Mathematics for Computer Science: Summative Assignment
12 December, 2018

Total hours spent on this assignment: 11

Discrete Mathematics and Linear Algebra

1. Prove, using induction, that the following holds for all $n \geq 1$:

$$2\left(\sqrt{n+1}-1\right) < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

When utilizing mathematical induction to prove some given statement $S(n)$, one must first complete the basis step by demonstrating that $S(1)$ is true. Then, one must perform the inductive step by demonstrating that $S(k+1)$ is true when $S(k)$ is assumed to be true.

Basis Step:

For $n = 1$,

$2\left(\sqrt{1+1}-1\right) < \frac{1}{\sqrt{1}} < 2\sqrt{1}$ (substituting $n = 1$ into given inequality)

= $2\left(\sqrt{2}-1\right) < 1 < 2$

$2\left(\sqrt{2}-1\right) \approx 0.83$

Thus, as the above inequality is true, the basis step holds and is complete.

Inductive Step: Let $k \geq 1$ be any integer.

Assume the claim holds for $n = k$, formulating the inductive hypothesis:

$$2\left(\sqrt{k+1}-1\right) < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} < 2\sqrt{k}$$

Therefore, assuming that the inductive hypothesis is true, it follows that:

$$2\left(\sqrt{(k+1)+1}-1\right) < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$$

Which, can also be expressed as:

$$2\left(\sqrt{k+1}-1\right)+\ \frac{1}{\sqrt{k+1}}<\ 1+\ \frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}+\ \frac{1}{\sqrt{k+1}}<2\sqrt{k}+\ \frac{1}{\sqrt{k+1}}$$

First, examining the right hand side of the inequality:

$$1+\ \frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}+\ \frac{1}{\sqrt{k+1}}<2\sqrt{k+1}$$

$$1+\ \frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}+\ \frac{1}{\sqrt{k+1}}<2\sqrt{k}+\ \frac{1}{\sqrt{k+1}}$$

Thus, it can be inferred that $1+\ \frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}+\ \frac{1}{\sqrt{k+1}}<2\sqrt{k+1}$ if and only if :

$$\frac{1}{\sqrt{k+1}}<2\sqrt{k+1}-2\sqrt{k}$$

$$=\frac{1}{\sqrt{k+1}}<2(\sqrt{k+1}-\sqrt{k})$$

Multiplying both sides of the inequality by $\sqrt{k+1}+\sqrt{k}$,

$$=\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}}<2(k+1-k)$$

$$=\frac{\sqrt{k+1}+\sqrt{k}}{\sqrt{k+1}}<2$$

$$=1+\ \frac{\sqrt{k}}{\sqrt{k+1}}<2$$

$$=\frac{\sqrt{k}}{\sqrt{k+1}}<1$$

$$=\sqrt{k}<\sqrt{k+1}$$

Which is a true statement, thereby proving the right hand side of the given inequality.

Now, proceeding to examine the left hand side of the given inequality:

$$2\left(\sqrt{(k+1)+1}-1\right) < 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}$$

$$2\left(\sqrt{k+1}-1\right)+\frac{1}{\sqrt{k+1}} < 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}$$

Using a similar line of reasoning as above, $2\left(\sqrt{(k+1)+1}-1\right) < 1+\frac{1}{\sqrt{2}}+\cdots+\frac{1}{\sqrt{k}}+\frac{1}{\sqrt{k+1}}$ if and only if:

$$\frac{1}{\sqrt{k+1}} > 2\left(\sqrt{(k+1)+1}-1\right)-2(\sqrt{k+1}-1)$$

$$=\frac{1}{2\sqrt{k+1}} > \left(\sqrt{k+2}-1\right)-(\sqrt{k+1}-1)$$

$$=\frac{1}{2\sqrt{k+1}} > \sqrt{k+2}-1-\sqrt{k+1}+1$$

$$=\frac{1}{2\sqrt{k+1}} > \sqrt{k+2}-\sqrt{k+1}$$

$$=\frac{1}{2\sqrt{k+1}} > \frac{1}{\sqrt{k+2}+\sqrt{k+1}}$$

Further, $\frac{1}{2\sqrt{k+1}}$ can be expressed as:

$$\frac{1}{2\sqrt{k+1}} = \frac{1}{\sqrt{k+1}+\sqrt{k+1}}$$

Thus,

$$\frac{1}{\sqrt{k+1}+\sqrt{k+1}} > \frac{1}{\sqrt{k+2}+\sqrt{k+1}}$$

Intuitively, the value of $\sqrt{k+2}+\sqrt{k+1}$ will always be greater than that of $\sqrt{k+1}+$

$\sqrt{k+1}$ for all integers $k \geq 1$. As dividing by an increasingly large denominator decreases the overall value of a fraction, $\frac{1}{\sqrt{k+1}+\sqrt{k+1}} > \frac{1}{\sqrt{k+2}+\sqrt{k+1}}$ will always hold true, as the denominator of the left-hand fraction is consistently less than that of the right hand side (e.g. $\frac{1}{n} > \frac{1}{n+1}$ for all positive n). Thus the left hand side of the given inequality is proved.

Upon completing the basis and inductive (for both right and left hand sides of the given inequality) steps, $2\left(\sqrt{n+1}-1\right) < 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$ is true for all $n \geq 1$ by mathematical induction.

2. Four fair dice are thrown. What is more likely: to get exactly one 6 or to get four different numbers?

First, computing the total possible outcomes among the 4 dice: $6^4 = 1296$.

Next, considering the probability of rolling exactly one 6:

Theorem (Lecture, Week 5): The probability of exactly k successes in n independent Bernoulli trials, with probability of success p and probability of failure $q = 1 - p$, is:

$$b(k;\ n, p) = C(n,k) * p^k * q^{n-k}$$

Thus, $P(exactly\ one\ 6) = C(4,1) * (\frac{1}{6})^1 * (\frac{5}{6})^3 = \frac{500}{1296}$

Next, considering the probability of rolling four different numbers:

The number of rolls containing different numbers in each position is equal to 6 * 5 * 4 * 3 = 360 rolls. In other words, there are 6 possible outcomes for the first roll, and 1 less possible outcome for each roll thereafter. Thus, $P(4\ different\ numbers) = \frac{360}{1296}$.

As $\frac{500}{1296} > \frac{360}{1296}$, $P(exactly\ one\ 6) > P(4\ different\ numbers)$. One is more likely to roll exactly one 6 when 4 fair dice are thrown.

3. Letters A, B, C, and D are listed in random order. Let X be the random variable equal to the number of pairs of letters not in alphabetical order (for example, in CADB, such pairs are CA, CB, and DB). Find the expected value and the variance of X.

Given 4 letters A, B, C, and D, the total number of possible arrangements is $4! = 4 * 3 * 2 * 1 = 24$.

Hence, the probability of each individual outcome is $\frac{1}{24}$.

Since, in terms of alphabetical ordering, A < B < C < D, general intuition suggests that the value of X will likely be greater in letter arrangements where D and C are listed first/closer to the front.

Next, calculating the number of possible outcomes of the random variable X can be formalized by the following 4 scenarios:

a) 1st letter chosen: A

2nd letter chosen	Random ordering of remaining letters	4 letter arrangement	$X =$ number of pairs not in alphabetical order
B	CD	ABCD	0
B	DC	ABDC	1
C	BD	ACBD	1
C	DB	ACDB	2
D	BC	ADBC	2
D	CB	ADCB	3

b) 1st letter chosen: B

2nd letter chosen	Random ordering of remaining letters	4 letter arrangement	$X =$ number of pairs not in alphabetical order
A	CD	BACD	1
A	DC	BADC	2
C	AD	BCAD	2
C	DA	BCDA	3
D	AC	BDAC	3
D	CA	BDCA	4

c) 1st letter chosen: C

2nd letter chosen	Random ordering of remaining letters	4 letter arrangement	$X =$ number of pairs not in

			alphabetical order
A	BD	CABD	2
A	DB	CADB	3
B	AD	CBAD	3
B	DA	CBDA	4
D	AB	CDAB	4
D	BA	CDBA	5

d) 1st letter chosen: D

2nd letter chosen	Random ordering of remaining letters	4 letter arrangement	$X =$ number of pairs not in alphabetical order
A	BC	DABC	3
A	CB	DACB	4
B	AC	DBAC	4
B	CA	DBCA	5
C	AB	DCAB	5
C	BA	DCBA	6

Hence, calculating the distribution of $p(X = r)$ for $0 \leq r \leq 6$:

$p(X = 6) = \frac{1}{24}$

$p(X = 5) = \frac{3}{24}$

$p(X = 4) = \frac{5}{24}$

$p(X = 3) = \frac{6}{24}$

$p(X = 2) = \frac{5}{24}$

$p(X = 1) = \frac{3}{24}$

$p(X = 0) = \frac{1}{24}$

Using these calculations, one can proceed to find the expected value and variance of X.

a. Expected value

Definition (Lecture, Week 6): The expected value of a random variable X on a sample space S with possible outcomes $s_1, \ldots\ s_n$ is equal to:

$$E(X) = \sum_{i=1}^{n} p(s_i)\, X(s_i)$$

Therefore, $E(X) = \left(\frac{1}{24} * 6\right) + \left(\frac{3}{24} * 5\right) + \left(\frac{5}{24} * 4\right) + \left(\frac{6}{24} * 3\right) + \left(\frac{5}{24} * 2\right) + \left(\frac{3}{24} * 1\right) + \left(\frac{1}{24} * 0\right)$
$= \frac{6}{24} + \frac{15}{24} + \frac{20}{24} + \frac{18}{24} + \frac{10}{24} + \frac{3}{24} = \frac{72}{24} = 3$

b. Variance

Theorem (Lecture, Week 6): If X is a random variable then $V(X) = E(X^2) - E(X)^2$

From part (a), $E(X) = 3$. Thus, $E(X)^2 = 3 * 3 = 9$.

Using the distribution of X from above,

$E(X^2) = \left(\frac{1}{24} * 36\right) + \left(\frac{3}{24} * 25\right) + \left(\frac{5}{24} * 16\right) + \left(\frac{6}{24} * 9\right) + \left(\frac{5}{24} * 4\right) + \left(\frac{3}{24} * 1\right) + \left(\frac{1}{24} * 0\right) =$
$\frac{36}{24} + \frac{75}{24} + \frac{80}{24} + \frac{54}{24} + \frac{20}{24} + \frac{3}{24} = \frac{268}{24} = \frac{67}{6}$

Thus, $V(X) = \frac{67}{6} - 9 \approx 2.17$

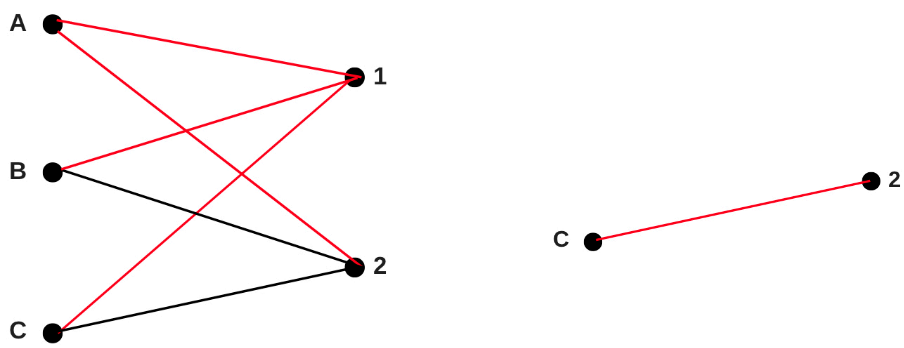
4. For each integer $q \geq 2$, determine whether the complete bipartite graph $K_{3,q}$ is KO-reducible, and if so find its KO-number.

A complete bipartite graph is one that can be described in the form $K_{p,q}$, which consists of two disjoint vertex sets on p and q vertices and all edges between the two vertex sets (and no other edges) (Lecture, Week 7).

In finding the KO-number(s) of the graph $K_{3,q}$, one can consider three cases: that where $q < p, p = q$, and $q > p$.

First, where $q < p$:

Given that $q \geq 2$, the only applicable instance in this category is the graph $K_{3,2}$ with KO-number 2, drawn below (lettered left-hand vertices A,B,C represent members of vertex set p, whereas numbered right-hand vertices 1,2,… represent members of vertex set q).

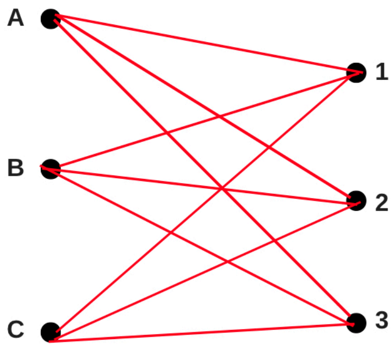


In the first round, the following selections are made between vertices: A-1, B-1, C-1, 1-B, 2-A. Upon eliminating vertices A, B, 1 and their incident edges, one is left with a $K_{1,1}$ graph between vertices C and 2; in the second round, C and 2 select each other, thereby eliminating all vertices in the graph. Intuitively, it is logical that a graph $K_{p,q}$ where $q < p$ would have a KO-number of at least 2, as it is not possible for all vertices of the set p to be selected in one round (there is not at least 1 corresponding vertex in q for each vertex in p).

Thus, $K_{3,2}$ is KO-reducible with KO-number 2.

Further, when $p = q$, the graph in question is that of $K_{3,3}$. As demonstrated below, $K_{3,3}$ is also KO-reducible, with KO-number 1.

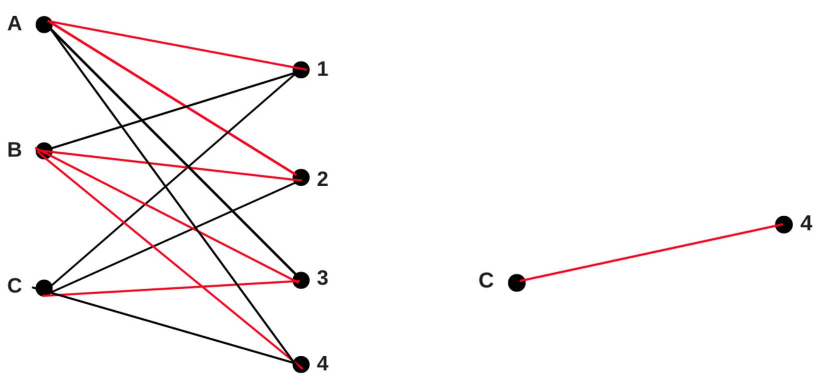
The following selections are made between vertices in an alternating, “zig-zag” pattern: A-1, 1-B, B-2, 2-C, C-3, 3-A. All vertices are therefore eliminated from the graph $K_{3,3}$ in one round, producing KO-number 1.



Lastly, where $q > p$:

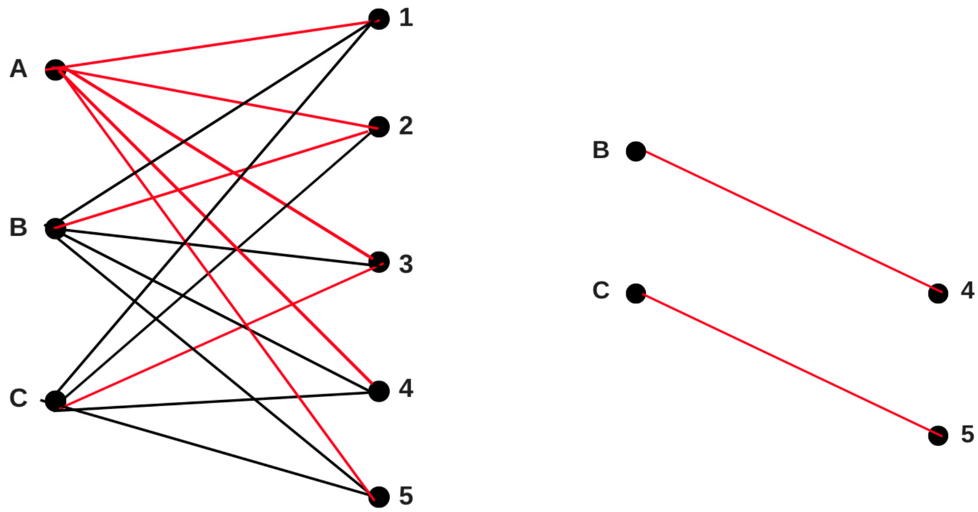
Jointly consider the graphs $K_{3,4}$ and $K_{3,5}$. Since one can eliminate at most 3 vertices on the right-hand side per application of the KO-scheme, a KO-reducible graph where $q > p$ must have a KO-number of at least 2. Graphs $K_{3,4}$ and $K_{3,5}$ can be shown to be KO-reducible in 2 rounds, as follows:

$K_{3,4}$



In the first round, vertices A, B, and C each select 1, 2, and 3, respectively. Vertex 1 selects A, and vertices 2,3,4 select B. In the second round, similarly to the graph $K_{3,2}$, vertices C and 4 select each other, thereby eliminating all vertices in the graph $K_{3,4}$ with KO-number 2.

$K_{3,5}$



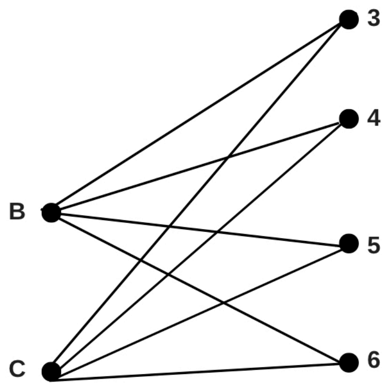
In the first round, similarly to the graph $K_{3,4}$, vertices A, B, and C each select 1, 2, and 3, respectively. Vertices 1, 2, 3, 4, and 5 all select vertex A, thereby producing a $K_{2,2}$ graph. In the second round, vertices B and 4 select each other, as do vertices C and 5. Hence, $K_{3,5}$ is KO-reducible with KO-number 2.

$K_{3,6}$

When examining the graph $K_{3,6}$, it is evident that applying the same reduction scheme as for $K_{3,5}$ will produce the graph $K_{2,3}$ after 1 round. Having shown that $K_{3,2}$ is KO-reducible with KO-number 2, it can therefore be inferred that the graph $K_{3,6}$ is KO-reducible with KO-number 1+2=3.

$K_{3,7}$

When examining the graph $K_{3,7}$, it is evident that applying the same reduction scheme as for $K_{3,5}$ and $K_{3,7}$ produces the graph $K_{2,4}$ after 1 round. However, unlike the graph $K_{2,3}$, $K_{2,4}$ is not KO-reducible, as demonstrated below:



At best, $K_{2,4}$ can be reduced to $K_{1,2}$; vertices B and C select vertices 3 and 4, respectively, and vertices 3,4,5,6 select vertex B. However, as $K_{1,2}$ is not KO-reducible (and, more generally, any graph $K_{1,q}$ for $q \geq 1$) , neither is $K_{3,7}$. Thus, one can conclude that for instances of $q \geq 7$, the graph $K_{p,q}$ is not KO-reducible.

In summary, for the complete bipartite graph $K_{3,q}$ where $q \geq 2$:

$q = 2$: KO-reducible, with KO-number 2
$q = 3$: KO-reducible, with KO-number 1
$q = 4$: KO-reducible, with KO-number 2
$q = 5$: KO-reducible, with KO-number 2
$q = 6$: KO-reducible, with KO-number 3
$q \geq 7$: Not KO-reducible

Logic and Discrete Structures

5. Let φ be the formula $((a \wedge b) \Rightarrow c) \wedge (a \vee b)$.

a. Reduce φ to disjunctive normal form using truth tables.

$$\varphi = ((a \wedge b) \Rightarrow c) \wedge (a \vee b)$$

a	b	c		$a \wedge b$	$\Rightarrow c$	$\wedge$	$a \vee b$		φ
T	T	T		T	T	T	T		T
F	T	F		T	F	F	T		F
T	F	T		F	T	T	T		T
T	F	F		F	T	T	T		T
F	T	T		F	T	T	T		T
F	T	F		F	T	T	T		T
F	F	T		F	T	F	F		F

F	F	F		F	T	F	F		F

Hence, in disjunctive normal form, φ is equivalent to

$$(a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge c) \vee (\neg a \wedge b \wedge \neg c)$$

b. Reduce ¬φ to conjunctive normal form (without using truth tables).

$\varphi = \big((a \wedge b) \Rightarrow c\big) \wedge (a \vee b)$

$\neg\varphi = \neg[\big((a \wedge b) \Rightarrow c\big) \wedge (a \vee b)]$

$\equiv \neg[(\neg(a \wedge b) \vee c) \wedge (a \vee b)]$

$\equiv \neg[\big((\neg a \vee \neg b) \vee c\big) \wedge (a \vee b)]$

$\equiv \neg(\neg a \vee \neg b \vee c) \vee \neg(a \vee b)$

$\equiv (a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b)$

Hence, $\varphi = \neg[(a \wedge b \wedge \neg c) \vee (\neg a \wedge \neg b)]$

$\equiv (\neg a \vee \neg b \vee c) \wedge (a \vee b)$ in conjunctive normal form.

6. Is the set $\{\wedge, \oplus\}$, where $p \oplus q \equiv \neg(p \Leftrightarrow q)$, functionally complete?

A set of logical connectives S is said to be functionally complete if any propositional formula is equivalent to one constructed using only the connectives from S.

Using truth tables, one can demonstrate that the logical connectives $\wedge$ and $\oplus$ are falsehood preserving; in other words, assigning a value of false to all of the variables in some propositional formula can never produce an output of true.

p	q	$\wedge$	$\oplus$
T	T	$p \wedge q$ = T	$p \oplus q$ =F
T	F	$p \wedge q$ = F	$p \oplus q$ =T
F	T	$p \wedge q$ = F	$p \oplus q$ =T
F	F	$p \wedge q$ = F	$p \oplus q$ =F

When both p and q are false, one cannot arrive at a true output regardless of how $\wedge$ and $\oplus$ are applied. For example, assuming some propositional formula Z which takes the value of false, $(p \wedge q) \oplus Z$= always false. Likewise, $(p \oplus q) \wedge Z$= always false. To demonstrate functional completeness, one would need at least ¬ to output a true value given two or more false inputs.

Hence, as ∧ and ⊕ are falsity preserving connectives, the set {∧, ⊕} is not functionally complete.

7. Prove the following sequents using natural deduction.

a. $\neg a \wedge b \wedge \big(a \vee (b \Rightarrow c)\big) \ \vdash c \vee d$

1	$\neg a \wedge b \wedge \big(a \vee (b \Rightarrow c)\big)$	premise
2	$\neg a \wedge b$	∧e1
3	b	∧e2
4	$\neg a$	∧e2
5	$a \vee (b \Rightarrow c)$	∧e1
6	a	assumption
7	$\perp$	¬e6,4
8	c	⊥e7
9	$b \Rightarrow c$	assumption
10	c	⇒e9,3
11	c	∨e6-8, 9-10
12	$c \vee d$	∨i11

b. $a \vee (\neg b \wedge \neg c \ \wedge \neg d\ ) \vdash (a \vee \neg b) \wedge (a \vee \neg c) \wedge (a \vee \neg d)$

1	$a \vee (\neg b \wedge \neg c \ \wedge \neg d\ )$	premise
2	a	assumption
3	$a \vee \neg b$	∨i2

4	$a \vee \neg c$	$\vee$i2
5	$a \vee \neg d$	$\vee$i2
6	$(a \vee \neg b) \wedge (a \vee \neg c)$	$\wedge$i3,4
7	$(a \vee \neg b) \wedge (a \vee \neg c) \wedge (a \vee \neg d)$	$\wedge$i5,6
8	$\neg b \wedge \neg c \wedge \neg d$	assumption
9	$\neg b$	$\wedge$e8
10	$\neg c$	$\wedge$e8
11	$\neg d$	$\wedge$e8
12	$a \vee \neg b$	$\vee$i9
13	$a \vee \neg c$	$\vee$i10
14	$a \vee \neg d$	$\vee$i11
15	$(a \vee \neg b) \wedge (a \vee \neg c)$	$\wedge$i12,13
16	$(a \vee \neg b) \wedge (a \vee \neg c)\ ) \wedge (a \vee \neg d)$	$\wedge$i14,15
17	$(a \vee \neg b) \wedge (a \vee \neg c)\ ) \wedge (a \vee \neg d)$	$\vee$e2-7,8-16

8. Let φ be the formula $\neg(a \wedge (\neg a \vee \neg b) \wedge (b \vee c) \wedge (\neg c \vee \neg d \vee e) \wedge (e \vee d) \wedge (\neg e \vee \neg c))$. Is φ a theorem? Use Resolution to answer this question. Mention on which clauses you use Resolution each time you apply it.

Given $\varphi = \neg(a \wedge (\neg a \vee \neg b) \wedge (b \vee c) \wedge (\neg c \vee \neg d \vee e) \wedge (e \vee d) \wedge (\neg e \vee \neg c))$

Hence, $\neg\varphi = \neg\neg\big(a \wedge (\neg a \vee \neg b) \wedge (b \vee c) \wedge (\neg c \vee \neg d \vee e) \wedge (e \vee d) \wedge (\neg e \vee \neg c)\big) = (a \wedge (\neg a \vee \neg b) \wedge (b \vee c) \wedge (\neg c \vee \neg d \vee e) \wedge (e \vee d) \wedge (\neg e \vee \neg c))$.

Thus, proceed by applying Resolution to $\neg\varphi = (a \wedge (\neg a \vee \neg b) \wedge (b \vee c) \wedge (\neg c \vee \neg d \vee e) \wedge (e \vee d) \wedge (\neg e \vee \neg c))$.

Set of clauses to which Resolution can be applied:

$$a \quad (\neg a \vee \neg b) \quad (b \vee c) \quad (\neg c \vee \neg d \vee e) \quad (e \vee d) \quad (\neg e \vee \neg c)$$

Subsequently, proceed by formally applying Resolution:

1	a	given
2	$\neg a \vee \neg b$	given
3	$b \vee c$	given
4	$\neg c \vee \neg d \vee e$	given
5	$e \vee d$	given
6	$\neg e \vee \neg c$	given
7	$\neg b$	1,2
8	c	3,7
9	$\neg d \vee e$	4,8
10	e	5,9
11	$\neg c$	6, 10
12	$\emptyset$	8, 11

Hence, upon inferring the empty clause, $\emptyset$, Resolution announces that φ is a theorem and therefore a tautology.