HOMEWORK 6

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Proposition 4.18. *Let $(x_j)_{j=1}^{\infty} and (y_j)_{j=1}^{\infty}$ be sequences in $\mathbb{Z}$ such that $x_j \leq y_j$ for all $j \in \mathbb{N}$. Then for all $k \in \mathbb{N}$, $\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j$.*

Proof. Let P(k) be the statement, "$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j$." Let's observe P(1).

Base. $\sum_{j=1}^{1} x_j \leq \sum_{j=1}^{1} y_j$. $x_1 \leq y_1$, thus P(1) holds.

Successor. Assume P(n) holds. That is, $\sum_{j=1}^{n} x_j \leq \sum_{j=1}^{n} y_j$. Consider $\sum_{j=1}^{n+1} x_j$ and $\sum_{j=1}^{n+1} y_j$. By definition, we can rewrite this as $\sum_{j=1}^{n} x_j + x_{n+1}$ and $\sum_{j=1}^{n} y_j + y_{n+1}$. By induction, we know $\sum_{j=1}^{n} x_j \leq \sum_{j=1}^{n} y_j$, and we know $x_{n+1} \leq y_{n+1}$, thus by proposition 2.7(ii), $\sum_{j=1}^{n+1} x_j \leq \sum_{j=1}^{n+1} y_j$. We have proven that the proposition holds by induction. □

Proposition 4.30. *For all k,m $\in \mathbb{N}$, where $m \geq 2$,*
$f_{m+k} = f_{m-1}f_k + f_m f_{k+1}$.

Proof. Let P(k) be the statement "$f_{m+k} = f_{m-1}f_k + f_m f_{k+1}$." Let's observe P(1) and P(2).
Base 1.k = 1. $f_{m-1}f_1 + f_m f_2$. By definition, we can rewrite this as $f_{m-1} \cdot 1 + f_m \cdot 1 = f_{m-1} + f_m = f_{m+1}$. Thus P(1) holds.
Base 2.k = 2. $f_{m-1}f_2 + f_m f_3$. By definition, we can rewrite this as $f_{m-1} \cdot 1 + f_m \cdot 2 = f_{m-1} + f_m + f_m = f_{m+1} + f_m = f_{m+2}$. Thus P(2) holds.
Successor. Assume P(k) holds for all k = 1,2,...,n for some n $\geq$ 2. That is,$f_{m+k} = f_{m-1}f_k + f_m f_{k+1}$. Consider
$f_{m-1}f_{(n+1)} + f_m f_{(n+1)+1} = f_{m-1}f_{(n+1)} + f_m f_{n+2}$. By definition, we can rewrite this as $f_{m-1}(f_n + f_{n-1}) + f_m(f_{n+1} + f_n)$. After distributing

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and commuting, we have $f_{m-1}(f_n + f_{n-1}) + f_m(f_{n+1} + f_n) = f_{m-1}f_n + f_{m-1}f_{n-1} + f_m f_{n+1} + f_m f_n = (f_{m-1}f_n + f_m f_{n+1}) + (f_{m-1}f_{n-1} + f_m f_n) = f_{m+n} + f_{m+n-1}$ by induction. By definition we have $f_{m+n} + f_{m+n-1} = f_{m+(n+1)}$. Thus P(n+1) holds and we have proven the proposition by the principle of induction. □

Proposition 4.31. *For all $k \in \mathbb{N}$, $f_{2k+1} = f_k^2 + f_{k+1}^2$.*

Proof. Let P(k) be the statement "$f_{2k+1} = f_k^2 + f_{k+1}^2$." Let's observe P(1).
Base. k=1. $f_{2+1} = f_3 = f_1^2 + f_{1+1}^2 = 1 + 1 = 2$. Thus P(1) holds.
Successor. Assume P(k) holds for k=1,2,...,n for some n $\in \mathbb{N}$. Consider $f_{(n+1)}^2 + f_{(n+1)+1}^2 = f_{(n+1)}^2 + f_{(n+2)}^2$. By definition we can rewrite this as $(f_{n+1} + f_n)^2 + f_{(n+1)}^2 = f_{n+1}^2 + 2f_{n+1}f_n + f_n^2 + f_{n+1}^2$. By induction, we have $f_{n+1}^2 + 2f_{n+1}f_n + f_n^2 + f_{n+1}^2 = f_{n+1}^2 + f_n^2 + 2f_{n+1}f_n + f_{n+1}^2 = f_{2n+1} + 2f_{n+1}f_n + f_{n+1}^2$. If we continue the computation, $f_{2n+1} + 2f_{n+1}f_n + f_{n+1}^2 = f_{2n+1} + f_{n+1}(2f_n + f_{n+1}) = f_{2n+1} + f_{n+1}(f_n + f_n + f_{n+1})$. By definition, $f_{2n+1} + f_{n+1}(f_n + f_n + f_{n+1}) = f_{2n+1} + f_{n+1}(f_n + f_{n+2}) = f_{2n+1} + f_{n+1}f_n + f_{n+1}f_{n+2}$. By proposition 4.30, we have $f_{2n+1} + f_{n+1}f_n + f_{n+1}f_{n+2} = f_{2n+1} + f_{2n+2} = f_{2n+3}$ by definition. Thus, P(n+1) holds and we have proven the proposition by induction. □

Proposition 5.1. *Let A,B,C be sets.*
(i) $A \subseteq A$.
(ii) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$.

(i) For any x $\in A$, then $x \in A$. Therefore A $\subseteq$ A.

(ii) Let $x \in A$, since $A \subseteq B$ we get x $\in$ B. Since $B \subseteq C$, this implies $x \in C$.

Proof. □

Proposition 5.4. *Let A,B,C be sets.*
(i) $A = A$.
(ii) if $A = B$ then $B = A$.
(iii) if $A = B$ and $B = C$ then $A = C$.

Proof. (i) For any x $\in A$, then $x \in A$. Therefore A $\subseteq$ A. Also, for any x $\in A$, then $x \in A$. Therefore A $\subseteq$ A. Thus, A = A.

(ii) It is given to us that $A \subseteq B$ and $B \subseteq A$. For any x $\in B$, then $x \in A$. Therefore B $\subseteq$ A. Also, for any x $\in A$, then $x \in B$. Therefore A $\subseteq$ B. Thus, B = A.

(iii) It is given to us that $A \subseteq B$ and $B \subseteq A$. It is also given to us that $B \subseteq C$ and $C \subseteq B$. Because $A \subseteq B$ and $B \subseteq C$, we have that $A \subseteq C$ by proposition 5.1. Because $C \subseteq B$ and $B \subseteq A$, we have that $C \subseteq A$ by proposition 5.1. Thus by definition we have that A = C. $\square$

Sources.
http://zimmer.csufresno.edu/ sdelcroix/sol111home6.pdf