HOMEWORK 12

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Proposition 10.13.
(i) $\lim_{k\to\infty} \dfrac{1}{k} = 0.$
(ii) $\lim_{k\to\infty} \dfrac{k-1}{k} = 1.$
(iii) $\lim_{k\to\infty} \dfrac{1}{4^k} = 0.$

Proof. (i) Fix $\epsilon > 0$. Let $N \in \mathbb{N}$ be such that $\forall n \geq \mathbb{N}$, $|0 - \frac{1}{n}| < \epsilon$. Take N large enough such that $\frac{1}{\epsilon} < N$. Therefore $\frac{1}{N} < \epsilon$. Now for any $n \geq N$, $\frac{1}{n} \leq \frac{1}{N}$. We now see that for $n \geq N$, $|0 - \frac{1}{n}| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$. Therefore $\frac{1}{n} \to 0$.

(ii) Fix $\epsilon > 0$. Let $N \in \mathbb{N}$ be such that $\forall n \geq \mathbb{N}$, $|0 - \frac{1}{4^n}| < \epsilon$. Take N large enough such that $\frac{1}{\epsilon} < N$. Therefore $\frac{1}{N} < \epsilon$. Now for any $n \geq N$, $\frac{1}{4^n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$. We now see that for $n \geq N$, $|0 - \frac{1}{4^n}| = \frac{1}{4^n} \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$. Therefore $\frac{1}{4^n} \to 0$.

(iii) Fix $\epsilon > 0$. Let $N \in \mathbb{N}$ be such that $\forall n \geq \mathbb{N}$, $|1 - \frac{n-1}{n}| < \epsilon$. Take N large enough such that $\frac{1}{\epsilon} < N$. Therefore $\frac{1}{N} < \epsilon$. Now for any $n \geq N$, $\frac{1}{n} \leq \frac{1}{N} < \epsilon$. We now see that for $n \geq N$, $|1 - \frac{n-1}{n}| = |1 - \frac{n}{n} + \frac{1}{n}| = |0 + \frac{1}{n}| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon$. Therefore $\frac{n-1}{n} \to 1$. □

Proposition 10.17. *Let $x \in \mathbb{R}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 0}$. Then*
$$(1+x)^k \geq 1 + kx.$$

Proof. By the binomial theorem $(1+x)^k = \sum_{m=0}^{k} \binom{k}{m} 1^m x^{k-m} \geq \binom{k}{k-1} 1^{k-1} x^{k-(k-1)} + \binom{k}{k} 1^k x^0 = kx + 1$. □

Proposition 10.27. *Given any $r \in \mathbb{R}_{>0}$, the number $\sqrt{r}$ is unique in the sense that, if x is a positive real number such that $x^2 = r$, then $x = \sqrt{r}$.*

Proof. Suppose u and v are such that $u^2 = r$ and $v^2 = r$. Let $w = sup\{x \in \mathbb{R} | x^2 < r\}$. We will show u = w = v. For any $x \in A := \{x \in$

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$\mathbb{R}|x^2 < r\}$ we see that $x^2 < u^2 = r$. If $x < 0$, then clearly $x < u$. If $x \geq 0$, then propositio 10.5 ensures that $x < u$. Since w is the least upper bound of A, we conclude that $w \leq u$. But $w^2 = r = u^2$. By proposition 10.5 again it must be the case that w = u. Similarly, v = w. □

Proposition 10.8. (iv) $|x + y| \leq |x| + |y|$.

Proof. Observe that $|x^2| = |x|^2$. That is, if $x \in \mathbb{R}_{\geq 0}$, then $|x| = x$, so $|x|^2 = x^2$. If $x < 0$, then $-x \in \mathbb{R}_{\geq 0}$, so $|x| = -x$. Thus, $|x|^2 = (-x)^2 = (-1)^2x^2 = x^2$. We now see that $|x + y|^2 = (x + y)^2 = x^2 + y^2 + 2xy \leq |x|^2 + |y|^2 + 2|x||y| = (|x| + |y|)^2$. Thus, $|x + y| \leq |x| + |y|$. □

Proposition 10.23. (v) *If $A \neq 0$, then* $\lim_{k\to\infty} \dfrac{1}{a_k} = \dfrac{1}{A}$.

Proof. We know that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N, |a_n - A| < \epsilon$. Fix $\eta > 0$. Take $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - A| < |a_n \cdot A|\eta$, and so $|\frac{1}{a_n} - \frac{1}{A}| = |\frac{a_n}{a_n \cdot A} - \frac{A}{a_n \cdot A}| = \frac{|a_n - A|}{|a_n \cdot A|} < \frac{|a_n \cdot A|\eta}{|a_n \cdot A|} = \eta$. We have thus proven $\frac{1}{a_n} \to \frac{1}{A}$. □

Proposition 10.26. *Given any $r \in \mathbb{R}_{>0}$, the real number $\sqrt{r}$ is well defined, positive, and satisfies $\sqrt{r}^2 = r$*

Proof. Put u = $\sqrt{r}$. Now for $1 \leq x$, we have that $x \leq x^2$. For each $x \in A$ either $x < 1$ or $x \geq 1$. If $x \geq 1$, then $x^2 > r$, so $x < r$. We conclude that r is an upper bound for A. Since $\mathbb{R}$ is complete the supremum of A is well-defined. That is to say u is well-defined. To see that u is positive, we note $1 \in A$ and $u \geq 1$. Thus, u is positive. We now argue that $u^2 = r$. We know that one of either $u^2 = r$, $u^2 < r$, or $u^2 > r$ holds.

Case 1 $u^2 > r$ Suppose toward a contradiction this case holds. Put $\delta = min\{1, u^2 - r\}$ and put h = $\frac{\delta}{4u}$. $u^2 - (u - h)^2 = u^2 - u^2 - h^2 + 2uh = h(2u - h) < h(2u) < \delta$. The distance between u^2 and $u - h^2$ is thus less than δ. We conclude that $r < u - h^2 < u^2$.

Fact. If $x, y \geq 1$, then $x \leq y$ if and only if $x^2 \leq y^2$.

We know that $\delta \leq 1$ and $u \geq 1$. Thus $u - h > 0$. For any $x \in A$ such that $x \geq 0$, we have $x^2 < r < (u - h)^2$. Thus by our fact, $x < u - h$. Since u - h is positive, u is larger than all $x \in A$. The element $u - h$ is thus an upper bound for A. We chose u to be the least upper bound of A. Thus we have reached an absurdity.

Case 2 Suppose toward a contradiction that $u^2 < r$. Put $\delta = r - u^2$ and $h = min\{\frac{\delta}{4u}, u\}$. We now see that $(u+h)^2 - u^2 = u^2 + h^2 + 2uh - u^2 = h(2u + h) \leq h3u < \delta$. So we now have $u^2 < u + h^2 < r$, but $u+h$ is a positive real number and $(u+h)^2 < r$. We conclude that $(u+h) \in A$. However, u is the least upper bound of A. So $u+h \leq u$, which is absurd. $\square$