1 First-Order Differential Equations

1.1 Numerical Methods

Euler Method

A first-order differential equation $dy/dx = f(x,y)$ with initial conditions $y(x_0) = y_0$ provides the slope $f(x_0, y_0)$ of the tangent line to the solution curve $y = y(x)$ at the point (x_0, y_0).

With a small step size $\triangle x = x_1 - x_0$, the initial condition (x_0, y_0) can be marched forward to (x_1, y_1) along the tangent line using Euler's method:

$$y_1 = y_0 + \triangle x f(x_0, y_0)$$

This solution (x_1, y_1) then becomes the initial cndition and is marched forward to (x_2, y_2) along a newly determined tangent line with slope given by $f(x_1, y_1)$.

For small enough $\triangle x$, the numerical solution converges to the unique solution, when such a solution exists.

Runge-Kutta Method

The Euler method for solving $dy/dx = f(x,y)$ can be rewritten as a first-order Runge-Kutta method

$$k_1 = \triangle x f(x_n, y_n), \qquad y_{n+1} = y_n + k_1$$

or a (more accurate) second-order Runge-Kutta method

$$k_1 = \triangle x f(x_n, y_n)$$

$$k_2 = \triangle x f(x_n + \alpha \triangle x, y_n + \beta k_1)$$

$$y_{n+1} = y_n + a k_1 + b k_2$$

with constraints

$$a + b = 1, \qquad \alpha b = \beta b = \frac{1}{2}$$

1.2 Separable First-Order Equations

Definition

A first-order differential equation is separable if it can be written as a **separated equation**

$$g(y)\, dy = f(x)\, dx, \qquad y(x_0) = y_0$$

where $g(y)$ is independent of x and $f(x)$ is independent of y, and which can therefore be integrated over y and x.

Example: $y' + y^2 \sin(x) = 0, \quad y(0) = 1.$

$$\frac{dy}{dx} = -y^2 \sin(x) \Rightarrow \frac{dy}{y^2} = -\sin(x)\, dx \Rightarrow \int_1^y \frac{dy}{y^2} = -\int_0^x \sin(x)\, dx \Rightarrow -\frac{1}{y}\Big|_1^y = \cos(x)\Big|_0^x \Rightarrow 1 - \frac{1}{y} = \cos(x) - 1 \Rightarrow y = \frac{1}{2 - \cos(x)}$$

1.3 Linear First-Order Equations

Definition

A linear first-order differential equation with initial condition can be written in standard form as

$$\frac{dy}{dx} + p(x)y = g(x), \qquad y(x_0) = y_0$$

All such linear first-order equations can be integrated using an integrating factor μ:

1. Multiply both sides by the yet unknown function $\mu = \mu(x)$ so that $\mu(x)\left(\frac{dy}{dx} + p(x)y\right) = \mu(x)g(x)$
2. Require $\mu(x)$ to satisfy the differential equation $\mu(x)\left(\frac{dy}{dx} + p(x)y\right) = \frac{d}{dx}(\mu(x)y)$
3. Thus, $\frac{d}{dx}(\mu(x)y) = \mu(x)g(x)$. Using $y(x_0) = y_0$ and choosing $\mu(x_0) = 1$,
$$\int_{x_0}^x \frac{d}{dx}(\mu(x)y)\, dx = \int_{x_0}^x \mu(x)g(x)\, dx \quad \Rightarrow \quad \mu(x)y - y_0 = \int_{x_0}^x \mu(x)g(x)\, dx \quad \Rightarrow \quad y(x) = \frac{1}{\mu(x)}\left(y_0 + \int_{x_0}^x \mu(x)g(x)\, dx\right)$$
4. By the product rule, $\mu\frac{dy}{dx} + \mu p y = \frac{d\mu}{dx}y + \mu\frac{dy}{dx}$, which gives the separable differential equation
$$\frac{d\mu}{dx} = p(x)\mu, \qquad \mu(x_0) = 1 \qquad \text{which can be integrated to obtain} \qquad \mu(x) = e^{\int_{x_0}^x p(x)\, dx}$$
5. Combining the previous two steps solves the differential equation.

Example: Consider the inseparable linear equation $\frac{dy}{dx} + 2y = e^{-x}, \quad y(0) = \frac{3}{4}$. Let $p(x) = 2, g(x) = e^{-x}$. Then

$$\mu(x) = e^{\int_0^x 2\, dx} = e^{2x}, \qquad y(x) = e^{-2x}\left(\frac{3}{4} + \int_0^x e^{2x}e^{-x}\, dx\right) = e^{-2x}\left(\frac{3}{4} + (e^x - 1)\right) = e^{-x}\left(1 - \frac{1}{4}e^{-x}\right)$$

Definition

A nonlinear differential equation can be transformed to a linear differential equation by a **change of variables**.

Example: Consider the nonlinear differential equation $\frac{dx}{dt} = x(1-x)$.

Let $z = \frac{1}{x}$. Then

$$x = \frac{1}{z}, \qquad \frac{dx}{dt} = \frac{dx}{dz}\frac{dz}{dt} = -\frac{1}{z^2}\frac{dz}{dt}$$

Thus,

$$\frac{dx}{dt} = x(1-x) \quad \Rightarrow \quad -\frac{1}{z^2}\frac{dz}{dt} = \frac{1}{z}\Big(1-\frac{1}{z}\Big) \quad \Rightarrow \quad \frac{dz}{dt}+z=1$$

1.4 Applications

Compound Interest

Let $S(t)$ be the value of an investment at time t, r the annual interest rate compounded every time interval $\triangle t$, k the annual deposit or withdrawal amount, and suppose that a fixed amount is deposited (or withdrawn) after every time interval $\triangle t$. Then

$$S(t+\triangle t) = S(t) + (r\triangle t)S(t) + k\triangle t$$

This gives a differential equation:

$$\lim_{\triangle t\to 0}\frac{S(t+\triangle t)-S(t)}{\triangle t} = \frac{dS}{dt} = rS(t)+k$$

with initial condition $S(0)=S_0$, i.e. the initial capital; and which can be written in standard form $dS/dt - rS = k$, so that the integrating factor is given by

$$\mu(t) = e^{-rt}$$

This gives the solution, and shows that compounding results in the exponential growth of an investment:

$$S(t) = e^{rt}\Big(S_0 + \int_0^t ke^{-rt}\,dt\Big)$$

1.5 Modelling with Differential Equations

Modelling Cycle

1. Problem definition
2. Model definition, e.g. $dy/dt = f(t,y)$
3. Computation, i.e. $y(t) = ...$
4. Verification, i.e. as $t\to\infty$

Example: You want to breed rainbowfish to sell to pet stores. You start with a nice big aquarium and 30 fish, half of them male, half of them female. You want to predict the number of fish after a number of days, to see how many you can sell.

In this particular case, we have the balance equation:

$$\triangle P = P(t+\triangle t) - P(t) = 0.7P(t)\triangle t$$

Which results in the differential equation for the problem:

$$\frac{dP}{dt} = \lim_{\triangle t\to 0}\frac{\triangle P}{\triangle t} == 0.7P(t), \quad P(0)=30$$

Definition

A differential equation is an equation involving a derivative:

$$\frac{df}{dx} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{\triangle x\to 0}\frac{\triangle f}{\triangle x}$$

If you use only words to describe the differential equation you would say: "The derivative of the function equals a multiple of the function." So the solution to the differential equation should be a function with this property.

Example continued: Let $P(t) = ce^{kt}$.

Given the differential equation and its initial condition, $k = 0.7$, and $c = 30$. Thus, the solution is $P(t) = 30e^{0.7t}$. (This simplified model only considers a birth rate $b = 0.7$ but not a death rate d.)

Example continued: A more realistic model is given by a model with **bounded growth**:

$$\frac{dP}{dt} = 0.7P\Big(1-\frac{P}{750}\Big) - 20, P(0) = 30$$

The differential equation for the rainbowfish that we have now, could still be solved by hand. It would give you the analytical solution, which is exact. In practice, for a more complicated model, you would probably use a numerical method like Euler's Method to approximate the solution.

Definition

For the general differential equation $\frac{dy}{dt} = f(t,y)$, the n^{th} step of Euler's method is given by

$$y\big((n+1)\triangle t\big) = y(n\triangle t) + \triangle t f\big(t, y(n\triangle t)\big)$$

in which $\triangle t$ is some step you have to choose.

2 Homogeneous Linear Differential Equations

2.1 Numerical Methods

Most higher-order ODEs are usually solved numerically.

Definition

In physics, **Newton's dot notation** is used to represent time derivatives, and can be applied to any dependent variable that is a function of time:

$$x = f(t), \qquad \dot{x} = f(t, x), \qquad \ddot{x} = f(t, x, \dot{x})$$

Euler Method for Higher-Order ODEs

Consider a general second-order ODE given by $\ddot{x} = f(t, x, \dot{x})$. To solve numerically:

1. convert the second-order ODE to a pair of first-order ODEs, by defining $u = \dot{x}$. Then
   1. $\dot{x} = u$ gives the slope of the tangent line to the curve $x = x(t)$.
   2. $\dot{u} = f(t, x, u)$ gives the slope of the tangent line to the curve $u = u(t) = \dot{x}(t)$.

2. Beginning at the initial values $(x, u) = (x_0, u_0)$ at time $t = t_0$, move along the tangent lines to determine
   1. $x_1 = x_0 + \triangle t u_0$
   2. $u_1 = u_0 + \triangle t f(t_0, x_0, u_0)$

3. The values x_1, u_1 at time $t_1 = t_0 + \triangle t$ are then used as new initial values to march the solution forward to time $t_2 = t_1 + \triangle t$.

When a unique solution of the ODE exists, the numerical solution converges to this unique solution as $\triangle t \to 0$.

2.2 Theory

Principle of Superposition

Consider a homogeneous linear second-order ODE

$$\ddot{x} + p(t)\dot{x} + q(t)x = 0$$

with solutions $x = x_1(t), x = x_2(t)$.

Any linear combination of the solutions to the homogeneous linear second-order ODE is also a solution.

Proof:

$$\begin{aligned}
\ddot{x} + &p(t)\dot{x} + q(t)x \\
&= c_1\ddot{x}_1 + c_2\ddot{x}_2 + p(c_1\dot{x}_1 + c_2\dot{x}_2) + q(c_1 x_1 + c_2 x_2) \\
&= c_1(\ddot{x}_1 + p\dot{x}_1 + q x_1) + c_2(\ddot{x}_2 + p\dot{x}_2 + q x_2) \\
&= c_1 \times 0 + c_2 \times 0, \text{ since } x_1, x_2 \text{ are solutions} \\
&= 0
\end{aligned}$$

Theorem

Consider a homogeneous linear second-order ODE $\ddot{x} + p(t)\dot{x} + q(t)x = 0$ with general solution $x = c_1 x_1(t) + c_2 x_2(t)$ for solutions $x = x_1(t)$, $x = x_2(t)$, and initial conditions $x(t_0) = x_0$, $\dot{x}(t_0) = u_0$ so that

$$c_1 x_1(t_0) + c_2 x_2(t_0) = x_0, \qquad c_1 \dot{x}_1(t_0) + c_2 \dot{x}_2(t_0) = u_0$$

There exists a unique solution if the determinant of the resulting linear system, called the **Wronskian**, is non-zero:

$$W = \begin{vmatrix} x_1(t_0) & x_2(t_0) \\ \dot{x}_1(t_0) & \dot{x}_2(t_0) \end{vmatrix} = x_1(t_0)\dot{x}_2(t_0) - \dot{x}_1(t_0)x_2(t_0) \neq 0$$

If so, the solutions $x_1(t)$, $x_2(t)$ are said to be **linearly independent**. (They span a two-dimensional vector space.)

Example: Given a homogeneous linear second-order ODE with solutions $x_1(t) = \cos(\omega t)$, $x_2(t) = \sin(\omega t)$, $\omega \neq 0$, the Wronskian $W = (\cos\omega t)(\omega\cos\omega t) - (-\omega\sin\omega t)(\sin\omega t) = \omega \neq 0$ for all t.

Theorem

Consider a homogeneous linear second-order ODE with constant coefficients

$$a\ddot{x} + b\dot{x} + cx = 0$$

Because of the differential properties of the exponential function, a natural **ansatz**, or educated guess, for the form of the solution is $x = e^{rt}$, $\dot{x} = re^{rt}$, $\ddot{x} = r^2 e^{rt}$, where r is a constant to be determined, and which gives the **characteristic equation** of the ODE:

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0 \qquad \Rightarrow \qquad ar^2 + br + c = 0, \qquad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case 1: Distinct Real Roots - $b^2 - 4ac > 0$

When the roots of the characteristic equation are distinct and real, then the general solution to the second-order ODE can be written as a linear superposition of the two solutions $e^{r_1 t}, e^{r_2 t}$:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

The unknown constants c_1, c_2 can then be determined by the given initial conditions $x(t_0) = x_0, \dot{x}(t_0) = u_0$.

Example: $\ddot{x} + 5\dot{x} + 6x = 0$, $x(0) = 2$, $\dot{x}(0) = 3$

Ansatz $x = e^{rt}$ gives characteristic equation $r^2 + 5r + 6 = 0$ which factors to $(r+3)(r+2) = 0$.

The general solution to the ODE is therefore $x(t) = c_1 e^{-2t} + c_2 e^{-3t}$, and by differentiation $\dot{x}(t) = -2c_1 c^{-2t} - 3c_2 e^{3t}$.

Plugging in the initial conditions gives $c_1 + c_2 = 2$, $-2c_1 - 3c_2 = 3$ with solution $c_1 = 9$, $c_2 = -7$.

Therefore, the unique solution that satisfies the ODE and its initial conditions is

$$x(t) = 9e^{-2t} - 7e^{-3t} = 9e^{-2t}\left(1 - \frac{7}{9}e^{-t}\right)$$

Case 2: Complex-Conjugate Roots - $b^2 - 4ac < 0$

When the roots of the characteristic equation are complex conjugates, there are real numbers λ, μ s.t.

$$r = \lambda + i\mu, \quad \bar{r} = \lambda - i\mu, \text{ or equivalently, } z(t) = e^{\lambda t} e^{i\mu t}, \quad \bar{z}(t) = e^{\lambda t} e^{-i\mu t}$$

By the principle of linear superposition, any linear combination of $z, \bar{z}$ is also a solution, i.e.

$$x_1(t) = \mathrm{Re}(z) = e^{\lambda t}\cos(\mu t), \quad x_2(t) = \mathrm{Im}(z) = e^{\lambda t}\sin(\mu t), \qquad x(t) = e^{\lambda t}(a\cos(\mu t) + b\sin(\mu t))$$

The real part of the roots of the characteristic equation appears in the exponential term, the imaginary part appears in the cosine and sine.

Example: $\ddot{x} + \dot{x} + x = 0$, $x(0) = 1$, $\dot{x}(0) = 0$ with characteristic equation $r^2 + r + 1 = 0$ and roots $r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

The general solution to the ODE is therefore $x(t) = e^{-t/2}\left(a\cos\left(\frac{\sqrt{3}}{2}t\right) + b\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$.

The derivative is $\dot{x}(t) = -\frac{1}{2}e^{-t/2}\left(a\cos\left(\frac{\sqrt{3}}{2}t\right) + b\sin\left(\frac{\sqrt{3}}{2}t\right)\right) + \frac{\sqrt{3}}{2}e^{-t/2}\left(-a\sin\left(\frac{\sqrt{3}}{2}t\right) + b\cos\left(\frac{\sqrt{3}}{2}t\right)\right)$.

Plugging in the initial conditions gives $a = 1$, $-\frac{1}{2}a + \frac{\sqrt{3}}{2}b = 0$ with solution $a = 1$, $b = \sqrt{3}/3$.

Therefore, the unique solution that satisfies the ODE and its initial conditions is

$$x(t) = e^{-t/2}\left(\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

Case 3: Repeated Roots - $b^2 - 4ac = 0$

For the case of repeated roots, the second solution is t times the first solution:

$$x(t) = (c_1 + c_2 t)e^{rt}$$

where r is the repeated root.

Example: $\ddot{x} + 2\dot{x} + x = 0$, $x(0) = 1$, $\dot{x}(0) = 0$

The characteristic equation $r^2 + 2r + 1 = (r+1)^2 = 0$ has a repeated root $r = 1$.

The general solution to the ODE is therefore $x(t) = (c_1 + c_2 t)e^{-t}$, $\dot{x}(t) = (c_2 - c_1 - c_2 t)e^{-t}$.

Plugging in the initial conditions gives $c_1 = 1$, $c_2 - c_1 = 0$ with solution $c_1 = c_2 = 1$.

Therefore, the unique solution that satisfies the ODE and its initial conditions is

$$x(t) = (1 + t)e^{-t}$$

3 Inhomogeneous Linear Differential Equations

We now add an inhomogeneous term to the second-order ode with constant coefficients. The in-homogeneous term may be an exponential, a sine or cosine, or a polynomial. A general solution will be the sum of a homogeneous and particular solution.

3.1 Theory

Theorem

Consider an inhomogeneous linear second-order differential equation

$$\ddot{x} + p(t)\dot{x} + q(t)x = g(t),\ g(t) \neq 0,\ \text{with initial conditions } x(t_0) = x_0, \dot{x}(t_0) = u_0$$

There is a three-step solution method:

1. Solve the homogeneous equation $\ddot{x} + p(t)\dot{x} + q(t)x = 0$ for two independent solutions $x = x_1(t), x = x_2(t)$ and form a linear superposition to obtain a *homogeneous solution*

$$x_h(t) = c_1 x_1(t) + c_2 x_2(t)$$

2. Find a *particular solution* $x = x_p(t)$ that solves the inhomogeneous equation.
3. Write the *general solution* of the inhomogeneous equation as the sum of the homogeneous and particular solutions, $x(t) = x_h(t) + x_p(t)$, and apply the initial conditions to determine c_1, c_2.

Note: The two free constants in x_h can be used to satisfy the two initial conditions because the sum of the homogeneous and particular soltuions solve the ODE, by linearity:

$$\begin{aligned}
\ddot{x} + p\dot{x} + qx &= \frac{d^2}{dt^2}(x_h + x_p) + p\frac{d}{dt}(x_h + x_p) + q(x_h + x_p) \\
&= (\ddot{x}_h + p\dot{x}_h + qx_h) + (\ddot{x}_p + p\dot{x}_p + qx_p) \\
&= 0 + g = g
\end{aligned}$$

3.2 Particular Solutions for Exponential, Sine / Cosine and Polynomial Inhomogeneous Terms

Example: $\ddot{x} - 3\dot{x} - 4x = 3e^{2t}, x(0) = 1, \dot{x}(0) = 0$

1. The characteristic equation of the homogeneous equation is $r^2 - 3r - 4 = (r-4)(r+1) = 0$ so that $x_h(t) = c_1 e^{4t} + c_2 e^{-t}$.
2. For the inhomogeneous solution, an Ansatz such that the exponential function cancels, $x(t) = Ae^{2t}$, where A is an undetermined coefficient, gives $4A - 6A - 4A = 3$ and consequently $A = -1/2$. Obtaining a solution for A independent of t justifies the Ansatz.
3. Plugging the initial conditions in $x(t) = x_h(t) + x_p(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2}e^{2t}$, $\quad \dot{x}(t) = \dot{x}_h(t) + \dot{x}_p(t) = 4c_1 e^{4t} - c_2 e^{-t} - e^{2t}$ gives $c_1 + c_2 = 3/2, \quad 4c_1 - c_2 = 1$ with solution $c_1 = 1/2, c_2 = 1$.

The solution is

$$x(t) = \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t} - e^{-t} = \frac{1}{2}e^{4t}\left(1 - e^{-2t} + 2e^{-5t}\right)$$

Example: $\ddot{x} + \dot{x} - 2x = t^2$

The Ansatz should be a polynomial in t of the same order as the inhomogeneous term, i.e. $x(t) = At^2 + Bt + C$.

This gives $2A + (2At + B) - 2(At^2 + Bt + C) = t^2$, or $-2At^2 + 2(A - B)t + (2A + B - 2C)t^0 = t^2$.

Equating powers of t, $-2A = 1, \quad 2(A - B) = 0, \quad 2A + B - 2C = 0$, gives $A = -\frac{1}{2}, \quad B = -\frac{1}{2}, C = -\frac{3}{4}$.

The particular solution is

$$x_p(t) = -\frac{1}{2}t^2 - \frac{1}{2}t - \frac{3}{4}$$

Example: $\ddot{x} - 3\dot{x} - 4x = 2\sin(t)$

Approach 1: Ansatz $x(t) = A\cos(t) + B\sin(t)$

The cosine term is required because it is the derivative of sine.

Substituting in the differential equation gives $(-A\cos(t)) - 3(-A\sin(t) + B\cos(t)) - 4(A\cos(t) + B\sin(t)) = 2\sin(t)$.

Regrouping terms gives $-(5A + 3B)\cos(t) + (3A - 5B)\sin(t) = 2\sin(t)$.

This equation is valid for all t, and in particular for $t = 0, t = \pi/2$ for which the sine and cosine functions vanish. For these two values of t, $5A + 3B = 0, \quad 3A - 5B = 2$, which gives $A = 3/17, B = -5/17$.

The particular solution is

$$x_p = \frac{1}{17}(3\cos(t) - 5\sin(t))$$

Approach 2: Converting the sine inhomogeneous term to an exponential term, given the relation $e^{it} = \cos(t) + i\sin(t)$.

That is, sine is the imaginary part of complex function $z = z(t)$: $\sin(t) = \mathrm{Im}\{e^{it}\}$. Therefore, $\ddot{z} - 3\dot{z} - 4z = 2e^{it}$, where $x = \mathrm{Im}\{z\}$ satisfies the original differential equation for x.

Substituting the Ansatz $z(t) = Ce^{it}$, where C is a complex constant, and using the fact that $i^2 = -1$, gives $-C - 3iC - 4C = 2$ with solution $C = \frac{-2}{5+3i} = \frac{-5+3i}{17}$.

$$\begin{aligned}
x_p = \mathrm{Im}\{z_p\} &= \mathrm{Im}\left\{\frac{1}{17}(-5 + 3i)(\cos(t) + i\sin(t))\right\} \\
&= \frac{1}{17}(3\cos(t) - 5\sin(t))
\end{aligned}$$

4 The Laplace Transform and Series Solution Methods

4.1 The Laplace Transform Method

Definition

The **Laplace transform** of a function $f(t)$, denoted by $F(s) = \mathcal{L}\{f(t)\}$, is defined by the integral

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

The values of s may be restricted to ensure convergence.

Theorem

There is a one-to-one correspondence between functions and Laplace transforms.

Theorem

The Laplace transform is a linear transformation.

Proof:

$$\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} = \int_0^\infty e^{-st}(c_1 f_1(t) + c_2 f_2(t))\, dt$$
$$= c_1 \int_0^\infty e^{-st} f_1(t)\, dt + c_2 \int_0^\infty e^{-st} f_2(t)\, dt$$
$$= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}$$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
$e^{at} f(t)$	$F(s-a)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
$\sin(bt)$	$\frac{b}{s^2+b^2}$
$\sinh(bt)$	$\frac{b}{s^2-b^2}$
$\cos(bt)$	$\frac{s}{s^2+b^2}$
$\cosh(bt)$	$\frac{s}{s^2-b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
$t\sin(bt)$	$\frac{2bs}{(s^2+b^2)^2}$
$t\cos(bt)$	$\frac{s^2-b^2}{(s^2+b^2)^2}$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$u_c(t) f(t-c)$	$e^{-cs} F(s)$
$\delta(t-c)$	e^{-cs}
$\dot{x}(t)$	$sX(s) - x(0)$
$\ddot{x}(t)$	$s^2 X(s)$ $-sx(0)$ $-\dot{x}(0)$

Laplace Transform Method for a Constant-Coefficient ODE

Consider the inhomogeneous constant-coefficient second-order differential equation

$$a\ddot{x} + b\dot{x} + cx = g(t), \quad x(0) = x_0, \quad \dot{x}(0) = u_0$$

which by linearity can be Laplace transformed s.t.

$$a\mathcal{L}\{\ddot{x}\} + b\mathcal{L}\{\dot{x}\} + c\mathcal{L}\{x\} = \mathcal{L}\{g\}$$

Let $X(s) = \mathcal{L}\{x(t)\}, G(s) = \mathcal{L}\{g(t)\}$. Then, by parts,

$$\int_0^\infty e^{-st}\dot{x}\, dt = xe^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st} x\, dt = sX(s) - x_0$$

$$\int_0^\infty e^{-st}\ddot{x}\, dt = \dot{x}e^{-st}\Big|_0^\infty + s\int_0^\infty e^{-st}\dot{x}\, dt = -u_0 + s(sX(s) - x_0) = s^2 X(s) - sx_0 - u_0$$

The resulting expression for the differential equation

$$a(s^2 X - sx_0 - u_0) + b(sX - x_0) + cX = G$$

is of a form that can then be solved by taking the inverse Laplace transform of $X = X(s)$ to obtain $x = x(t)$.

Example: $\ddot{x} + x = \sin 2t, x(0) = 2, \dot{x}(0) = 1$

Taking the Laplace transform of both sides, $s^2 X(s) - 2s - 1 + X(s) = \frac{2}{s^2+4}$. Thus, $X(s) = \frac{2s+1}{s^2+1} + \frac{2}{(s^2+1)(s^2+4)}$.

To determine the inverse Laplace transform from the table, perform a partial fraction expansion of: $\frac{2}{(s^2+1)(s^2+4)} = \frac{as+b}{s^2+1} + \frac{cs+d}{s^2+4}$.

Therefore, $a = c = 0, b = 2/3, d = -b$, and $X(s) = \frac{2s+1}{s^2+1} + \frac{2/3}{x^2+1} - \frac{2/3}{s^2+4} = \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$.

Taking the inverse Laplace transforms of the three terms separately, where the values in the table are $b = 1$ in the first two terms, and $b = 2$ in the third term:

$$x(t) = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$

4.2 The Heaviside Step Function and Dirac Delta Function

Definition

The Heaviside or unit step function, denoted here by $u_c(t)$, is zero for $t < c$ and one for $t \geq c$:

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

Definition

The Laplace transform of the Heaviside function is determined by integration:

$$\mathcal{L}\{u_c(t)\} = \int_0^\infty e^{-st} u_c(t)\, dt = \frac{e^{-cs}}{s}$$

Definition

The Heaviside function be used to represent a translation of a function $f(t)$ a distance c in the positive t direction:

$$u_c(t)f(t-c) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}$$

Definition

The Laplace transform is

$$\mathcal{L}\{u_c(t)f(t-c)\} = \int_0^\infty e^{-st} u_c(t) f(t-c)\, dt = e^{-cs}F(s)$$

That is, the translation of $f(t)$ a distance c in the positive t direction corresponds to the multiplication of $F(s)$ by the exponential e^{-cs}.

Theorem

Piecewise-defined inhomogeneous terms can be modelled using Heaviside functions.

Example: Consider the general case of a piecewise function defined on two intervals:

$$f(t) = \begin{cases} f_1(t), & \text{if } t < c \\ f_2(t), & \text{if } t \geq c \end{cases}$$

Using the Heaviside function u_c, the function $f(t)$ can be written in a single line as

$$f(t) = f_1(t) + \big(f_2(t) - f_1(t)\big)u_c(t)$$

Definition

The Dirac delta function, denoted as $\delta(t)$, is zero everywhere except at $t = 0$, at which it is infinite in such a way that the integral is one; s.t. for any function $f(t)$,

$$\int_{-\infty}^{\infty} f(t)\delta(t)\, dt = f(0)$$

Definition

The shifted Dirac delta function can be written as a limit:

$$\delta(t-c) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon}\big(u_{c-\epsilon}(t) - u_{c+\epsilon}(t)\big)$$

The integral of this function is one, independent of the value of ϵ.

Theorem

The Laplace transform of the Dirac delta function is found by integration using the definition of the delta function. With $c > 0$,

$$\mathcal{L}\{\delta(t-c)\} = \int_0^\infty e^{-st}\delta(t-c)\, dt = e^{-cs}$$

Example: Solution of a discontinuous inhomogeneous term: $\ddot{x} + 3\dot{x} + 2x = 1 - u_1(t), x(0) = \dot{x}(0) = 0$

The inhomogeneous term is a step-down function, from one to zero.

Taking the Laplace transform, $s^2X(s) + 3sX(s) + 2X(s) = \frac{1}{s}(1 - e^{-s})$ with solution for $X = X(s)$ given by $X(s) = \frac{1-e^{-s}}{s(s+1)(s+2)}$.

Defining $F(s) = \frac{1}{s(s+1)(s+2)}$, the inverse Laplace transform of $X(s)$ can be written as $x(t) = f(t) - u_1(t)f(t-1)$, where $f(t)$ is the inverse Laplace transform of $F(s)$.

To determine $f(t)$, the partial fraction expansion of $F(s)$ is $\frac{1}{s(s+1)(s+2)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+2}$ with $a = 1/2, b = -1, c- = 1/2$.

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2}\mathcal{L}^{-1}\Big\{\frac{1}{s}\Big\} - \mathcal{L}^{-1}\Big\{\frac{1}{s+1}\Big\} + \frac{1}{2}\mathcal{L}^{-1}\Big\{\frac{1}{s+2}\Big\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

$$x(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} - u_1(t)\Big(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\Big)$$

4.3 The Series Solution Method

Example: $y'' + y = 0$

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \qquad y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} n a_n x^{n-1} = 0$$

$$\sum_{n=0}^{\infty} \big((n+2)(n+1) a_{n+2} + a_n\big) x^n = 0$$

For the equality to hold, the coefficient of each power of x must vanish separately. Therefore,

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}, n = 0, 1, 2, \ldots$$

Even and odd coefficients decouple. Thus, there are two independent sequences:

$$a_0, \quad a_2 = -\frac{1}{2} a_0, \quad a_4 = -\frac{1}{d \cdot 3} a_2 = \frac{1}{4!} a_0, \quad \ldots$$

$$a_1, \quad a_3 = -\frac{1}{3 \cdot 2} a_1, \quad a_5 = -\frac{1}{5 \cdot 4} a_3 = \frac{1}{5!} a_1, \quad \ldots$$

By the principle of superposition, the general is

$$\begin{aligned} y(x) &= a_0 \Big(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\Big) + a_1 \Big(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\Big) \\ &= a_0 \cos x + a_1 \sin x \end{aligned}$$

5 Systems of Differential Equations

More than one dependent variables $x_1, x_2, \dots$ give rise to a system of differential equations.

5.1 Eigenvalues and Eigenvectors

The eigenvalue problem for an $n \times n$ matrix A is given by

$$A\mathbf{x} = \lambda x \quad \Longleftrightarrow \quad (A - \lambda I)\mathbf{x} = \mathbf{0}$$

where the scalar λ is called the eigenvalue and the $n \times 1$ column vector x is called the eigenvector.

When A is a 2×2 matrix, then

$$A - \lambda I = \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix}$$

A solution other than $\mathbf{x} = \mathbf{0}$ of the eigenvalue equation exists provided

$$\det(A - \lambda I) = 0$$

This equation is called the characteristic equation of A, and is given by

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = 0$$

The eigenvalues can be real and distinct, complex conjugates, or repeated.

After determining an eigenvalue, say $\lambda = \lambda_1$, the corresponding eigenvector $\mathbf{v}_1$ can be found by solving

$$(A - \lambda_1 I)\mathbf{v}_1 = \mathbf{0}$$

5.2 Systems of First-Order Linear Ordinary Differential Equations

Definition

A system of homogeneous linear differential equations with constant coefficients

$$\dot{x}_1 = ax_1 + bx_2, \quad \dot{x}_2 = cx_1 + dx_2$$

can be written in matrix form as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \text{ or } \dot{\mathbf{x}} = A\mathbf{x}$$

Example: $\dot{x}_1 = x_1 + x_2, \dot{x}_2 = 4x_1 + x_2$

With the ansatz $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$,

$$\det(A - \lambda I) = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$$

and the eigenvalues and eigenvectors are

$$\lambda_1 = -1, \quad \mathbf{v} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \quad \lambda_2 = 3, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

By the principle of superposition,

$$x(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

or explicitly writing out the components,

$$x_1(t) = c_1 e^{-t} + c_2 e^{3t}, \quad x_2(t) = -2c_1 e^{-t} + 2c_2 e^{3t}$$

Theorem

An ansatz $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$, where $\mathbf{v}$ and λ are independent of t and $\mathbf{v}$ is a column matrix, s.t. $\lambda \mathbf{v}e^{\lambda t} = A\mathbf{v}e^{\lambda t}$ gives the eigenvalue problem

$$A\mathbf{v} = \lambda \mathbf{v}$$

with characteristic equation $\det(A - \lambda I) = \lambda^2 - (a+d)\lambda + (ad - bc) = 0$.

Example: $\dot{x}_1 = -\frac{1}{2}x_1 + x_2, \dot{x}_2 = -x_1 - \frac{1}{2}x_2$

With the ansatz $\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$, the characteristic equation is $\det(A - \lambda I) = \lambda^2 + \lambda + \frac{5}{4} = 0$, which has **complex-conjugate roots**: $\lambda = -\frac{1}{2} + i, \quad \bar{\lambda} = -\frac{1}{2} - i$.

We can form a linear combination of the two complex eigenvectors $\mathbf{v}e^{\lambda t}, \bar{\mathbf{v}}e^{\bar{\lambda} t}$ to construct two independent real solutions:

$$\mathrm{Re}\{\mathbf{v}e^{\lambda t}\} = \mathrm{Re}\{\begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-\frac{1}{2}+i)t}\} = e^{-t/2}\begin{pmatrix} \cos t & -\sin t \end{pmatrix}$$

$$\mathrm{Im}\{\mathbf{v}e^{\lambda t}\} = \mathrm{Im}\{\begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-\frac{1}{2}+i)t}\} = e^{-t/2}\begin{pmatrix} \sin t & \cos t \end{pmatrix}$$

Taking a linear superposition of these two real solutions,

$$\mathbf{x}(t) = e^{-t/2}\left(A\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + B\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right)$$

5.3 Phase Portraits

Definition

The solution of two first-order differential equations for x_1 and x_2 can be visualised by drawing a **phase portrait**, with x-axis x_1 and y-axis x_2.

Each curve drawn on the phase portrait corresponds to a different initial condition, and can be viewed as the trajectory of a particle at position (x_1, x_2) moving with a velocity given by $(\dot{x}_1, \dot{x}_2)$.

Definition

If there are two distinct real eigenvalues of the same sign, the fixed point is a **node**.

1. When the eigenvalues are both negative, the fixed point is a stable node.
2. When the eigenvalues are both positive, the fixed point is an unstable node.
3. If the eigenvalues have opposite sign, the fixed point is a saddle point.

Example: Consider the differential equations given by $\dot{x}_1 = -3x_1 + \sqrt{2}x_2, \dot{x}_2 = \sqrt{2}x_1 - 2x_2$. This system has eigenvalues and eigenvectors $\lambda_1 = -4, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -\sqrt{2}/2 \end{pmatrix}, \lambda_2 = -1, \mathbf{v}_2 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$.

Because $\lambda_1, \lambda_2 < 0$, both exponential solutions for $\mathbf{x} = \mathbf{x}(t)$ decay in time and $\mathbf{x} \to (0,0)$ as $t \to \infty$.

The node is stable.

Example: Consider the differential equations given by $\dot{x}_1 = x_1 + x_2, \dot{x}_2 = 4x_1 + x_2$. This system has eigenvalues and eigenvectors $\lambda_1 = -1, \mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \lambda_2 = 3, \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Because $\lambda_1 < 0$, trajectories approach the fixed point along the direction of the first eigenvector, and because $\lambda_2 > 0$, trajectories move away from the fixed point along the direction of the second eigenvector.

Ultimately, a saddle point is an unstable equilibrium because for any initial conditions such that $c_2 \neq 0, |x(t)| \to \infty$ as $t \to \infty$.

The node is stable.

Definition

For a 2×2 system $\dot{\mathbf{x}} = A\mathbf{x}$, the point $\mathbf{x} = (0,0)$ is called an **equilibrium point**, or fixed point, of the system.

If $\mathbf{x}$ is at the fixed point initially, then $\mathbf{x}$ remains there for all time because $\dot{\mathbf{x}} = \mathbf{0}$ at the fixed point.

Definition

If there are complex-conjugate eigenvalues, the fixed point is a **spiral**.

1. If the real part of the eigenvalues is negative, the solution decays exponentially and the fixed point corresponds to a stable spiral.
2. If the real part of the eigenvalues is positive, the solution grows exponentially and the fixed point corresponds to an unstable spiral.
3. Alternatively, a spiral may wind around the fixed point clockwise or anticlockwise.

Example: Consider the system of differential equations given by $x_1 = -\frac{1}{2}x_1 + x_2, \quad x_2 = -x_1 - \frac{1}{2}x_2$. This system has complex eigenvalue and eigenvector

$$\lambda = -\frac{1}{2} + i, \mathbf{v} = \begin{pmatrix} 1 \\ i \end{pmatrix},$$

and their complex conjugates.

The general solution is written as

$$\mathbf{x}(t) = e^{-t/2}\left[A\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + B\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right]$$

The trajectories in the phase portrait are spirals centred at the fixed point.

If $\mathrm{Re}(\lambda) > 0$, the trajectories spiral out; if $\mathrm{Re}(\lambda) < 0$, they spiral in.

The spirals around the fixed point may be clockwise or counterclockwise, depending on the governing equations.

Here, since $\mathrm{Re}(\lambda) = -1/2 < 0$, the trajectories spiral into the origin.

To determine whether the spiral is clockwise or counterclockwise, examine the time derivatives at the point $(x_1, x_2) = (0, 1)$.

At this point in the phase space, $(x_1, x_2) = (1, -1/2)$, and a particle on this trajectory moves to the right and downward, indicating a clockwise spiral.

5.4 Normal Modes

Definition — Theorem

Description: Text

$$\mathbf{v} \cdot \mathbf{w} = 0 \iff \alpha = \frac{\pi}{2} \iff \mathbf{v} \perp \mathbf{w}$$

5.5 Modelling with Differential Equations

Definition

A solution $P(t) = P_e$ which neither increases nor decreases is an **equilibrium solution** of the differential equation. Because an equilibrium solution does not change, it has the property

$$\frac{dP}{dt} = 0$$

In other words, the equilibrium solution is constant in time.

Definition

We call an equilibrium point **stable** if any initial value close to the equilibrium point gives solutions that always remain close to the equilibrium point.

Any equilibrium point which is not stable we call **unstable**, so there is at least one initial value close to the equilibrium which will give a solution that moves away from the equilibrium point.

Definition

For the general differential equation $\frac{d\vec{X}}{dt} = \vec{F}(t, \vec{X})$, the n^{th} step of Euler's method is given by

$$\vec{X}\big((n+1)\triangle t\big) = \vec{X}(n\triangle t) + \triangle t\vec{F}\big(t, \vec{X}(n\triangle t)\big)$$

in which $\triangle t$ is some step you have to choose.

Definition

An equilibrium point of a system of differential equations $\frac{d\vec{X}}{dt} = \vec{F}(t, \vec{X})$ is a point $\vec{X}_0$ where

$$\frac{d\vec{X}_0}{dt} = \vec{F}(t, \vec{X}_0) = \vec{0}$$

- An equilibrium point $\vec{X}_0$ is called a saddle point if the Jacobian matrix $J(\vec{X}_0)$ has one negative and one positive eigenvalue.
- An equilibrium point $\vec{X}_0$ is called a stable node if the Jacobian matrix $J(\vec{X}_0)$ has two negative eigenvalues: all solutions that start near the equilibrium point stay near the equilibrium point.
- An equilibrium point $\vec{X}_0$ is called an unstable node if the Jacobian matrix $J(\vec{X}_0)$ has two positive eigenvalues: all solutions that start near the equilibrium point stay near the equilibrium point.
- An equilibrium point $\vec{X}_0$ is called a stable spiral point if the Jacobian matrix $J(\vec{X}_0)$ has two complex eigenvalues $\lambda = a \pm bi$ with negative real parts: $a < 0$.
- An equilibrium point $\vec{X}_0$ is called an unstable spiral point if the Jacobian matrix $J(\vec{X}_0)$ has two complex eigenvalues $\lambda = a \pm bi$ with positive real parts: $a > 0$.
- An equilibrium point $\vec{X}_0$ is called a circle point if the Jacobian matrix $J(\vec{X}_0)$ has two complex eigenvalues with zero real parts: $\lambda = \pm bi$.

6 Partial Differential Equations

Differential equations with more than one independent variables $x, y, \ldots$ s.t. $f = f(x, y, \ldots)$ are partial differential equations.

6.1 Fourier Series

Definition

A periodic function $f(x)$ with period $2L$ can be represented as a Fourier series in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right.$$

Definition — Theorem

Description: Text

$$\mathbf{v} \cdot \mathbf{w} = 0 \iff \alpha = \frac{\pi}{2} \iff \mathbf{v} \perp \mathbf{w}$$

6.2 The Diffusion Equation

Definition — Theorem

Description: Text

$$\mathbf{v} \cdot \mathbf{w} = 0 \iff \alpha = \frac{\pi}{2} \iff \mathbf{v} \perp \mathbf{w}$$

References

[1] The Hong Kong University of Science and Technology. "Differential Equations for Engineers". In: (2024). URL: https://www.coursera.org/learn/differential-equations-engineers/.

[2] TU Delft. "Modelling with Differential Equations". In: (2024). URL: https://online-learning.tudelft.nl/courses/modelling-with-differential-equations/.