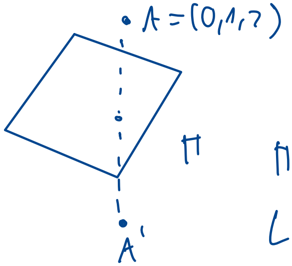
1st Exam Practice: Curves

1. $\pi: \ x+3y+2z+7=0$

   $A(0,1,2)$

   Halla el punto simétrico de A respecto a π



Let's construct a perpendicular line to π which contains A.

$\pi \equiv x+3y+2z+7=0$

$L=\{(0,1,2)+t(1,3,2) \ / \ t\in\mathbb{R}\}$

Let's see where π and L intersect:

$0+t+3(1+3t)+2(2+2t)+7=0 \iff 14t+14=0 \iff t=-1 \implies$

$L\cap\pi=\{(-1,-2,0)\}=P$

$\frac{A+A'}{2}=P \implies A'=2P-A=(-2,-4,0)-(0,1,2)=(-2,-5,-2)$

2. Halla la distancia entre ℓ_1 y ℓ_2:

$\ell_1 = \{(1,2,3) + t\cdot[1,7,2] : t\in\mathbb{R}\}$

$\ell_2 = \{(2,0,7) + s\cdot[2,0,9] : s\in\mathbb{R}\}$

$$\left.\begin{array}{l} \ell_1 = (1+t,\ 2+7t,\ 3+2t) \\ \ell_2 = (2+2s,\ 0,\ 7+9s) \end{array}\right\} \Rightarrow \ell_1 \cap \ell_2 \equiv \begin{cases} 1+t = 2+2t \iff t=-1 \\ 2+7t = 0 \iff t = -2/7 \ !!! \Rightarrow \\ 3+2t = 7+9t \iff t = -4/7 \end{cases}$$

$$\left.\begin{array}{l} \ell_1 \cap \ell_2 = \emptyset \\ \text{Besides, } \nexists\, x\in\mathbb{R} \,/\, (1,7,2) = x(2,0,9) \Rightarrow \ell_1 \nparallel \ell_2 \end{array}\right\} \Rightarrow \ell_1, \ell_2 \text{ are skew lines}$$

Let $P_1 = (1,2,3)$, $P_2 = (2,0,7)$, $V_1 = (1,7,2)$, $V_2 = (2,0,9)$

Let $\pi_2 = \{(2,0,7) + t(1,7,2) + s(2,0,9) \,/\, t,s\in\mathbb{R}\}$

$$V_1 \times V_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 7 & 2 \\ 2 & 0 & 9 \end{vmatrix} = 63\vec{i} + 4\vec{j} - 14\vec{k} - 9\vec{j} = 63\vec{i} - 5\vec{j} - 14\vec{k}$$

$$= (63,-5,-14) \Rightarrow \pi_2 \equiv 63x - 5y - 14z + D = 0$$

$$P_2 \in \pi_2 \Rightarrow 63\cdot 2 - 5\cdot 0 - 14\cdot 7 + D = 0 \Rightarrow D = -28 \Rightarrow$$

$$\pi_2 \equiv 63x - 5y - 14z - 28 = 0$$

$$d(\ell_1,\ell_2) = d(P_1,\pi_2) = \frac{|63\cdot 1 - 5\cdot 2 - 14\cdot 3 - 28|}{\sqrt{63^2 + 5^2 + 14^2}} = \frac{17}{\sqrt{4190}}$$

3. Halla la longitud de la curva $\varphi(t) = \left(t, \frac{e^t + e^{-t}}{2}\right)$ en $[-1,1]$.

$\varphi(t) = (t, \cosh t) \Rightarrow \varphi'(t) = (1, \sinh t) \Rightarrow$

$\|\varphi'(t)\| = \sqrt{1 + \sinh^2 t} = \cosh t$

$L(\varphi) = \int_{-1}^{1} \|\varphi'(t)\| dt = [\sinh t]_{-1}^{1} = \frac{1}{2}(e - e^{-1} - e^{-1} + e) = e - e^{-1} = \frac{e^2 - 1}{e}$

4. Calcula la curvatura de $\gamma(t) = (t, t^2, \cos t)$ en el punto $(0,0,1)$.

$\gamma'(t) = (1, 2t, -\sin t) \Rightarrow \|\gamma'(t)\| = \sqrt{1 + 4t^2 + \sin^2 t} \Rightarrow \gamma$ isn't unit speed

$\gamma''(t) = (0, 2, -\cos t)$

$$\gamma'(t) \times \gamma''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & -\sin t \\ 0 & 2 & -\cos t \end{vmatrix} = -2t\cos t\, \vec{i} + 2\vec{k} + 2\sin t\, \vec{i} + \cos t\, \vec{j}$$

$= (2\sin t - 2t\cos t, \cos t, 2)$

$\|\gamma'(t) \times \gamma''(t)\| = \sqrt{4\sin^2 t + 4t^2\cos^2 t - 4t\sin 2t + \cos^2 t + 4}$

$\|\gamma'(t)\|^3 = (1 + 4t^2 + \sin^2 t)^{3/2}$

then $k(t) = \dfrac{\|\gamma'(t) \times \gamma''(t)\|}{\|\gamma'(t)\|^3}$

$\gamma(t) = (t, t^2, \cos t) = (0,0,1) \Rightarrow t = 0$

$k(0) = \frac{\sqrt{5}}{1} = \sqrt{5}$

5. Sea el paraboloide $z = x^2 + y^2$ y la parábola $y^2 = x$. Considera la curva que resulta de la intersección de estos dos y calcula $t(s), n(s), b(s), k(s)$ y $\tau(s)$.

$y^2 = x \Rightarrow y^4 = x^2 \Rightarrow z = y^4 + y^2$.

Besides, we know $y^2 = x \Rightarrow x, z > 0$, $y \in \mathbb{R}$.

As a result, our curve is defined as

$\alpha(y) = (y^2, y, y^4 + y^2)$

$t(t) = \alpha'(t) = (2t, 1, 4t^3 + 2t)$

$\|\alpha'(t)\| = \sqrt{4t^2 + 1 + 16t^6 + 4t^2 + 16t^4} = \sqrt{16t^6 + 16t^4 + 8t^2 + 1}$

$\Rightarrow \alpha$ isn't unit speed

We know for any regular curve α, $\exists\, \hat{\alpha}$ regular curve with unit speed such that α is equivalent to $\hat{\alpha}$, where

$\hat{\alpha}(t) = \alpha\left(\left[\int_a^t \|\alpha'(t)\| dt\right]^{-1}\right)$

Let $f(t) = \int_a^t \|\alpha'(t)\|$, $g(t) = f^{-1}(t)$

$\hat{\alpha}'(t) = \alpha'(g(t)) \cdot g'(t) = (2g(t), 1, 4g(t)^3 + 2g(t)) \cdot \dfrac{1}{f'(f^{-1}(t))} =$