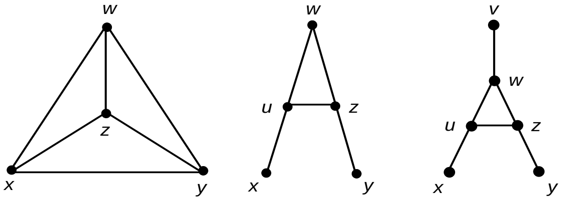
Graph and Network Theory

A **hamiltonian path** in a graph G is a path which contains all the vertices of G exactly once.

A **hamiltonian cycle** in a graph G is a cycle which contains all the vertices of G.

A graph G which contains a hamiltonian cycle is called the **hamiltonian graph**, a graph which contains a hamiltonian path is called the **semihamiltonian graph.**



a) b) c)

A sequence of all possible binary sequences of length n where any two successive sequences differ in only one bit (binary digit) is called the **binary Gray code** of order n.

How to construct binary Gray code?

A sequence of all possible binary sequences of length n where any two successive sequences differ in only one bit (binary digit) is called the **binary Gray code** of order n.

How to construct binary Gray code?
We start with the following observation:

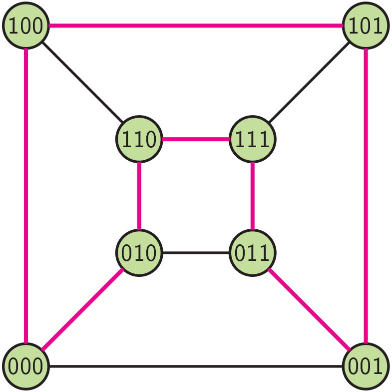
Suppose that a sequence $(C_1, C_2, ..., C_m)$ contains all the binary sequences C_i of length k, where C_i differs from C_{i+1} at exactly one position. Then a sequence

$$C_1 0, C_2 0, ..., C_m 0, C_m 1, C_{m-1} 1, ... C_1 1$$

contains all the binary sequences of length $k+1$, where every two sequences differ in only one bit.

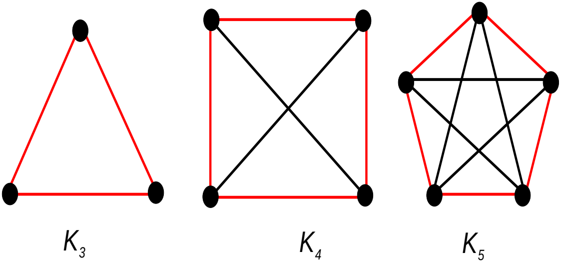
In this way, we construct a simple recursive algorithm for generating all binary strings of length n.

In the n−dimensional hypercube Q_n, the binary Gray code of order n can be obtained by finding a hamiltonian cycle.



If a graph with n vertices is hamiltonian then it has at least n edges.

The complete graph K_n is hamiltonian.

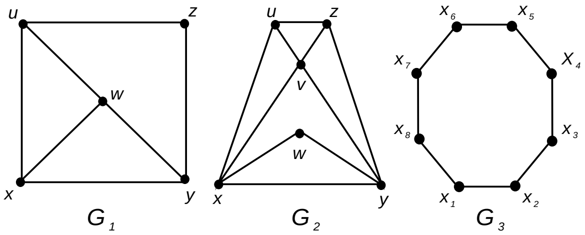


Theorem, Ore (1960)

Let $G = \langle V, E \rangle$ be a simple and connected graph, let $|V| = n \geq 3$ (i.e. G has at least tree vertices). If

$$deg(u) + deg(w) \geq n$$

for each pair of vertices $u, w \in V$ which are not connected with edge, then G is a **hamiltonian graph**.



a) b) c)

Let $G = \langle V, E \rangle$ be a simple and connected graph, let $|V| = n \geq 3$.If $deg(u) + deg(w) \geq n$ for each pair of vertices $u, w \in V$ which are not connected with edge, then G is a **hamiltonian graph**

.

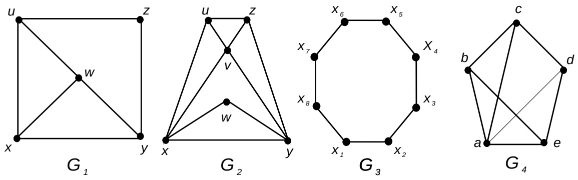
Proof. Let us assume that G has no Hamiltonian cycle. Consider the longest possible Hamiltonian path in G, denote it as $P = v_1, v_2, ...v_k$, where $k < n$ (since we assume no Hamiltonian cycle).
Since P is the longest path, there is no vertex outside P that is a neighbour of both v_1 and v_k, because then we could extend the path or close the cycle.
In particular, since v_1 and v_k are not connected by an edge (otherwise we would have a cycle), then $deg(v_1) + deg(v_k) \geq n$. So the vertices v_1 and v_k together have **at least n neighbours**.
Since G has only n vertices and P contains $k < n$ vertices, this implies that there is at least one vertex x outside P that is a neighbour of both v_1 and v_k.
However, this leads to a contradiction, because we could attach x to P, which would imply the existence of a longer path than P, which contradicts the assumption of the maximum length of P. ∎

Theorem, Dirac (1952)

If $G = \langle V, E \rangle$ is a simple and connected graph with n vertices ($n \geq 3$) and if $deg(u) \geq \frac{1}{2}n$ for every $u \in V$, then G is hamiltonian.

Proof. If the graph $G = \langle V, E \rangle$ is a simple graph and $|V| \geq 3$ and $deg(u) \geq \frac{1}{2}n$ for each $u \in V$, then $deg(u) + deg(w) \geq n$ for all $u, w \in V$, regardless of whether these vertices are adjacent or not. Thus, the assumptions of Ore's theorem is true, and hence the graph G is hamiltonian.

■



a) b) c) d)

If a simple and connected graph with n vertices has at least $\frac{1}{2}(n-1)(n-2)+2$ edges, then G is hamiltonian.

Proof. Let's take any two vertices $u, w \in V$ of a graph G, not connected by an edge, i.e. edge $\{u, w\} \notin E$. Let $G_1 = G \setminus \{u, w\}$. Because of $\{u, w\} \notin E$, so we removed $deg(u) + deg(w)$ edges from G.

$$|E(G_1)| \geq \frac{1}{2}(n-1)(n-2) + 2 - (deg(u) + deg(w))$$

The graph G_1 is a subgraph of the complete graphK_{n-2}. Graph K_{n-2} has $\frac{1}{2}(n-2)(n-3)$ edges. Because of $|E(K_{n-2})| \geq |E(G_1)|$, so

$$\frac{1}{2}(n-2)(n-3) \geq \frac{1}{2}(n-1)(n-2) + 2 - (deg(u) + deg(w))$$

Therefore

$$deg(u) + deg(w) \geq \frac{1}{2}(n-1)(n-2) - \frac{1}{2}(n-2)(n-3) + 2 =$$
$$\frac{1}{2}(n-2)(n-1-n+3) + 2 = n$$

hence $deg(u) + deg(w) \geq n$. This means that the graph G satisfies the assumptions of Ore's theorem, so it is a Hamiltonian graph. ■

Theorem

Let $G = (V_1 \cup V_2, E)$ be a bipartite graph

- If G is hamiltonian then $|V_1| = |V_2|$.
- If G is semihamiltonian (it has a hamiltonian path) then $||V_1| - |V_2|| \leq 1$.

Proof. Let $v_1, v_2, v_3, v_4, ..., v_{n-1}, v_n, v_1$ is a Hamiltonian cycle in a bipartite graph. Since the cycle length is even, so n is even. Therefore,

$$v_1, v_3, ..., v_{n-1} \in V_1$$
$$v_2, v_4, ..., , v_n \in V_2$$

so the number of vertices from each subset is the same. Hence $|V_1| = |V_2|$. The 2) condition is proved by analogy. ∎

The above theorem is also true for bipartite complete graphs.

Theorem

Let $G = (V_1 \cup V_2, E)$ be a complete bipartite graph

- If $|V_1| = |V_2|$ then G is hamiltonian.
- If $||V_1| - |V_2|| \leq 1$ then G is semihamiltonian.

Proof. Let $V_1 = \{v_1^1, v_2^1, \ldots, v_{|V_1|}^1\}$ and $V_2 = \{v_1^2, v_2^2, \ldots, v_{|V_2|}^2\}$. If $k = |V_1| = |V_2|$, then

$$v_1^1, v_1^2, v_2^1, v_2^2, \ldots, v_k^1, v_k^2, v_1^1$$

is a Hamiltonian cycle in a bipartite complete graph.
If $|V_1| = |V_2| + 1$, then

$$v_1^1, v_1^2, v_2^1, v_2^2, \ldots, v_{|V_1|-1}^1, v_{|V_2|}^2, v_{|V_1|}^1$$

is a Hamiltonian path in a complete bipartite graph. ∎

Two cycles are **edge-disjoint** if each edge of the graph belongs to only one of these cycles.

Theorem

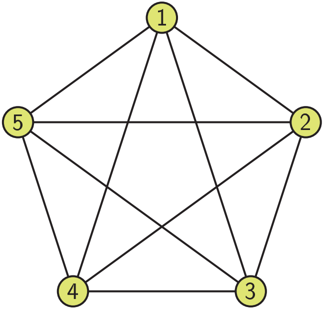
A complete graph K_n has

$$\left\lfloor \frac{n-1}{2} \right\rfloor$$

edge-disjoint hamiltonian cycles.

Floor and ceil

Graph K_5 has $\left[\dfrac{5-1}{2}\right] = 2$ edge-disjoint Hamiltonian cycles.



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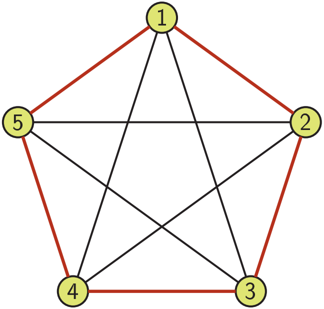
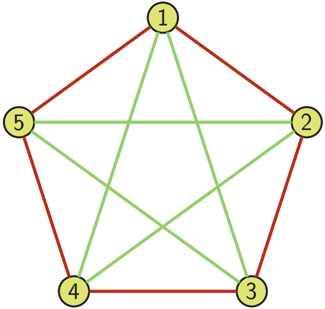


1. **1,2,3,4,5,1**
2.

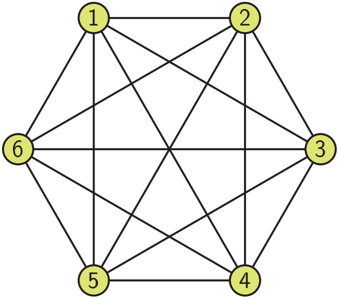
Graph K_5 has $\left[\dfrac{5-1}{2}\right] = 2$ edge-disjoint Hamiltonian cycles.



1. **1,2,3,4,5,1**
2. **1,3,5,2,4,1**

Edge-disjoint Hamiltonian cycles

Graph K_6 has $\left[\dfrac{6-1}{2}\right] = 2$ edge-disjoint Hamiltonian cycles.



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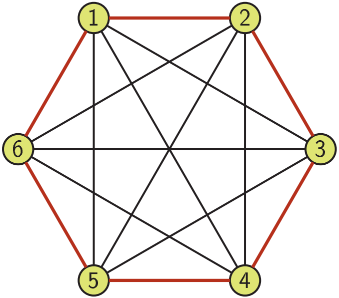

1. **1,2,3,4,5,6,1**
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Edge-disjoint Hamiltonian cycles

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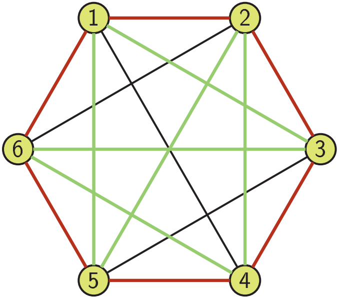
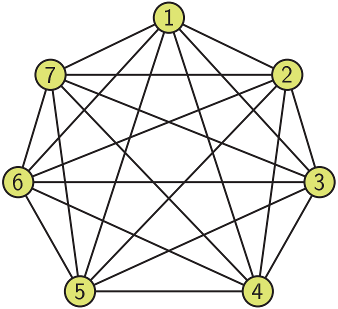


1. 1,2,3,4,5,6,1
2. 1,3,6,4,2,5,1

Graph K_7 has $\left[\dfrac{7-1}{2}\right] = 3$ edge-disjoint Hamiltonian cycles.



Edge-disjoint Hamiltonian cycles

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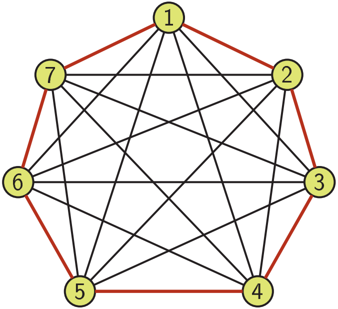
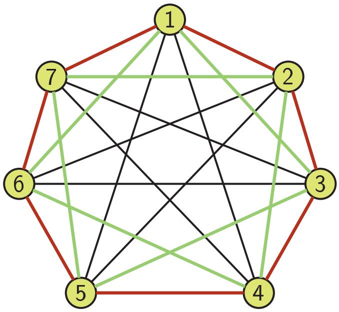


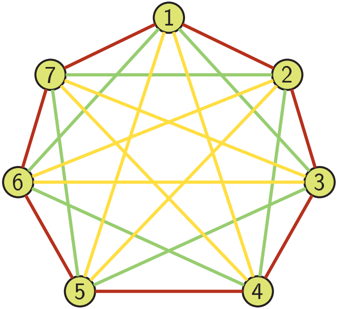
1. **1,2,3,4,5,6,7,1**
2.
3.

Graph K_7 has $\left[\frac{7-1}{2}\right] = 3$ edge-disjoint Hamiltonian cycles.



1. **1,2,3,4,5,6,7,1**
2. **1,3,5,7,2,4,6,1**
3.

Graph K_7 has $\left\lceil \frac{7-1}{2} \right\rceil = 3$ edge-disjoint Hamiltonian cycles.



1. 1,2,3,4,5,6,7,1
2. 1,3,5,7,2,4,6,1
3. 1,4,7,3,6,2,5,1

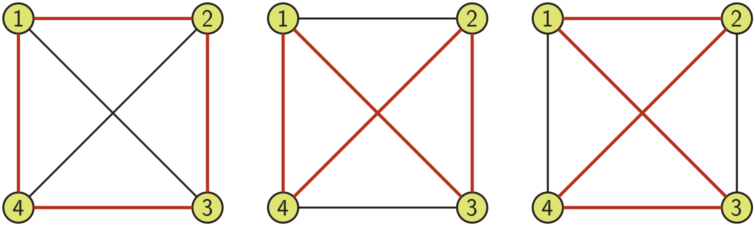
Theorem

A complete graph K_n has

$$\frac{(n-1)!}{2}$$

hamiltonian cycles.

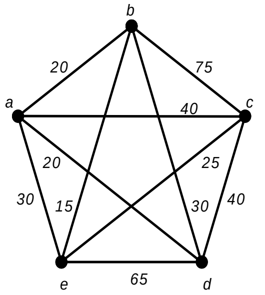
For K_4 we have $\frac{(4-1)!}{2} = 3$ hamiltonian cycles.



- For K_5 we have $\frac{(5-1)!}{2} = 12$ hamiltonian cyles.
- For K_{20} we have $\dfrac{19!}{2} > 10^{17}$ hamiltonian cyles.

The salesman has to visit several cities (each exactly once) and return to the starting city. The distance (cost or time) of routing between each pair of cities is known. **How to find the route which is optimal (shortest cheapest or fastest traveled)**?

The problem can be formulated in the theory of $n-$nodes network which is the complete graph, and then find the shortest (cheapest or fastest) hamiltonian cycle.



The cycle a, b, c, d, e, a costs 230, and the cycle a, b, e, c, d, a costs 110.

Theoretically, TSP can be solved by designating of $\frac{1}{2}(n-1)!$ hamiltonian cycles and choosing that which has the smallest total weight. Of course, this method is very ineffective. Indeed, if you have a computer checking a million permutations per second then:

- for $n = 10$ the number of cycles is $(10-1)!/2 = 181440$ and the time of counting is 1.8 s
- for $n = 20$ the number of cycles is $(20-1)!/2 = 60822550204416000$ and the time of counting is about 29330 of years.

Traveling salesman problem is NP-complete problem.

The history of TSP

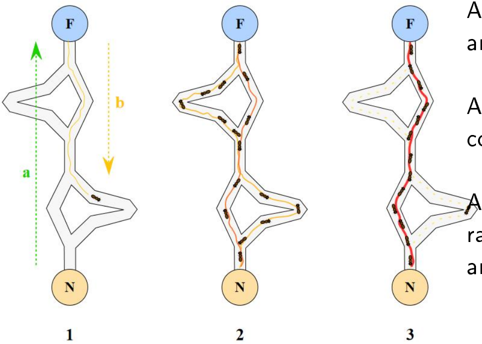
Heuristic (gr. heuriskein – find, discover) is a practical, based on the experience, way of solving a problem, which can significantly simplify and shorten the process of solving the problem under consideration.

For tasks where there is a lot of solutions, it is very important to reject directions of searching which are "pessimistic".
Heuristic methods allow to find, in an acceptable time, at least approximate solution, but does not guarantee that in all cases.
The effectiveness of heuristic steps can not be proved, but their accuracy is usually obvious.

Ants move from anthill to a given point (e.g. the place with food).

Returning each of them moves exactly the same way but in the opposite direction.

When the ant has possibility of choosing of two ways it chooses the way randomly (the probability of every choice is the same at the beginning).



As ants walk the route they leave a certain amount of pheromone.

As a result, over time, shorter routes are covered with more pheromone.

After some time ants select the route randomly, but taking into account the amount of pheromone left.

In the end, the ants choose the shortest routes.

Given a complete graph $G = \langle N, E \rangle$, where:

$N = \{1, 2, ..., n\}$ – the set of vertices (cities),

E — the set of edges (direct connections between cities).

Let d_{ij} be the length of edge $(i, j) \in E$, that is the distance between cities i and j.

In **AS** algorithm, in every iteration t $(1 \leq t \leq t_{max})$, $k-$th ant $(k = 1, ..., m)$ passes a graph building a cycle

Assume that in the $t - th$ iteration $k-$th ant is at the vertex i. The probability of choosing of edge to the vertex j is defined by the formula:

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Assume that in the $t - th$ iteration $k-$th ant is at the vertex i. The probability of choosing of edge to the vertex j is defined by the formula:

$$p_{ij}^k(t) = \frac{(\tau_{ij}(t))^\alpha (\eta_{ij})^\beta}{\sum_{l \in N_i^k} (\tau_{il}(t))^\alpha (\eta_{il})^\beta)}$$

$$p_{ij}^k(t) = \frac{(\tau_{ij}(t))^\alpha (\eta_{ij})^\beta}{\sum_{l \in N_i^k} (\tau_{il}(t)^\alpha (\eta_{il})^\beta)}$$

where:

$\tau_{ij}(t)$ – the amount of the pheromone at the edge (i, j)

η_{ij} – heuristic quantity describing "attractiveness" of the choice of (i, j) e.g.

$$\eta_{ij} = \frac{1}{d_{ij}}$$

N_i^k – the list of unvisited vertices adjacent to i.

After passing the graph by $k-th$ ant we count the amount of pheromone $\Delta\tau_{ij}^{k}(t)$, left by this ant on every edge (i,j) of the cycle.

It can be defined e.g. as

$$\Delta\tau_{ij}^{k}(t) = \frac{Q}{L^{k}(t)}$$

where Q is so called and $L^{k}(t)$ is the length of the route found.

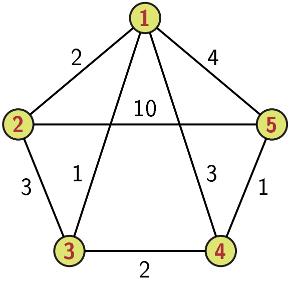
To ensure „forgetting" of pheromone dose for longer cycles, the algorithm takes into account the natural phenomenon of evaporation of pheromone. As a result, at the end of the iteration (ie, passing of m ants) the final amount of the pheromone for each edge is defined and this amount will be applied in the next iteration:

$$\tau_{ij}(t+1) = \gamma\tau_{ij}(t) + \Delta\tau_{ij}(t)$$

where $\Delta\tau_{ij}(t) = \Sigma_{k=1}^{m}\Delta\tau_{ij}^{k}(t)$ and m is the number of ants and γ is an "evaporation" factor.

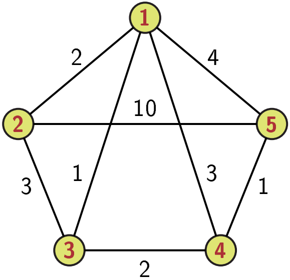
AS(t_{max})

for $t \leftarrow 1$ **to** t_{max}
    do
        for every ant $k = 1, ..., m$
            do
                choose a city
                for every unvisited city i
                    do
                        chose a city j from the list N_i^k, according to $p_{ij}^k(t)$
                for every edge (i, j)
                    do
                        find $\Delta\tau_{ij}^k(t)$ for the route $T^k(t)$
        for every edge (i, j)
            do
                count the dose of pheromone $\tau_{ij}(t+1) = \gamma\tau_{ij}(t) + \Delta\tau_{ij}(t)$



$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g \in N_i} \tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

$e \in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1



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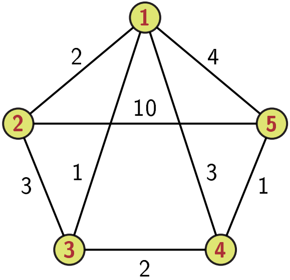
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

In the city of 1 we can choose cities $N_1 = \{2, 3, 4, 5\}$.

	2	3	4	5
p_{1j}				

Ant Colony Algorithm $\alpha = 1$, $\beta = 1$, $\tau_{ij} = 1$



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d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

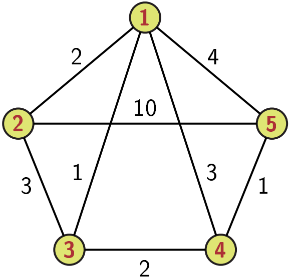
In the city of 1 we can choose cities $N_1 = \{2, 3, 4, 5\}$.

	2	3	4	5
p_{1j}				

$$p_{12} = \frac{\tau_{12}\eta_{12}}{\sum_{g \in N_1} \tau_{1g}\eta_{1g}} = \frac{\frac{1}{2}}{\frac{1}{2} + 1 + \frac{1}{3} + \frac{1}{4}} = \frac{\frac{1}{2}}{\frac{25}{12}} = 0.24$$

$$p_{13} = \frac{\tau_{13}\eta_{13}}{\sum_{g \in N_1} \tau_{1g}\eta_{1g}} = \frac{1}{\frac{25}{12}} = 0.48 \quad p_{14} = \frac{\tau_{14}\eta_{14}}{\sum_{g \in N_1} \tau_{1g}\eta_{1g}} = \frac{\frac{1}{3}}{\frac{25}{12}} = 0.16$$

$$p_{15} = \frac{\tau_{15}\eta_{15}}{\sum_{g \in N_1} \tau_{1g}\eta_{1g}} = \frac{\frac{1}{4}}{\frac{25}{12}} = 0.12$$



$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g\in N_i}\tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

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d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

In the city of 1 we can choose cities $N_1 = \{2, 3, 4, 5\}$.

	2	3	4	5
p_{1j}	0.24	0.48	0.16	0.12

$$p_{12} = \frac{\tau_{12}\eta_{12}}{\sum_{g\in N_1}\tau_{1g}\eta_{1g}} = \frac{\frac{1}{2}}{\frac{1}{2}+1+\frac{1}{3}+\frac{1}{4}} = \frac{\frac{1}{2}}{\frac{25}{12}} = 0.24$$

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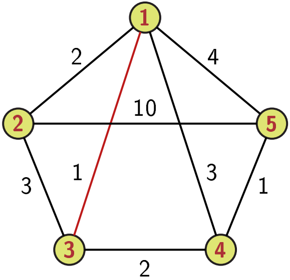


$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g\in N_i}\tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

$e \in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

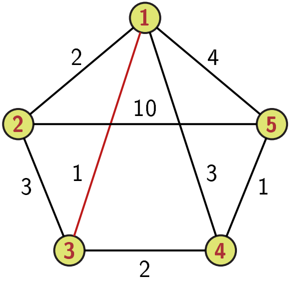
In the city of 1 we can choose cities $N_1 = \{2, 3, 4, 5\}$.

	2	3	4	5
p_{1j}	0.24	0.48	0.16	0.12

$$p_{12} = \frac{\tau_{12}\eta_{12}}{\sum_{g\in N_1}\tau_{1g}\eta_{1g}} = \frac{\frac{1}{2}}{\frac{1}{2}+1+\frac{1}{3}+\frac{1}{4}} = \frac{\frac{1}{2}}{\frac{25}{12}} = 0.24$$

$$p_{13} = \frac{\tau_{13}\eta_{13}}{\sum_{g\in N_1}\tau_{1g}\eta_{1g}} = \frac{1}{\frac{25}{12}} = 0.48 \quad p_{14} = \frac{\tau_{14}\eta_{14}}{\sum_{g\in N_1}\tau_{1g}\eta_{1g}} = \frac{\frac{1}{3}}{\frac{25}{12}} = 0.16$$

$$p_{15} = \frac{\tau_{15}\eta_{15}}{\sum_{g\in N_1}\tau_{1g}\eta_{1g}} = \frac{\frac{1}{4}}{\frac{25}{12}} = 0.12$$

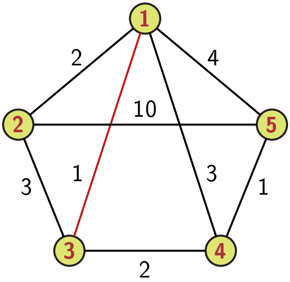


$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g\in N_i}\tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

$e \in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

In the city of **3** we can choose cities $N_3 = \{2, 4\}$.

	2	**4**
p_{3j}		



$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g\in N_i}\tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

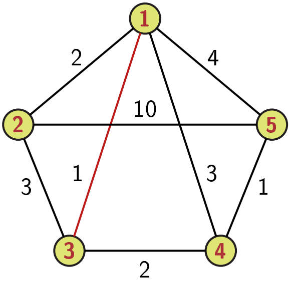
$e\in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

In the city of **3** we can choose cities $N_3 = \{2, 4\}$.

	2	**4**
p_{3j}		

$$p_{32} = \frac{\tau_{32}\eta_{32}}{\sum_{g\in N_3}\tau_{3g}\eta_{3g}} = \frac{\frac{1}{3}}{\frac{1}{3}+1+\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{5}{6}} = 0.4$$

$$p_{34} = \frac{\tau_{34}\eta_{34}}{\sum_{g\in N_3}\tau_{3g}\eta_{3g}} = \frac{\frac{1}{2}}{\frac{5}{6}} = 0.6$$



$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g\in N_i}\tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

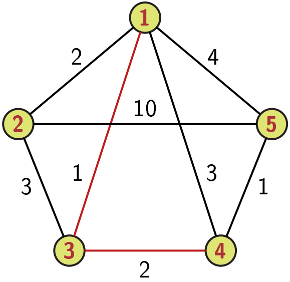
$e\in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

In the city of **3** we can choose cities $N_3 = \{2, 4\}$.

	2	**4**
p_{3j}	0.4	0.6

$$p_{32} = \frac{\tau_{32}\eta_{32}}{\sum_{g\in N_3}\tau_{3g}\eta_{3g}} = \frac{\frac{1}{3}}{\frac{1}{3}+1+\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{5}{6}} = 0.4$$

$$p_{34} = \frac{\tau_{34}\eta_{34}}{\sum_{g\in N_3}\tau_{3g}\eta_{3g}} = \frac{\frac{1}{2}}{\frac{5}{6}} = 0.6$$



$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g\in N_i}\tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

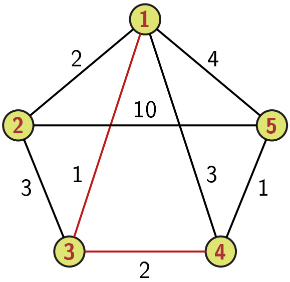
$e \in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

In the city of **3** we can choose cities $N_3 = \{2, 4\}$.

	2	**4**
p_{3j}	0.4	0.6

$$p_{32} = \frac{\tau_{32}\eta_{32}}{\sum_{g\in N_3}\tau_{3g}\eta_{3g}} = \frac{\frac{1}{3}}{\frac{1}{3}+1+\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{5}{6}} = 0.4$$

$$p_{34} = \frac{\tau_{34}\eta_{34}}{\sum_{g\in N_3}\tau_{3g}\eta_{3g}} = \frac{\frac{1}{2}}{\frac{5}{6}} = 0.6$$

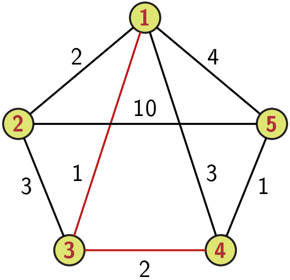


$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g \in N_i} \tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

$e \in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

In the city of **4** we can choose cities $N_4 = \{5\}$.

	5
p_{45}	



$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g\in N_i}\tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

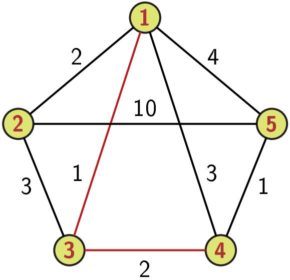
$e\in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

In the city of **4** we can choose cities $N_4 = \{5\}$.

	5
p_{45}	

$$p_{45} = \frac{\tau_{45}\eta_{45}}{\tau_{45}\eta_{45}} = 1$$

Ant Colony Algorithm $\alpha = 1$, $\beta = 1$, $\tau_{ij} = 1$



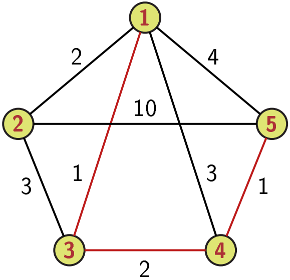
$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g \in N_i} \tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

$e \in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

In the city of **4** we can choose cities $N_4 = \{5\}$.

	5
p_{45}	1

$$p_{45} = \frac{\tau_{45}\eta_{45}}{\tau_{45}\eta_{45}} = 1$$



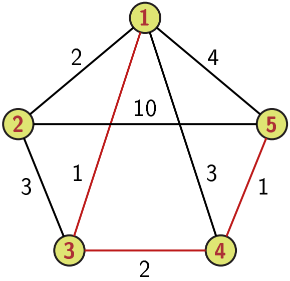
$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g \in N_i}\tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

$e \in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

In the city of **4** we can choose cities $N_4 = \{5\}$.

	5
p_{45}	1

$$p_{45} = \frac{\tau_{45}\eta_{45}}{\tau_{45}\eta_{45}} = 1$$

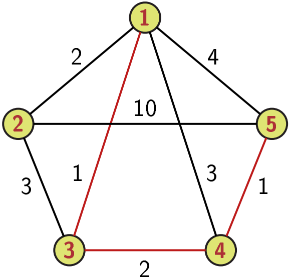


$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g \in N_i} \tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

$e \in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

In the city of **5** we can choose cities $N_5 = \{2\}$.

	2
p_{52}	



$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g\in N_i}\tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

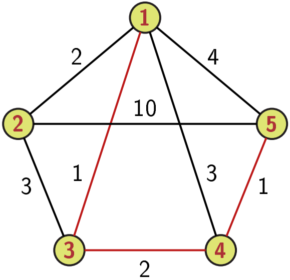
$e\in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

In the city of **5** we can choose cities $N_5 = \{2\}$.

	2
p_{52}	

$$p_{52} = \frac{\tau_{52}\eta_{52}}{\tau_{52}\eta_{52}} = 1$$

Ant Colony Algorithm $\alpha = 1$, $\beta = 1$, $\tau_{ij} = 1$



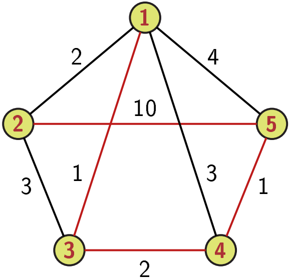
$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g \in N_i} \tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

$e \in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

In the city of **5** we can choose cities $N_5 = \{2\}$.

	2
p_{52}	1

$$p_{52} = \frac{\tau_{52}\eta_{52}}{\tau_{52}\eta_{52}} = 1$$



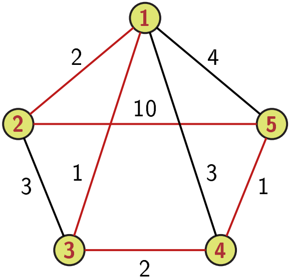
$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g \in N_i} \tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

$e \in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

In the city of **5** we can choose cities $N_5 = \{2\}$.

	2
p_{52}	1

$$p_{52} = \frac{\tau_{52}\eta_{52}}{\tau_{52}\eta_{52}} = 1$$



$$p_{ij} = \frac{\tau_{ij}^{\alpha}\eta_{ij}^{\beta}}{\sum_{g \in N_i}\tau_{ig}^{\alpha}\eta_{ig}^{\beta}}$$

$e \in E$	{1,2}	{1,3}	{1,4}	{1,5}	{2,3}	{2,5}	{3,4}	{4,5}
d_{ij}	2	1	3	4	3	10	2	1
η_{ij}	$\frac{1}{2}$	1	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{10}$	$\frac{1}{2}$	1

the total path found is

$$l = 1 + 2 + 1 + 10 + 2 = 16$$

http://demonstrations.wolfram.com/AntColonyOptimizationACO/

Thank you for your attention!!!