Summary of 'What is a Toric Variety' by David Cox

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Definition 0.1. A *toric variety* is an irreducible variety (in this, usually affine or projective) V such that

(i) $(\mathbb{C}^*)^n$ is an open subset of V and

(ii) the action of $(C^*)^n$ on itself extens to an action of $(\mathbb{C}^*)^n$ on V.

Examples are $\mathbb{C}^n$, $(\mathbb{C}^*)^n$ and $\mathbb{P}^n$, for the latter, use that $(\mathbb{C}^*)^n$ can be identified with one of the open subsets $U_i = \{(a_0 \colon \ \dots : a_{n+1}) \mid a_i \neq 0\}$.
Let $\boldsymbol{a} \in \mathbb{Z}^n$.
The map $(\mathbb{C}^*)^n \to \mathbb{C}^*$ given by $\boldsymbol{t} \mapsto \boldsymbol{t}^{\boldsymbol{a}}$ is called a *character*.
A *1-parameter subgroup* $\boldsymbol{\lambda}^{\boldsymbol{a}} \colon \mathbb{C}^* \to (\mathbb{C}^*)^n$ is given by $\boldsymbol{\lambda}^{\boldsymbol{a}}(t) = (t^{a_1}, \dots, t^{a_n})$.

Definition 0.2. A *rational polyhedral cone* $\sigma \subseteq \mathbb{R}^n$ is something of the form

$$\sigma = \{\lambda_1 \boldsymbol{u}_1 + \dots + \lambda_n \boldsymbol{u}_l \in \mathbb{R}^n \mid \lambda_1, \dots, \lambda_l \geq 0\},$$

for $\boldsymbol{u}_1, \dots, \boldsymbol{u}_l \in \mathbb{Z}^n$. We further define:

- σ is *strongly convex* if $\sigma \cap -\sigma = \{0\}$.
- The dimension of σ is the dimension of the smallest subspace of $\mathbb{R}^n$ that contains σ.
- A *face* of σ is the intersection of the set $\{\ell = 0\}$ with σ where ℓ is a linear form that is nononnegative on σ.
- 1-dimensional faces are called *edges*. For an edge ρ, its *primitive element* is the unique generator $\boldsymbol{n}_\rho$ of $\rho \cap \mathbb{Z}^n$. The cone is generated by the $\boldsymbol{n}_\rho$ for all its edges ρ.
- Codimension-1 faces are called *facets*.

Definition 0.3. For a cone (strongly convex, rational, polyhedral) σ, define its dual cone as

$$\sigma^\vee \{\boldsymbol{m} \in \mathbb{R}^n \mid \langle \boldsymbol{m}, \boldsymbol{u} \rangle \geq 0 \text{ for all } \boldsymbol{u} \in \sigma\}.$$

Here $\langle \cdot, \cdot \rangle$ is the usual scalar product on $\mathbb{R}^n$.

Using the dual cone $\sigma^\vee$, we can construct a variety U_σ as follows. Consider the lattice points of $\sigma^\vee$: $\sigma^\vee \cap \mathbb{Z}^n$. The lattice points are finitely generated (Gordan's Lemma). Let $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_l$ be generators, and consider the map

$$\varphi\colon \left(\mathbb{C}^*\right)^n \to \mathbb{C}^l$$

defined by $\varphi(\boldsymbol{t}) = \boldsymbol{t}^{m_1} \ldots, \boldsymbol{t}^{m_l}$ and let U_σ be the Zariski closure of φ.
One can prove that this is a toric variety, and that the $\boldsymbol{t}^{\boldsymbol{m}}$ are defined everywhere on U_σ.
The coordinate ring of U_σ is given by $\mathbb{C}\left[\sigma^\vee \cap \mathbb{Z}^n\right]$, which is a notation for the ring of Laurent polynomials over $\mathbb{C}$ generated by the $\boldsymbol{t}^{\boldsymbol{m}}$ for $\boldsymbol{m} \in \sigma^\vee \cap \mathbb{Z}^n$.

Theorem 0.4 (Theorem 7.2)**.** *The Zariski closure of the image of the map φ from above is the normal afine toric variety U_σ determined by σ and $\mathbb{Z}^n$ if and only if $\sigma^\vee \cap \mathbb{Z}^n$ is generated over $\mathbb{Z}_{\geq 0}$ by $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_l$.*

Definition 0.5. A *fan* is a finite collection Σ of cones in $\mathbb{R}^n$ such that :

- Each $\sigma \in \Sigma$ is a strongly convex rational polyhedral cone.
- If $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$.
- If $\sigma, \tau \in \Sigma$, then $\sigma \cap \tau$ is a face of each cone.