CHALLENGE PROBLEM 1

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Axiom (†). There exists a subset $\mathbb{N} \subseteq \mathbb{Z}$ with the following properties:

(1) If $m, n \in \mathbb{N}$, then $m + n \in \mathbb{N}$.
(2) If $m, n \in \mathbb{N}$, then $mn \in \mathbb{N}$.
(3) $0 \notin \mathbb{N}$.
(4) For every $m \in \mathbb{Z}$, we have $m \in \mathbb{N}$, $m = 0$, or $-m \in \mathbb{N}$.

Definition. The **successor function** on the integers is defined by $s(x) := x + 1$. For $m \geq 1$, we define $s^{m+1}(x)$ recursively by $s^{m+1}(x) := s(s^m(x))$.

Axiom(∗). For all $m \geq 1$, $s^m(0) \neq 0$.

Axiom(∗∗). For all x $\in \mathbb{Z}$, there is $m \geq 0$ such that $s^m(x) = 0$ or $s^m(0) = x$.

Definition. The set of **successors of zero** is defined to be

$$N := \{x \in \mathbb{Z} | \exists m \geq 1 s^m(0) = x\}$$

Prove the following proposition.

Proposition. *Assume that the integers satisfy* $(*)$ *and* $(**)$*. Then, the set* N *satisfies axiom* (†)*. That is to say,* N *satisfies the four conditions of the axiom.*

Proof. (1) Let P(n) be the statement "if $m, n \in \mathbb{N}$, then $m + n \in \mathbb{N}$" Let's first observe P(1)
Base. n = 1. $m + 1 = s(m) = s(s^m(0))$. Thus, the proposition holds.
Successor. Assume P(n) holds. That is, $m + n = s^m(n) \in N$. Consider $m + n + 1$. $m + n + 1 = s^{m+1}(n) = s(s^m(x))$ by definition. By induction, we know $s^m(n) \in N$. Thus, $s(s^m(x)) = s^m(x) + 1 = s^m(x) + s(0) \in N$ by the principal of mathematical induction.

(2) Let P(n) be the statement " If $m, n \in \mathbb{N}$, then $mn \in \mathbb{N}$."Observe P(1):
Base.n = 1. $m \cdot 1 = m \in \mathbb{N}$.
Successor. Assume P(n) holds. That is, $mn \in \mathbb{N}$. Consider $m(n+1)$.

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By our axioms for the integers, we may rewrite this as $mn + m$. By induction, $mn \in N$, and we already know $m \in N$. Thus, by the first part of this proposition and induction, $mn + n \in N$.

(3) By our axiom $(*)$, we know $0 \notin N$.

(4) Take $m \in \mathbb{Z}$. By part 3 of this proposition, if $m = -m = 0$, $m \notin N$.

Now assume, $m \in N$. Suppose toward a contradiction that $-m \in N$. By our first part of this proposition, this would mean $m + (-m) = 0 \in N$, which is absurd. $\square$

Sources.