Actividad EVALUABLE 3 (Varianza total)
Responder de forma razonada a las cuestiones que se plantean.

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1. Para cualesquiera $(a_1,\ldots,a_n) \in \mathbb{R}^n$, calcular detalladamente, indicando las fórmulas aplicadas, la expresión de la varianza:

$$\exists E[X_i^2],\ i=1,\ldots,n, \Rightarrow \exists \text{Var}\left[\sum_{i=1}^{n} a_i X_i\right]$$

$$\text{Var}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + \sum_{i\neq j}^{n} a_i a_j \text{Cov}(X_i, X_j).$$

$$\text{Var}\left[\sum_{i=1}^{n} a_i X_i\right] \overset{\text{Var}[X] = E[X^2] - E[X]^2}{=} E\left[\left(\sum_{i=1}^{n} a_i X_i\right)^2\right] - E\left[\sum_{i=1}^{n} a_i X_i\right]^2 \overset{\text{Linealidad}}{=}$$

$$E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j X_i X_j\right] - \left(\sum_{i=1}^{n} a_i E[X_i]\right)^2 \overset{\text{Linealidad}}{=}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j E[X_i X_j] - \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j E[X_i]E[X_j] =$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \left(E[X_i X_j] - E[X_i]E[X_j]\right) \overset{\text{COV}(X_i,X_j) = E[X_i X_j] - E[X_i]E[X_j]}{=}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \text{COV}(X_i, X_j)$$

- Cuando $i \neq j$, $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \text{COV}(X_i, X_j) = \sum_{\substack{i=1\\ i\neq j}}^{n} a_i a_j \text{COV}(X_i, X_j)$
- Cuando $i = j$, $\displaystyle\sum_{i=j=1}^{n} a_i a_j \text{COV}(X_i, X_j) = \sum_{i=1}^{n} a_i^2 E\left[(X_i - E[X_i])^2\right] = \sum_{i=1}^{n} a_i^2 \text{Var}[X_i]$

Por tanto, $\displaystyle\text{Var}\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i \text{Var}[X_i] + \sum_{\substack{i=1\\ i\neq j}} a_i a_j \text{COV}(X_i, X_j) \quad \forall a_i \in \mathbb{R}$

Además,

$\exists E[X_i^2]\ \forall i=1..n \implies \exists E[X_i X_j]\ \forall i,j=1..n$, pues por Des. Cauchy-Schwarz,

$E[X_i X_j]^2 \leq E[X_i^2]E[X_j^2] \iff E[X_i X_j] \leq \sqrt{E[X_i^2]E[X_j^2]} < \infty$