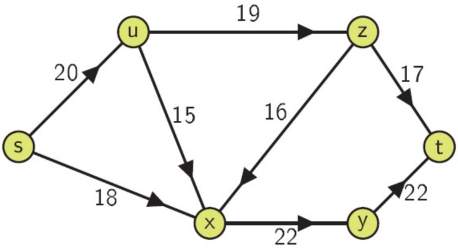
Graph and Network Theory

How much can be transported in a network from a source s to a sink t if the capacities of the connections are given?

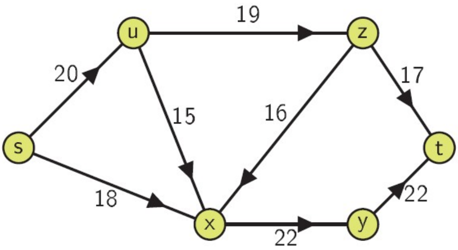


A flow network $N = (G, s, t, c)$, is a directed graph $G = (V, E)$ with two distinguished vertices: a **source** $s \in V$ and a **sink** $t \in V$ and with a given **capacity function** $c : V \times V \to \mathbb{R}^+ \cup \{0\}$ which satisfies the following conditions

$$c(u, v) = \begin{cases} c(u, v) > 0 & \text{for } (u, v) \in E \\ c(u, v) = 0 & \text{for } (u, v) \notin E \end{cases}$$

Remark

In the flow network we disallow self-loops and assume, for convenience, that each vertex lies on some path from the source to the sink.



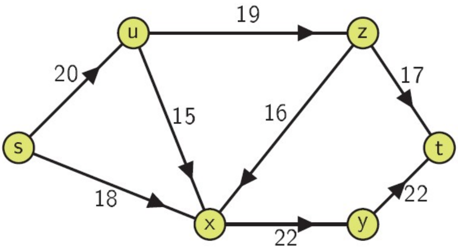
An admissible flow or, for short, a **flow on** N is a mapping $f : E \to \mathbb{R}^+ \cup \{0\}$ satisfying the following two conditions

[F1] $0 \leq f(e) \leq c(e)$ for each edge $e \in E$

[F2] for each vertex $v \in V$, different from source and sink

$$\sum_{e^+=v} f(e) = \sum_{e^-=v} f(e)$$

where e^- and e^+ denote the start and end vertex of e, respectively.



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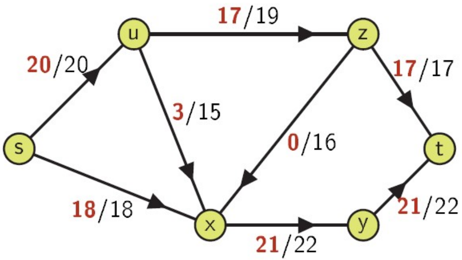

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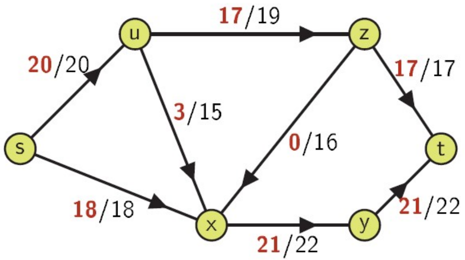

- **The feasibility condition (F1)** requires that each edge carries a nonnegative amount of flow which may not exceed the capacity of the edge;

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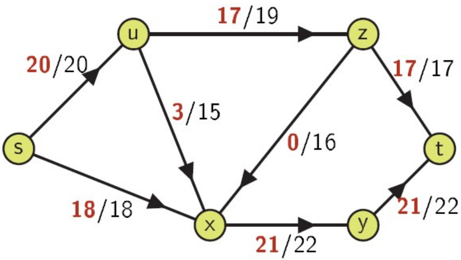

- **The flow conservation condition (F2)** means that flows are preserved: at each vertex, except for the source and the sink, the amount that **flows in** also **flows out**.

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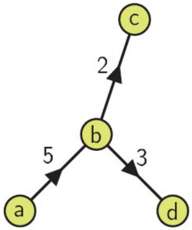
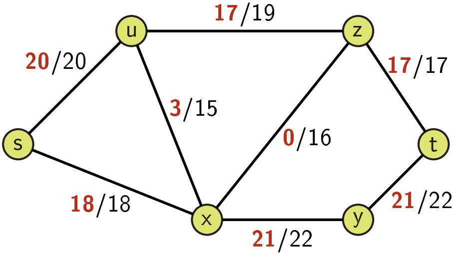
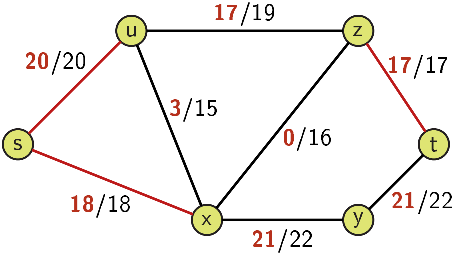


for $b \in V$ we have

$$\begin{aligned} \sum_{e^+=b} f(e) &= f(a,b) = 5 \\ \sum_{e^-=b} f(e) &= f(b,c) + f(b,d) = 2 + 3 \end{aligned}$$



- The edge e is called saturated, if we have $f(e) = c(e)$



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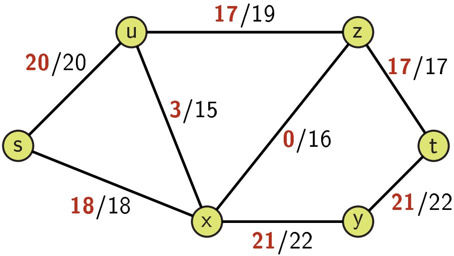

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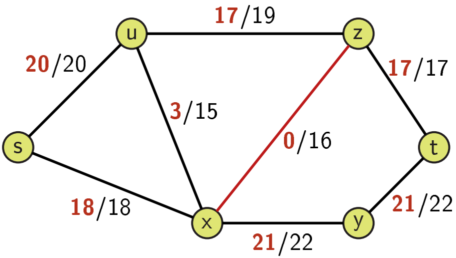
- The edge e is called empty(void), if we have $f(e) = 0$



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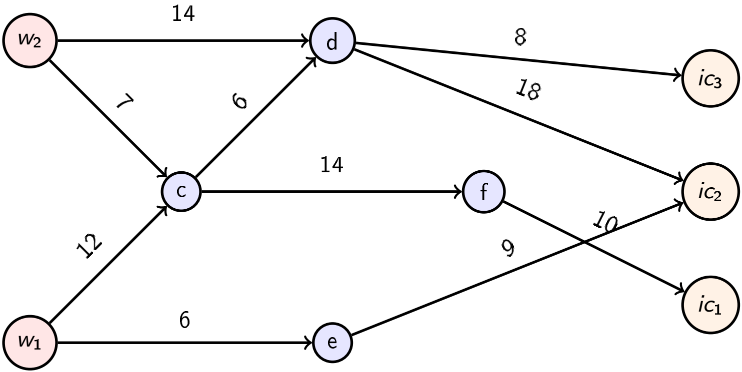
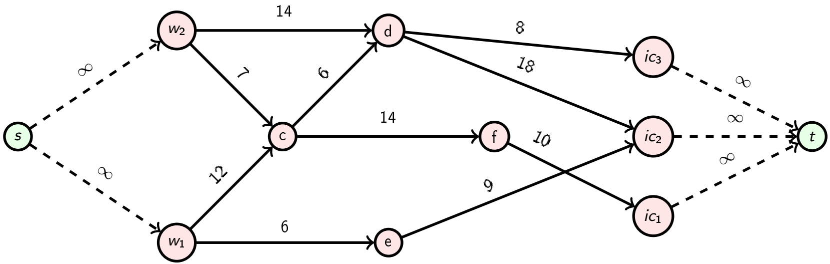
Consider a network of pipelines that transport natural gas from wells to industrial customers. A network is shown in Fig.



Each pipe has a finite maximum capacity. How can we determine the maximum network capacity of the network between wells w_1, w_2 and industrial customers ic_1, ic_2, ic_3?



The way of transformation the network with two sources and three sinks, by adding supersource and supersink so as the new network meets the assumptions of the definition of flow network.

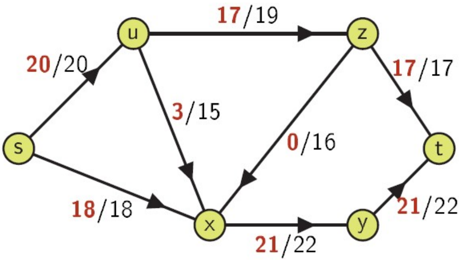
The function $b_f : V \to \mathbb{R}$ assigning to each vertex $v \in V$ real number of the form

$$b_f(v) = \sum_{e^- = v} f(e) - \sum_{e^+ = v} f(e)$$

is called a vector of flow conservation.

where e^- and e^+ denote the start and end vertex of e, respectively.

- for $v = s$ we have $b_f(v) > 0$
- for $v = t$ we have $b_f(v) < 0$
- for $v \neq s, t$ we have $b_f(v) = 0$

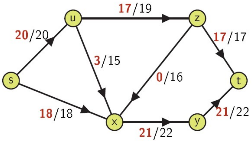


$b_f(s) = \sum_{e^-=s} f(e) - \sum_{e^+=s} f(e) = f(s,x) + f(s,u) = 18 + 20 = 38$

Let $N = (G, s, t, c)$ be a flow network with flow f. Then $b_f(s) = -b_f(t)$

Proof.

We have $b_f(s) = \sum_{e^- = s} f(e) - \sum_{e^+ = s} f(e)$
and $b_f(t) = \sum_{e^- = t} f(e) - \sum_{e^+ = t} f(e)$.



On the other hand

$$\sum_{e} f(e) = \sum_{e^- = s} f(e) + \sum_{e^- = t} f(e) + \sum_{v \neq s,t} \sum_{e^- = v} f(e)$$

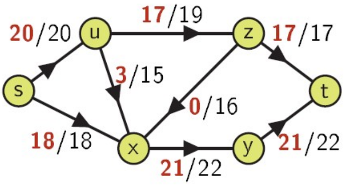
$$\sum_{e} f(e) = \sum_{e^+ = s} f(e) + \sum_{e^+ = t} f(e) + \sum_{v \neq s,t} \sum_{e^+ = v} f(e)$$

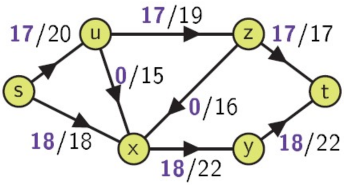
Hence

$$\begin{aligned} 0 &= \sum_{e^- = s} f(e) - \sum_{e^+ = s} f(e) + \sum_{e^- = t} f(e) - \sum_{e^+ = t} f(e) + \sum_{v \neq s,t} \left(\sum_{e^- = v} f(e) - \sum_{e^+ = v} f(e) \right) \\ &= b_f(s) + b_f(t) + \sum_{v \neq s,t} b_f(v) \end{aligned}$$

■

The quantity $b_f(s)$ is called the **value of f**. It is denoted by $|f|$. A flow f is said to be **maximal** if $|f| \geq |f'|$ holds for every flow f'on given network N.





$|f_1| = b_{f_1}(s) = 38$

$|f_2| = b_{f_2}(s) = 35$

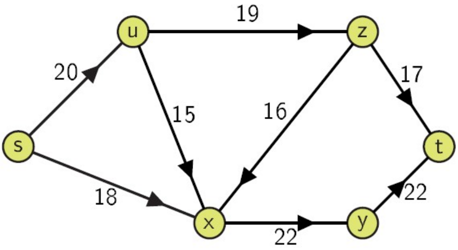
Remark

In all the examples of the capacity of the cut and flow through the cut, the same network is considered, unfortunately, for technical reasons sometimes tex does not generate arrows on the edges.

Let $N = (G, c, s, t)$ be a flow network. A **cut of N** is a partition $V = S \cup T$ of the vertex set V of G into two disjoint sets S and T with $s \in S$ and $t \in T$; thus cuts in flow networks constitute a special case of the cuts of G.

The **capacity of a cut (S,T)** is defined as

$$c(S,T) = \sum_{e^- \in S, e^+ \in T} c(e) = \sum_{x \in S} \sum_{y \in T} c(x,y)$$



Let $S = \{s, u, x\}$ i $T = \{z, y, t\}$.
Then capacity of a cut (S,T)

$c(S,T) = c(u,z) + c(x,y) = 19 + 22 = 41$

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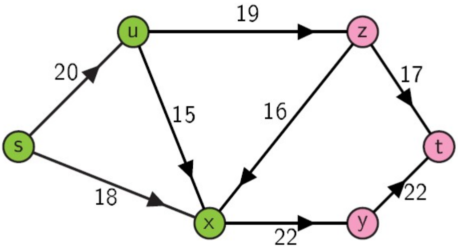

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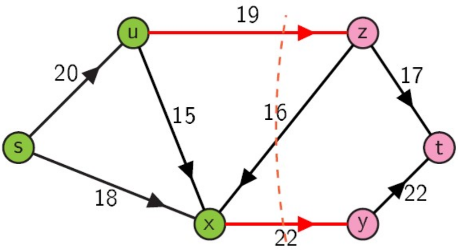

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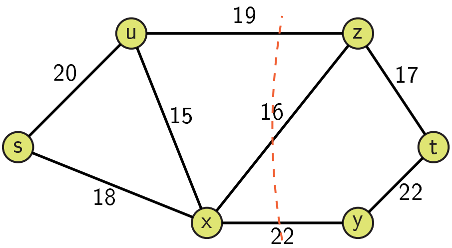

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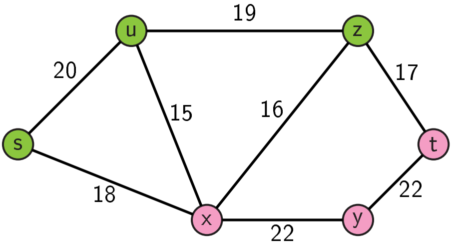

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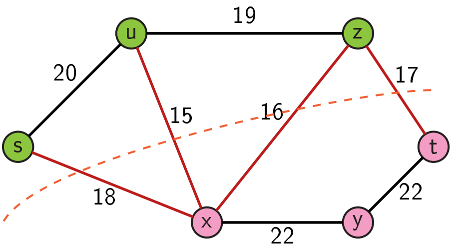

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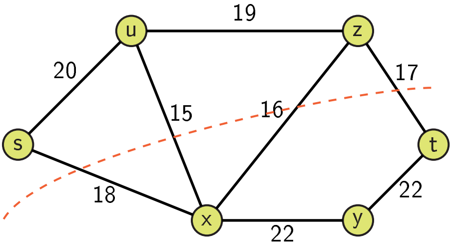

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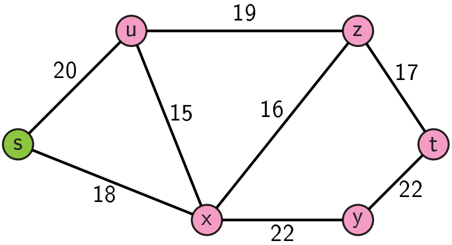

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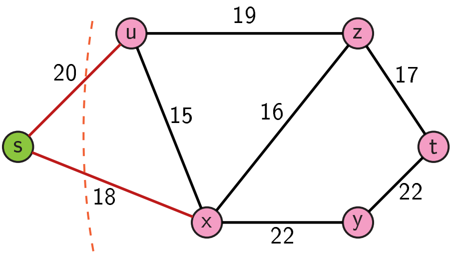

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$c(S,T) = c(s,x) + c(s,u) = 20 + 18 = 38$

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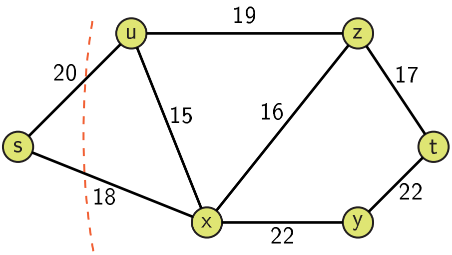

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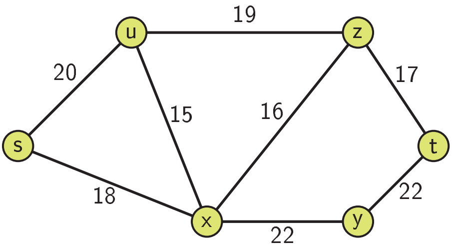
The cut (S, T) is said to be **minimal** if $c(S, T) \leq c(S', T')$ holds for every cut (S', T') in a given network.



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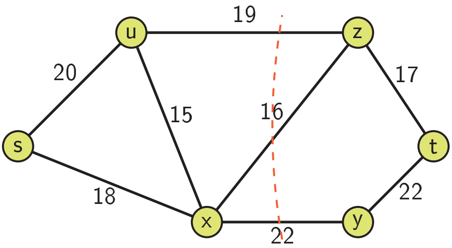

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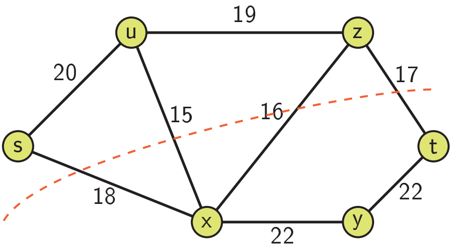

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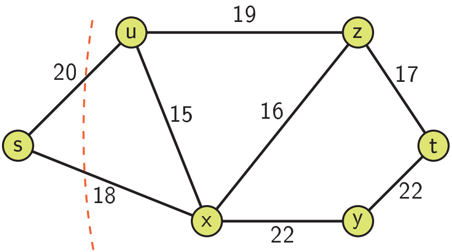

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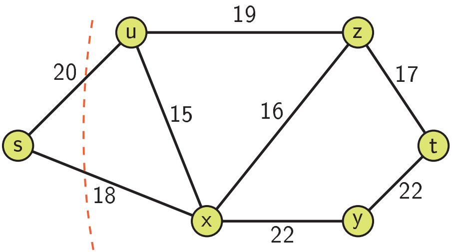

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In this network, minimal cut is cut $S = \{s\}$ i $T = \{u, x, z, y, t\}$.

Let $N = (G, s, t, c)$ be a flow network, (S, T) a cut, and f a flow. Then, the value of the flow through the cut (S, T) is of the form

$$f(S, T) = \sum_{e^- \in S, e^+ \in T} f(e) - \sum_{e^+ \in S, e^- \in T} f(e)$$

where e^- and e^+ denote the start and end vertex of e, respectively.

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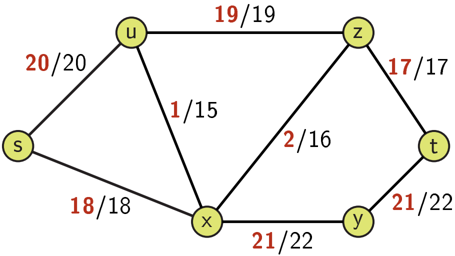

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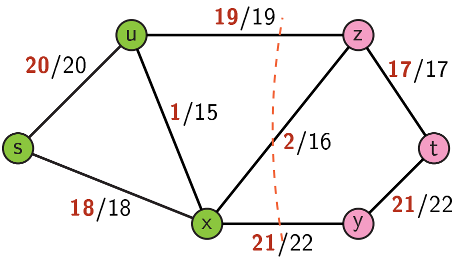
$$f(S,T) = \sum_{e^- \in S, e^+ \in T} f(e) - \sum_{e^+ \in S, e^- \in T} f(e)$$

where e^- and e^+ denote the start and end vertex of e, respectively.



The cut $S = \{s, u, x\}$ i
$T = \{z, y, t\}$ of **capacity**

$c(S,T) = c(u,z) + c(x,y) = 19 + 22 = 41$

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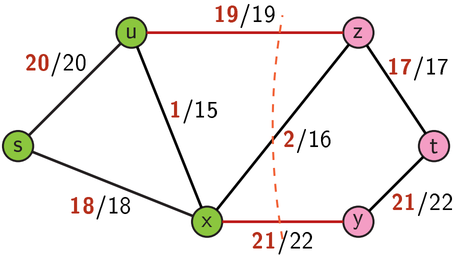

The cut $S = \{s, u, x\}$ i
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The value of the flow through the cut (S, T) is equal

$$\begin{aligned} f(S, T) &= f(u, z) + f(x, y) - f(x, z) \\ &= 19 + 21 - 2 = 38 \end{aligned}$$

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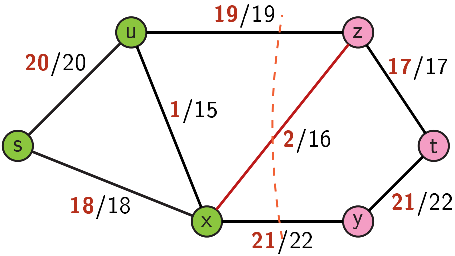

The cut $S = \{s, u, x\}$ i
$T = \{z, y, t\}$ of **capacity**

$c(S,T) = c(u,z)+c(x,y) = 19+22 = 41$

The value of the flow through the cut (S, T) is equal

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$$f(S,T) = \sum_{e^- \in S, e^+ \in T} f(e) - \sum_{e^+ \in S, e^- \in T} f(e)$$

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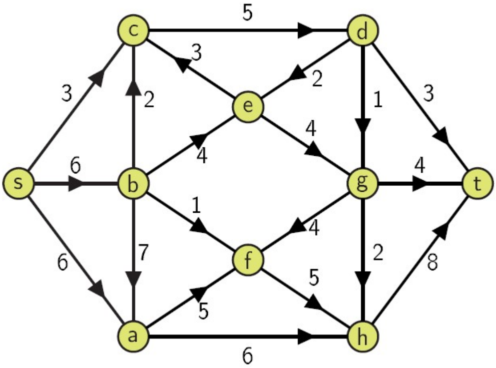
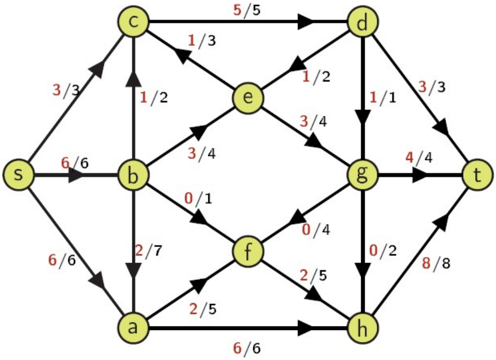


The value of the flow through the cut - example 2

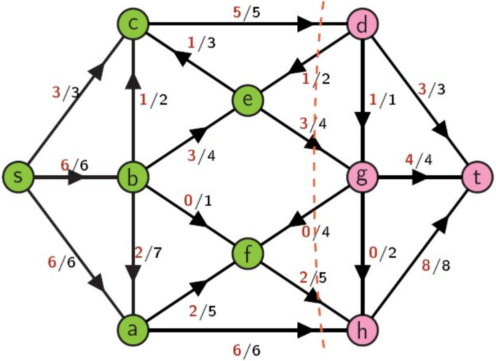
$$f(S,T) = \sum_{e^- \in S, e^+ \in T} f(e) - \sum_{e^+ \in S, e^- \in T} f(e)$$

where e^- and e^+ denote the start and end vertex of e, respectively.



$$f(S,T) = \sum_{e^- \in S, e^+ \in T} f(e) - \sum_{e^+ \in S, e^- \in T} f(e)$$

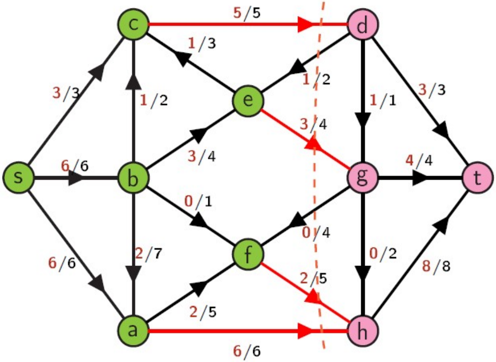
where e^- and e^+ denote the start and end vertex of e, respectively.



- the cut
  $S = \{s, a, b, c, e, f\}$ i
  $T = \{d, g, h, t\}$

$$f(S,T) = \sum_{e^- \in S, e^+ \in T} f(e) - \sum_{e^+ \in S, e^- \in T} f(e)$$

where e^- and e^+ denote the start and end vertex of e, respectively.

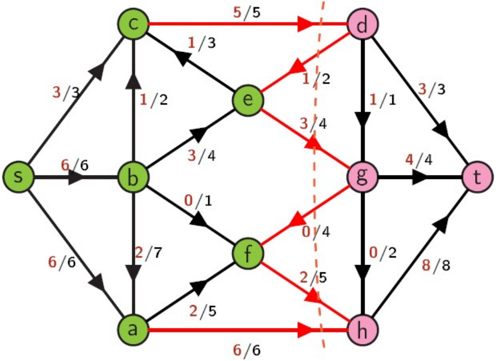


- the cut
  $S = \{s, a, b, c, e, f\}$ i
  $T = \{d, g, h, t\}$
- the capacity of cut

$$c(S,T) = 20$$

$$f(S,T) = \sum_{e^- \in S, e^+ \in T} f(e) - \sum_{e^+ \in S, e^- \in T} f(e)$$

where e^- and e^+ denote the start and end vertex of e, respectively.



- the cut
  $S = \{s, a, b, c, e, f\}$ i
  $T = \{d, g, h, t\}$
- the capacity

$$c(S,T) = 20$$

- the value of the flow through the cut

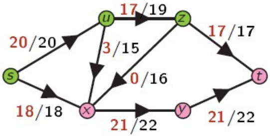
$$f(S,T) = 16 - 1 = 15$$

Let $N = (G, s, t, c)$ be a flow network, (S, T) a cut, and f a flow. Then the capacity of a minimal cut gives the upper bound on the value of a flow

$$|f| \leq c(S, T)$$

Equality holds if and only if each edge e with $e^- \in S$ and $e^+ \in T$ is saturated, whereas each edge e with $e^- \in T$ and $e^+ \in S$ is void. **where e^- and e^+ denote the start and end vertex of e, respectively.**

Let's consider any cut (S, T) in N. Then for any $v \in S$, $v \neq s$ we have $b_f(v) = 0$. Zatem $|f| = b_f(s) = \sum_{v \in S} b_f(v) = \sum_{v \in S} \left(\sum_{e^- = v} f(e) - \sum_{e^+ = v} f(e) \right)$



$$= \sum_{e^- \in S, e^+ \in S} f(e) - \sum_{e^+ \in S, e^- \in S} f(e) + \sum_{e^- \in S, e^+ \in T} f(e) - \sum_{e^+ \in S, e^- \in T} f(e)$$

$$\leq \sum_{e^- \in S, e^+ \in T} c(e) = c(S, T)$$

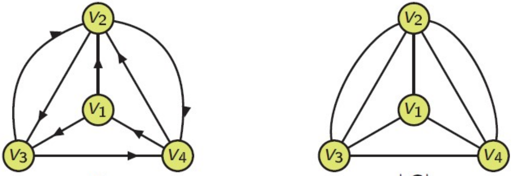
Path in directed graph

Let $G = (V, E)$ be a directed graph.

A sequence of edges $(e_1, e_2, ..., e_n)$ is called a a path in G if the sequence of corresponding edges is a path in the corresponding graph $|G|$.

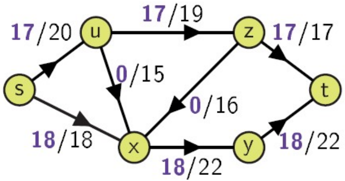
If a sequence of vertices $(v_0, v_1, ..., v_n)$ corresponds to a path in the graph G, then either

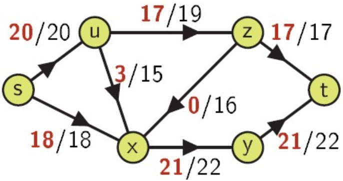
- (v_{i-1}, v_i) is an edge in graph G. In that case, we call it (v_{i-1}, v_i) a **forward edge**. Albo
- or (v_i, v_{i-1}) is an edge in graph G. In that case, we call it it a backward edge.



G $|G|$

Let f be a flow in the network $N = (G, s, t, c)$. A path W from s to t is called an **augmenting path with respect to f** if $f(e) < c(e)$ holds for every forward edge $e \in W$, whereas $f(e) > 0$ for every backward edge $e \in W$.





The path (s, u, x, y, t) is the augmenting path of the f flow.

A flow f on a flow network $N = (G, c, s, t)$ is **maximal** if and only if there are no augmenting paths with respect to f.

Proof. $\Rightarrow$ Let f be a maximal flow. Suppose there is an augmenting path W. Let $d = \min_{e \in W, forward}(c(e) - f(e))$. Then $d > 0$. by definition of an augmenting path. Now we define a mapping $f' : E \to \mathbb{R}^+ \cup \{0\}$

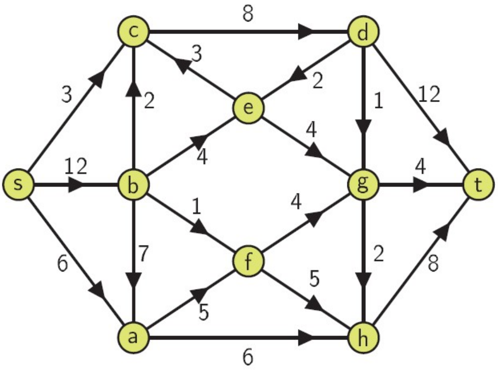
$$f'(e) = \begin{cases} f(e) + d \text{ if } e \in W \text{ is a forward edge in} \\ f(e) - d \text{ if } e \in W \text{ is a backward edge in} \\ f(e) \text{ otherwise.} \end{cases}$$

It is easily checked that f' is a flow on N with value $|f'| = |f| + d$contradicting the maximality of f. ■

A flow f on a flow network $N = (G, c, s, t)$ is **maximal** if and only if there are no augmenting paths with respect to f.

Proof. $\Leftarrow$ Conversely, suppose there are no augmenting paths in N with respect to f. Let S be the set of all vertices v such that there exists an augmenting path from s to v (including s itself), and put $T = V \setminus S$. By hypothesis, (S, T) is a cut of N. Note that each edge $e = (u, v)$, such, that $e^- = u \in S$ and $e^+ = v \in T$ has to be saturated: otherwise, it could be appended to an augmenting path from s to u to reach the point $v \in T$, a contradiction. Similarly, each edge e with $e^- \in T$ and $e^+ \in S$ has to be void. (if it were not, then $e^- \in S$.) From the properties of flow in network we have $|f| = c(S, T)$, so f must be maximal. ■

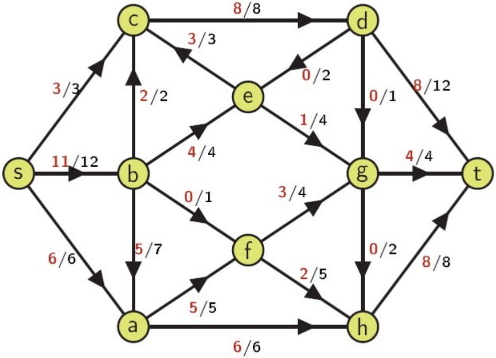
The maximal value of a flow on a flow network N is equal to the minimal capacity of a cut for N.



- each edge $e = (u, v)$, such, that $e^- = u \in S$ and $e^+ = v \in T$ has to be saturated: otherwise, it could be appended to an augmenting path from s to u to reach the point $v \in T$, a contradiction. Similarly, each edge e with $e^- \in T$ and $e^+ \in S$ has to be void. (if it were not, then $e^- \in S$.)

$$\begin{aligned} S &= \{s, b, a, f, h, g, e\} \\ T &= \{c, d, t\} \end{aligned}$$

- $$f(S, T) = c(S, T) = 20$$



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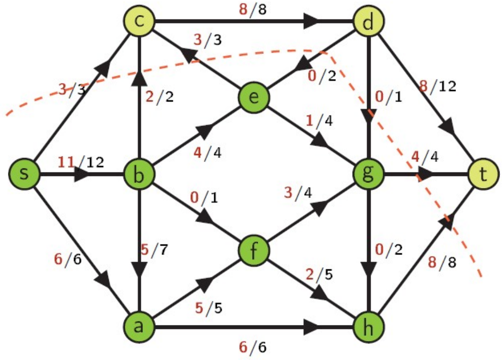

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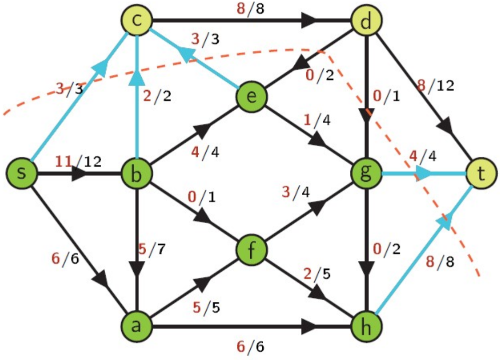
$$
\begin{aligned}
S &= \{s, b, a, f, h, g, e\} \\
T &= \{c, d, t\}
\end{aligned}
$$

- $$f(S, T) = c(S, T) = 20$$

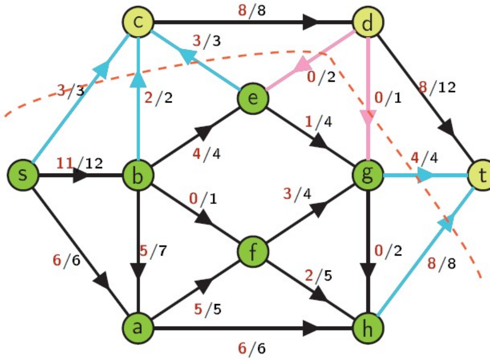


- each edge $e = (u, v)$, such, that $e^{-} = u \in S$ and $e^{+} = v \in T$ has to be saturated: otherwise, it could be appended to an augmenting path from s to u to reach the point $v \in T$, a contradiction. Similarly, each edge e with $e^{-} \in T$ and $e^{+} \in S$ has to be void. (if it were not, then $e^{-} \in S$.)

$$\begin{aligned} S &= \{s, b, a, f, h, g, e\} \\ T &= \{c, d, t\} \end{aligned}$$

-

$$f(S, T) = c(S, T) = 20$$



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\begin{aligned}
S &= \{s, b, a, f, h, g, e\} \\
T &= \{c, d, t\}
\end{aligned}
$$

- $$f(S, T) = c(S, T) = 20$$

A **flow** in (G, s, t, c) is a real-valued function $f : V \times V \to \mathbb{R}$ that satisfies the following three properties

capacity constrains for $u, v \in V$, we require

$$f(u, v) \leq c(u, v)$$

skew symmetry for all $u, v \in V$ we require

$$f(u, v) = -f(v, u)$$

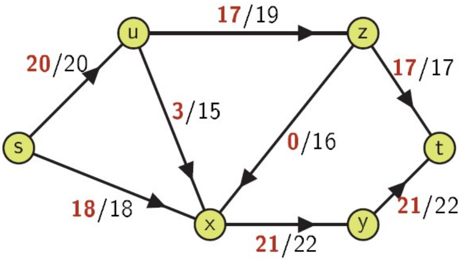
flow conservation for $u \in V \setminus \{s, t\}$, we require

$$\sum_{v \in V} f(v, u) = 0$$

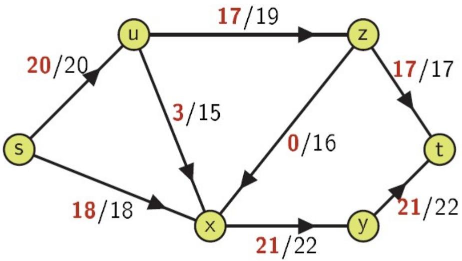
The quantity $f(u, v)$, which can be positive, zero, or negative, is called the **flow from vertex u to vertex v**.

The **value** $|f|$ **of a flow** f is defined as

$$|f| := \sum_{v \in V} f(s, v).$$



$|f| = 38$



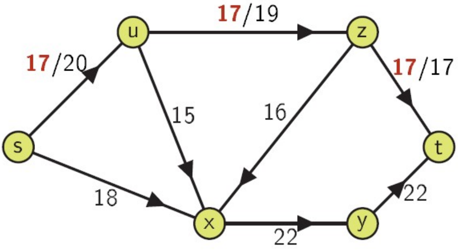
	c	f		f
(s,u)	20	20	(u,s)	-20
(s,x)	18	18	(x,s)	-18
(u,z)	19	19	(z,u)	-19
(u,x)	15	1	(x,u)	-1
(z,x)	16	2	(x,z)	-2
(x,y)	22	21	(y,x)	-21
(y,t)	22	21	(t,y)	-21
(z,t)	17	17	(t,z)	-17

$$\sum_{v\in V} f(z,v) \;=\; f(z,t) + f(z,u) + f(z,x) \;=\; 17 - 19 + 2 \;=\; 0$$

Given a flow network (G, s, t, c) with a flow f, the **residual network** of G induced by f is $G_f = (V, E_f)$, where

$$E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$$

and $c_f(u, v) = c(u, v) - f(u, v)$ is called the residual capacity.



$$s \to u \to z \to t$$

$$\begin{aligned}
c_f(s, u) &= 20 - 17 = 3 \\
c_f(u, s) &= 0 - (-17) = 17 \\
c_f(u, z) &= 19 - 17 = 2 \\
c_f(z, u) &= 0 - (-17) = 17 \\
c_f(z, t) &= \mathbf{17\text{-}17{=}0} \\
c_f(t, z) &= 0 - (-17) = 17
\end{aligned}$$

Given a flow network (G, s, t, c) with a flow f, the **residual network** of G induced by f is $G_f = (V, E_f)$, where

$$E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$$

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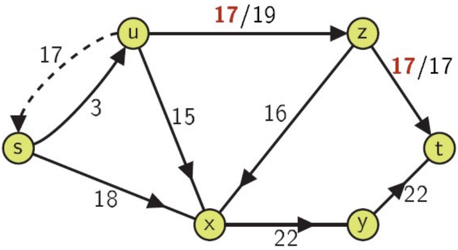

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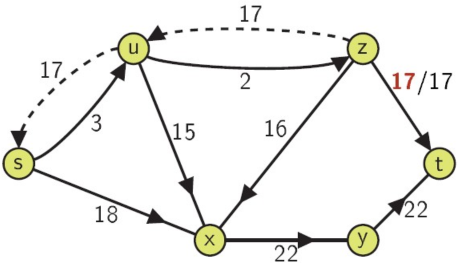

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Given a flow network (G, s, t, c) with a flow f, the **residual network** of G induced by f is $G_f = (V, E_f)$, where

$$E_f = \{(u, v) \in V \times V : c_f(u, v) > 0\}$$

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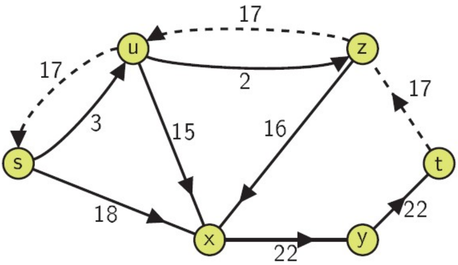

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c_f(z, t) &= \textbf{17-17=0} \\
c_f(t, z) &= 0 - (-17) = 17
\end{aligned}$$

If f is a flow in a flow network $N = (G, s, t, c)$, then the following conditions are equivalent:

1. f **is the maximum flow in** N
2. N_f **residual network contains no augmenting paths**
3. $|f| = c(S, T)$ **for a some cut** (S, T) **in** N

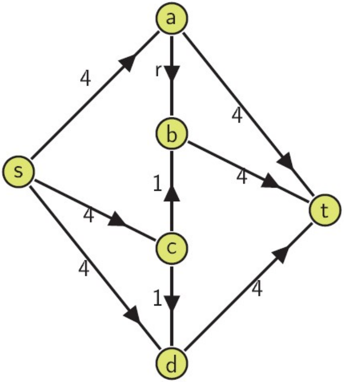
Algorithm

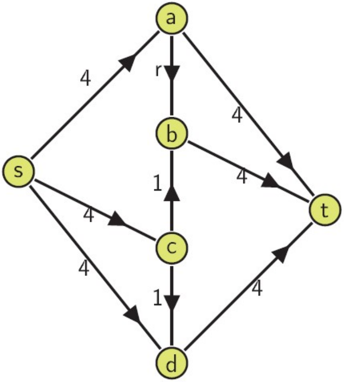
```
FORD-FULKERSON(G, s, t, c)
for each edge (u, v) ∈ E_G
        do f(u, v) ← 0
           f(v, u) ← 0
while there is augmenting path p from s to t in residual network (G_f, s, t, c_f)
        do c_f(p) ← min{c_f(u, v) : (u, v) is on path p}
           for each edge (u, v) na p
               do f(u, v) ← f(u, v) + c_f(p)
                  f(v, u) = −f(v, u)
```

A Bad Example for Ford Fulkerson

We will show an example of a network for which the Ford-Fulkerson algorithm does not converge to an optimal solution.

Let $r = \dfrac{\sqrt{5} - 1}{2}$. Let's consider a network



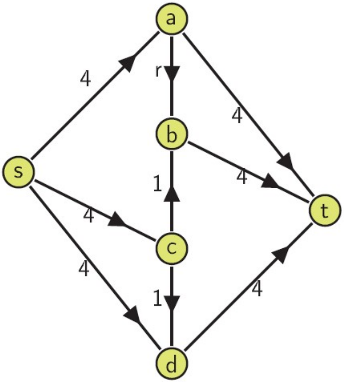


In the Ford-Fulkerson algorithm, let's choose the following augmenting paths

- **(s,a,t)** $|f| = 4$;
- **(s,c,b,t)** $|f| = 4 + 1 = 5$;
- **(s,d,t)** $|f| = 5 + 4 = 9$;

It is easy to see that the value of the maximum flow is 9.

Let's choose the following augmenting paths



Let's us denote

- **p0=(s,c,b,t)**
- **p1=(s,a,b,c,d,t)**
- **p2=(s,c,b,a,t)**
- **p3=(s,d,c,b,t)**

In the Ford-Fulkerson algorithm, let's choose the following infinite sequence of augmenting paths

p0(p1,p2,p1,p3)*=

=p0,p1,p2,p1,p3,p1,p2,p1,p3,...

Then the maximum flow obtained using the above method, **is not convergent to 9.**

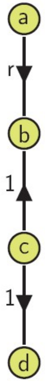
$$r = \frac{\sqrt{5} - 1}{2}$$

Zauważmy, że

- $r \approx 0.618034$
- $1 - r = r^2$

We choose the paths according to the scheme
p0(p1,p2,p1,p3)*=p0,p1,p2,p1,p3,p1,p2,p1,p3,...

- **p0=(s,c,b,t)**
- **p1=(s,a,b,c,d,t)**
- **p2=(s,c,b,a,t)**
- **p3=(s,d,c,b,t)**



A Bad Example for Ford Fulkerson

We choose the paths according to the scheme
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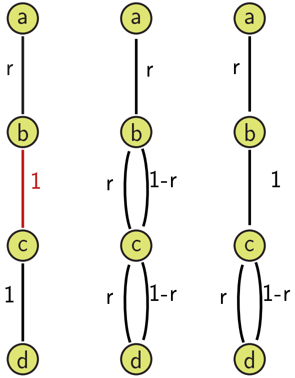

p0,f=1 p1,f=1+r p2,f=1+r+r

- path **p1 - a,b,c,d**. Here the maximum you can send $1-r=r^2$. Then

$$f=1+r+r+r^2$$

- next path **p3 - d,c,b**. Here the maximum you can send r^2.Then

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- in the next steps we will send $r^3, r^3, r^4, r^4, \ldots$

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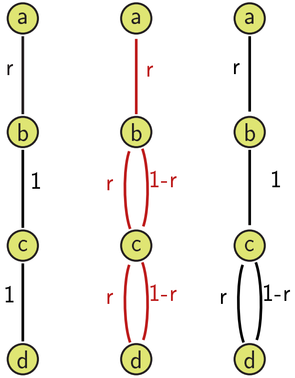

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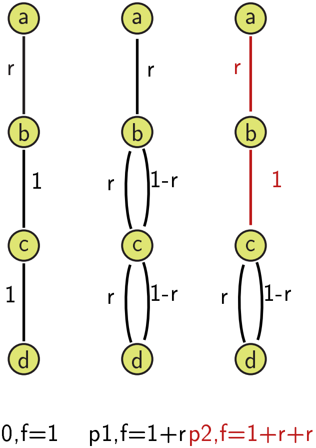

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A Bad Example for Ford Fulkerson

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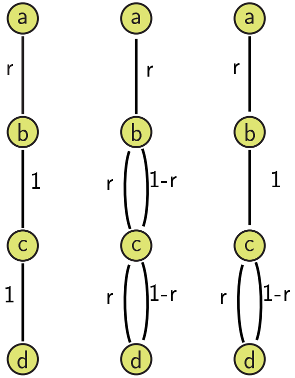

p0,f=1   p1,f=1+rp2,f=1+r+r

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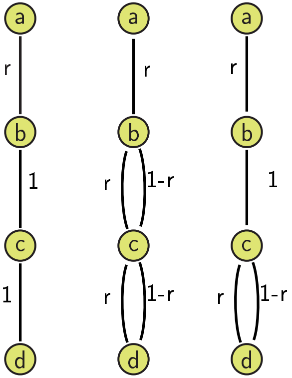

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- in the next steps we will send $r^3, r^3, r^4, r^4, \ldots$

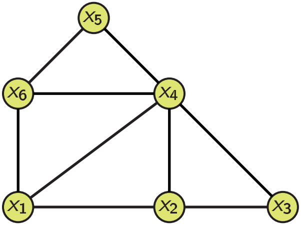
Zatem mamy

$$|f| = 1 + r + r + r^2 + r^2 + r^3 + r^3 + ... = 1 + 2(r + r^2 + r^3 + ...)$$

Using the formula for the sum of a geometric series $S = \frac{a_1}{1-q}$

$$|f| = 1 + \frac{2 \cdot 2}{\sqrt{5} - 1} = 1 + \frac{4 \cdot (\sqrt{5} + 1)}{4} = \sqrt{5} + 2 \approx 4.26$$

Let $G = (V, E)$ be a graph. A **matching** is a subset of edges $M \subset E$ such that for all vertices $v \in V$, at most one edge of M is incident on v. We say that a vertex $vinV$ is **matched** by matching M if some edge in M is incident on v; otherwise, v is **unmatched**.



- edges $\{\{x_1, x_2\}, \{x_5, x_6\}\}$ is a matching
- edges $\{\{x_1, x_4\}, \{x_2, x_3\}, \{x_5, x_6\}\}$ tworzą skojarzenie
- edges $\{\{x_1, x_6\}, \{x_6, x_4\}\}$ **is not** a matching
- **every single edge makes a matching.**

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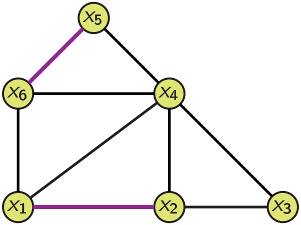

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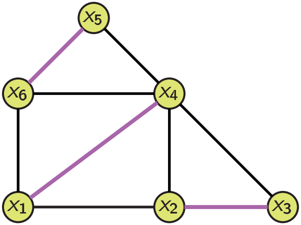

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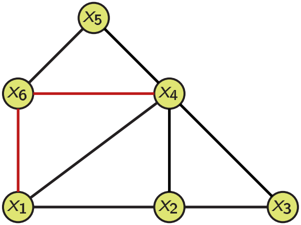

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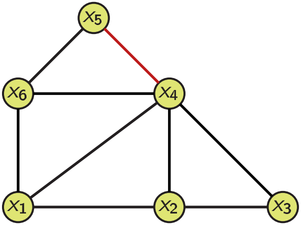

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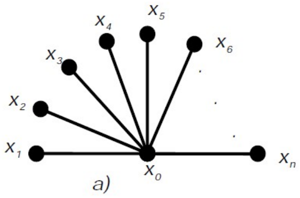
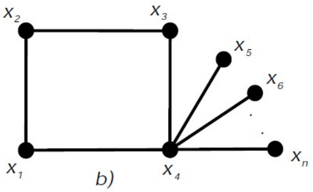
- **every single edge makes a matching.**

A maximum matching is a matching of maximum cardinality, that is, a matching M such that for any matching M', we have $|M| \geq |M'|$. The size of maximum matching is called a **matching number** of a graph G and is denoted by $\lambda(G)$.





Maximum matchings are made by

one edge

$$\{x_0, x_i\}, \;\; i = 1, 2, ..., n$$

two edges

$$\{\{x_1, x_2\}, \{x_4, x_i\}\} \text{ dla } i = 3, 4, ..., n$$

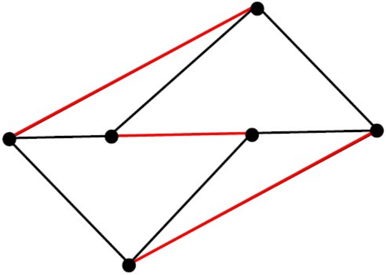
$$\{\{x_2, x_3\}, \{x_4, x_i\}\} \text{ dla } i = 5, 6, ..., n$$

Properties

In any graph with n vertices a size of maximum matching is not greater than $\left\lfloor \frac{1}{2} n \right\rfloor$ edges.

Definicja

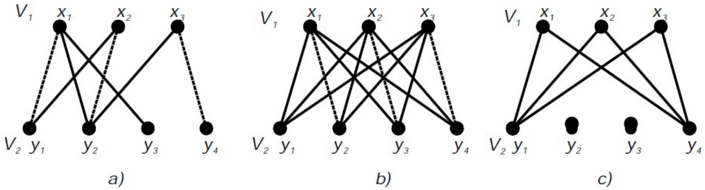
A **perfect matching** is a matching which matches all vertices of a graph (its size is $\left\lfloor \frac{1}{2} n \right\rfloor$).



Due to the wide range of applications, our considerations consider will be limited to bipartite graphs. Let $G = (V_1 \cup V_2, E)$, be a bipartite graph. We are especially interested in finding complete matchings

Definicja

A **complete matching** from V_1 to V_2 in a bipartite graph $G = (V_1 \cup V_2, E)$ is a matching that matches all vertices from V_1.



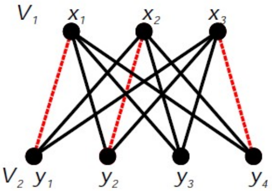
a) a complete matching: $\{\{x_1, y_1\}, \{x_2, y_2\}, \{x_3, y_4\}\}$,

b) a complete matching: $\{\{x_1, y_2\}, \{x_2, y_3\}, \{x_3, y_4\}\}$,

c) there is no complete matching from V_1 to V_2, since three vertices in V_1 are incident to only two vertices from V_2.

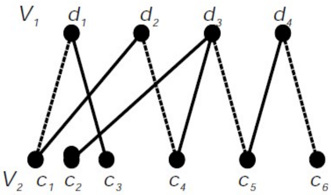
Theorem

In a complete bipartite graph $K_{m,n}$, where $|V_1| = m$, $|V_2| = n$ and $m \leq n$ there exists a complete matching from V_1 to V_2.



Given a finite set of girls V_1, such that each of them knows a given number of boys from the set V_2, for every girl find a husband (a husband for a given girl can be only the boy that she knows).

The set of girls $\{d_1, d_2, d_3, d_4\}$
The set of boys $\{c_1, c_2, c_3, c_4, c_5, c_6\}$

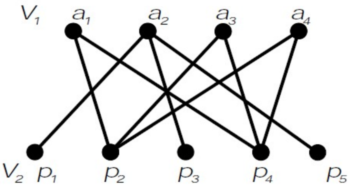


The existence of an edge $\{d_i, c_j\}$ means that girl d_i knows boy c_j.
The solution is e.g. the following complete matching

$$\{\{d_1, c_1\}, \{d_2, c_4\}, \{d_3, c_5\}, \{d_4, c_6\}\}$$

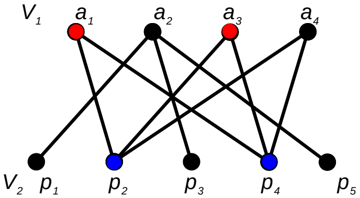
We have four candidates a_1, a_2, a_3, a_4 for five jobs p_1, p_2, p_3, p_4, p_5. Each candidate must have appropriate qualifications to a given job. Can all candidates be employed?

The edge $\{a_i, p_k\}$ means that a candidate a_i has a qualification for a job p_k ($i \in \{1, 2, 3, 4\}$, $j \in \{1, 2, 3, 4, 5\}$.



Here, the complete matching is not possible, because three candidates a_1, a_3, a_4 is qualified to only two jobs. In fact, only two of them may be employed.

Let $G = (V_1 \cup V_2, E)$ be bipartite graph. Let for any set of vertices U included in V_1 $(U \subset V_1)$ the set $\varphi(U)$ is the set of vertices $v_2 \in V_2$, for which there exist a vertex $v_1 \in U$, such that $\{v_1, v_2\} \in E$.



$$U = \{a_1, a_3\} \subset V_1 \qquad \varphi(U) = \{p_2, p_4\} \subset V_2$$

Hall's marriage theorem, 1935.

In a bipartite graph $G = (V_1 \cup V_2, E)$ there exists a complete matching from V_1 to V_2 if and only if for any $U \subset V_1$ we have that

$$|U| \leq |\varphi(U)|$$

Marriage version

Imagine, that we have a group of m girls. Each girl likes a certain number of boys. A marriage matching between girls and boys exists (which means that every girl can married a boy which she knows) if and only if every $k \leq m$ girls knows at least k boys.

If in a bipartite graph the complete matching is not possible, we are interested in the maximum set of V_1 that might be matched with vertices from V_2

Definicja

In bipartite graph $G = (V_1 \cup V_2, E)$ maximal difference

$$|U| - |\varphi(U)|$$

found for all subsets U of V_1 is called ***graph defect*** grafu of G and is denoted by $\delta(G)$.

Remark

Let for $U \subset V_1$, $|U| = r$ and $|\varphi(U)| = q$. Then the maximal difference $|U| - |\varphi(U)|$ is easy to find considering of values $r \in \{1, 2, ..., |V_1|\}$ and all subsets V_1 with cardinality of r

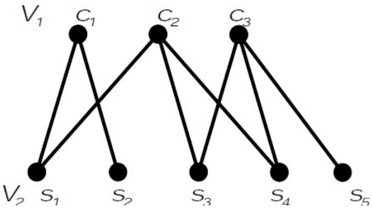
Theorem

The complete matching in bipartite graph exists if and only if $\delta(G) \leq 0$.

Theorem

Maximum number of vertices in V_1, which can be matched with vertices from V_2 equals $\min\{|V_1|, |V_1| - \delta(G)\}$.

Five PMs (Parliament Members) s_1, s_2, s_3, s_4, s_5 are members of three commission c_1, c_2, c_3. One member of each commission can be a representative of the main committee. Is it possible to choose from each commission one (different) representative?

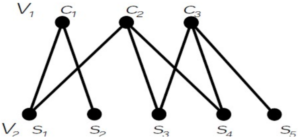


$\lvert U\rvert$	$U \subset V_1$	$\varphi(U) \subset V_2$	$q = \lvert\varphi(U)\rvert$	$\delta(G) = r - q$
$r = 1$	$\{c_1\}$	$\{s_1, s_2\}$	2	-1
	$\{c_2\}$	$\{s_1, s_3, s_4\}$	3	-2
	$\{c_3\}$	$\{s_3, s_4, s_5\}$	3	-2
$r = 2$	$\{c_1, c_2\}$	$\{s_1, s_2, s_3, s_4\}$	4	-2
	$\{c_1, c_3\}$	$\{s_1, s_2, s_3, s_4, s_5\}$	5	-3
	$\{c_2, c_3\}$	$\{s_1, s_3, s_4, s_5\}$	4	-2
$r = 3$	$\{c_1, c_2, c_3\}$	$\{s_1, s_2, s_3, s_4, s_5\}$	5	-2

Since $\delta(G) = -1 < 0$ complete matching exists, so it is possible to choose from each commissiion another representative.

Example 2

$\lvert U \rvert$	$U \subset V_1$	$\varphi(U) \subset V_2$	$q = \lvert\varphi(U)\rvert$	$\delta(G) = r - q$
$r = 1$	$\{a_1\}$	$\{p_1, p_4\}$	2	-1
	$\{a_2\}$	$\{p_1, p_3, p_5\}$	3	-2
	$\{a_3\}$	$\{p_2, p_4\}$	2	-1
	$\{a_4\}$	$\{p_2, p_4\}$	2	-1
$r = 2$	$\{a_1, a_2\}$	$\{p_1, p_2, p_4, p_5, p_3\}$	5	-3
	$\{a_1, c_3\}$	$\{p_2, p_4\}$	2	0
	$\{a_1, a_4\}$	$\{p_2, p_4\}$	2	0
	$\{a_2, c_3\}$	$\{p_1, p_2, p_4, p_5, p_3\}$	5	-3
	$\{a_2, a_4\}$	$\{p_1, p_2, p_4, p_5, p_3\}$	5	-3
	$\{a_3, a_4\}$	$\{p_2, p_4\}$	2	0
$r = 3$	$\{a_1, a_2, a_3\}$	$\{p_1, p_2, p_4, p_5, p_3\}$	5	-2
	$\{a_1, a_2, a_4\}$	$\{p_1, p_2, p_4, p_5, p_3\}$	5	-2
	$\{a_1, a_3, a_4\}$	$\{p_1, p_4\}$	2	1
	$\{a_2, a_3, a_4\}$	$\{p_1, p_2, p_4, p_5, p_3\}$	5	-2
$r = 4$	$\{a_1, a_2, a_3, a_4\}$	$\{p_1, p_2, p_3, p_4, p_5\}$	5	-1



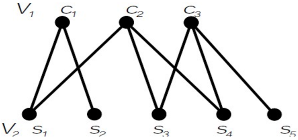
We have:

$$\delta(G) = 1, |V_1| - \delta(G) = 4 - 1 = 3$$

Maximum number of candidates which can be employed is 3.

Example 2

$\lvert U\rvert$	$U \subset V_1$	$\varphi(U) \subset V_2$	$q = \lvert\varphi(U)\rvert$	$\delta(G) = r - q$
$r = 1$	$\{a_1\}$	$\{p_1, p_4\}$	2	-1
	$\{a_2\}$	$\{p_1, p_3, p_5\}$	3	-2
	$\{a_3\}$	$\{p_2, p_4\}$	2	-1
	$\{a_4\}$	$\{p_2, p_4\}$	2	-1
$r = 2$	$\{a_1, a_2\}$	$\{p_1, p_2, p_4, p_5, p_3\}$	5	-3
	$\{a_1, c_3\}$	$\{p_2, p_4\}$	2	0
	$\{a_1, a_4\}$	$\{p_2, p_4\}$	2	0
	$\{a_2, c_3\}$	$\{p_1, p_2, p_4, p_5, p_3\}$	5	-3
	$\{a_2, a_4\}$	$\{p_1, p_2, p_4, p_5, p_3\}$	5	-3
	$\{a_3, a_4\}$	$\{p_2, p_4\}$	2	0
$r = 3$	$\{a_1, a_2, a_3\}$	$\{p_1, p_2, p_4, p_5, p_3\}$	5	-2
	$\{a_1, a_2, a_4\}$	$\{p_1, p_2, p_4, p_5, p_3\}$	5	-2
	$\{a_1, a_3, a_4\}$	$\{p_1, p_4\}$	2	1
	$\{a_2, a_3, a_4\}$	$\{p_1, p_2, p_4, p_5, p_3\}$	5	-2
$r = 4$	$\{a_1, a_2, a_3, a_4\}$	$\{p_1, p_2, p_3, p_4, p_5\}$	5	-1



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Thank you for your attention!!!