# 1st presentation

Leandro Jorge Fernández Vega

## 2.3 Homomorphisms, kernels and normal subgroups.

Let $G, H$ be groups. We say a function $f: G \to H$ is an homomorphism $\iff f(g_1 g_2) = f(g_1) f(g_2) \quad \forall g_1, g_2 \in G$.

We define the kernel and the image of $f$ as:

$\operatorname{Ker} f = \{ g \in G \,/\, f(g) = e_H \}$ $\qquad \operatorname{Im} f = \{ h \in H \,/\, \exists g \in G$ so that $f(g) = h \}$

**2.3.1 Theorem:** Let $f: G \to H$ be an homomorphism. Then:

- $f(e_G) = e_H$
- $f(g^{-1}) = f(g)^{-1} \quad \forall g \in G$
- $\operatorname{Ker} f \leq G$, $\operatorname{Im} f \leq H$
- $\forall K \leq H$, $f^{-1}(K) = \{ g \in G \,/\, f(g) \in K \} \leq G$
- $f$ is injective $\iff \operatorname{Ker} f = \{ e_G \}$

**2.3.2 Remark**

We say $N \leq G$ is normal $\iff N \trianglelefteq G \iff \forall g \in G,\ gNg^{-1} = N \iff$
$\forall g \in G, n \in N,\ gng^{-1} \in N$

Trivially, if $G$ is abelian $\Rightarrow \forall H \leq G,\ H \trianglelefteq G$.

**2.3.3 Proposition** Let $f: G \to H$ be an homomorphism. Then $\operatorname{Ker} f \trianglelefteq G$

## 2.5 Quotient Groups

Let $G$ be a group, $H \trianglelefteq G$. We can now define the quotient group $G/H = \{xH \,/\, x \in G\}$ with internal operation defined as:

$xHyH = (xy)H$

If $H$ isn't normal we can't guarantee $Hx = xH$ $\forall x \in G$, so $G/H$ wouldn't have a group structure.

**Proof:**

Let $h \in H$, $x \in G \,/\, xH \neq Hx \Rightarrow \exists h \in H \,/\, hx \notin xH \Rightarrow$
$hHxH = hxH \neq xH$, but $hH = eH \Rightarrow hHxH = eHxH = exH = xH$ !!!

**2.5.2 Proposition (1st Isomorphism Theorem):**

Let $f : G \to H$ be a group isomorphism. Then, $\exists \bar{f} : G/\ker f \xrightarrow{\cong} \operatorname{Im} f \,/$ $\bar{f}(x \ker f) = f(x)$ $\forall x \in G$, so that the next diagram commutes:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "pr"'] & H \\ G/\ker f \arrow[r, "\cong", "\bar{f}"'] & \operatorname{Im} f \arrow[u, hook, "i"'] \end{tikzcd}$$

where $i$ is the inclusion function and $pr : G \to G/\ker f \mid pr(x) = x\ker f$ is the quotient map homomorphism.

**2.5.4 Corollary (2nd Isomorphism Theorem):**

Let $G$ be a group, $N \trianglelefteq G$, $H \leq G$. We define $HN = \{hn \,/\, h \in H, n \in N\}$. Then, $H \cap N \trianglelefteq H$ and $\frac{H}{H \cap N} \cong \frac{HN}{N}$

- **Proposition:** Let $G, H, N$ be groups, $H \leq G$, $K \trianglelefteq G$. Then $HK = KH$.

• **Direct Sum:**

Given $G, H$ groups, we define their direct product $G \times H = \{(g,h) / g \in G, h \in H\}$, with internal operation $(g,h)(g',h') = (gg', hh')$

When $G, H$ are abelian, we write their direct product as a direct sum $G \oplus H$.

• **Characterization:** $G \cong K \times H \iff \begin{cases} - K, H \trianglelefteq G \\ - G = KH \\ - K \cap H = \{e\} \end{cases}$

## 2.8 Trying to clasify finite groups, part 1.

**2.8.2 Proposition:** Let $G, H$ be cyclic groups / $\gcd(|G|, |H|) = 1 \implies G \times H$ cyclic

• **Proposition:** $Aut(G) = \{f : G \to G \,/\, f \text{ isomorphism}\}$ is a group for composition and identity the identity map $1_G$ ($1_G(g) = g \ \forall g \in G$)

**2.8.6 Proposition:** $Aut(\mathbb{Z}_n) \cong \mathbb{Z}_n^\times$.

**Definition:** Given groups $H, N$, we consider $f : H \to Aut(N)$, $h \mapsto f_h$

We denote the semi-direct product group of $H$ and $N$ as $H \times_f N$, the set $H \times N$ with group operation defined as:

$$(h,n)(h',n') = (hh', f_{h'^{-1}}(n)\, n')$$

**Proposition:** $G \cong H \times_f N \iff \begin{cases} H \trianglelefteq G, N \leq G \\ - H \cap N = \{e\} \\ - G = HN \end{cases}$, where $f$ is the conjugate action.

Let $G$ be a group.

1) $|G| = p$, $p$ prime.

By Lagrange's Theorem, there are no proper subgroups. Necessarily, given $g \in G$, $g \neq e_G$, necessarily means $G = \langle g \rangle = C_p$.

2) $|G| = pq$, $p, q$ primes.

We can suppose $p < q$.

$n_q | p$ $n_q \equiv 1 \mod q$ $\Rightarrow n_q = 1 \Rightarrow$

$\exists_1 P_q \trianglelefteq G$ $q$-ss / $|P_q| = q$

$n_p | q$ $n_p \equiv 1 \mod p \Rightarrow n_p = 1, q$

- $n_p = 1 \Rightarrow \exists_1 P_p$ $p$-ss / $|P_p| = p \Rightarrow G \cong C_p \times C_q \cong C_{pq}$
- $n_p = q \Rightarrow p | (q-1)$

We have an isomorphism $P = \mathbb{Z}_p \cong Aut(\mathbb{Z}_q) \cong \mathbb{Z}_q^\times$

$\mathbb{Z}_q^\times$ is an abelian group of order divisible by $p \Rightarrow$

$\exists z \in \mathbb{Z}_q^\times / |z| = p$

We can take $f: \mathbb{Z}_p \to Aut(\mathbb{Z}_q) \mid f(x)(y) = z^x y$, which gives a semidirect product $\mathbb{Z}_p \times_f \mathbb{Z}_q$, as it can't be abelian (there are elements which don't commute).

How many different semi-products are there?

We know $\mathbb{Z}_q^\times = \langle z \rangle$ is cyclic $\Rightarrow \forall z' \in \mathbb{Z}_q^\times, \exists l \in \{1 \dots p-1\} / z^l = z'$

Now we consider the isomorphism $i: \mathbb{Z}_p \to P_p$ and $f: P_p \to \mathrm{Aut}(P_q) \,/\, f(i(1)) = z'$

Since $\ell$ is prime to $p$,

$$f(i(k)) = f(k\,i(1)) = f(i(1))^k = (z')^k = z^{\ell k} = z$$

As a result, up to isomorphism there is a unique semidirect product.

3) $|G| = pqr$

we cannot conclude anything about the normality of Sylow groups.

We can establish these hypothesis:

| | | |
|---|---|---|
| $q \neq 1 \bmod p$ | $r \neq 1 \bmod p$ | $qr \neq 1 \bmod p$ |
| $p \neq 1 \bmod q$ | $r \neq 1 \bmod q$ | $pr \neq 1 \bmod q$ |
| $p \neq 1 \bmod r$ | $q \neq 1 \bmod r$ | $pq \neq 1 \bmod r$ |

They are enough to prove $P_p, P_q, P_r$ are normal. We can conclude $G = \mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r \cong \mathbb{Z}_{pqr}$

4) $|G| = p^2$

these groups are either cyclic of order $p^2$, $G \cong \mathbb{Z}_{p^2}$, or $G \cong \mathbb{Z}_p \times \mathbb{Z}_p$

5) $|G| = pq^2$

- If $G$ abelian $\Rightarrow G = \mathbb{Z}_{pq^2} \vee G = \mathbb{Z}_{pq} \times \mathbb{Z}_q$
- If $G$ not abelian $\Rightarrow$
  - $p \mid (q-1) \Rightarrow G \cong \mathbb{Z}_p \times_f \mathbb{Z}_{q^2} \vee G \cong \mathbb{Z}_p \times_f (\mathbb{Z}_q \times \mathbb{Z}_q)$
  - $p \mid (q+1)$, $p \neq 2 \wedge q \neq 3 \Rightarrow G \cong \mathbb{Z}_p \times_f (\mathbb{Z}_q \times \mathbb{Z}_q)$
  - $q \nmid (p-1) \Rightarrow G \cong \mathbb{Z}_{pq^2} \vee G \cong \mathbb{Z}_{pq} \times \mathbb{Z}_q$

- Other important Theorems:

1) Every group $G$ / $|G| = p^2$, $p$ prime, is abelian.
2) Every group $G$ / $|G| = 2$, is normal.
3) Every group $G$ whose $p$-groups are normal is expressed as a direct product of them all.

- Groups up to order 15:

| Orden | Abelianos | no Abelianos |
|---|---|---|
| 1 | 1 | −− |
| 2 | $C_2$ | −− |
| 3 | $C_3$ | −− |
| 4 | $C_2 \times C_2,\ C_4$ | −− |
| 5 | $C_5$ | −− |
| 6 | $C_6$ | $S_3$ |
| 7 | $C_7$ | −− |
| 8 | $C_8,\ C_2 \times C_4,\ C_2 \times C_2 \times C_2.$ | $D_4,\ \mathbb{Q}_2$ |
| 9 | $C_9,\ C_3 \times C_3$ | −− |
| 10 | $C_{10}$ | $D_5$ |
| 11 | $C_{11}$ | −− |
| 12 | $C_{12},\ C_2 \times C_6$ | $Q_3,\ D_6,\ A_4$ |
| 13 | $C_{13}$ | −− |
| 14 | $C_{14}$ | $D_7$ |
| 15 | $C_{15}$ | −− |

# 2.9 Worked examples

**2.9.1 Example:** Let $G, H$ be finite groups with relatively prime orders. Show that any group homomorphism $f : G \longrightarrow H$ is necessarily trivial (that is, sends every element of $G$ to the identity in $H$.)

The 1st Isomorphism Theorem implies that

$G/\ker f \cong \operatorname{Im} f \Rightarrow |G/\ker f| = |\operatorname{Im} f| \Rightarrow \frac{|G|}{|\ker f|} = |\operatorname{Im} f| \Rightarrow$

$|\ker f| \mid |G|$.

Besides, $\operatorname{Im} f \leq H \Rightarrow |\operatorname{Im} f| \mid |H|$. and $|\operatorname{Im} f| \mid |G|$

As $\gcd(|G|, |H|) = 1 \Rightarrow |\operatorname{Im} f| = 1 \Rightarrow \operatorname{Im} f = \{e\}$

**2.9.6 Example:** Exhibit a non-abelian group of order $3 \cdot 7$.

$|G| = 3 \cdot 7$

$n_3 \mid 7 \quad n_3 \equiv 1 \bmod 3 \Rightarrow n_3 = 1, 7$

$n_7 \mid 3 \quad n_7 \equiv 1 \bmod 7 \Rightarrow n_7 = 1$

- $n_3 = n_7 = 1 \Rightarrow G = P_3 \times P_7$
- $n_3 = 7 \Rightarrow$

we consider $\theta_i : P_3 \to \operatorname{Aut}(P_7) \;/\; \theta_i(y)(x) = yxy^{-1} = x^i$
$i = 1..6$

$|\theta_i(y)| \mid 3 \;\wedge\; |\theta_i(y)| \mid \varphi(7) = 6 \Rightarrow |\theta_i(y)| = 1, 3$

So, we have $P_3 \times_{\theta_1} P_7 = P_3 \times P_7 \;\vee\; P_3 \times_{\theta_2} P_7$, since
$\operatorname{Aut}(P_7) \cong \mathbb{Z}_7^\times$ and if $2 \in \mathbb{Z}_7^\times$, $|2| = 3 \Rightarrow$

$P_3 \times_{\theta_2} P_7 = \langle x, y \,/\, x^7 = 1,\; y^3 = 1 \,/\, yxy^{-1} = x^2 \rangle$

# EXERCISES

- theorems used:

A) Sylow theorems

B) If $G, H$ are cyclic groups, $G \times H$ cyclic $\iff \gcd(|G|, |H|) = 1$

C) $G \cong H \times N \iff \begin{cases} \bullet\ H, N \trianglelefteq G \\ \bullet\ H \cap N = \{e\} \\ \bullet\ G = HN \end{cases}$

D) $G \cong H \rtimes_\theta N \iff \begin{cases} \bullet\ H \leq G,\ N \trianglelefteq G \\ \bullet\ H \cap N = \{e\} \\ \bullet\ G = HN \end{cases}$, where $\theta$ is the conjugate action.

F) If all Sylow subgroups of a group $G$ are normal $\Rightarrow$ $G$ is direct product of them all.

G) p-Sylow and q-Sylow subgroups have trivial intersection.

**2.3** Classify groups of order 9 and 10.

A) $|G| = 9 = 3^2$

We know groups of order $p^2$, $p$ prime can either be

$G \cong C_p \times C_p \lor G \cong C_{p^2} \Rightarrow G \cong C_3 \times C_3 \lor G \cong C_9$

B) $|G| = 10 = 2 \cdot 5$

$n_5 | 2$ $n_5 \equiv 1 \mod 5 \Rightarrow n_5 = 1 \Rightarrow \exists_1 P_5 \trianglelefteq G$ 5-SS/ $|P_5| = 5$

$n_2 | 5$, $n_2 \equiv 1 \mod 2 \Rightarrow n_2 = 1, 5$

- $n_2 = 1 \Rightarrow \exists_1 P_2$ 2-SS / $|P_2| = 2 \Rightarrow G \cong P_2 \times P_5 \cong C_2 \times C_5 \cong C_{10}$
- $n_2 = 5$, we consider the homomorphism

$\theta_i : P_2 \cong C_2 \to Aut(P_5) \cong Aut(C_5)$ / $\theta_i(h)(y) = hyh^{-1} = y^i$ $i \in \mathbb{Z}_5^\times$

$|\theta_i(h)| \,|\, 2 \land |\theta_i(h)| \,|\, \varphi(5) = 4 \Rightarrow |\theta_i(h)| = 1, 2 \Rightarrow$

$G \cong P_2 \times_{\theta_1} P_5 \cong C_2 \times C_5 \cong C_{10} \lor G \cong C_2 \times_{\theta_4} C_5 = \langle x, y \mid x^2 = 1, y^5 = 1, xyx^{-1} = y^4 \rangle$,

since $Aut(P_5) \cong \mathbb{Z}_5^\times$, and if $4 \in \mathbb{Z}_5$, $|4| = 2$

**2.4** Classify groups of order 12.

$|G| = 12 = 2^2 \cdot 3$

$n_3 | 4$, $n_3 \equiv 1 \mod 3 \Rightarrow n_3 = 1, 4$

$n_2 | 3$, $n_2 \equiv 1 \mod 2 \Rightarrow n_2 = 1, 3$

- $n_3 = 1 = n_2 \Rightarrow |G| \cong P_3 \times P_2 \cong \begin{cases} C_3 \times C_4 \cong C_{12} \\ \lor \\ C_3 \times C_2 \times C_2 \cong C_2 \times C_6 \end{cases}$
- $n_3 = 1$, $n_2 = 3 \Rightarrow G \cong P_2 \times_{\theta_i} P_3$ (analog to previous exercises)
- $n_3 = 4$, $n_2 = 1$ Don't know the elements of $Aut(C_2 \times C_2)$. If $P_2 \cong C_4$, then it's analog to previous exercises.

- $n_3 = 4$, $n_2 = 3$ : can't count elements to find contradiction. ??

**2.5** Classify groups of order 21.

$|G| = 21 = 3 \cdot 7$

$n_3 | 7 \quad n_3 \equiv 1 \mod 3 \quad \Rightarrow n_3 = 1, 7$

$n_7 | 3 \quad n_7 \equiv 1 \mod 7 \quad \Rightarrow n_7 = 1$

- $n_3 = 1 \Rightarrow G = P_3 \times P_7$
- $n_3 = 7 \Rightarrow G = P_3 \times_{\theta_i} P_7$, $\theta_i : P_3 \to Aut(P_7)$ / $\theta_i(h)(y) = h y h^{-1} = y^i$, $i \in \mathbb{Z}_7^\times$

$|\theta_i(h)| \,\big|\, |P_3|, |Aut(P_7)| = \varphi(7) = 6 \Rightarrow |\theta_i(h)| = 1, 3$

So, we have $P_3 \times_{\theta_1} P_7 = P_3 \times P_7 \vee P_3 \times_{\theta_2} P_7$, since
$Aut(P_7) \cong \mathbb{Z}_7^\times$ and if $2 \in \mathbb{Z}_7^\times$, $|2| = 2 \Rightarrow$
$P_3 \times_{\theta_2} P_7 = \langle x, y / x^7 = 1, y^3 = 1 / y x y^{-1} = x^2 \rangle$

**2.8** Classify groups of order 77.

$|G| = 77 = 7 \cdot 11$

$n_7 | 11, n_7 \equiv 1 \mod 7 \Rightarrow n_7 = 1 \Rightarrow \exists_1 P_7 \trianglelefteq G$ 7-SS / $|P_7| = 7$
$n_{11} | 7, n_{11} \equiv 1 \mod 11 \Rightarrow n_{11} = 1 \Rightarrow \exists_1 P_{11} \trianglelefteq G$ 11-SS / $|P_{11}| = 11$ $\Big\} \Rightarrow$

$G \cong P_7 \times P_{11} \cong C_7 \times C_{11} \cong C_{77}$

**2.10** Let $N$ be a normal subgroup of a group $G$, and let $H$ be a subgroup of $G$ such that $G = H \cdot N$, that is, such that the collection of all products $h \cdot n$ with $h \in H$ and $n \in N$ is the whole group $G$. Show that $G/N \approx H/(H \cap N)$.

$N \trianglelefteq G, H \leq G, G = HN$. The 2nd Isomorphism Theorem states

$$\frac{G}{N} = \frac{HN}{N} \cong \frac{H}{H \cap N}$$

**2.12** Let $G$ be a group in which $g^2 = 1$ for every $g \in G$. Show that $G$ is abelian.

$G$ abelian $\Leftrightarrow xy = yx \quad \forall x, y \in G$

As $xy, yx \in G$, $(xy)^2 = (yx)^2 = 1 \Rightarrow (xy)^{-1} = xy$, $(yx)^{-1} = yx$

$(xy)^2 = xyxy = xyx^{-1}y^{-1} = (xy)(yx)^{-1} \Leftrightarrow xy = (yx)^{-1} = yx$