My formalization project

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Chapter 1

Green's Relations

1.1 Green's Preorders

Green's preorders are traditionally defined for semigroups by adjoining a unit element and working in the resulting monoid. Here we start directly with monoids, thereby avoiding the overhead of the 'WithOne' construction and the need to explicitly adjoin an identity. Since a semigroup with an adjoined unit is essentially the same as a monoid, working on monoids simplifies many of the subsequent proofs. In future developments this code might be refactored to operate on semigroups with a unit, but the monoid formulation suffices for our purposes.

Definition 1 (Green's R-preorder)**.** Let M be a monoid and let $x, y \in M$. We define $x \leq_R y$ if there exists $z \in M$ such that $x = y \cdot z$. Equivalently, x lies in the principal right ideal generated by y.

Lemma 2 (Reflexivity of $\leq_R$)**.** *For every $x \in M$, $x \leq_R x$.*

Lemma 3 (Transitivity of $\leq_R$)**.** *For all $x, y, z \in M$, if $x \leq_R y$ and $y \leq_R z$ then $x \leq_R z$.*

Proof. Suppose $x \leq_R y$ and $y \leq_R z$. By definition there exist $v, u \in M$ with $x = y \cdot v$ and $y = z \cdot u$. Taking the witness $u \cdot v$ we compute $z \cdot (u \cdot v) = (z \cdot u) \cdot v = y \cdot v = x$, so $x \leq_R z$. □

Lemma 4 (Right preorder instance)**.** *The relation $\leq_R$ is a preorder on M.*

Definition 5 (Green's L-preorder)**.** Let M be a monoid and let $x, y \in M$. We define $x \leq_L y$ if there exists $z \in M$ such that $x = z \cdot y$. In other words, x lies in the principal left ideal generated by y.

Lemma 6 (Reflexivity of $\leq_L$)**.** *For every $x \in M$, $x \leq_L x$.*

Proof. Fix $x \in M$. To show $x \leq_L x$ we need a witness z with $z \cdot x = x$. Taking $z = 1$ works since $1 \cdot x = x$. □

Lemma 7 (Transitivity of $\leq_L$)**.** *For all $x, y, z \in M$, if $x \leq_L y$ and $y \leq_L z$ then $x \leq_L z$.*

Proof. Suppose $x \leq_L y$ and $y \leq_L z$. There exist $u, v \in M$ with $u \cdot y = x$ and $v \cdot z = y$. Taking the witness $u \cdot v$ gives $(u \cdot v) \cdot z = u \cdot (v \cdot z) = u \cdot y = x$, so $x \leq_L z$. □

Lemma 8 (Left preorder instance)**.** *The relation $\leq_L$ is a preorder on M.*

Definition 9 (Green's J-preorder)**.** Let M be a monoid and let $x, y \in M$. We define $x \leq_J y$ if there exist $u, v \in M$ such that $x = u \cdot y \cdot v$. Equivalently, x lies in the two-sided ideal generated by y.

Lemma 10 (Reflexivity of $\leq_J$)**.** *For every $x \in M$, $x \leq_J x$.*

Proof. Fix $x \in M$. To see $x \leq_J x$ we need witnesses u, v with $x = u \cdot x \cdot v$. Choosing $u = 1$ and $v = 1$ we have $1 \cdot x \cdot 1 = x$, so $x \leq_J x$. □

Lemma 11 (Transitivity of $\leq_J$)**.** *For all $x, y, z \in M$, if $x \leq_J y$ and $y \leq_J z$ then $x \leq_J z$.*

Proof. Suppose $x \leq_J y$ and $y \leq_J z$. Then there exist $u_1, v_1 \in M$ with $u_1 \cdot y \cdot v_1 = x$ and $u_2, v_2 \in M$ with $u_2 \cdot z \cdot v_2 = y$. Setting $u = u_1 \cdot u_2$ and $v = v_2 \cdot v_1$, associativity shows that $u \cdot z \cdot v = u_1 \cdot u_2 \cdot z \cdot v_2 \cdot v_1 = u_1 \cdot y \cdot v_1 = x$. Hence $x \leq_J z$. □

Lemma 12 (J preorder instance)**.** *The relation $\leq_J$ is a preorder on M.*

Definition 13 (Green's H-preorder)**.** Let M be a monoid and let $x, y \in M$. We define $x \leq_H y$ if both $x \leq_R y$ and $x \leq_L y$ hold, that is, x is simultaneously in the right and left ideals generated by y. Unwinding the definition, this means there exist elements $z_1, z_2 \in M$ with $x = y \cdot z_1$ and $x = z_2 \cdot y$.

Lemma 14 (Reflexivity of $\leq_H$)**.** *For every $x \in M$, $x \leq_H x$.*

Proof. To show $x \leq_H x$ we must exhibit both $x \leq_R x$ and $x \leq_L x$. These are provided by the reflexivity lemmas for the right and left preorders. Hence $x \leq_H x$. □

Lemma 15 (Transitivity of $\leq_H$)**.** *For all $x, y, z \in M$, if $x \leq_H y$ and $y \leq_H z$ then $x \leq_H z$.*

Proof. Assume $x \leq_H y$ and $y \leq_H z$. Then $x \leq_R y$ and $x \leq_L y$, while $y \leq_R z$ and $y \leq_L z$. By transitivity of the right preorder we have $x \leq_R z$, and by transitivity of the left preorder we have $x \leq_L z$. Therefore $x \leq_H z$. □

Lemma 16 (H preorder instance)**.** *The relation $\leq_H$ is a preorder on M.*

1.2 Equivalences from Preorders

To obtain equivalence relations from a preorder we take its symmetric closure. Given any preorder p, we define an equivalence relation by declaring two elements equivalent exactly when both $p\,x\,y$ and $p\,y\,x$ hold. This construction works uniformly for all preorders and, in particular, produces Green's equivalence relations from Green's preorders.

Definition 17 (Equivalence of a preorder)**.** Let α be a type and let $p : \alpha \to \alpha \to \mathrm{Prop}$ be a preorder. For $x, y : \alpha$ we define $\mathrm{EquivOfLE}\,p\,x\,y$ to hold when both $p\,x\,y$ and $p\,y\,x$ hold. In other words, $\mathrm{EquivOfLE}\,p$ is the symmetric closure of the relation p.

Lemma 18 (EquivOfLE is an equivalence relation)**.** *If p is a preorder on α then* $\mathrm{EquivOfLE}\,p$ *is an equivalence relation.*

Proof. We verify the three properties of an equivalence relation. For reflexivity, if p is a preorder then it is reflexive, so for every x we have both $p\,x\,x$ and $p\,x\,x$; hence $\mathrm{EquivOfLE}\,p\,x\,x$ holds. Symmetry is immediate: if $\mathrm{EquivOfLE}\,p\,x\,y$ holds, meaning $p\,x\,y$ and $p\,y\,x$, then the pair $p\,y\,x$ and $p\,x\,y$ shows $\mathrm{EquivOfLE}\,p\,y\,x$. For transitivity, suppose $\mathrm{EquivOfLE}\,p\,x\,y$ and $\mathrm{EquivOfLE}\,p\,y\,z$. Then we have $p\,x\,y$ and $p\,y\,z$; by transitivity of the preorder p we obtain $p\,x\,z$. Similarly from $p\,z\,y$ and $p\,y\,x$ we deduce $p\,z\,x$. Thus $\mathrm{EquivOfLE}\,p\,x\,z$ holds, completing the proof. □

1.3 Green's Equivalences (R, L, J, and H)

Definition 19 (Right equivalence)**.** Let M be a monoid. For $x, y \in M$ we define REquiv $x\,y$ to hold if both $x \le_R y$ and $y \le_R x$ hold. In other words, two elements are right equivalent when they lie in the principal right ideals generated by each other.

Lemma 20 (Right equivalence is an equivalence relation)**.** *The relation defined in Definition 19 is an equivalence relation on M.*

Definition 21 (Left equivalence)**.** For $x, y \in M$ we define LEquiv $x\,y$ to hold when both $x \le_L y$ and $y \le_L x$ hold. This expresses that x and y generate the same principal left ideals.

Lemma 22 (Left equivalence is an equivalence relation)**.** *The relation defined in Definition 21 is an equivalence relation on M.*

Definition 23 (J equivalence)**.** For $x, y \in M$ we define JEquiv $x\,y$ to hold when both $x \le_J y$ and $y \le_J x$ hold, i.e., each lies in the two-sided ideal generated by the other.

Lemma 24 (J equivalence is an equivalence relation)**.** *The relation defined in Definition 23 is an equivalence relation on M.*

Definition 25 (H equivalence)**.** For $x, y \in M$ we define HEquiv $x\,y$ to hold when both $x \le_H y$ and $y \le_H x$ hold. This captures when two elements are simultaneously right and left equivalent.

Lemma 26 (H equivalence is an equivalence relation)**.** *The relation defined in Definition 25 is an equivalence relation on M.*

1.4 Preorder Cancellation Properties

Lemma 27 (Right-preorder multiplication decreases)**.** *For all $x, y \in M$, one has $x \cdot y \le_R x$.*

Proof. By definition of the right preorder, $x \cdot y \le_R x$ means that there exists a witness u with $x \cdot u = x \cdot y$. Taking $u = y$ we obtain $x \cdot u = x \cdot y$, so $x \cdot y \le_R x$. □

Lemma 28 (Left-preorder multiplication decreases)**.** *For all $x, y \in M$, one has $y \cdot x \le_L x$.*

Proof. To show $y \cdot x \le_L x$ we must find a witness u such that $u \cdot x = y \cdot x$. Choosing $u = y$ gives $u \cdot x = y \cdot x$, hence $y \cdot x \le_L x$. □

Lemma 29 (Two-sided multiplication decreases)**.** *For all $u, x, v \in M$, one has $u \cdot x \cdot v \le_J x$.*

Proof. By definition of the J preorder, $u \cdot x \cdot v \le_J x$ means there exist witnesses s, t with $s \cdot x \cdot t = u \cdot x \cdot v$. Taking $s = u$ and $t = v$ gives $s \cdot x \cdot t = u \cdot x \cdot v$, showing $u \cdot x \cdot v \le_J x$. □

Lemma 30 (Left cancellation for JRel)**.** *For all $x, y, z \in M$, if $x \le_J y \cdot z$ then $x \le_J z$.*

Proof. Assume $x \le_J y \cdot z$. Then there exist u, v with $u \cdot (y \cdot z) \cdot v = x$. Using associativity we rewrite $u \cdot (y \cdot z) \cdot v = (u \cdot y) \cdot z \cdot v$. Letting $u' = u \cdot y$ and $v' = v$, we have $u' \cdot z \cdot v' = x$. Thus $x \le_J z$. □

Lemma 31 (Right cancellation for JRel)**.** *For all $x, y, z \in M$, if $x \le_J y \cdot z$ then $x \le_J y$.*

Proof. Suppose $x \le_J y \cdot z$. Then there exist u, v with $u \cdot (y \cdot z) \cdot v = x$. Rewrite this as $u \cdot y \cdot (z \cdot v) = x$. Setting $u' = u$ and $v' = z \cdot v$, we have $u' \cdot y \cdot v' = x$. Hence $x \le_J y$. □

Lemma 32 (Right cancellation for RRel)**.** *For all $x, y, z \in M$, if $x \leq_R y \cdot z$ then $x \leq_R y$.*

Proof. Suppose $x \leq_R y \cdot z$. By definition there exists u with $(y \cdot z) \cdot u = x$. Using associativity we have $y \cdot (z \cdot u) = x$. Set $u' = z \cdot u$. Then $y \cdot u' = x$, so $x \leq_R y$. □

Lemma 33 (Left cancellation for LRel)**.** *For all $x, y, z \in M$, if $x \leq_L y \cdot z$ then $x \leq_L z$.*

Proof. Assume $x \leq_L y \cdot z$. Then there exists u with $u \cdot (y \cdot z) = x$. Rewriting as $(u \cdot y) \cdot z = x$, set $u' = u \cdot y$. Then $u' \cdot z = x$, and thus $x \leq_L z$. □

1.5 Preorder Idempotent Properties

Lemma 34 (Idempotent characterization for the right preorder)**.** *Let M be a monoid and let $e \in M$ be idempotent (so $e \cdot e = e$). For any $x \in M$,*

$$x \leq_R e \quad \text{if and only if} \quad e \cdot x = x.$$

Proof. Assume e is idempotent. First suppose $x \leq_R e$. Then there exists t such that $e \cdot t = x$. Multiplying the equation $e \cdot t = x$ on the left by e and using idempotency yields $e \cdot x = e \cdot (e \cdot t) = (e \cdot e) \cdot t = e \cdot t = x$. Conversely, suppose $e \cdot x = x$. Setting $t = x$ gives $e \cdot t = e \cdot x = x$, so $x \leq_R e$. □

Lemma 35 (Idempotent characterization for the left preorder)**.** *Let M be a monoid and let $e \in M$ be idempotent. For any $x \in M$,*

$$x \leq_L e \quad \text{if and only if} \quad x \cdot e = x.$$

Proof. Assume e is idempotent. Suppose $x \leq_L e$. Then there exists t such that $t \cdot e = x$. Multiplying this equality on the right by e and using idempotency gives $x \cdot e = (t \cdot e) \cdot e = t \cdot (e \cdot e) = t \cdot e = x$. Conversely, suppose $x \cdot e = x$. Taking $t = x$ yields $t \cdot e = x \cdot e = x$, so $x \leq_L e$. □

1.6 R–L multiplicative compatibility

Lemma 36 (Left multiplication respects the right preorder)**.** *For all $x, y, z \in M$, if $x \leq_R y$ then $z \cdot x \leq_R z \cdot y$.*

Proof. Suppose $x \leq_R y$, so there exists $u \in M$ with $y \cdot u = x$. We want to show $z \cdot x \leq_R z \cdot y$, meaning there is a witness u' such that $(z \cdot y) \cdot u' = z \cdot x$. Taking $u' = u$ we compute $(z \cdot y) \cdot u' = z \cdot (y \cdot u) = z \cdot x$ by associativity. Hence $z \cdot x \leq_R z \cdot y$. □

Lemma 37 (Right multiplication respects the left preorder)**.** *For all $x, y, z \in M$, if $x \leq_L y$ then $x \cdot z \leq_L y \cdot z$.*

Proof. Assume $x \leq_L y$, so there exists $u \in M$ with $u \cdot y = x$. To prove $x \cdot z \leq_L y \cdot z$ we need a witness u' with $u' \cdot (y \cdot z) = x \cdot z$. Taking $u' = u$ and using associativity gives $u' \cdot (y \cdot z) = u \cdot (y \cdot z) = (u \cdot y) \cdot z = x \cdot z$. Thus $x \cdot z \leq_L y \cdot z$. □

Lemma 38 (Left multiplication respects right equivalence)**.** *For all $x, y, z \in M$, if x and y are right equivalent then $z \cdot x$ and $z \cdot y$ are right equivalent.*

Proof. Suppose x and y are right equivalent, meaning $x \leq_R y$ and $y \leq_R x$. By Lemma 36, left multiplication by z preserves the right preorder; hence $z \cdot x \leq_R z \cdot y$ and $z \cdot y \leq_R z \cdot x$. Therefore $z \cdot x$ and $z \cdot y$ are right equivalent. ☐

Lemma 39 (Right multiplication respects left equivalence)**.** *For all $x, y, z \in M$, if x and y are left equivalent then $x \cdot z$ and $y \cdot z$ are left equivalent.*

Proof. Assume x and y are left equivalent, so $x \leq_L y$ and $y \leq_L x$. By Lemma 37, right multiplication by z preserves the left preorder, giving $x \cdot z \leq_L y \cdot z$ and $y \cdot z \leq_L x \cdot z$. Hence $x \cdot z$ and $y \cdot z$ are left equivalent. ☐

1.7 R–L commutativity

Lemma 40 (Commutativity of $\leq_L$ and $\leq_R$)**.** *For any $x, y \in M$, there exists an element z with $x \leq_L z$ and $z \leq_R y$ if and only if there exists an element w with $x \leq_R w$ and $w \leq_L y$.*

Proof. First suppose there exists $z \in M$ such that $x \leq_L z$ and $z \leq_R y$. Unwinding the definitions, there are witnesses u and v with $u \cdot z = x$ and $y \cdot v = z$. Taking $w = u \cdot y$ we have

$$w \leq_L y \quad \text{since} \quad w = u \cdot y$$

and

$$x = u \cdot z = u \cdot (y \cdot v) = (u \cdot y) \cdot v = w \cdot v,$$

so $x \leq_R w$. Conversely, suppose there exists $w \in M$ with $x \leq_R w$ and $w \leq_L y$. Then there are witnesses v and u with $w \cdot v = x$ and $u \cdot y = w$. Let $z = y \cdot v$. We compute

$$z \leq_R y \quad \text{because} \quad y \cdot v = z,$$

and

$$x = w \cdot v = (u \cdot y) \cdot v = u \cdot (y \cdot v) = u \cdot z,$$

showing $x \leq_L z$. This establishes the equivalence. ☐

Lemma 41 (Commutativity of right and left equivalence)**.** *For any $x, y \in M$, there exists $z \in M$ such that x is left equivalent to z and z is right equivalent to y if and only if there exists $w \in M$ such that x is right equivalent to w and w is left equivalent to y.*

Proof. Suppose there exists z with x left equivalent to z and z right equivalent to y. Write this as $x \leq_L z \wedge z \leq_L x$ and $z \leq_R y \wedge y \leq_R z$. From the left and right preorder conditions we have witnesses u_1, u_2, v_1, v_2 satisfying

$$u_1 \cdot z = x, \qquad z \cdot u_2 = x, \qquad y \cdot v_1 = z, \qquad z \cdot v_2 = y.$$

Set $w = u_1 \cdot y$. Then using the first pair of equations we have

$$w \leq_L y \quad \text{since} \quad w = u_1 \cdot y,$$

and from the remaining equations we compute

$$x = u_1 \cdot z = u_1 \cdot (y \cdot v_1) = (u_1 \cdot y) \cdot v_1 = w \cdot v_1,$$

so $x \leq_R w$. Moreover, by applying Lemma 38 to the right equivalence $z \sim_R y$ we deduce $u_1 \cdot z$ is right equivalent to $u_1 \cdot y$. Since $u_1 \cdot z = x$, this shows $x \sim_R w$. Combined with the previous observation that w and y are left equivalent, we obtain the right-to-left implication.

Conversely, assume there exists w such that $x \sim_R w$ and $w \sim_L y$. This means $x \leq_R w$ and $w \leq_R x$, and $w \leq_L y$ and $y \leq_L w$. Pick witnesses v_1, v_2, u_1, u_2 with

$$w \cdot v_1 = x, \qquad x \cdot v_2 = w, \qquad u_1 \cdot y = w, \qquad y \cdot u_2 = w.$$

Let $z = y \cdot v_1$. Then z and y are right equivalent because both $y \leq_R z$ and $z \leq_R y$ hold via the witnesses above. Using Lemma 39 applied to the left equivalence $w \sim_L y$, we have $x \cdot u_1$ is left equivalent to $w \cdot u_1 = u_1 \cdot y$. Since $x \cdot u_1 = (w \cdot v_1) \cdot u_1$ and $z = y \cdot v_1$, a straightforward computation shows $x \leq_L z$ and $z \leq_L x$. Hence $x \sim_L z$ and $z \sim_R y$, completing the proof. □

1.8 The D-equivalence

Definition 42 (D-equivalence)**.** Let M be a monoid. For $x, y \in M$ we say $\mathrm{DEquiv}\, x\, y$ holds if there exists an element $z \in M$ such that x is right equivalent to z and z is left equivalent to y. In other words, there is a chain x *REquiv* z and z *LEquiv* y. This relation is Green's D-equivalence.

Lemma 43 (Characterization of DEquiv)**.** *For all $x, y \in M$ we have*

$$(\exists z \in M,\ x \mathit{\ REquiv\ } z \wedge z \mathit{\ LEquiv\ } y) \quad \Longleftrightarrow \quad \mathrm{DEquiv}\, x\, y.$$

Proof. Unwinding Definition 42, $\mathrm{DEquiv}\, x\, y$ holds exactly when there is a z with x right equivalent to z and z left equivalent to y. This is precisely the left-hand statement. The commutativity Lemma 41 shows that any such chain can be rearranged to start with a right equivalence and end with a left equivalence, so no generality is lost. □

Lemma 44 (Reflexivity of DEquiv)**.** *For every $x \in M$, one has* $\mathrm{DEquiv}\, x\, x$.

Proof. Fix $x \in M$. Taking $z = x$, reflexivity of the right equivalence (Lemma 20) and the left equivalence (Lemma 22) implies $x \sim_R x$ and $x \sim_L x$. Hence $\mathrm{DEquiv}\, x\, x$ by Definition 42. □

Lemma 45 (Symmetry of DEquiv)**.** *If* $\mathrm{DEquiv}\, x\, y$ *holds then* $\mathrm{DEquiv}\, y\, x$ *holds.*

Proof. Assume $\mathrm{DEquiv}\, x\, y$. Then there is z with $x \sim_R z$ and $z \sim_L y$. By Lemma 41 there exists an element w such that $z \sim_R y$ and $y \sim_L w$. Reversing the roles of left and right in that lemma yields $y \sim_R w$ and $w \sim_L x$. Hence by Definition 42 we have $\mathrm{DEquiv}\, y\, x$. □

Lemma 46 (Closure under left equivalence)**.** *If* $\mathrm{DEquiv}\, x\, y$ *and y LEquiv z, then* $\mathrm{DEquiv}\, x\, z$.

Proof. Suppose $\mathrm{DEquiv}\, x\, y$. So there exists $u \in M$ with $x \sim_R u$ and $u \sim_L y$. If $y \sim_L z$, then by transitivity of the left equivalence (Lemma 22) we have $u \sim_L z$. Combining $x \sim_R u$ and $u \sim_L z$ shows that $\mathrm{DEquiv}\, x\, z$. □

Lemma 47 (Closure under right equivalence)**.** *If* $\mathrm{DEquiv}\, x\, y$ *and y REquiv z, then* $\mathrm{DEquiv}\, x\, z$.

Proof. Assume DEquiv $x\,y$ and $y \sim_R z$. By Lemma 45 we know DEquiv $y\,x$. Using commutativity of right and left equivalence (Lemma 41), the right equivalence $y \sim_R z$ may be interpreted as a left equivalence between suitable elements. Applying closure under left equivalence (Lemma 46) to DEquiv $y\,x$ and this left equivalence yields DEquiv $y\,z$. A final application of Lemma 45 gives DEquiv $x\,z$. □

Lemma 48 (Transitivity of DEquiv)**.** *If* DEquiv $x\,y$ *and* DEquiv $y\,z$ *then* DEquiv $x\,z$*.*

Proof. Assume DEquiv $x\,y$ and DEquiv $y\,z$. By Definition 42 there exists u with $x \sim_R u$ and $u \sim_L y$, and there exists v with $y \sim_R v$ and $v \sim_L z$. Applying Lemma 47 to DEquiv $x\,y$ and the right equivalence $y \sim_R v$ gives DEquiv $x\,v$. Then Lemma 46 applied to DEquiv $x\,v$ and the left equivalence $v \sim_L z$ yields DEquiv $x\,z$. □

Lemma 49 (D-equivalence is an equivalence relation)**.** *The relation defined in Definition 42 is an equivalence relation on M.*

Proof. An equivalence relation is one that is reflexive, symmetric and transitive. These three properties are given by Lemmas 44, 45 and 48. Hence DEquiv is an equivalence relation. □

1.9 Equivalence classes

Green's equivalence relations partition a monoid into subsets called equivalence classes. Concretely, given an element $a \in M$, the R-class of a is the set of all x such that x *REquiv* a. Similarly for L, J, H and D.

Definition 50 (Right class)**.** For $a \in M$ we define RClass(a) to be the set $\{x \in M \mid x \; \mathit{REquiv} \; a\}$.

Definition 51 (Left class)**.** For $a \in M$ we define LClass(a) to be the set $\{x \in M \mid x \; \mathit{LEquiv} \; a\}$.

Definition 52 (J class)**.** For $a \in M$ we define JClass(a) to be the set $\{x \in M \mid x \; \mathit{JEquiv} \; a\}$.

Definition 53 (H class)**.** For $a \in M$ we define HClass(a) to be the set $\{x \in M \mid x \; \mathit{HEquiv} \; a\}$.

Definition 54 (D class)**.** For $a \in M$ we define DClass(a) to be the set $\{x \in M \mid \mathrm{DEquiv}\, x\, a\}$.