Exercise 3.1 Let $\{U_s\}_{s\in S}$ be a family of sets such that $\bigcup_{s\in S} U_s = X$. Show that the family of sets: $V_{s_1,\ldots,s_n} = U_{s_1} \cap \ldots \cap U_{s_n}$, i.e., intersections of finitely many sets from $\{U_s\}_{s\in S}$, can be a base of X. You can use e.g., Theorem 2.3. from the lectures.

Exercise 3.2 For any two compact sets F, K in a metric space (X, d) we define the Hausdorff distance between those sets as:

$$d_H(F,K) = \max\{\sup_{x\in F}\inf_{y\in K} d(x,y), \sup_{x\in K}\inf_{y\in F} d(x,y)\}.$$

For any set U we define U_ϵ as:

$$U_\epsilon = \bigcup_{x\in U} B(x,\epsilon).$$

Show that

$$d_H(F,K) = \inf\{\epsilon \geq 0 : F \subset K_\epsilon \text{ and } K \subset F_\epsilon\}$$

for any two compact sets F, K.

Exercise 3.3 Let F be a compact set in a metric space (X, d) and let $x \in X \setminus F$. Find relation between:

1. Hausdorff distance $d_H(\{x\}, F)$
2. distance of the point x from the set F in metric space X, i.e.,
$$d(x,F) = \inf_{y\in F} d(x,y)$$
3. diameter of the set F, i.e.,
$$\operatorname{diam}(F) = \sup_{y,z\in F} d(y,z).$$

See what happens if diameter of F is big and distance from x to F is small.

Let $\sim$ be equivalence relation:

$$x \sim y \quad \text{if } x = y \text{ or } x, y \in F$$

How does a neighbourhood of $[z]$ look like in the quotient space $X/\sim$, for $z \in F$?

Exercise 3.4 Let (X, T_X) and (Y, T_Y) be topological spaces. For a compact set $K \subset X$ and an open set $U \subset Y$ we define $V(K, U)$ as the set of all continuous functions $f : (X, T_X) \to (Y, T_Y)$ such that $f(K) \subset U$.

Let $F(X, Y)$ be the set of all continuous functions $f : (X, T_X) \to (Y, T_Y)$. Let T_F be the topology on $F(X, Y)$ with base defined as the family of finite intersections $V(K_1, U_1) \cap \ldots \cap V(K_n, U_n)$. T_F is the compact-open topology on $F(X, Y)$.

We say that a sequence $\{f_n\}$, where all $f_n \in F(X, Y)$, converges to a function $f \in F(X, Y)$ in compact-open topology, if for every open set $W \in T_F$ such that $f \in W$ there exists $N \in \mathbb{N}$ such that $f_n \in W$ for all $n > N$.

Let (X, d), (Y, ρ) be metric spaces. Prove that if $C \subset X$ is a compact set and f_n converges to f in compact-open topology, then f_n converges to f uniformly, i.e.,

$$\forall_{\epsilon>0} \quad \exists_{N\in\mathbb{N}} \quad \forall_{n>N} \quad \forall_{x\in C} \quad \rho(f(x), f_n(x)) < \epsilon.$$

You might need to use the uniform continuity of f on C, i.e., the following property:

$$\forall_{\epsilon>0} \quad \exists_{\delta>0} \quad d(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < \epsilon.$$

To see that the above is true: let $\epsilon > 0$, for all $x \in C$ let U_x be an open set such that $f(U_x) \in B(f(x), \epsilon/2)$ (how do we know that such U_x exists?). Then $\{U_x\}_{x\in C}$ is open cover of C, let δ be the Lebesgue number of this cover.