Lecture 8.
Graph coloring

Vertex coloring - definition

Let G be a graph without loops.

Definition

We say that G is $k-$***coloring*** if it is possible to assign to each vertex one of $k-$colors in such a way that no two vertices sharing the same edge (incident vertices) have the same color. If G is $k-$***coloring***, but it is not $(k-1)-$***coloring*** then we say that G is $k-$***chromatic***.

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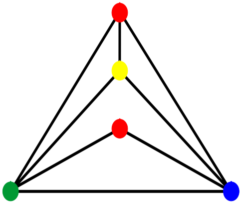

$\chi(G) = 4$, 4-chromatic

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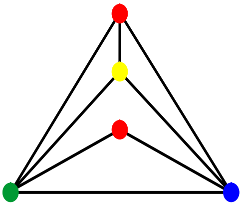

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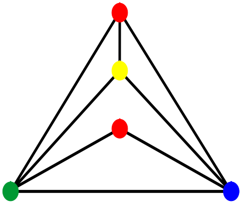

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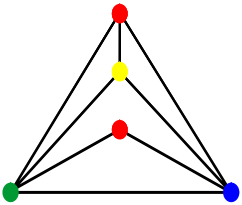

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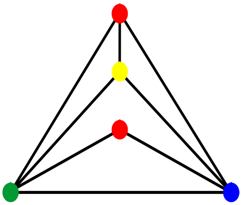

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The problem of vertex coloring will be considered only for simple and connected graphs.

Chromatic number for some classes of graphs

Theorem

For complete graph one has $\chi(K_n) = n$.

Chromatic number for some classes of graphs

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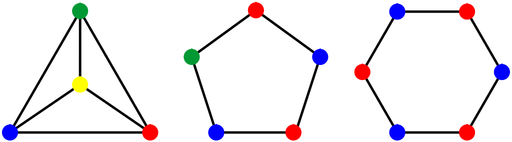
Theorem

For circular graph C_n one has

$$\chi(C_n) = \begin{cases} 2 & \text{whenever } n \text{ is even} \\ 3 & \text{whenever } n \text{ is odd,} \end{cases}$$

in other words $\chi(C_{2i+1}) = 3$, $\chi(C_{2i}) = 2$ for $i \in \{1, 2, \ldots\}$.



a) $\chi(K_4) = 4$ b) $\chi(C_5) = 3$ c) $\chi(C_6) = 2$

Chromatic number for some classes of graphs

Theorem

- $\chi(G) = 1$ if G is an empty graph (consisting of only one vertex).
- $\chi(G) = 2$, if G is a bipartite graph, in particular any tree is 2-chromatic.

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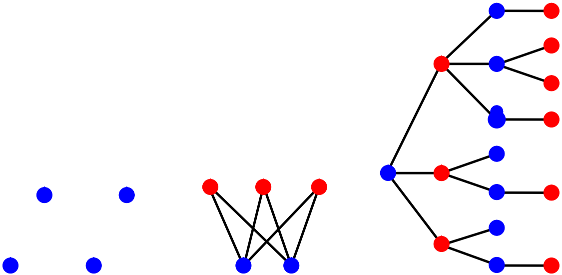


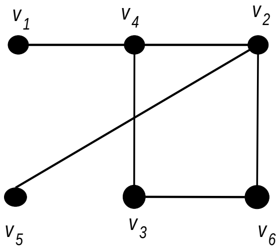
$\chi(G) = 1$ $\chi(K_{3,2}) = 2$ $\chi(T) = 2$

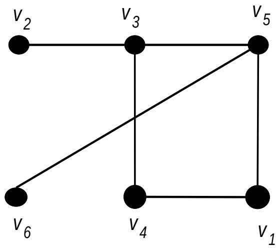
Greedy algorithm for vertex coloring

Inputs: a simple graph G with n vertices $\{v_1, v_2, \ldots, v_n\}$, the colors are denoted 1...n
Result: colored graph G

```
ZAKG(G)
assign color 1 to vertex v1
for k ← 2 to n
    do
        for vertex vk use the first available color,
        which is not used for coloring any neighbour of vk
```

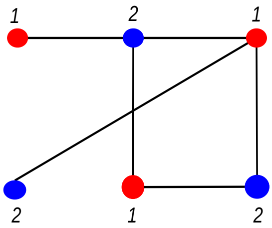
Example

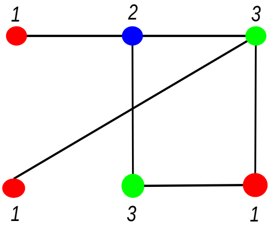




We start with coloring v_1.

as a result we have





Bounds on $\chi(G)$

Remark

From the definition of $\chi(G)$ it follows, that if G has n vertices, then its chromatic number is not greater than n i.e..

$$|V| = n \Rightarrow \chi(G) \leq n$$

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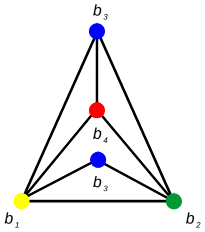

$$4 \leq \chi(G) \leq 5$$

$$\chi(G) = 4$$

Definition

A clique of G is a subgraph of G which is complete.

Theorem

For any graph G

$$\chi(G) \geq \omega$$

where ω is the number of vertices of a largest clique.

Brook's Theorem

If $G = \langle V, E \rangle$ is a simple graph, with the largest degree of vertex equal to $\Delta(G)$ then

$$\chi(G) \leq \Delta(G) + 1 \tag{1}$$

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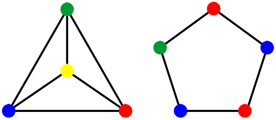
Remark

The equality in the above inequality is possible in only two cases:

- if G is complete K_n $(n \geq 3)$. We know, that $\Delta(K_n) = n - 1$ i $\chi(K_n) = n$ thus
$$\Delta(K_n) + 1 = n - 1 + 1 = n = \chi(K_n)$$
- if G is a circular graph with even number of vertices i.e. $G = C_{2i+1}$, $(i = 1, 2, \ldots)$. We have $\Delta(C_{2i+1}) = 2$, $\chi(C_{2i+1}) = 3$, thus
$$\Delta(C_{2i+1}) + 1 = 2 + 1 = 3 = \chi(C_{2i+1})$$



Remark

Form Brooks theorem it follows that $K_{2,s}$ is $s+1-$coloring i.e. $\chi(K_{2,s}) \leq s+1$, but as it can be easily concluded $\chi(K_{2,s}) = 2$.

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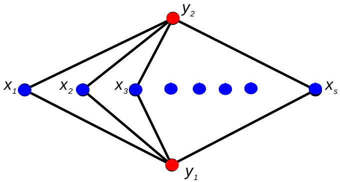

Definition

A subset $S \subset V$ of vertices of graph $G = (V, E)$, is said to be ***independent***, if no two vertices from S are incident. I particular any singleton and the empty set are independent sets.

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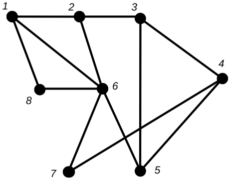

independent sets: $\{2, 5, 7, 8\}$, $\{1, 3, 7\}$, $\{2, 4, 8\}$, $\{4, 6\}$, $\{3, 6\}$, $\{1, 5, 7\}$, $\{1, 4\}$, $\{3, 7, 8\}$

Definition

A number

$$\alpha(G) = \max_i |S_i|$$

where S_i are independent sets of G, is called the ***independence number*** of G. The set S^*, for which the maximum is attained is called ***maximal independent set***.

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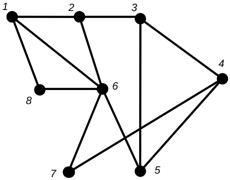
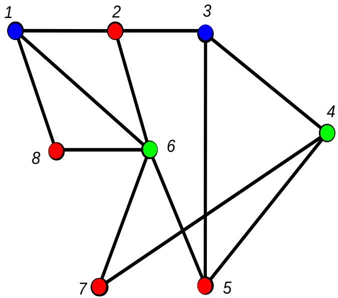


For the previous example (picture) we have $S^* = \{2, 5, 7, 8\}$ oraz $\alpha(G) = 4$

The problem of coloring vertices is equivalent to the problem of division of the set of vertices V into the family of k disjoint independent sets such that $\bigcup_{i=1}^{k} V_i = V$. Then $\chi(G) = k$,



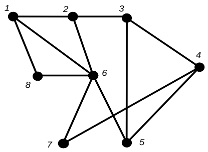
$V_1 = \{1, 3\}$, $V_2 = \{4, 6\}$, $V_3 = \{2, 5, 7, 8\}$.
The above graph is 3-coloring i $\chi(G) = 3$.

Bounds on $\chi(G)$

Property

We have

$$\chi(G) \geq \lceil \frac{n}{\alpha(G)} \rceil$$

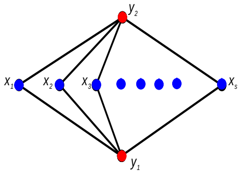


$$\chi(G) \geq \lceil \frac{8}{4} \rceil = 2$$

and from Brook's theorem

$$6 \geq \chi(G) \geq 2$$

one can see that $\chi(G) = 3$.



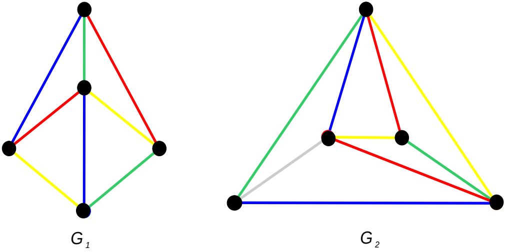
For $K_{2,s}$ we have $\alpha(G) = s$ and

$$\chi(G) \geq \lceil \frac{s+2}{\alpha(G)} \rceil = \lceil \frac{s+2}{s} \rceil = 2$$

Edge coloring – the definition

Definition

We say that a graph G is ***edge $k-$coloring*** if its edges can be colored with k colors in such a way that no two adjacent edges have the same color. If G is edge $k-$coloring and is not edge $(k-1)-$coloring, then we say that its ***chromatic index*** $\bar{\chi}(G)$ is k.



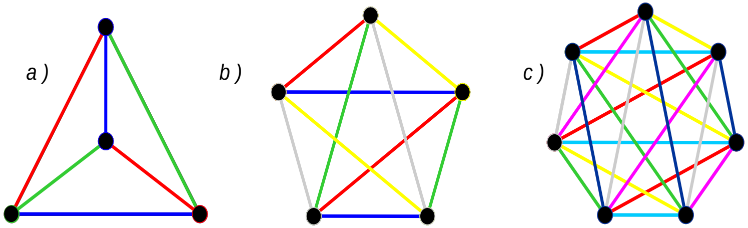
$$\bar{\chi}(G_1) = 4 \text{ i } \bar{\chi}(G_2) = 5$$

Chromatic index of complete graphs

Theorem

Chromatic index of K_n:

$$\bar{\chi}(K_n) = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$$

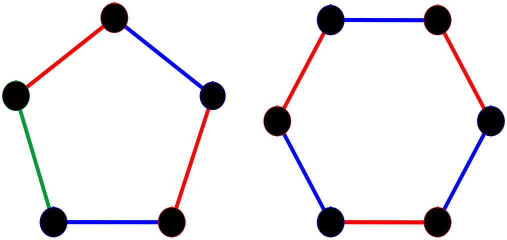


$\bar{\chi}(K_4) = 3,\ \bar{\chi}(K_5) = 5,\ \bar{\chi}(K_7) = 7$

Chromatic index of circular graphs

It is easy to see that chromatic index of C_n:

$$\bar{\chi}(C_n) = \begin{cases} 2 & \text{if } n - \text{is even,} \\ 3 & \text{if } n - \text{is odd.} \end{cases}$$



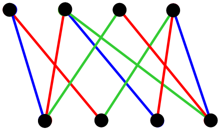
$$\bar{\chi}(C_5) = 3,\ \bar{\chi}(C_6) = 2$$

Chromatic index of bipartite graphs

König's Theorem, 1916

In a bipartite graph $G = (V, E)$, if the largest degree of vertex is Δ, then its chromatic index

$$\bar{\chi}(G) = \Delta$$



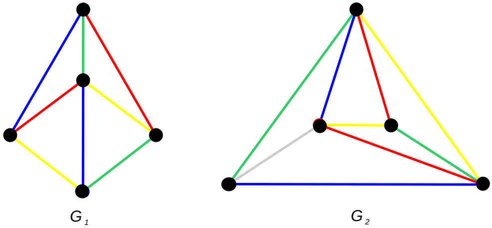
$\Delta = 3$ therefore $\bar{\chi}(G) = 3$

Vizing's Theorem

Vizing's Theorem, 1964

If $G = (V, E)$ is a simple graph, with the largest degree of vertex Δ, then the chromatic index should satisfy

$$\Delta \leq \bar{\chi}(G) \leq \Delta + 1.$$

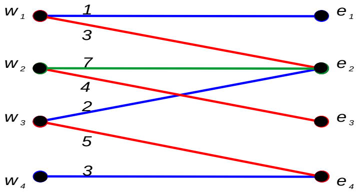


In G_1 and G_2 presented in the above picture we have $\Delta(G_1) = \Delta(G_2) = 4$, whereas $\bar{\chi}(G_1) = 4$ i $\bar{\chi}(G_2) = 5$. Thus

$$4 \leq \bar{\chi}(G_1) \leq 4 + 1 = 5 \text{ i } 4 \leq \bar{\chi}(G_2) \leq 4 + 1 = 5$$

Example

A service center has four different workshops $\{w_1, w_2, w_3, w_4\}$ and four different teams of specialists $\{e_1, e_2, e_3, e_4\}$. Service has to make seven various repairs at various locations. Any repair needs respective teams, and a suitable workshop, as shown in graph G in the picture.



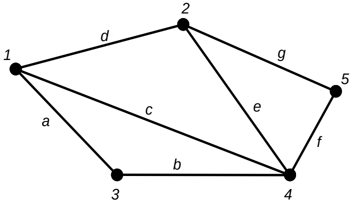
In this case one team, equipped with one workshop, may one day perform at most one repair. The goal is to make such a schedule for repairs that the number number of days for repairing is minimal .
The solution of the above task can be based on coloring of the edges of a graph G so that the number of colors is minimal. Repairs assigned to the edges of the same color can be implemented on the same day. The minimum number of days for the task is equal to chromatic index of the graph shown.

Another solution is: repair $(1, 2, 3, 6)$ will be implemented one day, the next day $(4, 5)$, and third day – repair 7.

Planar graphs

Definition

A graph G is said to be ***planar*** iff it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other. Such a drawing is called a ***plane graph*** or ***planar embedding of the graph***.



Regions

A planar embedding of the planar graph always divides a plane into disjoint sets. These sets are called ***regions of the graph***. The regions will be denoted by f_i dla $i \in \{1, 2, ..., n\}$.

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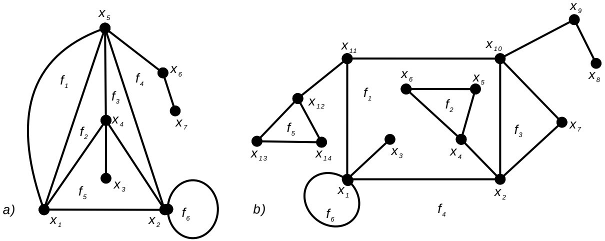
It follows from the definition that the region is uniquely define by a specific planar embedding rather that an abstract notion of a graph.



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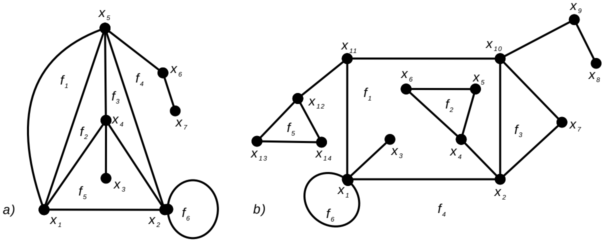

Region f_4 is a ***infinite region***. Every planar graph (and every embedding) has exactly one infinite region. Changing, however the embedding can transform the infinite region to the finite one and vice versa.

Euler's formula for planar graphs

Eulers' formula, 1750

Let G be a finite connected planar graph with n vertices, m edges and f regions. Then

$$f = m - n + 2$$

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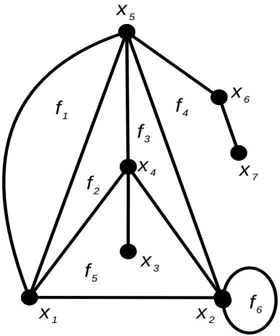

$$\begin{aligned} n &= 7 \\ m &= 11 \\ f &= m - n + 2 = \\ &= 11 - 7 + 2 = 6 \end{aligned}$$

Corollary – Euler's inequalities for planar graphs

If G is a planar simple connected graph with n vertices and m edges then

$$m \leq 3n - 6.$$

If additionally, G does not have triangles (circuits consisting of 3 edges), then

$$m \leq 2n - 4.$$

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Theorem

K_5 is not a planar graph.

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Theorem

K_5 is not a planar graph.

Theorem

$K_{3,3}$ is not a planar graph.

Kuratowski's theorem

Definition

Subdivision of an edge (a, b) of a graph G is an operation involving the addition of a new vertex v, the removal of (a, b), and the addition of two new edges (a, v) and (v, b). Geometrically, this operation consists of addition of some (interior) point v on the line (a, b); this point then becomes a new vertex. A graph G' is called a ***subdivision of a graph*** G if it can be obtained from G by repeating the operation of edge subdivision several times.

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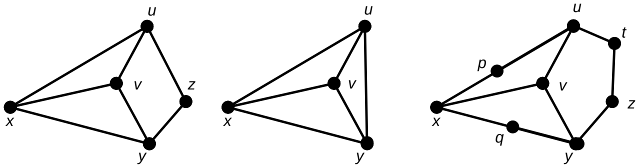

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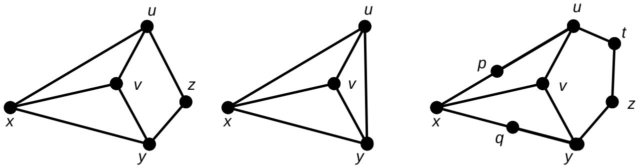
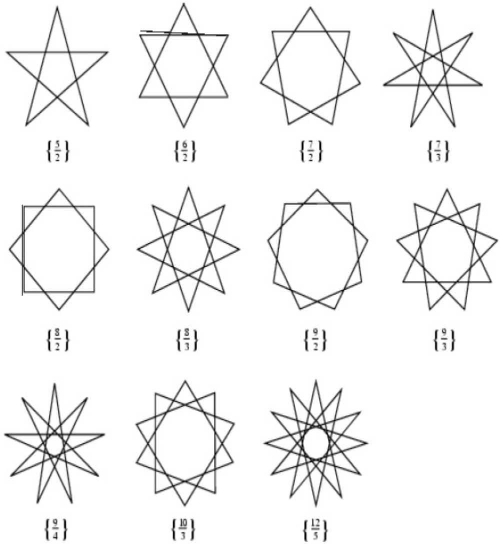


Kuratowski's theorem, 1930

A finite graph is planar if and only if it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$.

Generalised Petersen graphs



Definition

A ***star polygon*** $\{n/k\}$ is a graph with n vertices connected every k vertices.

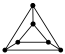
Without loss of generality we can assume that $k < \frac{n}{2}$.

Generalised Petersen graphs

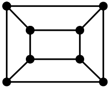
Definition

A ***generalised Petersen graph*** $P_{n,k}$ is a graph that is formed by connecting the vertices of a regular polygon C_n to the corresponding vertices of a star polygon $\{n/k\}$.

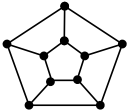
graph $P_{3,1}$

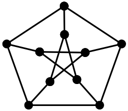


graph $P_{4,1}$



graphs $P_{5,1}$ and $P_{5,2}$





Petersen graph

Definition

Graph $P_{5,2}$ is called a Petersen graph.

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Theorem

Petersen graph is not a planar graph.

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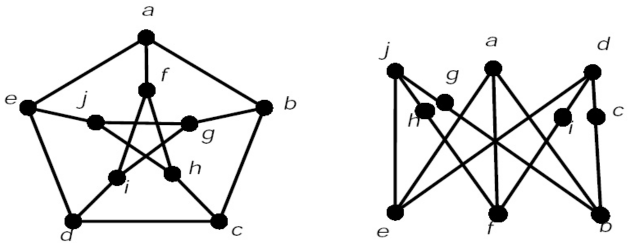
Theorem

Petersen graph is not a planar graph.

It foollows directly from the Kuratawski's theorem.



Regions coloring

Definition

We say that the regions of a planar graph are $k-$***colorable*** (that is it is possible to color them with k colors), if none two regions which are adjacent share the same color.

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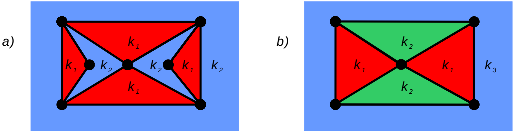

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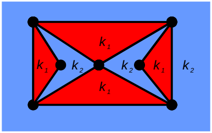
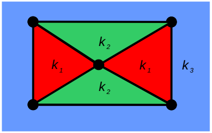
a)



b)



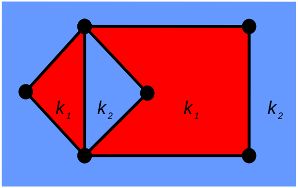
Definition

The minimal number k that the regions of planar graph are $k-$colorable is called a ***chromatic number for regions*** of this graph and is denoted by $\mu(G)$.

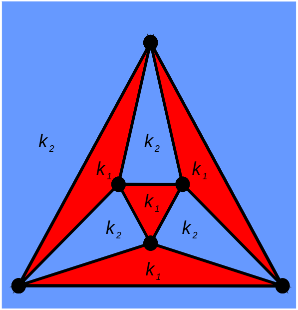
Theorem

Regions of a planar graph can be colored wit two colors iff every vertex has even degree.

a)



b)



Theorem

A cubic graph can be colored with two colors iff it is bipartite.

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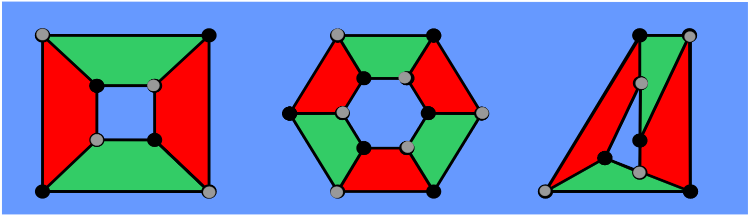

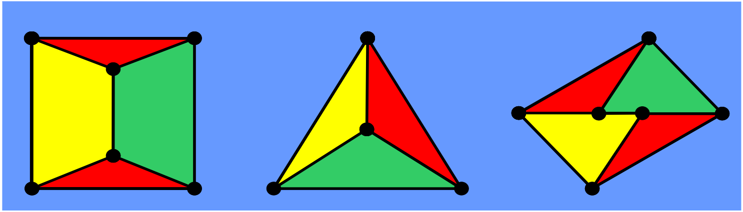

Four color problem

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- The four-color theorem was not proven until 1976 when two American mathematicians K. Appel and W. Haken proved it in a very complicated way which was computer-aided.
- In 2004 there a new proof was published. This proof also uses a computer, but covering only 600 reducible configuration that you can check on a laptop in few hours. Its authors are Robertson, Sanders, Seymour and Thomas from Atlanta

Dziękuję za uwagę!!!