PDE EXERCISE

Assume $n = 1$ and $u(x,t) = v\left(\frac{x^2}{t}\right)$.

(a) Show

$$u_t = u_{xx}$$

if and only if

$$(*) \qquad 4zv''(z) + (2+z)v'(z) = 0 \quad (z > 0).$$

let's take the region $\Omega = \{(x,t) \in \mathbb{R}^2 \mid t, x > 0\}$

let $z(x,t) = \frac{x^2}{t} \Rightarrow u(x,t) = v(z(x,t))$

$$z_t = -\frac{x^2}{t^2} \qquad z_{tt} = \frac{2x^2}{t^3}$$

$$z_x = \frac{2x}{t} \qquad z_{xx} = \frac{2}{t}$$

$$u_t = \frac{dv}{dz}\frac{\partial z}{\partial t} = -v'(z)\cdot\frac{x^2}{t^2}$$

$$u_x = \frac{dv}{dz}\frac{\partial z}{\partial x} = v'(z)\cdot\frac{2x}{t}$$

$$u_{xx} = \frac{d^2v}{dz^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{dv}{dz}\frac{\partial^2 z}{\partial x^2} = v''(z)\, z_x^2 + v'(z)\, z_{xx}$$

$\boxed{\Rightarrow}$

$$u_t = u_{xx} \Rightarrow -v'(z)\cdot\frac{x^2}{t^2} = v''(z)\, z_x^2 + v'(z)\, z_{xx} \Rightarrow$$

$$-v'(z)\frac{x^2}{t^2} = v''(z)\cdot\frac{4x^2}{t^2} + v'(z)\cdot\frac{2}{t} \Rightarrow -v'(z)\cdot\frac{z}{\cancel{t}} = v''(z)\cdot\frac{4z}{\cancel{t}} + v'(z)\cdot\frac{2}{\cancel{t}}$$

$$\Rightarrow 4zv''(z) + (2+z)v'(z) = 0$$

$\boxed{\Leftarrow}$

$$u_{xx} = v''(z)\, z_x^2 + v'(z)\, z_{xx} = v''(z)\cdot\frac{4x^2}{t^2} + v'(z)\cdot\frac{2}{t} =$$

$$4z\, v''(z)\cdot\frac{1}{t} + v'(z)\cdot\frac{2}{t} = \frac{1}{t}\left(4z\, v''(z) + 2v'(z)\right) =$$

$$\frac{1}{t}\left(4z\, v''(z) + 2v'(z) + zv'(z) - zv'(z)\right) =$$

$$\frac{1}{t}\left(\underbrace{4z\, v''(z) + (2+z)v'(z)}_{0} - z\, v'(z)\right) = -\frac{z\, v'(z)}{t} = -\frac{x^2}{t^2}\, v'(z) = u_t$$