DEFINING THE NATURAL NUMBERS

Our text book introduces the natural numbers via the following axiom.

Axiom. There exists a subset $\mathbb{N} \subseteq \mathbb{Z}$ with the following properties:

(1) If $m, n \in \mathbb{N}$, then $m + n \in \mathbb{N}$.
(2) If $m, n \in \mathbb{N}$, then $mn \in \mathbb{N}$.
(3) $0 \notin \mathbb{N}$.
(4) For every $m \in \mathbb{Z}$, we have $m \in \mathbb{N}$, $m = 0$, or $-m \in \mathbb{N}$.

As I explained in class, it is poor mathematical form to introduce an axiom for something that can be defined from existing axioms. (The axiom is also a statement in second order logic, which makes it even worse!) This challenge problem shows how to define $\mathbb{N}$, in a sensible manner.

0.1. **Challenge problem.**

Definition. The **successor function** on the integers is defined by $s(x) := x + 1$. The set of **successors of zero** is $N := \{n \in \mathbb{Z} \mid \exists m \geq 0 \; s^m(0) = n\}$

Prove the following proposition.

Proposition. *The set N satisfies the above axiom. That is to say, N satisfies the four conditions of the axiom.*