HOMEWORK 10

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Proposition 8.40. *(i)* $x \in \mathbb{R}_{>0}$ *if and only if* $\frac{1}{x} \in \mathbb{R}_{>0}$.
(ii) Let $x, y \in \mathbb{R}_{>0}$ *. If* $x < y$ *then* $0 < \frac{1}{y} < \frac{1}{x}$ *.*

Proof. (i) ($\Rightarrow$) Let's first assume $x \in \mathbb{R}_{>0}$. We must show $\frac{1}{x} \in \mathbb{R}_{>0}$. By the axiom 8.26(iv) either $\frac{1}{x} = 0$, $\frac{1}{x} \in \mathbb{R}_{>0}$, or $-\frac{1}{x} \in \mathbb{R}_{>0}$.
Case 1. $\frac{1}{x} = 0$. We know $1 \neq 0$ and we know $x \neq 0$ since $x \in \mathbb{R}_{>0}$. Let's consider $x \cdot \frac{1}{x}$. $1 = x \cdot \frac{1}{x} = x \cdot 0 = 0$, which is absurd.
Case 2. $\frac{1}{x} \in \mathbb{R}_{>0}$. This is what we want, if this case holds, we're done.
Case 3. Say $-\frac{1}{x} \in \mathbb{R}_{>0}$. Thus, $(-1)(\frac{1}{x}) \in \mathbb{R}_{>0}$. Since $x \in \mathbb{R}_{>0}$ and $(-1)(\frac{1}{x}) \in \mathbb{R}_{>0}$, $x \cdot (-1) \cdot (\frac{1}{x}) \in \mathbb{R}_{>0}$ since $\mathbb{R}_{>0}$ is closed under multiplication. Commuting, we see that $-1 \cdot 1 = -1 \in \mathbb{R}_{>0}$. On the other hand, $1 \in \mathbb{R}_{>0}$, so $1 + -1 = 0$ is in $\mathbb{R}_{>0}$, which is absurd.

We may now infer that case 2 must hold. Hence $\frac{1}{x} \in \mathbb{R}_{>0}$.

($\Leftarrow$)Now, let's assume $\frac{1}{x} \in \mathbb{R}_{>0}$. We must show $x \in \mathbb{R}_{>0}$. By the axiom 8.26(iv) either $x \in \mathbb{R}_{>0}$, $x = 0$, or $-x \in \mathbb{R}_{>0}$.
Case 1. $x = 0$. We know $1 \neq 0$ and we know $\frac{1}{x} \neq 0$ since $\frac{1}{x} \in \mathbb{R}_{>0}$. Let's consider $\frac{1}{x} \cdot x$. $1 = \frac{1}{x} \cdot x = \frac{1}{x} \cdot 0 = 0$, which is absurd.
Case 2. $x \in \mathbb{R}_{>0}$. This is what we want, if this case holds, we're done.
Case 3. Say $-x \in \mathbb{R}_{>0}$. Thus, $(-1)(x) \in \mathbb{R}_{>0}$. Since $\frac{1}{x} \in \mathbb{R}_{>0}$ and $(-1)(x) \in \mathbb{R}_{>0}$, $\frac{1}{x} \cdot (-1) \cdot (x) \in \mathbb{R}_{>0}$ since $\mathbb{R}_{>0}$ is closed under multiplication. Commuting, we see that $-1 \cdot 1 = -1 \in \mathbb{R}_{>0}$. On the other hand, $1 \in \mathbb{R}_{>0}$, so $1 + -1 = 0$ is in $\mathbb{R}_{>0}$, which is absurd.

(ii) Consider $x < y$. Let's multiply each side by $\frac{1}{x}$. We obtain $1 < \frac{y}{x}$. By 8.40(i), $\frac{1}{x} \in \mathbb{R}_{>0}$; this is why $1 < \frac{y}{x}$. We can multiply both sides by $\frac{1}{y}$, and since $\frac{1}{y} \in \mathbb{R}_{>0}$, by 8.40(i), we have $\frac{1}{y} < \frac{1}{x}$. Finally, since $\frac{1}{y} \in \mathbb{R}_{>0}$, $0 < \frac{1}{y} < \frac{1}{x}$. □

Theorem 8.41. *Let* $x \in \mathbb{R}_{>0}$*. Then* $x^2 < x^3$ *if and only if* $x > 1$.

Proof. ($\Rightarrow$) We assume $x^2 < x^3$. Let's observe that $x^3 = x \cdot x^2$. On the other hand, $x^2 = x^2 \cdot 1$, so we have $1 \cdot x^2 < x \cdot x^2$. If $x^2 \in \mathbb{R}$, then $\frac{1}{x^2} \in \mathbb{R}_{>0}$. Since $x^2 < x^3$, we conclude that $x \neq 0$. Since $x \neq 0$, $x^2 \in \mathbb{R}_{>0}$. In this case, $\frac{1 \cdot x^2}{x^3} < \frac{x^3}{x^2}$, hence $1 < x$.

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($\Leftarrow$) Suppose $1 < x$. Since $x > 0$, we have $x \in \mathbb{R}_{>0}$. Thus $x^2 \in \mathbb{R}_{>0}$. Multiplying both sides by x^2, we deduce that $x^2 < x^3$. $\square$

Proposition 8.43. *Let $x, y \in \mathbb{R}$ such that $x < y$. There exists $z \in \mathbb{R}$ such that $x < z < y$.*

Proof. By definition of $<$, $0 < y - x$ and $y - x \in \mathbb{R}_{>0}$. Suppose toward a contradiction there's no $z \in \mathbb{R}$ such that $x < z < y$. Thus, there is no real number s such that $0 < s < y - x$. For every $w \in \mathbb{R}_{>0}$, it is then the case that $y - x \leq w$. Hence $y - x$ is the last element of $\mathbb{R}_{>0}$, which contradicts theorem 8.42. $\square$

Proposition 8.50. *If the sets A and B are bounded above and $A \subseteq B$, then $sup(A) \leq sup(B)$.*

Proof. Sup(B) is an upper bound for B. Since $A \subseteq B$, sup(B) is also an upper bound for A. Since sup(A) is the least upper bound, $sup(A) \leq sup(B)$. $\square$

Proposition 8.53. *Every nonempty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound.*

Proof. Let $B \subseteq \mathbb{R}$ be a non-empty set. bounded from below. Consider A := $\{r \in \mathbb{R} \mid \forall b \in B \ r \leq b\}$. A is the set of lower bounds of B. The set B is non-empty, so there is some $b \in B$. This b is clearly an upper bound for A. Hence, A admits a least upper bound. Applying our axiom, sup(A) exists. Consider $b \in B$. Either $sup(A) \leq b$ or $b < sup(A)$. Suppose toward a contradiction $b < sup(A)$. In this case, b is an upper bound of A. This contradicts that sup(A) is the least upper bound. We thus can eliminate $b < sup(A)$. Thus, for all $b \in B$, $sup(A) \leq b$. We conclude that sup(A) is a lower bound for B. $\square$

Lemma. *Let p be a prime. If $p|(a_1 \dots a_n)$, then $p|a_i$ for some $1 \leq i \leq n$.*

Proof. Suppose toward a contradiction that the lemma fails. Let $n \geq 1$ be least $\sum$:= $\{$ $n \in \mathbb{N}$ | there are $a_1 \dots a_n$ integers such that $p|a_1 \dots a_n$ but $p \nmid a_1$ and $p \nmid a_2$ and $\dots p \nmid a_n\}$. By the well-ordering principle there is a least element (the set is non-empty since we assume the lemma fails).

By Euclid's lemma, we know that $n > 2$ (i.e. Euclid's lemma tells us the lemma holds for n = 2). Consider $p|a_1 \dots a_n$. We may rewrite $a_1 \dots a_n = b \cdot c$ where b:= $a_1 \dots a_{n-1}$, and c := a_n. By Euclid's lemma, $p|b \cdot c$ implies $p|b$ or $p|c$. We know $p \nmid a_n$, so $p \nmid c$. Hence, $p|b$. But now $p|a_1 \dots a_{n-1}$ and $p \nmid a_1 \dots p \nmid a_{n-1}$. Since $n - 1 \in \sum \in \mathbb{N}$, this contradicts that n is the least counter example. Thus, the lemma holds. $\square$