previous problem we showed that an element of U(1) takes the form $e^{i\theta}$. A matrix in the group of rotation matrices in $\mathbb{R}^2$ (i.e., SO(2)) can be written as

$$[\mathrm{Rot}]_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for any choice of θ. Argue that U(1) and SO(2) are isomorphic. *Hint: Go back to previous homeworks to see how we can faithfully represent complex numbers using matrices. This will give you a way to see that these groups behave analogously.*

Problem 4. Another way to think about SO(2) is to consider it as the rotational symmetry group of the unit circle in the real plane $\mathbb{R}^2$.

Next, consider a cyclohexane C_6H_{12} molecule:



This molecule also has rotational symmetry, but it is a smaller symmetry group than SO(2). We want to determine this rotational symmetry *subgroup*.

(a) This molecule looks much like a hexagon. Determine the external angles of a hexagon.

(b) Note that if we rotate a hexagon (or cyclohexane) by an external angle, then this leaves the molecule invariant (i.e., it looks no different). Using the external angle you found, write the rotation matrix for that angle and name this matrix $[g]$.

(c) We can *generate* this group C_6 from $[g]$ by repeatedly multiplying $[g]$ with itself. Show that there are only six elements in this group C_6.

(d) These are not all the symmetries of cyclohexane! Explain another symmetry operation that we could use that isn't captured by the rotations above.

If you're interested, look up the group D_{12} which is the *dihedral group of order 12*. Or, taking it further, look at `https://en.wikipedia.org/wiki/Cyclohexane_conformation`