Summary of the T-varieties seminar

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1 Toric Varieties

1.1 Affine Toric Varieties

Definition 1.1. A *toric variety* is an irreducible variety (in this, usually affine or projective) V such that

(i) $(\mathbb{C}^*)^n$ is an open subset of V and

(ii) the action of $(C^*)^n$ on itself extens to an action of $(\mathbb{C}^*)^n$ on V.

Examples are $\mathbb{C}^n$, $(\mathbb{C}^*)^n$ and $\mathbb{P}^n$, for the latter, use that $(\mathbb{C}^*)^n$ can be identified with one of the open subsets $U_i = \{(a_0\colon \dots : a_{n+1}) \mid a_i \neq 0\}$.
Let $\boldsymbol{a} \in \mathbb{Z}^n$.
The map $(\mathbb{C}^*)^n \to \mathbb{C}^*$ given by $\boldsymbol{t} \mapsto \boldsymbol{t}^{\boldsymbol{a}}$ is called a *character*.
A *1-parameter subgroup* $\boldsymbol{\lambda}^{\boldsymbol{a}} \colon \mathbb{C}^* \to (\mathbb{C}^*)^n$ is given by $\boldsymbol{\lambda}^{\boldsymbol{a}}(t) = (t^{a_1}, \dots, t^{a_n})$.

Definition 1.2. A *rational polyhedral cone* $\sigma \subseteq \mathbb{R}^n$ is something of the form

$$\sigma = \{\lambda_1 \boldsymbol{u}_1 + \dots + \lambda_n \boldsymbol{u}_l \in \mathbb{R}^n \mid \lambda_1, \dots, \lambda_l \geq 0\},$$

for $\boldsymbol{u}_1, \dots, \boldsymbol{u}_l \in \mathbb{Z}^n$. We further define:

- σ is *strongly convex* if $\sigma \cap -\sigma = \{0\}$.
- The dimension of σ is the dimension of the smallest subspace of $\mathbb{R}^n$ that contains σ.
- A *face* of σ is the intersection of the set $\{\ell = 0\}$ with σ where ℓ is a linear form that is nononnegative on σ.
- 1-dimensional faces are called *edges*. For an edge ρ, its *primitive element* is the unique generator $\boldsymbol{n}_\rho$ of $\rho \cap \mathbb{Z}^n$. The cone is generated by the $\boldsymbol{n}_\rho$ for all its edges ρ.

- Codimension-1 faces are called *facets.*

Definition 1.3. For a cone (strongly convex, rational, polyhedral) $\sigma \subseteq \mathbb{N}_{\mathbb{R}}$, define its dual cone

$$\sigma^{\vee} = \{\boldsymbol{m} \in M_{\mathbb{R}} \mid \langle \boldsymbol{m}, \boldsymbol{u} \rangle \geq 0 \text{ for all } \boldsymbol{u} \in \sigma\}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the natural pairing between N and its dual lattice M. In the case of $N \cong M \cong \mathbb{Z}^n$, this is just the usual scalar product.

Using the dual cone $\sigma^{\vee}$, we can construct a variety U_σ as follows. Consider the lattice points of $\sigma^{\vee}$: $\sigma^{\vee} \cap \mathbb{Z}^n$. The lattice points are finitely generated (Gordan's Lemma). Let $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_l$ be generators, and consider the map

$$\varphi\colon \left(\mathbb{C}^*\right)^n \to \mathbb{C}^l$$

defined by $\varphi(\boldsymbol{t}) = \boldsymbol{t}^{m_1} \ldots, \boldsymbol{t}^{m_l}$ and let U_σ be the Zariski closure of φ.
One can prove that this is a toric variety, and that the $\boldsymbol{t}^{\boldsymbol{m}}$ are defined everywhere on U_σ.
The coordinate ring of U_σ is given by $\mathbb{C}\left[\sigma^{\vee} \cap \mathbb{Z}^n\right]$, which is a notation for the ring of Laurent polynomials over $\mathbb{C}$ generated by the $\boldsymbol{t}^{\boldsymbol{m}}$ for $\boldsymbol{m} \in \sigma^{\vee} \cap \mathbb{Z}^n$.

Theorem 1.4 (Theorem 7.2)**.** *The Zariski closure of the image of the map φ from above is the normal afine toric variety U_σ determined by σ and $\mathbb{Z}^n$ if and only if $\sigma^{\vee} \cap \mathbb{Z}^n$ is generated over $\mathbb{Z}_{\geq 0}$ by $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_l$.*

Alternative construction of U_σ: the Spectrum of the semigroup algebra $\mathbb{C}\left[\sigma^{\vee} \cap M\right]$. We also refer to the affine toric variety thus obtained by V_σ.

1.2 The Toric Variety of a Fan

Definition 1.5. A *fan* is a finite collection Σ of cones in $\mathbb{R}^n$ such that :

- Each $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.
- If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$.
- If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of each cone.

Let σ, τ be cones in a fan Σ and τ a face og σ. This implies that $\sigma^{\vee} \subseteq \tau^{\vee}$ and thus also $\mathbb{C}\left[s_\sigma\right] \subseteq \mathbb{C}\left[S_\tau\right]$, hence we get an embedding of V_τ as an open subset of V_σ. More precisely, V_τ is naturally isomorphic to the open subset defined by χ^minside V_σ, since $\mathbb{C}\left[S_\tau\right] = \mathbb{C}\left[S_\sigma\right]_{\chi^m}$, where $m \in M$ is such that $H_m \cap \sigma = \tau$.
The fact that any two cones in Σ intersect in a common face yields immmersions

$$V_\sigma \cap V_{\sigma'} \to V_\sigma \text{ and } V_\sigma \cap V_{\sigma'} \to V_{\sigma'}.$$

Denoting the images of these immersions by $V_{\sigma\sigma'}$ and $V_{\sigma'\sigma}$, respectively, there are isomorphisms between them. Now we have all the data we need to glue an abstract variety, according to the next definition.

Definition 1.6. Given a fan Σ in $N_{\mathbb{R}}$, X_Σ is the abstract toric variety constructed using the gluing data $\left\{\{V_\sigma\}_{\sigma \in \Sigma}, \{V_{\sigma\sigma'}\}_{\sigma,\sigma' \in \Sigma}, \{g_{\sigma\sigma'}\}_{\sigma,\sigma' \in \Sigma}\right\}$.

Theorem 1.7. *The construction above yields a is a normal separated toric variety X_Σ.*

In fact, the torus $T(N) = N \otimes_{\mathbb{Z}} C^* \cong (C^*)^n$ is the open subset obtained from the cone $\{0\}$.

Definition 1.8. A variety Y is called

- *normal*, if each local ring $\mathcal{O}_{Y,p}$ is normal, i.e. integrally closed in its field of fractions. (should we assume Y irreducible here? yes, but toric varieties are irreducible)
- *separated*, if the diagonal map $\Delta \colon Y \to Y \times Y$ is closed. (With respect to which topology on $Y \times Y$?)

Some examples like $\mathbb{P}^2$, to be added.

2 Divisors

3 Affine T-varieties

We use different notation for the concepts known from the very first part: M, N are still mutually dual lattices. A cone in $N_{\mathbb{Q}}$ spanned by elements $a_1, \ldots, a_m$ is now denoted by $\sigma = \langle a_1, \ldots, a_m \rangle$ and $\sigma^\vee$ lives inside $M_{\mathbb{Q}}$. We assume that the a_i are primitive elements of $N_{\mathbb{Q}}$, and we denote the affine troic variety obtained from σ by $\mathrm{TV}(\sigma)$.

3.1 Toric Bouquets

Let Δ be a polyhedron, i.e. a finite intersection of hyperplanes, in $N_{\mathbb{Q}}$. We define its *tail cone*

$$\mathrm{tail}\,(\Delta) = \{a \in N_{\mathbb{Q}} \mid a + \Delta \subseteq \Delta\}\,.$$

Assume now that Δ is pointed, (i.e. strongly convex), then we can write it as the Minkowski sum $\Delta = \Delta^c + \mathrm{tail}\,(\Delta)$, where Δ^c denotes the convex hull of the vetices of Δ, which in this case exist.