STOCHASTIC PROCESSES EXERCISES

① Let $\{A_n\}_{n\in\mathbb{N}} \,/\, P(A_n)=1 \;\; \forall n\in\mathbb{N}$. Show that $P(\bigcap_{n=1}^{\infty} A_n)=1$.

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{m=1}^{\infty} \underbrace{\bigcap_{n=1}^{m} A_n}_{B_m} = \bigcap_{m=1}^{\infty} B_m . \qquad B_{m+1} = A_1 \cap \dots \cap A_{m+1} \subseteq A_1 \cap \dots \cap A_m = B_m$$

$$P(\bigcap_{n=1}^{\infty} A_n) = P(\bigcap_{m=1}^{\infty} B_m) = \lim_{m\to\infty} P(B_m) = \lim_{m\to\infty} P(\bigcap_{n=1}^{m} A_n)$$

Let's prove $P(A_1)=P(A_2)=1 \Rightarrow P(A_1\cap A_2)=1$

$$P(A_1\cap A_2) = P(A_1)+P(A_2)-P(A_1\cup A_2) = 2 - P(A_1 \underset{*}{\cup} A_2) = 2-1=1$$

$*$ $A_1 \subseteq A_1\cup A_2$ and $P(A_1)=1$

By induction, $P(\bigcap_{i=1}^{m+1} A_i) = P(\bigcap_{i=1}^{m} A_i \cap A_{m+1}) =$

$$P(\bigcap_{i=1}^{m} A_i) + P(A_{m+1}) - P(\bigcap_{i=1}^{m} A_i \cup A_{m+1}) = 1+1-1=1$$

As a result, $P(\bigcap_{n=1}^{\infty} A_n) = \lim_{m\to\infty} P(\bigcap_{n=1}^{m} A_n) = \lim_{m\to\infty} 1 = 1$

Another solution:

$$P((\bigcap_{n=1}^{\infty} A_n)') = P(\bigcup_{n=1}^{\infty} A_n') \le \sum_{n=1}^{\infty} P(A_n') = 0 \Rightarrow P((\bigcap_{n=1}^{\infty} A_n)') = 0 \Rightarrow P(\bigcap_{n=1}^{\infty} A_n) = 1$$

② Let $\Omega = \{w_1, w_2 \dots\}$, $\mathcal{F} = \mathcal{P}(\Omega)$, $P(\{w_i\}) > 0 \quad \forall i = 1 \dots$

Prove $\{X_n\} \to X$ in probability $\Longrightarrow$ $\{X_n\} \to X$ with probability 1.

First, we observe $\{X_n\} \to X$ with probability 1 means $\{X_n(w)\} \to X(w)$ $\forall w \in \Omega$

By contradiction, let's assume, $\exists w_{i_0} \in \Omega \,/\, \{X_n(w_{i_0})\} \not\to X(w_{i_0}) \Longrightarrow$

$\exists \varepsilon_0 > 0$ and a subsequence $\{X_{k_n}\} \,/\, |X_{k_n}(w_{i_0}) - X(w_{i_0})| \geq \varepsilon_0$

$\{X_n\} \to X$ in prob. $\Rightarrow \{X_{k_n}\} \to X$ in prob.

Let $A_n = \{w \in \Omega \,|\, |X_{k_n}(w) - X(w)| < \varepsilon_0\} \Longrightarrow P(A_n) \to 1$

$\Longrightarrow w_{i_0} \notin A_n \Longrightarrow$

$P(\Omega) = 1 \geq P(A_n \cup \{w_{i_0}\}) = P(A_n) + P(\{w_{i_0}\}) \to 1 + \underset{\vee}{P(\{w_{i_0}\})}_{0}$!!!

contradiction

③ Let X, Y be random variables,

$$d(X,Y)=\int_\Omega \frac{|X-Y|}{1+|X-Y|}\,dP=\int_\Omega \frac{|X(w)-Y(w)|}{1+|X(w)-X(Y)|}\,P(dw)$$

Prove $\{X_n\}\to X$ in probability $\Longleftrightarrow \{d(X_n,X)\}\to 0$

Let's start with some inequalities. Let $Z\geq 0$ be a random variable.

$$\forall\varepsilon>0,\ \int_\Omega \frac{Z}{1+Z}\,dP=\int_{\{w\in\Omega/Z(w)<\varepsilon\}}\frac{Z(w)}{1+Z(w)}\,P(dw)+\int_{\{w\in\Omega/Z(w)\geq\varepsilon\}}\frac{Z(w)}{1+Z(w)}\,P(dw)\leq$$

$$\int_{\{w\in\Omega/Z(w)<\varepsilon\}} Z(w)\,P(dw)+\int_{\{w\in\Omega/Z(w)\geq\varepsilon\}} P(dw)\leq\int_{\{w\in\Omega/Z(w)<\varepsilon\}}\varepsilon\,P(dw)+\int_{\{w\in\Omega/Z(w)\geq\varepsilon\}}P(dw)=$$

$$\varepsilon P(\{w\in\Omega/Z(w)<\varepsilon\})+P(\{w\in\Omega/Z(w)\geq\varepsilon\})\leq\varepsilon+P(\{w\in\Omega/Z(w)\geq\varepsilon\})$$

$$\int_\Omega\frac{Z}{1+Z}\,dP\geq\int_{\{w\in\Omega/Z(w)\geq\varepsilon\}}\frac{Z(w)}{1+Z(w)}\,P(dw)$$

Let $f(t)=\frac{1}{1+t}$ $\forall t\in[\varepsilon,+\infty[$. We have $f\nearrow\Rightarrow f(t)\geq\frac{\varepsilon}{1+\varepsilon}\Rightarrow$

$$\int_{\{w\in\Omega/Z(w)\geq\varepsilon\}}\frac{Z(w)}{1+Z(w)}\,P(dw)\geq\int_{\{w\in\Omega/Z(w)\geq\varepsilon\}}\frac{\varepsilon}{1+\varepsilon}\,P(dw)=\frac{\varepsilon}{1+\varepsilon}P[Z\geq\varepsilon]$$

As a result, $\frac{\varepsilon}{1+\varepsilon}P[Z\geq\varepsilon]\leq\int\frac{Z}{1+Z}\,dP\leq\varepsilon+P[Z\geq\varepsilon]$

$\boxed{\Leftarrow}$ Let $Z=|X_n-X|$.

$$0\leftarrow d(X_n,X)=\int_\Omega\frac{Z}{1+Z}\,dP\geq\frac{1}{1+\varepsilon}P[|X_n-X|\geq\varepsilon]\geq 0\Rightarrow$$

$P[|X_n-X|\geq\varepsilon]\to 0\Rightarrow\{X_n\}\to X$ in probability

$\boxed{\Rightarrow}$ $\forall\varepsilon>0$, $Z=|X_n-X|$,

$$\varepsilon+P[|X_n-X|\geq\varepsilon]\geq\int_\Omega\frac{|X_n-X|}{1+|X_n-X|}\,dP=d(X_n,X)\geq 0\Rightarrow$$

$$0\leq\limsup d(X_n,X)\leq\varepsilon+\limsup P[|X_n-X|\geq\varepsilon]=\varepsilon\Rightarrow\limsup d(X_n,X)=0,$$

$$0\leq\liminf d(X_n,X)\leq\limsup d(X_n,X)=0\Rightarrow\liminf d(X_n,X)=0$$

$$\Rightarrow\lim_{n\to+\infty}d(X_n,X)=0$$

Another way:

Let $\eta>0$, $\varepsilon=\frac{\eta}{3}$ then $\exists n_0\in\mathbb{N}/\ \forall n\geq n_0$ $P[|X_n-X|\geq\frac{\eta}{3}]<\frac{\eta}{2}$

$$\Rightarrow d(X_n,X)\leq\frac{\eta}{3}+\frac{\eta}{2}<\eta\Rightarrow\{d(X_n,X)\}\to 0$$

④ Let X_n $n=1 \dots$ be random variables.
Assume $\sum_{n=1}^{\infty} P[|X_n - X| \geq \varepsilon_n] < +\infty$, for some $\{\varepsilon_n\} \to 0$.
Prove $\{X_n\} \to X$ with probability 1.

Let $A_n = \{w \,/\, |X_n(w) - X(w)| \geq \varepsilon_n\} \Rightarrow \sum_{n=1}^{\infty} P(A_n) < +\infty \Rightarrow$

from Borel-Cantelli Lemma, $P[\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n] = 0 \Rightarrow$

$P[(\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n)'] = 1 \Rightarrow P[\underbrace{\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} \{w \,/\, |X_n(w) - X(w)| < \varepsilon_n\}}_{=B}] = 1$

Let $w_0 \in B \Rightarrow \exists k_0 \in \mathbb{N} \,/\, w_0 \in \bigcap_{n=k_0}^{\infty} \{w \,/\, |X_n(w) - X(w)| < \varepsilon_n\} \Rightarrow$

$\exists k_0 \in \mathbb{N} \,/\, \forall n \geq k_0$ $|X_n(w_0) - X(w_0)| < \varepsilon_n \Rightarrow \{X_n(w_0)\} \to X(w_0) \Rightarrow$

$X_n \to X$ with probability 1.

⑤ Prove that if Ω is discrete, then the continuity in probability of the process $(X_t \mid t\in T)$ is equivalent to the continuity of all samples.

Exercise 2)

$\Rightarrow$) Let $t_0 \in T$

$(X_t \mid t\in T)$ continuous in probability $\Rightarrow X_t \xrightarrow[t\to t_0]{} X_{t_0}$ in prob. $\Rightarrow$

as Ω is discrete, $X_t \xrightarrow[t\to t_0]{} X_{t_0}$ with prob. 1 $\Rightarrow$

$\exists A$ event $/ P(A)=1 \wedge \forall w\in A,\ X_t(w) \xrightarrow[t\to t_0]{} X_{t_0}(w)$

Let $\Omega=\{w_1,w_2,\dots\} \mid P(w_i)>0 \quad \forall i\geq 1$

Suppose $w_{i_0}\notin A \Rightarrow A\cap\{w_{i_0}\}=\emptyset \Rightarrow$

$1\geq P(A\cup\{w_{i_0}\}) = P(A)+P(\{w_{i_0}\}) = 1+\underset{\vee \atop 0}{P(\{w_{i_0}\})}$!!! $\Rightarrow$

$\Omega=A \Rightarrow X(t,w)=X_t(w) \xrightarrow[t\to t_0]{} X_{t_0}(w)=X(t_0,w) \quad \forall w\in\Omega \Rightarrow$

$X(\cdot,w)$ is continuous at $t_0 \Rightarrow X(\cdot,w)$ is continuous as t_0 was arbitrary.

$\Leftarrow$) Continuity of all samples $\Rightarrow (X_t \mid t\in T)$ cont. with prob. 1 $\Rightarrow$ cont. in probability.

⑥ Let $\Omega=[0,1]$, $F=B([0,1])$, P-Lebesgue Measure λ, $T=[0,1]$

$X_t(w)=\begin{cases}1 & w=t\\ 0 & w\neq t\end{cases}$ Show $(X_t \mid t\in T)$ is continuous in probability.

We know $\lambda([a,b])=b-a$, $\lambda(\mathbb{R})=+\infty$, $\lambda(\{a\})=0$

Let $t_0\in[0,1]$, $\varepsilon\in]0,1[$, $t\neq t_0$

$\emptyset$ for $\varepsilon>1$

$\{w/\ |X_t(w)-X_{t_0}(w)|\geq\varepsilon\}=\{w/\ |X_t(w)-X_{t_0}(w)|=1\}=$

$\{w/\ X_t(w)=1, X_{t_0}(w)=0\}\cup\{w/\ X_t(w)=0, X_{t_0}(w)=1\}=\{t\}\cup\{t_0\} \Rightarrow$

$P[\{w/\ |X_t(w)-X_{t_0}(w)|\geq\varepsilon\}]=P[\{t\}\cup\{t_0\}]=P[\{t\}]+P[\{t_0\}]=0$

$\Rightarrow X_t \xrightarrow[t\to t_0]{} X_{t_0}$ in probability

Observe $(X_t \mid t\in T)$ is not continuous with probability 1 because no sample is continuous.

(7) Let X be a symmetric random variable / $P[X=0]=0$, Y an arbitrary random variable. Define a stochastic process $(Z_t \mid t\geq 0)$ / $Z_t = t(X+t)+Y$. Find the probability that the samples of $(Z_t \mid t\geq 0)$ are increasing functions.

Let $-B=\{-x \mid x\in B\}$

X symmetric $\Rightarrow \mu_X(B) = P[X\in B] = P[X\in -B] = \mu_X(-B)$

We are interested in $P[\{\omega / Z(\cdot,\omega) \text{ increasing}\}]$.

$P[\{\omega / Z(\cdot,\omega) \text{ increasing}\}] = P[\{\omega / \forall 0\leq t_1<t_2,\ Z(t_1,\omega)<Z(t_2,\omega)\}] =$

$P[\{\omega / \forall 0\leq t_1<t_2,\ Z(t_2,\omega)-Z(t_1,\omega)>0\}] =$

$P[\{\omega / \forall 0\leq t_1<t_2,\ t_2(X(\omega)+t_2)-t_1(X(\omega)+t_1)>0\}] =$

$P[\{\omega / \forall 0\leq t_1<t_2,\ (t_2-t_1)X(\omega)+(t_2-t_1)(t_2+t_1)>0\}] =$

$P[\{\omega / \forall 0\leq t_1<t_2,\ \underbrace{(t_2-t_1)}_{>0}[X(\omega)+(t_2+t_1)]>0\}] =$

$P[\{\omega / \forall 0\leq t_1<t_2,\ X(\omega)+(t_2+t_1)>0\}] = P[\{\omega / X(\omega)\geq 0\}]$

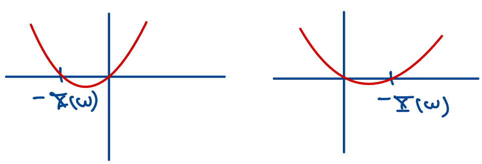
We have

$1 = P[X<0]+\cancel{P[X=0]}^{0}+P[X>0] = P[X\in]-\infty,0[\,]+P[X\in]0,+\infty[\,]$

$= 2P[X\geq 0] \Rightarrow P[X\geq 0] = 1/2$

Another solution:

$Z(t,\omega) = t^2+tX(\omega)+Y(\omega)$. Fixing ω, we obtain a quadratic function.

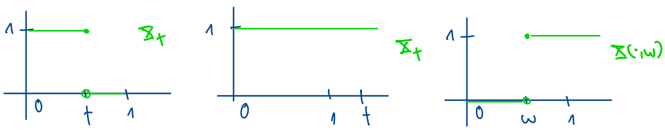


As a result, the probability of Z increasing is $P[X\geq 0]=\frac{1}{2}$

8) Let $\Omega =]0,1[$, $\mathcal{F} = \mathcal{B}(]0,1[)$, P – Lebesgue measure. $T = \mathbb{R}_0^+$

$$X_t(\omega) = \begin{cases} 0 & \omega > t \\ 1 & \omega \leq t \end{cases} \quad \omega \in \Omega.$$

A) Show $(X_t \mid t \in T)$ is continuous in probability.



$$|X_t(\omega) - X_{t_0}(\omega)| = \begin{cases} 1 & t < \omega \leq t_0 \ \vee \ t_0 < \omega \leq t \\ 0 & \text{otherwise} \end{cases}$$

Let $\varepsilon \in]0,1[$

- $t_0 < 1$
  - $t < \omega \leq t_0 < 1 \Rightarrow \lim\limits_{t \to t_0^-} P[\omega \in \Omega / |X_t(\omega) - X_{t_0}(\omega)| \geq \varepsilon] =$
    $\lim\limits_{t \to t_0^-} P[\{\omega \in \Omega / \omega \in [t, t_0]\}] = \lim\limits_{t \to t_0^-} t_0 - t = 0$
  - $t_0 < \omega \leq t < 1 \Rightarrow \lim\limits_{t \to t_0^+} P[\omega \in \Omega / |X_t(\omega) - X_{t_0}(\omega)| \geq \varepsilon] =$
    $\lim\limits_{t \to t_0^+} P[\{\omega \in \Omega / \omega \in [t_0, t]\}] = \lim\limits_{t \to t_0^+} t - t_0 = 0$

- $t_0 > 1 \Rightarrow \omega \notin]t, t_0],]t_0, t] \Rightarrow |X_t - X_{t_0}| = 0 \Rightarrow$
  $P[\{\omega \in \Omega / |X_t(\omega) - X_{t_0}(\omega)| \geq \varepsilon\}] \xrightarrow[t \to t_0]{} 0$

- $t_0 = 1$
  - $t < \omega \leq t_0 = 1 \Rightarrow P[\{\omega \in \Omega / |X_t(\omega) - X_{t_0}(\omega)| \geq \varepsilon] =$
    $P[\{\omega \in \Omega / \omega \in [t_0, t]\}] = t - t_0 \xrightarrow[t \to t_0]{} 0$
  - $1 = t_0 < t \Rightarrow |X_t - X_{t_0}| = 0 \Rightarrow P[|X_t - X_{t_0}| \geq \varepsilon] \xrightarrow[t \to t_0]{} 0$

As a result, $(X_t / t \in T)$ is cont. in prob.

B) Show $(X_t \mid t \in T)$ is continuous in the mean.

- $t_0 < 1$

  - $t < w \le t_0 < 1 \Rightarrow E[(X_t - X_{t_0})^2] = \int_t^{t_0} 1\,dw = t_0 - t$

  - $t_0 < w \le t < 1 \Rightarrow E[(X_t - X_{t_0})^2] = \int_{t_0}^{t} 1\,dw = t - t_0$

- $t_0 > 1 \Rightarrow w \notin]t, t_0],]t_0, t] \Rightarrow E[(X_t - X_{t_0})^2] = 0$

- $t_0 = 1$

  - $t < w \le t_0 = 1 \Rightarrow E[(X_t - X_{t_0})^2] = \int_t^1 dw = 1 - t$

  - $1 = t_0 < t \Rightarrow (X_t - X_{t_0})^2 = 0 \Rightarrow E[(X_t - X_{t_0})^2] = 0$

We can see that in every case, $E[(X_t - X_{t_0})^2] \xrightarrow[t \to t_0]{} 0$

⑨ Let $\Omega = [0,1]$ $\mathcal{F} = \mathcal{B}([0,1])$ P-Lebesgue Measure, $T = [0,+\infty[$

Find the covariance function of $(X_t \mid t \geq 0) \mid X_t(\omega) = \begin{cases} 1 & \omega \leq t \\ 0 & \omega > t \end{cases}$

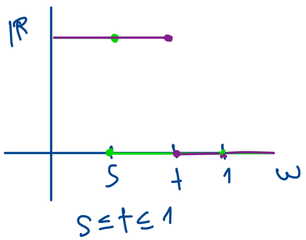
$$K(s,t) = \operatorname{COV}(X_s, X_t) = E[X_s X_t] - E[X_s]E[X_t]$$

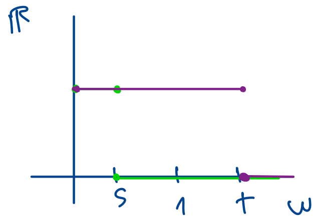
let $t \in T$.

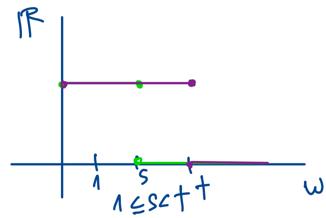
$$E[X_t] = \int_\Omega X_t(\omega)\, d\omega = \int_0^1 X_t(\omega)\, d\omega = \begin{cases} \int_0^t 1\, d\omega + \int_t^1 0\, d\omega = t & t \in [0,1] \\ \int_0^1 1\, d\omega = 1 & 1 < t \end{cases}$$

Let $s < t$, $s,t \in T$.

$$E[X_s X_t] = \int_\Omega X_s(\omega) X_t(\omega)\, d\omega = \begin{cases} \int_0^s 1\, d\omega = s & s \leq t \quad s \leq 1 \\ \int_0^1 1\, d\omega = 1 & 1 \leq s < t \end{cases}$$



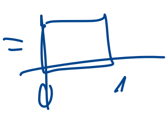




Avoiding the assumption $s < t$,

$$E[X_s X_t] = \begin{cases} s & s \leq t \quad s \leq 1 \\ t & t \leq s \quad t \leq 1 \\ 1 & s,t \geq 1 \end{cases} \qquad E[X_s]E[X_t] = \begin{cases} st & s,t \leq 1 \\ s & s \leq 1 < t \\ t & t \leq 1 < s \\ 1 & 1 < s,t \end{cases}$$

$$K(s,t) = \begin{cases} \min(s,t) - st & s,t \leq 1 \\ 0 & t < 1 \leq s \\ 0 & s < 1 \leq t \\ 0 & 1 < s,t \end{cases} = \begin{cases} \min(s,t) - st & (s,t) \in K \\ 0 & (s,t) \notin K \end{cases}$$

$K =$ 

10 Let $(X_t | t \in T)$ a stochastic process / X_t non-degenerated.
Prove that if samples are pairwise disjoint $\Rightarrow$ the process isn't continuous in probability.

By contradiction assume $(X_t | t\in T)$ is cont. in prob. at $t_0 \in T \Rightarrow$
Let $\Delta_\varepsilon = \{(x,y) \in \mathbb{R}^2 \,/\, |x-y| < \varepsilon\}$, μ the common distribution of all X_t's.

$\forall \varepsilon > 0$, $1 \underset{t\to t_0}{\leftarrow} P[|X_t - X_{t_0}| < \varepsilon] = P[\{w \in \Omega \,/\, (X_t(w), X_{t_0}(w)) \in \Delta_\varepsilon\}] =$

$\mu_{X_t, X_{t_0}}(\Delta_\varepsilon) \underset{X_t, X_{t_0} \text{ ind.}}{=} \mu_{X_t} \otimes \mu_{X_{t_0}}(\Delta_\varepsilon) \underset{\text{common distribution of all } X_t\text{'s}}{=} \mu\otimes\mu(\Delta_\varepsilon)$

Since $\mu\otimes\mu$ is ind. from t and t_0, $\mu\otimes\mu(\Delta_\varepsilon) = 1$

Let $\varepsilon = \frac{1}{n}$, $n \in \mathbb{N}$. The sequence $\{\Delta_{\frac{\varepsilon}{n}}\}$ is descending and

$\bigcap_{r=1}^{\infty} \Delta_{\frac{\varepsilon}{r}} = \{(x,y) \in \mathbb{R}^2 \,/\, x = y\} = D \Rightarrow \mu\otimes\mu(D) = \mu\otimes\mu\left(\bigcap_{r=1}^{\infty} \Delta_{\frac{\varepsilon}{r}}\right) =$

$\lim_{r\to+\infty} \mu\otimes\mu\left(\Delta_{\frac{\varepsilon}{r}}\right) = 1$

Let F be the distribution function of μ, which by definition, makes F not only take the values $0, 1$.

We can see $\forall a \in \mathbb{R}$, $]-\infty, a[\times]a, +\infty[\cap D = \emptyset$, but

$\mu\otimes\mu(]-\infty, a] \times]a, +\infty[) = \mu(]-\infty, a])\,\mu(]a, +\infty[) = F(a)(1-F(a)) > 0$

$1 = \mu\otimes\mu(\mathbb{R}^2) \geq \mu\otimes\mu(]-\infty, a] \times]a, +\infty[\cup D) =$

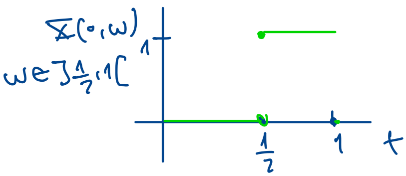
$\mu\otimes\mu(]-\infty, a] \times]a, +\infty[) + \mu\otimes\mu(D) = F(a)(1-F(a)) + 1 > 1$!!!

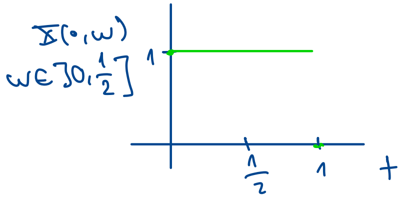
$\Rightarrow (X_t | t\in T)$ isn't continuous in probability.

11) Let $\Omega =]0,1[$, $T = [0,1]$ P-Lebesgue Measure

$$X_t(w) = \begin{cases} 1 & w \in]0,\tfrac{1}{2}] \, , \, t \in [\tfrac{1}{2},1] \\ 0 & \text{otherwise} \end{cases}$$

A) Show the samples are cont. with prob. $\frac{1}{2}$.





We can see $\forall w \in]\frac{1}{2},1[$, $X(\cdot,w)$ is discontinuous at $t = \frac{1}{2}$ $\Rightarrow$

$P[\{w \in \Omega / X(\cdot,w) \text{ cont}\}] = P[\{w \in \Omega / w \leq \frac{1}{2}\}] = \frac{1}{2}$

B) Show the process isn't continuous in probability.

Let $t_0 = \frac{1}{2}$

$$|X_t - X_{\frac{1}{2}}| = \begin{cases} 1 & w \in]0,\tfrac{1}{2}] \; t \in]0,\tfrac{1}{2}[\\ 0 & \text{otherwise} \end{cases}$$

Let $\varepsilon \in]0,1]$

$P[|X_t - X_{\frac{1}{2}}| \geq \varepsilon] = P[|X_t - X_{\frac{1}{2}}| = 1] = P(]0,\frac{1}{2}]) = \frac{1}{2} \Rightarrow$

$\exists t_0 \in T \,/\, X_t \underset{t \to t_0}{\not\to} X_{t_0} \Rightarrow (X_t | t \in T)$ isn't continuous in prob.

C) Is the process continuous in the mean?

As $(X_t | t \in T)$ is cont. in prob $\Rightarrow$ it isn't cont. in the mean.

D) Find correlation function.

let's see $(X_t \mid t\in T)$ is of 2^{nd} order.

$$E[X_t] = \int_\Omega X_t(\omega)\,P(d\omega) = \begin{cases} \int_0^{1/2} 1\,d\omega = \frac{1}{2} & t\in[\frac{1}{2},1] \\ 0 & t\notin[1,\frac{1}{2}[\end{cases} \quad < \infty \quad \forall t\in T$$

$\forall s,t\in T,$

$$X_s(\omega)X_t(\omega) = \begin{cases} 1 & \omega\in]0,\frac{1}{2}] \quad s,t\in[\frac{1}{2},1] \\ 0 & \text{otherwise} \end{cases}$$

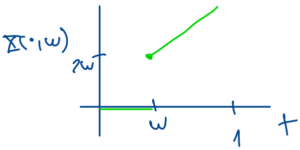
$$E[X_sX_t] = \int_\Omega X_s(\omega)X_t(\omega)\,P(d\omega) = \begin{cases} \int_0^{1/2} 1\,d\omega = \frac{1}{2} & s,t\in[\frac{1}{2},1] \\ 0 & \text{otherwise} \end{cases}$$

$$K(s,t) = \begin{cases} \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4} & s,t\in[\frac{1}{2},1] \\ 0 & \text{otherwise} \end{cases}$$

(12) Let $\Omega =]0,1[$, $T = [0,1]$ P-Lebesgue Measure

$$X_t(\omega) = \begin{cases} 0 & \omega > t \\ \omega + t & \omega \leq t \end{cases}$$

A) Is the process cont. with prob. 1 ?



We can see that $\forall \omega \in \Omega$, $\lim\limits_{t \to \omega} X_t(\omega) = \begin{cases} \lim\limits_{t \to \omega^-} 0 = 0 \\ \qquad\qquad \neq \\ \lim\limits_{t \to \omega^+} \omega + t = 2\omega \end{cases}$

$\Rightarrow X(\cdot,\omega)$ isn't cont. $\Rightarrow (X_t \mid t \in T)$ isn't cont. with probability 1.

B) Is the process continuous in the mean?

Let $t_0 \in T$

• $t_0 < t$

$$|X_t(\omega) - X_{t_0}(\omega)| = \begin{cases} \omega + t & t_0 < \omega \leq t \\ t - t_0 & \omega < t_0 < t \\ 0 & \text{otherwise} \end{cases}$$

$$E[(X_t - X_{t_0})^2] = \int_\Omega (X_t(\omega) - X_t^2(\omega))^2 P(d\omega) = \int_0^{t_0} (t - t_0)^2 d\omega + \int_{t_0}^{t} (\omega + t)^2 d\omega$$

$$= t_0 (t - t_0)^2 + \left[\frac{\omega^3}{3} + \omega t^2 + \omega^2 t\right]_{t_0}^{t} \xrightarrow[t \to t_0]{} 0$$

• $t < t_0$

$$|X_t(\omega) - X_{t_0}(\omega)| = \begin{cases} \omega + t_0 & t_0 < \omega \leq t \\ t_0 - t & \omega < t_0 < t \\ 0 & \text{otherwise} \end{cases}$$

$$E[(X_t - X_{t_0})^2] = \int_\Omega (X_t(\omega) - X_t^2(\omega))^2 P(d\omega) = \int_0^{t_0} (t_0 - t)^2 d\omega + \int_{t_0}^{t} (\omega + t_0)^2 d\omega$$

$$= t_0 (t_0 - t)^2 + \left[\frac{\omega^3}{3} + \omega t_0^2 + \omega^2 t_0\right]_{t_0}^{t} \xrightarrow[t \to t_0]{} 0$$

(13) Let $(X_t / t \in T)$ be a stochastic process cont. in prob. /
$1 \le X_t(\omega) \le 2$. Show $(X_t^2 / t \in T)$ is cont. in prob.
Hint: Estimate $X_t + X_{t_0}$

Let $t_0 \in T$, $\varepsilon > 0$.

$(X_t / t \in T)$ cont. in prob. $\Rightarrow P[|X_t - X_{t_0}| \ge \varepsilon/4] \xrightarrow[t \to t_0]{} 0$

$|X_t + X_{t_0}| \le 4 \Rightarrow P[|X_t + X_{t_0}| \le 4] = 1 \Rightarrow$

$P[|X_t - X_{t_0}| < \frac{\varepsilon}{4}] = P[|X_t - X_{t_0}| < \frac{\varepsilon}{4}]\, P[|X_t + X_{t_0}| \le 4] =$
$P[|X_t - X_{t_0}|\,|X_t + X_{t_0}| < \frac{\varepsilon}{4} \cdot \varepsilon] = P[|X_t^2 - X_{t_0}^2| < \varepsilon]$
$\Rightarrow (X_t^2 / t \in T)$ is cont. in prob.

(14) Let $\Omega =]0,1[$, $\mathcal{F} = \mathcal{B}(]0,1[)$, $T = [0,1]$, P-Lebesgue measure.
Let $X_t(\omega) = \omega + 2t$. Show $(X_t / t \in T)$ is cont. in prob.

Let $t_0 \in T$, $\varepsilon > 0$.

$P[|X_t - X_{t_0}| < 2\varepsilon] = P[\{\omega \in \Omega / |\omega + 2t - \omega - 2t_0| < 2\varepsilon\}] =$
$P[\{\omega \in \Omega / |t - t_0| < \varepsilon\}] \xrightarrow[t \to t_0]{} 1$ (since $\lim_{t \to t_0} |t - t_0| = 0$) $\Rightarrow$
$(X_t / t \in T)$ is cont. in prob.