Graph and Network Theory

- Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein, Introduction to Algorithms, third edition.
- Dieter Jungnickel, Graphs, Networks and Algorithms, third edition.
- Mark Newman, Networks.
- K. Erciye, Guide to Graph Algorithms.
- C.Stamile,A.Marzullo,E.Deusebio Graph Machine Learning.
- M.Zuhair Al-Taie, S.Kadry, Python for Graph and Network Analysis.

The **undirected graph** (or shortly graph) is a pair:

$$G = \langle V, E \rangle$$

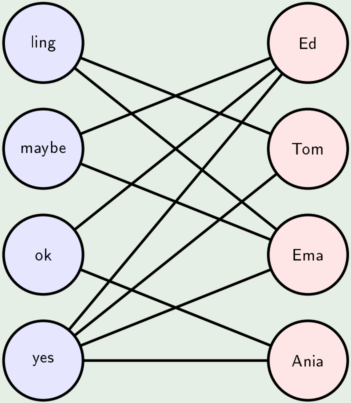
where V is a nonempty set (a set of **vertices** of G) and
$E = \{\{u, v\} : u, v \in V\}$ is a set of **edges** G.

Undirected Graphs

Consider the undirected graph $G = \langle V, E \rangle$, where $V = \{a, b, c, d, e\}$,
$E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{e, b\}, \{d, b\}, \{c, d\}, \{d, e\}\}$.

We usually use a graphical representation of the graph. The vertices are represented by points and the edges by lines connecting the points.

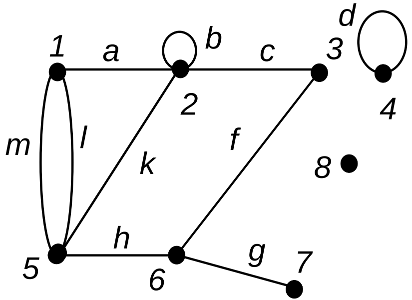
Consider schools of languages $SH = \{\text{yes}, \text{ok}, \text{maybe}, \text{ling}\}$ and students $S = \{\text{Ania}, \text{Ema}, \text{Tom}, \text{Ed}\}$ the set of potential students. Every student is interested in some schools $SH_i \subset SH$. The situation can be easily modelled by the graph $G = \langle SH \cup S, E \rangle$.



Let $m = \{p, q\} \in E$. This means that the edge m **connects** the vertex p and the vertex q. Moreover, in such a case p and q are called the **endpoints** of m and we say, that p and q are **incident** with m and p and q are **adjacent** or **neighbours** of each other.

In many applications we use a special types of edges: multiedges and loops.

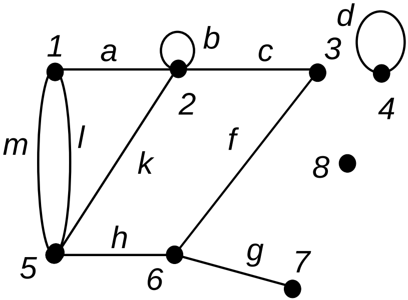
Multiedges are edges which connect the same pair of vertices and a **loop** connects the vertex with itself.



In this picture edges *m* and *l* are multiedges and *d* and *b* are loops.

However, from the formal point of view, to consider multiedges we need to modify a definition of the graph.

An undirected multigraph is a triple: $G = \langle V, E, \gamma \rangle$ where: V – nonempty ***set of vertices*** G, E – ***the set of edges*** G, γ – a function acting from the set E to the set $\{\{u, v\} : u, v \in V\}$ of all singletones or two-members subsets of V.



Often, we associate weights to edges of the graph. These weights can represent cost, profit or loss, length, capacity etc. of given connection.

The **weight** is a mapping from the set of edges to the set of real numbers

$$w : E \to \mathbb{R}$$

.

The graph with weight function is called the **network** and denoted by $G = \langle V, E, w \rangle$.

Let us consider the graph $G = \langle V, E, w \rangle$, where $V = \{a, b, c, d, e\}$, $E = \{\{a, b\}, \{a, c\}, \{a, e\}, \{e, b\}, \{d, b\}, \{c, d\}, \{d, e\}\}$ and the weight function

edge $e \in E$	{a,b}	{a,c}	{a,e}	{d,b}	{e,b}	{d,e}	{d,c}
weight $w(e)$	-9	3	6	0	12	8	0.7

A **directed graph** (**digraph**) is a pair:

$$G = \langle V, E \rangle$$

where V is a nonempty set of vertices and $E = \{(u, v) : u, v \in V\}$ is a set of directed edges. In digraphs, E is a set of ordered pairs. If $(p, q) \in E$, q is the **head** of edge and p is the **tail** of edge.

Consider the directed graph $G = \langle V, E \rangle$, where $V = \{a, b, c, d, e\}$,
$E = \{(a, b), (a, c), (a, e), (b, e), (b, d), (c, d), (d, e)\}$.

Consider the directed graph $G = \langle V, E \rangle$, where $V = \{a, b, c, d, e\}$,
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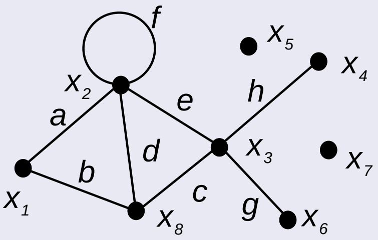
The **degree of the vertex** v in a graph G is the number of edges incident to the vertex, with loops counted twice. The degree of a vertex is denoted $\deg(v)$. A vertex v with degree 0 is called an **isolated vertex**, a vertex with degree 1 is called an **endvertex**, a **hanging vertex**, or a **leaf**.

The **degree of the graph** G, $\Delta(G)$ is the maximum degree of graph G

$$\Delta(G) = \max_{v \in V} \deg(v).$$

Consider the graph



then

1. isolated vertices are x_5 and x_7,
2. the endvertices are x_4 and x_6,
3. $\deg(x_1) = 2$ and $\deg(x_2) = 5$, $\deg(x_3) = 4$ and $\deg(x_8) = 3$
4. degree of graph $\Delta(G) = 5$.

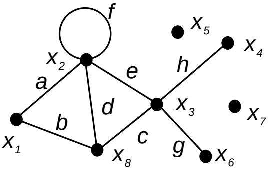
The distribution of degrees in a graph is called a sequence of

$$(p(0), p(1), \dots, p(k), \dots)$$

where $p(k) = P(X = k) = \frac{D_k}{n}$, $|V| = n$ and D_k denotes the number of vertices of degree k.

The average degree of a vertex is called the number

$$\langle k \rangle = \sum_k k p(k)$$



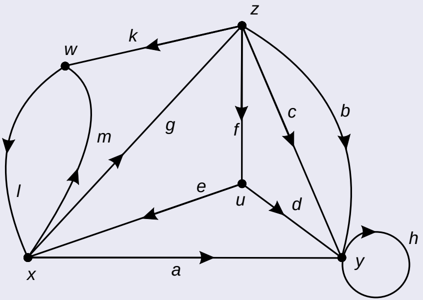
$$(0.25, 0.25, 0.125, 0.125, 0.125, 0.125)$$

Consider a digraph G.

- The **indegree** of a vertex v $degin(v)$, is the number of head endpoints adjacent to v.
- The **outdegree** of a vertex v $degout(v)$, is the number of tail endpoints adjacent from v.
- The **degree** of a vertex $deg(v)$ in digraph is equal

$$deg(v) = degout(v) + degin(v)$$

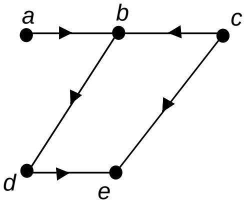
Consider the digraph



$$degout(x) = 3, \qquad degin(x) = 2,$$
$$degout(u) = 2, \qquad degin(u) = 1,$$
$$degout(z) = 4, \qquad degin(z) = 1,$$
$$degout(w) = 1, \qquad degin(w) = 2,$$
$$degout(y) = 1, \qquad degin(y) = 5.$$

- A vertex v for which $degin(v) = 0$ and $degout(v) > 0$ is called a **source**;
- A vertex v for which $degout(v) = 0$ and $degin(v) > 0$ is called a **sink**.

Consider the digraph

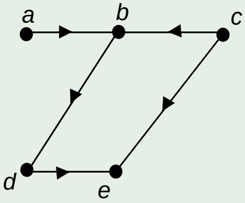


We have the following degrees of vertices

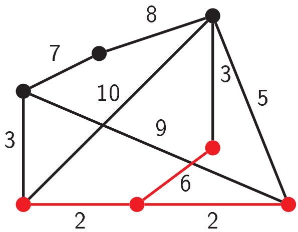
$$
\begin{aligned}
deg(a) &= degout(a) + degin(a) = 1 + 0 = 1 \\
deg(b) &= degout(b) + degin(b) = 1 + 2 = 3 \\
deg(c) &= degout(c) + degin(c) = 2 + 0 = 2 \\
deg(d) &= degout(d) + degin(d) = 1 + 1 = 2 \\
deg(e) &= degout(e) + degin(e) = 0 + 2 = 2
\end{aligned}
$$

The degree of digraph

$$\Delta(G) = \max_{u \in V} deg(u) = 3$$



Sources in digraph are vertices a and c. The sink is the vertex e.



$$deg(v) = 3$$

$$deg_w(v) = 2 + 2 + 6 = 10$$

Let $G = (V, E, w)$ be a weighted graph with a weight function w, then the **weighted degree** of a vertex we call the sum of the weights of the edges leaving the vertex $v \in V$ and denote by $deg_w(v)$.

$$deg_w(v) = \sum_{i=1}^{deg(v)} w(v, u_i)$$

Thank you for your attention!!!