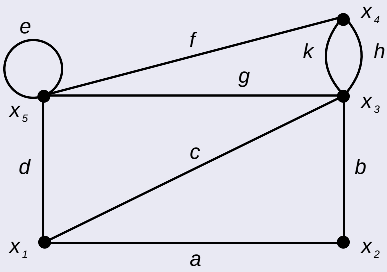
Graph and Network Theory

- A **path** in a graph (digraph or multigraph) is a sequence of edges which connect a sequence of vertices.
- To be more specific, a sequence $(e_1, e_2, \ldots, e_n)$ is called a **path of the length n** if there are vertices $v_0, \ldots, v_n$ such that $e_i = \{v_{i-1}, v_i\}$ (or $e_i = (v_{i-1}, v_i)$), $i = 1, \ldots, n$.
- The first vertex v_0 of the path, is called its **start vertex**, and the last vertex v_n, is called its **end vertex**. Both of them are called terminal vertices of the path.
- The other vertices in the path are **internal vertices**.
- If $v_0 = v_n$ then, the sequence is called a **closed path**.
- A closed path for which the edges and vertices are distinct (except the start and the end vertices) is called a **cycle**.

Consider the graph



- the path (d, e, g, b, a) is closed path,
- the path (d, g, b, a) is cycle.

Let G be an undirected (multi)graph without loops, with n vertices $(x_1, x_2, ..., x_n)$ and m edges $(e_1, e_2, ..., e_m)$.

Define $A(G) = [a_{ij}]_{n\times m}$ for $i = 1, ..., n$, $j = 1, ..., m$ in the following way:

$$a_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident to } i-\text{th vertex } x_i, \\ 0 & \text{otherwise.} \end{cases}$$

$A(G)$ is called the **incidence matrix**

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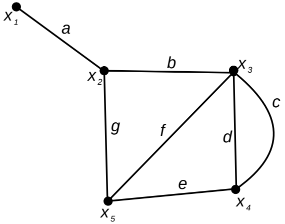
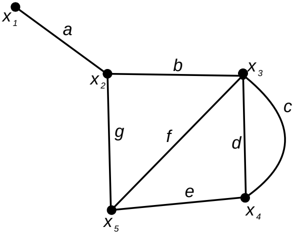


$$A(G) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

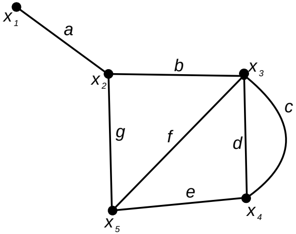
- each column of $A(G)$ contains exactly two unities (numbers one), because each edge is incident with the exactly two vertices



Edge a is incident with vertices x_1 i x_2.

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

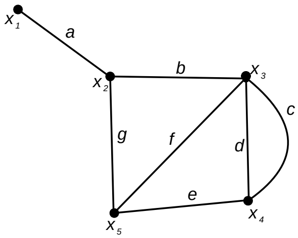
- each column of $A(G)$ contains exactly two unities (numbers one), because each edge is incident with the exactly two vertices
- the number of ones in each row is equal to the degree of corresponding vertex



$deg(x_2) = 3$

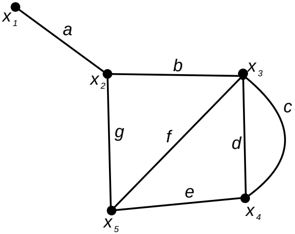
$$\begin{array}{c|ccccccc} & a & b & c & d & e & f & g \\ \hline x_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_2 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ x_3 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ x_4 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ x_5 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

- each column of $A(G)$ contains exactly two unities (numbers one), because each edge is incident with the exactly two vertices
- the number of ones in each row is equal to the degree of corresponding vertex
- row consisting of zeros represents an isolated vertex



$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

- each column of $A(G)$ contains exactly two unities (numbers one), because each edge is incident with the exactly two vertices
- the number of ones in each row is equal to the degree of corresponding vertex
- row consisting of zeros represents an isolated vertex
- parallel edges form identical columns



Edges c i d are parallel

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} a & b & c & d & e & f & g \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

From an algorithmic point of view, the matrix of incidence is the worst form of representation of the graph.

1. it requires defining of an array with $n \cdot m$ entries, most of which is filled with zeros,
2. the answer to the question whether there is an edge connecting given vertices requires m steps (one has to search m columns).

If the graph G has loops, the incidence matrix $A(G)$ can be defined as follows:

$$a_{ij} = \begin{cases} 1 & \begin{array}{l} \text{if edge } e_j \text{ is incident with} \\ i - \text{th vertex } x_i \\ \text{and it is not a loop around } x_i, \end{array} \\ \\ 2 & \text{if edge } e_j \text{ is a loop around } x_i, \\ \\ 0 & \text{otherwise} \end{cases}$$

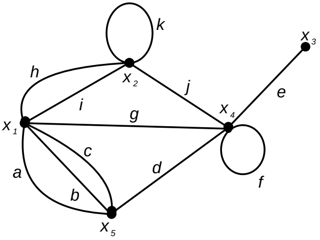
Adjacency matrix

Let G be an undirected (multi)graph with n vertices $(x_1, x_2, ..., x_n)$.

We define the matrix $A(G) = [a_{ij}]_{n\times n}$ as follows:

$$a_{ij} = \begin{cases} \text{the number of edges connecting vertex } x_i \text{ with vertex } x_j \\ 0 \text{ if there is not any edge from } x_i \text{ to } x_j \end{cases}$$

A square matrix defined in the above way is called **the adjacency matrix**.



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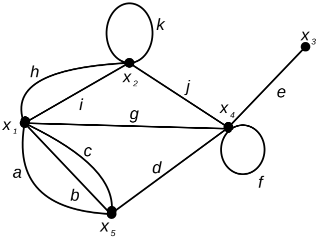
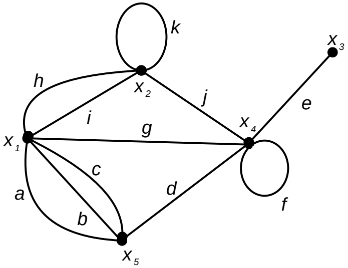


$A(G) =$

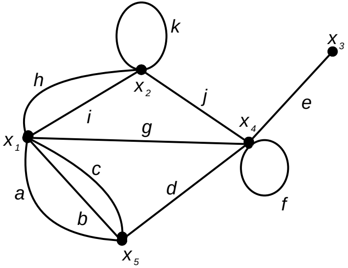
- along the main diagonal all elements are zeros if and only if the graph has no loops.



Edge f is a loop with x_4, so $a_{44} = 1$ and k is a loop with x_2, so $a_{22} = 1$.

$$
\begin{array}{c|c}
 & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} &
\begin{bmatrix}
0 & 2 & 0 & 1 & 3 \\
2 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 \\
3 & 0 & 0 & 1 & 0
\end{bmatrix}
\end{array}
$$

- along the main diagonal all elements are zeros if and only if the graph has no loops.
- if the graph does not have any loop (or on the diagonal it has zeros), the degree of a given vertex is equal to the sum of the elements in the corresponding row or column of matrix $A(G)$



$deg(x_5) = 4$

$$\begin{array}{c c} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} & \begin{bmatrix} 0 & 2 & 0 & 1 & 3 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

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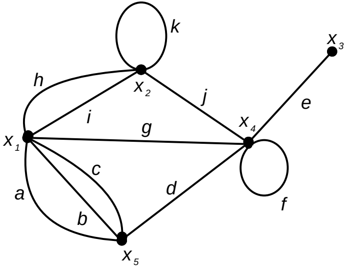

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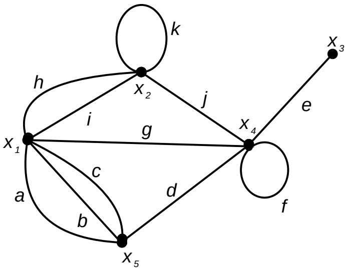
- if the graph does not have any loop (or on the diagonal it has zeros), the degree of a given vertex is equal to the sum of the elements in the corresponding row or column of matrix $A(G)$
- any adjacency matrix of undirected graph is a symmetric matrix



$$\begin{array}{c c} & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} & \begin{bmatrix} 0 & 2 & 0 & 1 & 3 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Properties of adjacency matrix

Advantage

The answer to the question of whether there is an edge from x_i to x_j one gets in one step.

Disadvantage

Regardless of the number of edges in the graph an adjacency matrix stores n^2 units of memory.

Theorem

Let $A(G)$ be adjacency matrix of a graph G. Then the number of paths of the length k between vertices x_i and x_j is equal to b_{ij}, where $B = A^k$.

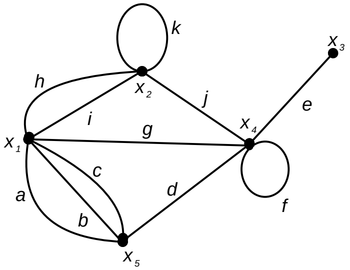
We will prove the theorem by induction with respect to k.
Let us notice, that matrix A^1 is a adjacency matrix, which gives the theorem for the $k = 1$.
Let $k > 1$. Then

$$A^k = A^{k-1}A$$

So, if $[a'_{i,j}] = A^{k-1}$, then

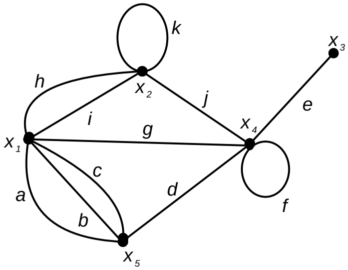
$$b_{i,j} = a'_{i,1}a_{1,j} + \cdots + a'_{i,n}a_{n,j}$$

Value $b_{i,j}$ is equal to the number of all paths from the vertex v_i to v_j, with a length $k-1$. On the other hand, the product of $a'_{i,l}a_{l,j}$ is equal to the number of all paths of length k, of the form $v_i, ...v_l, v_j$. By summing after all $l = 1, ..., n$, we get the thesis.



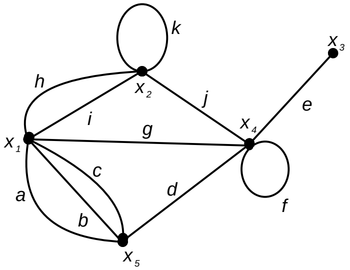
Paths of the length 2 we obtain from A^2.

Example



Paths of the length 2 we obtain from A^2.

$$B = A^2 = \begin{bmatrix} 0 & 2 & 0 & 1 & 3 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 & 1 & 3 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 1 & 6 & 1 \\ 3 & 6 & 1 & 4 & 7 \\ 1 & 1 & 1 & 1 & 1 \\ 6 & 4 & 1 & 5 & 4 \\ 1 & 7 & 1 & 4 & 10 \end{bmatrix}$$



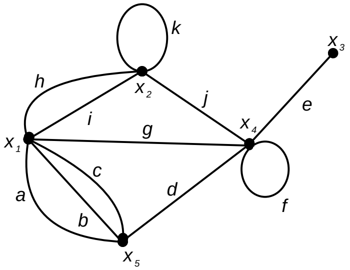
For instance

$$b_{42} = a_{41}a_{12} + a_{42}a_{22} + a_{43}a_{32} + a_{44}a_{42} + a_{45}a_{52}$$
$$= 1 \cdot 2 + 1 \cdot 1 + 1 \cdot 0 + 1 \cdot 1 + 1 \cdot 0 = 4$$

These are paths *gh*, *gi*, *jk*, *fj*

Paths of the length 2 we obtain from A^2.

$$B = A^2 = \begin{bmatrix} 0 & 2 & 0 & 1 & 3 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 0 & 1 & 3 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 1 & 6 & 1 \\ 3 & 6 & 1 & 4 & 7 \\ 1 & 1 & 1 & 1 & 1 \\ 6 & 4 & 1 & 5 & 4 \\ 1 & 7 & 1 & 4 & 10 \end{bmatrix}$$



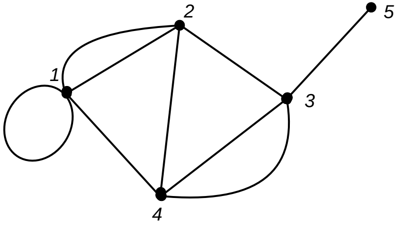
These are not the shortest paths.

We get

$$A^3 = \begin{bmatrix} 15 & 37 & 6 & 25 & 48 \\ 37 & 16 & 4 & 21 & 13 \\ 6 & 4 & 1 & 5 & 4 \\ 25 & 21 & 5 & 20 & 23 \\ 48 & 13 & 4 & 23 & 7 \end{bmatrix} A^4 = \begin{bmatrix} 243 & 92 & 25 & 131 & 70 \\ 92 & 111 & 21 & 91 & 132 \\ 25 & 21 & 5 & 20 & 23 \\ 131 & 91 & 20 & 94 & 95 \\ 70 & 132 & 23 & 95 & 167 \end{bmatrix}$$

The adjacency list is built for each vertex of the (multi)graph.

For a given vertex $x_i \in V$ we define the list of such vertices u, that $\{x_i, u\}$ is an edge in the graph G.



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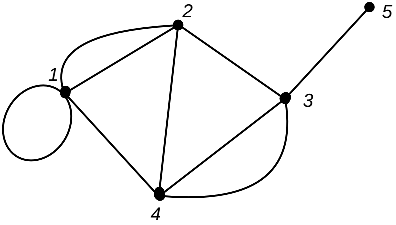


Adjacency lists:

1 :

2 :

3 :

4 :

5 :

Properties of adjacency lists

- any adjacency list consists of vectors of lists
- each edge in any undirected graph is stored in two different places, if we consider the edge $\{u, w\}$, w is the vertex on the list of vertex v and vice versa (this observation does not apply to the loop)

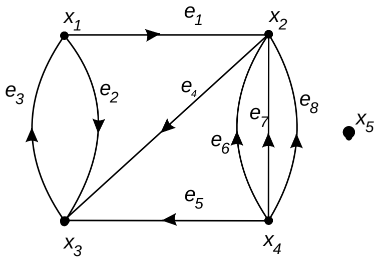
Advantages

- the number of memory units needed to store the graph with n vertices and m edges is, in the worst case, proportional to $m + n$ (generally it is enough to use $2m$ units of memory),
- testing of the existence of a single edge needs the time proportional to n,
- adjacency list allows to track all the links form a given vertex.

Let G be a digraph without loops with vertices $x_1, x_2, \ldots, x_n$ and edges $e_1, e_2, \ldots, e_m$.

An incidence matrix $A(G) = [a_{ij}]_{n \times m}$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$ is defined as follows:

$$a_{ij} = \begin{cases} 1 & \text{if } x_i \text{ is the tail of } e_j \\ -1 & \text{if } x_i \text{ is the head of } e_j \\ 0 & \text{otherwise} \end{cases}$$



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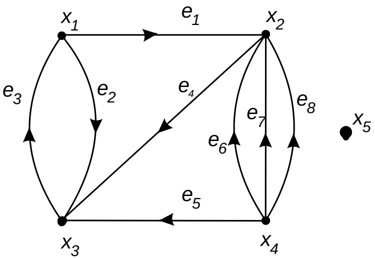

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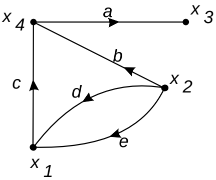
$$A(G) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & -1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Adjacency matrix for a directed graph is defined identically as for the undirected graph. Recall that

$$a_{ij} = \begin{cases} \text{the number of edges connecting vertex } x_i \text{ with vertex } x_j \\ 0 \text{ if there is not any edge from } x_i \text{ to } x_j \end{cases}$$

Similarly as for undirected graphs, successive powers of digraph adjacency matrix contains a number of paths between vertices.

Adjacency matrix is of the form



$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

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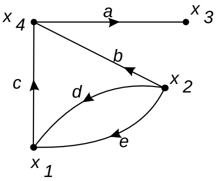

Adjacency matrix is of the form

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Then

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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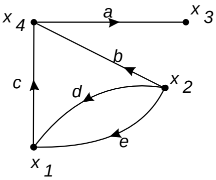

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

There are 4 paths of the length of 2:

$a_{13}^2 = 1$ - a path ca

$a_{23}^2 = 1$ - a path ba

$a_{24}^2 = 2$ - two paths dc and ec

Then

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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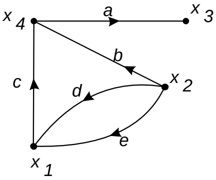


$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

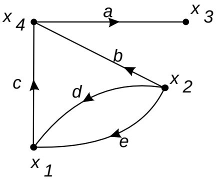
There are 2 paths of the length of 3:

$a^3_{23} = 2$ - two paths *dca* and *eca*

Then

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly as for undirected graphs, successive powers of digraph adjacency matrix contains a number of paths between vertices.



Adjacency matrix is of the form

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

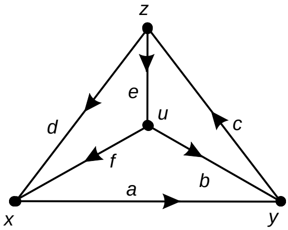
In the digraph there is no any path of the length of 4 or more

Then

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly as for a undirectec graph, the adjacency list is built for every vertex. For vertex $x_i \in V$, it consists of such vertices u, such that (x_i, u) is an edge in G.

Adjacency list



x : y

y : z

z : x, u

u : x, y

In the adjacency list for digraphs every edge is stored only once.

Degree-sum Formula

For any graph $G = \langle V, E, \gamma \rangle$ we have

$$\sum_{v \in V} \deg(v) = 2\,|E|$$

Proof. Every nonloop edge is incident to exactly two distinct vertices of G. On the other hand, every loop edge is counted twice in the degree of its incident vertex in G. Thus, every edge, whether it is loop or not, contributes a two to the summation of the degrees of the vertices of G. ■

- The above theorem, due to **Euler**, is an essential tool of graph theory and is sometimes refer to as the **First Theorem of Graph Theory** or the **Handshaking Lemma.**
- It implies that if some people shake hands, then the total number of hands shaken must be even since each handshake involves exactly two hands.

The number of odd degree vertices in a graph is an even number.

Proof. Let $|V| = n$. By previous lemma, we have $\sum_{i=1}^{n} \deg(v_i) = 2|E|$, so $\sum_{i=1}^{n} \deg(v_i)$ is even.

Next, let us see, that

$$\sum_{i=1}^{n} \deg(v_i) = \sum_{\deg(v_j)\text{ - even.}} \deg(v_j) + \sum_{\deg(v_k)\text{ - odd.}} \deg(v_k),$$

hence

$$\sum_{\deg(v_k)\text{ - odd.}} \deg(v_k) = \sum_{i=1}^{n} \deg(v_i) - \sum_{\deg(v_j)\text{ - even.}} \deg(v_j)$$

Therefore $\sum_{\deg(v_k)\text{ - odd.}} \deg(v_k)$ is an even number as the difference of two even numbers. Since each component $\deg(v_k)$ of the sum $\sum_{\deg(v_k)\text{ - odd.}} \deg(v_k)$ is odd, so the number of its components must be even.

■

Thank you for your attention!!!