Proposition 4.31. *For all $k \in \mathbb{N}$, $f_{2k+1} = f_k^2 + f_{k+1}^2$.*

Proof. Let P(k) be the statement "$f_{2k+1} = f_k^2 + f_{k+1}^2$." Let's observe P(1).
Base. k=1. $f_{2+1} = f_3 = f_1^2 + f_{1+1}^2 = 1 + 1 = 2$. Thus P(1) holds.
Successor. Assume P(k) holds for k=1,2,...,n for some n $\in \mathbb{N}$. Consider $f_{(n+1)}^2 + f_{(n+1)+1}^2 = f_{(n+1)}^2 + f_{(n+2)}^2$. By definition we can rewrite this as $(f_{n+1} + f_n)^2 + f_{(n+1)}^2 = f_{n+1}^2 + 2f_{n+1}f_n + f_n^2 + f_{n+1}^2$. By induction, we have $f_{n+1}^2 + 2f_{n+1}f_n + f_n^2 + f_{n+1}^2 = f_{n+1}^2 + f_n^2 + 2f_{n+1}f_n + f_{n+1}^2 = f_{2n+1} + 2f_{n+1}f_n + f_{n+1}^2$. If we continue the computation, $f_{2n+1} + 2f_{n+1}f_n + f_{n+1}^2 = f_{2n+1} + f_{n+1}(2f_n + f_{n+1}) = f_{2n+1} + f_{n+1}(f_n + f_n + f_{n+1})$. By definition, $f_{2n+1} + f_{n+1}(f_n + f_n + f_{n+1}) = f_{2n+1} + f_{n+1}(f_n + f_{n+2}) = f_{2n+1} + f_{n+1}f_n + f_{n+1}f_{n+2}$. By proposition 4.30, we have $f_{2n+1} + f_{n+1}f_n + f_{n+1}f_{n+2} = f_{2n+1} + f_{2n+2} = f_{2n+3}$ by definition. Thus, P(n+1) holds and we have proven the proposition by induction. $\square$