# HOMEWORK 4

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**Proposition 2.18.** *(i) For all $k \in \mathbb{N}$, $k^3$ + 2k is divisible by 3.*
*(ii) For all $k \in \mathbb{N}$, $k^4$ - 6$k^3$ + 11$k^2$ - 6k is divisible by 4.*
*(iii) For all $k \in \mathbb{N}$, $k^3$ + 5k is divisible by 6.*

*Proof.* (i) Let's first identify P(n). We put P(n) to be the sentence, "$n^3$ + 2n is divisible by 3." Let us first consider P(1).
**Base.** P(1) asserts $1^3$ + 2 = 3, we conclude that P(1) holds.
**Successor** We have that P(n) holds. That is, 3 divides $n^3$ +2n. Consider $(n+1)^3$ + 2(n + 1). $(n+1)^3$ + 2n + 2 = ... = $n^3$ + 3$n^2$ + 2n + 1 = $n^3$ + 2n + 3$n^2$ + 1. Now our inductive hypothesis tells us that 3 divides $n^3$ + 2n = 3 · j. We now see that $(n+1)^3$ + 2(n + 1) = 3j + 3($n^2$ + 1) = 3(j + $n^2$ + 1). Since j + $n^2$ + 1 is an integer, we conclude that 3 divides (n + 1)$^3$ + 2(n + 1). That is to say, P(n + 1) holds. By the principle of induction the proposition holds.

(ii) Let's first identify P(n). We put P(n) to be the sentence, "$n^4$ - 6$n^3$ + 11$n^2$ - 6n is divisible by 4". Let's consider P(1).
**Base.** P(1) asserts 1 - 6 + 11 - 6 = 0. Proposition 1.17 tells us 0 is divisible by any integer, so we conclude that P(1) holds.
**Successor.** Suppose P(n) holds. That is, $n^4$ - 6$n^3$ + 11$n^2$ - 6n is divisible by 4. Consider $(n+1)^4$ - 6$(n+1)^3$ + 11$(n+1)^2$ - 6(n + 1). $(n+1)^4$ - 6$(n+1)^3$ + 11$(n+1)^2$ - 6(n + 1) = $(n+1)^4$ - 6$(n+1)^3$ + 11$(n+1)^2$ - 6n + 6 = ... = ($n^4$ - 6$n^3$ + 11$n^2$ - 6n) + 4$n^3$ - 12$n^2$ + 8n. Now our inductive hypothesis tells us that 4 divides $n^4$ - 6$n^3$ + 11$n^2$ - 6n = 4 · j. We now see that ($n^4$ - 6$n^3$ + 11$n^2$ - 6n) + 4$n^3$ - 12$n^2$ + 8n = 4j + 4$n^3$ - 12$n^2$ + 8n = 4j + 4($n^3$ - 3$n^2$ + 2n). Letting $n^3$ - 3$n^2$ + 2n = i, we have 4j + 4($n^3$ - 3$n^2$ + 2n) = 4j + 4i = 4(j + i).
Thus 4 divides $(n+1)^4$ - 6$(n+1)^3$ + 11$(n+1)^2$ - 6(n + 1), verifying that P(n + 1) holds. By the principle of induction we conclude $n^4$ - 6$n^3$ + 11$n^2$ - 6n is divisible by 4.

(iii) Let's first identify P(n). We put P(n) to be the sentence "$n^3$ + 5n is divisible by 6". Let's first consider P(1).
**Base.** P(1) asserts $1^3$ + 5 is divisible by 6. Since $1^3$ + 5 = 6, we

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conclude that P(1) holds.
**successor** Assume P(n) holds. That is, $n^3$ + 5n is divisible by 6. Consider $(n+1)^3$ + 5(n + 1). $(n+1)^3$ + 5n + 5 = $n^3$ + $3n^2$ + 3n + 1 + 5n + 5 = ($n^3$ + 5n) + ($3n^2$ + 3n + 6) = ($n^3$ + 5n) + 3($n^2$ + n + 2). In order to move forward, we first need to prove $n^2$ + n + 2 is divisible by 2 (proven below). Now that we know $n^2$ + n + 2 is even, we have 2 · j = $n^2$ + n + 2 for some j. We thus have that 3($n^2$ + n + 2) = 3 · 2 · j = 6 · j. Letting $n^3$ + 5n = 6 · i, we see that $(n+1)^3$ + 5n = 6i + 6j = 6(i + j). Thus 6 divides $(n+1)^3$ + 5(n + 1), verifying that P(n + 1) holds. By the principle of induction we conclude $n^3$ + 5n is divisible by 6. □

**Proposition.** *For all $n \in \mathbb{N}$, $n^2$ + n + 2 is divisible by 2.*

*Proof.* Let's first identify P(n). We put P(n) to be the sentence "$n^2$ + n + 2 is divisible by 2". Let's first consider P(1).
**Base.** P(1) asserts $1^2$ + 1 + 2 is divisible by 2. Since $1^2$ + 1 + 2 = 4, and 4 is divisible by 2, we conclude that P(1) holds.
**Successor.** Assume P(n) holds. That is $n^2$ + n + 2 is divisible by 2. Consider $(n+1)^2$ + n + 1 + 2. $(n+1)^2$ + n + 3 = $n^3$ + 2n + 1 + n + 3 = $n^3$ + 3n + 4. Rewriting we have ($n^2$ + n + 2) + (2n + 2). Since P(n) holds, $n^2$ + n + 2 = 2j for some j. We have 2(j + n + 1). We conclude that $(n+1)^2$ + n + 1 + 2 is divisible by 2, verifying that P(n + 1) holds. By the principle of induction the proposition holds. □

**Proposition 2.21.** *There exists no integer x such that $0 < x < 1$.*

*Proof.* Suppose toward a contradiction there exists an integer x such that 0 < x < 1. By definition, x - 0 ∈ ℕ and by our axioms for the integers x ∈ ℕ. x < 1, but by Proposition 2.20 (proven below), x ≥ 1 which is absurd. □

**Proposition 2.20.** *For all $k \in \mathbb{N}$, $k \geq 1$.*

*Proof.* Let's formulate P(n) . "n ≥ 1"
**Base.** For this case, 1 = 1, so 1 ≥ 1. Hence P(1) holds.
**Successor.** Suppose P(n) holds. That is 1 ≤ n. Consider n + 1 - n. Commuting we see that 1 + n - n = 1. Therefore n + 1 - n ∈ ℕ. We deduce that n < n + 1 since P(n) holds 1 ≤ n. The relation ≤ is transitive, so 1 ≤ n ≤ n + 1 implies 1 ≤ n + 1. We have proven the successor case by the principle of mathematical induction, we are done. □

**Proposition 2.23.** *Let m, $n \in \mathbb{N}$. If n is divisible by m then $m \leq n$.*

*Proof.* We argue by induction on the claim P(n) "m ≤ n."
**Base.** n = 1. In this case 1 = j · m for some j ∈ ℤ by definition of

divisibilty. j $\in \mathbb{N}$ by proposition 2.11. m = j = n = 1, Thus P(1) holds.
**Successor.** Suppose P(n) holds. That is, m $\leq$ n. By definition, this means n - m $\in \mathbb{N}$ or n - m = 0. If n - m $\in \mathbb{N}$, n - m + 1 $\in \mathbb{N}$ because $\mathbb{N}$ is closed under addition, and 1 $\in \mathbb{N}$ by proposition 2.14(i). Otherwise, if n - m = 0, then n - m + 1 = 0 + 1 = 1 $\in \mathbb{N}$ by proposition 2.14.(i). We have proven the successor case by the induction. □

**Proposition 2.24.** *For all $k \in \mathbb{N}$, $k^2$ + 1 > k.*

*Proof.* We argue by induction the claim P(n) "$n^2$ + 1 > n." Let's first observe P(1)
**Base.** n = 1. In this case $1^2$ + 1 = 2 > 1. Thus P(1) holds.
**Successor.** Suppose P(n) holds. That is, $n^2$ + 1 > n. Consider $(n+1)^2$ + 1. $(n+1)^2$ + 1 = $n^2$ + 2n + 2 = $n^2$ + 2 + 2n = $n^2$ + 1 + 1 + 2n > n + 1 + 2n. n + 1 + 2n > n + 1 by proposition 2.7(i), since 2n > 0. By transitivity of >, $(n+1)^2$ + 1 > (n + 1). By induction, we've proven the proposition. □

**Proposition 2.27.** *For all integers $k \geq 2$, $k^2 < k^3$.*

*Proof.* We argue by induction the claim P(n) "$n^2 < n^3$." Let's first observe P(2)
**Base.** n = 2. In this case, $2^2 < 2^3$. $2^2 = 4 < 2^3 = 8$. Thus P(2) holds.
**Successor** Suppose P(n) holds. That is, $n^2 < n^3$. Consider $(n+1)^3$. $(n+1)^3 = n^3$ + 3$n^2$ + 3n + 1 > $n^2$ + 3$n^2$ + 3n + 1 = 4$n^2$ + 3n + 1 = 3$n^2$ + n + $n^2$ + 2n + 1. 3$n^2$ + n + $n^2$ + 2n + 1 > $n^2$ + 2n + 1 by proposition 2.7(i), since 3$n^2$ + n > 0. $n^2$ + 2n + 1 = $(n+1)^2$, so by transitivity of >, $(n+1)^2 < (n+1)^3$. We have thus proven this proposition by induction. □