1 Distance, balls, convergence, diameter

Definition 1. Let X be a set. A function $d : X \times X \to [0, +\infty)$ is called a *distance function* (or a *metric*) on X if

1. $d(x, y) = 0 \Leftrightarrow x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, y) \leq d(x, z) + d(z, y)$

for all $x, y, z \in X$. We say that (X, d) is a *metric space*. The inequality 3. above is called the *triangle inequality*.

Definition 2. Let (X, d) be a metric space. An *open ball* with center x and radius r is the set

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

A *closed ball* with center x and radius r is the set

$$\overline{B}(x, r) = \{y \in X : d(x, y) \leq r\}.$$

Definition 3. Let (X, d) be a metric space. We say that a sequence $\{x_n\} \subset X$ converges to x (or: has the limit x) if $d(x_n, x) \to 0$, i.e.,

$$\forall_{\epsilon>0} \quad \exists_N \quad \forall_{n>N} \quad d(x_n, x) < \epsilon.$$

We write then $x_n \to x$, or $\lim_{n\to\infty} x_n = x$.

Theorem 4. *If $x_n \to x$ and $x_n \to y$ then $x = y$.*

Definition 5. Let $A \subset X$. We define the *diameter* of the set A by the formula

$$\mathrm{diam}(A) = \sup_{x,y\in A} d(x, y).$$

If $\mathrm{diam}(A)$ is finite, we say that the set A is *bounded*.

Theorem 6. *Let (X, d) be a metric space. A set A is bounded if and only if there exist $x \in X$ and $R > 0$ such that $A \subset B(x, R)$.*

Definition 7. For all $x \in X$ we define the *distance of x from the set A* by the formula:

$$\mathrm{dist}(x, A) = \inf_{y\in A} d(x, y).$$

2 Metric spaces

Theorem 8. *If (X,d) is a metric space and $A \subset X$ then $(A, d|_{A\times A})$ is a metric space. Here, $d|_{A\times A}$ denotes restriction of the function d to $A \times A$.*

Theorem 9. *If (X,d) and (Y,ρ) are metric spaces then $X \times Y$ with one of the distance functions:*

$$\sigma_1((x,y),(z,w)) = d(x,z) + \rho(y,w),$$
$$\sigma_2((x,y),(z,w)) = \sqrt{(d(x,z))^2 + (\rho(y,w))^2},$$
$$\sigma_p((x,y),(z,w)) = ((d(x,z))^p + (\rho(y,w))^p)^{1/p}, \text{ where } p \geq 1,$$
$$\sigma_\infty((x,y),(z,w)) = \max\{d(x,z), \rho(y,w)\}$$

is a metric space.

Theorem 10. *Let (X,d) and (Y,ρ) be metric spaces. In the metric space $X \times Y$ with the metric*

$$\sigma_1((x,y),(z,w)) = d(x,z) + \rho(y,w)$$

we have $(x_n, y_n) \to (x,y)$ if and only if $d(x_n, x) \to 0$ and $\rho(y_n, y) \to 0$.

Theorem 11. *The function $d(x,y) = |x-y|$ is a distance function on $\mathbb{R}$.*

Theorem 12. *Let X be a non-empty set. Let*

$$d(x,y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$$

for all $x, y \in X$. Then d is a distance function on X and we say that (X,d) is a discrete space.

Definition 13. Let X be a set. We say that distance functions d and ρ on X are *equivalent* if there exist $c_1 > 0$ and $c_2 > 0$ such that for all $x, y \in X$ we have

$$c_1 \cdot d(x,y) \leq \rho(x,y) \leq c_2 \cdot d(x,y).$$

Theorem 14. *Let d and ρ be equivalent distance functions on X. Then $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $\lim_{n\to\infty} \rho(x_n, x) = 0$.*

3 Open and closed sets, interior and closure

Definition 15. Let (X,d) be a metric space. We say that a set $A \subset X$ is *open* if

$$\forall_{x\in A} \quad \exists_{r_x>0} \quad B(x, r_x) \subset A.$$

Definition 16. Let (X, d) be a metric space. We say that a set $F \subset X$ is *closed* if the set $X \setminus F$ is open.

Theorem 17. *Union of any family of open sets is open. Intersection of any family of closed sets is closed.*

Union of a finite family of closed sets is closed. Intersection of a finite family of open sets is open.

Theorem 18. *An open ball is open. A closed ball is closed.*

Definition 19. The interior $Int(A)$ of a set A is the union of all open sets contained in A, i.e.,

$$Int(A) = \bigcup\{U : U \text{ is open}, U \subset A\}$$

Definition 20. The closure $Cl(A)$ of a set A is the intersection of all closed sets containing A, i.e.,

$$Cl(A) = \bigcap\{F : F \text{ is closed}, A \subset F\}$$

Theorem 21. $x \in Int(A)$ *if and only if*

$$\exists_{r>0} \quad B(x, r) \subset A$$

Proof. If $B(x, r) \subset A$ for some $r > 0$, then $x \in B(x, r)$ and $B(x, r)$ is open, so $x \in Int(A)$. On the other hand, if $x \in Int(A)$ then $x \in U$ for some open set U such that $U \subset A$, hence there exists $r > 0$ such that $B(x, r) \subset U \subset A$. $\square$

Theorem 22. *A set A is open if and only if $Int(A) = A$.*

A set F is closed if and only if $Cl(F) = F$.

Theorem 23. $Cl(A) = X \setminus Int(X \setminus A)$

Proof. ($A \subset F$ and F is closed) if and only if ($X \setminus F \subset X \setminus A$ and $X \setminus F$ is open). Since for all families of sets $\{F_\alpha\}$ we have $X \setminus \bigcap_\alpha F_\alpha = \bigcup_\alpha (X \setminus F_\alpha)$, we have $X \setminus Cl(A) = Int(X \setminus A)$. $\square$

Theorem 24. $Int(A) = X \setminus Cl(X \setminus A)$

Proof. ($U \subset A$ and U is open) if and only if ($X \setminus A \subset X \setminus U$ and $X \setminus U$ is closed). Since for all families of sets $\{U_\alpha\}$ we have $X \setminus \bigcup_\alpha U_\alpha = \bigcap_\alpha (X \setminus U_\alpha)$, we have $X \setminus Int(A) = Cl(X \setminus A)$. $\square$

Theorem 25. $x \in Cl(A)$ *if and only if*

$$\forall_{\epsilon>0} \quad \exists_{y \in A} \quad d(x, y) < \epsilon.$$

Theorem 26. *$x \in Cl(A)$ if and only if there exists a sequence $\{x_n\} \subset A$ such that $x_n \to x$.*

Theorem 27. *Let d and ρ be equivalent distance functions on X. Then a set F in (X, d) is closed if and only if F is closed in (X, ρ). A set A in (X, d) is open if and only if A is open in (X, ρ).*

Theorem 28. *Let (X, d) be a metric space and let $A \subset X$. Then a set U is open in the metric space $(A, d_{A\times A})$ if and only if $U = W \cap A$ for some open set W in (X, d).*

Proof. It is enough to observe that every open set is a union of open balls and every open ball in $(A, d_{A\times A})$ is the intersection of A and an open ball (of same center and radius) in (X, d). $\square$

4 Dense and nowhere dense sets

Definition 29. We say that a set $A \subset X$ is dense in X if $Cl(A) = X$.

Definition 30. We say that a set $A \subset X$ is nowhere dense if $Int(Cl(A)) = \emptyset$.

Theorem 31. *A set $A \subset X$ is nowhere dense if and only if $X \setminus Cl(A)$ is dense in X.*

Proof. $X \setminus Int(Cl(A)) = Cl(X \setminus Cl(A))$. Hence $Int(Cl(A)) = \emptyset$ if and only if $X = Cl(X \setminus Cl(A))$, i.e., $X \setminus Cl(A)$ is dense in X. $\square$

Theorem 32. *Let $A \subset X$ be nowhere dense. Then for every $x \in A$ for all $r > 0$ there exist $y \in B(x, r)$ and $\epsilon > 0$ such that $B(y, \epsilon) \subset X \setminus A$.*

Definition 33. We say that a set $A \subset X$ is a first category set if it is a countable union of nowhere dense sets, i.e., $A = \bigcup_{n=1}^{\infty} A_n$, where every A_n is nowhere dense.

Theorem 34. *Let (X, d) be a metric space. If $A \subset X$ is dense and $U \subset X$ is open, then $Cl(U) = Cl(U \cap A)$.*

Proof. Since $U \cap A \subset U$, we have $Cl(U \cap A) \subset Cl(U)$. On the other hand, let $x \in Cl(U)$ and let $\epsilon > 0$. Then there exists $u \in U$ such that $d(u, x) < \epsilon/2$. Since U is open, there exists $0 < r < \epsilon/2$ such that $B(u, r) \subset U$. Since A is dense in X there exists $a \in A$ such that $d(a, u) < r$, but then $a \in A \cap B(u, r) \subset A \cap U$ and $d(x, a) \leq d(x, u) + d(u, a) < \epsilon$. Hence, $x \in Cl(A \cap U)$. $\square$

5 Functions

Definition 35. Let (X, d) and (Y, ρ) be metric spaces. We say that a function $f : X \to Y$ is continuous at a point $x_0 \in X$ if

$$\forall_{\epsilon>0} \quad \exists_{\delta>0} \quad \forall_{x\in X} \quad d(x, x_0) < \delta \Rightarrow \rho(f(x), f(x_0)) < \epsilon.$$

We say that f is continuous, if f is continuous at every point of its domain. The set of all continuous functions from (X, d) to (Y, ρ) will be denoted by $C(X, Y)$, or $C((X, d), (Y, \rho))$.

Theorem 36. *A function $f : X \to Y$ is continuous at a point $x_0 \in X$ if and only if for every sequence $\{x_n\}$ such that $x_n \to x_0$ we have $f(x_n) \to f(x_0)$.*

Theorem 37. *A function $f : X \to Y$ is continuous at every point of X if and only if $f^{-1}(U)$ is open for every open set $U \subset Y$.*

A function $f : X \to Y$ is continuous at every point of X if and only if $f^{-1}(F)$ is closed for every closed set $F \subset Y$.

Definition 38. Let (X, d) and (Y, ρ) be metric spaces. We say that a function $f : X \to Y$ is a *Lipschitz function* with Lipschitz constant L if there exists $L > 0$ such that for all $x, y \in X$ we have

$$\rho(f(x), f(y)) \leq L \cdot d(x, y).$$

Theorem 39. *Every Lipschitz function is continuous.*

Definition 40. We say that $f : (X, d) \to (Y, \rho)$ is a homeomorphism if f is one-to-one (i.e., if $f(x) = f(y)$ we have $x = y$), $f(X) = Y$, and both f and its inverse, f^{-1} (i.e., $f^{-1}(y) = x$ if and only if $f(x) = y$) are continuous.

Theorem 41. *If $f : (X, d) \to (Y, \rho)$ is a homeomorphism then for all open $U \subset X$ and all closed $F \subset X$ we have: $f(U)$ is open and $f(F)$ is closed.*

Definition 42. We say that $f : (X, d) \to (Y, \rho)$ is bounded if $f(X)$ is a bounded set. The set of all bounded functions from (X, d) to (Y, ρ) will be denoted by $B(X, Y)$, or $B((X, d), (Y, \rho))$.

Theorem 43. *For all $f, g \in B((X, d), (Y, \rho))$ let*

$$\sigma(f, g) = \sup_{x\in X} \rho(f(x), g(x)).$$

Then σ is a distance function on $B((X, d), (Y, \rho))$.

6 Compactness

Recall that a sequence $\{x_n\}$ of elements of a set X is a function $x : \mathbb{N} \to X$, we use notation $x_n = x(n)$. Let $k : \mathbb{N} \to \mathbb{N}$ be an increasing function (i.e., $k(l) < k(m)$ for all $l < m$), then the composition $x \circ k : \mathbb{N} \to X$ is a subsequence of the sequence x, we use notation $x(n(k)) = x_{n_k}$. If $\lim_{n\to\infty} x_n = x$ then for every subsequence $\{x_{n_k}\}$ of the sequence $\{x_n\}$ we have $\lim_{k\to\infty} x_{n_k} = x$.

Definition 44. We say that a metric space (X, d) is compact if for every sequence $\{x_n\}$ of elements of X, i.e., $\{x_n\} \subset X$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in X$ such that $\lim_{k\to\infty} x_{n_k} = x$. In other words, a space is compact if and only if every sequence of its elements has a convergent subsequence.

Definition 45. We say that a set $F \subset X$ is compact if for every sequence $\{x_n\}$ of elements of F, i.e., $\{x_n\} \subset F$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x \in F$ such that $\lim_{k\to\infty} x_{n_k} = x$. In other words, a set F is compact if and only if every sequence of its elements has a convergent subsequence, converging to some element of F.

Note that a set $F \subset X$ is compact if and only if the metric space $(F, d|_{F\times F})$ is compact.

Theorem 46. *If $F \subset X$ is closed and X is compact, then F is compact.*

Theorem 47. *Consider $\mathbb{R}$ with the distance function $d(x, y) = |x - y|$, then $F \subset \mathbb{R}$ is compact if and only if F is closed and bounded.*

Theorem 48. *Let (X, d) be compact. Let $\{U_\alpha\}$ be an open cover of X, i.e., a family of open sets in X such that $X \subset \bigcup_\alpha U_\alpha$. Then there exists $\delta > 0$ such that for every $x \in X$ there exists α_x such that $B(x, \delta) \subset U_{\alpha_x}$. Such δ is called a* Lebesgue number *of the cover $\{U_\alpha\}$.*

Theorem 49. *$F \subset X$ is compact if and only if for every open cover of F, i.e., a family $\{U_\alpha\}$ of open sets in X such that $F \subset \bigcup_\alpha U_\alpha$, there exists a finite subcover, i.e., a family $U_{\alpha_1}, \dots, U_{\alpha_N}$ for some N such that $F \subset \bigcup_{n=1}^N U_{\alpha_n}$.*

Theorem 50. *If $F \subset X$ is compact, then F is closed and bounded.*

Definition 51. We say that a family $\{F_\alpha\}$ has a *finite intersection property* if every finite subfamily $\{F_{\alpha_1}, \dots, F_{\alpha_N}\}$ has non-empty intersection, i.e., $\bigcap_{n=1}^N F_{\alpha_n} \neq \emptyset$.

Theorem 52. *A space (X, d) is compact if and only if for every family $\{F_\alpha\}$ of closed sets in X that has a finite intersection property, we have $\bigcap_\alpha F_\alpha \neq \emptyset$.*

Theorem 53. *Let $f : (X, d) \to (Y, \rho)$ be continuous. If (X, d) is a compact metric space, then $f(X)$ is a compact set in (Y, ρ).*

Theorem 54. *Let X be the space of all bounded real-valued sequences, i.e., $\{x_n\} \in X$ if $\sup_{n\in\mathbb{N}} |x_n| < +\infty$. Let $d(\{x_n\},\{y_n\}) = \sup_{n\in\mathbb{N}} |x_n - y_n|$. Let $\theta = (0,0,\dots,)$ be a sequence such that $\theta_n = 0$ for all $n \in \mathbb{N}$. Then (X,d) is a metric space, and $B(\theta,1)$ is not compact.*

Theorem 55. *Let (X,d) and (Y,ρ) be compact spaces. Then $X \times Y$ with metric $\sigma((x,y),(z,w)) = d(x,z) + \rho(y,w)$ is compact.*

7 Completeness

Definition 56. Let (X,d) be a metric space. We say that a sequence $\{x_n\} \subset X$ is a *Cauchy sequence* if

$$\forall_{\epsilon>0} \quad \exists_{N\in\mathbb{N}} \quad \forall_{n,m>N} \quad d(x_n,x_m) < \epsilon.$$

Theorem 57. $\{x_n\} \subset X$ *is a* Cauchy sequence *if and only if*

$$\forall_{\epsilon>0} \quad \exists_{N\in\mathbb{N}} \quad \forall_{n,m\geq N} \quad d(x_n,x_m) < \epsilon.$$

$\{x_n\} \subset X$ *is a* Cauchy sequence *if and only if*

$$\forall_{\epsilon>0} \quad \exists_{N\in\mathbb{N}} \quad \forall_{n>N} \quad d(x_n,x_N) < \epsilon.$$

Theorem 58. *Every convergent sequence is a Cauchy sequence.*

Theorem 59. *Every Cauchy sequence is bounded, i.e., for every Cauchy sequence $\{x_n\}$ there exists an open ball $B(x,r)$ such that $x_n \in B(x,r)$ for all $n \in \mathbb{N}$.*

Theorem 60. *If $\{x_n\}$ is a Cauchy sequence, and there exists a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = x$, then $\lim_{n\to\infty} x_n = x$.*

Definition 61. We say that a metric space (X,d) is complete if every Cauchy sequence in X has a limit in X.

Theorem 62. *The metric space $(\mathbb{R},d)$ with $d(x,y) = |x-y|$ is complete.*

Theorem 63. *If (Y,ρ) is a complete metric space, then the metric space $(B((X,d),(Y,\rho)),\sigma)$ with $\sigma(f,g) = \sup_{x\in X} \rho(f(x),g(x))$ is complete.*

If (Y,ρ) is a complete metric space, then the metric space $(B(X,Y) \cap C(X,Y),\sigma)$ with $\sigma(f,g) = \sup_{x\in X} \rho(f(x),g(x))$ is complete.

Definition 64. Let (X,d) be a metric space. We say that a function $f : X \to X$ is a *contraction* if there exists $0 < M < 1$ such that for all $x,y \in X$ we have

$$d(f(x),f(y)) \leq M \cdot d(x,y).$$

Definition 65. We say that x is a *fixed point* of a function $f : X \to X$ if $f(x) = x$.

Theorem 66. *Let (X, d) be a complete metric space. If $f : X \to X$ is a contraction then there exists exactly one fixed point x_0 of f in X. For all $x \in X$ we have $x_0 = \lim_{n\to\infty} f^n(x)$, where $f^n = f \circ f \circ \ldots \circ f$ (the composition of f with itself, n times).*

Theorem 67. *A metric space (X, d) is complete if and only if for every family $\{F_n\}$ of closed sets in (X, d) such that $F_{n+1} \subset F_n$ and $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$ we have $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.*

Proof. Let (X, d) be complete. Let $\{F_n\}$ be a family of closed sets such that $F_{n+1} \subset F_n$ and $\lim_{n\to\infty} \operatorname{diam}(F_n) = 0$. Take $x_n \in F_n$. Then $\{x_n\}$ is a Cauchy sequence, since for all $\epsilon > 0$ there exists N such that $\operatorname{diam}(F_N) < \epsilon$ and $F_{N+k} \subset F_N$ for all $k \in \mathbb{N}$. From completeness of (X, d) we have $x_n \to x$. Since every F_n is closed, $x \in F_n$ for all n, and hence $x \in \bigcap_{n=1}^{\infty} F_n$.

On the other hand, let $\{x_n\}$ be a Cauchy sequence, let $F_n = Cl(\{x_n, x_{n+1}, \ldots\})$. Then $\operatorname{diam}(F_n) \to 0$, because $\{x_n\}$ is a Cauchy sequence - indeed: for every $\epsilon > 0$ there exists N such that $d(x_n, x_m) < \frac{1}{2}\epsilon$ for all $m, n \geq N$, hence $x_n \in B(x_N, \frac{1}{2}\epsilon)$ for all $n \geq N$ and hence $F_N \subset B(x_N, \frac{1}{2}\epsilon)$ and so $\operatorname{diam}(F_n) \leq \operatorname{diam}(B(x_N, \frac{1}{2}\epsilon)) \leq \epsilon$. Hence, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$, so let $x \in \bigcap_{n=1}^{\infty} F_n$. For every $\epsilon > 0$ there exists N such that $\operatorname{diam}(F_n) < \epsilon$ for all $n > N$, hence $d(x_n, x) < \epsilon$ for all $n > N$ and so $x_n \to x$ and (X, d) is complete. □

Theorem 68. *Let (X, d) be a complete metric space. Then for every countable family $\{A_n\}$ of nowhere dense sets, the set $X \setminus \bigcup_{n=1}^{\infty} A_n$ is dense.*

Proof. Let B_0 be an open ball in (X, d). By results from Section 4, since $X \setminus Cl(A_1)$ is dense in X, we have $Cl(B_0) = Cl(B_0 \cap (X \setminus Cl(A_1))) = Cl(B_0 \setminus Cl(A_1))$. It follows that $B_0 \setminus Cl(A_1)$ is open and non-empty. There exists an open ball B_1 in X such that $Cl(B_1) \subset B_0 \setminus Cl(A_1)$ and $\operatorname{diam}(B_1) < 1$. Continuing this process, we obtain a sequence of open balls $\{B_n\}$ in X such that $\operatorname{diam}(B_n) < \frac{1}{n}$ and $Cl(B_n) \subset B_{n-1} \setminus Cl(A_n)$. The family of closed sets $\{Cl(B_n)\}$ has non-empty intersection, hence there exists $x \in X$ such that

$$x \in \bigcap_{n=1}^{\infty} (B_{n-1} \setminus Cl(A_n)) \subset B_0 \cap \bigcap_{n=1}^{\infty} (X \setminus Cl(A_n)) = B_0 \cap (X \setminus \bigcup_{n=1}^{\infty} Cl(A_n)).$$

□

8 Topology, separable spaces, connected spaces

Definition 69. Let X be a set. A *topology* on X is a family $\mathcal{T}$ of subsets of X such that

1. $\emptyset, X \in \mathcal{T}$,

2. if $U, V \in \mathcal{T}$ then $U \cap V \in \mathcal{T}$,

3. if $U_s \in \mathcal{T}$ for all $s \in S$ (where S is any set of indices), then $\bigcup_{s \in S} U_s \in \mathcal{T}$.

A pair $(X, \mathcal{T})$ is called a *topological space*. Sets from the family $\mathcal{T}$ are called *open sets* in $(X, \mathcal{T})$.

Definition 70. In a topological space $(X, \mathcal{T})$ we define:

closed sets as in Definition 16

interior of a set as in Definition 19

closure of a set as in Definition 20

dense set as in Definition 29.

Definition 71. Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. We say that a function $f : X \to Y$ is continuous if it satisfies conditions in Theorem 37.

Definition 72. Let $(X, \mathcal{T})$ be a topological space and let $M \subset X$. Then the family $\mathcal{T}_{M,X} = \{U \cap M, U \in \mathcal{T}\}$ is a topology on M. We call it the *induced topology* (or the topology induced from X).

Theorem 73. *Theorems: 17, 22, 23, 24 remain true in topological spaces.*

Theorem 74. *The family of all open sets in a metric space (X, d) is a topology on X. We will call this topology the* metric topology *on (X, d).*

Definition 75. A family $\mathcal{B} \subset \mathcal{T}$ is called a base of topology $\mathcal{T}$ if every non-empty set that belongs to $\mathcal{T}$ is a union of some sets from $\mathcal{B}$.

Theorem 76. *A family of open balls in a metric space (X, d) is a base of the metric topology.*

Definition 77. A topological space $(X, \mathcal{T})$ is called *second countable* if it has a countable base, i.e., there exists a sequence of open sets $\{B_n\}$ such that every open sets is a union of some sets from $\{B_n\}$.

Definition 78. A topological space $(X, \mathcal{T})$ is called *separable* if there exists a countable dense set in $(X, \mathcal{T})$.

Theorem 79. *Every second countable space is separable.*

A metric space is second countable if and only if it is separable.

Theorem 80. *A discrete metric space (X, d) is separable if and only if the set X has countably many elements.*

Definition 81. A topological space $(X, \mathcal{T})$ is called *connected* if it cannot be presented as union of two open, non-empty, disjoint sets.

In other words: $(X, \mathcal{T})$ is connected if and only if for all $U, V \in \mathcal{T}$ such that $X = U \cup V$ we have: $U = \emptyset$, or $V = \emptyset$, or $U \cap V \neq \emptyset$.

Theorem 82. *If $(X, \mathcal{T})$ is a connected topological space and $f : (X, \mathcal{T}) \to (Y, \mathcal{T}_Y)$ is continuous (where $(Y, \mathcal{T}_Y)$ is a topological space), then $(f(X), \mathcal{T}_{f(X),Y})$ is connected.*

Note that $f(X)$ is a subset of Y and $\mathcal{T}_{f(X),Y}$ is the topology induced on it from Y.

Shortly: continuous image of a connected space is connected.

Definition 83. A path in a topological space $(X, \mathcal{T})$ is a continuous function $c : [0, 1] \to X$, where $[0, 1]$ is considered with the topology induced by metric $d(x, y) = |x - y|$.

Definition 84. We say that a topological space $(X, \mathcal{T})$ is *path-connected* if for every $x, y \in X$ there exists a path $c : [0, 1] \to X$ such that $c(0) = x$ and $c(1) = y$.

Theorem 85. *Every path-connected space is connected.*

Definition (Ideal of sets): $\mathcal{J}$ is an ideal of sets in X if:

$$\left.\begin{array}{l} 1)\ \emptyset \in \mathcal{J} \\ 2)\ \forall A,B \in \mathcal{J},\ A \cup B \in \mathcal{J} \\ 3)\ \forall A \subseteq X, \forall C \in \mathcal{J},\ A \cap C \in \mathcal{J} \end{array}\right\} \underset{\Longleftrightarrow}{\text{or equivalently}} \left\{\begin{array}{l} 1)\ A \in \mathcal{J}, B \subseteq A \Rightarrow B \in \mathcal{J} \\ 2)\ A, B \in \mathcal{J} \Rightarrow A \cup B \in \mathcal{J} \end{array}\right.$$

Example: Let (X, d) be a metric space, $X_F = \{F \subseteq X \,/\, F \text{ compact}\}$

Let $F_\varepsilon = \bigcup_{x \in F} B(x, \varepsilon)$

We define the distance $\sigma(F_1, F_2) = \inf_{\varepsilon > 0} \{F_1 \subseteq F_{2,\varepsilon} \wedge F_2 \subseteq F_{1,\varepsilon}\}$

Then, we have $\sigma(F_1, F_2) = \max \left\{ \sup_{x \in F_1} \left\{ \inf_{y \in F_2} \{d(x,y)\} \right\}, \sup_{y \in F_2} \left\{ \inf_{x \in F_1} \{d(x,y)\} \right\} \right\}$ and (X_F, σ) is a metric space.

1. Topological spaces

Definition 1.1. Topological space is an ordered pair (X, T), where X is a non-empty set and T is a family of its subsets, satisfying the following conditions:

1. $\emptyset, X \in T$
2. if $U, V \in T$, then $U \cap V \in T$;
3. if $\{U_s : s \in S\} \subset T$, then $\bigcup_{s \in S} U_s \in T$

The set X is then called a space, elements of X are called points of X (or elements of X), the family T is called a topology, and elements of the family T are called open sets.

Remarks:

0. S in property 3 above is a set of indices - it may be $\mathbb{N}$ (then U_s is a sequence), $\mathbb{R}$ (then U_s is a family parametrized by one real parameter), a set of points (then we have a family of sets U_x, where $x \in X$), or any other set. We use these indices mainly to easily denote the union of all sets from the family: $\bigcup_{s \in S} U_s$ or their intersection $\bigcap_{s \in S} U_s$.

1. Whenever it will not be misleading, we will write X instead of (X, T).

2. Using induction, one can easily conclude from 2, that an intersection of any finite family of open sets is an open set.

3. It follows directly from the properties of open sets in metric spaces, that if (X, d) is a metric space, then the family of all open subsets of X is a topology - we say that this topology is induced (or generated) by the metric d. In further parts of the lecture, whenever we consider a metric space, we consider it as a topological space with the topology induced by its metric. If for a topological space (X, T) there exists a metric d such that the family of all open subsets of the metric space (X, d) coincides with T, then we say that (X, T) a metrizable space and that d generates topology T. In particular, for $X = \mathbb{R}^m$, $m = 1, 2, \ldots$ (or $X = [0, 1]$), the topology generated by the natural metric is called the natural topology of X.

Definition 1.2. Let (X, T) be a topological space. A neighbourhood of a point $x \in X$ is any set $U \in T$ such that $x \in U$.

Additional information:

Some authors use the following definition: A neighbourhood of a point $x \in X$ is any set U, such that there exists $V \in T$, such that $x \in V \subset U$.

Theorem 1.3. *Let (X, T) be a topological space. A set W is open (i.e., $W \in T$) if and only if for every point $x \in W$ there exists a neighbourhood V_x of x such that $V_x \subset W$.*

Proof. If W is open and $x \in W$ then W is a neighbourhood of x. On the other hand, if every x has its neighbourhood $V_x \subset W$, then $\bigcup_{x \in W} V_x = W$ and W is open as union of open sets. $\square$

Definition 1.4. Let (X, T) be a topological space.

a) A set $F \subset X$ is called a closed set if $X \setminus F \in T$.

b) A set $K \subset X$ is called clopen set, if it is both open and closed set.

Theorem 1.5. *Let (X, T) be a topological space. Then*

(D_1) Sets $\emptyset$ and X are closed

(D_2) Union of two (or any finite number) of closed sets is a closed set

(D_3) Intersection of any family of closed sets is a closed set.

Proof. This follows immediately from definition of closed set and properties 1-3 of open sets in Definition 1.1. Some authors, because of this duality between open and closed sets, consider a closed set as the fundamental object of topology, so they take properties 1-3 from Theorem 1.5. as the definition of a closed set (analogous to Definition 1.1.).

$\square$

Remark. We also have the following property:

(D_4) If $U \in T$ then $X \setminus U$ is a closed set, because $X \setminus (X \setminus U) = U \in T$.

Definition 1.6. A family of open sets $B \subset T$ is called a base of topological space (X, T) if every non-empty open set in X is a union of some subfamily of B.

Theorem 1.7. *Let (X, T) be a topological space. A family $B \subset P(X)$ is a base of (X,T) if and only if $B \subset T$ and for every $x \in X$ and every neighbourhood V_x of x there exists a set $U_x \in B$ such that $x \in U_x \subset V_x$.*

Proof. If for every neighbourhood V_x of x there exists a set $U_x \in B$ such that $x \in U_x \subset V_x$, then you can modify the proof of Theorem 1.3., taking $B \ni U_x \subset V_x$ to prove that any open set W is union $\bigcup_{x \in W} U_x$.

On the other hand, if every open set is union of sets from the base B, then every neighbourhood $V_x = \bigcup_{s \in S} U_s$ where $U_s \in B$ for all $s \in S$. Then we have $x \in U_{s_0}$ for some $s_0 \in S$ and so $x \in U_{s_0} \subset V_x$. □

Theorem 1.8. *Every base B of a topological space (X,T) satisfies the following conditions:*

(B_1) for all $U, V \in B$ and for every point $x \in U \cap V$ there exists a set $W \in B$ such that $x \in W \subset U \cap V$

(B_2) for every $x \in X$ there exists $H \in B$ such that $x \in H$.

Proof of B_1: Note that $U \cap V$ is open (as intersection of two open sets) and by Theorem 1.7 there exists $W \in B$ such that $x \in W \subset (U \cap V)$.

Proof of B_2: We have $x \in X$, but X is open so it is a union of some sets from B. So there is $U \in B$ such that $x \in U \subset X$ (it may happen that $U = X$, but only if the set X is one of the sets of the base B - it doesn't have to be so)

Definition 1.9. Let (X, T) be a topological space and let $x \in X$. A *base of* (X, T) *at* x is any family $B(x)$ of neighbourhoods of x satisfying the following condition: for every neighbourhood V of x there exists $U \in B(x)$ such that $U \subset V$.

Obviously, for any $x \in X$ the family $\{U \in T : x \in U\}$ is a base of (X, T) at x.

Theorem 1.10. *If B is a base of the topological space (X, T), then the family $B(x) = \{U \in B : x \in U\}$ is a base of (X, T) at x.*

Proof. Let V be a neighbourhood of x, take $V_x = V$ in Theorem 1.7., from this theorem we get $U_x \in B$ such that $x \in U_x \subset V$. Clearly, $U_x \in B(x)$ with $B(x)$ defined above. □

Definition 1.11. A family $\{B(x) : x \in X\}$, where $B(x)$ is a base of (X, T) at x is called a neighbourhood system of (X, T).

Theorem 1.12. *A neighbourhood system $B(x) : x \in X$ of the topological space (X,T) has the following properties:*

(PO_1) for all $x \in X$ we have $B(x) \neq \emptyset$ and for all $U \in B(x)$ we have $x \in U$.

(PO_2) if $x \in V \in B(y)$, then there exists $U \in B(x)$ such that $U \subset V$.

(PO_3) for all $W, V \in B(x)$ there exists $U \in B(x)$ such that $U \subset V \cap W$.

Proof. For PO_1 use the fact that X is a neighbourhood of x and Definition 1.9. to show $B(x) \neq \emptyset$. The second statement in PO_1 is trivial, as any neighbourhood of x contains x by definition of neighbourhood.

For PO_2 use the fact that such V is a neighbourhood of x and Definition 1.9.

For PO_3 use the fact that such $V \cap W$ is neighbourhood of x and Definition 1.9.

$\square$

Definition 1.13. a) We say that a topological space (X, T) is a first countable space, if for every point $x \in X$ there exists a countable base of (X, T) at x.

b) We say that a topological space (X, T) is a second countable space, if there exists a countable base of (X, T).

Definition 1.14. Let (X, T) be a topological space. Interior of a set $A \subset X$ is the union of all open sets contained in A (hence, it is the largest open set contained in A), it is denoted as $Int(A)$.

Theorem 1.15. *Let (X, T) be a topological space and let $A, B \subset X$. Then:*

a) $Int(A)$ is an open set;

b) $A \in T$ if and only if $A = Int(A)$;

c) $x \in Int(A)$ if and only if there exists a neighbourhood U of x such that $U \subset A$;

d) if $A \subset B$ then $Int(A) \subset Int(B)$.

Proof. a) follows from the fact that union of open sets in open

b) if $A = Int(A)$ then A is open because $Int(A)$ open. On the other hand, if A is open, then A is one of all open sets contained in A, so $Int(A)$, which is the union of such sets, must be equal to A.

c) if $x \in Int(A)$ then we can take $Int(A)$ as the neighbourhood U. On the other hand, if there exists such neighbourhood U then $U \subset Int(A)$ by definition of $Int(A)$, and we have $x \in U \subset Int(A)$ so $x \in Int(A)$.

d) follows from Definition 1.14 and the fact that if for an open set U we have $U \subset A$ and $A \subset B$ then $U \subset B$. Taking unions of such sets U also preserves this inclusion.

□

Theorem 1.16. *Taking the interior of a set in a topological space (X,T) has the following properties:*

(W_1) Int(X)=X;

(W_2) for all $A \subset X$ we have $Int(A) \subset A$;

(W_3) for all $A, B \subset X$ we have $Int(A \cap B) = Int(A) \cap Int(B)$;

(W_4) for all $A \subset X$, $Int(Int(A)) = Int(A)$.

Proof. W_1 follows from the fact that X is open and X is the only set in X that includes X.

W_2 follows from Definition 1.14 and the fact that union of sets contained in A is still contained in A.

For W_3: if U is an open set and $U \subset A \cap B$ then $U \subset A$ and $U \subset B$. By taking union of such open sets U we obtain $Int(A \cap B) \subset (IntA) \cap (IntB)$.

On the other hand, $Int(A) \cap Int(B)$ is open (as intersection of two open sets) and contained in $A \cap B$, so $Int(A) \cap Int(B) \subset Int(A \cap B)$.

For W_4: $Int(A)$ is open, so use Theorem 1.15b.

□

Definition 1.17. Let (X, T) be a topological space. The closure of a set

$A \subset X$ is the intersection of all closed sets containing A (hence, it is the smallest closed set containing A), it is denoted as $cl(A)$.

Remark. It is common to write $\bar{A}$, instead of $cl(A)$.

Theorem 1.18. *Let (X,T) be a topological space and let $A, B \subset X$. Then:*

a) $cl(A)$ is a closed set;

b) A is a closed set if and only if $A = cl(A)$;

c) $x \in cl(A)$ if and only if for every neighbourhood U of x we have: $U \cap A \neq \emptyset$

d) if $A \subset B \subset X$, then $cl(A) \subset cl(B)$.

Proofs of a,b,d are analogous to Theorem 1.15.

For c, Suppose that $x \in cl(A)$ and $U \cap A = \emptyset$ for some neighbourhood U of x. Then $x \notin A$ (because $x \in U$ and $U \cap A = \emptyset$). The set $F = X \setminus U$ is closed and contains A. As $x \in cl(A)$, x belongs to F (because it belongs to intersection of all closed sets containing A). But then $x \in X \setminus U$ contradicts the assumption that U is a neighbourhood of x.

On the other hand, if for every neighbourhood U of x we have $U \cap A \neq \emptyset$, let F be a closed set such that $A \subset F$ - we will show that $x \in F$. Suppose that $x \notin F$, then $W = X \setminus F$ is an open neighbourhood of x and $W \cap A \neq \emptyset$ by the assumption. But $A \subset F$ implies $W \subset X \setminus A$ so $W \cap A = \emptyset$ - a contradiction.

Theorem 1.19. *If U is an open set and $A \cap U = \emptyset$, then $cl(A) \cap U = \emptyset$.*

Suppose U is open, $A \cap U = \emptyset$ and $x \in cl(A) \cap U$. Then U is a neighbourhood of x so we can repeat the argument from proof of Theorem 1.18c.

Theorem 1.20. *Let (X, T) be a topological space. Then for all $A \subset X$ we have $cl(A) = X \setminus Int(X \setminus A)$.*

Proof. $X \setminus Int(X \setminus A)$ is closed, as a complement of an open set. It also contains A, so $cl(A) \subset X \setminus Int(X \setminus A)$. On the other hand, if $x \in X \setminus Int(X \setminus A)$ and W is a neighbourhood of x, then $W \cap A \neq \emptyset$ (otherwise x would have a neighbourhood contained in $X \setminus A$, so we would have $x \in Int(X \setminus A)$ by Theorem 1.15c) - and we obtain $x \in cl(A)$ by Theorem 1.18c.

□

Theorem 1.21. *Taking the closure of a set in a topological space (X, T) has the following properties:*

d_1 *$cl(\emptyset) = \emptyset$;*

d_2 *for all $A \subset X$ we have $A \subset cl(A)$;*

d_3 *for all $A, B \subset X$ we have $cl(A \cup B) = cl(A) \cup cl(B)$;*

d_4 *for all $A \subset X$ we have $cl(cl(A)) = cl(A)$.*

Proof is similar to Theorem 1.16.

Theorem 1.22. *Let (X, T) be a topological space. For all $A \subset X$ we have $Int(A) = X \setminus cl(X \setminus A)$.*

Proof is similar to Theorem 1.20.

2. Various ways of generating a topology

Theorem 2.1. *Let X be a non-empty set and let D be a family of its subsets, satisfying the following conditions (see Theorem 1.5):*

(D_1) $\emptyset, X \in D$

(D_2) if $A, B \in D$, then $A \cup B \in D$;

(D_3) if $A_s \in D$ for all $s \in S$, then $\bigcap_{s\in S} A_s \in D$.

Then the family $T_D = \{X \setminus F : F \in D\}$ satisfies conditions for being a topology (we call T_D the topology generated by the family of closed sets *D). The family of closed sets in (X, T_D) coincides with the family D.*

Proof. do it yourself. It is easy to show that D_1-D_3 for $A, B, A_s \in D$ are equivalent to conditions 1-3 in Definition 1.1 for $X \setminus A$, $X \setminus B$, $X \setminus A_s$. $\square$

Theorem 2.2. *Let X be a non-empty set. Consider an operator that assigns to every set $A \subset X$ set $\overline{A} \subset X$ such that*

(d_1) $\overline{\emptyset} = \emptyset$;

(d_2) for every $A \subset X$ we have $A \subset \overline{A}$;

(d_3) for all $A, B \subset X$ we have $\overline{A \cup B} = \overline{A} \cup \overline{B}$;

(d_4) for every $A \subset X$ we have $\overline{\overline{A}} = \bar{A}$.

Then the family of sets $T_d = \{X \setminus F : F = \bar{F}\}$ satisfies conditions for being a topology (we call T_D the topology generated by the closure operator*). In the topological space (X, T_d) for all sets $A \subset X$ we have $cl(A) = \overline{A}$.*

Proof. It is enough to prove that if d_1-d_4 hold, then the family $D = \{A \subset X : \overline{A} = A\}$ (i.e., the family D of sets A such that $\overline{A} = A$) satisfies D_1-D_3

in Theorem 2.1.. Indeed, we have $\emptyset \in D$ by d_1, and $X \in D$ by d_2 (and the fact that operator in Theorem 2.2. acts on 2^X). For D_2, let $A = \overline{A}$, $B = \overline{B}$. Then by d_2 and d_3 we have $A \cup B \subset \overline{A \cup B} = A \cup B$, so $A \cup B \in D$.

For D_3, first we show that if $A \subset B$, then $\overline{A} \subset \overline{B}$. Indeed, for $A \subset B$ by d_3 we have $\overline{B} = \overline{A \cup B} = \overline{A} \cup \overline{B}$ and it follows that $\overline{A} \subset \overline{B}$.

Now we prove D_3. Let $A_s \in D$ for $s \in S$. We have for all $t \in S$: $\bigcap_{s\in S} A_s \subset A_t$, so for all $t \in S$ we have $\overline{\bigcap_{s\in S} A_s} \subset \overline{A_t} = A_t$ (the last equality holds by $A_t \in D$). But because it holds for all $t \in S$, we have $\overline{\bigcap_{s\in S} A_s} \subset \bigcap_{t\in S} A_t = \bigcap_{s\in S} A_s$ which, together with d_2 gives D_3.

$\square$

Theorem 2.3. *Let X be a non-empty set and let B be a family of its subsets, satisfying the following conditions (see Theorem 1.8):*

(B_1) for all sets $U, V \in B$, for every $x \in U \cap V$ there exists a set $W \in B$ such that $x \in W \subset U \cap V$;

(B_2) for every $x \in X$ there exists $H \in B$ such that $x \in H$.

Denote by T_B the family of all sets which are unions of some families of sets from B (i.e., $A \in T_B \Leftrightarrow \exists_{B_0 \subset B}\ A = \bigcup_{U\in B_0} U$). Then the family of set T_B satisfies conditions for being a topology (we call it the topology generated by the base *B). The family B is a base of (X, T_B).*

Proof. Consider $\emptyset$ as the union of an empty family of sets from B, then $\emptyset \in T_B$. $X \in T_B$ by B_2, because union of H_x obtained for all $x \in X$ by (B_2) is X.

For $U, V \in T_B$ we have $U \cap V = \cup_{x\in U\cap V} W_x$ where $W_x \in B$ exists by (B_1).

For $\{U_s, s \in S\}$ where every U_s is union of some sets from B, the union $\bigcup_{s\in S} U_s$ is clearly also a union of some sets from B (because union of unions of sets is a union of sets).

$\square$

Theorem 2.4. *Let X be a set and let $\{B(x) : x \in X\}$ be a class of families of subsets of X satisfying the following conditions (see Theorem 1.12):*

(PO_1) for every $x \in X$ we have $B(x) \neq \emptyset$ and for all $U \in B(x)$ we have $x \in U$;

(PO_2) if $x \in V \in B(y)$, then there exists $U \in B(x)$ such that $U \subset V$;

(PO_3) for all $V, W \in B(x)$ there exists $U \in B(x)$ such that $x \in U \subset V \cap W$.

Denote by T_{ot} the family of all sets which are unions of some families of sets from $\bigcup_{x\in X} B(x)$. Then the family of sets T_{ot} satisfies conditions for being a topology (we call it the topology generated by the neighbourhood system $\{B(x) : x \in X\}$*). The family $\{B(x) : x \in X\}$ is a neighbourhood system in the topological space (X, T_{ot}).*

Proof. do it yourself. The main difficulty is to prove that if $U, V \in T_{ot}$ then $U \cap V \in T_{ot}$. But if $U = \bigcup_{s\in S} U_s$ and $V = \bigcup_{t\in S} V_t$ where $U_s, V_t \in \bigcup_{x\in X} B(x)$, then $U \cap V = \bigcup_{s\in S} U_s \cap \bigcup_{t\in S} V_t = \bigcup_{s,t\in S} U_s \cap V_t$. Then each $U_s \cap V_t$ is either empty, or for every $x_{st} \in U_s \cap V_t$ there are x_s, x_t such that $U_s \in B(x_s)$ and $V_t \in B(x_t)$. By (PO_2) and (PO_3), we can find $W_{st} \in B(x)$ such that $x \in W_{st}$ and $U \cap V = \bigcup_{s\in S, t\in T} W_{st} \in T_{ot}$.

$\square$

Theorem 2.5. *Let (X,T) be a topological space and let $\emptyset \neq M \subset X$. Then the family $T_M = \{M \cap U : U \in T\}$ satisfies the conditions for being a topology in M, i.e. $(M.T_M)$ is a topological space (we call T_M the* subspace topology *or the* topology induced by the topology of (X,T)*).*

Proof. do it yourself. If a family T of sets satisfies conditions in Definition 1.1., then sets of the form $U \cap M$, where $U \in T$ satisfy analogous conditions (with $X = M$). To prove it, use $(U \cap M) \cap (V \cap M) = (U \cap V) \cap M$ and a similar formula for union in condition 3. from Definition 1.1.

□

Theorem 2.6. *Let (X,T) be a topological space, let (M,T_M) be its subspace and let $A \subset M$. The closure $cl_M(A)$ of a set A in (M,T_M) is given by the formula $cl_M(A) = cl(A) \cap M$.*

Proof. Let $K \subset X$ be closed in (X,T). Then $X \setminus K \in T$ and $M \setminus (M \cap K) = M \cap (M \setminus K) = M \cap (X \setminus K) \in T_M$ by Theorem 2.5. This argument proves that if K is closed in (X,T), then $M \cap K$ is closed in (M,T_M). On the other hand, if a set K_M is closed in (M,T_M), then $K_M = M \cap K$, where K is closed in X. To see that, observe that $K_M = M \setminus V_M$, where $V_M = V \cap M$ and $V \in T$. We can write $K = M \setminus V_M = M \cap (X \setminus V_M) = M \cap (X \setminus (M \cap V)) = M \cap ((X \setminus M) \cup (X \setminus V)) = M \cap (X \setminus V)$ and $X \setminus V = K$ is closed.

Recall that $cl(A) = \bigcap_{s \in S} F_s$ where every F_s is closed in (X,T) and satisfies $A \subset F_s$. Then every closed set in M that contains A is of the form $F_s \cap M$. So $cl_M(A) = \bigcap_{s \in S} F_s \cap M = (\bigcap_{s \in S} F_s) \cap M = cl(A) \cap M$.

□

Definition 2.7. Let X be a non-empty set and let T_1, T_2 be topologies in X. We say that topology T_1 is coarser than T_2 (or, equivalently, that T_2 is finer than T_1) if $T_1 \subset T_2$. Among all topologies that can be defined for a non-empty set X there exists the coarsest one (indiscrete: $\{\emptyset, X\}$) and the finest one (discrete: 2^X).

Example (Compact – open Topology). Let $(X, \tau_X), (Y, \tau_Y)$ be topological space. Let $F(X, Y) = \{f : X \to Y \mid f \text{ continuous}\}$

Let $K \subseteq X$ compact, $U \in \tau_Y$, $W_{K,U} = \{f \in F(X,Y) \mid f(K) \subseteq U\}$

An open set in $F(X, Y) := \bigcap_{i=1}^{n} W_{K_i, U_i}$

- $W_{\{x\}, \emptyset} = \emptyset$
- $X = W_{\{x\}, Y}$???

Exercise: Compact-open Topology is a topology.

3. Various types of sets

Definition 3.1. Let (X, T) be a topological space and let $A \subset X$. Then the set $Fr(A) = cl(A) \cap cl(X \setminus A)$ is called the *boundary* of the set A.

Example 3.I. In $\mathbb{R}$ we have $Fr(\mathbb{Q}) = \mathbb{R}$.

(Here we consider the natural topology of $\mathbb{R}$, because we didn't write otherwise).

Theorem 3.2. *Let (X, T) be a topological space and let $A \subset X$. Then $Fr(A) = cl(A) \setminus Int(A)$.*

Proof. Recall Theorem 1.22, which said that $Int(A) = X \setminus cl(X \setminus A)$. So we have $X \setminus Int(A) = cl(X \setminus A)$ (you can also get the same writing Theorem 1.20 for $X \setminus A$ instead of A). Using this we obtain $Fr(A) = cl(A) \cap cl(X \setminus A) = cl(A) \cap (X \setminus Int(A)) = cl(A) \setminus Int(A)$. $\square$

Theorem 3.3. *Let (X, T) be a topological space and let $A \subset X$. Then $x \in Fr(A)$ if and only if for every neighbourhood U of x we have $U \cap A \neq \emptyset \neq U \setminus A$.*

Proof. By Theorem 3.2., $x \in Fr(A)$ if and only if $x \in cl(A)$ and $x \notin Int(A)$. By Theorem 1.18c., we have $x \in cl(A)$ if and only if $U \cap A \neq \emptyset$ for every neighbourhood U of x. This proves the first inequality.

On the other hand, by Theorem 1.15c, we have $x \in Int(A)$ if and only if there exists a neighbourhood W of x such that $W \subset A$. By elementary logic:

$x \notin Int(A)$ if and only if for every neigbourhood U of x we have $U \not\subset A$. Since $U \cap A \neq \emptyset$, condition $U \not\subset A$ means that $U \setminus A \neq \emptyset$.

(If the last part is hard to understand, you can try drawing a simple picture of A and U. We needed $U \cap A \neq \emptyset$ because for disjoint U and A we would have $U \not\subset A$ but $U \setminus A = \emptyset$).

$\square$

Note that the main idea of this statement is often written in a more convenient form:

Theorem 3.3'. Let (X, T) be a topological space and let $A \subset X$. Then the following statements are equivalent:

(i) $x \in Fr(A)$.

(ii) for every base $B(x)$ and every set $U \in B(x)$ we have $U \cap A \neq \emptyset \neq U \setminus A$.

(iii) there exists a base $B(x)$ such that for every set $U \in B(x)$ we have $U \cap A \neq \emptyset \neq U \setminus A$.

(iv) for every neighbourhood U of x we have $U \cap A \neq \emptyset \neq U \setminus A$.

Proof (similar to the proof of Theorem 3.3.) - do it yourself. model: Theorem 3.6.

Theorem 3.4. *For every topological space* (X, T) *and for all sets* $A \subset X$ *all the following statements hold:*

(a) $Int(A) = A \setminus Fr(A)$.

(b) $cl(A) = A \cup Fr(A)$.

(c) $Fr(A) = Fr(X \setminus A)$.

(d) A is open if and only if $Fr(A) = cl(A) \setminus A$.

(e) A is closed if and only if $Fr(A) = A \setminus Int(A)$.

(f) A is clopen (i.e. A is both open and closed) if and only if $Fr(A) = \emptyset$.

I'm leaving the proof of this to you as an exercise. You don't need to consider particular points and neighbourhoods for this, just some set operations should be enough.

Definition 3.5. Let (X, T) be a topological space and let $A \subset X$. A point $x \in X$ is called an *accumulation point of the set* A if $x \in cl(A \setminus \{x\})$. The set of all accumulation points of A is called the *derived set* of A, we denote it by A^d. If $x \in X^d$ we will call x an accumulation point.

Theorem 3.6. *Let* (X, T) *be a topological space and let* $A \subset X$. *Then the following statements are equivalent:*

(i) $x \in A^d$.

(ii) for every base $B(x)$ *and for every set* $U \in B(x)$ *there exists* $y \in A$ *such that* $y \neq x$ *and* $y \in U$.

(iii) there exists a base $B(x)$ *such that for every set* $U \in B(x)$ *there exists* $y \in A$ *such that* $y \neq x$ *and* $y \in U$.

(iv) for every neighbourhood U *of* x *there exists* $y \in A$ *such that* $y \neq x$ *and* $y \in U$.

Proof. I will only prove (i) $\Leftrightarrow$ (iv) in detail, because this is the most useful of the above. The other conditions are easy to prove, but here are some hints if you need them.

In general, if you have a condition that holds for every neighbourhood of x, then it clearly holds for every set from any base at the point x.

To prove (iii) $\Rightarrow$ (ii): if there exists a base $B(x)$ such that for every $U \in B(x)$ there exists $y \in A$ such that $y \neq x$ and $y \in U$, then for any neighbourhood W of x you can take $U \in B(x)$ small enough (so that $U \subset W$) and find y as in (iii).

Now about (i) $\Leftrightarrow$ (iv): by Definition 3.5 and Theorem 1.17c $x \in A^d \Leftrightarrow x \in cl(A \setminus \{x\})$ if and only if for every neighbourhood U of x we have $U \cap (A \setminus \{x\}) \neq \emptyset$. And $U \cap (A \setminus \{x\}) \neq \emptyset$ if and only if there exists $y \in U \cap (A \setminus \{x\})$ and this y satisfies all these conditions: $y \in A$, $y \neq x$ and $y \in U$.

$\square$

Theorem 3.7. *The operation of obtaining the derived set has the following properties.*

(a) $cl(A) = A \cup A^d$.

(b) if $A \subset B$ *then* $A^d \subset B^d$.

(c) $(A \cup B)^d = A^d \cup B^d$

(d) $\bigcup_{s \in S} A_s^d \subset (\bigcup_{s \in S} A_s)^d$.

Proof. For volunteers.

The most useful here is probably (a). To prove it: $A \subset cl(A)$ is obvious, $A^d \subset cl(A)$ follows from Theorem 3.6.(iv), which is a stronger condition than

Theorem 1.18c. This gives $cl(A) \supset A \cup A^d$.

On the other hand, suppose that $x \in cl(A)$.

If $x \in A$ then clearly $x \in A \cup A^d$.

If $x \notin A$, then from Theorem 1.18c for every neighbourhood U of x we have $U \cap A \neq \emptyset$, so there exists $y \in U \cap A$. Since $x \notin A$ and $y \in A \cap U$, we have $y \neq x$ and by Theorem 3.6 (iv) $x \in A^d$ and so $x \in A \cup A^d$.

$\square$

Theorem 3.8. *Let (X, ρ) be a metric metric space. Then $x \in A^d$ if and only if x is the limit of a sequence $\{x_n\} \subset A \setminus \{x\}$.*

Proof. You should have seen something similar in the class about metric spaces. To translate what you know from there to the language of topology: use (iv) in Theorem 3.6. with the ball $B(x, \frac{1}{n})$ as the neighbourhood U, obtaining $y_n \in A \setminus \{x\}$. Then $y_n \to x$.

$\square$

Definition 3.9. (a) A set A is called *dense in itself* if $A \subset A^d$.

(b) A set A is called *perfect* if $A = A^d$.

Example 3.III. (Cantor set)

Consider a segment $I = [0, 1]$ with its natural topology. Cantor set is the set of all numbers from I that can be written in the ternary system without using the number 1. In other words: Cantor set is the set of all numbers $c \in I$ of the following form:

$$c = \frac{c_1}{3} + \frac{c_2}{3^2} + \frac{c_3}{3^3} + \dots,$$

where $c_n \in \{0, 2\}$ for $n = 1, 2, \ldots$.

Cantor set has the following geometric presentation. Divide the segment I into three parts of equal length, between points $\frac{1}{3}$ and $\frac{2}{3}$ and remove the open interval $(\frac{1}{3}, \frac{2}{3})$.

We obtain the set $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.

Divide each of the two segments in C_1 into three equal parts and remove the interiors of the middle parts.

We obtain the set $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.

Continuing this process, we obtain the sequence of closed sets $\{C_k\}$. Cantor set is $\bigcap_{k=1}^{\infty} C_k$.

Cantor set is a perfect, uncountable set.

Theorem 3.10. *Let (X, T) be a topological space. A set A is perfect if and only if it is closed and dense in itself.*

Proof. Follows from Theorem 3.7a. □

Definition 3.11. Let (X, T) be a topological space. A point $x \in X$ is called *isolated point* (of the space X) if $x \notin X^d$.

Theorem 3.12. *A point $x \in X$ is an isolated point of a topological space (X, T) if and only if $\{x\} \in T$ (i.e. $\{x\}$ is an open set).*

Proof. $x \notin X^d \Leftrightarrow x \notin cl(X \setminus \{x\}) \Leftrightarrow x \in X \setminus cl(X \setminus \{x\}) \Leftrightarrow x \in Int(\{x\})$, the last step uses Theorem 1.22.

So $\{x\} \subset Int(\{x\})$ and obviously $Int(\{x\}) \subset \{x\}$, so $\{x\} = Int(\{x\})$ and that means $\{x\}$ is open.

□

Remark: Set of the form $\{x\}$ (which has only one element: x) is sometimes called a singleton.

The discrete topology can be characterized by the condition that every singleton is open.

Definition 3.13. Let (X, T) be a topological space and let $A \subset X$.

(a) A is called a *dense set* if $cl(A) = X$.

(b) A is called a *boundary set* if $cl(X \setminus A) = X$.

(c) A is called a *nowhere dense set* if $cl(X \setminus cl(A)) = X$.

Example 3.IV. Cantor set is a nowhere dense set in $[0, 1]$.

Example 3.V. In $\mathbb{R}$ with its natural topology, the sets $\mathbb{Q}$ (of all rational numbers) and $\mathbb{Q}'$ (of all irrational numbers) are both dense and boundary sets.

Theorem 3.14. *Let (X, T) be a topological space. A set $A \subset X$ is a dense set if and only if for every $U \in T \setminus \{\emptyset\}$ we have $U \cap A \neq \emptyset$.*

Proof. Do it yourself. It follows immediately from Theorem 1.15c. □

Theorem 3.15. *Let (X, T) be a topological space. A set $A \subset X$ is a boundary set if and only if $Int(A) = \emptyset$.*

Proof. Do it yourself. Use Theorem 1.22.: $cl(X \setminus A) = X \setminus Int(A)$. $\square$

Theorem 3.16. *Let (X,T) be a topological space and let $A \subset X$. Then the folloiwng statements are equivalent:*

(i) A is a nowhere dense set.

(ii) $cl(A)$ is a boundary set.

(iii) for every $U \in T \setminus \{\emptyset\}$ there exists a set $V \in T \setminus \{\emptyset\}$ such that $V \subset U$ and $V \cap A = \emptyset$.

I'm leaving the proof of this theorem as an exercise.

Definition 3.17. Let (X,T) be a topological space and let $A \subset X$.

(a) A is called a *set of first category* (or: a *meager set*) if A is a countable union of nowhere dense sets.

(b) A is called a *set of second category* if A is not a first category set.

Example 3.VII. In $\mathbb{R}$ with its natural topology the set $\mathbb{Q}$ of all rational numbers is a set of first category, and the set $\mathbb{Q}'$ of all irrational numbers is a set of second category.

Example 3.VIII. In the space $X = \mathbb{Q}$ with the topology induced from $\mathbb{R}$ the set $(0,1) \cap \mathbb{Q}$ is an open set of first category.

Theorem (Baire): Every complete or compact metric space is a 2nd category set.

4. Continuous functions and homeomorphisms

Recall the definition of a function $f : \mathbb{R} \to \mathbb{R}$ continuous at x_0:

$$\forall_{\epsilon>0} \quad \exists_{\delta>0} \quad \forall_{x\in\mathbb{R}} \qquad (|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \epsilon)$$

or, using a different notation:

$$\forall_{\epsilon>0} \quad \exists_{\delta>0} \qquad (f(x_0-\delta, x_0+\delta)) \subset (f(x_0)-\epsilon, f(x_0)+\epsilon).$$

Definition 4.1. Let (X, T_X) and (Y, T_Y) be topological spaces.

(i) A function $f : X \to Y$ is called *continuous at* x_0 if for every neighbourhood V of $f(x_0)$ (i.e. $f(x_0) \in V \in T_Y$) there exists a neighbourhood U of x_0 (i.e. $x_0 \in U \in T_X$) such that $f(U) \subset V$. The set of all points at which f is continuous (in other words, the *set of all continuity points of* f) will be denoted by $C(f)$.

(ii) A function $f : X \to Y$ is called *continuous* if it is continuous at every point of its domain.

The set of all continuous functions $f : X \to Y$ will be denoted by $C(X, Y)$.

Remark: if $f : X \to Y$ and x_0 is an isolated point in (X, T_X) (by Theorem 3.12, it means that $\{x_0\} \in T_X$), then f is continuous at x_0 ($x_0 \in C(f)$).

Theorem 4.2. *Let $(X, T_X), (Y, T_Y)$ be topological spaces and let $f : X \to Y$. Then the following statements are equivalent:*

(i) f is a continuous function.

(ii) $f^{-1}(V) \in T_X$ for every $V \in T_Y$ (preimage of an open set in Y is an open set in X)

(iii) $f^{-1}(F)$ is a closed set in (X, T_X) for every closed set F in (Y, T_Y) (preimage of a closed set in Y is a closed set in X).

Proof. $(i) \Rightarrow (ii)$.

Fix an arbitrary $V \in T_Y$. We must prove that

(1) $f^{-1}(V) \in T_X$.

In order to do so, we will use Theorem 1.3:

Theorem 1.3. Let (X,T) be a topological space. A set W is open (i.e., $W \in T$) if and only if for every point $x \in W$ there exists a neighbourhood V_x of x such that $V_x \subset W$.

Let $x \in f^{-1}(V)$. By Definition 4.1(a) there exists a neighbourhood W of x such that $f(W) \subset V$. Observe, that then

$$W \subset f^{-1}(f(W)) \subset f^{-1}(V).$$

By Theorem 1.3., $f^{-1}(V) \in T_X$.

$(ii) \Rightarrow (iii)$.

Let F be a closed set in (Y, T_Y). Then $Y \setminus F \in T_Y$. By the assumption (ii), we have then $f^{-1}(Y \setminus F) \in T_Y$. We have

$$X \setminus f^{-1}(F) = f^{-1}(Y) \setminus f^{-1}(F) = f^{-1}(Y \setminus F) \in T_X.$$

It follows that $f^{-1}(F)$ is a closed set in (X, T_X).

$(iii) \Rightarrow (i)$.

We will show that f is continuous at every point of its domain. Let $x_0 \in X$ and let V be a neighbourhood of $f(x_0)$. Then $Y \setminus V$ is a closed set, hence by the assumption (iii)

$f^{-1}(Y \setminus V)$ is a closed set in (X, T_X).

Set $U = X \setminus f^{-1}(Y \setminus V)$. Then U is an open set (see the Remark D_4 after Theorem 1.5.) and $x_0 \in U$, because $f(x_0) \notin Y \setminus V$ and therefore $x_0 \notin f^{-1}(Y \setminus V)$. It means that U is a neighbourhood of x_0. To finish the proof, it is enough to show that

(2) $f(U) \subset V$.

Let $\alpha \in f(U)$. Then there exists $t \in U = X \setminus f^{-1}(Y \setminus V)$ such that $f(t) = \alpha$. Since $t \notin f^{-1}(Y \setminus V)$, we have $\alpha = f(t) \notin Y \setminus V$ and so $\alpha \in V$.

$\square$

Remark: Prove the following statement:

Let $(X, T_X), (Y, T_Y)$ be topological spaces, let B be a base of (Y, T_Y) and let $f : X \to Y$. Then f is a continuous function if and only if the preimage of any set from B is an open set.

Example 4.I. Let $(X, T_X), (Y, T_Y)$ be topological spaces and let $y \in Y$. The constant function $f(x) = y$ for all $x \in X$ is continuous.

Remark: There exist topological spaces (X, T_X) such that the only continuous functions $f : X \to \mathbb{R}$ are constant functions.

Example 4.II. Let $X = Y = \{0\} \cup [1, 2] \subset \mathbb{R}$. Let T_X be the topology induced from the natural topology of $\mathbb{R}$, let T_Y be the discrete topology and let $f : X \to Y$ be given by the formula: $f(x) = x$. Then $C(f) = \{0\}$.

Example 4.V. Let $X = (-\frac{\pi}{2}, \frac{\pi}{2})$ and let $Y = \mathbb{R}$. Consider the topology in X induced from the natural topology of $\mathbb{R}$ and let the topology of Y be the natural topology of $\mathbb{R}$. Then $\tan : X \to Y$ is a continuous function, recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

Theorem 4.3. *Let $(X, \rho_X), (Y, \rho_Y)$ be metric spaces and let $f : X \to Y$. Then the following statements are equivalent:*

(i) f is continuous at x_0.

(ii) For every sequence $\{x_n\}$ if $\lim_{n\to\infty} x_n = x_0$, then $\lim_{n\to\infty} f(x_n) = f(x_0)$ (Heine condition),

(iii) For every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x \in X$ if $\rho_X(x, x_0) < \delta$, then $\rho_Y(f(x), f(x_0)) < \epsilon$. (Cauchy condition)

Proof. do it yourself, using what you know from lectures in metric topology. □

Theorem 4.4. *Let (X, ρ) be a metric space and let $A \neq \emptyset$ be a subset of X. Then the function $\rho_A : X \to \mathbb{R}$ defined by the formula:*

$$\rho_A(x) = \rho(x, A) = \inf_{a \in A} \rho(x, a)$$

(ρ_A *is the distance from the set* A*) is continuous.*

Proof. do it yourself, using what you know from lectures in metric topology.

$\square$

Definition 4.5. Let $f_n, f : X \to Y$, where X is a set and (Y, ρ) is a metric space. We say that the sequence $\{f_n\}$ is

(a) *(pointwise) convergent to* f (or that f is the *pointwise limit of the sequence* $\{f_n\}$), which we denote by $\lim_{n\to\infty} f_n = f$) if for every $x \in X$ we have $\lim_{n\to\infty} f_n(x) = f(x)$;

$$\forall_{x\in X} \quad \forall_{\epsilon>0} \quad \exists_N \quad \forall_{n>N} \quad \rho(f_n(x), f(x)) < \epsilon.$$

(b) *uniformly convergent to* f (or that f is the *uniform limit of the sequence* $\{f_n\}$) if for every $\epsilon > 0$ there exists a natural number N such that for every $n > N$ and every $x \in X$ we have $\rho(f_n(x), f(x)) < \epsilon$,

$$\forall_{\epsilon>0} \quad \exists_N \quad \forall_{n>N} \quad \forall_{x\in X} \quad \rho(f_n(x), f(x)) < \epsilon.$$

Theorem 4.6. *Let* X *be a set, let* (Y, ρ) *be a metric space and let* $f_n, f : X \to Y$*. If the sequence* $\{f_n\}$ *is uniformly convergent to* f*, then it is also pointwise convergent.*

Proof. do it yourself, using what you know from lectures in metric topology.

$\square$

Theorem 4.7. *Let* (X, T) *be a topological space and let* (Y, ρ) *be a metric space. Then the limit of a uniformly convergent sequence of continuous functions* $f_n : X \to Y$ *is a continuous function.*

Proof. Let f be the limit of a uniformly convergent sequence of continuous functions $f_n : X \to Y$ and let $x_0 \in X$. It is enough to show that f is continuous at x_0.

Let V be a neighbourhood of $f(x_0)$ and let $B(f(x_0), \epsilon) \subset V$. From the fact that $\{f_n\}$ is uniformly convergent to f, it follows that

(1) there exists a natural number n^* such that $\rho(f_{n^*}(x), f(x)) < \epsilon/4$ for every $x \in X$.

Since f_{n^*} is continuous at x_0, there exists a neighbourhood U of x_0 such that

$$f_{n^*}(U) \subset B(f_{n^*}(x_0), \epsilon/4) \subset V$$

which implies that

(2) $\rho(f_{n^*}(x), f_{n^*}(x_0)) < \epsilon/4$ for all $x \in U$.

From (1) we obtain in particular (taking $x = x_0$):

(3) $\rho(f_{n^*}(x_0), f(x_0)) < \epsilon/4$.

Using (1),(2),(3) for every $x \in U$ we have:

$$\begin{aligned} \rho(f(x), f(x_0)) &\leq \rho(f(x), f_{n^*}(x)) + \rho(f_{n^*}(x), f(x_0)) \\ &\leq \rho(f(x), f_{n^*}(x)) + \rho(f_{n^*}(x), f_{n^*}(x_0)) + \rho(f_{n^*}(x_0), f(x_0)) \\ &\leq \frac{3\epsilon}{4} < \epsilon. \end{aligned}$$

It follows that $f(U) \subset B(f(x_0), \epsilon) \subset V$.

□

Definition 4.8. Let (X, T_X) and (Y, T_Y) be topological spaces. A function $f : X \to Y$ is called a *homeomorphism* if f is a bijection (i.e. f is one-to-one and $f(X) = Y$) and both f, f^{-1} are continuous.

Example 4.VI. Consider the topological spaces from Example 4.V.: $X = (-\frac{\pi}{2}, \frac{\pi}{2})$ and $Y = \mathbb{R}$. Then $\tan : X \to Y$ is a homeomorphism.

Example 4.VII. (a continuous, one-to-one function that is not a homeomorphism)

Let $X = Y = \{a, b\}$, consider $T_X = 2^X$ (the discrete topology) as the topology in X and let $T_Y = \{\emptyset, Y\}$ (the antidiscrete topology) be the topology in Y. Consider a function $f : X \to Y$ given by the formula $f(x) = x$. Then f is one-to-one. Obviously, f is continuous and f^{-1} is not continuous.

Definition 4.9. Let (X, T_X) and (Y, T_Y) be topological spaces. If there exists a homeomorphism $h : X \to Y$, then we say that the spaces (X, T_X) and (Y, T_Y) are *homeomorphic* or that they are *topologically equivalent*, we sometimes denote it by

$$(X, T_X) \ \textit{hom} \ (Y, T_Y).$$

Example 4.VIII. According to Example 4.VI the spaces considered in Example 4.V. are homeomorphic.

Example 4.IX. Let $X = (a, b), Y = (c, d)$, where $-\infty \leq a < b \leq +\infty$ and $-\infty \leq c < d \leq +\infty$. Let both T_X and T_Y be topologies induced by the natural topology of $\mathbb{R}$. Then (X, T_X) and (Y, T_Y) are homeomorphic.

You can easily find a homeomorphism between intervals (a, b) and (c, d) with their induced topology, e.g., take a function of the form $f(x) = Ax + B$ with appropriate A, B.

Theorem 4.10. *For every continuous bijection f from (X, T_X) onto (Y, T_Y) the following statements are equivalent:*

(i) f is a homeomorphism;

(ii) f^{-1} is a homeomorphism;

(iii) a set $A \subset X$ is closed if and only if $f(A)$ is closed in (Y, T_Y);

(iv) a set $A \subset X$ is open if and only if $f(A)$ is open in (Y, T_Y).

Proof. Equivalence of (i) and (ii) follows immediately from the definition of homeomorphism, and to show (iii) and (iv) it is enough to notice that for $g = f^{-1}$ we have $g^{-1} = f$ and so $f(A) = g^{-1}(A)$.

□

Theorem 4.11. *Let $(X, T_X), (Y, T_Y), (Z, T_Z)$ be topological spaces and let $f : X \to Y$ and $g : Y \to Z$. Then:*

(a) if f and g are continuous, then the composition $h = g \circ f : X \to Z$ is continuous.

(b) if f and g are homeomorphisms, then the composition $h = g \circ f : X \to Z$ is a homeomorphism.

Proof. We will only show (a). (b) easily follows from (a) - do it yourself.

Let $U \in T_Z$. According to Theorem 4.2. it is enough to show that

(1) $h^{-1}(U) \in T_X$.

Observe that $h^{-1}(U) = f^{-1}(g^{-1}(U))$. From the continuity of the function g it follows that $g^{-1}(U) \in T_Y$, which - by the continuity of f - implies: $h^{-1}(U) = f^{-1}(g^{-1}(U)) \in T_X$ and therefore (1) holds.

□

Theorem 4.12. Relation *hom* (of being homeomorphic) is an equivalence relation in the class of topological spaces.

Proof. Obviously, the relation is reflexive, because for every topological space (X, T_X) function $f(x) = x$ is a homeomorphism of X onto X.

Symmetry of the relation follows from the equivalence of conditions (i) and (ii) in Theorem 4.10., and its transitivity follows easily from Theorem 4.11(b).

5. Separation axioms

In this lecture we assume that every considered space has at least 2 distinct elements.

We shall not discuss T_0-spaces. A topological space (X, T) is called a T_0-space if for every pair of distinct points of X there exists an open set containing only one of the points from the pair. The antidiscrete space (X, T), where $T = \{\emptyset, X\}$ is not a T_0-space.

Mathematically, (X,T) is a T_0-space $\iff \forall x_1, x_2 \in X,\ x_1 \neq x_2,\ \exists U \in T \,/\, x_1 \in U \not\ni x_2 \vee x_1 \notin U \ni x_2$

Definition 5.1. A topological space (X, T) is called a T_1-space if for every pair of distincts points $x_1, x_2 \in X$ there exists an open set U such that $x_1 \in U$ and $x_2 \notin U$.

Theorem 5.2. *A topological space (X, T) is a T_1-space if and only if every singleton (i.e. a set that contains exactly one element) is closed.*

Proof. Necessity.

Let $x \in X$. We assume that (X, T) is a T_1-space, so there exists at least one set $U \in T$ such that $x \notin U$. Then

$$\{x\} = \bigcap_{x \notin U \in T} (X \setminus U) \qquad (1)$$

Indeed, obviously $\{x\} \subset \bigcap_{x \notin U \in T}(X \setminus U)$. We will now show the opposite inclusion. Let $y \in \bigcap_{x \notin U \in T}(X \setminus U)$ (obviously, $x \in \bigcap_{x \notin U \in T}(X \setminus U)$). We will show that $y = x$. Suppose the opposite, i.e. that $y \neq x$. From the assumption (that X is a T_1-space), there exists an open set W such that $y \in W$ and $x \notin W$. But that implies $y \notin X \setminus W$ and hence $y \notin \bigcap_{x \notin U \in T}(X \setminus U)$ which contradicts our assumption. This proves the formula (1). By (1), $\{x\}$ is a closed set, as an intersection of closed sets.

Sufficiency.

Let x_1, x_2 be two distinct points of X. Then, according to our assumption, $\{x_2\}$ is a closed set. Let $U = X \setminus \{x_2\}$. Then U is open and $x_1 \in U$, $x_2 \notin U$.

$\square$

Example 5.I. Consider the space (X, T_d) from Example 2.II. (see below). According to the last theorem, (X, T_d) is not a T_1-space.

Example 2.II. Let $X = \mathbb{R}$ and for any $A \subset X \setminus \{\emptyset\}$, $\overline{A} = A \cup \{0\}$ for $A \neq \emptyset$ and $\overline{\emptyset} = \emptyset$. Then $T_d = \{\mathbb{R} \setminus F : F = \overline{F}\}$ is the topology generated by a closure operation. It can be shown that in (X, T_d) a set $\{x\}$ is closed if and only if $x = 0$.

Example 5.II. Consider the space (X, T_D) from Example 2.I. (see below). According to the last theorem and the remark after Example 2.I., (X, T_D) is a T_1-space.

Example 2.I. Let X be an infinite set and let $D = \{A \subset X : \#(A) <$

$\aleph_0\} \cup \{X\}$. Then $T_D = \{X \setminus F : F \in D\} = \{X \setminus A : \#(A) < \aleph_0\} \cup \{\emptyset\}$ is the topology generated by the family of closed sets D. Observe that in this space every single-element set is closed.

Definition 5.3. A topological space (X, T) is called a T_2-space (or a *Hausdorff space*) if for every pair of distinct points $x_1, x_2 \in X$ there exist open sets U_1, U_2 such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 = \emptyset$.

Theorem 5.4. *Every T_2-space is a T_1-space.*

Proof. Obvious. □

Remark. Hausdorff spaces are very important. We do not consider special sequences in topological spaces, but in a Hausdorff space a sequence that is convergent has exactly one limit.

Example 5.III. Let (X, T) be the space from Example 5.II. Then (X, T) is a T_1-space but not a T_2-space.

Example 5.IV. Consider the space from Example 1.IV (see below) under the assumptions that X is an infinite set and J is an ideal of finite sets. Then it is a Hausdorff space.

Example 1.IV. Let $X \neq \emptyset$, $x_0 \in X$ and let J be an ideal of subsets of X. Then (X, T), with $T = \{X \setminus A : A \in J\} \cup \{A \subset X : x_0 \notin A\}$ is a topological space.

Definition 5.5. A topological space (X, T) is called a T_3-space (or a *regular space*) if it is a T_1-space and for every $x_0 \in X$ and every closed set $F \subset X$ such that $x_0 \notin F$ there exist open sets U_1 and U_2 such that $x_0 \in U_1$, $F \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

Example 5.V. Consider the space from Example 2.VII (see below). Then it is a Hausdorff space that is not a regular space.

Example 2.VII. Let $X = [0, +\infty)$. Define $B(x)$ in the following way: $B(x) = \{(x - \frac{1}{n}, x + \frac{1}{n}) \cap [0, +\infty) : n = 1, 2, \ldots\}$ when $x \in (0, +\infty)$ and $B(0) = \{[0, \frac{1}{n}) \setminus \{\frac{1}{m} : m > n\} : n = 1, 2, \ldots\}$. Then in X we can consider the topology generated by the neighbourhood system $\{B(x) : x \in X\}$.

Example 5.VI. Consider the space from Example 1.IV (see below), under assumptions that X is an infinite set and J is the ideal of finite sets. Then it is a regular space.

Example 1.IV. Let $X \neq \emptyset$, $x_0 \in X$ and let J be an ideal of subsets of X. Then (X, T), with $T = \{X \setminus A : A \in J\} \cup \{A \subset X : x_0 \notin A\}$ is a topological space.

Theorem 5.6. *Every T_3-space is a T_2-space.*

Proof. Do it yourself. Perhaps the easiest way is to observe that T_3-space is by definition a T_1-space and use Theorem 5.2. $\square$

Theorem 5.7. *If (X, T) is a T_1-space, then it is a T_3-space if and only if for every $x \in X$ and every closed set F such that $x \notin F$ there exists an open set U such that $x \in U$ and $cl(U) \cap F = \emptyset$.*

Proof. Necessity.

Let $x \in X$ and let F be a closed set such that $x \notin F$. By Definition 5.5., there exist open sets U_1 and U_2 such that $x \in U_1$, $F \subset U_2$ and $U_1 \cap U_2 = \emptyset$. Let $U = U_1$ and $K = X \setminus U_2$. Then K is a closed set such that $U \subset K$ (because $U_1 \cap U_2 = \emptyset$). From the definition of the closure of a set it follows that $cl(U) \subset K$. Since $F \subset U_2$, we have $K \cap F = \emptyset$, which together with the previous inclusion yields $cl(U) \cap F = \emptyset$.

Sufficiency.

Let $x \in X$ and let F be a closed set such that $x \notin F$. From our assumption, there exists an open set U such that $x \in U$ and $cl(U) \cap F = \emptyset$. Let $U_1 = U$ and $U_2 = X \setminus cl(U) \in T$. Then $x \in U_1$, $F \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

$\square$

Recall the notation from the first lecture: $X = I$ and the *natural distance function on I* is $\rho_E(x, y) = |x{-}y|$. If we write I and do not specify the distance function, then we mean the natural distance function.

Definition 5.8. A topological space (X, T) is called a $T_{3\frac{1}{2}}$-space (or a *completely regular space*, or a *Tychonoff space*) if it is a T_1-space and for every point $x_0 \in X$ and every closed set $F \subset X$ such that $x_0 \notin F$ there exists a continuous function $f : X \to I$ such that $f(x_0) = 0$ and $f(x) = 1$ for all $x \in F$.

Examples of T_3-spaces which are not completely regular can be constructed using more advanced techniques.

Theorem 5.9. *Every completely regular topological space is a T_3-space.*

Proof. Let (X, T) be a completely regular topological space. Obviously, by the definition, (X, T) is a T_1-space. Let $x_0 \in X$ and let F be a closed set such that $x_0 \notin F$. By the definition, there exists a function $f : X \to I$ such that $f(x_0) = 0$ and $f(x) = 1$ for $x \in F$. Let $U_1 = f^{-1}([0, \frac{1}{2}))$ and $U_2 = f^{-1}((\frac{1}{2}, 1])$. Then U_1 and U_2 are open sets such that $x_0 \in U_1$, $F \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

□

Definition 5.10. A topological space (X, T) is called a T_4-space (or a *normal space*) if it is a T_1-space and for every pair of closed sets F_1, F_2 such that $F_1 \cap F_2 = \emptyset$ there exist open sets U_1, U_2 such that $F_1 \subset U_1$, $F_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

Theorem 5.11 (Urysohn lemma)**.** *For every pair A, B of disjoint, non-empty closed subsets of a normal space (X, T) there exists a continuous function $f : X \to I$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof. For volunteers. See e.g., Bredon, "Topology and Geometry", Springer 1993, Lemma 10.1 and 10.2. For explanations of some equalities in Lemma 10.1, see the following:

1. If $x \in \bigcup\{U_r, r < \alpha\}$ then there exists $t < \alpha$ such that $x \in U_t$; let $t < \beta < \alpha$, then $x \in U_\beta$ and hence $f(x) \leq \beta < \alpha$.

On the other hand, if $f(x) < \alpha$ then $x \in U_\alpha$, because $f(x) = \inf\{s : x \in U_s\}$.

2. Next, $\bigcup\{X \setminus cl(U_r), r > \beta\} \subset \bigcup\{X \setminus U_r, r > \beta\}$ by $U_r \subset cl(U_r) \Rightarrow X \setminus cl(U_r) \subset X \setminus U_r$.

On the other hand, if $x \in \bigcup\{X \setminus U_r, r > \beta\}$ then there exists $s > \beta$ such that $x \in X \setminus U_s$, let $\beta < t < s$ then $U_t \subset cl(U_t) \subset U_s$, hence $X \setminus U_s \subset X \setminus cl(U_t)$.

3. If $f(x) > \beta$ then there exists $f(x) > r > \beta$ so $x \notin U_r$ and hence $\{f(x) > \beta\} \subset \bigcup\{X \setminus U_r, r > \beta\}$.

On the other hand, if there exists $t > \beta$ such that $x \in X \setminus U_t$, suppose that $f(x) \leq \beta$. Then for all $r > \beta$ we would have $x \in U_r$ and since $t > \beta$ also $x \in U_t$ - a contradiction.

□

Theorem 5.12. *Every T_4-space (X, T) is a completely regular space (hence, also T_3-space and T_2-space).*

Proof. Let $x \in X$ and let F be a closed set such that $x \notin F$. Since (X, T) is a T_1-space, $\{x\}$ is a closed set and $\{x\} \cap F = \emptyset$. By the Urysohn lemma, there exists a continuous function $f : X \to I$ such that $f(x) = 0$ and $f(F) = \{1\}$.

$\square$

Theorem 5.13. *Every metric space (X, ρ) is a normal space.*

Proof. It is easy to see that (X, ρ) is a T_1-space. Let A, B be closed sets such that $A \cap B = \emptyset$. We will show that (1) there exist open sets U_1 and U_2 such that $A \subset U_1$, $B \subset U_2$ and $U_1 \cap U_2 = \emptyset$. Let $U_1 = \{x \in X : \rho(x, A) < \rho(x, B)\}$ and $U_2 = \{x \in X : \rho(x, B) < \rho(x, A)\}$ We will show that U_1 and U_2 are open.

Recall Theorem 4.4.:

Theorem 4.4. Let (X, ρ) be a metric space and let $A \neq \emptyset$ be a subset of X. Then the function $\rho_A : X \to \mathbb{R}$ defined by the formula:

$$\rho_A(x) = \rho(x, A) = \inf_{a \in A} \rho(x, a)$$

(ρ_A is the distance from the set A) is continuous.

By Theorem 4.4., functions ρ_A and ρ_B (functions measuring distance from sets A and B, respectively) are continuous. Hence, functions $\rho_A - \rho_B$ and $\rho_B - \rho_A$ are also continuous. Therefore,

$$\begin{aligned} U_1 &= \{x \in X : \rho(x, A) < \rho(x, B)\} = \{x \in X : \rho(x, A) - \rho(x, B) < 0\} \\ &= (\rho_A - \rho_B)^{-1}((-\infty, 0)) \end{aligned}$$

is an open set and

$$\begin{aligned} U_2 &= \{x \in X : \rho(x, B) < \rho(x, A)\} = \{x \in X : \rho(x, B) - \rho(x, A) < 0\} \\ &= (\rho_B - \rho_A)^{-1}((-\infty, 0)) \end{aligned}$$

is an open set.

Obviously, $A \subset U_1$ and $B \subset U_2$. We will show only the first of these inclusions, the proof of the second one is analogous. If $x \in A$, then $\rho_A(x) = \rho(x, A) = 0$. Also, $x \notin B = cl(B)$. It follows that $\rho_B(x) = \rho(x, B) > 0$, hence $x \in U_1$.

To finish the proof, we need to show that $U_1 \cap U_2 = \emptyset$. Suppose the contrary, i.e. that exists $x \in U_1 \cap U_2$. Then $\rho(x, A) < \rho(x, B)$ (because $x \in U_1$) and $\rho(x, B) < \rho(x, A)$ (because $x \in U_2$), which is impossible for any real numbers $\rho(x, A)$ and $\rho(x, B)$.

$\square$

Theorem 5.14.

(a) A topological space (X, T) is regular if and only if it is a T_1-space and for every $x \in X$ and every open set V such that $x \in V$ there exists an open set U such that $x \in U \subset cl(U) \subset V$.

(b) A topological space (X, T) is normal if and only if it is a T_1-space and for every closed set F and every open set V such that $F \subset V$ there exists an open set U such that $F \subset U \subset cl(U) \subset V$.

Proof. Do it yourself. For (a), let $x \in V$ and V be an open set. Take a closed set $K = X \setminus V$, use definition of T_3-space to find open sets A, W such that $x \in W \subset A \subset V$. Then use Theorem 5.7 for $F = X \setminus A$ to find a set U such that $x \in U \subset cl(U) \subset V$.

For (b) - try to prove it analogously as Theorem 5.7., I'm leaving it at as exercise.

$\square$

Theorem 5.15 (Tietze extension theorem for continuous functions)**.** *Let (X, T) be a normal space, let $F = cl(F) \subset X$ and let $f : F \to I$ ($f : F \to \mathbb{R}$) be a continuous function. Then there exists a continuous function $f^* : X \to I$ ($f^* : X \to \mathbb{R}$) such that $f^*(x) = f(x)$ for all $x \in F$.*

Example 5.IX. Let X be a non-empty set, let T^* be the discrete topology on X and let T^{**} be the antidiscrete topology on X. Let $f : (X, T^*) \to (X, T^{**})$ be given by the formula $f(x) = x$. Then f is continuous, one-to-one function mapping a normal space onto a topological space that is not T_1. This means that continuous functions do not preserve the separation axioms.

Definition 5.16. Let (X, T) be a topological space.

(a) We say that A is an F_σ-set if

$$A = \bigcup_{n=1}^{\infty} F_n$$

where all F_n are closed sets.

(b) We say that A is an G_δ-set if

$$A = \bigcap_{n=1}^{\infty} G_n$$

where all G_n are open sets.

Definition 5.17. A normal space (X, T) is called *perfectly normal* if it is normal and every closed set in it is a G_δ-set.

Theorem 5.18. *A normal space (X, T) is a perfectly normal space if and only if every open set in it is a F_σ-set.*

Proof. Do it yourself. It is enough to observe that the complement of G_δ set is an F_σ set and the complement of an open set is closed. $\square$

Theorem 5.19. *Every metric space (X, ρ) is a perfectly normal space.*

Proof. Perhaps the easiest way is to use Theorem 5.18. Observe that just as the open set in a metric space can be presented as a union of open balls (taking small enough ball around its every point), it can be also presented as a union of closed balls (taking even smaller, closed ball around its every point). However, this works only for separable metric spaces (i.e., those with countable dense set).

In general, we can prove the following: if $A = f^{-1}(\{0\})$, where $f : X \to \mathbb{R}$ is continuous, then A is a closed, G_δ set. Then observe that for a closed set A in a metric space (X, ρ) we have $A = g^{-1}(\{0\})$ for $g(x) = \inf_{y \in A} \rho(x, y)$. $\square$

Example 5.20. Consider the space from Example 1.IV (see below), where X is an uncountable set and J is the ideal of all finite sets. Then (X, T) is normal and not perfectly normal, because the open set $X \setminus \{x_0\}$ is not a F_σ-set.

Example 1.IV. Let $X \neq \emptyset$, $x_0 \in X$ and let J be an ideal of subsets of X. Then (X, T), with $T = \{X \setminus A : A \in J\} \cup \{A \subset X : x_0 \notin A\}$ is a topological space.

Definition 5.21*.

(i) A subset A of a topological space (X, T) is called *functionally closed* if there exists a continuous function $f : X \to I$ such that $A = f^{-1}(\{0\})$.

(ii) A subset A of a topological space (X, T) is called *functionally open* if it is the complement of a functionally closed set.

Remark: Every functionally closed (open) set is closed (open). Union of a finite family of functionally closed sets is a functionally closed set. Intersection of *countable* family of functionally closed sets is a functionally closed set.

Theorem 5.22 (Vedenisov).

For a T_1-space (X, T) the following statements are equivalent:

(i) (X, T) is a perfectly normal space.

(ii) All open subsets of X are functionally open.

(iii) All closed subsets of X are functionally closed.

(iv) For every pair A, B of disjoint, closed sets in X there exists a continuous function $f : X \to I$ such that $A = f^{-1}(0)$ and $B = f^{-1}(1)$.

Theorem 5.23.

(a) Every subspace of a T_i-space is a T_i-space for $i \leq 3\frac{1}{2}$.

(b) Every closed subspace of a normal space is a normal space.

(c) Every subspace of a perfectly normal space is a perfectly normal space.

Recall Example 5.IX., that shows that in general, continuous functions do not preserve the separation axioms.

Definition 5.24*. Let X, Y be topological spaces and let $f : X \to Y$ be a continuous function. We say that f is a *closed map* if the image of every closed set (in the map f) is a closed set.

Theorem 5.25*. If (X, T_X), (Y, T_Y) are topological spaces, $f : X \to Y$ is a closed map and (X, T_X) is a T_i-space, where $i \in \{1, 4\}$, or a perfectly normal space; then (Y, T_Y) also has this property.

6. Cartesian product of topological spaces

Reminder: Cartesian product has the following properties:

$(A \cup B) \times C = (A \times C) \cup (B \times C)$

$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

$(A \backslash B) \times C = (A \times C) \backslash (B \times C)$

We shall considere here Cartesian product of two spaces (you can generalize results presented here to the case of Cartesian product of any finite number of spaces)

Lemma. Let (X, T_X), (Y, T_Y) be topological spaces. Then the family $B_{X\times Y} = \{U_1 \times U_2 : U_1 \in T_X, U_2 \in T_Y\}$ of subsets of $X \times Y$ satisfies conditions (B_1) and (B_2) of Theorem 2.3.

Proof. We write conditions (B_1) and (B_2) in our case:

(B_1) for all $U_1 \times U_2, V_1 \times V_2 \in B_{X\times Y}$ and every point $x \in (U_1 \times U_2) \cap (V_1 \times V_2)$ there exists a set $W_1 \times W_2 \in B_{X\times Y}$ such that $x \in W_1 \times W_2 \subset (U_1 \times U_2) \cap (V_1 \times V_2)$

(B_2) For every $x \in X \times Y$ there exists $U_1 \times U_2 \in B_{X\times Y}$ such that $x \in U_1 \times U_2$.

To prove (B_1) it is enough to observe that for all sets $U_1 \times U_2, V_1 \times V_2 \in B_{X\times Y}$ we have $(U_1 \times U_2) \cap (V_1 \times V_2) = (U_1 \cap V_1) \times (U_2 \cap V_2)$, so (B_1) holds for $W_1 = U_1 \cap V_1$, $W_2 = U_2 \cap V_2$.

Condition (B_2) obviously holds, because $X \in T_X$ and $Y \in T_Y$, and so $X \times Y \in B_{X \times Y}$.

$\square$

Definition 6.1. *Cartesian product* of topological spaces (X, T_X) and (Y, T_Y) is the topological space $(X \times Y, T_{X \times Y})$, where $T_{X \times Y}$ is the topology introduced by the base $B_{X \times Y} = \{U_1 \times U_2 : U_1 \in T_X, U_2 \in T_Y\}$ (this topology is called *Tychonoff topology*). The sets X, Y are then called *axes* of the Cartesian product.

Definition 6.2. *Projection* of the Cartesian product of topological spaces (X, T_X) and (Y, T_Y) onto axis X (resp. Y) is the mapping $proj_X : X \times Y \to X$ (resp. $proj_Y : X \times Y \to Y$) given by the formula $proj_X(x, y) = x$ (resp. $proj_Y(x, y) = y$).

Theorem 6.3. *Projection of a Cartesian product onto any of its axes is a contnuous map.*

Proof. Let (X, T_X) and (Y, T_Y) be topological spaces, and let $proj_X : X \times Y \to X$ and $proj_Y : X \times Y \to Y$ be projections onto axis X and axis Y, respectively.

Let $U \in T_X$ and $V \in T_Y$. Then

$(proj_X)^{-1}(U) = U \times Y \in B_{X \times Y}$, hence $(proj_X)^{-1}(U) \in T_{X \times Y}$

and

$(proj_Y)^{-1}(V) = X \times V \in B_{X\times Y}$, hence $(proj_Y)^{-1}(V) \in T_{X\times Y}$.

$\square$

Theorem 6.4. *Let (X, T_X), (Y, T_Y) be topological spaces and let $F \subset X$, $K \subset Y$ be closed sets. Then $F \times K$ is closed set in $(X \times Y, T_{X\times Y})$.*

Proof. It is easy to check that

$$(X \times Y)\backslash(F \times K) = [(X\backslash F) \times Y] \cup [X \times (Y\backslash K)].$$

But then it follows that $(X \times Y)\backslash(F \times K)$ is a union of two open sets in $(X \times Y, T_{X\times Y})$. $\square$

Theorem 6.5. *Cartesian product of two T_i-spaces is a T_i-space for $i \leqslant 3\frac{1}{2}$.*

Proof. We prove only the case $i = 2$. Let $(X, T_X), (Y, T_Y)$ be T_2-spaces, let $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in X \times Y$ and let $z_1 \neq z_2$. Then $x_1 \neq x_2$ or $y_1 \neq y_2$. Without loss of generality, we can assume that $x_1 \neq x_2$. Since (X, T_X) is a T_2-space, there exist open sets U_1 and U_2 such that $x_1 \in U_1$, $x_2 \in U_2$ and $U_1 \cap U_2 \neq \varnothing$. Take $W_1 = U_1 \times Y$ and $W_2 = U_2 \times Y$. Then we have $z_1 \in W_1$, $z_2 \in W_2$ and $W_1 \cap W_2 = \varnothing$. $\square$

Cartesian product of two normal spaces (”Sorgenfrey lines”) discussed in Example 2.4 is not a normal space.

Definition 6.6. Let (X, T_X), (Y, T_Y) be topological spaces and let $f : X \to Y$. The *graph* of function f is the set $\Gamma(f) = \{(x, y) \in X \times Y : y = f(x)\}$.

Theorem 6.7. *Let (X, T_X) be a topological space, let (Y, T_Y) be a Hausdorff space and let $f : X \to Y$ be a continuous function. Then $\Gamma(f)$ is closed set in $X \times Y$.*

Proof. Let $(x, y) \in cl(\Gamma(f))$, where $cl(\Gamma(f))$ denotes the closure of $\Gamma(f)$. We will show that $(x, y) \in \Gamma(f)$, i.e. that $cl(\Gamma(f)) \subset \Gamma(f)$ (because we already know that $\Gamma(f) \subset cl(\Gamma(f))$ - this is always true for closures of sets).

Suppose that $(x, y) \notin \Gamma(f)$, this is equivalent to saying that $y \neq f(x)$ (see Definition 6.6). We will prove Theorem 6.7 by getting a contradiction.

(Y, T_Y) is a Hausdorff space, so we can find open sets $V_y, U_y \in T_Y$ such that $f(x) \in V_y$, $y \in U_y$ and

$$V_y \cap U_y = \varnothing. \tag{6.1}$$

The function f is continuous (see Definition 4.1), so for V_y as above, there exists an open set $V_x \in T_X$ such that for all $z \in X$ we have:

$$z \in V_x \Rightarrow f(z) \in V_y. \tag{6.2}$$

Now we recall Theorem 1.18c), which says

$x \in cl(A)$ if and only if for every neighbourhood U of x we have $U \cap A \neq \varnothing$.

Take the neighbourhood $V_x \times U_y$ of the point (x, y), with V_x and U_y as above in this proof. This is indeed a neighbourhood of (x, y), because $x \in V_x \in T_X$ and $y \in U_y \in T_Y$.

By Theorem 1.18c, there exists a point $(a, f(a)) \in \Gamma(f)$ such that $(a, f(a)) \in V_x \times U_y$, i.e., $a \in V_x$ and $f(a) \in U_y$. But by (6.2), we have $f(a) \in V_y$ - which contradicts (6.1).

$\square$

Homotopy

Recall that $I = [0, 1]$ is the interval in $\mathbb{R}$.

Definition 6.8. Let (X, T_X), (Y, T_Y) be topological spaces and let $f, g : X \to Y$ be continuous functions. We say that the map f is *homotopic* to g if there exists a continuous map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. Then H is called a *homotopy* between f and g. We denote this writing $f \sim_H g$

Theorem 6.9. *Relation of being homotopic is an equivalence relation in the set of continuous maps between two given spaces.*

Proof. To show $f \sim_H f$ simply take $H(x, t) = f(x)$ for all $t \in [0, 1]$.

If $f \sim_H g$, let H be as in Definition 4.13., $H(x, 0) = f(x)$, $H(x, 1) = g(x)$. Let $H_0(x, t) = H(x, 1 - t)$. Then $H_0(x, 0) = g(x)$ and $H_0(x, 1) = f(x)$ and we obtain $g \sim_H f$.

If $f \sim_H g$, $g \sim_H h$, let H_{fg} be the homotopy between f and g and H_{gh} be the homotopy between g and h. Take H_{fh} as

$$H_{fh}(x, t) = \begin{cases} H_{fg}(x, 2t) & \text{for } 0 \leqslant t \leqslant \frac{1}{2} \\ H_{gh}(x, 2(t - \frac{1}{2})) & \text{for } \frac{1}{2} < t \leqslant 1. \end{cases}$$

Then H_{fh} is a homotopy between f and h.

□

Definition 6.10. A map $f : X \to Y$ is said to be a *homotopy equivalence* with *homotopy inverse* $g : Y \to X$ if $g \circ f \sim Id_X$ and $f \circ g \sim Id_Y$. Then we say that X and Y have the same homotopy type.

Theorem 6.11. *Let (X, T_X) be a topological space and let E be a convex subset of $\mathbb{R}^n$ (considered with the natural topology). If $f, g : X \to E$, then f and g are homotopic.*

Proof. Do it yourself. Hint: take as the homotopy H the function $H(x, r) = (1 - r) \cdot f(x) + r \cdot g(x)$, for $x \in X$ and $r \in I$. □

Definition 6.12. We say that a topological space (X, T) is *contractible* to a point $x_0 \in X$ if the identity map $id_X : X \to X$ (i.e., $id_X(x) = x$ for all $x \in X$) is homotopic to the constant map $t(x) = x_0$

Fact 4.17*. Sphere S^1 is not a contractible space.

Definition 4.13. A subset M of a topological space (X, T) is called a *retract of* X if there exists a continuous transformation $r : X \to M$ such that $r(x) = x$ for all $x \in M$. The transformation r is then called a *retraction*.

Definition 4.18. Let (X, T) be a topological space and let $A \subset X$. A continuous map $F : X \times I \to X$ is called a *deformation retraction* of the space X onto A if for all $x \in X$, $a \in A$:

$$F(x, 0) = x, \quad F(x, 1) \in A, \quad F(a, 1) = a.$$

Then A is called a *deformation retract* of X.

If, additionally, $F(a, t) = a$ for all $t \in I$, then F is called a *strong deformation retraction.*

Theorem 4.19. *A metric space X is contractible to x_0 if and only if $\{x_0\}$ is a deformation retract of X.*

Proof. It follows from comparing Definitions 6.12 and 4.18. □

Remark. From the fact 4.17* it follows that a single-point set is not a deformation retract of S^1, however it is a retract of S^1.

Generalized Cartesian product

Definition 4.20*. Let $\{X_s\}_{s \in S}$ be an indexed family of sets. The Cartesian product of the sets of the family $\{X_s\}_{s \in S}$ is the set of all functions $\phi : S \to \bigcup_{s \in S} X_s$ satisfying the following condition: $\phi(s) \in X_s$ for every $s \in S$. This Cartesian product we be denoted by $\times_{s \in S} X_s$. If $X_s = X$ for all $s \in S$, then we shall write X^S instead, and if $S = \mathbb{N}$ (the set of natural numbers), we shall write $X^{\aleph_0}$

Definition 4.21*. Let $\{(X_s, T_s)\}_{s\in S}$ be an indexed family of topological spaces. The Cartesian product of topological spaces $\{(X_s, T_s)\}_{s\in S}$ is the topological space $(\times_{s\in S} X_s, T)$, where T is the topology generated by the base consisting of the sets $\times_{s\in S} V_s$, where $V_s \in T_s$ for all $s \in S$ and $V_s \neq X_s$ only for a finite number of indices $s \in S$. The sets X_s are called axes of the Cartesian product.

Definition 4.22*. The space $I^{\aleph_0}$ is called the *Hilbert cube.*

Theorem 4.23*. *Hilbert cube is a metrizable space.*

Definition 4.24*. A topological space Y is embeddable in the topological space X if there exists a subspace A of the space X and a homeomorphism $h : Y \to A$ onto A.

Definition 4.25*. A topological space (X, T) is called a universal space for the property W if the space (X, T) has the property W and every space that has the property W is embeddable in (X, T).

Theorem 4.26*. *Hilbert cube is a universal space for second countable metrizable spaces.*

7. Separable spaces. Second-countable spaces.

Recall Definition 1.6. A family $B \subset T$ is called a *base* of a topological space (X, T) if every non-empty open set is a union of some sets from B.

Recall Definition 1.13(b). A topological space (X, T) is called *second-countable* if it has a countable base.

Recall Definition 3.13(a). A set A is called *dense* if $cl(A) = X$.

Recall Theorem 3.14. Let (X, T) be a topological space. A set $A \subset X$ is dense in (X, T) if and only if for all $U \in T \backslash \{\varnothing\}$ we have $U \cap A \neq \varnothing$.

General assumption: we say that a set is *countable* if it is finite or of the same cardinality as the set of natural numbers.

Definition 7.1. A topological space (X, T) is called a *separable* space if there exists a countable set A dense in X.

Example 7.I. The Sorgenfrey line from Example 2.IV. is a separable space.

Example 7.II. Let X be an uncountable set and let J be the ideal of countable subsets of X. Then the space (X, T_J) constructed as in Example 1.I. is not a separable space.

If J is the ideal of finite sets, then (X, T_J) is a separable space.

Example 7.III. A topological space (X, T), where T is the discrete topology, is separable if and only if X is a countable set.

Theorem 7.2. *If a topological space (X, T) is second-countable, then it is separable.*

Proof. Let β be a countable base of (X, T). Let A be a set obtained by taking one element from each of the sets $B \in \beta$. Then, obviously, A is a countable set.

To finish the proof, we need to show that A is a dense set in (X, T). Let $U \in T \backslash \{\varnothing\}$. According to the definition of a base, there exists a set $B \in \beta$ such that $B \subset U$. Let $b \in B \cap A$ (such element exists, by the construction of the set A). It follows that $U \cap A \neq \varnothing$, which, by Theorem 3.14, completes the proof.

Example 7.IV. Sorgenfrey line (from Example 2.IV) is a separable space (see Example 7.I), which is not second-countable.

Theorem 7.3. *A metric space (X, ρ) is separable if and only if it is second-countable.*

This theorem should be known from the metric topology, perhaps formulated in the following way: a metric space (X, ρ) is separable if and only if it has a countable base.

Theorem 7.4. *If a topological space (X, T) is second-countable, then the cardinality of the set of all open subsets of (X, T) is not greater than continuum.*

Proof. Since (by Definition 1.6) every open set is a union of some sets from the base, there cannot be more open sets than distinct subsets of the set of natural numbers.

$\square$

Definition 7.5. Let (X, T) be a topological space.

(a) A family $\{A_s\}_{s \in S}$ is called a *cover* of a set A if $A \subset \bigcup_{s \in S} A_s$. If, furhermore, the family $\{A_s\}_{s \in S}$ consists of open (closed) sets, then we call it an *open* (*closed*) *cover.* A cover of the space X is shortly called: a cover.

(*b*) Let $\{A_s\}_{s\in S}$ and $\{B_t\}_{t\in T}$ be covers of A. We say that a cover $\{A_s\}_{s\in S}$ is a *refinement* of a cover $\{B_t\}_{t\in T}$ if for every $s \in S$ there exists $t(s) \in T$ such that $A_s \subset B_{t(s)}$.

(*c*) A cover $\{A_s\}_{s\in S}$ is called a *subcover* of a cover $\{B_t\}_{t\in T}$ if for every $s \in S$ there exists $t(s) \in T$ such that $A_s = B_{t(s)}$.

Theorem 7.6 (Lindeloef). *If (X, T) is a second-countable topological space, then from every cover of an open set $A \subset X$ we can choose a countable subcover.*

Remark. In particular, from every open cover (of the space (X, T)) we can choose a countable subcover.

Proof. Let $\{A_s\}_{s\in S}$ be an open cover of a set A and let $\beta = \{B_1, B_2, \ldots\}$ be a base of (X, T).

For every $x \in A \subset \bigcup_{s\in S} A_s$ there exists $s(x) \in S$ such that $x \in A_{s(x)}$. On the other hand, $A_{s(x)}$ is an open set, so there exists $n(x)$ such that

(1) $\qquad B_{n(x)} \in \beta \quad$ and $\quad x \in B_{n(x)} \subset A_{s(x)}$.

This way, to every $x \in A$ we can assign $B_{n(x)} \in \beta$ such that (1) holds.

$x \mapsto B_{n(x)} \in \beta$

We have

(2) $\qquad A = \bigcup_{x\in A}\{x\} \subset \bigcup_{x\in A} B_{n(x)}$.

Denote by $N^* = \{n = n(x) : x \in A\}$. Then $A \subset \bigcup_{n\in N^*} B_n$. Obviously, N^* is a countable set. For every $n \in N^*$ let us choose (using (1)) a set $A_{s(x)}$ that contains $B_{n(x)} = B_n$ and denote it by A_n. Then $A \subset \bigcup_{n\in N^*} B_n \subset \bigcup_{n\in N^*} A_n$ and it follows that $\{A_n\}_{n\in N^*}$ is a countable subcover of the cover $\{A_s\}_{s\in S}$ of the set A.

□

Theorem 7.7. *If (X, T) is a separable space, then every family of open, non-empty and pairwise disjoint sets is countable.*

Proof. Assume the contrary, i.e. that there exists an uncountable family $\{V_t\}_{t\in T}$ of open, non-empty and pairwise disjoint sets. Let A be a dense set in X. We will show that

(1) A is an uncountable set.

Assume the contrary. Then, by assumptions that the family $\{V_t\}_{t\in T}$ is uncountable and that the sets of this family are pairwise disjoint, there exists $t_0 \in T$ such that $V_{t_0} \cap A = \varnothing$, which contradicts the assumption that A is dense. This contradiction proves (1). From (1) it follows that (X, T) is not separable, contrary to the assumption of the theorem.

□

Theorem 7.8. *If (X, T_X) is a separable space and (Y, T_Y) is a topological space and there exists a continuous function $f : X \xrightarrow{onto} Y$, then (Y, T_Y) is separable.*

Proof. Let A be a countable set dense in (X, T_X). We will show that $f(A)$ is a countable set dense in (Y, T_Y). Since A is countable, so is $f(A)$.

We will show that

(1) $f(A)$ is a dense set in (Y, T_Y).

In order to do this, we will first prove that

(2) $f(cl(B)) \subset cl(f(B))$, for every set $B \subset X$.

Remark: Condition (2) is equivalent to the continuity of the function f.

Let $B \subset X$. Then

$B \subset f^{-1}(f(B)) \subset f^{-1}(cl(f(B)))$.

Observe that $cl(f(B))$ is a closed set, so by Theorem 4.2., $f^{-1}(cl(f(B)))$ is also a closed set. From the definition of the closure of a set we can obtain that $cl(B) \subset f^{-1}(cl(f(B)))$.

Hence $f(cl(B)) \subset f(f^{-1}(cl(f(B)))) \subset cl(f(B))$. This proves condition (2).

From (2) we obtain $Y = f(X) = f(cl(A)) \subset cl(f(A))$.

□

Remark. Let (X, ρ_X) and (Y, ρ_Y) be metric spaces, let (X, ρ_X) be a second-countable space and let $f : X \xrightarrow{\text{onto}} Y$ be a continuous function. Then (Y, ρ_Y) is a second-countable space.

Lemma 7.9. *If (X, T_X) and (Y, T_Y) are topological spaces, A is a dense set in (X, T_X) and B is a dense set in (Y, T_Y), then $A \times B$ is a dense set in $X \times Y$.*

Proof. Recall

Theorem 3.14. Let (X, T) be a topological space. A set $A \subset X$ is a dense set if and only if for every $U \in T \backslash \{\emptyset\}$ we have $U \cap A \neq \emptyset$.

Definition 6.1. *Cartesian product* of topological spaces (X, T_X) and (Y, T_Y) is the topological space $(X \times Y, T_{X \times Y})$, where $T_{X \times Y}$ is the topology introduces by the base $B_{X \times Y} = \{U_1 \times U_2 : U_1 \in T_X, U_2 \in T_Y\}$ (this topology is called *Tychonoff topology*). The sets X, Y are then called *axes* of the Cartesian product.

Let $W \neq \emptyset$ be an open set in $X \times Y$. By Definition 6.1. there exist non-empty sets $U_1 \in T_X$ and $U_2 \in T_Y$ such that $U_1 \times U_2 \subset W$. From Theorem 3.14. it follows that there exist $x_1 \in U_1 \cap A$ and $x_2 \in U_2 \cap B$. Then $(x_1, x_2) \in A \times B$ and $(x_1, x_2) \in U_1 \times U_2 \subset W$.

$\square$

Theorem 7.10. *Finite Cartesian product of separable spaces is a separable space.*

Remark: Cartesian product of a continuum of separable spaces is a separable space (but the proof is much more difficult). Recall Theorem 7.6 (Lindeloef).

Theorem 7.6 (Lindeloef). If (X, T) is a second-countable topological space, then from every cover of an open set $A \subset X$ we can choose a countable subcover.

Remark. In particular, from every open cover (of the space (X, T)) we can choose a countable subcover.

Definition 7.11. A topological space (X, T) is called a *Lindeloef space* if it is a regular space and every open cover of this space has a countable subcover.

Cartesian product of Lindeloef spaces does not need to be a Lindeloef space.

Theorem 7.12. *Every regular, second-countable space is a Lindeloef space.*

Proof follows directly from Theorem 7.6.

Example 7.IV. Niemycki plane is a separable space, but is not a Lindeloef space.

Example 7.V. Let X be an uncountable set and let J be the ideal of countable subsets of X. Then the space (X, T_J) constructed as in Example 1.I. (see below) is not a separable space, but is a Lindeloef space.

Example 1.I. Let $X \neq \emptyset$ and let J be an ideal of subsets of X. Then (X, T_J), where $T_J = \{X \backslash A : A \in J\} \cup \{\emptyset\}$ is a topological space, we call it the topological space generated by the ideal J.

Theorem 7.13. *Every Lindeloef space is normal.*

The proof of the above theorem follows from regularity of a Lindeloef space and the following lemma.

Lemma Let X be a T_1-space such that for every closed set $F \subset X$ and open set W such that $F \subset W$ there exists a sequence of open sets $W_1, W_2, \ldots$ such that $F \subset \bigcup_{n=1}^{\infty} W_n$ and $cl(W_n) \subset W$ for all $n = 1, 2, \ldots$. Then X is a normal space.

Proof of the Lemma. Let A, B be closed, disjoint subsets of X. Setting $F = A$ and $W = X \backslash B$, we can find W_n such that $A \subset \bigcup_{n=1}^{\infty} W_n$ and $B \cap cl(W_n) = \varnothing$ for all n. Analogously, setting $F = B$ and $W = X \backslash A$, we get another sequence of open sets V_n such that $B \subset \bigcup_{n=1}^{\infty} V_n$ and $A \cap cl(V_n) = \varnothing$ for all n.

Let $G_n = W_n \backslash \bigcup_{i=1}^{n} cl(V_i)$ and $H_n = V_n \backslash \bigcup_{i=1}^{n} cl(W_i)$.

Then all sets G_n, H_n are open and $A \subset \bigcup_{n=1}^{\infty} G_n = U$ and $B \subset \bigcup_{n=1}^{\infty} H_n = V$. Now it is enough to prove that U, V are disjoint.

Since $G_i \cap V_j = \varnothing$ for $j \leqslant i$, we have also $G_i \cap H_j = \varnothing$ for $j \leqslant i$. Similarly, $H_j \cap W_i = \varnothing$ for $j \geqslant i$, so also $G_i \cap H_j = \varnothing$ for $j \geqslant i$. Hence, $G_i \cap H_j = \varnothing$ for all $i, j = 1, 2, \ldots$ and so $U \cap V = \varnothing$.

$\square$

Example 7.VI. Sorgenfrey line is a Lindeloef space. Indeed, consider an open cover $\{U_s\}_{s \in S}$ of the Sorgenfrey line, for $a \in \mathbb{R}$ let C_a be the set of all $x > a$ such that there exists a countable subcover of $\{U_s\}_{s \in S}$ which covers the interval $[a, x]$. Clearly $C_a \neq \varnothing$ because some set U_{s_a} covers $\{a\}$ and U_{s_a} is open, so it contains a set from the base: $[a, a + \epsilon)$ for some $\epsilon > 0$.

We show that $\sup\{x \in C_a\} = +\infty$. Indeed, suppose that $\sup\{x \in C_a\} = b < +\infty$, then for every $b_n = b - \frac{1}{n}$ we have $b_n \in C_a$ so for every $n \in \mathbb{N}$ we have a countable subcover $\{W_m^{(n)}\}_{m=1}^{\infty}$ of the cover $\{U_s\}_{s \in S}$, such that $[a, b_n]$ is covered by $\{W_m^{(n)}\}_{m=1}^{\infty}$. Now, let W be any set from $\{U_s\}_{s \in S}$ such that $b \in W$. Then there exists $\epsilon > 0$ such that $[b, b + \epsilon) \subset W$, and hence $b + \epsilon/2 \in W$. Then $W \cup \bigcup_{n,m=1}^{\infty} W_m^{(n)}$ is a countable subcover of the cover $\{U_s\}_{s \in S}$, which covers the interval $[a, b + \epsilon/2]$, hence $\sup C_a \geqslant b + \epsilon/2 > b$.

Therefore $\sup\{x \in C_a\} = +\infty$ and hence every interval $[-n, n]$ has a countable subcover $\mathcal{U}_n$ of the cover $\{U_s\}_{s\in S}$. Then $\bigcup_{n=1}^{\infty} \mathcal{U}_n$ is a countable subcover of the Sorgenfrey line.

Hence, by Theorem 7.13, Sorgenfrey line is a normal space.

Theorem 7.14. *If (X, T_X) is a Lindeloef space, (Y, T_Y) is a regular space and there exists a continuous function $f : X \xrightarrow{onto} Y$, then (Y, T_Y) is a Lindeloef space.*

A family $\{A_s\}_{s\in S}$ of subsets of X is called a *locally finite family of sets* if for every point $x \in X$ there exists a neighbourhood U_x of this point such that $\{s \in S : A_s \cap U_x \neq \varnothing\}$ is a finite set.

Theorem 7.15. *If (X, T_X) is a Lindeloef space, then every open cover of X has a refinement that is a locally finite family of open sets.*

Theorem 7.16 (*). *For every cardinal number k and every metrizable space (X, T) the following statements are equivalent:*

(i) The space (X, T) has a base of cardinality k.

(ii) Every open cover of the space (X, T) has a subcover of cardinality k.

(iii) Every discrete subspace of (X, T) has cardinality not greater than k.

(iv) There exists a dense subset in (X, T) of cardinality not greater than k.

Theorem 7.17. *For every metrizable space (X, T) the following statements are equivalent:*

(i) The space (X, T) is second-countable.

(ii) The space (X, T) is a Lindeloef space.

(iii) The space (X, T) is separable.

Proof. For all topological spaces *(i)* implies *(ii)* and *(iii)*, according to Theorems 7.6 and 7.2.

For a separable metric space with a countable dense set A, a family of open balls $\{B(x, \frac{1}{n}) : x \in A, n \in \mathbb{N}\}$ is a countable base, hence a separable metric space is second countable.

Let (X, d) be a Lindeloef metric space. For an open cover of X by open balls $\{B(x, \frac{1}{n}) : x \in X, n \in \mathbb{N}\}$ take a countable subcover $\{S_m^{(n)}\}_{m=1}^{\infty}$ and for every $m \in \mathbb{N}$ choose any point $x_m^{(n)} \in S_m^{(n)}$. Then the set $\{x_m^{(n)}, n \in \mathbb{N}, m \in \mathbb{N}\}$ is countable and dense, hence a Lindeloef metric space is separable. $\square$

8. Complete spaces.

Definition 8.1. Let (X, ρ) be a metric space and let $\{x_n\}$ be a sequence of its elements. We say that $\{x_n\}$ is a *Cauchy sequence* if

$$\forall_{\epsilon>0} \quad \exists_{n^*\in\mathbb{N}} \quad \forall_{n,m>n^*} \quad \rho(x_n, x_m) < \epsilon.$$

Definition 8.2. A metric space (X, ρ) is called *complete* if every Cauchy sequence in it is convergent.

Remark: A complete metric space may be homeomorphic to a space that is not complete, e.g. $\mathbb{R}$ is homeomorphic to $(0, 1)$ with natural metric.

Theorem 8.1 (Cantor)**.** *A metric space (X, ρ) is complete if and only if for every sequence of closed, non-empty sets $\{F_n\}$ such that $F_{n+1} \subset F_n$ and* $\operatorname{diam} F_n = \sup_{x,y\in F_n} \rho(x, y) \to 0$ *we have $\bigcap_n F_n$ is a one-element set.*

Proof. Take $x_n \in F_n$, then from $\operatorname{diam} F_n \to 0$ we get that $\{x_n\}$ is a Cauchy sequence. Hence it has a limit x. Since $x_n \in F_n$ and limit of elements of a closed set belongs to that closed set, we have $x \in \bigcap_n F_n$. From $\operatorname{diam} F_n \to 0$ we get $\{x\} = \bigcap_n F_n$.

On the other hand, for a Cauchy sequence $\{x_n\}$ consider $F_n = cl(\{x_n, x_{n+1}, \ldots\})$. Then $F_{n+1} \subset F_n$ and $\operatorname{diam} F_n \to 0$ (from $\{x_n\}$ being a Cauchy sequence). Since $\bigcap_n F_n = x$ we get $x = \lim_{n\to\infty} x_n$.

$\square$

Theorem 8.2 (Baire)**.** *Let (X, ρ) be a complete metric space. If $A \subset X$ is a first category set, then it is a boundary set (i.e., $X \setminus A$ is dense in X).*

Proof. Let B_1 be an open ball in X. We will show that $B_1 \cap (X \backslash A) \neq \varnothing$.

Since A is first category set, we have $A = \bigcup_{n=1}^{\infty} A_n$ for some nowhere dense sets A_n. Since A_1 is nowhere dense, there is an open ball $B_2 \subset B_1$ such that $B_2 \cap A_1 = \varnothing$. Analogously, there is $B_3 \subset B_2$ such that $B_3 \cap (A_1 \cup A_2) = \varnothing$ etc. We can assume that $\mathrm{diamB_n} < \frac{1}{n}$ for all n. Considering smaller, closed balls $F_n \subset B_n$ and using Cantor theorem, we get $\bigcap_{n=1}^{\infty} F_n = x \notin \bigcup_{n=1}^{\infty} A_n = A$. Since $x \in B_1$ and B_1 was a ball in X, we see that every open set contains an element from $X \backslash A$.

$\square$

Definition 8.3. Let (X, ρ) be a metric space and let $f : X \to X$. We say that f is a *contraction* if there exists a constant $M \in [0, 1)$ such that for all $x, y \in X$ we have $\rho(f(x), f(y)) \leqslant M\rho(x, y)$.

Theorem 8.3 (Banach fixed point theorem). *Let (X, ρ) be a complete metric space and let $f : X \to X$ be a contraction. Then f is a continuous function and there exists exactly one fixed point y_0 of f (i.e. $f(y_0) = y_0$, see Definition 4.15).*

Proof.

First we show that f is continuous. Let $z_0 \in X$ and let $\{z_n\}$ be a sequence in X convergent to z_0. Since f is a contraction, there exists a constant $M \in [0, 1)$ such that for all n we have $\rho(f(z_n), f(z_0)) \leqslant M\rho(z_n, z_0)$. Therefore, we have $\rho(f(z_n), f(z_0)) \to 0$ for $n \to \infty$, which proves that f is continuous.

We will now construct (using induction) a sequence $\{x_n\}$. Let x_0 be any element of X. Assume that we have defined $x_0, x_1, \ldots, x_n \in X$. Set $x_{n+1} =$

$f(x_n)$. Thus we obtain a sequence $(x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \ldots)$, that we can shortly write as $(x_0, f(x_0), f^2(x_0), f^3(x_0), \ldots)$. We will show (using induction) that

(1) $\rho(x_{n+1}, x_n) \leqslant \rho(x_1, x_0)M^n$ for all $n = 0, 1, 2, \ldots$

Indeed, for $n = 0$ we have $\rho(x_1, x_0) = \rho(x_1, x_0)M^0$. Suppose that

$$\rho(x_{n+1}, x_n) \leqslant \rho(x_1, x_0)M^n,$$

we will show that

$$\rho(x_{n+2}, x_{n+1}) \leqslant \rho(x_1, x_0)M^{n+1}.$$

We compute

$$\begin{aligned} \rho(x_{n+2}, x_{n+1}) &= \rho(f(x_{n+1}), f(x_n)) \leqslant M\rho(x_{n+1}, x_n) \\ &\leqslant M\rho(x_1, x_0)M^n = \rho(x_1, x_0)M^{n+1}. \end{aligned}$$

(in the first line we used that f is a contraction, in the second line we used the assumption from our induction) This proves (1).

Now we show that

(2) $\rho(x_n, x_{n+k}) \leqslant \rho(x_1, x_0)M^n \frac{1}{1-M}$.

Using the triangle inequality and (1), we obtain

$$\begin{aligned} \rho(x_n, x_{n+k}) &\leqslant \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \ldots + \rho(x_{n+k-1}, x_{n+k}) \\ &\leqslant \rho(x_1, x_0)M^n + \rho(x_1, x_0)M^{n+1} + \ldots + \rho(x_1, x_0)M^{n+k-1} \\ &= \rho(x_1, x_0)M^n(1 + M + \ldots + M^{k-1}) \\ &= \rho(x_1, x_0)M^n \frac{1-M^k}{1-M} \leqslant \rho(x_1, x_0)M^n \frac{1}{1-M} \end{aligned}$$

This proves (2).

We will now show that

(3) $\{x_n\}$ is a Cauchy sequence.

Let $\epsilon > 0$. Notice that

$$\rho(x_1, x_0)M^n \frac{1}{1-M} \to 0$$

Hence, there exists $n^* \in \mathbb{N}$ such that

$$\rho(x_1, x_0)M^n \frac{1}{1-M} < \epsilon$$

for $n > n^*$. Let $n, m > n^*$ and assume that $m > n$. Then $m = n + k$ for some $k \in \mathbb{N}$, and from (2) it follows that

$$\rho(x_n, x_m) = \rho(x_n, x_{n+k}) \leqslant \rho(x_1, x_0)M^n \frac{1}{1-M} < \epsilon$$

which proves that $\{x_n\}$ is a Cauchy sequence. Since (X, ρ) is a complete space, there exists $y_0 \in X$ such that $x_n \to y_0$. We will show now that y_0 is the only fixed point of f.

First we show that

(4) $f(y_0) = y_0$.

Consider a sequence $\{f(x_n)\}$. Since $x_n \to y_0$ and f is continuous, it follows that $f(x_n) \to f(y_0)$. On the other hand, since $f(x_n) = x_{n+1}$, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = y_0$, which proves (4). To finish the proof, it is enough to show that f has no other fixed points. Suppose that for y_1 we have $f(y_1) = y_1$. We will show that then

(5) $y_0 = y_1$.

Since f is a contraction and y_0, y_1 are its fixed points, we have

$$\rho(f(y_0), f(y_1)) \leqslant M\rho(y_0, y_1) \leqslant M\rho(f(y_0), f(y_1)),$$

but since $M < 1$, the above inequality can be satisfied only with

$$\rho(f(y_0), f(y_1)) = 0,$$

i.e., $f(y_0) = f(y_1)$. Using again the assumption that both y_0 and y_1 are fixed points of f, we obtain $y_0 = y_1$, which proves (5).

Theorem 8.7. Let A be a set and let (X, ρ) be a complete metric space. We denote by $M(A, X)$ the set of all bounded functions $\tau : A \to X$. Then $d : M(A, X) \times M(A, X) \to \mathbb{R}$ given by the formula

$$d(\tau_1, \tau_2) = \sup_{a \in A} \rho(\tau_1(a), \tau_2(a))$$

for $\tau_1, \tau_2 \in M(A, X)$ is a metric on $M(A, X)$, and $(M(A, X), d)$ is a complete metric space.

For every metric space X there exists a complete metric space $\tilde{X}$ such that X is isometric to a dense subset of $\tilde{X}$.

Indeed, consider space C of Cauchy sequences in X and the equivalence relation $\{x_n\} \sim \{y_n\} \Leftrightarrow d(x_n, y_n) \to 0$. Then in $C/\sim$ we can define metric $d(\{x_n\}, \{y_n\}) = \lim_{n\to\infty} d(x_n, y_n)$. To show that it is indeed a metric, we use completeness of $\mathbb{R}$, since $\{d(x_n, y_n)\}$ is a Cauchy sequence of real numbers.

Every $x \in X$ can be identified with the constant sequence $\{x, x, x, \ldots\} \in C$.

Elements $\{x_n\}$ of a Cauchy sequence in X will be represented by constant sequences $\{x_1\} = (x_1, x_1, \ldots)$ etc., their limit in $C/\sim$ will be the equivalence class of $(x_1, x_2, x_3, \ldots)$.

Let X be a linear space over a field K of real or complex numbers. A mapping $\|\cdot\| : X \to [0, \infty)$ that satisfies for all $x, y \in X$ and for all $\alpha \in K$ the following conditions:

(i) non-degeneracy: $\|x\| = 0 \Rightarrow x = 0$.

(ii) absolute homogenity: $\|\alpha x\| = |\alpha| \cdot \|x\|$,

(iii) trangle inequality (sub-additivity): $\|x + y\| \leqslant \|x\| + \|y\|$,

is called a *norm* in the space X, and a pair $(X, \|\cdot\|)$ is called a *normed space*.

A *Banach space* is a normed space $(X, \|\cdot\|)$ such that for the metric d defined by the formula $d(x, y) = \|x - y\|$ for all $x, y \in X$, the space (X, d) is a complete metric space.

Definition 8.8*. A topological space (X, T) is called a *completely metrizable space* if there exists in X a metric ρ such that the family of all open sets in the metric space (X, ρ) coincides with T.

Theorem 8.9*. If (X, T) is a completely metrizable space and $A \subset X$ is a G_δ-set, then A as a subspace of (X, T) is a completely metrizable space.

Theorem 8.10* If (X, T) is a metrizble space (not necessarily completely metrizable) and $A \subset X$ as a subspace of (X, T) is a completely metrizable space, then A is a G_δ-set.

Theorem 8.11*.(Lavrentiev). Let X, Y be completely metrizable spaces and let $\varnothing \neq A \subset X$, $\varnothing \neq B \subset Y$ (we do not assume anything else about A, B). Then every homeomorphism $h : A \to B$ can be extended to a homeomorphism $H : C \to D$, where C, D are some G_δ-sets, such that $A \subset C \subset X$, $B \subset D \subset Y$.

Definition 8.12.* A topological space (X, T) that is separable and completely metrizable is called a *Polish space*. If we further assume that (X, T) has no isolated points, then it is called a *perfect Polish space*.

Theorem 8.13* (Alexandrov). If (X, T) is a Polish space and $A \subset X$ is a G_δ-set, then A as a subspace of (X, T) is a Polish space.

Definition 8.14*. Let (X, T) be a topological space. A family M of its subsets is called a σ*-algebra* of sets if it satisfies the following conditions

(i) if $A \in M$, then $X \backslash A \in M$,

(ii) if $A_n \in M$ for $n = 1, 2, \ldots$, then $\bigcup_{n=1}^{\infty} A_n \in M$.

Remark: Condition (ii) can be replaced by:

(ii)' if $A_n \in M$ for $n = 1, 2, \ldots$, then $\bigcap_{n=1}^{\infty} A_n \in M$

Definition 8.15*. Let (X, T) be a topological space. The smallest σ-

algebra that contains all open sets in (X, T) is called the σ-algebra of Borel sets. It is denoted by $\mathcal{B}(X, T)$ (or $\mathcal{B}(X)$, or $\mathcal{B}$).

Remark: It can be proved, that such σ-algebra exists.

Theorem 8.16*. Let (X, T) be a topological space. Then open, closed, G_δ- and F_σ-sets are Borel sets.

Theorem 8.17*. (Kuratowski). If (X, T_X) and (Y, T_Y) are perfect Polish spaces, then there exist Borel first category sets $A_X \subset X$ and $A_Y \subset Y$ such that the subspaces $X \backslash A_X$ and $Y \backslash A_Y$ are homeomorphic.

Recall Theorem 8.4.(Baire)

Definition 8.18*. A topological space (X, T) is called a *Baire space* if every first category set in (X, T) is a boundary set.

Theorem 8.19*. Let (X, T) be a topological space. Then (X, T) is a Baire space if and only if the intersection of any sequence of dense, open sets is dense.

Definition 8.20*. We say that a topological space (X, T) has the *Blumberg property* if for every function (it is enough to consider here discontinuous functions) $f : X \to \mathbb{R}$ there exists a set B dense in X such that $f|_B : B \to \mathbb{R}$ is a continuous function.

Theorem 8.21*. If (X, T) has the Blumberg property, then it is a Baire space.

Definition 8.22*. Let (X, T) be a topological space.

(i) We say that a function $f : X \to \mathbb{R}$ is upper semi-continuous, if for every $\alpha \in \mathbb{R}$ the set $f^{-1}((-\infty, \alpha))$ is open.

(ii) We say that a function $f : X \to \mathbb{R}$ is lower semi-continuous, if for every $\alpha \in \mathbb{R}$ the set $f^{-1}((\alpha, \infty))$ is open.

Exercise: think about the relationship between semi-continuity and continuity.

Theorem 8.23*. A topological space (X, T) is a Baire space if and only if every lower semi-continuous function has a dense set of points of continuity.

Exercise: think if we can replace lower semi-continuous by upper semi-continuous in the theorem above. Hint: examine the relationship of different types of semi-continuity for f and $-f$.

Theorem 8.24*. (sandwich theorem) Let $f, g : \mathbb{R} \to \mathbb{R}$. If f is a upper semi-continuous function, g is a lower semi-continuous function and $f(x) \leqslant g(x)$ for all $x \in \mathbb{R}$, then there exists a continuous function $\phi : \mathbb{R} \to \mathbb{R}$ such that $f(x) \leqslant \phi(x) \leqslant g(x)$ for all $x \in \mathbb{R}$.

Banach-Mazur game *

Let (X, ρ) be a metric space. Let $\Gamma_0 = X$. Assume that there are two players A and B that make subsequent choices.

First player chooses a non-empty, open set $\Gamma_1 \subset \Gamma_0$.

Then, second player chooses non-empty, open set $\Gamma_2 \subset \Gamma_1$

Then, first player chooses a non-empty, open set $\Gamma_3 \subset \Gamma_2$.

etc.

We obtain a sequence of open sets $\{\Gamma_n\}$ such that $\Gamma_{n+1} \subset \Gamma_n$ and

$$\bigcap_{n=0}^{\infty} \Gamma_n = \bigcap_{n=0}^{\infty} \Gamma_{2n} = \bigcap_{n=1}^{\infty} \Gamma_{2n-1}.$$

Player A wins the game if $\bigcap_{n=0}^{\infty} \Gamma_n \neq \emptyset$.

Player B wins the game if $\bigcap_{n=0}^{\infty} \Gamma_n = \emptyset$.

A partial Banach-Mazur game for a player A (resp. B) is a finite sequence of players' choices, i.e. a finite sequence of non-empty, open sets $\{\Gamma_0, \Gamma_1, \dots, \Gamma_{n-1}, \Gamma_n\}$ such that $\Gamma_0 \supset \Gamma_1 \supset, \dots, \supset \Gamma_{n-1} \supset \Gamma_n$ and Γ_n was a set chosen by player B (resp. A). We adopt a convention that if the player A (resp. B) chooses first, then $\{\Gamma_0\}$ is a partial game for this player.

The set of all partial games for the player A (resp. B) will be denoted by $P(A)$ (resp. $P(B)$).

A strategy for the player A (resp. B) in the Banach-Mazur game is a function η that maps $P(A)$ (resp. $P(B)$) into a family of non-empty open sets such that $\eta(\{\Gamma_0, \Gamma_1, \ldots, \Gamma_{n-1}, \Gamma_n\}) \subset \Gamma_n$.

We say that a strategy for the player A (resp. B) in the Banach-Mazur game is *winning* (for the player A (resp. B)) if for every sequence of non-empty, open sets Γ_n such that $\Gamma_{n+1} \subset \Gamma_n$ that satisfy the following condition:

for every $i \in \mathbb{N}$ if $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_{i-1}\} \in P(A)$ (resp. $\{\Gamma_0, \Gamma_1, \ldots, \Gamma_{i-1}\} \in P(B)$), then $\Gamma_i = \eta(\{\Gamma_0, \Gamma_1, \ldots, \Gamma_{i-1}\})$

we have $\bigcap_{i=1}^{\infty} \Gamma_i \neq \varnothing$ (resp. $\bigcap_{i=1}^{\infty} \Gamma_i = \varnothing$).

Theorem 8.25*. A metric space (X, ρ) is a Baire space if and only if there is *no* winning strategy for player A, if player A starts the game.

Completion of a Metric Space

Definition. A *completion* of a metric space (X, d) is a pair consisting of a complete metric space (X^*, d^*) and an isometry $\varphi\colon X \to X^*$ such that $\varphi[X]$ is dense in X^*.

Theorem 1. *Every metric space has a completion.*

Proof. Let (X, d) be a metric space. Denote by $\mathcal{C}[X]$ the collection of all Cauchy sequences in X. Define a relation $\sim$ on $\mathcal{C}[X]$ by

$$(x_n) \sim (y_n) \iff \lim_{n\to\infty} d(x_n, y_n) = 0.$$

It is easy to see that this is an equivalence relation on $\mathcal{C}[X]$. Let X^* be the set of all equivalence classes for $\sim$:

$$X^* \;=\; \{\, [(x_n)] \;:\; (x_n) \in \mathcal{C}[X]\,\}.$$

Define $d^*\colon X^* \times X^* \to [0, \infty)$ by

$$d^*\left([(x_n)], [(y_n)]\right) = \lim_{n\to\infty} d(x_n, y_n),$$

where $[(x_n)]$, $[(y_n)] \in X^*$. To show that d^* is well-defined, let (x'_n) and (y'_n) be two Cauchy sequences in X such that $(x_n) \sim (x'_n)$ and $(y_n) \sim (y'_n)$. Then

$$\lim_{n\to\infty} d(x_n, x'_n) = \lim_{n\to\infty} d(y_n, y'_n) = 0.$$

By the triangle inequality,

$$\begin{aligned} d(x_n, y_n) \;&\le\; d(x_n, x'_n) + d(x'_n, y'_n) + d(y'_n, y_n) \quad \text{and} \\ d(x'_n, y'_n) \;&\le\; d(x'_n, x_n) + d(x_n, y_n) + d(y_n, y'_n). \end{aligned}$$

Hence,

$$|d(x_n, y_n) - d(x'_n, y'_n)| \;\le\; d(x_n, x'_n) + d(y_n, y'_n) \;\longrightarrow\; 0.$$

Since both $(d(x_n, y_n))$ and $(d(x'_n, y'_n))$ are convergent, this shows that

$$\lim_{n\to\infty} d(x_n, y_n) \;=\; \lim_{n\to\infty} d(x'_n, y'_n).$$

Thus d^* is well-defined.

Next, we show that d^* is a metric on X^*. Let $[(x_n)]$, $[(y_n)]$, $[(z_n)] \in X^*$. Then

$$d^*([(x_n)], [(y_n)]) = 0 \Leftrightarrow \lim_{n\to\infty} d(x_n, y_n) = 0 \Leftrightarrow (x_n) \sim (y_n) \Leftrightarrow [(x_n)] = [(y_n)].$$

Also,

$$d^*([(x_n)], [(y_n)]) = \lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(y_n, x_n) = d^*([(y_n)], [(x_n)]).$$

Since $d(x_n, z_n) \le d(x_n, y_n) + d(y_n, z_n)$,

$$\lim_{n\to\infty} d(x_n, z_n) \ \le \ \lim_{n\to\infty} d(x_n, y_n) + \lim_{n\to\infty} d(y_n, z_n).$$

Thus

$$d^*([(x_n)], [(z_n)]) \ \le \ d^*([(x_n)], [(y_n)]) + d^*([(y_n)], [(z_n)]).$$

Hence d^* is a metric on X^*.

For each $x \in X$, let $\hat{x} = [(x, x, \dots)] \in X^*$, the equivalence classes of the constant sequence $(x, x, \dots)$. Define $\varphi\colon X \to X^*$ by $\varphi(x) = \hat{x}$. Then for any x, $y \in X$,

$$d^*(\varphi(x), \varphi(y)) = d^*(\hat{x}, \hat{y}) = \lim_{n\to\infty} d(x, y) = d(x, y).$$

Hence φ is an isometry from X into X^*. To show that $\varphi[X]$ is dense in X^*, let $x^* = [(x_n)] \in X^*$ and let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence, there exists an $N \in \mathbb{N}$ such that for any m, $n \ge N$, $d(x_m, x_n) < \dfrac{\varepsilon}{2}$. Let $z = x_N$. Then $\hat{z} \in \varphi[X]$ and

$$d^*(x^*, \hat{z}) \ = \ \lim_{n\to\infty} d(x_n, z) \ = \ \lim_{n\to\infty} d(x_n, x_N) \ \le \ \frac{\varepsilon}{2} \ < \ \varepsilon.$$

Thus $\hat{z} \in B_{d^*}(x^*, \varepsilon) \cap \varphi[X]$. Hence, $\varphi[X]$ is dense in X^*.

Finally we show that (X^*, d^*) is complete. To establish this, we apply the following lemma of which proof is left as an exercise:

Lemma. *Let (X, d) be a metric space and A a dense subset such that every Cauchy sequence in A converges in X. Prove that X is complete.*

Hence, it suffices to show that every Cauchy sequence in the dense subspace $\varphi[X]$ converges in X^*. Let $(\widehat{z}_k)$ be a Cauchy sequence in $\varphi[X]$, where each $\widehat{z}_k$ is represented by the Cauchy sequence $(z_k, z_k, \dots)$. Since φ is an isometry,

$$d(z_n, z_m) \ = \ d^*(\widehat{z}_n, \widehat{z}_m) \qquad \text{for each } m,\, n.$$

Hence, $(z_1, z_2, z_3, \dots)$ is a Cauchy sequence in X. Let $z^* = [(z_1, z_2, z_3, \dots)] \in X^*$. To show that $(\widehat{z}_k)$ converges to z^*, let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that $d(z_k, z_n) < \dfrac{\varepsilon}{2}$ for any k, $n \ge N$. Hence, for each $k \ge N$,

$$d^*(\widehat{z}_k, z^*) \ = \ \lim_{n\to\infty} d(z_k, z_n) \ \le \ \frac{\varepsilon}{2} \ < \ \varepsilon.$$

This shows that $(\widehat{z}_k)$ converges to a point z^* in X^* and that X^* is complete. $\square$

Theorem 2. *A completion of a metric space is unique up to isometry. More precisely, if* $\{\varphi_1, (X_1^*, d_1^*)\}$ *and* $\{\varphi_2, (X_2^*, d_2^*)\}$ *are two completions of* (X, d)*, then there is a unique isometry* f *from* X_1^* *onto* X_2^* *such that* $f \circ \varphi_1 = \varphi_2$*.*

Proof. Since φ_1 is an isometry, φ_1 is 1-1. Thus $\varphi_1^{-1}\colon \varphi_1[X] \to X$ is an isometry from $\varphi_1^{-1}[X]$ onto X. Since φ_2 is an isometry from X onto $\varphi_2[X] \subseteq X_2^*$, it follows that $\varphi_2 \circ \varphi_1^{-1}\colon \varphi_1[X] \to \varphi_2[X]$ is a surjective isometry. Let $h = \varphi_2 \circ \varphi_1^{-1}$. Then

$$h \circ \varphi_1 = (\varphi_2 \circ \varphi_1^{-1}) \circ \varphi_1 = \varphi_2 \circ (\varphi_1^{-1} \circ \varphi_1) = \varphi_2 \circ i_X = \varphi_2.$$

Hence there exists a unique isometry f from X_1^* into X_2^* which is an extension of h. For each $x \in X$,

$$f \circ \varphi_1(x) = f(\varphi_1(x)) = h(\varphi_1(x)) = h \circ \varphi_1(x) = \varphi_2(x).$$

Thus $f \circ \varphi_1 = \varphi_2$. Similarly, there exists a unique isometry g from X_2^* into X_1^* such that $g \circ \varphi_2 = \varphi_1$. Therefore

$$g \circ f \circ \varphi_1 = g \circ \varphi_2 = \varphi_1 \qquad \text{and} \qquad f \circ g \circ \varphi_2 = f \circ \varphi_1 = \varphi_2.$$

Hence $g \circ f = i_{\varphi_1[X]}$ and $f \circ g = i_{\varphi_2[X]}$. Since $\varphi_1[X]$ is dense in X_1^*, we have $g \circ f = i_{X_1^*}$. Similarly, $f \circ g = i_{X_2^*}$. Thus $f = g^{-1}$. Hence, f is a unique isometry from X_1^* onto X_2^* such that $f \circ \varphi_1 = \varphi_2$. $\square$

9. Compact spaces.

Definition 9.1.

(a) A topological space (X, T) is called a *compact space*, if it is a Hausdorff space and every open cover of X has a finite subcover (*Borel-Lebesgue condition*).

(b) A subset M of a topological space (X, T) is called a *compact set* if M as a subspace (see Theorem 2.5) of the space (X, T) is a compact space.

Example 9.I. Let X be an infinite set and let J be the ideal of finite subsets of X. Then the space from Example 1.IV. (see below) is compact.

Example 1.IV. Let $X \neq \varnothing$, $x_0 \in X$ and let J be an ideal of subsets of X. Then (X, T_J) where $T_J = \{X \backslash A : A \in J\} \cup \{A \subset X : x_0 \notin A\}$ is a topological space.

Example 9.II. The Sorgenfrey line is not a compact space.

Example 9.III. Let $X = \mathbb{R}$ and let J be the ideal of finite subsets of X. Then the space (X, T_J) from Example 1.I. (see below) satisfies the Borel-Lebesgue condition, but it is not a compact space.

Example 1.I. Let $X \neq \varnothing$ and let J be an ideal of subsets of X. Then (X, T_J) where $T_J = \{X \backslash A : A \in J\} \cup \{\varnothing\}$ is a topological space.

Example 9.IV. The space I is a compact space.

Theorem 9.2. *Every compact subset of a Hausdorff space is a closed set.*

Proof. Let F be a compact set. To prove the Theorem, we will show that $X \backslash F \in T$.

Recall Theorem 1.3: $W \subset X$ is an open set if and only if for every $x \in W$ there exists a neighbourhood V_x of x such that $V_x \subset W$.

Let $x_0 \in X \backslash F$. Since (X, T) is a T_2-space, for every $y \in F$ there exist open sets U_y,V_y such that $x_0 \in U_y$, $y \in V_y$ and $U_y \cap V_y = \varnothing$.

Consider the family of sets $\{F \cap V_y\}_{y \in F}$. It is an open cover of the topological space (F, T_F) (T_F is the topology induced on F from (X, T)), so by the assumption that F is compact, there exists a finite set $y_1, y_2, \ldots y_n$ such that

$$F = (F \cap V_{y_1}) \cup (F \cap V_{y_2}) \cup \ldots \cup (F \cap V_{y_n})$$

It follows that

$$F \subset V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_n}$$

Let

$$U_{x_0} = U_{y_1} \cap U_{y_2} \cap \ldots \cap U_{y_n} \in T.$$

Then, by the definition of U_y, we have $x_0 \in U_{x_0}$ and since U_y and V_y are disjoint,

$$\varnothing = U_{x_0} \cap (V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_n}) \supset U_{x_0} \cap F$$

and it follows that $U_{x_0} \subset X \backslash F$. $\square$

Example: consider $X = I \cup \{x_0\}$, $T = \varnothing \cup X \cup T_I$, where T_I is the natural topology of I. Then X is not a Hausdorff space, I is a compact set (see Example 9.IV), I is not closed in X (because $X \backslash I$ is not an open set).

Theorem 9.3. *Let (X, T) be a compact space and let $F \subset X$ be a closed set. Then F is a compact set.*

Proof. Recall Theorem 2.5., which says that the family of sets $T_F = \{F \cap U : U \in T\}$ is the induced topology on F.

Obviously, Theorem 9.3. is true for $F = X$. In what follows we consider only the case $F \neq X$. Consider the topological space (F, T_F). First, observe that

(1) (F, T_F) is a T_2-space.

Indeed. Let $x, y \in F \subset X$ be distinct points of the space. Since (X, T) is a T_2-space, there exist open sets U and V such that $x \in U$, $y \in V$, $U \cap V = \varnothing$. Then $U \cap F, V \cap F \in T_F$ and

$x \in (U \cap F)$, $y \in V \cap F$, $(U \cap F) \cap (V \cap F) = \varnothing$

which proves (1). Now we will show that

(2) (F, T_F) satisfies the Borel-Lebesgue condition

Let $\{U_s\}_{s \in S}$ be an open cover of (F, T_F), i.e.,

$\forall_{s \in S} \quad U_s \in T_F$ and $F \subset \bigcup_{s \in S} U_s$.

For every $s \in S$ since $U_s \in T_F$, we have $U_s = V_s \cap F$ for some $V_s \in T$. Let

$W = X\backslash F \in T$. Consider the family $\{V_s\}_{s\in S} \cup \{W\}$. Since

$$F \subset \bigcup_{s\in S} U_s \subset \bigcup_{s\in S} V_s,$$

we have $X = \bigcup_{s\in S} V_s \cup W$, and hence the family $\{V_s\}_{s\in S} \cup \{W\}$ is an open cover of X. The space (X, T) satisfies the Borel-Lebesgue condition, so we can choose a finite subcover of this cover, $V_{s_1}, \ldots V_{s_n}, W$, i.e., $X = V_{s_1} \cup \ldots \cup V_{s_n} \cup W$. Then

$$F = F \cap (V_{s_1} \cup \ldots \cup V_{s_n} \cup W) = (F \cap V_{s_1}) \cup \ldots \cup (F \cap V_{s_n}) \cup (F \cap W).$$

Since $W \cap F = \varnothing$, it follows from $(U_s = V_s \cap F)$ that $F \subset U_{s_1} \cup \ldots \cup U_{s_n}$. $\square$

Theorem 9.4. *Let (X, T_X) be a compact space, let (Y, T_Y) be a T_2-space and let $f : X \xrightarrow{onto} Y$ be a continuous function. Then (Y, T_Y) is a compact space.*

Proof. We assumed that (Y, T_Y) is a T_2-space, so it is enough to show that it satisfies the Borel-Lebesgue condition.

Let $\{V_s\}_{s\in S}$ be an open cover of (Y, T_Y). By Theorem 4.2. (which says that the pre-image of an open set by a continuous function is open), we have $f^{-1}(V_s) \in T_X$ for all $s \in S$. It follows that

$$X = f^{-1}(Y) = f^{-1}\big(\bigcup_{s\in S} V_s\big) = \bigcup_{s\in S} f^{-1}(V_s),$$

which implies that $\{f^{-1}(V_s)\}_{s\in S}$ is an open cover of (X, T_X). By the definition of a compact space, we can take a finite subcover

$$\{f^{-1}(V_{s_i})\}_{i=1,\ldots,n}$$

Then, since f maps X onto Y, we have

$$Y = f(X) = f(\bigcup_{i=1}^{n} f^{-1}(V_{s_i})) = \bigcup_{i=1}^{n} f(f^{-1}(V_{s_i})) = \bigcup_{i=1}^{n} V_{s_i}.$$

□

Theorem 9.5. *Every compact space is normal.*

Proof. Let (X, T) be a compact space. By definition (X, T) is a Hausdorff space, so it is a T_1-space. Let A, B be non-empty, disjoint closed subsets of (X, T). Let $a \in A$, then (since (X, T) is a Hausdorff space):

(1) for every $y \in B$ there exist sets $U_y, V_y \in T$ such that $a \in U_y, y \in V_y$ and $U_y \cap V_y = \varnothing$.

Consider a family of sets $\{B \cap V_y : y \in B\}$. This family is an open cover of the space (B, T_B) (prove it yourself!). By Theorem 9.3., (B, T_B) is a compact space, so there exists a finite set of elements $y_1, y_2, \ldots, y_n \in B$ such that

$$B = (B \cap V_{y_1}) \cup (B \cap V_{y_2}) \cup \ldots \cup (B \cap V_{y_n}).$$

Let $H_a = V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_n} \in T$. By the equality above, we have $B \subset H_a$. By (1), for every V_{y_i}, $i = 1, 2, \ldots, n$ there exists a corresponding U_{y_i}. Let $G_a = U_{y_1} \cap U_{y_2} \cap \ldots \cap U_{y_n} \in T$. Obviously, $a \in G_a$ and $G_a \cap H_a = \varnothing$.

Since we can choose arbitrary $a \in A$ above, we can repeat the same construction for every element of A. This way we obtain two families of sets $\{G_a\}_{a \in A}$ and $\{H_a\}_{a \in A}$ such that

for every $a \in A$ we have $a \in G_a$, $B \subset H_a$ and $G_a \cap H_a = \varnothing$.

Consider the family of sets $\{A \cap G_a\}_{a \in A}$. Then this family is an open cover of the space (A, T_A) (prove it yourself!). By Theorem 9.3., (A, T_A) is a compact space, so there exists a finite set of elements $a_1, a_2, \ldots, a_m \in A$ such that

$$A = (A \cap G_{a_1}) \cup (A \cap G_{a_2}) \cup \ldots \cup (A \cap G_{a_m}).$$

Let $U = G_{a_1} \cup G_{a_2} \cup \ldots \cup G_{a_m} \in T$ and let $V = H_{a_1} \cap H_{a_2} \cap \ldots \cap H_{a_m} \in T$. Then $A \subset U$, $B \subset V$ and $U \cap V = \varnothing$.

□

Definition 9.6. A family $\{F_s\}_{s \in S}$ of subsets of a topological space (X, T) has the *finite intersection property* if it is empty, or for every finite set of indices $s_1, s_2, \ldots, s_n \in S$ we have $\bigcap_{i=1}^{n} F_{s_i} \neq \varnothing$.

Theorem 9.7. *A Hausdorff space (X, T) is compact if and only if every non-empty family of its closed sets that has the finite intersection property has non-empty intersection.*

Proof. Necessity.

Let $\{F_s\}_{s \in S}$ be a non-empty family of closed sets, that has the finite intersection property. Suppose that

(1) $\bigcap_{s \in S} F_s = \varnothing$.

Let $U_s = X \backslash F_s$ for $s \in S$. Then the sets U_s are open and from (1) we obtain

$$\bigcup_{s \in S} U_s = \bigcup_{s \in S} X \backslash F_s = X \backslash (\bigcap_{s \in S} F_s) = X \backslash \varnothing = X,$$

hence $\{U_s\}_{s \in S}$ is an open cover of X. By the definiton of a compact space, there exists a finite set of indices $s_1, s_2, \ldots, s_n$ such that $X = \bigcup_{i=1}^{n} U_{s_i}$. Then, by the above equality, we have

$$\bigcap_{i=1}^{n} F_{s_i} = \bigcap_{i=1}^{n} X \backslash U_{s_i} = X \backslash \bigcup_{i=1}^{n} U_{s_i} = X \backslash X = \varnothing,$$

which contradicts the assumption that $\{F_s\}_{s \in S}$ has the finite intersection property. This proves that $\bigcap_{s \in S} F_s \neq \varnothing$.

Sufficiency.

Suppose that (X, T) is not a compact space. Then (note that we still assume that it is a Hausdorff space) there exists an open cover $\{U_s\}_{s \in S}$ of (X, T) such that for every finite set of indices $s_1, s_2, \ldots, s_n$ we have $X \neq U_{s_1} \cup U_{s_2} \cup \ldots \cup U_{s_n}$. It follows that for every finite set of indices $s_1, s_2, \ldots, s_n$ we have $\varnothing \neq X \backslash (U_{s_1} \cup U_{s_2} \cup \ldots \cup U_{s_n}) = (X \backslash U_{s_1}) \cap (X \backslash U_{s_2}) \cap \ldots \cap (X \backslash U_{s_n})$. From the above we obtain that the family of closed sets $\{X \backslash U_s\}_{s \in S}$ has the finite intersection property, so by the assumption $\varnothing \neq \bigcap_{s \in S} (X \backslash U_s) = X \backslash \bigcup_{s \in S} U_s$, which contradicts the assumption that the family $\{U_s\}_{s \in S}$ is an open cover of (X, T).

□

Theorem 9.8. *Let (X, ρ) be a metric space. Then (X, ρ) is a compact space if and only if it satisfies the Bolzano-Weierstrass condition, i.e., every sequence of points of X has a convergent subsequence.*

Proof - omitted.

Theorem 9.9. *Let (X, ρ) be a compact metric space. If $\{F_n\}$ is sequence of non-empty closed sets in (X, T), such that $F_{n+1} \subset F_n$, then $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$.*

Proof - omitted.

Theorem 9.10. *Every compact subset of a metric space (X, ρ) is closed and bounded.*

Proof - omitted.

Theorem 9.11 (Weierstrass)**.** *Let (X, T) be a compact topological space and let $f : X \to \mathbb{R}$ be a continuous function. Then $f(X)$ is a closed and bounded set.*

Proof. By Theorem 9.4. $f(X)$ is a compact subset of $\mathbb{R}$, so by the previous theorem it is a closed and bounded set.

Theorem 9.12. *A subset $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof - omitted.

Theorem 9.13. *Let (X, T_X) be a compact topological space and let (Y, T_Y) be a T_2-space. If $f : X \xrightarrow{onto} Y$ is a continuous, one-to-one function, then f is a homeomorphism.*

Proof. We will show that the function $g = f^{-1} : Y \to X$ is continuous. Indeed, let F be a closed subset of X. Then by Theorem 9.3., F is compact, so from Theorem 9.4. we obtain that $f(F)$ is compact. It follows from Theorem 9.2., that $f(F)$ is a closed set. Observe that $g^{-1}(F) = (f^{-1})^{-1}(F) = f(F)$, which completes the proof.

$\square$

Theorem 9.14 (Tichonov). *The Cartesian product of compact spaces is a compact space.*

Proof - we omit it.

Theorem 9.15. *Let (X, ρ) be a compact metric space and let λ be an open cover of this space. Then there exists a number $l > 0$ (called a* Lebesgue number of the cover λ*) such that for every element $x \in X$ we have $B(x, l)$ is contained in one of the sets from λ.*

Proof. Suppose that there is no such number $\lambda > 0$. This means that for every natural number n there exists an element $x_n \in X$ such that $B(x_n, \frac{1}{n})$ is not contained in any of the sets from λ. We will show that then

$(*)$ the sequence $\{x_n\}$ has no convergent subsequence.

Let $y \in X$. To prove $(*)$ we will show that y is not a limit of any subsequence of $\{x_n\}$. Since λ is an open cover of X, there exists an open set $V \in \lambda$ such that $y \in V$. Let $\epsilon > 0$ be such that $B(y, \epsilon) \subset V \in \lambda$. We consider now, which elements of $\{x_n\}$ are contained in $B(y, \epsilon/2)$? Suppose that $x_n \in B(y, \frac{\epsilon}{2})$. Then it easily follows that

(1) $B(x_n, \frac{\epsilon}{2}) \subset V$.

But according to our assumption,

(2) $B(x_n, \frac{1}{n}) \not\subset V$.

From (1) and (2) we conclude that $\frac{1}{n} > \frac{\epsilon}{2}$.

It follows that $x_n \notin B(y, \frac{\epsilon}{2})$ for $n \geqslant \frac{2}{\epsilon}$, which proves $(*)$, and $(*)$ contradicts the assumption that (X, ρ) is a compact metric space.

$\square$

One-point compactification: We say that a topological space X is locally compact if every point has a neighbourhood which has compact closure. If X is locally compact, we can make a new space $X^+ = X \cup \{\infty\}$ (here ∞ is just some new point, which does not belong to X), with topology defined as follows: U is open in X^+ if $U \subset X$ and U is open, or if $U = X^+ \backslash C$ for some compact set $C \subset X$. Then X^+ with the above topology is compact.

Other concepts of compactness

Definition 9.15. Let (X, T_X) be a topological space. A family $\{A_s\}_{s \in S}$ of subsets of X is called a *locally finite family of sets* if for every $x \in X$ there exists a neighbourhood U_x of x such that $\{s \in S : A_s \cap U_x \neq \emptyset\}$ is a finite set.

Theorem 9.16. For every locally finite family of sets $\{R_s\}_{s \in S}$ we have $cl(\bigcup_{s \in S} R_s) = \bigcup_{s \in S} cl(R_s)$.

$\square$

Definition 9.17. A topological space (X, T) is called a *paracompact space* if it is a T_2-space and every open cover of (X, T) has an open refinement that is a locally finite family of sets (shortly: every open cover has a locally finite open refinement).

Example 9.V. Sorgenfrey line is a paracompact space that is not compact.

Theorem 9.18.

(a) Every compact space is paracompact

(b)* Every metrizable space is paracompact

Theorem 9.19*. Every paracompact space is normal.

Note that in the chapter about Cartesian products we observed that the Cartesian product of two Sorgenfrey lines (which are normal spaces) is not a normal space. In the context of Example 9.4. it means that the Cartesian product of two paracompact spaces does not have to be a paracompact space.

Definition 9.20. A topological space (X, T) is called a *countably compact space* if it is a T_2-space and every countable open cover of (X, T) has a finite subcover.

Theorem 9.21*. For every T_2-space (X, T) the following statements are equivalent:

(i) (X, T) is a countably compact space

(ii) Every countable family of closed sets in (X, T) that has the finite intersection property, has nonempty intersection

(iii) For every sequence $\{F_n\}$ of non-empty closed sets in (X, T) such that

$F_{n+1} \subset F_n$, the intersection $\bigcap_{n=1}^{\infty} F_n$ is non-empty.

$\square$

Theorem 9.22*. For every T_2-space (X, T) the following statements are equivalent

(a) (X, T) is a countably compact space

(b) Every locally finite family of non-empty sets is finite

(c) Every infinite set has an accumulation point

$\square$

Definition 9.23. A topological space (X, T) is called a *pseudocompact space* if it is a Tichonov space and every continuous function $f : X \to \mathbb{R}$ is bounded.

Recall that Tichonov space is called $T_{3\frac{1}{2}}$-space, or completely regular space.

Theorem 9.24*. Every countably compact space that is a Tichonov space is pseudocompact.

Theorem 9.25*. Every Lindeloef space (see Definition 7.11.) is paracompact.

Theorem 9.26*. A closed subspace of a paracompact space is paracompact.

Theorem 9.27* (Michael). Being paracompact is a property invariant under closed transformations.

Recall: in context of Example 9.4. (Sorgenfrey line is a paracompact space) this means that the Cartesian product of two paracompact space does not have to be a paracompact space.

Theorem 9.28*. The Cartesian product $X \times Y$ of a paracompact space X and a compact space Y is a paracompact space.

Theorem 9.29*. For a regular space (X, T) the following statements are equivalent:

(i) (X, T) is a paracompact space

(ii) Every open cover of X has a refinement (consisting of arbitrary sets, e.g. not necessarily open or closed) that is locally finite.

(iii) Every open cover of X has a refinement consisting of closed sets that is locally finite.

Theorem 9.30*. A topological space (X, T) is compact if and only if it is a countably compact Lindeloef space.

Theorem 9.31*. For a Hausdorff space (X, T) the following statements are equivalent:

(i) (X, T) is a countably compact space

(ii) Every countable family of closed sets that has the finite intersection property has non-empty intersection

(iii) For every sequence $\{F_n\}$ of non-empty closed subsets of (X, T) such that $F_{n+1} \subset F_n$ we have $\bigcap_{n=1}^{\infty} F_n \neq \varnothing$.

□

Theorem 9.32*. For a Hausdorff space (X, T) the following statements are equivalent:

(i) (X, T) is a countably compact space

(ii) Every locally finite family of non-empty subsets (not necessarily closed) of (X, T) is finite

(iii) Every infinite subset of (X, T) has an accumulation point

Theorem 9.33*. A closed subspace of a countably compact space is a countably compact space.

Theorem 9.34*. The Cartesian product $X \times Y$ of a countably compact space X and a compact space Y is a countably compact space.

Theorem 9.35*. If $f : X \xrightarrow{\text{onto}} Y$ is a continuous function, X is a countably compact space and Y is a Hausdorff space, then Y is a countably compact space.

Theorem 9.36*. If $f : X \to \mathbb{R}$ is a continuous function and X is a countably compact space, then f is bounded.

Theorem 9.37*.

(a) Every countably compact Tichonov space is a pseudocompact space.

(b) Every pseudocompact normal space is countably compact.

Theorem 9.38*. For a Tichonov space (X, T) the following statements are equivalent:

(i) (X, T) is a pseudocompact space

(ii) Every locally finite family of non-empty open subsets of (X, T) is finite

(iii) Every locally finite open cover of (X, T) consisting of non-empty sets is finite

(iv) Every locally finite open cover of (X, T) has a finite subcover

Theorem 9.39*. For a Tichonov space (X, T) the following statements are equivalent:

(i) (X, T) is a pseudocompact space

(ii) For every sequence $\{V_i\}$ of non-empty open subsets of (X, T) such that $V_{i+1} \subset V_i$ we have $\bigcap_{i=1}^{\infty} cl(V_i) \neq \varnothing$.

Theorem 9.40*. If $f : X \xrightarrow{\text{onto}} Y$ is a continuous function, X is a pseudocompact space and Y is a Tichonov space, then Y is a pseudocompact space.

□

Theorem 9.41*. The Cartesian product $X \times Y$ of a pseudocompact space X and a compact space Y is a pseudocompact space.

□

10. Connected spaces.

Definition 10.1.

(a) A topological space (X, T) is called a *connected space* if X cannot be presented as the union of two non-empty, closed, disjoint sets.

(b) A subset C of a topological space (X, T) is called a *connected set* if it is a connected space as a subspace of (X, T).

Remark: Equivalent statement to (a) is the following:

(a') A topological space (X, T) is called a *connected space* if X cannot be presented as the union of two non-empty, open, disjoint sets.

Indeed, if $X = A \cup B$ for A, B - closed, then $B = X \backslash A$ and $A = X \backslash B$ are open.

Reminder: the only connected subsets of $\mathbb{R}$ are intervals (also one-point sets)

Example 10.I. Let $X = \mathbb{R}$ and let J be the ideal of sets of Lebesgue measure 0. Then the space from Example 1.I. (see below) is connected.

Example 1.I. Let $X \neq \varnothing$ and let J be an ideal of subsets of X. Then (T, T_J), where $T_J = \{X \backslash A : A \in J\} \cup \{\varnothing\}$ is a topological space, we call it the topological space generated by the ideal J.

Example 10.II. Sorgenfrey line is not a connected space.

Theorem 10.2. *Let (X, T_X), (Y, T_Y) be topological spaces and let $f : X \to^{onto} Y$ be a continuous function. If (X, T_X) is a connected space, then (Y, T_Y) is also a connected space.*

Proof. Assume the contrary, i.e., that (Y, T_Y) is not a connected space.

Then there exist non-empty, closed and disjoint sets A and B such that $Y = A \cup B$. Consider sets $K = f^{-1}(A)$ and $F = f^{-1}(B)$. By the continuity of f and the fact that it maps X onto Y, we obtain that K, F are non-empty closed sets. Observe that K and F are disjoint (prove it yourself) and that

$$X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) = K \cup F.$$

It follows that (X, T_X) is not a connected space, which contradicts the assumption. $\square$

Definition 10.3. Sets A and B are called *separated* if

$$(cl(A) \cap B) \cup (A \cap cl(B)) = \varnothing.$$

Note that sets A and B are separated if and only if

$$(cl(A) \cap B) = \varnothing = (A \cap cl(B)).$$

Example 10.III. In every topological space, if A, B are open and disjoint sets, then they are separated. Indeed, if $A \cap B = \varnothing$, then $B \subset X \setminus A$, and since A is open, $X \setminus A$ is closed.

Theorem 10.4. *A subset C of a topological space (X, T) is connected if and only if it cannot be presented as a union of two non-empty, separated sets in (X, T)*

Proof. Necessity.

Suppose the contrary, i.e., thet $C = A \cup B$, where A, B are non-empty, separated sets in (X, T). Then

(1) A, B are closed sets in (C, T_C), considered as a subspace of (X, T).

We only prove (1) for the set A, the proof for B is analogous. We will denote by $cl_C(Z)$ the closure of a set Z in the space (C, T_C).

Observe that (see Theorem 2.6.) $cl_C(A) = C \cap cl(A)$. Hence, $cl_C(A) \cap B = C \cap cl(A) \cap B = \varnothing$, (because A and B are separated sets in (X, T)), so

$$cl_C(A) \subset C \backslash B = A,$$

and it follows (because $A \subset cl_C(A)$) that $cl_C(A) = A$. This proves (1).

Statement (1) implies that C, as a subspace, can be presented as a union of two non-empty, closed, disjoint sets, hence C is not a connected set - which contradicts our assumption.

Sufficiency.

Suppose that C is not a connected set. Then there exist sets A, B such that $A \neq \varnothing \neq B$, $A = cl_C(A)$, $B = cl_C(B)$, $A \cap B = \varnothing$ and $C = A \cup B$. Observe that $cl_C(A) \cap B = A \cap B = \varnothing$, which impies (together with $B \subset C$, and hence $B \cap C = B$) that

$$\varnothing = cl_C(A) \cap B = cl(A) \cap C \cap B = cl(A) \cap B.$$

(we used Theorem 2.6. and then the equality $B \cap C = B$). Analogously, we can obtain $A \cap cl(B) = \varnothing$.

We have $C = A \cup B$ and $A \neq \varnothing \neq B$, moreover A and B are separated sets, which leads to the contradiction.

$\square$

Theorem 10.5. *The following statements are equivalent:*

(1) (X, T) is connected, i.e., it cannon be decomposed as $X = X_1 \cup X_2$ where X_1, X_2 are both open, non-empty and disjoint.

(2) The only subsets of X that are both open and closed are X and $\varnothing$.

(3) If $X = X_1 \cup X_2$, where X_1, X_2 are separated, i.e., $cl(X_1) \cap X_2 = \varnothing = cl(X_2) \cap X_1$, then $X_1 = \varnothing$ or $X_2 = \varnothing$.

(4) If $f : (X, T) \to (\{0, 1\}, T_d)$, where T_d is the discrete topology, is continuous, then $f(X) = \{1\}$ or $f(X) = \{0\}$.

Proof. (1) $\Rightarrow$ (2).

If $X_1 \subset X$ is both open and closed, then $X = X_1 \cup (X \backslash X_1)$

(2) $\Rightarrow$ (3).

Suppose that $X = X_1 \cup X_2$, where X_1, X_2 are separated. Then $cl(X_1) \cap X_2 = \varnothing$, so $cl(X_1) \subset X \backslash X_2 = X_1$, so $cl(X_1) = X_1$. Similarly, $cl(X_2) = X_2$. But then $X_1 = X \backslash X_2$ is open, and similarly X_2 is open. So both X_1, X_2 are both open and closed.

(3) $\Rightarrow$ (4).

If $f : (X, T) \to (\{0, 1\}, T_d)$ is continuous, then both $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are separated (they are disjoint and closed), and $X = f^{-1}(\{0\}) \cup f^{-1}(\{1\})$.

(4) $\Rightarrow$ (1).

If (1) is not true, i.e., $X = X_1 \cup X_2$, where X_1, X_2 are both open and disjoint, then the function:

$$f = \begin{cases} 0 & \text{for all } x \in X_1 \\ 1 & \text{for all } x \in X_2 \end{cases}$$

is continuous.

□

Corollary 10.6. *If $C \subset X$ is connected, then for every pair of sets X_1, X_2 separated in (X, T) such that $C \subset X_1 \cup X_2$ we have $C \subset X_1$ or $C \subset X_2$.*

Proof. $C \cap X_1$ and $C \cap X_2$ are separated in C.

□

Theorem 10.7. *Let (X, T) be a topological space and let $C \subset X$ be a connected set. If $A \subset X$ is satisfies $C \subset A \subset cl(C)$, then A is a connected set.*

Proof. Let $A = M \cup N$, where M, N are any separated sets in X, i.e., $cl(M) \cap N = M \cap cl(N) = \varnothing$. We will show that from our assumptions it follows that $M = \varnothing$ or $N = \varnothing$.

From the previous corollary and the assumption $C \subset A$, we have $C \subset M$ or $C \subset N$. We can assume (without loss of generality) that $C \subset M$. Then $cl(C) \subset cl(M)$, hence $cl(C) \cap N = \varnothing$ (because $cl(M) \cap N = \varnothing$), and therefore (from $A \subset cl(C)$) it follows that $A \cap N = \varnothing$. Since (from the assumption that $A = M \cup N$) we have $N \subset A$, we obtain $N = \varnothing$.

□

Theorem 10.8. *Let (X, T) be a topological space and let $\{C_s\}_{s \in S}$ be a family of connected subsets of the space (X, T). If there exists $s_0 \in S$ such that C_{s_0} is not separated from any of the sets of the family $\{C_s\}_{s \in S}$, then $\bigcup_{s \in S} C_s$ is a connected set.*

Proof. Let $C = \bigcup_{s \in S} C_s$. Suppose that $C \subset X_1 \cup X_2$, where X_1, X_2 are separated in (X, T). Then also $C_{s_0} \subset X_1 \cup X_2$, and by previous results, we can assume that $C_{s_0} \subset X_1$. Similarly, for every $s_1 \in S$ we have either $C_{s_1} \in X_1$ or $C_{s_1} \in X_2$. But since $C_{s_0} \subset X_1$ and C_{s_0}, C_{s_1} are not separated, we have $C_{s_1} \subset X_1$.

Indeed, suppose that $C_{s_1} \subset X_2$. If $x \in cl(C_{s_1}) \cap C_{s_0}$ then from $C_{s_1} \subset X_2$ we get $cl(C_{s_1}) \subset cl(X_2)$ and from $C_{s_0} \subset X_1$ we get $x \in cl(X_2) \cap X_1 \neq \varnothing$. Analogously, if $x \in C_{s_1} \cap cl(C_{s_0})$ then $x \in X_2 \cap cl(X_1) \neq \varnothing$.

Hence, for all $s \in S$ we have $C_s \subset X_1$ and $\bigcup_{s \in S} C_s \subset X_1$ which proves that $\bigcup_{s \in S} C_s$ is connected.

□

Definition 10.9. Let (X, T) be a topological space. Every connected set $S \subset X$ that has the following property:

for every connected set C such that $S \subset C$ we have $S = C$ (i.e., S is the maximal connected set)

is called a *connected component* of the space (X, T).

In other words, connected components are maximal (in the sense of inclusion) connected subsets of (X, T).

Theorem 10.10. *Connected components of a topological space (X, T) are closed and pairwise disjoint sets.*

Proof. Let S be a connected component of (X, T). By Theorem 10.7., $cl(S)$ is a connected set and $S \subset cl(S)$, which together with the definition of connected component implies that $S = cl(S)$ and hence S is a closed set.

Let S and C be two distinct connected components of (X, T). Suppose that $S \cap C \neq \emptyset$. Then, by Theorem 10.8., $S \cup C$ is a connected set and

$S \subset S \cup C$, so (by the definition of connected component) $S = S \cup C$,

$C \subset S \cup C$, so (by the definition of connected component) $C = S \cup C$.

It follows that $S = C$, which contradicts the assumption that S and C are two distinct connected components of (X, T).

□

Theorem 10.11. *Every connected Tichonov ($T_{3\frac{1}{2}}$) space that contains at least two points, is a set of cardinality not lower than continuum.*

Proof. Let (X, T) be a connected Tichonov space that contains at least two points - let us denote them by x and y (we have $x \neq y$). Since (X, T) is a T_1-space, it follows that $\{x\}$ and $\{y\}$ are closed and disjoint sets. Then there exists a continuous function $f : X \to I$ such that $f(x) = 0$ and $f(y) = 1$.

We will show that

(1) $\quad f(X) = I$.

Indeed, if there existed an element $\alpha \in I$ such that $\alpha \notin f(X)$, then (see the reminder after Definition 10.1) $f(X)$ would not be a connected subset of I, which would contradict Theorem 10.2. This argument proves (1). From (1) it follows that the cardinality of $f(X)$ is continuum, and so X is a set of cardinality not lower than continuum. □

Theorem 10.12. *Cartesian product of non-empty connected spaces is a connected space.*

□

Theorem 10.13. *For every compact metric space (X, ρ) there exists a continuous mapping of this space into the Cantor set, that maps different connected components into different points of the Cantor set.*

□

Definition 10.14. A topological space (X, T) is called *locally connected at a point* x_0 if for every $V \in T$ such that $x_0 \in V$ there exists a connected set $C \subset V$ such that $x_0 \in Int(C)$. A space that is locally connected at every of its points is called a *locally connected space.*

The space called *broom* is not locally connected at one point.

Definition 10.15. An open and connected set is called a *domain*.

Theorem 10.16. *A topological space (X, T) is locally connected if and only if it has a base that consists only of domains.*

Definition 10.17. Let (X, T) be a topological space and let $x, y \in X$.

(a) A space (X, T) is called *path-connected* (also: *roadconnected, path-wiseconnected* or *0-connected*) if for every two points $x, y \in X$ there exists a continuous function $d : I \to X$ such that $d(0) = x$ and $d(1) = y$. $d(I)$ is called a *path* (also: *curve* or *road*).

(b) A space (X, T) is called *arcwise connected* if for every two distinct points $x, y \in X$ there exists a homeomorphism $l : I \to X$ such that $l(0) = x$ and $l(1) = y$. $l(I)$ is called an *arc*.

Theorem 10.18.

(i) Every path-connected space is connected.

(ii) Every arcwise connected space is path-connected.

shortly: arcwise connected $\Rightarrow$ path-connected $\Rightarrow$ connected

$\square$

Example 10.III. Let $X = \{a, b\}$ and $T = \{\varnothing, X, \{a\}\}$ (Alexandrov-Sierpinski space). This space is path-connected but it is not arcwise connected. $f(x) = a$ for $x \in [0, 1)$ and $f(1) = b$ (the only neighbourhood of b is X).

Example 10.IV. Consider $X = \{(x, \sin\frac{1}{x}) : x \in (0,1]\} \cup \{0\} \times [0,1]$ with the natural topology induced from $\mathbb{R}^2$. Then this space is connected but not path-connected.

Theorem 10.19. *If (X, T) is a Hausdorff space, then it is path-connected if and only if it is arcwise connected.*

Theorem 10.20. *If (X, T) is an open subspace of $\mathbb{R}^2$ (i.e., X is an open subset of $\mathbb{R}^2$ and T is the topology induced from $\mathbb{R}^2$), then being path-connected is equivalent to being arcwise connected.*

Definition 10.21.

(a) A topological space (X, T) is called a *continuum* if it is compact and connected. (plural form for "continuum" is "continua")

(b) A set A is called a *continuum* if it is a continuum (in the sense above) when considered as a subspace.

Theorem 10.22. *If a family of continua $\{K_s\}_{s \in S}$ is closed with respect to taking finite intersections (i.e., for every finite set of indices $s_1, \ldots, s_k \in S$ the set $\bigcap_{i=1}^{k} K_{s_i}$ is a continuum), then $\bigcap_{s \in S} K_s$ is a continuum.*

$\square$

As a simple corollary from the above theorem we obtain

Theorem 10.23. *The intersection $\bigcap_{i=1}^{\infty} K_i$ of a sequence of continua $K_1 \supset K_2 \supset K_3 \supset \ldots$ is a continuum.*

Definition 10.21*. Let (X, T) be a topological space. We say that a point $x \in X$ is a *cut point* (or that it *cuts the space* into two sets W i V) if $X \setminus \{x\} = W \cup V$, where W and V are open, disjoint, non-empty sets.

Theorem 10.22*. If a point x cuts a continuum (X, T) into sets W and V, then $W \cup \{x\}$ and $V \cup \{x\}$ are both continua. $\square$

Theorem 10.23*. In every continuum that has more than one element there exist at least two points that do not cut the continuum ("*non-cut points*"). $\square$

Theorem 10.23* (Moore). Every separable continuum that contains exactly two non-cut points is an arc (i.e. it is homeomorphic to I). $\square$

Fundamental Group:

The fundamental group $\pi_1(X, x_0)$ can be considered as $[I, \partial I; X, x_0]$ i.e., as the set of homotopy classes of closed paths or loops in X at the base point x_0.

Conjunto de clases de equivalencia de caminos cerrados en X, que empiezan y acaban en x_0. Los bucles son equivalentes si son homotópicos.

Otra definición: El conjunto de lazos $\Omega(X, x_0)$ en un espacio topológico X basado en un punto x_0 bajo la relación de ser homotópicos tiene estructura de grupo algebraico y se denomina grupo fundamental. Su operación se define:

$$*: \Omega(X, x, y) \times \Omega(X, y, z) \to \Omega(X, x, z) \;/\; *(\alpha, \beta)(t) = (\alpha * \beta)(t) = \begin{cases} \alpha(2t) & t \in [0, \frac{1}{2}] \\ \beta(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

Simply connected:

(X, T) is simply connected $\iff$ It's arcwise-connected $\wedge$ $\pi_1(X, x_0) = \{1\}$ $\forall x_0 \in X$