HOMEWORK 11

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Proposition 9.12. *Let A and B be sets. There exists an injection from A to B if and only if there exists a surjection from B to A.*

Proof. ($\Rightarrow$) Suppose there's an injection $f : A \to B$. Fix $a_o \in A$. Observe that for $b \in f(a)$ there's a unique $a_b \in A$ such that $f(a_b) = b$. Define $g : B \to A$ by $g(b) = \begin{cases} a_o, \, b \notin f(a) \\ a_b, \, b \in f(a) \end{cases}$ is surjective by definition.

($\Leftarrow$) Suppose $g : B \to A$ is surjective. For each $a \in A$, there is at least one $b \in B$ such that $g(b) = a$. For each $a \in A$, fix some such $b_a \in B$. Define $f : A \to B$ by $f(a) = b_a$. Let's check if f is injective. Suppose $a_1 \neq a_2$. Then, $g(a_1) \neq g(a_2)$. $\square$

Proposition 10.9. *Let $x \in \mathbb{R}$ be such that $0 \leq x \leq 1$, and let $m, n \in \mathbb{N}$ be such that $m \geq n$. Then $x^m \leq x^n$.*

Proof. Here we have three cases. $x = 0$, $x = 1$, and $0 < x < 1$.
Case 1. x = 0. $0^m = 0 = 0^n$.
Case 2. x = 1. $1^m = 1 = 1^n$
Case 3. $0 < x < 1$. We have two different subcases here, since either $m = n$ or $m > n$.

Subcase 1. m = n. Here $x^m = x^n$

Subcase 2. $m > n$. Take $y > 1$ such that $\frac{1}{y} = x$. Because $m > n$, we have $y^m > y^n$. By proposition 8.40 (ii), we know $\frac{1}{y^m} < \frac{1}{y^n}$. Thus, $\frac{1}{y^m} = \frac{1^m}{y^m} = (\frac{1}{y})^m = x^m < \frac{1}{y^n} = \frac{1^n}{y^n} = (\frac{1}{y})^n = x^n$. $\square$

Proposition 10.16. *If the sequence (x_k) converges to L, then*

$$\lim_{k\to\infty} x_{k+1} = L.$$

Proof. The sequence (x_k) converging to L means that $\lim_{k\to\infty} x_k = L$. Letting $\epsilon > 0$, this also means ther is some $N' \in \mathbb{N}$ such that $|x_{k'} - L| < \epsilon \; \forall k' \geq N'$. Let $N := N' - 1$. Then if $k \geq N$ we have $k + 1 \geq N'$, and because it's given that $\lim_{k\to\infty} x_k = L$. Applying this to $k' = k+1$, we have that $|x_{k+1} - L| < \epsilon$. This by definition means that $\lim_{k\to\infty} x_{k+1} = L$. $\square$

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Proposition 10.14. *If (x_k) converges to L and to L' then $L = L'$.*

Proof. Suppose towards a contradiction, that $L \neq L'$.
Let $\epsilon = \frac{1}{2}|L - L'|$. Because (x_k) converges to L, by definition there's some $N \in \mathbb{N}$ such that $|x_k - L| < \epsilon$ for all $k \geq N$. Also, by definition we have some $N' \in \mathbb{N}$ such that $|x_k - L'| < \epsilon$ for all $k \geq N'$. Let $k > max\{N, N'\}$. By the triangle inequality, we have $|L - L'| \leq |L - x_k| + |x_k - L'|$. By our assumptions this gives $|L - L'| < 2\epsilon = |L - L'|$. This implies a real number is less than itself, which is absurd. □

Sources.
http://users.math.msu.edu/users/duncan42/LimitsHW(Solutions).pdf