Exercise 2.1 Let (X, T) be a topological space. Prove that $x \in X$ is an isolated point if and only if $\{x\} \in T$.

Exercise 2.2 (Example 4.I) Let (X, T_X) and (Y, T_Y) be topological spaces and let $y \in Y$. Prove that the constant function $f(x) = y$ for all $x \in X$ is continuous.

Exercise 2.3 (Example 4.II) Let $X = Y = \{0\} \cup [1, 2]$. Let T_X be the topology induced from the Euclidean space $\mathbb{R}$ and let T_Y be the discrete topology. Let $f : X \to Y$ be given by the formula $f(x) = x$. Prove that $\mathcal{C}(f) = \{0\}$ (i.e., that f is continuous only at the point $x = 0$).

Exercise 2.4 Let $(X, \mathcal{T}_X)$ and (Y, T_Y) be as in Exercise 2.3. Find the set of points of continuity $\mathcal{C}(g)$ of the function $g : Y \to X$, given by the formula $g(s) = s$ for $s \in Y$.

Exercise 2.5 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $\mathcal{B}$ be a base of the space $(Y, \mathcal{T}_Y)$. Let $f : X \to Y$. Prove that f is continuous if and only if the pre-image of every set from $\mathcal{B}$ is open.

Exercise 2.6 Let $X_i = (\mathbb{R}, \mathcal{T}_i)$, where $\mathcal{T}_1$ - the natural topology (of the Euclidean space), $T_2 = 2^{\mathbb{R}}$, $\mathcal{T}_3 = \{\emptyset, \mathbb{R}\}$, $\mathcal{T}_4 = \{(a, +\infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ and let $f_1 = 0$ for all $x \in \mathbb{R}$, $f_2 = x$ for all $x \in \mathbb{R}$, $f_3 = -x$ for all $x \in \mathbb{R}$,

$$f_4(x) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases} \qquad f_5(x) = \begin{cases} 1 & \text{for } x \in \mathbb{Q} \\ 0 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Consider the above functions $f_k : X_i \to X_j$ for all $k \in \{1, \ldots, 5\}$ and $i, j \in \{1, \ldots, 4\}$. Check which of these functions are continuous and where (i.e., find the sets of their points of continuity)

Exercise 2.7 (Example 4. III) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let P be an equivalence relation in X. Let $f : X|P \to Y$. Show that f is continuous if and only if $f \circ \xi : X \to Y$ is continuous.

Exercise 2.8 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f : X \to Y$ be continuous, one-to-one function. Let $A \in \mathcal{T}_X$ and $f(A) \in T_Y$. Prove that then $f|A : (A, \mathcal{T}_X^{ind}) \to (f(A), \mathcal{T}_Y^{ind})$ is continuous. (i.e., restriction of continuous function is continuous in the induced topology)

Exercise 2.9 (Example 4.IX) Let $X = (a, b)$ and $Y = (c, d)$, where $-\infty \leq a < b \leq +\infty$ i $-\infty \leq c < d \leq +\infty$. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be the topologies on X, Y induced from $\mathbb{R}$ (with its natural topology of Euclidean space). Prove that X and Y are homeomorphic.

Exercise 2.10 Let $X = \mathbb{R}$ and $\mathcal{T}$ be the topology of Sorgenfrey line. Let $f(x) = -2x$ for all $x \in \mathbb{R}$. Is f a homeomorphism from X onto X? Explain, why.

Exercise 2.11 Let X be an infinite set and let $\mathcal{J}$ be the ideal of all finite subsets of X. Show that in the topological space $(X, \mathcal{T}_{\mathcal{J}})$, where $\mathcal{T}_{\mathcal{J}} = \{X \setminus A : A \in \mathcal{J}\} \cup \{\emptyset\}$, every infinite set is dense in X. Let $A \subset X$ be an infinite set. Find the set A^d.

Exercise 2.12 Let X be an infinite set, $x_0 \in X$ and let $\mathcal{J}$ be the ideal of all finite subsets of X. Show that in the topological space $(X, \mathcal{T})$, where $\mathcal{T} = \{X \setminus A : A \in \mathcal{J}\} \cup \{A \subset X : x_0 \notin A\}$, the set $\{x_0\}$ is the only nowhere dense set.

Exercise 2.13 Let X be an infinite set, $x_0 \in X$ and let $\mathcal{J}$ be the ideal of all finite subsets of X. Show that in the topological space $(X, \mathcal{T})$, where $\mathcal{T} = \{X \setminus A : A \in \mathcal{J}\} \cup \{A \subset X : x_0 \notin A\}$, the set $\{x_0\}$ is the only first category set. (it actually immediately follows from the previous Exercise)

Exercise 2.14 (Exanole 5.III) Let X be an infinite set and let $\mathcal{J}$ be the ideal of all finite subsets of X. Show that the topological space $(X, \mathcal{T})$, where $\mathcal{T} = \{X \setminus A : A \in \mathcal{J}\} \cup \{\emptyset\}$, is a T_1-space, but not a T_2-space.

Exercise 2.15 (Example 5.IV) Let X be an infinite set, $x_0 \in X$ and let $\mathcal{J}$ be the ideal of all finite subsets of X. Show that the topological space $(X, \mathcal{T})$, where $\mathcal{T} = \{X \setminus A : A \in \mathcal{J}\} \cup \{A \subset X : x_0 \notin A\}$ is a Hausdorff space.

Exercise 2.16 (Example 5.VI) Let X be an infinite set, $x_0 \in X$ and let $\mathcal{J}$ be the ideal of all finite subsets of X. Show that the topological space $(X, \mathcal{T})$, where $\mathcal{T} = \{X \setminus A : A \in \mathcal{J}\} \cup \{A \subset X : x_0 \notin A\}$, is a regular space.

Exercise 2.17 Let $X = \mathbb{R}$ and $\mathcal{T} = \{(a, +\infty) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. Which separation axiom does the space $(X, \mathcal{T})$ satisfy?

Exercise 2.18 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $(Y, \mathcal{T}_Y)$ be a Hausdorff space. Let $f, g : X \to Y$ be continuous functions. Show that the set $\{x \in X : f(x) = g(x)\}$ is closed. (note: this exercise is actually useful sometimes!)

Exercise 2.19 Show that the Sorgenfrey line is a regular space.

Exercise 2.20 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be homeomorphic topological spaces. Prove that if $(Y, \mathcal{T}_Y)$ is a T_3-space then $(X, \mathcal{T}_X)$ is also a T_3-space.

Exercise 2.21 Prove that a closed subset of a normal topological space is a normal space (in the induced topology).

Exercise 2.22 Prove that if $f_1 : X \to Y$ and $f_2 : X \to Y$ are continuous functions from a topological space $(X, \mathcal{T}_X)$ to a Hausdorff space $(Y, \mathcal{T}_Y)$, and the set $A = \{x \in X : f_1(x) = f_2(x)\}$ is dense in X, then $f_1 = f_2$ on X.

Exercise 2.23 Let $X_i = (\mathbb{R}, \mathcal{T}_i)$, where $\mathcal{T}_1$ - the natural topology (of the Euclidean space), $\mathcal{T}_2 = 2^{\mathbb{R}}$, $\mathcal{T}_3 = \{\emptyset, \mathbb{R}\}$, $\mathcal{T}_4 = \{(a, +\infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ and let
$A_1 = \{(x, y) \in \mathbb{R}^2 : x = 0\}$
$A_2 = \{(x, y) \in \mathbb{R}^2 : x > 0\}$
$A_3 = \{(x, y) \in \mathbb{R}^2 : x = y\}$
$A_4 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
$A_5 = \{(0, 0)\}$
Check which sets A_k are open, and which are closed in spaces $X_i \times X_j$ for all $k = 1, 2, \ldots, 5$ and all $i, j \in \{1, \ldots, 4\}$ (with the topology of Cartesian product).

Exercise 2.24 (see Theorem 6.3) For topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ and their Cartesian product $X \times Y$, prove that each projection $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ is continuous.

2. Show that a finite T_1-space is discrete.

Exercise 2.1 Let (X, T) be a topological space. Prove that $x \in X$ is an isolated point if and only if $\{x\} \in T$.

$x \in X^d \Leftrightarrow x \notin \mathrm{cl}(X \setminus \{x\}) \Leftrightarrow x \in X \setminus \mathrm{cl}(X \setminus \{x\}) = \mathrm{Int}(\{x\})$

Exercise 2.3 (Example 4.II) Let $X = Y = \{0\} \cup [1, 2]$. Let T_X be the topology induced from the Euclidean space $\mathbb{R}$ and let T_Y be the discrete topology. Let $f : X \to Y$ be given by the formula $f(x) = x$. Prove that $\mathcal{C}(f) = \{0\}$ (i.e., that f is continuous only at the point $x = 0$).

$\forall U \in T_Y,\ f^{-1}(U) = U$

f cont $\Leftrightarrow f^{-1}(U) \in T_X \quad \forall U \in T_Y$

f cont at $x_0 \in X \Leftrightarrow \forall V \in N_{f(x_0)},\ \exists U \in N_{x_0} / U \subseteq f^{-1}(V)$

- $x = 0$

  let $V \in N_0 \Rightarrow f^{-1}(U) = f^{-1}(\{0\}) = \{0\} \in T_X$

- $x \neq 0 \Rightarrow x \in [1,2] \in T_Y$

  let $U \in T_Y$ / U is countable $\Rightarrow f^{-1}(U) = U \notin T_X$, since $\nexists V \in T_X / V \subseteq U$

Exercise 2.5 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $\mathcal{B}$ be a base of the space $(Y, \mathcal{T}_Y)$. Let $f : X \to Y$. Prove that f is continuous if and only if the pre-image of every set from $\mathcal{B}$ is open.

let $\mathcal{B} = \{B_s / s \in S\} \subseteq \mathcal{T}_Y$, $U \in \mathcal{T}_Y \Rightarrow \exists T \subseteq S / \ U = \bigcup_{t \in T} B_t$

f cont. $\Leftrightarrow \forall U \in \mathcal{T}_Y,\ f^{-1}(U) \in \mathcal{T}_X$

$\Rightarrow)$ f cont $\Rightarrow f^{-1}(B_t) \in \mathcal{T}_X \quad \forall t \in S$

$\Leftarrow)$ $f^{-1}(B_t) \in \mathcal{T}_X \ \forall t \in S \Rightarrow f^{-1}(U) = f^{-1}\left(\bigcup_{t \in S} B_t\right) = \bigcup_{t \in S} f^{-1}(B_t) \in \mathcal{T}_X$

$\Rightarrow f$ cont.

Exercise 2.8 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f : X \to Y$ be continuous, one-to-one function. Let $A \in \mathcal{T}_X$ and $f(A) \in T_Y$. Prove that then $f|A : (A, \mathcal{T}_X^{ind}) \to (f(A), \mathcal{T}_Y^{ind})$ is continuous. (i.e., restriction of continuous function is continuous in the induced topology)

f cont. $\Leftrightarrow \forall U \in \mathcal{T}_Y,\ f^{-1}(U) \in \mathcal{T}_X$

$f|_A$ cont $\Leftrightarrow \forall U \cap f(A) \in \mathcal{T}_Y^{ind},\ U \in \mathcal{T}_Y,\ f|_A^{-1}(U \cap f(A)) \in \mathcal{T}_X^{ind}$

We have $f(x) = f|_A(x) \ \forall x \in A \Rightarrow f^{-1}(B) = f|_A^{-1}(B) \quad \forall B \subseteq X$

$f|_A^{-1}(U \cap f(A)) = f^{-1}(U \cap f(A)) = f^{-1}(U) \cap f^{-1}(f(A)) = f^{-1}(U) \cap f|_A^{-1}(f(A)) =$

$\underbrace{f^{-1}(U)}_{\in \mathcal{T}_X} \cap A \in \mathcal{T}_X^{ind}$

Exercise 2.10 Let $X = \mathbb{R}$ and $\mathcal{T}$ be the topology of Sorgenfrey line. Let $f(x) = -2x$ for all $x \in \mathbb{R}$. Is f a homeomorphism from X onto X? Explain, why.

$f : (\mathbb{R}, \mathcal{T}) \to (\mathbb{R}, \mathcal{T}) \mid f(x) = -2x$

$\mathcal{T}$ is given by the base $\mathcal{B} = \{ [a,b[\mid a,b \in \mathbb{R} \}$

- f is clearly bijective.
- f^{-1} isn't continuous, as $f([a,b[) =]-2b, -2a] \notin \mathcal{T}$

As a result, f isn't a homeomorphism

Exercise 2.15 (Example 5.IV) Let X be an infinite set, $x_0 \in X$ and let $\mathcal{J}$ be the ideal of all finite subsets of X. Show that the topological space $(X, \mathcal{T})$, where $\mathcal{T} = \{X \setminus A : A \in \mathcal{J}\} \cup \{A \subset X : x_0 \notin A\}$ is a Hausdorff space.

$(X, \mathcal{T})$ $T_2 \iff \forall x_1, x_2 \in X, x_1 \neq x_2, \exists U_1, U_2 \in \mathcal{T} \mid x_1 \in U_1, x_2 \in U_2, U_1 \cap U_2 = \emptyset$

Let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 = \{ X \setminus A \mid A \in \mathcal{J} \} \cup \{ A \subseteq X \mid x_0 \notin A \}$

- Take $x_1, x_2 \in X \mid x_1 \neq x_2, x_1, x_2 \neq x_0$. Let $U_1 = \{x_1\}, U_2 = \{x_2\} \in \mathcal{T}_2 \Rightarrow$

  $U_1 \cap U_2 = \emptyset$
- Take $x_0, x_1 \in X$, $x_0 \neq x_1$. Let $U_1 = A \setminus \{x_1\} \in \mathcal{T}_1$, $U_2 = \{x_1\} \in \mathcal{T}_2$

$\Rightarrow U_1 \cap U_2 = \emptyset$

Exercise 2.19 Show that the Sorgenfrey line is a regular space.

Let $(\mathbb{R}, \mathcal{T})$ be our Sorgenfrey line.

$\mathcal{T}$ is given by the base $\mathcal{B} = \{[a,b[\ / \ a,b \in \mathbb{R}\}$

$(\mathbb{R}, \mathcal{T})$ regular $\iff$ it's $T_1 \wedge \forall x \in \mathbb{R},\ \forall V \in \mathcal{T} / x \in V,\ \exists U \in \mathcal{T} / x \in U \subseteq cl(U) \subseteq V$

- Let $x_1, x_2 \in \mathbb{R},\ \varepsilon = |x_1 - x_2|,\ U = [x_1, x_1 + \varepsilon[\implies x_1 \in U \not\ni x_2 \implies (\mathbb{R}, \mathcal{T})$ is T_1.
- Let $U = [x, x + \frac{\varepsilon}{2}[\overset{*}{=} cl(U) \subseteq V = [x, x+\varepsilon[\quad \forall \varepsilon > 0$

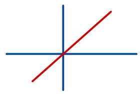
* In this topology open sets are also closed:

$\mathbb{R} \setminus [x, x+\varepsilon[=]-\infty, x[\cup [x+\varepsilon, +\infty[= \bigcup_{y \in \mathbb{R}} [-y, x[\cup [x+\varepsilon, y[\in \mathcal{T}$

As a result, $(\mathbb{R}, \mathcal{T})$ is a T_3-space.

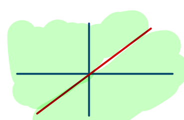
Exercise 2.23 Let $X_i = (\mathbb{R}, \mathcal{T}_i)$, where $\mathcal{T}_1$ - the natural topology (of the Euclidean space), $T_2 = 2^{\mathbb{R}}$, $\mathcal{T}_3 = \{\emptyset, \mathbb{R}\}$, $\mathcal{T}_4 = \{(a, +\infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ and let
$A_1 = \{(x, y) \in \mathbb{R}^2 : x = 0\}$
$A_2 = \{(x, y) \in \mathbb{R}^2 : x > 0\}$
$A_3 = \{(x, y) \in \mathbb{R}^2 : x = y\}$
$A_4 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$
$A_5 = \{(0, 0)\}$
Check which sets A_k are open, and which are closed in spaces $X_i \times X_j$ for all $k = 1, 2, \ldots, 5$ and all $i, j \in \{1, \ldots, 4\}$ (with the topology of Cartesian product).

$A_3 = \{(x,y) \in \mathbb{R}^2 / x = y\}$



- $\mathcal{T}_2 \times \mathcal{T}_1$
  - Open?
    A_3 open $\iff \forall (x,x) \in A_3, \exists U \in \mathcal{T}_2 \times \mathcal{T}_1 / (x,x) \in U \subseteq A_3$.
    However, $\forall V \subseteq \mathbb{R}, \varepsilon > 0, V \times]x-\varepsilon, x+\varepsilon[\not\subseteq A_3 \Rightarrow A_3 \notin \mathcal{T}_2 \times \mathcal{T}_1$
  - Closed?
    let $f : (\mathbb{R}, \mathcal{T}_2) \longrightarrow (\mathbb{R}, \mathcal{T}_1) / f(x) = x$.

    It's clearly continuous and surjective. Besides, $(\mathbb{R}, \mathcal{T}_1)$ is $T_2 \overset{\text{Theorem 6.7}}{\Longrightarrow}$
    $\Gamma(f) = A_3$ is closed in $(\mathbb{R}^2, \mathcal{T}_2 \times \mathcal{T}_1)$
- $\mathcal{T}_2 \times \mathcal{T}_3$
  - Open?
    Clearly A_3 isn't open, as $\exists (x,x) \in A_3 / \forall U \in \mathcal{T}, (x,x) \in U, U \not\subseteq A_3$
    Let $x \in \mathbb{R}$, $A \subseteq \mathbb{R} / x \in A \Rightarrow (x,x) \in A \times \mathbb{R} \not\subseteq A_3$
  - Closed?



    $\mathbb{R}^2 \setminus A_3 = \{(x,y) \in \mathbb{R}^2 / x \neq y\} \in \mathcal{T}_2 \times \mathcal{T}_3$?

    let $(x,y) \in \mathbb{R}^2 \setminus A_3$, $A \times \mathbb{R} \in \mathcal{T}_2 \times \mathcal{T}_3 / x \in A, y \in \mathbb{R}$
    clearly $A \times \mathbb{R} \not\subseteq \mathbb{R}^2 \setminus A_3$, since $\exists (z,z) \in A \times \mathbb{R} / (z,z) \notin \mathbb{R}^2 \setminus A_3 \Rightarrow$
    $\mathbb{R}^2 \setminus A_3 \notin \mathcal{T}_2 \times \mathcal{T}_3 \Rightarrow A_3$ isn't closed

• $T_3 \times T_4$

- open?

$\forall a\in\mathbb{R}, \varepsilon>0, \ \mathbb{R}\times]a-\varepsilon,+\infty[\ \not\subseteq A_3 \Rightarrow A_3 \notin T_3\times T_4$

- closed?

Let $(x,y)\in\mathbb{R}^2\setminus A_3$, $\mathbb{R}\times B\in T_3\times T_4 / x\in\mathbb{R}, y\in B=]y-\varepsilon,+\infty[, \varepsilon>0$.

Clearly $\mathbb{R}\times B \not\subseteq \mathbb{R}^2\setminus A_3$, since $\exists (z,z)\in \mathbb{R}\times B / (z,z)\notin \mathbb{R}^2\setminus A_3 \Rightarrow$

$\mathbb{R}^2\setminus A_3\notin T_3\times T_4 \Rightarrow A_3$ isn't closed

2. Show that a finite T_1-space is discrete.

Let (X,T) be a finite T_1 space. Let's show $T=P(X)$

(X,T) $T_1 \Leftrightarrow \forall x_1,x_2\in X, x_1\neq x_2, \exists U\in T / x_1\notin U\ni x_2 \Leftrightarrow$

$\{x\}$ closed $\forall x\in X$.

We can see $\{x_i\} = \underbrace{\bigcap_{\substack{x\in X \\ x\neq x_i}} \underbrace{X\setminus\{x\}}_{\in T}}_{\in T}$ $i=1..n \Rightarrow \{x_i\}\in T \ \forall i=1..n$

$\Rightarrow \bigcup_{i\in I\subseteq P(\{1..n\})}\{x_i\}\in T \Rightarrow T=P(X)$