HOMEWORK 2

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Proposition 1.24. *Let $x \in \boldsymbol{Z}$. If $x \cdot x = x$ then $x = 0$ or 1.*

Proof. By proposition 1.7, 1 · x = x. We may thus write x · x = x · 1. If x = 0, we are done. Otherwise, we assume x ≠ 0. Since x ≠ 0, axiom 1.5 (cancellation) implies x = 1. □

Proposition 1.25. *For all m,n $\in \boldsymbol{Z}$:*
(i) -(m + n) = (-m) + (-n).
(ii) -m = (-1)m.
(iii) (-m)n = m(-n) = -(mn).

Proof. (i) (-1) · -(-m) = 1 · (-m) by proposition 1.20. 1 · (-m) = -m by proposition 1.7. On the other hand, -(-m) = m by proposition 1.22. Thus -m = (-1)m. Because of this, we can say -(m + n) = (-1)(m + n). Distributivity (axiom 1.1.3) shows (-1)(m + n) = -1·m + -1·n. Using what we just recently proved again, -1·m + -1·n = (-m) + (-n). We conclude -(m + n) = (-m) + (-n).

(ii) (-1) · -(-m) = 1 · (-m) by proposition 1.20. 1 · (-m) = -m by proposition 1.7. On the other hand, -(-m) = m by proposition 1.22. Thus -m = (-1)m.

(iii) By proposition 1.20, we have (-m) · -(-n) = m · (-n). Also thanks to proposition 1.20, (-1) · -(-n) = 1 · (-n). Then by proposition 1.7, we can say that 1(-n) = -n. On the other hand, -(-n) = n by proposition 1.22. Thus (-1)n = -n. Because of this, we can say that m · (-n) = m · (-1)n. By commutativity, m · (-1)n = (-1)mn. Then by associativity, (-1)mn = -1(mn). Because we just proved (-1)n = -n ∀ n $\in \mathbf{Z}$, we can say that -1(mn) = -(mn). We conclude that (-m)n = m(-n) □

Proposition 1.26. *Let m,n $\in \boldsymbol{Z}$. If mn = 0, then m = 0 or n = 0.*

Proof. If we look at m · n, then by proposition 1.14, m · 0 = 0, so m · n = m · 0. Because of axiom 1.5, n = 0. On the other hand, commutativity shows m · n = n · m. If we look at this case, by proposition 1.14, n · 0

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= 0, so n $\cdot$ m = n $\cdot$ 0. Because of axiom 1.5, m = 0. We conclude if m $\cdot$ n = 0, then m = 0 or n = 0. $\square$

Project 3.1. *Express each of the following statements using quantifiers:*
(i) There exists a smallest natural number.
(ii) There exists no smallest integer.
(iii) Every integer is the product of two integers.
(iv) The equation x^2 - $2y^2$ = 3 has an integer solution.

Answer.
(i) $\exists$ m $\in$ **N** $\forall$ n $\in$ **N** (m < n $\vee$ m = n).
(ii) $\forall$ m $\in$ **Z** $\exists$ n $\in$ **Z** n < m.
(iii) $\forall$ m,n $\in$ **Z** $\exists$ p $\in$ **Z** m $\cdot$ n = p.
(iv) $\exists$ x,y $\in$ **Z** x^2 - $2y^2$ = 3

Project 3.2. *In each of the following cases explain what is meant by the statement and decide whether it is true or false.*
(i) For each $x \in Z$ there exists $y \in Z$ such that $x + y = 1$.
(ii) There exists $y \in Z$ such that for each $x \in Z$, $x + y = 1$.
(iii) For each $x \in Z$ there exists $y \in Z$ such that $xy = x$.
(iv) There exists $y \in Z$ such that for each $x \in Z$, $xy = x$.

Answer.
(i) This statement is saying that every integer x has an integer y that when added to it is equal to 1. This is true.
(ii) This statement is saying that there is some universal integer y that when added to any integer x the sum is 1. This is false.
(iii) This statement is saying that every integer x has an integer y that when multiplied together is equal to x. This statement is false, since 1 is the only integer capable of this (proposition 1.18).
(iv) This statement is saying that there is a universal integer y that when multiplied with any integer x the product is x. This statement is true.

Project 3.3. *Construct two more mathematical if-then statements that are true, but whose converses are false.*

Answer.
(i) There exists y $\in$ **Z** such that for each x $\in$ **Z** x $\cdot$ y = 0.
For each x $\in$ **Z** There exists y $\in$ **Z** such that x $\cdot$ y = 0.
(ii) For each x $\in$ Z there exists y $\in$ Z such that x + y = 5.
There exists y $\in$ Z such that For each x $\in$ Z x + y = 5.

1 The classical logical connectives are and, or, not, and (if then). These are denoted, respectively, by $\wedge$, $\vee$, $\neg$, and $\to$. Write out the truth tables for $\wedge$, $\vee$, $\neg$, and $\to$.

Answer.

P	$\wedge$	Q
T	T	T
T	F	F
F	F	T
F	F	F

P	$\vee$	Q
T	T	T
T	T	F
F	T	T
F	F	F

P	$\neg$P
T	F
F	T

P	$\to$	Q
T	T	T
T	F	F
F	T	T
F	T	F

2 Find an expression using only $\vee$ and $\neg$ with the same truth table as $\to$.

Answer.

$\neg$P	$\vee$	Q
T	T	T
T	F	F
F	T	T
F	T	F