

A formalization of Dedekind domains and class groups of global fields

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Abstract

Dedekind domains and their class groups are notions in commutative algebra that are essential in algebraic number theory. We formalized these structures and several fundamental properties, including number theoretic finiteness results for class groups, in the Lean prover as part of the **mathlib** mathematical library. This paper describes the formalization process, noting the idioms we found useful in our development and ‘mathlib’s decentralized collaboration processes involved in this project.

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Supplementary Material Full source code of the formalization is part of **mathlib**. Copies of the source files relevant to this paper are available in a separate repository.

Software: <https://github.com/lean-forward/class-number>

archived at **Software Heritage Identifier**

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1 Introduction

In its basic form, number theory studies properties of the integers \mathbb{Z} and its fraction field, the rational numbers \mathbb{Q} .¹ Both for the sake of generalization, as well as for providing powerful techniques to answer questions about the original objects \mathbb{Z} and \mathbb{Q} , it is worthwhile to study finite extensions of \mathbb{Q} , called *number fields*, as well as their *rings of integers* (defined in Section 2 below), whose relations mirror the way \mathbb{Q} contains \mathbb{Z} as a subring. In this paper, we describe our project aiming to formalize these notions and some of their important properties. Our goal, however, is not to get to the definitions and properties as quickly as possible, but

¹ From a classical point of view, one could even argue that the positive, or perhaps nonnegative, integers and rational numbers are the most basic objects of study of number theory. From an algebraic point of view, this would still quickly lead into studying \mathbb{Z} and \mathbb{Q} .



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rather to lay the foundations for future work, as part of a natural and more general theory as we shall explain below.

In particular, our project resulted in formalized definitions and elementary properties of number fields and their rings of integers (described in Section 3.3), Dedekind domains (Section 4), and the ideal class group and class number (Section 7). The main proofs that we formalized show that two definitions of Dedekind domains are equivalent (Section 4.3), that the ring of integers (or more generally: the integral closure of a Dedekind domain in a finite separable field extension) is a Dedekind domain (Section 6) and that the class group of a number field is finite (Section 7). In fact, most of our results for number fields are actually obtained in the more general setting of so-called *global fields*, i.e. number fields together with finite field extensions of $\mathbb{F}(t)$ with \mathbb{F} a finite field (restricting to $\mathbb{F} \simeq \mathbb{Z}/p\mathbb{Z}$ with p prime yields no loss of generality here).

Apart from the achievement of formalizing a non-trivial amount of mathematical theory, our formal definition of the class number is an essential requirement for the use of theorem provers in modern number theory research.

Our work is developed as part of the mathematical library `mathlib` [21] for the Lean 3 theorem prover [6]. The formal system of Lean is a dependent type theory based on the calculus of inductive constructions, with a proof-irrelevant impredicative universe `Prop` at the bottom of a noncumulative hierarchy of universes `Prop : Type : Type 1 : Type 2 : ...`.² Other important characteristics of Lean as used in `mathlib` are the use of quotient types, ubiquitous classical reasoning and the use of typeclasses to define the hierarchy of algebraic structures.

Organizationally, `mathlib` is characterized by a distributed and decentralized community of contributors, a willingness to refactor its basic definitions, and a preference for small yet complete contributions over larger projects added all at once. Our own project, being part of the development of `mathlib`, follows this philosophy by contributing pieces of our work as they are finished, in turn taking advantage of results contributed by others after the start of the project. At several points, we had just merged a formalization into `mathlib` that another contributor needed, immediately before they contributed a result that we needed. Due to the decentralized organization and fluid nature of contributions to `mathlib`, its contents are built up of many different contributions from many different authors. Attributing each formalization to a single set of main authors would not do justice to all others whose additions and tweaks are essential to its current use. Therefore, we will make clear whether a contribution is part of our project or not, but we will not stress whom we consider to be the main author(s).

The source files of the formalization are currently in the process of being merged into `mathlib`, an up-to-date branch being available <https://github.com/leanprover-community/mathlib/tree/dedekind-domain-dev>. We also maintain a repository containing the files relevant to this paper, available at <https://github.com/lean-forward/class-number>.

2 Mathematical background

Let us now introduce some of the main objects we study, described in a “standard” mathematical way.

A *number field* K is a finite extension of \mathbb{Q} , and as such has the structure of a finite dimensional vector space over \mathbb{Q} ; its dimension is called the *degree* of K . The smallest

² In our code samples, we use `Type*` as abbreviation of “`Type u` for an arbitrary choice of `u`”.

example is \mathbb{Q} itself, and the two-dimensional cases are given by the quadratic number fields $\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} : a, b \in \mathbb{Q}\}$ with $d \in \mathbb{Z}$ not a square (which additionally may be taken squarefree). For an interesting cubic example, let α be the unique real number satisfying $\alpha^3 + \alpha^2 - 2\alpha + 8 = 0$: it gives rise to the number field $\mathbb{Q}(\alpha) = \{a + b\alpha + c\alpha^2 : a, b, c \in \mathbb{Q}\}$. In general, taking any (say) complex root α of an irreducible polynomial of degree n over \mathbb{Q} yields a number field of degree n : $\mathbb{Q}(\alpha) = \{c_0 + c_1\alpha + \dots + c_{n-1}\alpha^{n-1} : c_0, c_1, \dots, c_{n-1} \in \mathbb{Q}\}$, and, up to isomorphism, these are all the number fields of degree n .

The *ring of integers* \mathcal{O}_K of a number field K is defined as the integral closure of \mathbb{Z} in K , which boils down to

$$\mathcal{O}_K := \{x \in K : f(x) = 0 \text{ for some monic polynomial } f \text{ with integer coefficients}\},$$

where we recall that a polynomial is called *monic* if its leading coefficient equals 1. While it might not be immediately obvious that \mathcal{O}_K is a ring, this follows from general algebraic properties of integral closures. Some examples of \mathcal{O}_K are as follows. Taking $K = \mathbb{Q}$, we get $\mathcal{O}_K = \mathbb{Z}$ back. For $K = \mathbb{Q}(\sqrt{2})$ we get $\mathcal{O}_K = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. But for $K = \mathbb{Q}(\sqrt{5})$ we do *not* simply get $\mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$ as \mathcal{O}_K , since the golden ratio $\varphi := (1 + \sqrt{5})/2 \notin \mathbb{Z}[\sqrt{5}]$ satisfies the monic polynomial equation $\varphi^2 - \varphi - 1 = 0$, hence by definition $\varphi \in \mathcal{O}_K$; it turns out that $\mathcal{O}_K = \mathbb{Z}[\varphi] = \{a + b\varphi : a, b \in \mathbb{Z}\}$. Finally, if $K = \mathbb{Q}(\alpha)$ with α as before, then $\mathcal{O}_K = \{a + b\alpha + c(\alpha + \alpha^2)/2 : a, b, c \in \mathbb{Z}\}$, illustrating that explicitly writing down \mathcal{O}_K can quickly become complicated.

Thinking of \mathcal{O}_K as a generalization of \mathbb{Z} , it is natural to ask which of its properties still hold in \mathcal{O}_K and, when this fails, if a reasonable weakening does.

An important property of \mathbb{Z} is that every ideal is generated by one element, which implies that every nonzero nonunit element can be written as a finite product of prime elements, which is unique up to reordering and multiplying by ± 1 : a ring where this holds is called a unique factorization domain, or UFD. For example, 6 can be factorized in exactly 4 ways, namely $6 = 2 \cdot 3 = 3 \cdot 2 = (-2) \cdot (-3) = (-3) \cdot (-2)$. Some well-known rings of integers are the Gaussian integers $\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$ (where $i^2 = -1$), the Eisenstein integers $\mathbb{Z}[(1 + \sqrt{-3})/2]$, and the ring $\mathbb{Z}[\sqrt{2}]$. In fact, these examples are UFDs, but this is certainly not true for all rings of integers. For example, unique factorization *does not* hold in $\mathbb{Z}[\sqrt{-5}]$: it is easy to prove that $6 = 2 \cdot 3$ and $6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$ provide two essentially different ways to factor 6 into prime elements of $\mathbb{Z}[\sqrt{-5}]$.

As it turns out, there is a way to remedy this. Namely, by considering factorization of *ideals* instead of elements: given a number field K , with ring of integers \mathcal{O}_K , a beautiful and classical result by Dedekind shows that every nonzero ideal of \mathcal{O}_K can be factored as a product of prime ideals in a unique way, up to the reordering.

Although unique factorization in terms of ideals is of great importance and beauty, it is still very interesting, and for many arithmetic applications necessary, to also consider factorization properties in terms of elements. We mentioned that unique factorization in \mathbb{Z} follows from the fact that every ideal is generated by a single element. A convenient way to rephrase this algebraically is to first consider the notion of *fractional ideal* of \mathbb{Z} , i.e. a subgroup of \mathbb{Q} of the form $\frac{1}{d}I$ with I an ideal of \mathbb{Z} and d a nonzero integer. An advantage of this generalization of the notion of *ideal* of \mathbb{Z} is that nonzero fractional ideals naturally form a multiplicative group (whereas there is no ideal $I \subseteq \mathbb{Z}$ such that $I * (2\mathbb{Z}) = (1)$). With this notion at one's disposal, the statement that every ideal is generated by a single element translates to the fact that the quotient group of nonzero fractional ideals modulo \mathbb{Q}^\times (where $a/b \in \mathbb{Q}^\times$ corresponds to the fractional ideal $a\mathbb{Z} * (b\mathbb{Z})^{-1}$) is trivial.

It turns out that this procedure can be performed for every ring of integers \mathcal{O}_K . The fundamental theoretical notion beneath this construction is that of Dedekind domain: these

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are integral domains D which are Noetherian (every ideal of D is finitely generated), integrally closed (if an element x in the fraction field of D is a root of a monic polynomial with coefficients in D , then actually $x \in D$), and of Krull dimension at most 1 (every nonzero prime ideal of D is maximal). It can be proved that the nonzero fractional ideals of D again form a group, and the quotient of this group by the image of the natural embedding of $\text{Frac } D^\times$ is called the (ideal) class group \mathcal{Cl}_D .

What is arithmetically crucial is the theorem ensuring that the ring of integers \mathcal{O}_K of every number field K is a Dedekind domain, and that in this case the class group $\mathcal{Cl}_{\mathcal{O}_K}$ is actually *finite*. In particular, $\mathcal{Cl}_{\mathcal{O}_K}$ can be seen as “measuring” by what extent ideals of \mathcal{O}_K are far from being generated by a single element and hence, in turn, as a measure of the failure of unique factorization: somewhat intuitively, the smaller the class group, the fewer factorizations are possible. In particular, as long as we are concerned with “uniqueness” of factorization, and with measuring the lack thereof, already the *order* of $\mathcal{Cl}_{\mathcal{O}_K}$, called the *class number* of K , is a tremendously interesting arithmetic feature. Actually, the same statement holds for a larger class of fields, the so-called “global fields”. They encompass all number fields and all fields which are finite extensions of $\text{Frac } \mathbb{F}_q(t)$, where \mathbb{F}_q is a *finite field* with q elements. A prototypical example is $K = \mathbb{Z}/p\mathbb{Z}(t) = \mathbb{F}_p(t)$, or an extension of the form, for instance, $K = \mathbb{F}_{p^2}(t)[s]/s^2 - t$, obtained by adjoining to $\mathbb{F}_p(t)$ a square root of t as well as the $(p^2 - 1)$ -st roots of unity. The fields of this second class are called *function fields*.

In this project, we formalized Dedekind domains, their class group, number fields together with their ring of integers, and the definition of the class number, via the proof that the class group of a ring of integers is finite. In the next sessions we will describe this formalization.

3 Number fields, global fields and rings of integers

We refer the reader to Section 2 for the mathematical background needed in this section.

We formalized number fields as the following typeclass:

```
class is_number_field (K : Type*) [field K] :=
  [cz : char_zero K] [fd : finite_dimensional ℚ K]
```

The condition `[cz : char_zero K]` states that K has characteristic zero, so the canonical ring homomorphism $\mathbb{Z} \rightarrow K$ is an embedding. This implies that there is a \mathbb{Q} -algebra structure on K (found by typeclass search), and from this follows the vector space structure used in the `[fd : finite_dimensional ℚ K]` hypothesis.

We define the function fields K over a finite field \mathbb{F}_q using the following typeclass:

```
class is_function_field_over {F_q F : Type*} [field F_q] [fintype F_q] [field F]
  (f : fraction_map (polynomial F_q) F) (L : Type*) [field L]
  [algebra f.codomain L] : Prop :=
  [fd : finite_dimensional f.codomain L]
```

«««< HEAD The map `f` witnesses that F is a fraction field of the polynomial ring $\mathbb{F}_q[t]$, the notation `f.codomain` endows K with the algebra structure of $F = \mathbb{F}_q(t)$. We present a more detailed examination of `fraction_map` in Section 3.5. ===== The map `f` witnesses that L is the field of fractions of the polynomial ring $K[X]$, the notation `f.codomain` endows L with the algebra structure of $K(X)$. We present a more detailed examination of `fraction_map` in Section 3.5. »»»> d9fa398bca1dda9f9cda1ba2881dd2430c76c305

3.1 Field extensions

The definition of `is_number_field` illustrates our treatment of field extensions. In informal mathematics, a field L containing a subfield K is said to be a field extension L/K . Often we encounter towers of field extensions: we might have that \mathbb{Q} is contained in K , K is contained in L , L is contained in an algebraic closure \bar{K} of K , and \bar{K} is contained in \mathbb{C} . We might formalize this situation by viewing \mathbb{Q} , K , L and \bar{K} to be sets of complex numbers \mathbb{C} and defining field extensions as subset relations between these subfields. This way, no coercions need to be inserted to map elements of one field into a larger field. In type theory we cannot define \mathbb{Q} as a subset of \mathbb{C} since we need \mathbb{Q} to define \mathbb{C} . Thus, some coercion is always needed to go from the original definition of \mathbb{Q} to its image in \mathbb{C} ; and similar issues arise for other subfields that were not originally defined as such. Moreover, such an approach loses flexibility since we need to fix the top field, of which all others are subfields, at the start of our development and cannot adjoin more elements when needed.

Instead, we formalize results about field extensions by parametrization. The fields K and L are represented by arbitrary types and the hypothesis “ L is a field extension of K ” is represented by an instance parameter `[algebra K L]`. This provides us with a canonical ring homomorphism `algebra_map K L : K → L`; this map is injective because K and L are fields. In other words, field extensions are given by their canonical embeddings.

3.2 Scalar towers

The main drawback of using arbitrary embeddings to represent field extensions is that we need to prove that these maps commute. For example, we might start with a field extension L/\mathbb{Q} , then define a subfield K of L , resulting in a tower of extensions $L/K/\mathbb{Q}$. In such a tower, the map $\mathbb{Q} \rightarrow L$ should be equal to the composition $\mathbb{Q} \rightarrow K \rightarrow L$. The example has other maps depend on the map $\mathbb{Q} \rightarrow L$, so we cannot arrange the coherence condition by defining $\mathbb{Q} \rightarrow L$ after the fact.

The solution in `mathlib` is to parametrize over all three maps, as long as there is also a proof of coherency: a hypothesis of the form “ $L/K/F$ is a tower of field extensions” is translated to three instance parameters `[algebra F K]`, `[algebra K L]` and `[algebra F L]`, along with an additional parameter `[is_scalar_tower F K L]` expressing that the maps commute.

The `is_scalar_tower` typeclass derives its name from its applicability to any three types between which exist scalar multiplication operations:

```
class is_scalar_tower (M N α : Type*) [has_scalar M N] [has_scalar N α]
  [has_scalar M α] : Prop :=
  (smul_assoc : ∀ (x : M) (y : N) (z : α), (x · y) · z = x · (y · z))
```

For example, if R is a ring, A is an R -algebra and M an A -module, we can express the fact that M is also an R -module by adding a `[is_scalar_tower R A M]` parameter. Since $x \cdot y$ for an R -algebra A is defined as `algebra_map R A x * y`, applying `smul_assoc` for each x with $y = z = 1$ shows that the `algebra_map`s indeed commute.

The typeclass system is set up to automatically provide common `is_scalar_tower` instances, such as for the maps $R \rightarrow S \rightarrow A$ when S is a R -subalgebra of S . The effect is that almost all coherence proof obligations are automatically solved from known results or filled in from parameters. In our formalization, we found that the `is_scalar_tower` typeclass translates towers of field extension well.

223 3.3 Ring of integers

224 When K is a number field, the ring \mathcal{O}_K of integers in K is defined as the integral closure of
 225 \mathbb{Z} in K . This is the subring containing those $x : K$ that are the root of a monic polynomial
 226 with coefficients in \mathbb{Z} :

```
227
228 def number_field.ring_of_integers (K : Type*) [field K]
229   [is_number_field K] : subalgebra  $\mathbb{Z}$  K :=
230   integral_closure  $\mathbb{Z}$  K
231
```

232 where `integral_closure` was previously defined in `mathlib` as follows:

```
233
234 def integral_closure (R A : Type*) [comm_ring R] [comm_ring A]
235   [algebra R A] : subalgebra R A :=
236   { carrier := { r | is_integral R r }, .. /- proofs omitted -/ }
237
```

238 Some examples of rings of integers include \mathbb{Z} and $\mathbb{Z}[i]$.

239 When K is a function field over the finite field \mathbb{F}_q , we define \mathcal{O}_K analogously as
 240 `integral_closure (polynomial K) F`. In order to reason uniformly for both concepts
 241 of the ring of integers, we will work with the integral closure of any principal ideal domain
 242 when possible.

243 3.4 Subobjects

244 The ring of integers are one example of a subobject, such as a subfield, subring or subalgebra,
 245 defined through a characteristic predicate. In `mathlib`, a subobject is defined as a bundled
 246 structure comprising the carrier set, along with proofs showing the carrier set is closed under
 247 the relevant operations.

248 Two new subobjects we needed in our development were `subfield` and `intermediate-`
 249 `_field`. We define a subfield of a field K as a subset of K that contains 0 and 1 and is
 250 closed under addition, negation, multiplication, and taking inverses. If L is a field extension
 251 of K , we define an intermediate field as a subfield that is also a subalgebra: a subfield that
 252 contains the image of `algebra_map K L`. Other examples of subobjects available in `mathlib`
 253 are submonoids, subgroups and submodules (with ideals as a special case of submodules).

254 The new definitions found immediate use: soon after we contributed our definition of
 255 `intermediate_field` to `mathlib`, the Berkeley Galois theory group used it in a proof of
 256 the primitive element theorem. Soon after the primitive element theorem was merged into
 257 `mathlib`, we used it in our development of the trace form. This anecdote illustrates the
 258 decentralized development style of `mathlib`, with different groups and people building on each
 259 other's results in a collaborative process.

260 By providing a coercion from subobjects to types, sending a subobject S to the subtype
 261 of all elements of S , and putting typeclass instances on this subtype, we can reason about
 262 inductively defined rings such as \mathbb{Z} and subrings such as `integral_closure \mathbb{Z} K` uniformly. If
 263 $S : \text{subfield } K$, the map that sends $x : S$ to K by “forgetting” that $x \in S$ is a ring embedding,
 264 and we register this map as a `algebra S K` instance, also allowing us to treat field extensions
 265 of the form $\mathbb{Q} \rightarrow \mathbb{C}$ and subfields uniformly. Similarly, for $F : \text{intermediate_field } K L$,
 266 we defined the corresponding `algebra K F`, `algebra F L` and `is_scalar_tower K F L`
 267 instances.

3.5 Fields of fractions

The fraction field $\text{Frac } R$ of an integral domain R can be defined explicitly as a quotient type as follows: starting from the set of pairs (a, b) with $a, b \in R$ such that $b \neq 0$, one quotients by the equivalence relation stating that $(\alpha a, \alpha b) \sim (a, b)$ for all $\alpha \neq 0 : R$, writing the equivalence class of (a, b) as $\frac{a}{b}$. It can easily be proved that the ring structure on R extends uniquely to a ring structure turning $\text{Frac } R$ into a field. When $R = \mathbb{Z}$, this yields the traditional description of \mathbb{Q} as the set of equivalence classes of fractions, where $\frac{2}{3} = \frac{-4}{-6}$, etc. The drawback of this construction is that there are many other fields that can serve as the field of fractions for the same ring. For instance, although there is an isomorphism of $\text{Frac } \mathbb{C}[[t]]$ with the field

$$\mathbb{C}((t)) = \left\{ \sum_{i=a}^{+\infty} a_i t^i \quad \text{with } a \in \mathbb{Z} \right\}$$

of Laurent series, there is no (definitional) equality between the types. Another example comes from the field

$$\mathbb{Q}(i) = \{z \in \mathbb{C} : \Re z \in \mathbb{Q}, \Im z \in \mathbb{Q}\}$$

which is isomorphic to $\text{Frac}(\mathbb{Z}[i])$, but not definitionally equal to it. In fact, even the rational numbers in Lean are a counterexample: for computational efficiency, \mathbb{Q} is defined as a subtype where the numerator and denominator are coprime, instead of a quotient by “scalar multiplication”. A definition like

```
def fraction_field (R : Type*) : Type* :=
  {ab : R × R // ab.2 ≠ 0}
```

would require transferring results across isomorphisms as soon as one needs to handle a different construction of a field isomorphic to $\text{Frac } R$.

The strategy used in `mathlib` is to rather allow for many different *fraction fields* of our given integral domain R , as fields F along with an injective *fraction map* $f : R \rightarrow F$ which witnesses that all elements of F are “fractions” of elements of R ³, and to parametrize every result over the choice of f . The conditions on f imply that F is the smallest field containing R , by showing each injective map $g : R \rightarrow A$ to a ring A such that $g(x)$ has a multiplicative inverse for all $x \neq 0 : R$, can be extended uniquely to a map $F \rightarrow A$ compatible with f and g . In particular, if $f_1 : R \rightarrow F_1$ and $f_2 : R \rightarrow F_2$ are fraction maps, they induce an isomorphism $F_1 \simeq F_2$. The construction of $\text{Frac } R$ then results in *a* field of fractions rather than *the* field of fractions.

This came at a price: informally, at any given stage of one’s reasoning, the field F is fixed and the map $f : R \rightarrow F$ is applied implicitly, just viewing every $x : R$ as $x : F$. It is now impossible to view $\text{range } f \leq F$ as an inclusion of subalgebras, because the map f is needed explicitly to give the R -algebra structure on F . We use a type synonym `f.codomain` $:= F$ and instantiate the R -algebra structure given by f on this synonym.

3.6 Representing simple field extensions

We have seen in Section 2 that every number field K can be written as $K = \mathbb{Q}(\alpha)$ by adjoining to \mathbb{Q} the root of a polynomial: there is an irreducible polynomial $p \in \mathbb{Q}[X]$ such

³ In the definition used by `mathlib`, a fraction map is a special case of a *localization map*. Different localizations restrict the denominators to different subsets of $R \setminus \{0\}$.

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that $\mathbb{Q}[X]/p \simeq K$; we set α to be the image of X in $\mathbb{Q}[X]/p$. We can also take $\alpha : K$ and let $\mathbb{Q}(\alpha)$ to be the smallest subfield of K containing α ; then $K = \mathbb{Q}(\alpha)$ means that $\mathbb{Q}(\alpha)$, as a subfield of K , is equal to the subfield \top containing all elements of K . We could also view K and $\mathbb{Q}(\alpha)$ as subfields of an arbitrary larger field F . Because α is algebraic, the smallest subring containing α and \mathbb{Q} will be a field, thus we can add two more representations, replacing “smallest subfield” with “smallest subring”. Moreover, all subfields/subrings containing \mathbb{Q} are also \mathbb{Q} -algebras, so we can additionally replace “subfield” with “intermediate field” and “subring” with “ \mathbb{Q} -subalgebra”. The same continues to hold if we replace the base field \mathbb{Q} with F , thus considering extensions of the form $F(\alpha)$, now requiring that α be a root of some $p \in F[X]$.

The ability to switch between these representations is important: sometimes K and F are fixed and we want an arbitrary α ; sometimes α is fixed and we want an arbitrary type representing $F(\alpha)$. The different constructions of $F(\alpha)$ have already been formalized in `mathlib`,

To find a uniform way to reason about all these equivalent definitions, we chose to formalize the notion of *power basis* to represent simple field extensions, a basis of the form $1, x, x^2, \dots, x^{n-1} : K$ (viewing K as a F -vector space)⁴. We call x the *generator* and n the *dimension* of this power basis. We defined the following type of power basis, bundling the information of a power basis:

```
structure power_basis (F K : Type*) [field F] [field K] [algebra F K] :=
  (gen : S) (dim : ℕ)
  (is_basis : is_basis F (λ (i : fin dim), gen ^ (i : ℕ)))
```

We proved that the various notions of simple field extensions are equivalent to the existence of a power basis.

With the `power_basis` structure, we have the ability to parametrize our results, being able to choose the F and K in a simple field extension K/F , or being able to choose the α generating $F(\alpha)$ (by setting `power_basis.gen pb` equal to α). Specializing a result from an arbitrary K with a power basis over F , to a specific value of K such as $F(\alpha) = \text{algebra.adjoin } F \{ \alpha \}$, is a matter of applying the result to the power basis generated by α , and rewriting `power_basis.gen (adjoin.power_basis F α) = α` .

4 Dedekind domains

The aim of this section is to introduce the notion of *Dedekind domain* which, as discussed in Section 2 is the right setting to study algebraic properties of number fields.

4.1 Definitions

There are various equivalent conditions, used at various times, for an integral domain D to be a Dedekind domain, of which the following three have been formalized in `mathlib`:

- `is_dedekind_domain D`: D is a Noetherian integral domain, integrally closed in its fraction field and has Krull dimension at most 1;
- `is_dedekind_domain_inv D`: D is an integral domain and nonzero fractional ideals of D have a multiplicative inverse (we discuss the notion and formalization of fractional ideals in Section 4.2);

⁴ In the formalization we generalize this notion to any algebra A over a commutative ring R

352 ■ `is_dedekind_domain_dvr` D : D is a Noetherian integral domain and the localization of
 353 D at each prime ideal is a discrete valuation ring.

354 We did not use `is_dedekind_domain_dvr` in our project, so we will not discuss this definition
 355 further.

356 Some authors exclude fields from being Dedekind domains, a convention we initially
 357 followed. Since we did not encounter any cases where excluding fields was necessary to prove
 358 a theorem, we decided to simplify the definition of a Dedekind domain. It is still possible to
 359 exclude fields in a theorem by adding an extra hypothesis $\neg \text{is_field } D$.

360 The “main” definition was chosen to be `is_dedekind_domain`, since this condition
 361 is usually the one checked in practice [19]. The other two equivalent definitions were
 362 added `mathlib`, before the proof they are indeed equivalent. Having multiple definitions
 363 allowed us to do our work in parallel without depending on unformalized results. For
 364 example, the proof of unique ideal factorization in a Dedekind domain initially assumed
 365 `is_dedekind_domain_inv D`, and the proof that the ring of integers \mathcal{O}_K is a Dedekind
 366 domain concluded `is_dedekind_domain (ring_of_integers K)`. After the equivalence be-
 367 tween `is_dedekind_domain D` and `is_dedekind_domain_inv D` was formalized, we could
 368 painlessly replace usages of `is_dedekind_domain_inv R` with `is_dedekind_domain D`. Sep-
 369 arating the different definitions meshed well with the contribution philosophy followed
 370 by `mathlib`, preferring small, standalone additions over in-progress work or entire finished
 371 projects.

372 The conditions `is_dedekind_domain` and `is_dedekind_domain_inv` require a fraction
 373 field F , although the truth value of the predicates does not depend on the choice of F .
 374 For ease of use, we let the type of `is_dedekind_domain` only depend on the domain D by
 375 instantiating F in the definition as `fraction_ring D`. From now on, we fix a fraction map
 376 $f: D \rightarrow F$.

```
377
378 class is_dedekind_domain (D : Type*) [integral_domain D] : Prop :=
379   (to_is_noetherian_ring : is_noetherian_ring D)
380   (dimension_le_one : dimension_le_one D)
381   (is_integrally_closed : integral_closure D (fraction_ring D) = ⊥)
382
```

383 Applications of `is_dedekind_domain` can choose a specific fraction field through the following
 384 lemma exposing the alternate definition:

```
385
386 lemma is_dedekind_domain_iff (f : fraction_map D F) :
387   is_dedekind_domain D ↔
388     is_noetherian_ring D ∧ dimension_le_one D ∧
389     integral_closure D f.codomain = ⊥
390
```

391 We mark `is_dedekind_domain` as a typeclass by using the keyword `class` rather than
 392 `structure`, allowing the typeclass system to automatically infer the Dedekind domain
 393 structure when an appropriate instance is declared, such as for principal ideal domains or
 394 rings of integers.

395 4.2 Fractional ideals

396 The notion which is pivotal to the definition of the ideal class group of a Dedekind domain is
 397 that of *fractional ideals*: given any integral domain R with a field of fractions F , these are
 398 R -ideals divided by some $x : R$, or equivalently R -submodules J of F such that there is an
 399 $x : R$ with $xJ \subseteq R$. The reason for introducing them is that, unlike their subset of proper
 400 ideals, they form a group under multiplication. As it should be clear from Section 3.5, this

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notion depends on the field F as well as on the localization map $f: R \rightarrow F$ allowing to speak about R -submodules of F and, more importantly, to see an element $x: R$ as the element $fx: F$, so as to be able to write the inclusion $f(x)J \subseteq f(R)$. We formalized the definition of fractional ideals relative to a map $f: R \rightarrow F$ as a type `fractional_ideal f`. We encoded that the structure of fractional ideals does not depend on the choice of fraction map f , which we formalized as an isomorphism `fractional_ideal.canonical_equiv` between the fractional ideals relative to embeddings $f_1: R \rightarrow F_1$ and $f_2: R \rightarrow F_2$.

We defined the addition, multiplication and intersection operations on fractional ideals, by showing the corresponding operations on submodules map fractional ideals to fractional ideals. We also proved that these operations give a commutative semiring structure on the type of fractional ideals. For example, multiplication of fractional ideals is defined as:

```

412 lemma fractional_mul (I J : fractional_ideal f) :
413   is_fractional f (I.1 * J.1) := _ -- proof omitted
414
415
416 instance : has_mul (fractional_ideal f) :=
417   ⟨λ I J, ⟨I.1 * J.1, fractional_mul I J⟩⟩
418

```

Defining the quotient of two fractional ideals requires slightly more work. Consider any R -algebra A and an injection $R \hookrightarrow A$, allowing to look at $x: R$ as $x: A$: given ideals $I, J \leq R$, the submodule quotient $I/J \leq A$ is characterized by the property

```

422 lemma submodule.mem_div_iff_forall_mul_mem {x : A} {I J : submodule R A} :
423   x ∈ I / J ↔ ∀ y ∈ J, x * y ∈ I
424
425

```

In our setting, we can look at every ideal as the fractional ideal $I/1 \leq F$. The first main theoretical result that we need to define the ideal class group is to show that every non-zero ideal $0 < I \leq R$ becomes invertible when seen as a fractional ideal: this means, by definition, that the equality

$$f(I) * \frac{1}{f(I)} = 1 = f(R) \leq F \quad (1)$$

as R -submodules of F , holds. Beware that the notation $1/I$ might be misleading here: indeed, for general integral domains, equation (1) might not hold. An example comes from the product

$$\frac{1}{(X, Y)} * (X, Y) = (X, Y) < \mathbb{C}[X, Y]$$

of the fractional ideals $1/(X, Y)$ and (X, Y) in the fraction field $\mathbb{C}(X, Y)$ of $\mathbb{C}[X, Y]$. On the other hand, it can be proved that Dedekind domain are precisely the right class of integral domains for which (1) always holds. This was formalised as the following

```

438 lemma fractional_ideal.is_unit {hD : is_dedekind_domain D}
439   (I : fractional_ideal f) (hne : I ≠ ⊥) : is_unit I :=
440
441

```

together with

```

443 noncomputable instance [is_dedekind_domain D] (g : fraction_map D F) :
444   has_inv (fractional_ideal g) :=
445     ⟨λ I, 1 / I⟩
446
447

```

asserting that the inverse of any fractional ideal I (defined as another fractional ideal J such that $I * J = 1$)—which always exists thanks to the lemma `fractional_ideal.is_unit`—is unique and coincides with $1/I$.

Two remarks are in order. The first is that in lemma `fractional_ideal.is_unit` the hypothesis (`hne : I ≠ 0`) that I be non-zero is added, and apparently dropped in the `has_inv` instance: this reflects the existence of the typeclass `group_with_zero` in `mathlib`, consisting of groups endowed with an extra element 0 whose inverse is again 0 . In particular, the zero fractional ideal is invertible (in the `mathlib` sense) but is not a unit, leading to the strange phenomenon above. Even more fundamentally, the fact that (1) might fail to hold in certain circumstances shows that, for general domains, $1/I \neq I^{-1}$. Since `a / b` used to have the built-in definition $a/b = a * b^{-1}$, the notation $1/I$, defined for every non-zero I , was conflicting with the fact that I might not be invertible. Since, for Dedekind domains, we wanted to *define* I^{-1} as $1/I$, a major refactor of a core definition was needed. In particular, to break the circularity, we had to weaken the definitional equality to a proposition; this involved many small changes throughout `mathlib`.

4.3 Equivalence of the definitions

We now describe how we proved and formalized that the two definitions `is_dedekind_domain` and `is_dedekind_domain_inv` of being a Dedekind domain are equivalent. Let D be a Dedekind domain, and $f: D \rightarrow F$ a fraction map to a field of fractions F of D .

To show that `is_dedekind_domain_inv` implies `is_dedekind_domain`, we follow the proof given by Fröhlich in [13, Chapter 1, § 2, Proposition 1]. A constant challenge that was faced while coding this proof was already mentioned in Section 3.5, namely the fact that elements of the ring must be traced along the fixed morphism to the fields of fractions. The proofs for being integrally closed and of dimension being less than or equal to 1 are fairly straightforward.

Proving the Noetherian condition was the most challenging. In the original proof by Fröhlich, he considers elements $a_1, \dots, a_n \in I$ and $b_1, \dots, b_n \in I^{-1}$ for any nonempty fractional ideal I , satisfying $\sum_i a_i b_i = 1$. However, it is quite challenging to prove that an element of the product of two D -submodules M and N must be of the form $\sum_{i=1}^m a_i * b_i$, for $a_i \in A$ and $b_i \in B$ for all $1 \leq i \leq m$. Instead, we show that, for every element of an ideal, there exists a `s : finset D` whose span is contained in the ideal, and which contains the element. This is accomplished by the lemma `submodule.mem_span_mul_finite_of_mem_span_mul`. Now considering an ideal I of the ring D , due to its invertibility (as a fractional ideal), by `submodule.mem_span_mul_finite_of_mem_span_mul`, we obtain finite sets $T \subset I$ and $T' \subset 1/I$ of type `finset D`, such that 1 is contained in the D -span of $T * T'$. With coercions, the actual statement of the latter expression in Lean is $\uparrow T' \subseteq \uparrow\uparrow(1 / \uparrow s)$, which reads :

```
(T' : set (localization_map.codomain (fraction_ring.of D))) ⊆ (((1 / (s :
  fractional_ideal (fraction_ring.of D))) : submodule D (
    localization_map.codomain (fraction_ring.of D))) set (localization_map.
    codomain (fraction_ring.of D))) )
```

This is then sufficient to show that I is finitely generated, as shown in the lemma `fg_of_one_mem_span_mul`.

The theorem `fractional_ideal.mul_inv_cancel` proves the converse, namely that `is_dedekind_domain` implies `is_dedekind_domain_inv`. The classical proof first shows that every maximal ideal $M : \text{ideal } R$, seen as a fractional ideal, is invertible; from this, some work allows to show that every non-zero ideal is invertible, using that it is contained in

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a maximal ideals; and, finally, the fact that every fractional ideal $J : \text{fractional_ideal } f$ satisfies $xJ \leq I$ for a suitable $x : D$ and $I : \text{ideal } D$ allows to show that every fractional ideal is invertible, concluding the proof that non-zero fractional ideals form a group. We have found that formalizing the second step, so passing from the case where M is maximal to the general case, required more code that directly showing invertibility of arbitrary non-zero ideals. We have coded this in the following

```
lemma coe_ideal_mul_one_div [hD : is_dedekind_domain D] (hNF : ¬ is_field
  D)
  (I : ideal D) (hne : I ≠ ⊥) :
  ↑I * ((1 : fractional_ideal f) / ↑I) = (1 : fractional_ideal f) :=
```

from where it becomes apparent that, over and over again, we had to carefully distinguish between the ideal I , which is a term of type `ideal D`, and its coercion $\uparrow I$, which is of type `fractional_ideal f`, although these objects, from a mathematical point of view, are identical.

The proof of the above result relies on the lemma `exists_not_mem_one_of_ne_bot`, which says that for every non-trivial ideal $0 \leq I \leq D$, there exists an element in the field F which is not integral (so, not in `f.range`) but lies in $1/I$. This depends crucially on D being Noetherian, since the proof begins by invoking that every non-zero ideal in a Noetherian ring contains a product of non-zero prime ideals. This result was not previously available in `mathlib`, and we formalized it as `exists_prime_spectrum_prod_le_and_ne_bot_of_domain`. It is when applying this that the dimension condition shows its full force: the constructed prime ideal, being non-zero, will be maximal because the Krull dimension of D is at most 1; from this, the conclusion follows straightforwardly. Having the above lemma at our disposal, we can prove that every ideal $I \neq 0$ is invertible by arguing by contradiction: if $I * 1/I \leq D$, we can find an element $x \in F \setminus f(R)$ which is in $1/(1 * 1/I)$ thanks to `exists_not_mem_one_of_ne_bot` and some easy algebraic manipulation will imply that x is actually integral over D . Since D is integrally closed, it must lie in $f(D)$, contradicting its construction.

The final step, when we prove that invertibility of ideals implies that of fractional ones as well, was easy: the material developed for the general theory of `fractional_ideals f` allowed to smoothly deduce that a fractional ideal J must be invertible as soon as a certain multiple xJ of it is, and since there always exists a $x : D$ satisfying the latter condition (because xJ can be made into a “usual” ideal), this leads to the final lemma `fractional_ideal.is_unit` quoted above.

5 Principal ideal domains are Dedekind

As an example of our definitions, we will discuss in some detail our formalization of the fact that a principal ideal domain is a Dedekind domain. A principal ideal domain (PID) is an integral domain R such that each ideal is generated by one element. There is no explicit definition of PIDs in `mathlib`, rather it is split up into two hypotheses. One uses `[integral_domain R] [is_principal_ideal_ring R]` to denote a PID R , where `is_principal_ideal_ring` is a typeclass defined for all commutative rings:

```
class is_principal_ideal_ring (R : Type*) [comm_ring R] : Prop :=
  (principal : ∀ (I : ideal R), I.is_principal)
```

Our proof that the hypotheses `[integral_domain R] [is_principal_ideal_ring R]` imply `is_dedekind_domain R` is relatively short:

```

545
546 instance principal_ideal_ring.to_dedekind_domain (R : Type*)
547   [integral_domain R] [is_principal_ideal_ring R] :
548   is_dedekind_domain R :=
549   ⟨principal_ideal_ring.is_noetherian_ring,
550     dimension_le_one.principal_ideal_ring _,
551     unique_factorization_monoid.integrally_closed (fraction_ring.of R)⟩
552

```

553 Making this an `instance` instead of a `lemma` ensures that the typeclass system can now
 554 automatically infer a Dedekind domain structure whenever a principal ideal structure is
 555 already available.

556 The Noetherian property of a Dedekind domain follows easily by the previously defined
 557 lemma `principal_ideal_ring.is_noetherian_ring`, since, by definition, each ideal in a
 558 principal ideal ring is finitely generated (by a single element).

559 The lemma `dimension_le_one.principal_ideal_ring` is an instantiation of the existing
 560 result `is_prime.to_maximal_ideal` showing a nonzero prime ideal in a PID is maximal.
 561 The latter lemma uses the characterization that I is a maximal ideal if and only if any strictly
 562 larger ideal $J > I$ is the full ring \top . If I is a nonzero prime ideal and $J > I$ in the PID R ,
 563 we have that the generator j of J is a divisor of the generator i of I . Since I is prime, this
 564 implies that either $j \in I$, contradicting the assumption that $J > I$, $i = 0$, contradicting that
 565 I is nonzero, or that j is a unit, implying $J = \top$ as desired.

566 The final condition of a PID being integrally closed is the most challenging. We use the pre-
 567 viously defined instance `principal_ideal_ring.to_unique_factorization_monoid` that
 568 a PID is a unique factorisation monoid (UFM), to instantiate our proof that every UFM is
 569 integrally closed. In the same way that principal ideal domains are generalized to principal
 570 ideal rings, `mathlib` generalizes unique factorization domains to unique factorization monoids.
 571 A commutative monoid R with an absorbing element 0 and injectivity of multiplication is
 572 defined to be a UFM, if the relation “ x properly divides y ” is well-founded (implying each
 573 element can be factored as a product of irreducibles) and an element of R is prime if and
 574 only if it is irreducible (implying the factorization is unique). The first condition is satisfied
 575 for a PID since the Noetherian property implies that the division relation is well-founded.
 576 The second condition follows from `principal_ideal_ring.irreducible_iff_prime`. To
 577 prove that an irreducible element p is prime, the proof uses that prime elements generate
 578 prime ideals and irreducible elements of a PID generate maximal ideals. Since all maximal
 579 ideals are prime ideals, the ideal generated by p is maximal, hence prime, thus p is prime.
 580 The lemma `irreducible_of_prime` proves the converse holds in any commutative monoid
 581 with zero.

582 In order to show that a UFM is integrally closed, we first proved the Rational Root
 583 Theorem, named `denom_dvd_of_is_root`, which states that for polynomial $p : R[X]$ and x
 584 an element of the fraction field $\text{Frac } R$ such that $p(x) = 0$, the denominator of x divides the
 585 leading coefficient of p . If x is integral with minimal polynomial p , the leading coefficient is
 586 1, therefore the denominator is a unit and x is an element of R . This gives us the required
 587 lemma `unique_factorization_monoid.integrally_closed`, which states that the integral
 588 closure of R in its fraction field is R itself.

589 6 Rings of integers are Dedekind domains

590 Recall that we defined the ring of integers of a number field K as the integral closure of \mathbb{Z} in
 591 K . We proved a stronger result: give a Dedekind domain D and a field of fractions F , if L is

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a finite separable extension of F , then the integral closure of D in L is a Dedekind domain with fraction field L . Our approach adapts [19, Theorem 3.1]. Throughout this section, let D be a Dedekind domain with a field of fractions F (given by the map $f: D \rightarrow F$), L a field extension of F and let S denote the integral closure of D in L .

The first step is to show that L is a field of fractions for the integral closure, namely that there is a map `fraction_map_of_finite_extension` $f L : \text{fraction_map } S L$. We formalized the following definition, which implies the desired result:

```
def fraction_map_of_algebraic (alg : is_algebraic D L)
  (inj : function.injective (algebra_map D L)) :
  fraction_map S L
```

The main content of `fraction_map_of_algebraic` consists of showing that all elements $x : L$ can be written as y/z for elements $y \in S$, $z \in D \subseteq S$; the standard proof of this fact (see [7, Theorem 15.29]) formalizes readily.

Now we are ready to show that the integral closure of D in L is a Dedekind domain, by proving it is integrally closed in L , has Krull dimension at most 1 and is Noetherian. The fact that the integral closure is integrally closed is immediate.

To show the Krull dimension is at most 1, we needed to develop basic going-up theory for ideals. In particular, we show that an ideal I in an integral extension is maximal if it lies over a maximal ideal, and use a result already available in `mathlib` that a prime ideal I in an integral extension lies over a prime ideal.

```
lemma is_maximal_of_is_integral_of_is_maximal_comap
  {S : Type*} [integral_domain S] [algebra D S]
  (hDS : algebra.is_integral D S) (I : ideal S) [I.is_prime]
  (hI : is_maximal (I.comap (algebra_map D S))) : is_maximal I

theorem is_prime_comap [hI : I.is_prime] : (comap f I).is_prime
```

The final condition, that the integral closure S of D in L is a Noetherian ring, requires the most work. We start by following the first half of [7, Theorem 15.29], so that it suffices to find a nondegenerate bilinear form B such that all integral $x, y : L$ satisfy $B(x, y) \in \text{integral_closure } D L$. We formalized the results in [19, §§ 2.5–2.8,] to show the *trace form* is a bilinear form satisfying these requirements.

6.1 The trace form

Retaining notations from the previous section, we have a bilinear form $\text{lmul} = \lambda xy : S, xy$. The trace of the linear map $\text{lmul } x$ is called the *algebra trace* $\text{Tr}_{L/F}(x)$ of x . We define the algebra trace as a linear map from L to F :

```
noncomputable def trace : L →L[F] F :=
  (linear_map.trace F L).comp (lmul F L).to_linear_map
```

This definition is marked `noncomputable` since `linear_map.trace` makes a case distinction on the existence of a basis, choosing an arbitrary basis if one exists and returning 0 otherwise. This latter case does not occur in our development.

The *trace form* is a F -bilinear form on L , mapping $x, y : L$ to $\text{Tr}(xy)$.

```
noncomputable def trace_form : bilin_form F L :=
  { bilin := λ x y, trace F L (x * y), .. /- proofs omitted -/ }
```


In fact, we define the trace and trace form for any algebra over a commutative ring. For simplicity of exposition in this paper we will only consider finite extensions of fields. In the following, let $E/L/F$ be a tower of finite extensions of fields, namely we assume `[algebra E L] [algebra L F] [algebra E F] [is_scalar_tower E L F]`, as described in Section 3.2.

The value of the trace depends on the choice of E and L ; we formalized this as lemmas `trace_algebra_map x : trace E L (algebra_map E L x) = findim E L • x` and `trace_comp L x : trace E F x = trace E L (trace L F x)`. These results follow by direct computation.

To compute $\text{Tr}_{L/F}(x)$ it therefore suffices to consider the trace of x in the smallest field containing x and F , which is the simple extension $F(x)$ discussed in Section 3.6. There is a nice formula for the trace in $F(x)$, although the terms in this formula are elements in a larger field E (such as the *splitting field* of the minimal polynomial of x). In formalizing this formula, we must first map the trace to F using the canonical embedding `algebra_map E F`, giving the following lemma statement:

```
lemma power_basis.trace_gen_eq_sum_roots (pb : power_basis F L)
  (h : polynomial.splits (algebra_map F E) pb.minpoly_gen) :
  algebra_map F E (trace F L pb.gen) =
    (pb.minpoly_gen.map (algebra_map F E)).roots.sum
```

We formulate the lemma in terms of the power basis, since we will need to use it for $F(x)$ here and for an arbitrary finite separable extension L/F later in the proof.

The elements of `(pb.minpoly_gen.map (algebra_map F E)).roots` are called *conjugates* of x in E . Each conjugate of x is integral since it is a root of (the same) monic polynomial, and integer multiples and sums of integral elements are integral. Combining `trace_gen_eq_sum_roots` and `trace_algebra_map` together shows that the trace of x is an integer multiple (namely `findim F(x) L`) of a sum of conjugate roots, hence we conclude that the trace (and trace form) of an integral element is also integral.

Finally, we show the trace form is nondegenerate, following [19, Proposition 2.8]. Since L/F is a finite, separable field extension, it has a power basis `pb` generated by x . Letting x_k denote the k -th conjugate of x in an algebraically closed field $E/L/F$, the main difficulty lies in checking the equality $\sum_k x_k^{i+j} = \text{Tr}_{L/F}(x^{i+j})$. Directly applying `trace_gen_eq_sum_roots` is tempting, since we have a sum over conjugates of powers on both sides. However, the two expressions will not precisely match: the left hand side is a sum of conjugates of x , where each conjugate is raised to the power $i+j$, while the conclusion of `trace_gen_eq_sum_roots` results in a sum over conjugates of x^{i+j} .

Instead, the informal proof switches here to an equivalent definition of conjugate: the conjugates of x in E are the images (counted with multiplicity) of x under each embedding $\sigma: F(x) \rightarrow E$ that fixes F . This equivalence between the two notions of conjugate was contributed to `mathlib` by the Berkeley group in the week before we realized we needed it. Mapping `trace_gen_eq_sum_roots` through the equivalence gives $\text{Tr}_{L/F}(x) = \sum_{\sigma: L \rightarrow_a [F]E} \sigma x$. Since σ is a ring homomorphism, $\sigma x^{i+j} = (\sigma x)^{i+j}$, so the conjugates of x^{i+j} are the $(i+j)$ -th powers of conjugates of x , concluding the proof.

7 Class group and class number

Given a Dedekind domain with fraction map $f: D \rightarrow F$, we formalize the notion of class group in Lean by defining a map `to_principal_ideal f : units f.codomain →`

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units (fractional_ideal f), and define the class group to be the quotient group modulo to_principal_ideal.range. In general, Dedekind domains can have infinite class groups. However, as discussed in Section 2, the rings of integers of global field have finite class group.

We let K be a number field and K' be a function field, with ring of integers \mathcal{O}_K and $\mathcal{O}_{K'}$, respectively. Most proofs of the finiteness of $\mathcal{C}_{\mathcal{O}_K}$ one finds in a modern textbook (see [19, Theorems 4.4, 5.3, 6.3]) depend on Minkowski's lattice point theorem, a result from the geometry of numbers (which has been formalized in Isabelle/HOL [8]). Extending this proof to show the finiteness of $\mathcal{C}_{\mathcal{O}_{K'}}$ is quite involved and does not result in a uniform proof for $\mathcal{C}_{\mathcal{O}_K}$ and $\mathcal{C}_{\mathcal{O}_{K'}}$. Instead we were inspired by a more classical approach to the finiteness of $\mathcal{C}_{\mathcal{O}_K}$, where the use of Minkowski's theorem is replaced by the pigeonhole principle. This approach seems to go back to Kronecker and can be found, for instance, in [16]. Our formalization adapts and generalizes this approach. We note that some other “uniform” approaches can be found in [1] and [20].

Let D be a Dedekind domain with field of fractions F of D , and fraction map $f: D \rightarrow F$. Moreover, let L be a finite separable field extension of F . We prove in the theorem `class_group.finite_of_admissible` that the integral closure of D in L has a finite class group if it has an “admissible” absolute value `abs`. Informally, the admissibility conditions require that the remainder operator produces values that are not too far apart. Formally, we define the type of admissible absolute values on D as:

```
structure admissible_absolute_value (D : Type*) [euclidean_domain D]
  extends euclidean_absolute_value D ℤ :=
  (card : ℝ → ℕ) (exists_partition :
    ∀ (n : ℕ) (ε > (0 : ℝ)) (b ≠ (0 : D)) (A : fin n → D),
    ∃ (t : fin n → fin (card ε)), ∀ i0 i1, t i0 = t i1 →
    (to_fun (A i1 % b - A i0 % b) : ℝ) < to_fun b · ε)
```

Here, `to_fun` stands for an application of the absolute value operator.

The above condition formalizes an intermediate result in the typical finiteness proofs; the different proofs for number fields and function fields are the same after this point. We use division with remainder to replace the *fractional part* operator on F in the classical proof, allowing our proof to stay entirely within D to avoid casts.

The absolute value extends to a norm `abs_norm f abs : integral_closure D L → ℤ`. We use the admissibility of `abs` to find a finite set `finset_approx L f abs` of elements of R , such that the following generalization of [16, Theorem 12.2.1] holds.

```
theorem exists_mem_finset_approx' (a b : integral_closure R L) :=
  ∃ (q : integral_closure R L) (r ∈ finset_approx L f abs),
  abs_norm f abs (r · a - q * b) < abs_norm f abs b
```

After this, the classical approach mentioned above formalizes smoothly.

It remains to define an admissible absolute value for \mathbb{Z} and $K[t]$ (where K is a finite field). On \mathbb{Z} , the usual Archimedean absolute value does the job (which is straightforward to formalize). For $K[t]$, let q be the cardinality of K , then we show $|f|_{\deg} := q^{\deg f}$ for $f \in \mathbb{F}_q[t]$ works in the function field setting (which is somewhat more involved). (Note that these absolute values coincide with cardinalities in corresponding residue rings, but we do not need this result.) We conclude that the class group is finite:

```
noncomputable instance : fintype
  (class_group (number_field.ring_of_integers.fraction_map K)) :=
  class_group.finite_of_admissible K int.fraction_map int.admissible_abs
```

```

741 noncomputable instance : fintype
742   (class_group (function_field.ring_of_integers.fraction_map f F)) :=
743   class_group.finite_of_admissible F f polynomial.admissible_char_pow_degree
744
745

```

Finally, we define `number_field.class_number` and `function_field.class_number` as the cardinality of the respective class groups.

8 Discussion

8.1 Related work

Broadly speaking, one could see the formalization work as part of number theory. There are several formalization result in this direction; see e.g. [5, Section 6], most notably the formalization in Isabelle/HOL of a substantial part of analytic number theory [9]. Narrowing somewhat in on our more algebraic setting, we are not aware of any other formal developments of fractional ideals, Dedekind domains or class groups of global fields. Since our project touches upon the theories of field extensions, ideals, number fields and number rings, we provide here a partial overview of formalizations in these areas.

There are many libraries formalizing basic notions of commutative algebra such as field extensions and ideals, including the Mathematical Components library in Coq [17], the algebraic library for Isabelle/HOL [11], the `set.mm` database for MetaMath [18] and the Mizar Mathematical Library [15]. The field of algebraic numbers, or more generally algebraic closures of arbitrary fields, are also available in many provers, for example Coq [4, 17], Isabelle/HOL [23], MetaMath [2], and Mizar [24]. To our knowledge, the Coq Mathematical Components library is the only formal development except ours specifically dealing with number fields [17, `field/alnum.v`].

Apart from the general theory of algebraic numbers, there are formalizations of specific number rings: the Gaussian integers $\mathbb{Z}[i]$ are available in Isabelle/HOL [10], MetaMath [3] and Mizar [14]. The Isabelle/HOL formalization deserves special mention since it introduces techniques from algebraic number theory, defining the integer-valued norm on $\mathbb{Z}[i]$ and classifying the prime elements of $\mathbb{Z}[i]$.

8.2 Future directions

Having formalized the basics of algebraic number theory, there are several natural directions for future formalization work. These include the following.

- Finiteness of the class group for the ring of integers in all global fields. This would entail dropping the separability condition in the result mentioned in the third line of Section 6, and consequently adapt some of the details in the final steps for the finiteness of the class group in the admissible case. Some basic prerequisites would be setting up some field theory dealing with (finite) inseparable field extensions, especially the purely inseparable ones. All in all this seems a tedious though reasonable project.
- Finite generation of the group of units of the ring of integers in a number field, or slightly stronger, Dirichlet's unit theorem [19, Theorem 7.4]. This seems a somewhat more involved, but still reasonable, project. The finite generation result also holds for function fields, so actually it would be nice (and doable) to consider all global fields (which would involve finite inseparable field extensions, as in the previous item).
- Other finiteness results in algebraic number theory, most notably Hermitte's theorem about the existence of only finitely many number fields (up to isomorphism, or embedded

in a fixed algebraically closed field containing \mathbb{Q} , e.g. \mathbb{C}) with discriminant below a given bound [19, Theorem 2.16]. This would be significantly more involved than the previous items and would require, amongst other things, setting up a lot of ramification theory (which is very important for algebraic number theory). As usual, there are analogue results in the function field setting, though they are less straightforward. One reason being that the nondegenerateness of the trace form from Section 6.1 does not hold any more when the separability condition is dropped.

- Computational aspect. Our formalization lays some foundations to the verification of number theoretic software, such as KASH/KANT [12] and PARI/GP [22]. Verifying computations for class groups, or just class numbers, in the case of “small” (e.g. some quadratic) number fields, looks within reach. Of course, getting really efficient algorithms (or certificates), is a hard research topic.
- Number theoretic applications. All of the above items consider theoretical of computational aspects within algebraic number theory itself. There are many applications of these, e.g. solving Diophantine equations. Solving Mordell equations, i.e. for a given nonzero integer D determining all pairs of integers (x, y) such that $y^2 = x^3 + D$, could be an interesting first case study (dealing with some values of D where elementary methods fail).

8.3 Conclusion

In this project, we found that the rule holds that the hardest part of formalization is to get the definitions just right. Once this is accomplished, the informal proof almost always translated to a formal proof without too much effort. In particular, we regularly had to invent abstractions to treat instances the “same” situation uniformly. Instead of fixing a canonical representation such $K \subseteq L \subseteq F$ as subfields, $\text{Frac } F$ as the field of fractions, or $K(\alpha)$ as the simple field extension, we find that making the essence of the situation an explicit parameter, as in `is_scalar_tower`, `fraction_map` or `power_basis`, treats equivalent viewpoints uniformly without the need for transferring results.

The formalization efforts described in this paper cannot be cleanly separated from the development of `mathlib` as a whole. The decentralized organization and highly integrated design of `mathlib` meant we could contribute our formalizations as we completed them, resulting in a quick integration into the rest of the library. Other contributors building on these results often extended them to meet our requirements, before we could identify that we needed them, as the anecdote in Section 3.4 illustrates. In other words, the low barriers for contributions ensured mutually beneficial collaboration.

The formalization project described in this paper resulted in the contribution of thousands of lines of Lean code involving hundreds of declarations. We validated existing design choices used in `mathlib`, refactored those that did not scale well and contributed our own set of designs. The real achievement was not to complete each proof, but to build a better foundation for formal mathematics.

References

- 1 Emil Artin and George Whaples. Axiomatic characterization of fields by the product formula for valuations. *Bull. Amer. Math. Soc.*, 51(7):469–492, 07 1945. URL: <https://projecteuclid.org:443/euclid.bams/1183507128>.
- 2 Mario Carneiro. Definition `df-aa`. <http://us.metamath.org/mpeuni/df-aa.html>.
- 3 Mario Carneiro. Definition `df-gz`. <http://us.metamath.org/mpeuni/df-gz.html>.

- 831 4 Cyril Cohen. Construction of real algebraic numbers in Coq. In Lennart Beringer and
832 Amy P. Felty, editors, *Interactive Theorem Proving - Third International Conference, ITP*
833 *2012, Princeton, NJ, USA, August 13-15, 2012. Proceedings*, volume 7406 of *Lecture Notes in*
834 *Computer Science*, pages 67–82. Springer, 2012. doi:10.1007/978-3-642-32347-8_6.
- 835 5 Sander R. Dahmen, Johannes Hölzl, and Robert Y. Lewis. Formalizing the Solution to the
836 Cap Set Problem. In John Harrison, John O’Leary, and Andrew Tolmach, editors, *10th*
837 *International Conference on Interactive Theorem Proving (ITP 2019)*, volume 141 of *Leibniz*
838 *International Proceedings in Informatics (LIPIcs)*, pages 15:1–15:19, Dagstuhl, Germany, 2019.
839 Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik. URL: [http://drops.dagstuhl.de/opus/](http://drops.dagstuhl.de/opus/volltexte/2019/11070)
840 [volltexte/2019/11070](http://drops.dagstuhl.de/opus/volltexte/2019/11070), doi:10.4230/LIPIcs.ITP.2019.15.
- 841 6 Leonardo de Moura, Soonho Kong, Jeremy Avigad, Floris van Doorn, and Jakob von Raumer.
842 The lean theorem prover (system description). In A. P. Felty and A. Middeldorp, editors,
843 *Automated Deduction - CADE-25*, volume 9195 of *LNCS*, pages 378–388. Springer, Cham,
844 2015. doi:10.1007/978-3-319-21401-6_26.
- 845 7 David S. Dummit and Richard M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken,
846 NJ, third edition, 2004.
- 847 8 Manuel Eberl. Minkowski’s theorem. *Archive of Formal Proofs*, July 2017. [https://isa-afp.](https://isa-afp.org/entries/Minkowskis_Theorem.html)
848 [org/entries/Minkowskis_Theorem.html](https://isa-afp.org/entries/Minkowskis_Theorem.html), Formal proof development.
- 849 9 Manuel Eberl. Nine chapters of analytic number theory in Isabelle/HOL. In John Harrison,
850 John O’Leary, and Andrew Tolmach, editors, *Interactive Theorem Proving, ITP 2019*, volume
851 141 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 16:1–16:19. Schloss
852 Dagstuhl, Leibniz-Zentrum fuer Informatik, 2019. doi:10.4230/LIPIcs.ITP.2019.16.
- 853 10 Manuel Eberl. Gaussian integers. *Archive of Formal Proofs*, April 2020. [https://isa-afp.](https://isa-afp.org/entries/Gaussian_Integers.html)
854 [org/entries/Gaussian_Integers.html](https://isa-afp.org/entries/Gaussian_Integers.html), Formal proof development.
- 855 11 Clemens Ballarin (editor), Jesús Aransay, Martin Baillon, Paulo Emílio de Vilhena, Stephan
856 Hohe, Florian Kammüller, and Lawrence C Paulson. The Isabelle/HOL algebra library.
857 <http://isabelle.in.tum.de/dist/library/HOL/HOL-Algebra/index.html>.
- 858 12 M. E. Pohst et al. The computer algebra system KASH/KANT.
859 <http://www.math.tu-berlin.de/~kant>.
- 860 13 A. Fröhlich. Local fields. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton,*
861 *1965)*, pages 1–41. Thompson, Washington, D.C., 1967.
- 862 14 Y. Futa, D. Mizushima, and H. Okazaki. Formalization of Gaussian integers, Gaussian rational
863 numbers, and their algebraic structures with Mizar. In *2012 International Symposium on*
864 *Information Theory and its Applications*, pages 591–595, 2012.
- 865 15 Adam Grabowski, Artur Kornilowicz, and Christoph Schwarzweller. On algebraic hierarchies
866 in mathematical repository of mizar. In M. Ganzha, L. Maciaszek, and M. Paprzycki, editors,
867 *Proceedings of the 2016 Federated Conference on Computer Science and Information Systems*,
868 volume 8 of *ACSIS*, pages 363–371, 2016.
- 869 16 K. Ireland and M. Roosen. *A Classical Introduction to Modern Number Theory*. Springer-Verlag
870 New York, 2 edition, 1990.
- 871 17 Assia Mahboubi and Enrico Tassi. *The Mathematical Components Libraries*. [https://](https://math-comp.github.io/mcb/)
872 math-comp.github.io/mcb/, 2017.
- 873 18 Norman D. Megill and David A. Wheeler. *Metamath: A Computer Lan-*
874 *guage for Mathematical Proofs*. Lulu Press, Morrisville, North Carolina, 2019.
875 <http://us.metamath.org/downloads/metamath.pdf>.
- 876 19 Jürgen Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen*
877 *Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin,
878 1999. Translated from the 1992 German original and with a note by Norbert Schappacher,
879 With a foreword by G. Harder. doi:10.1007/978-3-662-03983-0.
- 880 20 Alexander Stasinski. A uniform proof of the finiteness of the class group of a global field. *to*
881 *appear in Amer. Math. Monthly*. URL: <https://arxiv.org/abs/1909.07121>.

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- 882 21 The mathlib Community. The lean mathematical library. In J. Blanchette and C. Hrițcu,
883 editors, *CPP 2020*, page 367–381. ACM, 2020. doi:10.1145/3372885.3373824.
- 884 22 The PARI Group, Univ. Bordeaux. *PARI/GP version 2.11.2*, 2019. available from [http:](http://pari.math.u-bordeaux.fr/)
885 [//pari.math.u-bordeaux.fr/](http://pari.math.u-bordeaux.fr/).
- 886 23 René Thiemann, Akihisa Yamada, and Sebastiaan Joosten. Algebraic numbers in Is-
887 abelle/HOL. *Archive of Formal Proofs*, December 2015. [https://isa-afp.org/entries/](https://isa-afp.org/entries/Algebraic_Numbers.html)
888 [Algebraic_Numbers.html](https://isa-afp.org/entries/Algebraic_Numbers.html), Formal proof development.
- 889 24 Yasushige Watase. Algebraic numbers. *Formalized Mathematics*, 24(4):291–299, 2016. doi:
890 10.1515/forma-2016-0025.