HOMEWORK 5

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Proposition 4.6. *Let* $b \in \mathbb{Z}$ *and* k, $m \in \mathbb{Z} \geq 0$.
(i) If $b \in \mathbb{N}$ *then* $b^k \in \mathbb{N}$.
(ii) $b^m b^k = b^{m+k}$.
(iii) $(b^m)^k = b^{mk}$.

Proof. (i) Isolate P(k): "$b^k \in \mathbb{N}$"
Base. k = 0, $b^0 = 1$. By definition, we have that $1 \in \mathbb{N}$, so P(0) holds.
successor Suppose P(n) holds. That is, $b^n \in \mathbb{N}$. $(b^n) \cdot$ b $\in \mathbb{N}$, since b $\in \mathbb{N}$ and $\mathbb{N}$ is closed under multiplication. By definition, $b^{n+1} = b^n \cdot$ b. Hence $b^{n+1} \in \mathbb{N}$. That is P(n + 1) holds. This completes the induction.

(ii) We argue by induction on the claim P(k) "$b^m b^k = b^{m+k}$."
Base. k = 0. In this case $b^0 = 1$, by definition. So $b^m b^k = b^m \cdot 1 = b^m$. Plainly, $b^{m+0} = b^m$.
Successor. Suppose P(n) holds. Consider $b^m b^{n+1}$. By definition, $b^{n+1} = b^n$b. So we may write $b^m b^{n+1} = (b^{m+n})$b. On the other hand, (b^{m+n})b $= b^{m+n+1}$. We conclude that $b^m b^{n+1} = b^{m+n+1}$. The induction is complete, so we conclude that P(n) holds for all n.

(iii) We argue by induction the claim P(k) "$(b^m)^k = b^{mk}$"
Base. k = 0. $(b^m)^0$. By definition, $(b^m)^0 = 1 = b^{m0} = b^0$ by our axioms for the integers.
Successor. Suppose P(n) holds. That is, $(b^m)^n = b^{mn}$. Consider $(b^m)^{n+1}$. By definition, $(b^m)^{n+1} = (b^m)^n b^m$. $(b^m)^n b^m = b^{mn} b^m$. $b^{mn} b^m = b^{mn+m}$ By definition. Finally, $b^{mn+m} = b^{m(n+1)}$. We conclude that $(b^m)^{n+1} = b^{m(n+1)}$. □

Proposition 4.7. *For all* $k \in \mathbb{N}$ *:*
(i) 5^{2k} *- 1 is divisible by 24;*
(ii) 2^{2k+1} *+ 1 is divisible by 3;*
(iii) 10^k *+ 3* $\cdot$ 4^{k+2} *+ 5 is divisible by 9.*

Proof. (i) Let P(k) be the sentence "5^{2k} - 1 is divisible by 24". Consider P(1):
Base. $5^{2(1)}$ - 1 = 5^2 - 1 = 5 $\cdot$ 5 - 1 = 24. Thus P(1) holds.
Successor. Suppose P(n) holds. That is, 5^{2n} - 1 is divisible by 24.

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Consider $5^{2(n+1)}$ - 1. $5^{2(n+1)}$ - 1 = 5^{2n+2} - 1. By definition, 5^{2n+2} - 1 = 5^{2n} 5^2 - 1 = (5^{2n} - 1)5^2 + 5^2 - 1. We've assumed that 5^{2n} - 1= 24j, so we can substitute (5^{2n} - 1)5^2 + 5^2 - 1 = 24j $\cdot$ 5^2 + 5^2 - 1 = 24j $\cdot$ 25 + 24 = 24(25j + 1). Hence $5^{2(n+1)}$ - 1 is divisible by 24. We conclude the proposition holds by the principle of induction.

(ii) Let P(k) be the sentence "2^{2k+1} + 1 is divisible by 3." Consider P(1):

Base. $2^{2(1)+1}$ + 1 = 2^3 + 1 = 8 + 1 = 9. 9 = 3(3). We have thus proven P(1).

Successor. We assume P(n) holds. That is, 2^{2n+1} + 1 is divisible by 3. Consider $2^{2(n+1)+1}$ + 1. $2^{2(n+1)+1}$ + 1 = 2^{2n+3} + 1. By definition, 2^{2n+3} + 1 = $2^{2n+1}2^2$ + 1 = (2^{2n+1} + 1)2^2 - 2^2 + 1. Since we assumed P(n), we can substitute (2^{2n+1} + 1)2^2 - 2^2 + 1 = 3j2^2 - 2^2 + 1 = 12j - 3 = 3(4j - 3). Hence $2^{2(n+1)+1}$ + 1 is divisible by 3. We have thus proven the proposition by induction.

(iii) Let P(k) be the sentence "10^k + 3 $\cdot$ 4 $^{k+2}$ + 5 is divisible by 9." Consider P(1):

Base. 10^1 + 3 $\cdot$ 4^{1+2} + 5 = 10 + 3 $\cdot$ 64 + 5 = 10 + 192 + 5 = 207 = 9(23). Hence, P(1) holds.

Successor. Assume P(n) holds. That is, 10^n + 3 $\cdot$ 4^{n+2} + 5 is divisible by 9. Consider $10^{(n+1)}$ + 3 $\cdot$ $4^{(n+1)+2}$ + 5. $10^{(n+1)}$ + 3 $\cdot$ $4^{(n+1)+2}$ + 5 = 10^n10 + 3 $\cdot$ 4^{n+2} 4 + 5 = 10^n + 3 $\cdot$ 4^{n+2} + 5 + 9 $\cdot$ 10^n + 3 $\cdot$ 3 $\cdot$ 4^{n+2}. Because we assumed P(n), we can substitute 10^n + 3 $\cdot$ 4^{n+2} + 5 + 9 $\cdot$ 10^n + 3 $\cdot$ 3 $\cdot$ 4^{n+2} = 9j + 9 $\cdot$ 10^n + 3 $\cdot$ 3 $\cdot$ 4^{n+2} = 9j + 9 $\cdot$ 10^n + 9$\cdot$ 4^{n+2} = 9(j + 10^n + 4^{n+2}). We have thus shown that P(n + 1) holds, and have proven the proposition by induction. $\square$

Proposition 4.8. *For all $k \in \mathbb{N}$, $4^k > k$.*

Proof. Let P(k) be the statement "4^k > k." Consider P(1):

Base. $4^1 = 4 > 1$. Hence, P(1) holds.

Successor. Assume P(n) holds. That is, 4^n > n. Consider 4^{n+1}. By definition, $4^{n+1} = 4^n4$ > n $\cdot$ 4 since we assumed P(n) and by proposition 2.7(iii). n $\cdot$ 4 = 4n = 3n + n. Because n $\in \mathbb{N}$, 3n $\geq$ 3(1) > 1. Therefore, 3n + n > 1 + n = n + 1. We can conclude that 4^{n+1} > n + 1, and have proven the proposition by induction. $\square$

Proposition 4.13. *For $x \neq 1$ and $k \in \mathbb{Z}_{\geq 0}$, $\sum\limits_{j=0}^{k} x^j = \dfrac{1 - x^{k+1}}{1 - x}$.*

Proof. Let P(k) be the statement "$\sum\limits_{j=0}^{k} x^j = \dfrac{1 - x^{k+1}}{1 - x}$."

Let's first observe P(0).

Base. k = 0. $\sum_{j=0}^{0} x^j = x^0 = 1 = \frac{1-x}{1-x}$. We have thus proven that P(0) holds.

textbfSuccessor Assume P(n) holds. Consider $\sum_{j=0}^{n+1} x^j$.

$\sum_{j=0}^{n+1} x^j := \sum_{j=0}^{n} x^j + x^{n+1}$. By induction,

$\sum_{j=0}^{n} x^j + x^{n+1} = \frac{1-x^{n+1}}{1-x} + x^{n+1} = \frac{1-x^{n+1}+(1-x)^{n+1}}{1-x} = \frac{1-x^{n+2}}{1-x}$.

Hence, P(n+1) holds, and we have proven the proposition by induction. □

Proposition 4.17. *Let* $(x_j)_{j=1}^{\infty}$ *be a sequence in* $\mathbb{Z}$*, and let* $a,b,r \in \mathbb{Z}$ *be such that* $a \leq b$*. Then* $\sum_{j=a}^{b} x_j = \sum_{j=a+r}^{b+r} x_{j-r}$.

Proof. Let P(a,b) be the statement "$\sum_{j=a}^{b} x_j = \sum_{j=a+r}^{b+r} x_{j-r}$".Let's first observe P(0,1).

Base. a = 0, b = 1. $\sum_{j=0}^{1} x_j = x_0 + x_1 = x_{r-r} + x_{1+r-r} = \sum_{j=0+r}^{1+r} x_{j-r}$

Successor. Suppose P(m,n) holds. That is, $\sum_{j=m}^{n} x_j = \sum_{j=m+r}^{n+r} x_{j-r}$. Consider $\sum_{j=m+1}^{n+1} x_j$. We can rewrite this as $\sum_{j=m}^{n} x_j + x_{n+1} - x_m$. By induction, we have $\sum_{j=m}^{n} x_j + x_{n+1} - x_m = \sum_{j=m+r}^{n+r} x_{j-r} + x_{n+1} - x_m$. By definition, we have

$\sum_{j=m+r}^{n+r} x_{j-r} + x_{n+1} - x_m := \sum_{j=m+1+r}^{n+1+r} x_{j-r}$. We have proven P(m+1,n+1) holds, and thus proven the proposition by the principle of induction. □

Sources.

https://www.cs.uaf.edu/ maxwell/math215/HW8Sols.pdf
http://www.voutsadakis.com/TEACH/LSSU/F03/LSSU215F03/hwk4sol.pdf