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Chapter 1

Green's Relations

Definition 1 (Green's Preorder and Equivalence Relations)**.** In a semigroup S, Green's relations are a set of five equivalence relations that characterize the elements of S in terms of the principal ideals they generate. These relations are fundamental to the study of semigroups.

First, we define four preorder relations: $\leq_{\mathcal{R}}$, $\leq_{\mathcal{L}}$, $\leq_{\mathcal{J}}$, and $\leq_{\mathcal{H}}$. Let S^1 be the monoid obtained by adjoining an identity element to S if it does not already have one. For any two elements $x, y \in S$:

- $x \leq_{\mathcal{R}} y$ if and only if the principal right ideal generated by x is a subset of the principal right ideal generated by y ($xS^1 \subseteq yS^1$). In Lean, we use the equivalent definition that there must exist some $z \in S^1$ such that $x = yz$.

- $x \leq_{\mathcal{L}} y$ if and only if the principal left ideal generated by x is a subset of the principal left ideal generated by y ($S^1x \subseteq S^1y$). In Lean, we use the equivalent definition that there must exist some $z \in S^1$ such that $x = zy$.

- $x \leq_{\mathcal{J}} y$ if and only if the principal two-sided ideal generated by x is a subset of the principal two-sided ideal generated by y ($S^1xS^1 \subseteq S^1yS^1$). In Lean, we use the equivalent definition that there must exist some $u, v \in S^1$ such that $x = uyv$.

- $x \leq_{\mathcal{H}} y$ if and only if $x \leq_{\mathcal{R}} y$ and $x \leq_{\mathcal{L}} y$.

These four relations are preorders (they are reflexive and transitive).

From these preorders, we define the equivalence relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, and $\mathcal{H}$ as the symmetric closures of their corresponding preorders. For example, $x\mathcal{R}y$ if and only if $x \leq_{\mathcal{R}} y$ and $y \leq_{\mathcal{R}} x$. The equivalence classes are denoted by $[x]_{\mathcal{R}}$, $[x]_{\mathcal{L}}$, etc.

Finally, the $\mathcal{D}$ relation is defined by composing $\mathcal{R}$ and $\mathcal{L}$: $x\mathcal{D}y$ if there exists an element $z \in S$ such that $x\mathcal{R}z$ and $z\mathcal{L}y$. It can be shown that $\mathcal{D}$ is an equivalence relation and that it can also be defined by composing $\mathcal{L}$ and $\mathcal{R}$.

Lemma 2 (Multiplication Compatibility of Green's Relations)**.** *The $\mathcal{R}$ and $\mathcal{L}$ relations exhibit compatibility with semigroup multiplication on one side. Specifically, the $\mathcal{R}$-preorder is compatible with left multiplication, and the $\mathcal{L}$-preorder is compatible with right multiplication. If $x \leq_{\mathcal{R}} y$, then for any $z \in S$, we have $zx \leq_{\mathcal{R}} zy$. This property extends to the equivalence relation $\mathcal{R}$. If $x\mathcal{R}y$, then $zx\mathcal{R}zy$. A similar argument holds for the $\mathcal{L}$-preorder and $\mathcal{L}$-equivalence, which are compatible with right multiplication. If $x \leq_{\mathcal{L}} y$, then $xz \leq_{\mathcal{L}} yz$ for any $z \in S$.*

Proof. Let $x, y, z \in S$. To prove left compatibility for $\leq_{\mathcal{R}}$, assume $x \leq_{\mathcal{R}} y$. By definition, there exists $a \in S^1$ such that $x = ya$. Multiplying by z on the left gives $zx = z(ya) = (zy)a$. This implies $zx \leq_{\mathcal{R}} zy$. For the equivalence $x\mathcal{R}y$, we have both $x \leq_{\mathcal{R}} y$ and $y \leq_{\mathcal{R}} x$. Applying the result for preorders, we get $zx \leq_{\mathcal{R}} zy$ and $zy \leq_{\mathcal{R}} zx$, which means $zx\mathcal{R}zy$. The proof for $\leq_{\mathcal{L}}$ and $\mathcal{L}$ with right multiplication is analogous. □

Lemma 3 (Commutation of R and L Relations)**.** *The composition of the relations $\mathcal{R}$ and $\mathcal{L}$ is commutative. That is, for any $x, y \in S$, there exists a z such that $x\mathcal{R}z$ and $z\mathcal{L}y$ if and only if there exists a w such that $x\mathcal{L}w$ and $w\mathcal{R}y$. This can be written as $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$. This property is crucial for proving that the relation $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$ is symmetric, and therefore an equivalence relation.*

Proof. Suppose there exists z with $x\mathcal{R}z$ and $z\mathcal{L}y$. From $x\mathcal{R}z$, we have $x = za$ and $z = xb$ for some $a, b \in S^1$. From $z\mathcal{L}y$, we have $z = cy$ and $y = dz$ for some $c, d \in S^1$. We need to find an element w such that $x\mathcal{L}w$ and $w\mathcal{R}y$. The Lean proof shows that in the non-trivial case, the element dza can be used for w. This commutation is essential for establishing that $\mathcal{D}$ is an equivalence relation, as it directly implies symmetry. □

Lemma 4 (Closure of D under R and L)**.** *The $\mathcal{D}$ relation is closed under composition with $\mathcal{R}$ and $\mathcal{L}$. If $x\mathcal{D}y$ and $y\mathcal{L}z$, then $x\mathcal{D}z$. Similarly, if $x\mathcal{D}y$ and $y\mathcal{R}z$, then $x\mathcal{D}z$. This property is used to prove the transitivity of $\mathcal{D}$.*

Proof. Suppose $x\mathcal{D}y$ and $y\mathcal{L}z$. By definition of $\mathcal{D}$, there exists an element w such that $x\mathcal{R}w$ and $w\mathcal{L}y$. Since $\mathcal{L}$ is an equivalence relation, from $w\mathcal{L}y$ and $y\mathcal{L}z$, we can deduce $w\mathcal{L}z$. Now we have $x\mathcal{R}w$ and $w\mathcal{L}z$, which by definition means $x\mathcal{D}z$. A similar argument holds for closure under $\mathcal{R}$. If $x\mathcal{D}y$ and $y\mathcal{R}z$, we use the commutation of $\mathcal{R}$ and $\mathcal{L}$ (3). $x\mathcal{D}y$ means there is a w with $x\mathcal{L}w$ and $w\mathcal{R}y$. Since $\mathcal{R}$ is an equivalence relation, $w\mathcal{R}y$ and $y\mathcal{R}z$ implies $w\mathcal{R}z$. So we have $x\mathcal{L}w$ and $w\mathcal{R}z$, which means $x\mathcal{D}z$. □