1 Relations

Definition

A **relation** on a set X is a property of an ordered pair of elements of X which can be true or false.

Example: $<$ is a relation on the set of natural numbers: if a and b are natural numbers then $a < b$ is either true or false.

Definition

Properties of relations: Let $\sim$ be a relation on a set X.

- $\sim$ is called **symmetric** if for any $x, y \in X$ if $x \sim y$ then $y \sim x$.
- $\sim$ is called **reflexive** if for any $x \in X$ we have $x \sim x$.
- $\sim$ is called **transitive** if for any $x, y, z \in X$ if $x \sim y$ and $y \sim z$ then $x \sim z$.
- $\sim$ is called an **equivalence relation** if it is reflexive, symmetric and transitive.

Definition

Let $\sim$ be an equivalence relation on a set X, and let $x \in X$. The **equivalence class** of x, written $[x]$ or $[x]_\sim$, is

$$[x] = \{y \in X | y \sim x\}$$

Theorem

Let $\sim$ be an equivalence relation on a set X. Then

- Every $x \in X$ belongs to some equivalence class.
- If two equivalence classes classes are not disjoint, then they are equal.

2 Functions

Definition

Let $f : X \to Y$ be a function. The set X is called the domain of f. The set Y is called the co-domain of f.

Definition

Let $f : X \to Y$ be a function.

- f is called **injective** or **one-to-one** if for all $a, b \in X$, if $f(a) = f(b)$ then $a = b$.
- The **image** of f, written $\operatorname{im} f$, is $\{f(x) : x \in X\}$.
- f is called **surjective** or **onto** if $\operatorname{im} f = Y$.
- f is called a **bijection** if it is injective and surjective.

Definition

Let $f : X \to Y$ and $g : Y \to Z$ be functions. The **composition** of g and f, written $g \circ f$, is the function $g \circ f : X \to Z$ such that $(g \circ f)(x) = g(f(x))$.

NB: Composition only makes sense when the co-domain of f is the same as the domain of g.

Theorem

Function composition is associative.

If $f : X \to Y, g : Y \to Z, h : Z \to W$, then

$$h \circ (g \circ f) = (h \circ g) \circ f$$

The reason this is true is because both sides send an input $x \in X$ to the output $h(g(f(x)))$.

Definition

The **identity function** id_X does nothing: it is defined by $\mathrm{id}_X(x) = x$ for all $x \in X$.

Definition

Let $f : X \to Y$ and $g : Y \to X$ be functions. Then

- g is a **left inverse** to f, and f is a **right inverse** to g, if $g \circ f = \mathrm{im}_x$.
- f is **invertible** if there is a function $h : Y \to X$ such that $f \circ h = \mathrm{id}_Y$ and $h \circ f = \mathrm{id}_X$.
- If f is invertible, then there is one and only one function which is a left and right inverse to f - its inverse f^{-1}.

Theorem

Let $f : X \to Y$ be a function.

- f has a left inverse if and only if it is injective.
- f has a right inverse if and only if it is surjective.
- f is invertible if and only if it is a bijection.

Theorem

If functions $f_1, f_2, \ldots, f_n$ are invertible and the composition $f_1 \circ f_2 \circ \cdots \circ f_n$ makes sense, then it is invertible with inverse $f_n^{-1} \circ f_{n-1}^{-1} \circ \cdots \circ f_1^{-1}$.

3 Permutations

Definition — Theorem

Description: Text

$$\mathbf{v}\cdot\mathbf{w}=0 \iff \alpha=\frac{\pi}{2} \iff \mathbf{v}\perp\mathbf{w}$$

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4 Groups

Definition — Theorem

Description: Text

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4.1 The Symmetric Group

Definition — Theorem

Description: Text

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4.2 Subgroups

Definition — Theorem

Description: Text

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4.3 Cosets and Lagrange's Theorem

Definition — Theorem

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4.4 The Dihedral Groups

Definition — Theorem

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4.5 Homomorphisms and Isomorphisms

Definition — Theorem

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5 Categories

Definition — Theorem

Description: Text

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References

[1] Matthew Towers, UCL. *MATH0007: Algebra for Joint Honours Students*. 2020. URL: https://www.ucl.ac.uk/~ucahmto/0007/_book/.