DEFINING THE NATURAL NUMBERS

Our text book introduces the natural numbers via the following axiom.

Axiom (†)**.** There exists a subset $\mathbb{N} \subseteq \mathbb{Z}$ with the following properties:

(1) If $m, n \in \mathbb{N}$, then $m + n \in \mathbb{N}$.
(2) If $m, n \in \mathbb{N}$, then $mn \in \mathbb{N}$.
(3) $0 \notin \mathbb{N}$.
(4) For every $m \in \mathbb{Z}$, we have $m \in \mathbb{N}$, $m = 0$, or $-m \in \mathbb{N}$.

As I explained in class, it is poor mathematical form to introduce an axiom for something that can be defined from more basic axioms. (The axiom is also a statement in second order logic, which makes it even worse!) This challenge problem shows how to define $\mathbb{N}$, in a sensible manner.

Remark 0.1. The approach you explore below requires induction. However, induction can be formalized without introducing the notion of the integers.

0.1. **Challenge problem.**

Definition. The **successor function** on the integers is defined by $s(x) := x + 1$. For $m \geq 1$, we define $s^{m+1}(x)$ recursively by $s^{m+1}(x) := s(s^m(x))$.

Assume that the integers satisfy the following two additional axioms.

Axiom ($*$)**.** For all $m \geq 1$, $s^m(0) \neq 0$.

Axiom ($**$)**.** For all $x \in \mathbb{Z}$, there is $m \geq 0$ such that $s^m(x) = 0$ or $s^m(0) = x$.

Definition. The **set of successors of** 0 is defined to be

$$N := \{x \in \mathbb{Z} \mid \exists m \geq 1\ s^m(0) = x\}.$$

Prove the following proposition.

Proposition. *Assume that the integers satisfy* ($*$) *and* ($**$)*. Then, the set* N *satisfies axiom* (†)*. That is to say,* N *satisfies the four conditions of the axiom.*