Actividad Obligatoria 2

Leandro Jorge Fdez. Vega DGIIM

Deducir las funciones generatrices de momentos de todas las distribuciones continuas.

$M_X :]-t_0, t_1[\to \mathbb{R} \;/\; t_0, t_1 \in \mathbb{R}^+ \cup \{+\infty\}$.

$M_X(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} f_X(x)\, dx \quad \forall t \in]-t_0, t_1[$

1) $X \rightsquigarrow U(a,b)$

$$f_X(x) = \frac{1}{b-a} \quad \forall x \in]a,b[$$

$$M_X(t) = \int_a^b \frac{e^{tx}}{b-a}\, dx = \left[\frac{e^{tx}}{t(b-a)}\right]_{x=a}^{x=b} = \frac{e^{tb} - e^{ta}}{t(b-a)} \quad \forall t \neq 0$$

Vemos $\lim_{t\to 0} M_X(t) = \lim_{t\to 0} \frac{e^{tb} - e^{ta}}{t(b-a)} = \left[\frac{0}{0}\right] \overset{L'H}{=}$

$$\lim_{t\to 0} \frac{be^{tb} - ae^{ta}}{b-a} = \frac{b-a}{b-a} = 1.$$

Por tanto,

$$M_X(t) = \begin{cases} 1 & t = 0 \\ \frac{e^{tb} - e^{ta}}{t(b-a)} & t \neq 0 \end{cases}$$

2) $X \rightsquigarrow N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad \forall x \in \mathbb{R}$$

Veamos $X \rightsquigarrow N(0,1) \Leftrightarrow M_X(t) = e^{t^2/2} \quad \forall t \in \mathbb{R}$

$$M_X(t) = E[e^{tX}] = \int_{\mathbb{R}} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \overset{\mu=0,\ \sigma^2=1}{=}$$

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{tx - \frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\frac{t^2}{2} - \frac{1}{2}(x-t)^2} dx =$$

$$tx - \frac{x^2}{2} = \frac{2tx - x^2}{2} = -\frac{(x-t)^2}{2} + \frac{t^2}{2}$$

$$\frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(x-t)^2} dx = e^{\frac{t^2}{2}} \int_{\mathbb{R}} f_X(x)\, dx = e^{t^2/2} \quad \forall t \in \mathbb{R}$$

f_X es fdd de $X \rightsquigarrow N(t,1)$, $\forall t \in \mathbb{R}$

Por tipificación, sabemos

$$X \rightsquigarrow N(\mu, \sigma^2) \Leftrightarrow Z = \frac{X-\mu}{\sigma} \rightsquigarrow N(0,1) \Rightarrow X = \sigma Z + \mu$$

$$M_X(t) = E[e^{tX}] = E[e^{t(\sigma Z + \mu)}] =$$

$$e^{t\mu} E[e^{(t\sigma)Z}] = e^{t\mu} M_Z(t\sigma) = e^{t\mu} e^{\frac{(t\sigma)^2}{2}} =$$

$$e^{t\mu + \frac{t^2\sigma^2}{2}}$$

3) $\underline{X} \sim \exp(\lambda) \quad \lambda > 0$

$$f_{\underline{X}}(x) = \lambda e^{-\lambda x} \qquad \forall x \in \mathbb{R}_0^+$$

$$M_{\underline{X}}(t) = \int_0^{+\infty} e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^{+\infty} e^{x(t-\lambda)} dx =$$

$$\frac{\lambda}{t-\lambda} \left[e^{x(t-\lambda)} \right]_{x=0}^{x=+\infty} = \frac{-\lambda}{t-\lambda} = \left(\frac{\lambda - t}{\lambda} \right)^{-1} = \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$\forall t < \lambda$$

4) $X \rightsquigarrow \mathcal{E}(n,\lambda)$, $n \in \mathbb{N}-\{0\}$, $\lambda > 0$

$$f_X(x) = \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x} \quad \forall x \in \mathbb{R}^+$$

$$M_X(t) = \frac{\lambda^n}{\Gamma(n)} \int_0^{+\infty} e^{tx} x^{n-1} e^{-\lambda x} dx =$$

$$\frac{\lambda^n}{\Gamma(n)} \int_0^{+\infty} x^{n-1} e^{x(t-\lambda)} dx = \left[\begin{array}{l} u = x^{n-1} \Rightarrow du = (n-1)x^{n-2} dx \\ dv = e^{x(t-\lambda)} \Rightarrow v = \dfrac{e^{x(t-\lambda)}}{t-\lambda} \end{array} \right] =$$

$$\frac{\lambda^n}{\Gamma(n)} \left(\left[\cancelto{0}{x^{n-1} \frac{e^{x(t-\lambda)}}{t-\lambda}} \right]_{x=0}^{x=+\infty} - \frac{n-1}{t-\lambda} \int_0^{+\infty} x^{n-2} e^{x(t-\lambda)} dx \right)$$

Es claro que:

$$\frac{\lambda^n}{\Gamma(n)} I_{n-1} = \frac{\lambda^n}{\Gamma(n)} \int_0^{+\infty} x^{n-1} e^{x(t-\lambda)} = \frac{\lambda^n}{\Gamma(n)} \frac{n-1}{\lambda-t} I_{n-2} =$$

$$\frac{\lambda^n}{\Gamma(n)} \frac{(n-1)!}{(\lambda-t)^{n-1}} I_0 = \frac{\lambda^n}{(\lambda-t)^{n-1}} \int_0^{+\infty} e^{x(t-\lambda)} dx =$$

$$\frac{\lambda^n}{(\lambda-t)^{n-1}} \left[\frac{e^{x(t-\lambda)}}{t-\lambda} \right]_0^{+\infty} = \left(\frac{\lambda}{\lambda-t} \right)^n = \left(1 - \frac{t}{\lambda} \right)^{-n} \quad \forall t < \lambda$$

5) $X \sim \Gamma(u, \lambda) \quad u \in \mathbb{R}, \ \lambda > 0$

$$f_X(x) = \frac{\lambda^u}{\Gamma(u)} x^{u-1} e^{-\lambda x} \qquad \forall x \in \mathbb{R}^+$$

$$M_X(t) = \frac{\lambda^u}{\Gamma(u)} \int_0^{+\infty} e^{tx} x^{u-1} e^{-\lambda x} dx =$$

$$\frac{\lambda^u}{\Gamma(u)} \int_0^{+\infty} x^{u-1} e^{x(t-\lambda)} dx = \begin{bmatrix} -y = x(t-\lambda) \\ -dy = (t-\lambda) dx \end{bmatrix} =$$

$$\frac{\lambda^u}{\Gamma(u)} \int_0^{+\infty} \left(\frac{y}{\lambda-t}\right)^{u-1} e^{-y} \cdot \frac{1}{\lambda-t} dy = \frac{\lambda^u}{(\lambda-t)\Gamma(u)} \cdot \frac{1}{(\lambda-t)^{u-1}} \Gamma(u)$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^u = \left(1 - \frac{t}{\lambda}\right)^{-u} \qquad \forall t < \lambda$$

6) $X \rightsquigarrow \beta(p,q)$, $p,q>0$

$$f_X(x) = \frac{1}{\beta(p,q)} x^{p-1}(1-x)^{q-1} \quad \forall x \in]0,1[$$

Desarrollo Maclaurin de e^{tx}

$$M_X(t) = E[e^{tX}] = \frac{1}{\beta(p,q)} \int_0^1 e^{tx} x^{p-1}(1-x)^{q-1}\,dx =$$

$$\frac{1}{\beta(p,q)} \int_0^1 \sum_{k=0}^{+\infty} \frac{(tx)^k}{k!} x^{p-1}(1-x)^{q-1}\,dx =$$

$$\frac{1}{\beta(p,q)} \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^1 x^{p+k-1}(1-x)^{q-1}\,dx = \frac{\sum_{k=0}^{+\infty} \frac{t^k}{k!}\beta(p+k,q)}{\beta(p,q)} =$$

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

$$= \frac{\sum_{k=0}^{+\infty} \frac{t^k}{k!} \frac{\Gamma(p+k)\cancel{\Gamma(q)}}{\Gamma(p+q+k)}}{\frac{\Gamma(p)\cancel{\Gamma(q)}}{\Gamma(p+q)}} = \sum_{k=0}^{+\infty} \frac{t^k}{k!} \frac{\Gamma(p+q)\Gamma(p+k)}{\Gamma(p)\Gamma(p+q+k)} \quad \forall t \in \mathbb{R}$$