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Chapter 1

Idempotent Elements in Finite Semigroups

Lemma 1 (Existence of Idempotent Powers)**.** *In a finite semigroup, for any element x, there exists a positive integer m such that x^m is an idempotent. This idempotent power is unique.*

Proof. Since the semigroup S is finite, for any $x \in S$, the set of its powers $\{x^1, x^2, x^3, \dots\}$ must also be finite. By the pigeonhole principle, there must exist distinct positive integers m, n such that $x^m = x^n$. Let's assume $m < n$. Then we can write $x^m = x^m x^{n-m}$. This shows that powers of x eventually enter a cycle. From this cyclic part, we can extract a power k such that x^k is idempotent. The proof of uniqueness follows by showing that if x^a and x^b are both idempotents, then $x^a = x^b$. □

Lemma 2 (Sandwich Property in Finite Monoids)**.** *In a finite monoid M, if an element a satisfies the property $a = xay$ for some $x, y \in M$, then there exist non-zero powers of x and y, say n_1 and n_2, such that $x^{n_1}a = a$ and $ay^{n_2} = a$.*

Proof. From $a = xay$, we can repeatedly substitute a into itself to get $a = x^k a y^k$ for any $k \geq 1$. Since M is a finite monoid, there exists a non-zero power n_1 such that x^{n_1} is an idempotent (by 1). Then we have $a = x^{n_1} a y^{n_1}$. Multiplying by x^{n_1} on the left gives $x^{n_1} a = x^{2n_1} a y^{n_1}$. Since x^{n_1} is idempotent, $x^{2n_1} = x^{n_1}$, so $x^{n_1} a = x^{n_1} a y^{n_1} = a$. A symmetric argument shows that $ay^{n_2} = a$ for some idempotent power y^{n_2}. □

Chapter 2

Green's Relations

Definition 3 (Green's Preorder and Equivalence Relations)**.** In a semigroup S, Green's relations are a set of five equivalence relations that characterize the elements of S in terms of the principal ideals they generate. These relations are fundamental to the study of semigroups.

First, we define four preorder relations: $\leq_{\mathcal{R}}$, $\leq_{\mathcal{L}}$, $\leq_{\mathcal{J}}$, and $\leq_{\mathcal{H}}$. Let S^1 be the monoid obtained by adjoining an identity element to S if it does not already have one. For any two elements $x, y \in S$:

- $x \leq_{\mathcal{R}} y$ iff the principal right ideal generated by x is a subset of the principal right ideal generated by y ($xS^1 \subseteq yS^1$). In Lean, we use the equivalent definition that there must exist some $z \in S^1$ such that $x = yz$.

- $x \leq_{\mathcal{L}} y$ iff the principal left ideal generated by x is a subset of the principal left ideal generated by y ($S^1x \subseteq S^1y$). In Lean, we use the equivalent definition that there must exist some $z \in S^1$ such that $x = zy$.

- $x \leq_{\mathcal{J}} y$ iff the principal two-sided ideal generated by x is a subset of the principal two-sided ideal generated by y ($S^1xS^1 \subseteq S^1yS^1$). In Lean, we use the equivalent definition that there must exist some $u, v \in S^1$ such that $x = uyv$.

- $x \leq_{\mathcal{H}} y \iff x \leq_{\mathcal{R}} y$ and $x \leq_{\mathcal{L}} y$.

These four relations are preorders (they are reflexive and transitive).

From these preorders, we define the equivalence relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$, and $\mathcal{H}$ as the symmetric closures of their corresponding preorders. For example, $x\mathcal{R}y \iff x \leq_{\mathcal{R}} y$ and $y \leq_{\mathcal{R}} x$. The equivalence classes are denoted by $[x]_{\mathcal{R}}$, $[x]_{\mathcal{L}}$, etc.

Finally, the $\mathcal{D}$ relation is defined by composing $\mathcal{R}$ and $\mathcal{L}$: $x\mathcal{D}y$ if there exists an element $z \in S$ such that $x\mathcal{R}z$ and $z\mathcal{L}y$. It can be shown that $\mathcal{D}$ is an equivalence relation and that it can also be defined by composing $\mathcal{L}$ and $\mathcal{R}$.

Lemma 4 (Multiplication Compatibility of Green's Relations)**.** *The $\mathcal{R}$ and $\mathcal{L}$ relations exhibit compatibility with semigroup multiplication on one side. Specifically, the $\mathcal{R}$-preorder is compatible with left multiplication, and the $\mathcal{L}$-preorder is compatible with right multiplication. If $x \leq_{\mathcal{R}} y$, then for any $z \in S$, we have $zx \leq_{\mathcal{R}} zy$. This property extends to the equivalence relation $\mathcal{R}$. If $x\mathcal{R}y$, then $zx\mathcal{R}zy$. A similar argument holds for the $\mathcal{L}$-preorder and $\mathcal{L}$-equivalence, which are compatible with right multiplication. If $x \leq_{\mathcal{L}} y$, then $xz \leq_{\mathcal{L}} yz$ for any $z \in S$.*

Proof. Let $x, y, z \in S$. To prove left compatibility for $\leq_{\mathcal{R}}$, assume $x \leq_{\mathcal{R}} y$. By definition, there exists $a \in S^1$ such that $x = ya$. Multiplying by z on the left gives $zx = z(ya) = (zy)a$. This implies $zx \leq_{\mathcal{R}} zy$. For the equivalence $x\mathcal{R}y$, we have both $x \leq_{\mathcal{R}} y$ and $y \leq_{\mathcal{R}} x$. Applying the result for preorders, we get $zx \leq_{\mathcal{R}} zy$ and $zy \leq_{\mathcal{R}} zx$, which means $zx\mathcal{R}zy$. The proof for $\leq_{\mathcal{L}}$ and $\mathcal{L}$ with right multiplication is analogous. □

Lemma 5 (Commutation of R and L Relations)**.** *The composition of the relations $\mathcal{R}$ and $\mathcal{L}$ is commutative. That is, for any $x, y \in S$, there exists a z such that $x\mathcal{R}z$ and $z\mathcal{L}y \iff$ there exists a w such that $x\mathcal{L}w$ and $w\mathcal{R}y$. This can be written as $\mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}$. This property is crucial for proving that the relation $\mathcal{D} = \mathcal{R} \circ \mathcal{L}$ is symmetric, and therefore an equivalence relation.*

Proof. Suppose there exists z with $x\mathcal{R}z$ and $z\mathcal{L}y$. From $x\mathcal{R}z$, we have $x = za$ and $z = xb$ for some $a, b \in S^1$. From $z\mathcal{L}y$, we have $z = cy$ and $y = dz$ for some $c, d \in S^1$. We need to find an element w such that $x\mathcal{L}w$ and $w\mathcal{R}y$. The Lean proof shows that in the non-trivial case, the element dza can be used for w. □

Lemma 6 (Closure of D under R and L)**.** *The $\mathcal{D}$ relation is closed under composition with $\mathcal{R}$ and $\mathcal{L}$. If $x\mathcal{D}y$ and $y\mathcal{L}z$, then $x\mathcal{D}z$. Similarly, if $x\mathcal{D}y$ and $y\mathcal{R}z$, then $x\mathcal{D}z$. This property is used to prove the transitivity of $\mathcal{D}$.*

Proof. Suppose $x\mathcal{D}y$ and $y\mathcal{L}z$. By definition of $\mathcal{D}$, there exists an element w such that $x\mathcal{R}w$ and $w\mathcal{L}y$. Since $\mathcal{L}$ is an equivalence relation, from $w\mathcal{L}y$ and $y\mathcal{L}z$, we can deduce $w\mathcal{L}z$. Now we have $x\mathcal{R}w$ and $w\mathcal{L}z$, which by definition means $x\mathcal{D}z$. A similar argument holds for closure under $\mathcal{R}$. If $x\mathcal{D}y$ and $y\mathcal{R}z$, we use the commutation of $\mathcal{R}$ and $\mathcal{L}$ (5). $x\mathcal{D}y$ means there is a w with $x\mathcal{L}w$ and $w\mathcal{R}y$. Since $\mathcal{R}$ is an equivalence relation, $w\mathcal{R}y$ and $y\mathcal{R}z$ implies $w\mathcal{R}z$. So we have $x\mathcal{L}w$ and $w\mathcal{R}z$, which means $x\mathcal{D}z$. □

Chapter 3

Basic Properties of Green's Relations

Lemma 7 (Characterization of Elements Below Idempotents)**.** *Let e be an idempotent element in a semigroup S. An element $x \in S$ is $\mathcal{R}$-below e if and only if $x = ex$. Similarly, x is $\mathcal{L}$-below e if and only if $x = xe$. It follows that x is $\mathcal{H}$-below e if and only if both conditions hold, i.e., $x = ex$ and $x = xe$.*

Proof. For the $\mathcal{R}$-preorder, if $x \leq_{\mathcal{R}} e$, then $x = ez$ for some $z \in S^1$. Since e is idempotent, $e = e^2$, so $ex = e(ez) = e^2 z = ez = x$. Conversely, if $x = ex$, then $x \leq_{\mathcal{R}} e$ by definition. The argument for the $\mathcal{L}$-preorder is analogous. The statement for $\mathcal{H}$ follows directly from the definitions. □

Lemma 8 (Preservation of Green's Relations by Morphisms)**.** *Green's relations are preserved under semigroup morphisms. Let $f : S \to T$ be a semigroup morphism. If two elements $x, y \in S$ are related by any of Green's preorders or equivalence relations, then their images $f(x), f(y)$ are related by the same relation in T.*

Proof. If $x \leq_{\mathcal{R}} y$, then $x = yz$ for some $z \in S^1$. Applying the morphism f gives $f(x) = f(yz) = f(y)f(z)$, so $f(x) \leq_{\mathcal{R}} f(y)$. This extends to the equivalence relation: if $x\mathcal{R}y$, then $x \leq_{\mathcal{R}} y$ and $y \leq_{\mathcal{R}} x$, which implies $f(x) \leq_{\mathcal{R}} f(y)$ and $f(y) \leq_{\mathcal{R}} f(x)$, so $f(x)\mathcal{R}f(y)$.

A similar argument holds for $\mathcal{L}$. The preservation of $\mathcal{J}$ and $\mathcal{H}$ follows from their definitions. For $\mathcal{D}$, if $x\mathcal{D}y$, there exists z such that $x\mathcal{R}z$ and $z\mathcal{L}y$. Then $f(x)\mathcal{R}f(z)$ and $f(z)\mathcal{L}f(y)$, which implies $f(x)\mathcal{D}f(y)$. □

Chapter 4

Green's Relations in Finite Semigroups

Important lemma: (2).

Lemma 9 (Equivalence of D and J Relations in Finite Semigroups)**.** *In a finite semigroup, the $\mathcal{D}$ and $\mathcal{J}$ relations are equivalent. That is, for any two elements $x, y \in S$, $x\mathcal{D}y$ if and only if $x\mathcal{J}y$.*

Proof. The forward direction, that $x\mathcal{D}y$ implies $x\mathcal{J}y$, holds in any semigroup, not just finite ones. This is because if $x\mathcal{D}y$, there exists z such that $x\mathcal{R}z$ and $z\mathcal{L}y$. These relations imply $x \leq_{\mathcal{J}} z$ and $z \leq_{\mathcal{J}} y$, and by transitivity, $x \leq_{\mathcal{J}} y$. A symmetric argument shows $y \leq_{\mathcal{J}} x$, so $x\mathcal{J}y$.

The reverse direction relies on the semigroup being finite. TODO: FINISH PROOF □

Lemma 10 (J-Equivalence Strengthening Preorders)**.** *In a finite semigroup, if two elements are $\mathcal{J}$-equivalent, then a one-sided preorder implies the corresponding one-sided equivalence. Specifically, if $x\mathcal{J}y$ and $x \leq_{\mathcal{R}} y$, then $x\mathcal{R}y$. Similarly, if $x\mathcal{J}y$ and $x \leq_{\mathcal{L}} y$, then $x\mathcal{L}y$.*

Proof. Suppose $x\mathcal{J}y$ and $x \leq_{\mathcal{R}} y$. Since we are in a finite semigroup, $x\mathcal{J}y$ implies $x\mathcal{D}y$ by 9. TODO: FINISH PROOF □

Lemma 11 (H-Equivalence from Sandwiching)**.** *In a finite semigroup, if an element x can be written as $x = uxv$ for some $u, v \in S$, then x is $\mathcal{H}$-equivalent to both ux and xv.*

Proof. TODO □