Exercise 4.1 Let $f : (X, T_X) \to (Y, T_Y)$ be a function. Prove that if $f(cl(A)) \subset cl(f(A))$ for every set A then f is continuous (for the opposite implication, see one of the proofs in Lecture 7).

Exercise 4.2 Let (X, T) be a Hausdorff (i.e., T_2) topological space and let $(X \times X, T_{X\times X})$ be the Cartesian product of X and X, with the product topology. Prove that the following set (called "the diagonal"):

$$D = \{(x, x) : x \in X\}$$

is closed in $(X \times X, T_{X\times X})$.

Exercise 4.3 A topological space (X, T) is compact if every open cover of X has a finite subcover. Let (X, T_X) and (Y, T_Y) be compact topological spaces. Prove (directly, i.e., without using equivalent definitions of compact spaces) that every open cover of their Cartesian product $X \times Y$, considered with the product topology, has a finite subcover.

Exercise 4.4 Prove Theorem 7.13 (using the Lemma below it).

Exercise 4.5 Let (X, ρ_X) and (Y, ρ_Y) be metric spaces, let (X, ρ_X) be a second-countable space and let $f : X \xrightarrow{\text{onto}} Y$ be a continuous function. Prove that (Y, ρ_Y) is a second-countable space.

Exercise 4.5 Let (X, ρ_X) and (Y, ρ_Y) be metric spaces, let (X, ρ_X) be a second-countable space and let $f : X \xrightarrow{\text{onto}} Y$ be a continuous function. Prove that (Y, ρ_Y) is a second-countable space.

By Theorem 7.8: $f: (X, \mathcal{T}_X) \xrightarrow{\text{onto}} (Y, \mathcal{T}_Y)$ is cont $\wedge$ $(X, \mathcal{T}_X)$ separable $\Rightarrow$ $(Y, \mathcal{T}_Y)$ also is

Let's prove Theorem 7.3: Every metric space is separable $\Leftrightarrow$ It's 2nd countable.

$\Rightarrow$ (Y, ρ_Y) separable $\Rightarrow \exists A \subseteq Y$ dense in Y and countable.

Let's see $\mathcal{B} = \{B(x, \frac{1}{n}) / x \in A, n \in \mathbb{N}\}$ is a base of (Y, ρ_Y).

Let $y \in Y$, $r > 0$. $\forall B(y, r) = \bigcup_{x \in B(y,r) \cap A} B(x, \frac{1}{\lceil r - \rho(x,y) \rceil})$ countable $\Rightarrow \mathcal{B}$ countable

$\Leftarrow$ (Y, ρ_Y) 2nd countable $\Rightarrow \exists A \subseteq Y$ numerable $/ \mathcal{B} = \{B(x, \frac{1}{n}) / x \in A, n \in \mathbb{N}\}$ is a countable base.

Let $B = \bigcup_{\substack{n \in \mathbb{N} \\ x_n \in B_n \in \mathcal{B}}} \{x_n\} \Rightarrow \forall U \in \mathcal{T} \setminus \{\emptyset\},\ B \cap U \neq \emptyset \Rightarrow B$ is dense $\wedge$ countable

$\Rightarrow (Y, \rho_Y)$ separable.