1 Complex Numbers

Definition

Complex numbers have a real part and an imaginary part:

$$z = a + bi,$$

$$\mathrm{Re}\{z\} = a, \mathrm{Im}\{z\} = b, i = \sqrt{-1}, i^2 = -1$$

Definition

The set of complex numbers is defined as:

$$\mathbb{C} = \{a + bi \quad | \quad a, b \in \mathbb{R}\}$$

Definition

The **complex conjugate** of a complex number $z = a+bi$ is defined as:

$$\bar{z} = a - bi$$

Definition

Properties of complex conjugates:

1. $\bar{\bar{z}} = z$
2. $\overline{z+w} = \bar{z} + \bar{w}$
3. $\overline{zw} = \bar{w}\bar{z}$
4. If $z \neq 0$, then $\overline{w/z} = \bar{w}/\bar{z}$
5. z is real if and only if $\bar{z} = z$.

Definition

The **absolute value** (or **modulus**) of a complex number $z = a + bi$ is its distance from the origin: $|z| = \sqrt{a^2 + b^2}$

Definition

Properties of absolute values:

1. $|z| = 0$ if and only if $z = 0$.
2. $|z| = |\bar{z}|$
3. $|zw| = |z||w|$
4. If $|z| \neq 0$, then $|\frac{1}{z}| = \frac{1}{|z|}$
5. $|z + w| = |z| + |w|$

Definition

Polar form:

1. $z = |z|\angle\varphi_z = r\angle\varphi_z = r(\cos(\varphi_z) + i\sin(\varphi_z))$
   - $\mathrm{Re}\{z\} = r\cos(\varphi_z)$
   - $\mathrm{Im}\{z\} = r\sin(\varphi_z)$
2. Magnitude $|z| = \sqrt{\bar{z}z} = \sqrt{a^2 + b^2}$
3. Phase / argument $\varphi_z = \tan^{-1}(b/a)$

Definition

The principal argument of z, $\arg(z)$ satisfies $-\pi < \varphi \leq \pi$.

Definition

The polar form of complex numbers can be used to give **geometric interpretations** of multiplication and division:

$$z_1 z_2 = r_1 r_2(\cos(\varphi_1\varphi_2) + i\sin(\varphi_1\varphi_2))$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\varphi_1\varphi_2) + i\sin(\varphi_1\varphi_2)), \text{ if } z \neq 0$$

Therefore,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \text{ and } \arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$$

If $z = r(\cos(\varphi) + i\sin(\varphi))$ is non-zero, then

$$\frac{1}{z} = \frac{1}{r}(\cos(\varphi) - i\sin(\varphi))$$

de Moivre's Theorem

If $z = r(\cos(\varphi)+i\sin(\varphi))$, and n is a positive integer, then $z^n = r^n(\cos(n\varphi) + i\sin(n\varphi))$.
Therefore, $|z^n| = |z|^n$ and $\arg(z^n) = n\arg(z)$.
Then z has exactly n distinct n^{th} roots, given by:

$$r^{\frac{1}{n}}\Big(\cos\big(\frac{\varphi + 2\pi k}{n}\big) + i\sin\big(\frac{\varphi + 2\pi k}{n}\big)\Big)$$

for $k = 0, 1, 2, \ldots, n-1$.

de Moivre's Formula, $e^{i\theta} \cdot \cdots \cdot e^{i\theta} = e^{i(\theta+\cdots+\theta)} = (e^{i\theta})^n = e^{i\cdot n\theta}$ can be used to derive equations for the sine and cosine:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Example:

$\cos 2\theta + i\sin 2\theta = (\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\sin\theta\cos\theta + (-1)\sin^2\theta = (\cos^2\theta - \sin^2\theta) + 2i\sin\theta\cos\theta$

Euler's Formula

For any real number θ, $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ s.t. $|e^{i\theta}| = 1$, $\overline{e^{i\theta}} = e^{-i\theta}$, $\frac{1}{e^{i\theta}} = e^{-i\theta}$, $e^{i(\theta+\omega)} = e^{i\theta}\cdot e^{i\omega}$, and,

$$z = r(\cos(\varphi) + i\sin(\varphi)) = re^{i\varphi}$$

Proof: The exponential form of a complex number can be determined from a Taylor series:

$$e^{i\varphi} = 1 + (i\varphi) + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^3}{3!} + \cdots = \Big(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \ldots\Big) + i\Big(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} + \ldots\Big) = \cos(\varphi) + i\sin(\varphi)$$

Euler's Identity

$$e^{i\varphi} + 1 = 0$$

2 Polynomials

Definition

A polynomial is a function p of a single variable x that can be written in the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

where $a_0, a_1, \ldots, a_n \in \mathbb{R}, a_n \neq 0$ are called the **coefficients** of p.

The integer n is called the **degree** of p, which is denoted by writing $\deg(p) = n$.

A polynomial of degree zero is called a constant polynomial.

Definition

Two polynomials are equal if the coefficients of corresponding powers of x are all equal.

In particular, equal polynomials must have the same degree.

Definition

For two polynomials p, q,

$$\deg(pq) = \deg(p) + \deg(q)$$

The Factor Theorem

Let f be a polynomial and let a be a constant. Then a is a zero of f if and only if $x - a$ is a factor of $f(x)$.

The Rational Roots Theorem

Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ be a polynomial with integer coefficients and let a/b be rational number written in lowest terms. If a/b is a zero of f, then a_0 is a multiple of a and a_n is a multiple of b.

If $f(x)$ is a quadratic polynomial,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Descartes' Rule of Signs

Let p be a polynomial with real coefficients that has k sign changes. Then the number of positive zeros of p (counting multiplicities) is at most k.

(That is, a real polynomial cannot have more positive zeros than it has sign changes.)

Let p be a polynomial with real coefficients. Then the number of negative zeros of p is at most the number of sign changes of $p(-x)$.

The Fundamental Theorem of Algebra

Every polynomial of degree n with real or complex coefficients has exactly n roots (counting multiplicities) in $\mathbb{C}$:

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = a_n (z - z_1)(z - z_2) \ldots (z - z_n), \quad a_n \neq 0$$

The complex roots of a polynomial with real coefficients occur in conjugate pairs.

Example: Consider the polynomial $p(x) = x^2 + 1$ in $\mathbb{R}$. It has no real roots. But in $\mathbb{C}$ it can be factored: $z^2 + 1 = (z + i)(z - i)$

Definition

The n^{th} roots of 1 are called the n^{th} **roots of unity**.

Example: Since $1 = 1e^{i \cdot 0}$, $1^{1/n} = \sqrt[n]{1} \cdot e^{i(\frac{0}{n} + \frac{2k\pi}{n})}$, $k = 0, 1, \ldots, n-1 = e^{i(\frac{2k\pi}{n})}$, $k = 0, 1, \ldots, n-1$

References

[1] Poole, David. “Linear Algebra: A Modern Introduction”. In: (2012).

[2] Wesleyan University. “Introduction to Complex Analysis”. In: (2024). URL: https://www.coursera.org/learn/complex-analysis/.