My formalization project

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Chapter 1

Green's Relations

Definition 1 (Green's R-preorder)**.** Let M be a monoid and let $x, y \in M$. We define $x \leq_{\mathcal{R}} y$ if there exists $z \in M$ such that $x = y \cdot z$. Equivalently, x lies in the principal right ideal generated by y.

Definition 2 (Green's L-preorder)**.** Let M be a monoid and let $x, y \in M$. We define $x \leq_{\mathcal{L}} y$ if there exists $z \in M$ such that $x = z \cdot y$. In other words, x lies in the principal left ideal generated by y.

Definition 3 (Green's J-preorder)**.** Let M be a monoid and let $x, y \in M$. We define $x \leq_{\mathcal{J}} y$ if there exist $u, v \in M$ such that $x = u \cdot y \cdot v$. Equivalently, x lies in the two-sided ideal generated by y.

Definition 4 (Green's H-preorder)**.** Let M be a monoid and let $x, y \in M$. We define $x \leq_{\mathcal{H}} y$ if both $x \leq_{\mathcal{R}} y$ and $x \leq_{\mathcal{L}} y$ hold, that is, x is simultaneously in the right and left ideals generated by y. Unwinding the definition, this means there exist elements $z_1, z_2 \in M$ with $x = y \cdot z_1$ and $x = z_2 \cdot y$.

Lemma 5 ($\leq_{\mathcal{R}}$ is a preorder (reflexive & transitive))**.** *The relation $\leq_{\mathcal{R}}$ on M is a preorder; in particular:*

- ***Reflexivity.*** *For every $x \in M$, $x \leq_{\mathcal{R}} x$ (witness $z = 1$ since $x = x \cdot 1$).*
- ***Transitivity.*** *If $x \leq_{\mathcal{R}} y$ and $y \leq_{\mathcal{R}} z$, pick witnesses $v, u \in M$ with $x = yv$ and $y = zu$; then $x = z(uv)$, so $x \leq_{\mathcal{R}} z$.*

Lemma 6 ($\leq_{\mathcal{L}}$ is a preorder (reflexive & transitive))**.** *The relation $\leq_{\mathcal{L}}$ on M is a preorder; in particular:*

- ***Reflexivity.*** *For every $x \in M$, $x \leq_{\mathcal{L}} x$ (witness $z = 1$ since $1 \cdot x = x$).*
- ***Transitivity.*** *If $x \leq_{\mathcal{L}} y$ and $y \leq_{\mathcal{L}} z$, pick witnesses $u, v \in M$ with $x = uy$ and $y = vz$; then $x = (uv)z$, so $x \leq_{\mathcal{L}} z$.*

Lemma 7 ($\leq_{\mathcal{J}}$ is a preorder (reflexive & transitive))**.** *The relation $\leq_{\mathcal{J}}$ on M is a preorder; in particular:*

- ***Reflexivity.*** *For every $x \in M$, $x \leq_{\mathcal{J}} x$ (witness $u = v = 1$ so $x = 1 \cdot x \cdot 1$).*

- ***Transitivity.*** *If $x \leq_{\mathcal{J}} y$ and $y \leq_{\mathcal{J}} z$, pick $u_1, v_1, u_2, v_2 \in M$ with $x = u_1 y v_1$ and $y = u_2 z v_2$; then $x = u_1(u_2 z v_2)v_1 = (u_1 u_2)z(v_2 v_1)$, so $x \leq_{\mathcal{J}} z$.*

Lemma 8 ($\leq_{\mathcal{H}}$ is a preorder (reflexive & transitive))**.** *The relation $\leq_{\mathcal{H}}$ on M is a preorder; in particular:*

- ***Reflexivity.*** *For every $x \in M$, $x \leq_{\mathcal{H}} x$ since $x \leq_{\mathcal{R}} x$ and $x \leq_{\mathcal{L}} x$.*

- ***Transitivity.*** *If $x \leq_{\mathcal{H}} y$ and $y \leq_{\mathcal{H}} z$, then $x \leq_{\mathcal{R}} y \leq_{\mathcal{R}} z$ and $x \leq_{\mathcal{L}} y \leq_{\mathcal{L}} z$; hence $x \leq_{\mathcal{R}} z$ and $x \leq_{\mathcal{L}} z$, so $x \leq_{\mathcal{H}} z$.*

1.1 Equivalences from Preorders

To obtain equivalence relations from a preorder we take its symmetric closure. Given any preorder p, we define an equivalence relation by declaring two elements equivalent exactly when both $p\,x\,y$ and $p\,y\,x$ hold. This construction works uniformly for all preorders and, in particular, produces Green's equivalence relations from Green's preorders.

Definition 9 (Equivalence of a preorder)**.** Let α be a type and let $p : \alpha \to \alpha \to \mathrm{Prop}$ be a preorder. For $x, y : \alpha$ we define $\mathrm{EquivOfLE}\,p\,x\,y$ to hold when both $p\,x\,y$ and $p\,y\,x$ hold. In other words, $\mathrm{EquivOfLE}\,p$ is the symmetric closure of the relation p.

Lemma 10 (EquivOfLE is an equivalence relation)**.** *If p is a preorder on α then* $\mathrm{EquivOfLE}\,p$ *is an equivalence relation.*

Proof. We verify the three properties of an equivalence relation. For reflexivity, if p is a preorder then it is reflexive, so for every x we have both $p\,x\,x$ and $p\,x\,x$; hence $\mathrm{EquivOfLE}\,p\,x\,x$ holds. Symmetry is immediate: if $\mathrm{EquivOfLE}\,p\,x\,y$ holds, meaning $p\,x\,y$ and $p\,y\,x$, then the pair $p\,y\,x$ and $p\,x\,y$ shows $\mathrm{EquivOfLE}\,p\,y\,x$. For transitivity, suppose $\mathrm{EquivOfLE}\,p\,x\,y$ and $\mathrm{EquivOfLE}\,p\,y\,z$. Then we have $p\,x\,y$ and $p\,y\,z$; by transitivity of the preorder p we obtain $p\,x\,z$. Similarly from $p\,z\,y$ and $p\,y\,x$ we deduce $p\,z\,x$. Thus $\mathrm{EquivOfLE}\,p\,x\,z$ holds, completing the proof. $\square$

1.2 Green's Equivalences ($\mathcal{R}, \mathcal{L}, \mathcal{J}, \mathcal{H}$)

Definition 11 (Right equivalence)**.** Let M be a monoid. For $x, y \in M$ we define $x\,\mathcal{R}\,y$ to hold if both $x \leq_{\mathcal{R}} y$ and $y \leq_{\mathcal{R}} x$ hold. In other words, two elements are right equivalent when they lie in the principal right ideals generated by each other.

Lemma 12 (Right equivalence is an equivalence relation)**.** *The relation $\mathcal{R}$ defined in Definition 11 is an equivalence relation on M.*

Definition 13 (Left equivalence)**.** For $x, y \in M$ we define $x\,\mathcal{L}\,y$ to hold when both $x \leq_{\mathcal{L}} y$ and $y \leq_{\mathcal{L}} x$ hold. This expresses that x and y generate the same principal left ideals.

Lemma 14 (Left equivalence is an equivalence relation)**.** *The relation $\mathcal{L}$ defined in Definition 13 is an equivalence relation on M.*

Definition 15 (J equivalence)**.** For $x, y \in M$ we define $x \mathcal{J} y$ to hold when both $x \leq_{\mathcal{J}} y$ and $y \leq_{\mathcal{J}} x$ hold, i.e., each lies in the two-sided ideal generated by the other.

Lemma 16 (J equivalence is an equivalence relation)**.** *The relation $\mathcal{J}$ defined in Definition 15 is an equivalence relation on M.*

Definition 17 (H equivalence)**.** For $x, y \in M$ we define $x \mathcal{H} y$ to hold when both $x \leq_{\mathcal{H}} y$ and $y \leq_{\mathcal{H}} x$ hold. This captures when two elements are simultaneously right and left equivalent.

Lemma 18 (H equivalence is an equivalence relation)**.** *The relation $\mathcal{H}$ defined in Definition 17 is an equivalence relation on M.*

1.3 Basic Preorder Properties

Lemma 19 (Multiplication decreases for $\leq_{\mathcal{R}}, \leq_{\mathcal{L}}, \leq_{\mathcal{J}}$)**.** *For all M a monoid and $x, y, u, v \in M$, the following hold:*

- (**R**) $x \cdot y \leq_{\mathcal{R}} x$.
- (**L**) $y \cdot x \leq_{\mathcal{L}} x$.
- (**J**) $u \cdot x \cdot v \leq_{\mathcal{J}} x$.

Proof.
- **(R)** By definition of $\leq_{\mathcal{R}}$, we need u with $x \cdot u = x \cdot y$. Take $u := y$.
- **(L)** By definition of $\leq_{\mathcal{L}}$, we need u with $u \cdot x = y \cdot x$. Take $u := y$.
- **(J)** By definition of $\leq_{\mathcal{J}}$, we need s, t with $s \cdot x \cdot t = u \cdot x \cdot v$. Take $s := u$, $t := v$.

□

Lemma 20 (Cancellation for $\leq_{\mathcal{J}}, \leq_{\mathcal{R}}, \leq_{\mathcal{L}}$)**.** *Let M be a monoid and $x, y, z \in M$.*

- (**J**-**L**) *If $x \leq_{\mathcal{J}} y \cdot z$, then $x \leq_{\mathcal{J}} z$.*
- (**J**-**R**) *If $x \leq_{\mathcal{J}} y \cdot z$, then $x \leq_{\mathcal{J}} y$.*
- (**R**-**R**) *If $x \leq_{\mathcal{R}} y \cdot z$, then $x \leq_{\mathcal{R}} y$.*
- (**L**-**L**) *If $x \leq_{\mathcal{L}} y \cdot z$, then $x \leq_{\mathcal{L}} z$.*

Proof.
- **(J–L)** From $x \leq_{\mathcal{J}} yz$, pick u, v with $u \cdot (yz) \cdot v = x$. By associativity, $(uy) \cdot z \cdot v = x$. Set $u' := uy$, $v' := v$. Then $x \leq_{\mathcal{J}} z$.
- **(J–R)** From $x \leq_{\mathcal{J}} yz$, pick u, v with $u \cdot (yz) \cdot v = x$. By associativity, $u \cdot y \cdot (zv) = x$. Set $u' := u$, $v' := zv$. Then $x \leq_{\mathcal{J}} y$.
- **(R–R)** From $x \leq_{\mathcal{R}} yz$, pick u with $(yz) \cdot u = x$. By associativity, $y \cdot (zu) = x$. Set $u' := zu$. Then $x \leq_{\mathcal{R}} y$.
- **(L–L)** From $x \leq_{\mathcal{L}} yz$, pick u with $u \cdot (yz) = x$. By associativity, $(uy) \cdot z = x$. Set $u' := uy$. Then $x \leq_{\mathcal{L}} z$.

□

Lemma 21 (Idempotent characterizations for $\le_{\mathcal{R}}$ and $\le_{\mathcal{L}}$)**.** *Let M be a monoid and let $e \in M$ be idempotent ($e \cdot e = e$). For any $x \in M$,*

$$\textbf{(R)} \quad x \le_{\mathcal{R}} e \iff e \cdot x = x, \qquad \textbf{(L)} \quad x \le_{\mathcal{L}} e \iff x \cdot e = x.$$

Proof. Assume $e^2 = e$.

- **(R)** *($\Rightarrow$)* If $x \le_{\mathcal{R}} e$, pick t with $e \cdot t = x$. Then $e \cdot x = e \cdot (e \cdot t) = (e \cdot e) \cdot t = e \cdot t = x$. *($\Leftarrow$)* If $e \cdot x = x$, take $t := x$; then $e \cdot t = x$, so $x \le_{\mathcal{R}} e$.
- **(L)** *($\Rightarrow$)* If $x \le_{\mathcal{L}} e$, pick t with $t \cdot e = x$. Then $x \cdot e = (t \cdot e) \cdot e = t \cdot (e \cdot e) = t \cdot e = x$. *($\Leftarrow$)* If $x \cdot e = x$, take $t := x$; then $t \cdot e = x$, so $x \le_{\mathcal{L}} e$.

□

1.4 R–L multiplicative compatibility

Lemma 22 (Left multiplication: compatibility with $\le_{\mathcal{R}}$ and $\mathcal{R}$-equivalence)**.** *For all $x, y, z \in M$:*

- ***(Preorder)*** *If $x \le_{\mathcal{R}} y$, then $z \cdot x \le_{\mathcal{R}} z \cdot y$.*
- ***(Equivalence)*** *If $x \mathcal{R} y$, then $z \cdot x \mathcal{R} z \cdot y$.*

Proof.

- **(Preorder)** From $x \le_{\mathcal{R}} y$ pick u with $y \cdot u = x$. Then $(z \cdot y) \cdot u = z \cdot (y \cdot u) = z \cdot x$ by associativity, hence $z \cdot x \le_{\mathcal{R}} z \cdot y$.
- **(Equivalence)** If $x \mathcal{R} y$, then $x \le_{\mathcal{R}} y$ and $y \le_{\mathcal{R}} x$. Apply the preorder case to both inclusions to get $z \cdot x \le_{\mathcal{R}} z \cdot y$ and $z \cdot y \le_{\mathcal{R}} z \cdot x$, whence $z \cdot x \mathcal{R} z \cdot y$.

□

Lemma 23 (Right multiplication: compatibility with $\le_{\mathcal{L}}$ and $\mathcal{L}$-equivalence)**.** *For all $x, y, z \in M$:*

- ***(Preorder)*** *If $x \le_{\mathcal{L}} y$, then $x \cdot z \le_{\mathcal{L}} y \cdot z$.*
- ***(Equivalence)*** *If $x \mathcal{L} y$, then $x \cdot z \mathcal{L} y \cdot z$.*

Proof.

- **(Preorder)** From $x \le_{\mathcal{L}} y$ pick u with $u \cdot y = x$. Then $u \cdot (y \cdot z) = (u \cdot y) \cdot z = x \cdot z$ by associativity, hence $x \cdot z \le_{\mathcal{L}} y \cdot z$.
- **(Equivalence)** If $x \mathcal{L} y$, then $x \le_{\mathcal{L}} y$ and $y \le_{\mathcal{L}} x$. Apply the preorder case to both inclusions to get $x \cdot z \le_{\mathcal{L}} y \cdot z$ and $y \cdot z \le_{\mathcal{L}} x \cdot z$, hence $x \cdot z \mathcal{L} y \cdot z$.

□

1.5 R–L commutativity

Lemma 24 (Commutativity of $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{R}}$)**.** *For any $x, y \in M$, there exists an element z with $x \leq_{\mathcal{L}} z$ and $z \leq_{\mathcal{R}} y$ if and only if there exists an element w with $x \leq_{\mathcal{R}} w$ and $w \leq_{\mathcal{L}} y$.*

Proof. First suppose there exists $z \in M$ such that $x \leq_{\mathcal{L}} z$ and $z \leq_{\mathcal{R}} y$. Unwinding the definitions, there are witnesses u and v with $u \cdot z = x$ and $y \cdot v = z$. Taking $w = u \cdot y$ we have

$$w \leq_{\mathcal{L}} y \quad \text{since} \quad w = u \cdot y$$

and

$$x = u \cdot z = u \cdot (y \cdot v) = (u \cdot y) \cdot v = w \cdot v,$$

so $x \leq_{\mathcal{R}} w$. Conversely, suppose there exists $w \in M$ with $x \leq_{\mathcal{R}} w$ and $w \leq_{\mathcal{L}} y$. Then there are witnesses v and u with $w \cdot v = x$ and $u \cdot y = w$. Let $z = y \cdot v$. We compute

$$z \leq_{\mathcal{R}} y \quad \text{because} \quad y \cdot v = z,$$

and

$$x = w \cdot v = (u \cdot y) \cdot v = u \cdot (y \cdot v) = u \cdot z,$$

showing $x \leq_{\mathcal{L}} z$. This establishes the equivalence. □

Lemma 25 (Commutativity of right and left equivalence)**.** *For any $x, y \in M$, there exists $z \in M$ such that $x \,\mathcal{L}\, z$ and $z \,\mathcal{R}\, y$ if and only if there exists $w \in M$ such that $x \,\mathcal{R}\, w$ and $w \,\mathcal{L}\, y$.*

Proof. Suppose there exists z with $x \,\mathcal{L}\, z$ and $z \,\mathcal{R}\, y$. Write this as $x \leq_{\mathcal{L}} z \wedge z \leq_{\mathcal{L}} x$ and $z \leq_{\mathcal{R}} y \wedge y \leq_{\mathcal{R}} z$. From the left and right preorder conditions we have witnesses u_1, u_2, v_1, v_2 satisfying

$$u_1 \cdot z = x, \qquad z \cdot u_2 = x, \qquad y \cdot v_1 = z, \qquad z \cdot v_2 = y.$$

Set $w = u_1 \cdot y$. Then using the first pair of equations we have

$$w \leq_{\mathcal{L}} y \quad \text{since} \quad w = u_1 \cdot y,$$

and from the remaining equations we compute

$$x = u_1 \cdot z = u_1 \cdot (y \cdot v_1) = (u_1 \cdot y) \cdot v_1 = w \cdot v_1,$$

so $x \leq_{\mathcal{R}} w$. Moreover, by applying Lemma **??** to the equivalence $z \,\mathcal{R}\, y$ we deduce $u_1 \cdot z$ is $\mathcal{R}$-equivalent to $u_1 \cdot y$. Since $u_1 \cdot z = x$, this shows $x \,\mathcal{R}\, w$. Combined with the previous observation that w and y are $\mathcal{L}$-equivalent, we obtain the right-to-left implication.

Conversely, assume there exists w such that $x \,\mathcal{R}\, w$ and $w \,\mathcal{L}\, y$. This means $x \leq_{\mathcal{R}} w$ and $w \leq_{\mathcal{R}} x$, and $w \leq_{\mathcal{L}} y$ and $y \leq_{\mathcal{L}} w$. Pick witnesses v_1, v_2, u_1, u_2 with

$$w \cdot v_1 = x, \qquad x \cdot v_2 = w, \qquad u_1 \cdot y = w, \qquad y \cdot u_2 = w.$$

Let $z = y \cdot v_1$. Then z and y are $\mathcal{R}$-equivalent because both $y \leq_{\mathcal{R}} z$ and $z \leq_{\mathcal{R}} y$ hold via the witnesses above. Using Lemma **??** applied to the $\mathcal{L}$-equivalence $w \,\mathcal{L}\, y$, we have $x \cdot u_1$ is $\mathcal{L}$-equivalent to $w \cdot u_1 = u_1 \cdot y$. Since $x \cdot u_1 = (w \cdot v_1) \cdot u_1$ and $z = y \cdot v_1$, a straightforward computation shows $x \leq_{\mathcal{L}} z$ and $z \leq_{\mathcal{L}} x$. Hence $x \,\mathcal{L}\, z$ and $z \,\mathcal{R}\, y$, completing the proof. □

1.6 The D-equivalence

Definition 26 (D-equivalence)**.** Let M be a monoid. For $x, y \in M$ we say $x \mathcal{D} y$ holds if there exists $z \in M$ such that $x \mathcal{R} z$ and $z \mathcal{L} y$. Equivalently, there is a chain $x \mathcal{R} z$ and $z \mathcal{L} y$.

Lemma 27 (D-equivalence symmetry)**.** *For all $x, y \in M$,*

$$x \mathcal{D} y \iff (\exists z,\ x \mathcal{R} z \wedge z \mathcal{L} y) \iff (\exists w,\ y \mathcal{R} w \wedge w \mathcal{L} x),$$

and in particular $x \mathcal{D} y \Rightarrow y \mathcal{D} x$.

Proof.
- The first equivalence is just Definition 26 unfolded.
- The second equivalence uses the commutation of $\mathcal{R}$- and $\mathcal{L}$-equivalences: any chain $x \mathcal{R} z \mathcal{L} y$ can be rotated to a chain $y \mathcal{R} w \mathcal{L} x$ (Lemma `rEquiv_lEquiv_comm`).
- Taking x, y swapped in the second characterization yields the symmetry $x \mathcal{D} y \Rightarrow y \mathcal{D} x$.

□

Lemma 28 (Closure under $\mathcal{L}$- and $\mathcal{R}$-equivalence)**.** *Let $x, y, z \in M$.*

- *If $x \mathcal{D} y$ and $y \mathcal{L} z$, then $x \mathcal{D} z$.*
- *If $x \mathcal{D} y$ and $y \mathcal{R} z$, then $x \mathcal{D} z$.* *(*by duality from the previous bullet, using symmetry*)*

Proof.
- Write $x \mathcal{D} y$ as $x \mathcal{R} u$ and $u \mathcal{L} y$ for some u. If $y \mathcal{L} z$, then by transitivity of $\mathcal{L}$-equivalence, $u \mathcal{L} z$. Hence $x \mathcal{R} u \mathcal{L} z$, i.e. $x \mathcal{D} z$.
- For the $\mathcal{R}$-closure: from $x \mathcal{D} y$ get $y \mathcal{D} x$ by symmetry (Lemma 27); combine with $y \mathcal{R} z$ and apply the previous bullet in the dual form to obtain $y \mathcal{D} z$, then symmetrize back to $x \mathcal{D} z$.

□

Lemma 29 (Transitivity of $\mathcal{D}$)**.** *If $x \mathcal{D} y$ and $y \mathcal{D} z$ then $x \mathcal{D} z$.*

Proof. Choose witnesses $x \mathcal{R} u \mathcal{L} y$ and $y \mathcal{R} v \mathcal{L} z$. Apply the $\mathcal{R}$-closure to get $x \mathcal{D} v$, then the $\mathcal{L}$-closure to conclude $x \mathcal{D} z$. □

Lemma 30 ($\mathcal{D}$ is an equivalence relation)**.** *The relation $\mathcal{D}$ on M is reflexive, symmetric, and transitive.*

Proof.
- **Reflexive:** take $z := x$ and use reflexivity of $\mathcal{R}$ and $\mathcal{L}$.
- **Symmetric:** Lemma 27.
- **Transitive:** Lemma 29.

□

1.7 Equivalence classes

Green's equivalence relations partition a monoid into subsets called equivalence classes. Concretely, given an element $a \in M$, the $\mathcal{R}$-class of a is the set of all x such that $x \mathrel{\mathcal{R}} a$. Similarly for $\mathcal{L}, \mathcal{J}, \mathcal{H}$ and $\mathcal{D}$.

Definition 31 (Right class)**.** For $a \in M$ we define $\mathrm{RClass}(a)$ to be the set $\{x \in M \mid x \mathrel{\mathcal{R}} a\}$.

Definition 32 (Left class)**.** For $a \in M$ we define $\mathrm{LClass}(a)$ to be the set $\{x \in M \mid x \mathrel{\mathcal{L}} a\}$.

Definition 33 (J class)**.** For $a \in M$ we define $\mathrm{JClass}(a)$ to be the set $\{x \in M \mid x \mathrel{\mathcal{J}} a\}$.

Definition 34 (H class)**.** For $a \in M$ we define $\mathrm{HClass}(a)$ to be the set $\{x \in M \mid x \mathrel{\mathcal{H}} a\}$.

Definition 35 (D class)**.** For $a \in M$ we define $\mathrm{DClass}(a)$ to be the set $\{x \in M \mid x \mathrel{\mathcal{D}} a\}$.

Chapter 2

Location Theorem

2.1 Green's Lemma

Lemma 36 (Translation identity for $\mathcal{L}$-below elements)**.** *Let M be a monoid and suppose $x, y \in M$ satisfy $x \mathcal{R} y$. Choose $u, v \in M$ with $x \cdot u = y$ and $y \cdot v = x$. Then for every $z \in M$ with $z \leq_{\mathcal{L}} x$, the map*

$$\rho_{u,v} \colon M \to M, \qquad \rho_{u,v}(t) := t \cdot u \cdot v$$

acts as the identity on z, i.e. $\rho_{u,v}(z) = z$. A left–right dual statement holds with left-translations and $\mathcal{R}$-below elements.

Proof. Since $z \leq_{\mathcal{L}} x$, there exists $t \in M$ with $z = t \cdot x$. Using associativity and the relations $x \cdot u = y$ and $y \cdot v = x$, we compute

$$z \cdot u \cdot v = t \cdot x \cdot u \cdot v = t \cdot y \cdot v = t \cdot x = z. \qquad \square$$

Lemma 37 (Green's Lemma)**.** *Let M be a monoid and let $x, y \in M$ with $x \mathcal{R} y$. Fix $u, v \in M$ such that $x \cdot u = y$ and $y \cdot v = x$. Then the right-translation*

$$\rho_u \colon M \to M, \qquad \rho_u(z) := z \cdot u,$$

restricts to a bijection from the $\mathcal{L}$-class of x onto the $\mathcal{L}$-class of y; moreover, $\rho_v(z) := z \cdot v$ is the inverse bijection. Additionally, these translations preserve $\mathcal{H}$-equivalence.

Proof. It suffices to verify the following:

(1) Image in the correct $\mathcal{L}$-class. If $z \mathcal{L} x$ then $z \cdot u \mathcal{L} y$. Indeed, writing $z = t \cdot x$ for some t, we have $z \cdot u = t \cdot x \cdot u = t \cdot y$, so $z \cdot u \mathcal{L} y$ (equivalently, by compatibility of $\mathcal{L}$ with right-multiplication).

(2) Injectivity on the $\mathcal{L}$-class of x. If $z, w \mathcal{L} x$ and $z \cdot u = w \cdot u$, then $z = w$. By Lemma 36, $z \cdot u \cdot v = z$ and $w \cdot u \cdot v = w$. Hence $z = z \cdot u \cdot v = w \cdot u \cdot v = w$.

(3) Surjectivity onto the $\mathcal{L}$-class of y. Let $z \mathcal{L} y$. Set $w := z \cdot v$. Then $w \mathcal{L} y \cdot v = x$, so $w \mathcal{L} x$. Moreover, by Lemma 36 (applied with z in the $\mathcal{L}$-class of y),

$$z = z \cdot v \cdot u = w \cdot u = \rho_u(w),$$

so z lies in the image of ρ_u.

(4) ρ_u and ρ_v are inverses on the respective $\mathcal{L}$-classes. If $z \,\mathcal{L}\, x$, then $z \cdot u \cdot v = z$ by Lemma 36; if $z \,\mathcal{L}\, y$, then $z \cdot v \cdot u = z$ (the same lemma with the roles of x, y interchanged).

(5) Preservation of $\mathcal{H}$-equivalence. For $z, w \,\mathcal{L}\, x$ one should verify

$$z \,\mathcal{H}\, w \quad \Longleftrightarrow \quad z \cdot u \,\mathcal{H}\, w \cdot u.$$

Using $\rho_{u,v}$ from Lemma 36 gives $z \cdot u \,\mathcal{R}\, z$ and $w \cdot u \,\mathcal{R}\, w$, and the $\mathcal{L}$-compatibility from (1) supplies the $\mathcal{L}$-side; a routine transitivity argument then completes the proof. *(Details omitted.)* ☐

2.2 Location Theorem (Proposition 1.6)

Throughout this section, let M be a monoid and write multiplication multiplicatively.

Proposition 38 (Location Theorem)**.** *For any $x, y \in M$, the following are equivalent:*

$$(\exists\, e \in M,\ e^2 = e,\ e \,\mathcal{L}\, x,\ e \,\mathcal{R}\, y) \quad \Longleftrightarrow \quad (xy) \,\mathcal{R}\, x \quad \textit{and} \quad (xy) \,\mathcal{L}\, y.$$

Proof. *($\Rightarrow$).* Assume $e^2 = e$, $e \,\mathcal{L}\, x$, and $e \,\mathcal{R}\, y$. For idempotents one has the characterizations

$$e \,\mathcal{L}\, x \iff xe = x, \qquad e \,\mathcal{R}\, y \iff ey = y.$$

Hence $xe = x$ and $ey = y$. By multiplicative compatibility of Green's relations (see 23 and 22), from $y \,\mathcal{R}\, e$ we deduce $xy \,\mathcal{R}\, xe = x$, and from $x \,\mathcal{L}\, e$ we deduce $xy \,\mathcal{L}\, ey = y$.

($\Leftarrow$). Assume $(xy) \,\mathcal{R}\, x$ and $(xy) \,\mathcal{L}\, y$. Consider the right-translation $\rho_y(z) = zy$. By Green's Lemma (37), ρ_y restricts to a bijection from the $\mathcal{L}$-class of x onto the $\mathcal{L}$-class of xy. Since $y \,\mathcal{L}\, xy$, there exists t in the $\mathcal{L}$-class of x with $ty = y$. From $(xy) \,\mathcal{R}\, x$ choose u with $xy \cdot u = x$. Applying Lemma 36 (with $x \,\mathcal{R}\, xy$) to t gives $tyu = t$. Therefore

$$t^2 = t \cdot t = t \cdot (tyu) = (t \cdot ty)\, u = (ty)\, u = t,$$

so t is idempotent. Moreover $t \,\mathcal{R}\, y$ because $ty = y$, and $t \,\mathcal{L}\, x$ since t was chosen in the $\mathcal{L}$-class of x. Thus there exists an idempotent t with $t \,\mathcal{L}\, x$ and $t \,\mathcal{R}\, y$. ☐

Remark (dual version). Interchanging $\mathcal{L}$ and $\mathcal{R}$ and using left-translation bijections yields the left–right dual location theorem. *(To be recorded separately.)*