1 Functions and Limits

1.1 Functions

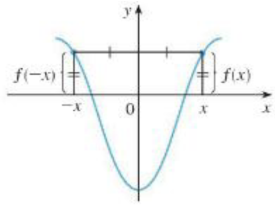

FIGURE 14 An even function

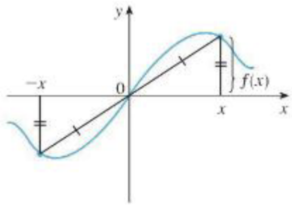

FIGURE 15 An odd function

The Vertical Line Test

A curve in the xy-plane is the graph of a function of x if and only if no vertical line intersects the curve more than once.

Definition

A function f is called **increasing** on an interval I if

$$f(x_1) < f(x_2) \text{ for all } x_1 < x_2 \text{ in } I$$

Definition

A function f is called **decreasing** on an interval I if

$$f(x_1) > f(x_2) \text{ for all } x_1 > x_2 \text{ in } I$$

Definition

A function f is called **even** if $f(-x) = f(x)$.

Definition

A function f is called **odd** if $f(-x) = -f(x)$.

Shifts

Suppose $c > 0$.

- $y = f(x) + c$ shifts the graph c units upward.
- $y = f(x) - c$ shifts the graph c units downward.
- $y = f(x + c)$ shifts the graph c units to the left.
- $y = f(x - c)$ shifts the graph c units to the right.

Reflections

Suppose $c > 1$.

- $y = -f(x)$ reflects the graph about the x-axis.
- $y = f(-x)$ reflects the graph about the y-axis.

Stretches

Suppose $c > 1$.

- $y = cf(x)$ stretches the graph vertically by a factor of c.
- $y = f(x/c)$ stretches the graph horizontally by a factor of c.

Shrinkages

Suppose $c > 1$.

- $y = \frac{1}{c}f(x)$ shrinks the graph vertically by a factor of c.
- $y = f(cx)$ shrinks the graph horizontally by a factor of c.

1.2 Limits

Limit Laws

Suppose that c is a constant and the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
2. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n$ if n is a positive integer
7. $\lim_{x\to a} c = c$
8. $\lim_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ if n is a positive integer
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ if n is a positive integer
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ if n is a positive integer and $\lim_{x\to a} f(x) > 0$

Direct Substitution Property

If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)$$

Theorem

$$\lim_{x\to a} f(x) = L \text{ if and only if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$$

Theorem

If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

The Squeeze Theorem

If $f(x) \leq f(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \text{ then } \lim_{x\to a} g(x) = L$$

1.3 Continuity

Definition

A function f is **continuous** at a number a if

$$\lim_{x\to a} f(x) = f(a)$$

Definition

A function f is **continuous from the right** at a if

$$\lim_{x\to a^+} f(x) = f(a)$$

A function f is **continuous from the left** at a if

$$\lim_{x\to a^-} f(x) = f(a)$$

Theorem

A function f is **continuous on an interval** it is continuous at every number in the interval.

Theorem

If f and g are continuous at a, and c is a constant, then the following functions are also continuous at a:

$$f+g \qquad f-g \qquad cf \qquad fg \qquad \frac{f}{g} \text{ if } g(a)\neq 0$$

Theorem

The following types of functions are continuous at every number in their domains:

- Polynomials.
- Rational functions.
- Root functions.
- Trigonometric functions.

Theorem

If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f\big(\lim_{x\to a} g(x)\big) = f(b)$

Theorem

If g is continuous at a and f is continuous at $g(a)$, then composite function $f\circ g$ given by $(f\circ g)(x) = f(g(x))$ is continuous at a.

The Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a)\neq f(b)$. Then there exists a number c in (a,b) such that $f(c)=N$.

That is, a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$ at least once.

1.4 Inverse Functions

Definition

A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1)\neq f(x_2) \text{ for all } x_1\neq x_2$$

Horizontal Line Test

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

Theorem

If f is a one-to-one continuous function defined on an interval, then its inverse function is also continuous.

Theorem

If f is a one-to-one differentiable function with inverse function f^{-1} and $f'(f^{-1}(a))\neq 0$, then the inverse function is differentiable at a and

$$(f^{-1})'(a) = \frac{1}{f'(f{-}1(a))}$$

Definition

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad\iff\quad f(x) = y$$

for any y in B.

1.5 Asymptotes

Definition

Infinite Limits:
The line $x=a$ is called a **vertical asymptote** of the curve $y=f(x)$ if at least one of the following is true:

$$\lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a} f(x) = \infty$$

$$\lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a} f(x) = -\infty$$

Definition

Limits at Infinity:

The line $y=L$ is called a **horizontal asymptote** of the curve $y=f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

The line $y=mx+b$ is called a **slant asymptote** if $\lim_{x\to\pm\infty} f(x)-(mx+b)=0$.

1.6 Indeterminate Forms and l'Hospital's Rule

Definition

Indeterminate Products:

$$fg = \frac{f}{1/g} \qquad \text{or} \qquad fg = \frac{g}{1/g}$$

Definition

Indeterminate Differences: If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit $\lim_{x\to a}[f(x) - g(x)]$ is called an indeterminate form of type $\infty - \infty$.

Definition

Indeterminate Powers: $\lim_{x\to a}[f(x)]^{g(x)}$

1. Type 0^0: $\lim_{x\to a} f(x) = 0, \lim_{x\to a} g(x) = 0$
2. Type ∞^0: $\lim_{x\to a} f(x) = \infty, \lim_{x\to a} g(x) = 0$
3. Type $1^{\pm\infty}$: $\lim_{x\to a} f(x) = 1, \lim_{x\to a} g(x) = \pm\infty$

Each of these can be converted to an indeterminate product $g(x)\ln f(x)$ of type $0\cdot\infty$, either by taking the natural logarithm, or by writing the function as an exponential:

- Let $y = [f(x)]^{g(x)}$, then $\ln y = g(x)\ln f(x)$
- $[f(x)]^{g(x)} = e^{g(x)\ln g(x)}$

L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a). Suppose that we have an indeterminate form of type $0/0$ or ∞/∞:

$$\lim_{x\to a} f(x) = 0, \lim_{x\to a} g(x) = 0 \qquad \text{or} \qquad \lim_{x\to a} f(x) = \pm\infty, \lim_{x\to a} g(x) = \pm\infty$$

Then, if the limit on the right side exists (or is $\pm\infty$),

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

2 Differentiation

2.1 Derivatives and Rates of Change

Definition

The **tangent line** to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

Definition

The difference quotient

$$\frac{\triangle y}{\triangle x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is the **average rate of change** with respect to x over the interval $[x_1, x_2]$.

Definition

The **instantaneous rate of change** is

$$\lim_{\triangle x\to 0} \frac{\triangle y}{\triangle x} = \lim_{x_2\to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Definition

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to $x = a$.

Definition

The **derivative** of a function f at a number a is

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

or equivalently

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists.

Theorem

The derivative is defined as a function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Definition

A function f is **differentiable** at a if $f'(a)$ exists. It is differentiable on an open interval (a, b), (a, ∞), $(-\infty, a)$ or $(-\infty, \infty)$ if it is differentiable at every number in the interval.

Theorem

If f is differentiable at a, then f is continuous at a.

2.2 Linear Approximations and Differentials

Definition

Linearisation:

$$L(x) = f(a) + f'(a)(x - a)$$

Definition

Differentials:

$$dy = df = f'(x)\,dx$$

2.3 Extreme Values

Definition

Let c be a number in the domain D of a function f. Then $f(c)$ is the

- **absolute maximum** value of f on D if $f(c) \geq f(x)$
- **absolute minimum** value of f on D if $f(c) \leq f(x)$

for all x in D.

- **local maximum** value of f if $f(c) \geq f(x)$
- **local minimum** value of f if $f(c) \leq f(x)$

when x is near c.

The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Fermat's Theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Definition

A **critical number** of a function f is a number c in the domain of f s.t. either $f'(c) = 0$ or $f'(c)$ does not exist.

Corollary

If f has a local maximum or minimum at c, then c is a critical number of f.

The Closed Interval Method

To find the *absolute* maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

2.4 The Mean Value Theorem

Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

Theorem

If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

Corollary

If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b); that is, $f(x) = g(x) + c$ where c is a constant.

The Mean Value Theorem

Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a,]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a)$$

Cauchy's Mean Value Theorem

Suppose that the functions f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for all x in (a, b). Then there is a number c in (a, b) such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Note: Given the special case in which $g(x) = x$, then $g'(c) = 1$ and Cauchy's Mean Value Theorem is the Mean Value Theorem.

2.5 Derivatives and the Shapes of Graphs

Increasing / Decreasing Test

- If $f'(x) > 0$, then f is increasing on that interval.
- If $f'(x) < 0$, then f is decreasing on that interval.

The First Derivative Test

Suppose c is a critical number of a continuous function f.

- Local max: f' changes from positive to negative.
- Local min: f' changes from negative to positive.
- Neither: f' does not change sign at c.

Definition

If the graph of f lies above / below all of its tangents on an interval I, it is **concave upward / downward** on I.

Concavity Test

- Concave upward on I: $f''(x) > 0$ for all x in I.
- Concave downward on I: $f''(x) < 0$ for all x in I.

Definition

A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test

- Local minimum at c: $f'(c) = 0$ and $f''(c) > 0$.
- Local maximum at c: $f'(c) = 0$ and $f''(c) < 0$.

3 Integration

3.1 Areas and Distances

Definition

The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\triangle x + f(x_2)\triangle x + \cdots + f(x_n)\triangle x]$$

3.2 The Definite Integral

Definition

If f is a function defined on $[a,b]$, the **definite integral** of f from a to b is the number

$$\int_a^b f(x)\,dx = \lim_{\max \triangle x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\triangle x_i$$

provided that this limit exists.

Theorem

If f is integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\triangle x$$

where $\triangle x = \frac{b-a}{n}$ and $x_i = a + i\triangle x$.

Definition

Properties of the integral:

1. $\int_a^b c\,dx = c(b-a)$, where c is any constant
2. $\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, where c is any constant
4. $\int_a^b [f(x)-g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
5. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$

Evaluation Theorem

If f is continuous on the interval $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any **antiderivative** of f, that is, $F'(x) = f(x)$ for all x in an interval I.

Theorem

If f is continuous on $[a,b]$, or if f has only a finite number of jump discontinuities, then f is **integrable** on $[a,b]$; that is, the definite integral $\int_a^b f(x)\,dx$ exists.

Midpoint Rule

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\bar{x}_i)\triangle x = \triangle x[f(\bar{x}_1) + \cdots + f(\bar{x}_n)]$$

where $\triangle x = \frac{b-a}{n}$

and $\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i)$ is the midpoint of $[x_{i-1}, x_i]$.

Definition

Comparison properties of the integral:

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$.
2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
3. If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

Net Change Theorem

The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

3.3 The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus

Suppose f is continuous on $[a,b]$.

1. If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, where F is any antiderivative of f; that is, $F' = f$.

The Mean Value Theorem for Integrals

If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that $f(c)$ is the **average value** of a function:

$$f(c) = f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

that is, $$\int_a^b f(x)\,dx = f(c)(b-a)$$

3.4 Improper Integrals

Definition

Improper integrals of type 1:

- If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

- If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

Theorem

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

Theorem

If $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are called convergent, then

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

Definition

Improper integrals of type 2:

- If f is continuous on $[a, b)$ and discontinuous at b,

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

- If f is continuous on $(a, b]$ and discontinuous at a,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

Theorem

The improper integrals $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

Theorem

If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Theorem

$\int_1^\infty \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leq 1$.

Proof: Suppose $T_n(x)$ is the n^{th} Taylor polynomial of a function $f(x)$. Then $\int T_n(x)\,dx$ is the $(n+1)^{\text{st}}$ Taylor polynomial of an anti-derivative $F(x)$ of $f(x)$. For $a > 0$:

$$\int_a^\infty \frac{1}{x^p}\,dx \begin{cases} \text{diverges} & \text{if } p \leq 1 \\ \text{converges to } \frac{a^{-p+1}}{p-1} & \text{if } p > 1 \end{cases} \qquad \int_0^a \frac{1}{x^p}\,dx \begin{cases} \text{diverges} & \text{if } p \geq 1 \\ \text{converges to } \frac{a^{1-p}}{1-p} & \text{if } p < 1 \end{cases}$$

Comparison Theorem

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

- If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.
- If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

3.5 Techniques of Integration

Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Integrals of Symmetric Functions

Suppose f is continuous on $[-a, a]$.

- If f is even, $f(-x) = f(x)$, then $\int_{-a}^a f(x)\,dx = 2\int_0^a f(x)\,dx$.
- If f is odd, $f(-x) = -f(x)$, then $\int_{-a}^a f(x)\,dx = 0$.

Integration by Parts

$$\int u\,dv = uv - \int v\,du \qquad \text{or, equivalently,} \qquad \int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b g(x)f'(x)\,dx$$

Expression	Trigonometric Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta, 0 \leq \theta \leq \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

3.6 Applications of Integration

Definition

The **area** A of the region bounded by the curves $y = f(x), y = g(x)$ and the lines $x = a, x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Definition

Surface area: Where f is positive and has a continuous derivative, the area of the surface obtained by rotating the curve $y = f(x), a \leq x \leq b$, about the x-axis as

$$S = \int_a^b 2\pi\, f(x)\, \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b 2\pi\, y\, \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If the curve is described as $x = g(y), c \leq y \leq d$, then the formula for the surface area becomes

$$S = \int_c^d 2\pi\, g(y)\, \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d 2\pi\, x\, \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Definition

If a curve C is described by the **parametric equations** $x = f(t), y = g(t)$ and is traversed once as t increases from α to β, then the area under the curve is

$$A = \int_a^b y\, dx = \int_\alpha^\beta g(t) f'(t)\, dt$$

Definition

In **polar coordinates**,

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2\, d\theta = \int_a^b \frac{1}{2} r^2\, d\theta$$

Definition

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is an integrable function, then the **volume** of S is

$$V = \lim_{\max \triangle x_i \to 0} \sum_{i=1}^n A(x_i^*) \triangle x_i = \int_a^b A(x)\, dx$$

Definition

Arc length: If f' is continuous on $[a, b]$, then the length of the curve $y = f(x), a \leq x \leq b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

If g' is continuous on $[c, d]$, then the length of the curve $x = g(y), c \leq x \leq d$, is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Definition

If a curve C is described by the **parametric equations** $x = f(t), y = g(t), \alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is transversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Definition

In **polar coordinates**,

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

4 Transcendental Functions

Transcendental functions are non-algebraic functions that cannot be expressed as a finite combination of algebraic operations such as addition, subtraction, multiplication, division, raising to a power, and extracting roots. In other words, they are functions that are not solutions to polynomial equations.

4.1 The Natural Logarithm

Definition

The natural logarithmic function is the function defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt, \quad x > 0$$

The existence of this function depends on the fact that the integral of a continuous function always exists.

- For $x > 1$, then $\ln x$ can be interpreted geometrically as the area under the hyperbola $y = \frac{1}{t}$ from $t = 1$ to $t = x$.
- For $x = 1$, $\ln 1 = \int_1^1 \frac{1}{t}\,dt = 0$
- For $0 < x < 1$, $lnx = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt < 0$ and so $\ln x$ is the negative of the area under the hyperbola on that interval.

Corollary

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Definition

$$\lim_{x\to\infty} \ln x = \infty \qquad \lim_{x\to 0^+} \ln x = -\infty$$

Definition

e is the number such that $\ln e = 1$.

$$e = \lim_{x\to 0}(1+x)^{1/x}$$

Definition

Laws of Logarithms:

$$\ln xy = \ln x + \ln y \qquad \ln \frac{x}{y} = \ln x - \ln y \qquad \ln x^r = r \ln x$$

4.2 The Natural Exponential Function

Since ln is an increasing function, it is one-to-one and therefore has an inverse function.

Definition

$$e^x = y \qquad \iff \qquad \ln y = x$$

Definition

$$e^{\ln x} = x, \quad x > 0 \qquad \ln e^x = x \text{ for all } x$$

Definition

Laws of Exponents:

If x, y are real numbers and r is rational, then

$$e^{x+y} = e^x e^y \qquad e^{x-y} = \frac{e^x}{e^y} \qquad (e^x)^r = e^{rx}$$

If x, y are real numbers and $a, b > 0$, then

$$a^{x+y} = a^x a^y \quad a^{x-y} = \frac{a^x}{a^y} \quad (a^x)^y = a^{xy} \quad (ab)^x = a^x b^x$$

Definition

Properties of the natural exponential function: The exponential function $f(x) = e^x$ is a continuous function with domain $\mathbb{R}$ and range $(0, \infty)$.
Thus, $e^x > 0$ for all x, $\lim_{x\to-\infty} e^x = 0$, $\lim_{x\to\infty} e^x = \infty$.
The x-axis is a horizontal asymptote of $f(x) = e^x$.

4.3 General Logarithmic Functions

Definition

If $a > 0$ and $a \neq 1$, then $f(x) = a^x$ is a one-to-one function. Its inverse function is called the logarithmic function with base a:

$$\log_a x = y \qquad \iff \qquad a^y = x$$

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

Definition

$$\log_a(a^x) = x \text{ for all } x \in \mathbb{R}$$

Definition

$$a^{\log_a x} = x \text{ for all } x > 0$$

Definition

$$\log_a(x/y) = \log_a(x) - \log_a(y)$$

Definition

$$\log_a(xy) = \log_a(x) + \log_a(y)$$

Change-of-Base Formula

For any positive number a ($a \neq 1$),

$$\log_a x = \frac{\ln x}{\ln a}$$

Definition

If $a > 1$, then

$$\lim_{x\to\infty} \log_a x = \infty \qquad \text{and} \qquad \lim_{x\to 0^+} \log_a x = -\infty$$

5 Sequences and Series

5.1 Sequences

Definition

A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges** (or is convergent). Otherwise, it **diverges** (or is divergent).

Theorem

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$.

Theorem

Limit Laws for Sequences: If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is a constant, then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c\lim_{n\to\infty} a_n \qquad\qquad \lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} \text{ if } \lim_{n\to\infty} b_n \neq 0$$

$$\lim_{n\to\infty} a_n^p = [\lim_{n\to\infty} a_n]^p \text{ if } p > 0 \text{ and } a_n > 0$$

Theorem

If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Theorem

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Continuity and Convergence Theorem

If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then

$$\lim_{n\to\infty} f(a_n) = f(L)$$

Theorem

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$ and divergent for all other values of r.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Definition

A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$, that is, $a_1 < a_2 < a_3 < \ldots$. It is called **decreasing** if $a_n > a_{n+1}$ for all $n \geq 1$.

A sequence is **monotonic** if it is either increasing or decreasing.

Definition

A sequence $\{a_n\}$ is **bounded above** if there is a number M such that

$$a_n \leq M \quad \text{for all } n \geq 1$$

A sequence $\{a_n\}$ is **bounded below** if there is a number m such that

$$m \leq a_n \quad \text{for all } n \geq 1$$

If a sequence is bounded above and below, then it is a **bounded sequence**.

Monotonic Sequence Theorem

Every bounded, monotonic sequence is convergent.

5.2 Series

Definition

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$, let s_n denote its n^{th} partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent**, and we write

$$a_1 + a_2 + \cdots + a_n + \cdots = s \qquad \text{or} \qquad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, the series is called **divergent**.

Definition

The **geometric series** $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots$ is divergent if $|r| \geq 1$; convergent if $|r| < 1$, and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

Proof: $s_n = 1 + a + a^2 + \cdots + a^n$, $s_{n+1} = 1 + a + a^2 + \cdots + a^n + a^{n+1}$. Then $s_n + a^{n+1} = 1 + as_n \quad \Rightarrow \quad s_n(1-a) = 1 - a^{n+1} \quad \Rightarrow \quad s_n = \frac{1-a^{n+1}}{1-a}$. Taking the limit,

$$\lim_{n\to\infty} \frac{1-a^{n+1}}{1-a} = \begin{cases} \infty & |a| \geq 1 \\ \frac{1}{1-a} & |a| < 1 \end{cases}$$

Example: $a = 1, r = 2$
$\sum_{n=1}^{\infty} ar^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2 = \frac{1}{1/2} = \frac{a}{1-r}$

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

With any series $\sum a_n$ we associate two sequences:

- the sequence $\{s_n\}$ of its partial sums, and
- the sequence $\{a_n\}$ of its terms.

If $\sum a_n$ is convergent, then the limit of the sequence $\{s_n\}$ is the sum of the series, s, and, as the theorem asserts, the limit of the sequence $\{a_n\}$ is 0.

However, the converse is not true in general: If $\lim_{n\to\infty} a_n = 0$, we cannot conclude that $\sum a_n$ is convergent. **Example**: For the harmonic series $\sum 1/n$, $a_n = 1/n \to 0$ as $n \to \infty$, but $\sum 1/n$ is divergent.

Test for Divergence

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

- If $\lim_{n\to\infty} a_n \neq$, $\sum a_n$ is divergent.
- If $\lim_{n\to\infty} a_n = 0$, the series $\sum a_n$ may be convergent or divergent.

Theorem

If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum(a_n + b_n)$, and $\sum(a_n - b_n)$; and

- $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$
- $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
- $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

5.3 Convergence Tests

The Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$.

Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ is convergent.

In other words:

- If $\int_1^{\infty} f(x)\, dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
- If $\int_1^{\infty} f(x)\, dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Theorem

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

- If $\sum b_n$ is convergent and the above limit equals 0, then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and the above limit equals ∞, then $\sum a_n$ is also divergent.

The Alternating Series Test

The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \ldots, b_n > 0$ is convergent if it satisfies

- $b_{n+1} \leq b_n$ for all n
- $\lim_{n\to\infty} b_n = 0$

Alternating Series Estimation Theorem

If $s = \sum(-1)^{n-1} b_n$ is the sum of an alternating series that satisfies $0 \leq b_{n+1} \leq b_n$ and $\lim_{n\to\infty} b_n = 0$, then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Definition

A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Definition

A series $\sum a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

Theorem

If a series $\sum a_n$ is absolutely convergent, it is convergent.

The Ratio Test

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test is inconclusive.

The Root Test

- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
- If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, the Root Test is inconclusive.

5.4 Power Series

Theorem

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

- The series converges only when $x = a$. $(R = 0)$
- The series converges for all x. $(R = \infty)$
- There is a positive number R, the **radius of convergence** of the power series, such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges:

- a single point a,
- the interval $(-\infty, \infty)$,
- in the third case, $|x-a| < R \quad \Rightarrow \quad a-R < x < a+R$, the series might converge at one or both endpoints or it might diverge at both endpoints:

$(a-R, a+R) \quad (a-R, a+R] \quad [a-R, a+R) \quad [a-R, a+R]$

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n!x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$

The Ratio Test (or sometimes the Root Test) should be used to determine the radius of convergence R in most cases. The Ratio and Root Tests always fail when x is an endpoint of the interval of convergence, so the endpoints must be checked with some other test.

Theorem

If the power series $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$; and

- $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
- $\int f(x)\,dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the power series in these two equations are both R, that is, the radius of convergence remains the same when a power series is differentiated or integrated.

Note: We know that for finite sums, the derivative of a sum is the sum of the derivatives and the integral of a sum is the sum of the integrals. The same is true for infinite sums, provided we are dealing with power series:

- $\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n]$
- $\int\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right]dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n\,dx$

Note: Although the theorem states that the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the interval of convergence remains the same. It may happen that the original series converges at an endpoint, whereas the differentiated series diverges there.

5.5 Taylor and Maclaurin Series

Theorem

If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, \quad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Definition

If f has a power series representation (expansion) at a, then it is called the **Taylor series** of the function f at a (or about a or centred at a):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

Definition

A Taylor series for which $a = 0$ is called a **Maclaurin series**:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots$$

Theorem

If $f(x) = T_n(x)+R_n(x)$, where T_n is the n^{th}-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

NB: If f can be represented as a power series about a, then f is equal to the sum of its Taylor series.

But there exist functions that are not equal to the sum of their Taylor series.

Taylor's Formula

If f has $n+1$ derivatives in an interval I that contains the number a, then for x in I there is a number z strictly between x and a such that the remainder term in the Taylor series can be expressed as

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$

The number z lies somewhere between x and a.

For the special case $n = 0$, with $x = b, z = c$,

$$f(b) = f(a) + f'(c)(b-a)$$

which is the Mean Value Theorem.

Theorem

For every real number x,

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

Definition

Multivariable Taylor / Maclaurin Series: $f(x,y) = f + f_x(x)x + f_y(y)y + \frac{1}{2!}\left(f_{xx}(x)x^2 + 2f_{xy}(x,y)xy + f_{yy}(y)y^2\right) + \dots$

Example: Important Maclaurin series and their radii of convergence:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots, \quad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, \quad R = 1$$

$$\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, \quad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, \quad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots, \quad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \quad R = \infty$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots, \quad R = 1$$

5.6 Approximating Functions by Polynomials

Definition

Suppose that $f(x)$ is equal to the sum of its Taylor series at a: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. Thus, its n^{th}-degree Taylor polynomial is $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}}{n!}(x-a)^n$.

Since f is the sum of its Taylor series, $T_n(x) \to f(x)$ as $n \to \infty$ and so T_n can be used as an approximation to f:

$$f(x) \approx T_n(x)$$

Notice that the first-degree Taylor polynomial is the same as the linearisation of the function $f(x)$:

$$T_1(x) = f(a) + f'(a)(x-a)$$

Notice also that T_1 and its derivative have the same values at a that f and f' have. In general, it can be shown that the derivatives of T_n at a agree with those of f up to and including derivatives of order n.

How good an approximation is it? How large should we take n to be in order to achieve a desired accuracy? To answer these questions we need to look at the absolute value of the remainder:

$$|R_n(x)| = |f(x) - T_n(x)|$$

There are three methods for estimating the size of the error:

1. If a graphing device is available, we can use it to graph $|R_n(x)|$ and thereby estimate the error.
2. If the series happens to be an alternating series, we can use the Alternating Series Estimation Theorem.
3. In all cases, we can use Taylor's Formula, which says that
$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$$
where z is a number that lies between x and a.

Taylor's Inequality

If $|f^{(n+1)}(x)| \le M$ for $|x-a| \le d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a| \le d$$

6 Partial Derivatives

6.1 Functions of Several Variables

Definition

A function of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) | (x, y) \in D\}$.

That is, $z = f(x, y)$, where x and y are the **independent variables** and z is the **dependent variable**. The domain is a subset of $\mathbb{R}^2$ and the range is a subset of $\mathbb{R}$.

Definition

Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then the **limit** of $f(x, y)$ as (x, y) approaches (a, b) is L

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \varepsilon$.

Theorem

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Definition

If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Definition

The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

Definition

A function f of two variables is continuous at (a, b) if

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$$

Definition

A function f of two variables is continuous on a disk D if it is continuous at every point (a, b) in D.

Definition

If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{x\to a} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\partial > 0$ such that

$$\text{if } \mathbf{x} \in D \text{ and } 0 < |\mathbf{x} - \mathbf{a}| < \partial \text{ then } |f(\mathbf{x} - L| < \varepsilon$$

6.2 Partial Derivatives

Definition

If f is a function of two variables, its **partial derivatives** are the rates of change of f in directions x and y:

$$f_x(x, y)\left(= \frac{\partial f}{\partial x}\right) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y(x, y)\left(= \frac{\partial f}{\partial y}\right) = \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

In general, if f is a function of n variables, $f = f(x_1, \dots, x_n)$, its partial derivative with respect to the i^{th} variable x_i is $f_{x_i} = \frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1,\dots,x_{i-1},x_i+h,x_{i+1},\dots,x_n) - f(x_1,\dots,x_i,\dots,x_n)}{h}$.

Clairaut's Theorem

Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy}, f_{yx} are both continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

The Chain Rule

Suppose that f is a differentiable function of the n variables $x_1, x_2, \dots, x_n$ and each x_j is a differentiable function of the m variables ${}_1, t_2, \dots, t_m$, $f(x_1(t_1, \dots, t_m), x_2(t_1, \dots, t_m), \dots, x_n(t_1, \dots, t_m))$. Then, for each $i = 1, 2, \dots, m$,

$$\frac{\partial f}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i}$$

Example: $z = f(x, y), x = g(t), y = h(t)$, $\frac{z}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

Implicit Differentiation

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}, \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

6.3 Tangent Planes and Linear Approximations

Definition

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Theorem

If the partial derivatives f_x, f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Definition

Linearisation:
$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

6.4 Directional Derivatives

Definition

The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Definition

If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$, and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)a + f_y(x, y)b$$

Definition

If f is a function of two variables x and y, then the **gradient** of f is the vector function

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Definition

The directional derivative in the direction of a unit vector $\mathbf{u}$ is the scalar projection of the gradient vector onto $\mathbf{u}$:

$$\begin{aligned} D_{\mathbf{u}} f(x, y) &= \nabla f(x, y) \cdot \mathbf{u} \\ &= ||\nabla|| \cos\theta \qquad (= ||\nabla||||\hat{\mathbf{u}}|| \cos\theta) \end{aligned}$$

since $\hat{\mathbf{u}}$ is a unit vector

Theorem

Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is

$$|\nabla f(\mathbf{x})|$$

and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

The directional derivative $D_{\mathbf{u}} f(x, y)$ is

- a maximum when $\cos\theta = 1 \iff \theta = 0 \iff \mathbf{u}$ has the same direction as the gradient.
- a minimum when $\cos\theta = -1 \iff \theta = \pi \iff \mathbf{u}$ has the opposite direction as the gradient.
- zero when $\cos\theta = 0 \iff \theta = \frac{\pi}{2} \iff \mathbf{u}$ is orthogonal.

6.5 Extreme Values

Theorem

If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Second Derivative Test

Suppose the second partial derivatives of f are continuous on a disk with centre (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ (that is, (a, b) is a critical point of f).

Let $D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - \big(f_{xy}(a, b)\big)^2$.

- Local minimum: $D > 0$ and $f_{xx}(a, b) > 0$
- Local maximum: $D > 0$ and $f_{xx}(a, b) < 0$
- Neither: $D < 0$

Bivariate Extreme Value Theorem

If f is continuous on a closed, bounded set D in $\mathbf{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

6.6 Constrained Optimisation

Method of Lagrange Multipliers

To find the maximum and minimum values of $f(x, y, z)$ **subject to the constraint** $g(x, y, z) = k$ (assuming that these extreme values exist, and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$):

1. Find all values of x, y, z and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z), \qquad g(x, y, z) = k$$

2. Evaluate f at all the points (x, y, z) that result from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Given two constraints of the form $g(x, y, z) = k, h(x, y, z) = c$, geometrically this means the extreme values of $f(x, y, z)$ when (x, y, z) is restricted to lie on the curve C of intersection of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$.

Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$.

Then ∇g is orthogonal to $g(x, y, z) = k$, and ∇h is orthogonal to $h(x, y, z) = c$, so ∇g and ∇h are both orthogonal to C.

Therefore, the gradient vector $\nabla f(x_0, y_0, z_0)$ is in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$. (Assuming these gradient vectors are not zero and not parallel.)

So there are numbers λ, μ (the Lagrange multipliers) s.t.

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

Writing this equation in terms of its components yields five equations in the five unknowns x, y, z, λ, μ.

7 Multiple Integrals and Curvilinear Coordinates

7.1 Double Integrals over Rectangles

Definition

The **double integral** of f over the rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i,y_j)\,\Delta A$$

if this limit exists.

Definition

If $f(x,y) \geq 0$, then the **volume** V of the solid that lies above the rectangle R and below the surface $z = f(x,y)$ is

$$V = \iint_R f(x,y)\,dA$$

Midpoint Rule

$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(\bar{x}_i,\bar{y}_j)\Delta A$$

where $\bar{x}_i$ is the midpoint of $[x_{i-1},x_i]$ and $\bar{y}_i$ is the midpoint of $[y_{j-1},y_j]$

Fubini's Theorem

If f is continuous on the rectangle $R = \{(x,y)|a \leq x \leq b, c \leq y \leq d\}$, then

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_a^b \int_c^d f(x,y)\,dy\,dx \\ &= \int_c^d \int_a^b f(x,y)\,dx\,dy\end{aligned}$$

More generally, this is true if f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Theorem

$$\iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

where $R = [a,b] \times [c,d]$

7.2 Double Integrals over General Regions

Definition

If f is continuous on a **type I region** D such that

$$D = \{(x,y)|a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Definition

If f is continuous on a **type II region** D such that

$$D = \{(x,y)|c \leq y \leq d, h_1(y) \leq x \leq h_2(y)\}$$

then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

7.3 Properties of Double Integrals

Theorem

Linearity:

$$\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

$$\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA, \quad c \in \mathbb{R}$$

Theorem

If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then

$$\iint_D f(x,y)\,dA \geq \iint_D g(x,y)\,dA$$

Theorem

The integral of the constant function $f(x,y) = 1$ over a region D is the area of D:

$$\iint_D 1\,dA = A(D)$$

Theorem

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries, then

$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

This property can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II.

Theorem

If $m \leq f(x,y) \leq M$ for all (x,y) in D, then

$$mA(D) \leq \iint_D f(x,y)\,dA \leq MA(D)$$

7.4 Double Integrals in Polar Coordinates

Polar to Cartesian conversion:

$$x = r\cos\theta \qquad y = r\sin\theta$$

Cartesian to Polar conversion:

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

Theorem

If f is continuous on a polar rectangle R given by $0 \le 0 \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Theorem

If f is continuous on a polar region of the form

$$D = \{(r,\theta) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$

then $\displaystyle \iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

7.5 Triple Integrals

Definition

The triple integral of f over the box B is

$$\iiint_B f(x,y,z)\,dV = \lim_{m,n,o\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o} f(x_i,y_j,z_k)\,\Delta V$$

if this limit exists.

Fubini's Theorem

If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Theorem

$$\begin{aligned}
&\iiint_E f(x,y,z)\,dV \\
&\quad = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA \\
&\quad = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx \\
&\quad = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy
\end{aligned}$$

7.6 Triple Integrals in Cylindrical Coordinates

Cylindrical to Cartesian conversion:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Cartesian to cylindrical conversion:

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Definition

$$\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta}^{u_2(r\cos\theta, r\sin\theta} f(r\cos\theta, r\sin\theta, z)\,r\,dz\,dr\,d\theta$$

7.7 Triple Integrals in Spherical Coordinates

Conversion:

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi \qquad \rho^2 = x^2 + y^2 + z^2$$

Definition

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho,\theta,\phi) \,|\, a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$$

7.8 Change of Variables in Multiple Integrals

Definition

Suppose T is a transformation from a region S in uv-space onto a region R in xy-space by means of the equations $x = g(u, v), y = h(u, v)$.

The **Jacobian** of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

Definition

Suppose T is a transformation from a region S in uvw-space onto a region R in xyz-space by means of the equations $x = g(u, v, w), y = h(u, v, w), z = k(u, v, w)$.

The **Jacobian** of the transformation is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Theorem

Suppose T is a transformation whose Jacobian is nonzero and that maps a region S in the uv-plane onto a region R in the xy-plane.
Suppose that f is continuous on R and that R and S are type I or type II plane regions.
Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x,y)\,dA = \iint_S f\big(x(u,v), y(u,v)\big) \left|\tfrac{\partial(x,y)}{\partial(u,v)}\,du\,dv\right|$$

Example: Polar coordinates:

$$dA = \left|\tfrac{\partial(x,y)}{\partial(r,\theta)}\,dr\,d\theta\right| = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}\,dr\,d\theta = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}\,dr\,d\theta = r\,dr\,d\theta$$

The region in the $r - \theta$ plane that defines a circle is $0 \le r \le R, 0 \le \theta \le 2\pi$. Its area is

$$A = \int_0^{2\pi}\int_0^R r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^R r\,dr = \pi R^2$$

Theorem

Suppose T is a transformation whose Jacobian is nonzero and that maps a region S in the uvw-hyperplane onto a region R in the xyz-hyperplane.
Suppose that f is continuous on R and that R and S are type I or type II plane regions.
Suppose also that T is one-to-one, except perhaps on the boundary of S. Then

$$\iint_R f(x,y,z)\,dV = \iint_S f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big) \left|\tfrac{\partial(x,y,z)}{\partial(u,v,w)}\,du\,dv\,dw\right|$$

7.9 Applications in Physics

Definition

The **centre of mass** $(\bar{x}, \bar{y}, \bar{z})$ is

$$\bar{x} = \frac{1}{m}\iiint_E x\,\rho(x,y,z)\,dV$$

$$\bar{y} = \frac{1}{m}\iiint_E y\,\rho(x,y,z)\,dV$$

$$\bar{z} = \frac{1}{m}\iiint_E z\,\rho(x,y,z)\,dV$$

Definition

$$\iint_R 1\;dA = A(R) \qquad \iiint_E 1\,dV = V(E)$$

$$\iiint_E \rho(x,y,z)\,dV = mass(E), \qquad \rho = \text{density}$$

Definition

The **moments of inertia** measure the spread of mass around an object's centre of mass determining how much an object will resist changes in its rotation (about an axis or the origin):

$$I_x = \iint_D y^2\,\rho(x,y),\,dx\,dy \qquad I_y = \iint_D x^2\,\rho(x,y),\,dx\,dy$$

$$I_0 = \iint_D (x^2+y^2)\,\rho(x,y),\,dx\,dy$$

In three dimensions:

$$I_x = \iiint_E (y^2+z^2)\,\rho(x,y,z),\,dx\,dy\,dz$$

$$I_y = \iiint_E (x^2+z^2)\,\rho(x,y,z),\,dx\,dy\,dz$$

$$I_z = \iiint_E (x^2+y^2)\,\rho(x,y,z),\,dx\,dy\,dz$$

$$I_0 = \iiint_E (x^2+y^2+z^2)\,\rho(x,y,z),\,dx\,dy\,dz$$

8 Vector Calculus

8.1 Vector Functions and Space Curves

Definition

If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then

$$\lim_{t\to a}\mathbf{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)\rangle$$

provided the limits of the component function exist.

Definition

The **derivative** of a vector function $\mathbf{r}$ is given by

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$

Example: Velocity and acceleration:

$\mathbf{v}(t) = \lim_{h\to 0}\frac{\mathbf{a}(t+h)-\mathbf{a}(t)}{h} = \mathbf{a}'(t)$

Theorem

If $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

Theorem

Suppose $\mathbf{u}, \mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$
3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
4. $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
5. $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

Definition

The **integral** of a vector function $\mathbf{r}$ is given by

$$\int_a^b \mathbf{r}(t)\,dt = \Big(\int_a^b f(t)\,dt\Big)\mathbf{i} + \Big(\int_a^b g(t)\,dt\Big)\mathbf{j} + \Big(\int_a^b h(t)\,dt\Big)\mathbf{k}$$

Fundamental Theorem of Calculus

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big|_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where $\mathbf{R}$ is an antiderivative of $\mathbf{r}$, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$.

Definition

Arc length:

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\,dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\ &= \int_a^b |\mathbf{r}'(t)|\,dt \end{aligned}$$

Definition

The **curvature** of a curve is

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|$$

where $\mathbf{T}$ is the unit tangent vector.

Theorem

The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

8.2 Vector Fields

Definition

In physics, scalars and vectors can be functions of both space and time. Such functions are called **fields**. The equations governing a field are called the **field equations**. These commonly take the form of partial differential equations.

Example: The temperature in some region of space is a scalar field, i.e. a function of space and time: $T(\mathbf{r}, t) = T(x, y, z; t)$

Example: Maxwell's equations govern electric and magnetic vector fields, the Navier-Stokes equation governs fluid velocity, the Schrödinger equation is the equation for the scalar field (called the wave function) in non-relativistic quantum mechanics.

Definition

Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field** on $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

A vector field can be expressed in terms of its component functions, e.g. in $\mathbb{R}^3$:

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + Q(x, y, z)\mathbf{k}$$

The component functions are **scalar fields**.

Definition

Let E be a subset of $\mathbb{R}^3$ (a plane region). A **vector field** on $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point (x, y, z) in E a three-dimensional vector $\mathbf{F}(x, y, z)$.

Definition

A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation, f is called a **potential function** for $\mathbf{F}$.

Conservative vector fields arise frequently in physics.

Theorem

Consider a three-dimensional scalar field $f(x,y,z)$ and the differential

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \nabla f \cdot \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \nabla f \cdot dr$$

"del-f" is the **gradient** of f : $\quad \nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix}$

The gradient ∇f points in the direction of maximally increasing f, and its magnitude gives the slope (or gradient) of f in that direction.

Example: The gradient of the scalar field $f(x,y,z) = xyz$ is the gradient vector field

$$\nabla f = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$$

Definition

Consider a three-dimensional vector field $\mathbf{F} = \big(P(x,y,z), Q(x,y,z), R(x,y,z)\big)$. The **divergence** of $\mathbf{F}$, "del-dot-F", is defined as the scalar field given by the dot product of the vector differential operator ∇ and the vector field $\mathbf{F}$:

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} P \\ Q \\ R \end{pmatrix} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The divergence measures how much a vector field spreads out, or diverges, from a point.

Example: The divergence of $\mathbf{F} = \frac{\mathbf{r}}{|\mathbf{r}|}, \mathbf{r} \neq \mathbf{0}$, is

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \frac{1}{(x^2+y^2+z^2)^{3/2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
&= \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} \\
&+ \frac{(x^2+y^2+z^2)^{3/2} - 3y^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} \\
&+ \frac{(x^2+y^2+z^2)^{3/2} - 3z^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3} \\
&= \frac{1}{|\mathbf{r}|^3} - \frac{3x^2}{|\mathbf{r}|^5} + \frac{1}{|\mathbf{r}|^3} - \frac{3y^2}{|\mathbf{r}|^5} + \frac{1}{|\mathbf{r}|^3} - \frac{3z^2}{|\mathbf{r}|^5} \\
&= \frac{3}{|\mathbf{r}|^3} - \frac{3(x^2+y^2+z^2)}{|\mathbf{r}|^5} \\
&= \frac{3}{|\mathbf{r}|^3} - \frac{3|\mathbf{r}|^2}{|\mathbf{r}|^5} \\
&= 0
\end{aligned}$$

Definition

If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and $\operatorname{curl}\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a **conservative vector field**.

Definition

Consider a three-dimensional vector field $\mathbf{F} = \big(P(x,y,z), Q(x,y,z), R(x,y,z)\big)$. The **curl** of $\mathbf{F}$, "del-cross-F", is defined as the vector field given by the cross product of the vector differential operator and the vector field $\mathbf{F}$:

$$\operatorname{curl}\mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \begin{pmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{pmatrix}$$

The curl measures how much a vector field rotates, or curls, around a point.

Theorem

If f is a function of three variables that has continuous second-order partial derivatives, then the curl of the gradient is zero: $\operatorname{curl}(\nabla f) = \mathbf{0}$.

Proof:

$$\nabla \times (\nabla f) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix} = \begin{pmatrix} \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \end{pmatrix} = \mathbf{0}$$

Theorem

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, R have continuous second-order partial derivatives, then the divergence of the curl is zero: $\operatorname{div}\operatorname{curl}\mathbf{F} = 0$.

Proof : $\nabla \cdot (\nabla \times \mathbf{F})$

$$\begin{aligned}
&= \frac{\partial}{\partial x}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \left(\frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 P}{\partial z \partial y}\right) + \left(\frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 Q}{\partial x \partial z}\right) + \left(\frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 R}{\partial y \partial x}\right) \\
&= 0
\end{aligned}$$

Definition

The **Laplacian**, "del-squared", is defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The Laplacian applied to a scalar field $f = f(x,y,z)$ can be written as the divergence of the gradient:

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The Laplacian applied to a vector field $\mathbf{F}$ acts on each component of the vector field separately:

$$\nabla^2 \mathbf{u} = \begin{pmatrix} \nabla^2 P(x,y,z) \\ \nabla^2 Q(x,y,z) \\ \nabla^2 R(x,y,z) \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} \\ \frac{\partial^2 Q}{\partial x^2} + \frac{\partial^2 Q}{\partial y^2} + \frac{\partial^2 Q}{\partial z^2} \\ \frac{\partial^2 R}{\partial x^2} + \frac{\partial^2 R}{\partial y^2} + \frac{\partial^2 R}{\partial z^2} \end{pmatrix}$$

Example: The Laplacian of the scalar field $f(x,y,z) = x^2 + y^2 + z^2$ is $\nabla^2 f = 2 + 2 + 2 = 6$.

8.4 Line Integrals

Definition

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t), a \le t \le b$. Then the **line integral** of $\mathbf{F}$ along C is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$
$$= \int_C P\,dx + Q\,dy + R\,dz,$$
$$\text{where } \mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

The Fundamental Theorem for Line Integrals

Let C be a smooth curve given by the vector function $\mathbf{r}(t), a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then $$\int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

That is, the line integral of a conservative vector field (the gradient field of the potential function f) can be evaluated simply by knowing the value of f at the endpoints of C. The line integral of ∇f is the net change in f.

Theorem

$\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path C in D.

Theorem

Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Theorem

If $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

Theorem

Let $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D. Then $\mathbf{F}$ is conservative.

8.5 Surface Integrals

Definition

If a smooth parametric surface S is given by the equation

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}, \quad (u,v)\in D$$

and S is covered just once as (u,v) ranges throughout the parameter domain D, then the surface area of S is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

where $\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}, \quad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}.$

Definition

The surface area of the graph of a function is

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}$$

Definition

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$
$$= \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\,dA$$

Definition

If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the surface integral of $\mathbf{F}$ over S is called the **flux** of $\mathbf{F}$ across S:

$$\iint_S \mathbf{F}\cdot\,dS = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS$$

Example: If $\mathbf{v}$ is the fluid velocity (length divided by time), and ρ is the fluid density (mass divided by volume), then the surface integral $\oint_S \rho\mathbf{v}\cdot dS$ computes the mass flux, that is, the mass passing through the surface S per unit time.

If S is a closed surface, then the normal vector $\mathbf{n}$ is assumed to be in the outward direction, and a positive value for the flux integral implies a net flux from inside the surface to outside; a negative value implies a net flux from outside to inside. If a fluid is incompressible, a positive mass flux indicates a source of fluid inside the closed surface, and a negative mass flux indicates a sink.

Definition

$$\iint_S \mathbf{F}\cdot\,dS = \iint_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA$$
$$= \iint_D \Big(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\Big)\,dA$$

Integration over a closed surface can be signified with $\oint_S \mathbf{r}\,dS$.

8.6 Fundamental Theorems

The Gradient Theorem

Let $\nabla\phi$ be the gradient of a scalar field $\phi = \phi(\mathbf{r})$, and let C be a directed curve C by $\mathbf{r} = \mathbf{r}(t)$, where $t_1 \le t \le t_2, \mathbf{r}(t_1) = \mathbf{r}_1, \mathbf{r}(t_2) = \mathbf{r}_2$. Then, for any closed curve C, $\oint_C \nabla\phi\cdot d\mathbf{r} = 0$.

That is, the line integral of the gradient of a function is path independent, depending only on the endpoints of the curve. The theorem is a generalisation of the FTC for line integrals.

Proof: Using the chain rule, $\frac{d}{dt}\phi(\mathbf{r}) = \nabla\phi(\mathbf{r})\cdot\frac{d\mathbf{r}}{dt}$, and FTC,

$$\int_C \nabla\phi\cdot d\mathbf{r} = \int_{t_1}^{t_2} \nabla\phi(\mathbf{r})\cdot\frac{d\mathbf{r}}{dt}dt = \int_{t_1}^{t_2} \frac{d}{dt}\phi(\mathbf{r})\,dt = \phi(\mathbf{r}(t_2)) - \phi(\mathbf{r}(t_1))$$

Theorem

For a vector field $\mathbf{F}$ defined on $\mathbb{R}^3$, except perhaps at isolated singularities, the following conditions are equivalent:

1. $\nabla \times \mathbf{F} = \mathbf{0}$.
2. $\mathbf{F} = \nabla\phi$ for some scalar field $\phi = \phi(r)$.
3. $\int_C \mathbf{F} \cdot dr$ is path independent for any curve C.
4. $\oint_C \mathbf{F} \cdot dr = 0$ for any closed curve C.
5. $\mathbf{F}$ is a **conservative vector field**.

Example: Let $\mathbf{F}(x, y) = x^2(1+y^3)\mathbf{i}+y^2(1+x^3)\mathbf{j}$. Show that $\mathbf{F}$ is a conservative vector field, and determine $\phi = \phi(x, y)$ such that $\nabla \times \mathbf{F} = \mathbf{0}$.

$\mathbf{F}$ is a conservative vector field if and only if $\nabla \times \mathbf{F} = 0$:

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2(1+y^3) & y^2(1+x^3) & 0 \end{vmatrix} = (3x^2y^2-3x^2y^2)\mathbf{k} = 0$$

To find the scalar field ϕ,

$$\frac{\partial\phi}{\partial x} = x^2(1+y^3), \qquad \frac{\partial\phi}{\partial y} = y^2(1+x^3)$$

Integrating the first equation with respect to x holding y fixed,

$$\phi = \int x^2(1+y^3)\,dx = \frac{1}{3}x^3(1+y^3) + f(y)$$

Differentiating ϕ with respect to y and using the second equation,

$$x^3y^2 + f'(y) = y^2(1+x^3) \qquad \text{or} \qquad f'(y) = y^2$$

One more integration results in $f(y) = y^3/3 + c$, and the scalar field is given by

$$\phi(x, y) = \frac{1}{3}(x^3 + x^3y^3 + y^3) + c$$

Example: Conservation of energy

The work-energy theorem states that the work done on a mass by a force is equal to the change in the kinetic energy of the mass, or $\int_C \mathbf{F} \cdot d\mathbf{r} = T_f - T_i$, where the kinetic energy of a mass m moving with velocity v is given by $T = \frac{1}{2}m|v|^2$.

If $\mathbf{F}$ is a conservative vector field, then $\mathbf{F} = -\nabla V$, where $V = V(\mathbf{r})$ is the potential energy.

Using the gradient theorem, $T_f - T_i = -\int_C \nabla V \cdot d\mathbf{r} = V_i - V_f$, where V_i, V_f are the initial and final potential energies of the masses.

The sum of the of the kinetic and potential energy is conserved:

$$T_i + V_i = T_f + V_f$$

The Divergence Theorem

Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\oint_S \mathbf{F} \cdot dS = \iiint_E (\nabla \cdot \mathbf{F})\,dV = \iiint_E \operatorname{div}\mathbf{F}\,dV$$

That is, the integral of the divergence of a vector field over the enclosed volume is equal to the vector field's flux through the bounding surface.

It is often used to derive a continuity equation, which expresses the local conservation of some physical quantity.

Example: Let the scalar function $\rho(\mathbf{r}, t)$ be the mass density of a fluid at position $\mathbf{r}$ and time t, and $\mathbf{F}(\mathbf{r}, t)$ be the fluid velocity. Consider a small test volume V in the fluid flow and consider the change in the fluid mass M inside V.

The fluid mass M in V varies because of the mass flux through the surface S surrounding V, $\frac{dM}{dt} = -\oint_S \rho\mathbf{F} \cdot dS$.

Now the mass of the fluid is given in terms of the mass density by $M = \int_V \rho\,dV$, and application of the divergence theorem to the surface integral gives $\frac{d}{dt}\int_V \rho\,dV = -\int_V \nabla \cdot (\rho\mathbf{F})\,dV$.

Taking the time derivative inside the integral on the left, and combining the two integrals yields $\int_V (\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{F}))\,dV = 0$.

Since this integral vanishes for any test volume placed in the fluid, the integrand itself must be zero everywhere, resulting in the continuity equation $\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho\mathbf{F}) = 0$.

For an incompressible fluid, for which the mass density ρ is uniform and constant, the continuity equation reduces to $\nabla \cdot \mathbf{F} = 0$. A vector field with zero divergence is called **incompressible** or solenoidal.

Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane, and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Green's theorem is the counterpart of the FTC ($F(b)-F(a) = \int_a^b F'(x)\,dx$) for double integrals.

Stokes' Theorem

Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with a positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iiint_S (\nabla \times \mathbf{F}) \cdot dS = \iiint_S \operatorname{curl}\mathbf{F} \cdot dS$$

Stoke's theorem is the extension of Green's Theorem to three dimensions.

- With $\mathbf{F} = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = (\nabla \times \mathbf{F}) \cdot \mathbf{k}$, i.e. the curl.
- And with $dS = \mathbf{k}dS$, $P\,dx + Q\,dy = \mathbf{F} \cdot dr$

References

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