Data Simulation

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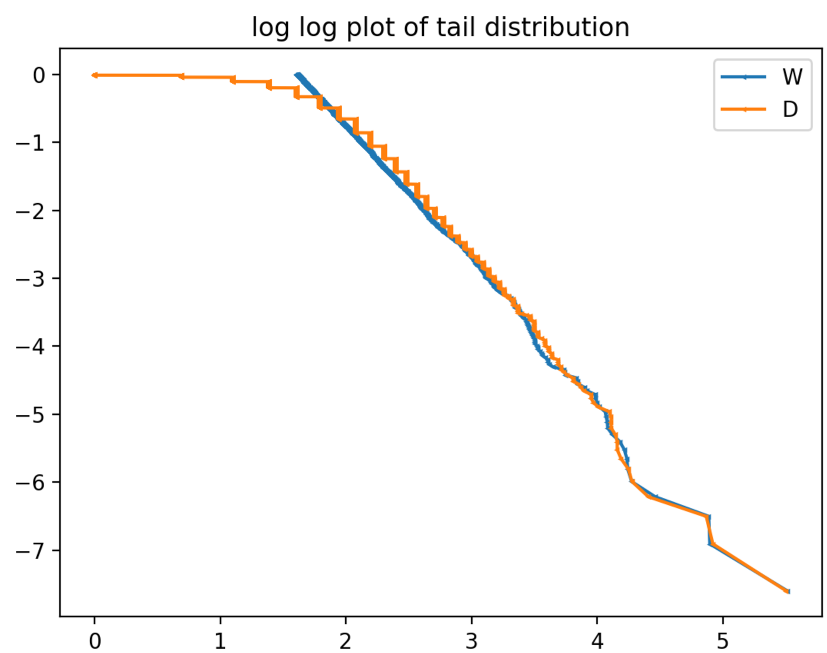
1. Degree sequence generation
   We generate W^- by
   $$U \sim \text{Uniform}(0,1), W^{-1} = F^{-1}(U) = c(1-U)^{-\frac{1}{\beta}}$$
   And then get generate W^+ by
   $$W^= = a(W^-)^d$$
   And then independently generate $(D^+, D^-) = (Poisson(W^+), Poisson(W^-))$.

2. Tail distribution test
   We set $c = 5, \beta = 2$, and W has the distribution
   $$P(W > x) = (\frac{x}{c})^{-\beta}, \qquad x > c$$
   And $D = Poisson(W)$.
   We generate independent samples of W and D of sample size 2000 and obtain the log-log plot of tail distribution below:

   We can see that the two tails almost coincide and are straight lines.

3. Tail distribution of page rank, betweenness centraliy and in/out degree sequence.

page_rank_degree_centrality.png

The in (out) degree centrality is (out) degree sequence didivded by max in (out) degree. We can clearly see that the page rank plot is parallel to degree centrality plot.

Below are calculations:

4. This is the basic set-up:

$$P(D^+ = d^+, D^- = d^-) = P(Poisson(W^+) = d^+, Poisson(W^-) = d^-)$$

where (W^+, W^-) are jointly regularly varying.

$$P(W^+ > x) = (\frac{x}{b})^{-\alpha}, \qquad x > b$$

$$P(W^- > x) = (\frac{x}{c})^{-\beta}, \qquad x > c$$

$$(\alpha, \beta > 1)$$

$$W^+ = a(W^-)^d$$

5. We can calcluate the distribution of W^- to be:

$$z = F(x) = 1 - P(W^- > x) = 1 - (\frac{x}{c})^{-\beta}$$

$$\Rightarrow x = F^{-1}(z) = c(1-z)^{-\frac{1}{\beta}}$$

So W^- can be generated by

$$U \sim \text{Uniform}(0,1), W^{-1} = F^{-1}(U) = c(1-U)^{-\frac{1}{\beta}}$$

The probablility density function of W^-: $p(x)$ is given by:

$$p(x) = -\beta c^\beta x^{-\beta-1}$$

6.

$$E(D^-) = E(W^-) = \int_c^\infty xp(W^-)dx = \frac{\beta b^\beta}{\beta - 1} b^{1-\beta} = \frac{c\beta}{\beta - 1}$$

$$E(D^+) = E(W^+) = \frac{b\alpha}{\alpha - 1}$$

7. Setting $ED^+ = ED^-$, we get

$$d = \frac{\beta}{\alpha}$$

If $\alpha \neq \beta$, $\qquad b = (\frac{\alpha(\beta-1)}{(\alpha-1)\beta})^{\frac{\beta}{\alpha-\beta}} a^{\frac{\alpha}{\alpha-\beta}}$

$$c = (\frac{\alpha(\beta-1)}{(\alpha-1)\beta})^{\frac{\alpha}{\alpha-\beta}} a^{\frac{\alpha}{\alpha-\beta}}$$

If $\alpha = \beta$, then $a = 1$, and we could arbitralily choose $b = c$.
Setting $Var(W^+ - W^-) = E[a(W^-)^d - W^-]^2 < \infty$, we get

$$\beta > 2, \beta > 2d, \beta > d + 1$$

8. Degree correlation

$$\begin{aligned} ED^+D^- &= E_{W^-}[E[D^+D^-|W^-]] \\ &= E_{W^-}a(W^-)^{d+1} \\ &= \int_c^\infty ax^{d+1}\beta c^\beta x^{-\beta-1}dx \\ &= \frac{a\beta c^{d+1}}{\beta-d-1} \end{aligned}$$

$$Var(D^-) = E(D^-)^2 - (ED^-)^2 = E[W^- + (W^-)^2] - (EW^-)^2$$

where

$$E(W^-)^2 = \int_c^\infty x^2\beta c^\beta x^{-\beta-1}dx = \frac{\beta c^2}{\beta-2}$$

$$EW^- = \frac{\alpha b}{\alpha-1}$$

Similarily we can get $Var(D^+)$.
So we can get the degree correlation fby above items:

$$\rho = corr(D^+, D^-) = \frac{Cov(D+, D^-)}{\sqrt{Var(D^+)}\sqrt{Var(D^-))}} = \frac{ED^+D^- - ED^+ED^-}{\sqrt{Var(D^+)}\sqrt{Var(D^-)}}$$