HOMEWORK 13

LEANDRO RIBEIRO

Proposition 10.27. *Given any $r \in \mathbb{R}_{>0}$, the number $\sqrt{r}$ is unique in the sense that, if x is a positive real number such that $x^2 = r$, then $x = \sqrt{r}$.*

Proof. Suppose u and v are such that $u^2 = r$ and $v^2 = r$. Let $w = sup\{x \in \mathbb{R} | x^2 < r\}$. We will show u = w = v. For any $x \in A := \{x \in \mathbb{R} | x^2 < r\}$ we see that $x^2 < u^2 = r$. If $x < 0$, then clearly $x < u$. If $x \geq 0$, then propositio 10.5 ensures that $x < u$. Since w is the least upper bound of A, we conclude that $w \leq u$. But $w^2 = r = u^2$. By proposition 10.5 again it must be the case that w = u. Similarly, v = w. □

Proposition 11.12. *If $r \in \mathbb{N}$ is not a perfect square, then $\sqrt{r}$ is irrational.*

Let us prove the contrapositive. Suppose $\sqrt{r}$ is rational. We aregue r is a perfect square. Suppose $\sqrt{r} = \frac{m}{n}$ with m and n in lowest terms, i.e. the $gcd(m, n) = 1$. Thus, $r = \frac{m^2}{n^2}$ and $rn^2 = m^2$. If p is prime and $k \geq 0$ is such that $p^k | n$ then $p^{2k} | n^2$. Since $rn^2 = m^2$, we conclude that $p^{2k} | m^2$. Thus, $p^k | m$. Since $n = p_1^{k_1} \dots p_l^{k_l}$, take $m = q_1^{a_1} \dots q_j^{a_j}$ with $q_i = p_i$ and $k_i \leq a_i$. $m = p_1^{k_1} \dots p_l^{k_l} \cdot c$ with $c = q_{l+1}^{a_{l+1}} \dots q_j^{a_j}$. Thus, $m = n \cdot c$. We conclude $n | m$, however $gcd(m, n) = 1$, thus $n = 1$. Hence, $\sqrt{r} = m$, and r is a perfect square.

Proof. □

Proposition 11.4. *Given a rational number $r \in \mathbb{Q}$, we can always write it as $r = \frac{m}{n}$, where $n > 0$ and m and n do not have any common factors.*

Proof. Suppose toward a contradiction there were $m, n \in \mathbb{Z}_{>0}$ such that the fraction $\frac{m}{n}$ cannot be written in lowest terms. Let C be the the set of positive integers that are numerators of such fractions. Then $m \in C$, so C is not empty. Therefore, by the well-ordering principle there must be a smallest integer m in C. There is an integer $n_0 > 0$ such that the

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fraction $\frac{m_0}{n_0}$ cannot be written in lowest terms by our definition of C. This means that m_0 and n_0 must have a common factor $p > 1$, however $(\frac{m_0}{p})/(\frac{n_0}{p}) = \frac{m_0}{n_0}$. Any way of expressing the left hand fraction in lowest terms would also work for $\frac{m_0}{n_0}$, which implies the fraction $(\frac{m_0}{p})/(\frac{n_0}{p})$ cannot be written in lowest terms either. By our definition of C, $\frac{m_0}{p}$, is in C, but $\frac{m_0}{p} < m_0$, which contradicts that m_0 is the smallest element of C. $\square$

proposition 11.13. *Let m and n be nonzero integers. Then $\frac{m}{n}\sqrt{2}$ is irrational.*

Proof. Suppose toward a contradiction $\frac{m}{n}\sqrt{2} \in \mathbb{Q}$. From our assumption, we may take $q, p \in \mathbb{Z}$ such that $\frac{m}{n}\sqrt{2} = \frac{p}{q}$. Multiplying by $\frac{n}{m}$ on both sides, we have $\frac{n}{m}\frac{m}{n}\sqrt{2} = \sqrt{2} = \frac{n}{m}\frac{p}{q} = \frac{np}{mq}$. Let $a = np$ and $b = mq$. Let's assume, by proposition 11.4, $\frac{a}{b}$ is in lowest terms. This means $gcd(a, b) = 1$. We see that $\frac{(a)^2}{(b)^2} = 2$, so $a^2 = 2b^2$. We deduce that a is even. $2|a \cdot a$, and by Euclid's lemma, $2|a$. Since $2|a$, we see $a = 2 \cdot k$. Hence $(2k)^2 = 2b^2 = 4k^2$. Therefore, $2k^2 = b^2$. As with a, we conclude that $2|b$. This is absurd since we assumed $gcd(a, b) = 1$. $\square$

Sources.

https://math.stackexchange.com/questions/463342/prove-that-theres-no-fractions-that-cant-be-written-in-lowest-term-with-well-o