CHALLENGE PROBLEM 2

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Project 4.23 (Leibniz's formula). Consider an operation denoted by $'$ that is applied to symbols such as u, v, w. Assume that the operation $'$ satisfies the following axioms:

$$\begin{aligned}(u+v)' &= u' + v' \\ (uv)' &= uv' + u'v \\ (cu)' &= cu', \text{ where } c \text{ is a constant.}\end{aligned}$$

Define $w^{(k)}$ recursively by

(i) $w^{(0)} := w$.
(ii) Assuming $w^{(n)}$ defined (where $n \in \mathbb{Z}_{\geq 0}$), define $w^{(n+1)} := (w^{(n)})'$
Prove:

$$(uv)^{(k)} = \sum_{m=0}^{k} \binom{k}{m} u^{(m)} v^{(k-m)}.$$

Proof. Take P(k) to be the statement "$(uv)^{(k)} = \sum_{m=0}^{k} \binom{k}{m} u^{(m)} v^{(k-m)}$."
Let's observe k=0.
Base. $(uv)^{(0)} = \sum_{m=0}^{0} \binom{0}{m} u^{(m)} v^{(0-m)}. = u^{(0)}v^{(0)} = uv$
Successor. Assume P(k) holds. Observe $\sum\limits_{m=0}^{k+1} u^{(m)} v^{(k+1-m)}$. By proposition 4.16(i) we may rewrite this as

$$\binom{k+1}{0} u^{(0)} v^{(k+1)} + \sum_{m=1}^{k} \binom{k+1}{m} u^{(m)} v^{(k+1-m)} + \binom{k+1}{k+1} u^{(k+1)} v^{0}.$$

By part (ii) of our recursive definition and Corollary 4.20, it follows that this is equal to

$$uv^{(k+1)} + \sum_{m=1}^{k} \left(\binom{k}{m-1} + \binom{k}{m} \right) u^{(m)} v^{(k+1-m)} + u^{(k+1)} v$$

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. By distributivity and Proposition 4.16(ii),

$$= uv^{(k+1)} + \sum_{m=1}^{k} \binom{k}{m-1} u^{(m)} v^{(k+1-m)} + \sum_{m=1}^{k} \binom{k}{m} u^{(m)} v^{(k+1-m)} + u^{(k+1)} v$$

Then, we apply proposition 4.17 to the first sum to get

$$uv^{(k+1)} + \sum_{m=0}^{k-1} \binom{k}{m} u^{(m+1)} v^{(k+1-(m+1))} + \sum_{m=1}^{k} \binom{k}{m} u^{(m)} v^{(k+1-m)} + u^{(k+1)} v$$

Since $\binom{k}{0} = 1$ and $\binom{k}{k} = 1$ we then combine $\binom{k}{0} uv^{(k+1)}$ to the second sum and $\binom{k}{k} u^{(k+1)} v$ to the first using proposition 4.16(i):

$$\sum_{m=0}^{k} \binom{k}{m} u^{(m+1)} v^{(k-m)} + \sum_{m=0}^{k} \binom{k}{m} u^{(m)} v^{(k+1-m)}$$

Applying 4.16(ii) again and part (ii) of our recursive definition, we have

$$\sum_{m=0}^{k} \binom{k}{m} (u^{(m)})' v^{(k-m)} + \binom{k}{m} u^{(m)} (v^{(k-m)})'.$$

Using our second and third axioms for $'$, we thus have

$$\sum_{m=0}^{k} \binom{k}{m} (u^{(m)} v^{(k-m)})'.$$

Finally, by our first axiom for $'$, this is equal to $(\sum_{m=0}^{k} \binom{k}{m} u^{(m)} v^{(k-m)})'$, and by induction and our recursive definition $(\sum_{m=0}^{k} \binom{k}{m} u^{(m)} v^{(k-m)})' = ((uv)^{(k)})' = (uv)^{(k+1)}$. This completes the induction. $\square$