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20

Interest Rate Derivatives and Volatility

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20.1 INTRODUCTION

Interest rate volatility (IRV) affects a wide base of individuals, investors, companies, and even governments. Individuals who have borrowed through adjustable-rate retail products, such as student loans and mortgages, are susceptible to greater uncertainty about the magnitude of their liabilities from one payment period to the next when short-term interest rates are volatile. For individual and institutional investors in fixed-income products like corporate bonds and mortgage-backed securities, IRV translates directly into undesirable, and sometimes devastating, portfolio volatility. Routine issuers of debt, such as financial institutions, corporations, and governmental agencies, are also forced to deal with the impact of IRV on their vital funding decisions.

As with other asset classes, IRV trading resides in the domain of interest rate derivative (IRD) markets. However, unlike volatility trading in spot instruments such as equity indexes, commodities, and currency cross rates, fixed-income assets and derivatives require significantly different mathematical treatments to address added complexities such as annuities and credit risk to name a few. This chapter aims to equip the reader with a foundational understanding of the vast IRD market and the quantitative tools for measuring and managing IRV.

Section 20.2 provides perspective on the immensity of over-the-counter (OTC) and listed IRD markets and gives context to how IRV affects market participants. Section 20.3 introduces notation and foundational concepts required to understand various IRD contracts and their risks. Section 20.4 surveys a portion of the large literature on IRD pricing methodologies to the extent relevant to the subject of IRV. Section 20.5 reviews existing volatility trading practices, introduces model-free option-based volatility measures for various interest rates, and covers recent developments in the standardization of IRV trading. Section 20.6 concludes. A technical appendix contains additional details aiming to make the chapter as self-contained as possible.

20.2 MARKETS AND THE INSTITUTIONAL CONTEXT

20.2.1 Market Size

IRDs constitute the largest segment of the OTC derivative market. At a notional amount outstanding of US \$561.3 trillion as of June 2013 as reported by the Bank for International Settlements (BIS), the IRD market dwarfs all other OTC markets, with foreign exchange derivatives being a distant second at US \$73.1 trillion (see Figure 20.1).

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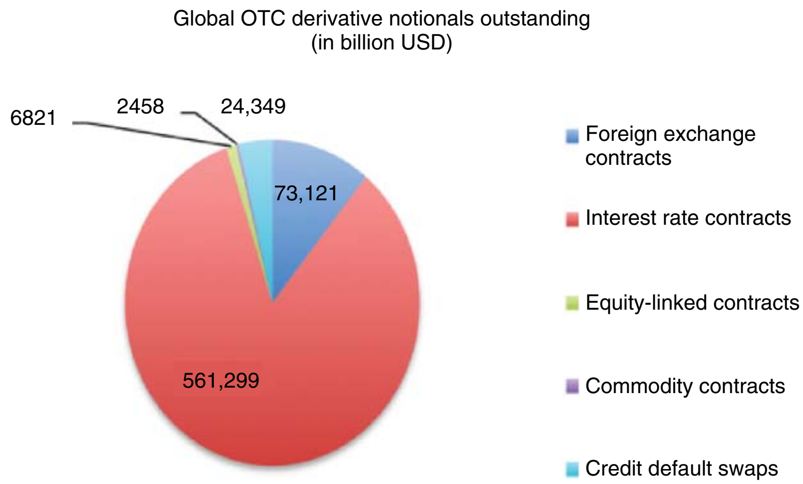

Figure 20.1 Global OTC derivative notional outstanding. Source: Bank of International Settlements.

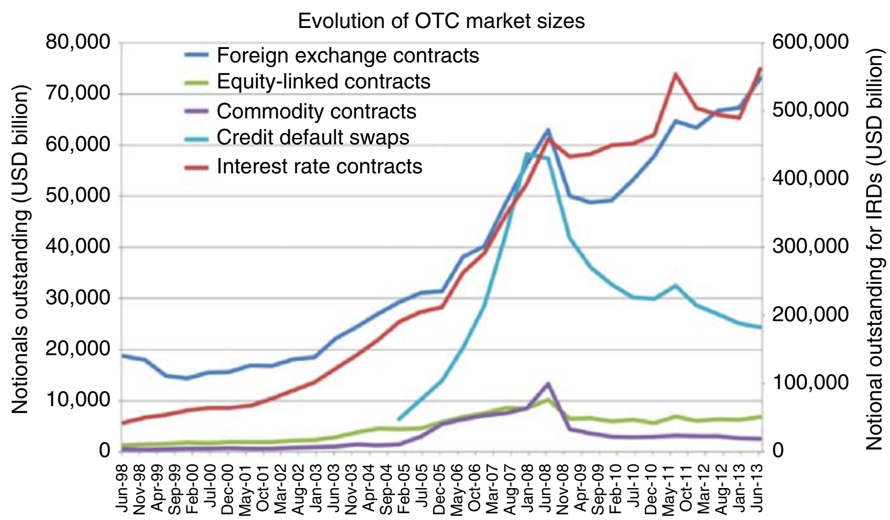

Figure 20.2 Evolution of OTC market sizes. Source: Bank of International Settlements.

Many factors come together to drive the activity and resulting size of the IRD market. The underlying fixed-income asset base has large notional amounts outstanding and a wide range of varied debt instruments such as interbank loans, mortgages, and corporate and government bonds to name just a few. Financial institutions are heavy users of IRDs for hedging against, or modifying the risk profile of, such assets on their books. IRDs such as fixed-for-floating swaps also serve as popular tools globally among large corporations, municipalities, and other nonfinancial and financial institutions alike for asset liability and cash flow management. IRDs moreover allow speculators to make leveraged bets to express their investment views on various interest rates and fixed-income asset prices.

Over the past 15 years, OTC derivativetrading activity has increased significantly across all asset classes as can be seen in Figure 20.2. The steady growth since the late 1990s hit a speed bump during the financial crisis of 2007–2008 as market makers and end users were forced to reduce risk. Notably, credit derivatives, commonly thought to have been at the epicenter of the global market collapse, has since had a sharp decline in market size to just about half its peak of US $58 trillion notional outstanding in 2007. In contracts, the IRD market has continued to grow in size postcrisis and stands at 13 times its size 15 years ago, which is more than double the growth rate of other OTC derivative markets.

While the bulk of IRD trading has traditionally taken place OTC, certain types of IRDs, such as futures and options on government bonds and time deposits, are actively traded on derivative exchanges such as Chicago Mercantile Exchange (CME) Group, Eurex, and NYSE LIFFE. Listed IRDs across exchanges have a total notional outstanding of US $24.2 trillion, which pales in comparison to their OTC counterparts, but are still an order of magnitude larger in notional outstanding compared to other listed derivatives such as those on currencies and equity indexes (see Figure 20.3).

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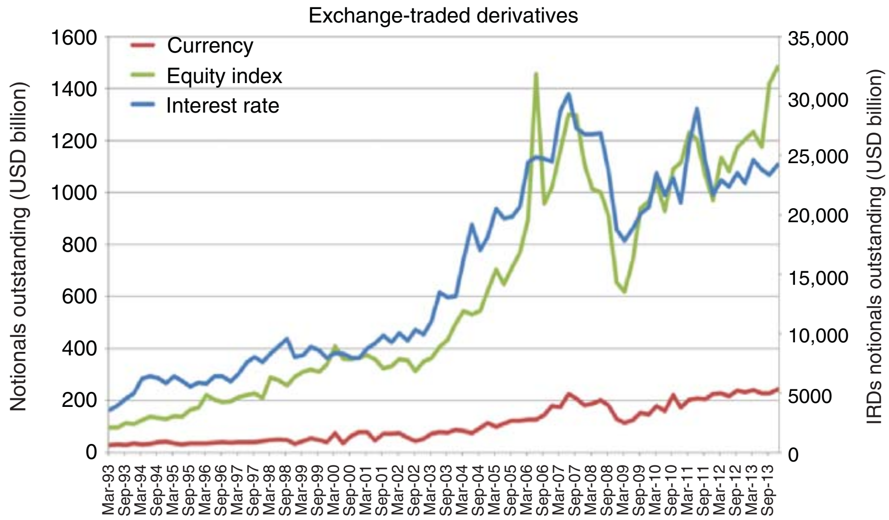

Figure 20.3 Exchange-traded derivatives. Source: Bank of International Settlements.

20.2.2 OTC IRD Trading and Volatility

The OTC fixed-for-floating interest rate swap (IRS) is by far the most liquid and actively traded IRD, with a notional amount outstanding of US $425.6 trillion as of June 2013. In a vanilla IRS, one party makes a stream of fixed-rate payments to its counterparty over a period of time, typically ranging from 1–30 years, in exchange for a stream of floating-rate payments. The floating leg of the IRS references a short-term interbank lending rate, most commonly the 3- and 6-month London Interbank Offered Rates (Libors). The British Bankers' Association had historically calculated Libor until its administration was recently transferred to ICE Benchmark Administration Limited in February 2014. Libor is designed to indicate the average rate at which a panel of banks is able to obtain unsecured funding in various currencies and terms and is calculated on a daily basis as the panel's interquartile average. Libor is frequently used as the reference rate for a wide range of debt instruments ranging from floating-rate notes issued by corporations, adjustable-rate mortgages, and student loans, which are quoted and traded at a spread to Libor.

The fixed leg of an IRS, known as the "swap rate," may be interpreted as the market's expectation of a break-even rate that provides the same net present value as the floating leg over the life of the IRS and thus closely relates to the term funding cost for highly rated financial institutions and corporations. In turn, the zero-coupon yield curve based on Libor/swaps, and variants thereof, are used to discount cash flows and drive the pricing of a large pool of fixed-income securities and derivatives.

At the most general level, volatility of swap rates affects individual and institutional investors in fixed-income assets by translating into volatility of their portfolios, which is a key risk driving investment decisions. Even with careful diversification, correlation across assets has the tendency to increase in times of heightened volatility to reduce the intended benefits of holding a mix of assets believed to smooth out returns. Figure 20.4 plots various historical realized basis point (BP) volatilities, the 1Y–10Y forward swap rate, and the Chicago Board Options Exchange (CBOE) SRVX Index, which is an implied volatility index based on 1Y–10Y swaptions. The "shrink fwd 1y10y BP vol" is the realized basis point volatility based on the 10-year swap rate 1 year forward, 1 year minus 1 day forward, 1 year minus 2 days forward, and so forth.

On the other side of the fence, issuers of Libor/swap-sensitive debt instruments are also affected by swap rate volatility. Greater volatility makes a corporate treasurer's job generally more difficult. For instance, in times of acute uncertainty about the future path of interest rates, corporate bond issuers have been known to delay or cancel the pricing of new issues in the primary market until volatility subsides. In other examples, companies with large fixed-income portfolios deliberately constructed to offset specific liabilities arising from their core businesses, such as insurance companies, or bond funds benchmarked to certain indexes face increased risk of tracking errors when interest rates experience large moves.

For both investors and issuers of fixed-income securities, spikes in volatility also have a secondary, but no less pernicious, effect of being accompanied by the evaporation of liquidityas dealers widen the width and reduce the depth of their markets or step to the sidelines altogether as seen during the crisis of 2007–2008. In extreme market panics, reduced liquidity leads to smaller trades having disproportionate price impact, and a vicious cycle between increased volatility and reduced liquidity can ensue.

Hedging and expressing views on swap rate volatility are traditionally done through trading swaptions, that is, option on IRS, which are part of the third largest OTC IRD category – interest rate options – with US $49.4 trillion outstanding as of June 2013. At maturity, a payer (receiver) swaption gives the buyer the right, but not obligation, to pay (receive) a predetermined fixed rate,

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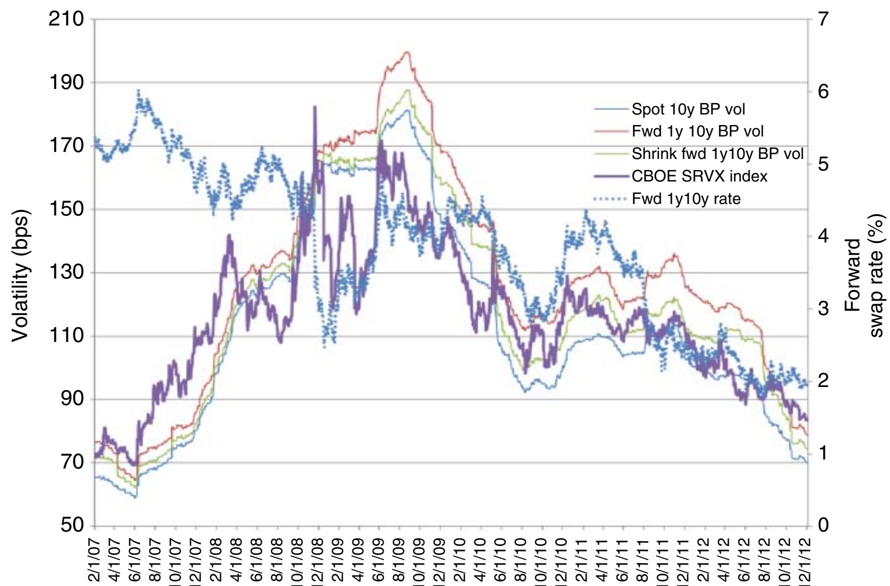

Figure 20.4 Realized and traded volatility indexes. Source: Chicago Board Options Exchange and Bloomberg.

that is, strike, on an IRS. Unlike equity options, which have only one temporal dimension, swaptions are defined by two time horizons – the maturity of the swaption and the tenor of the underlying swap – and therefore give rise to a volatility cube defined by (maturity, tenor, strike) coordinates, which offers a rich set of trading opportunities. Some maturity–tenor combinations, such as USD 1Y–10Y, are generally more active than others, but the distribution of liquidity morphs through time depending on the nature of events driving the market.

The most common directional volatility trades are at-the-money (ATM) swaptionstraddles in which one buys (sells) a payer and a receiver swaption, both struck ATM, to be long (short) volatility. According to a major global interdealer broker (IDB), such trades constitute the majority of swaption trades they broke between dealers on most days. To remain delta neutral throughout the trade, ATM straddles require dynamic hedging using, for example, forward swaps, which are part of the US $88.3 trillion OTC forward rate agreement (FRA) market. Even with delta hedging, there can be significant P&L noise arising from the path-dependent nature of the strategy's payoffs, as explained in Section 20.5.1, which has been cited by some funds as a reason, among others, such as collateral requirements, for abstaining from OTC swap volatility trading.

Swap rate volatility trading is also done in the form of forward volatility Agreements (FVAs)whereby one party agrees to buy or sell on some future date an ATM swaption straddle at a predetermined strike. This is a form of forward volatility trading in which one takes a view on the change in implied volatility over time. The perception of the FVA market's activeness varies significantly depending on which dealer one speaks to, ranging from nearly nonexistent to a burgeoning business. More obscure forms of OTC IRV trading include variance swaps on bond exchange-traded funds (ETFs) in which one party agrees to pay another the realized variance of daily returns of the reference ETF.

A comprehensive survey of the entire OTC IRD landscape is outside the scope of this chapter, but the essential building blocks for OTC IRV trading have been introduced.

20.2.3 Exchange-Listed IRD Trading and Volatility

Some of the larger derivative exchanges have succeeded in capturing the market for certain types of IRDs, the most historically successful of which are futures and options on government bonds and time deposits.

Government bond futures and options include those listed on the U.S. Treasuries at the CME, on Euro Bund/Schatz/Bobl at Eurex, on Gilts at NYSE LIFFE, and on Japanese Government Bonds at the Osaka Securities Exchange. In a departure from the rate-based OTC IRDs described earlier, futures and options on government bonds are quoted and traded in terms of prices as opposed to yields. Moreover, unlike OTC FRA, listed bond futures involve additional contract specifications such as the cheapest-to-deliver option, which adds a layer of complexity to the pricing of futures but is a topic that is by now well understood. Liquidity in options on government bond futures is concentrated in the short term, such as the front 3 serial months, and does not go nearly as far out as in the swaption market. Such shorter-dated options are commonly thought to be useful for trading volatility around macro announcements and policy events and are also used by OTC IRD dealers as additional sources of liquidity to hedge against their OTC positions.

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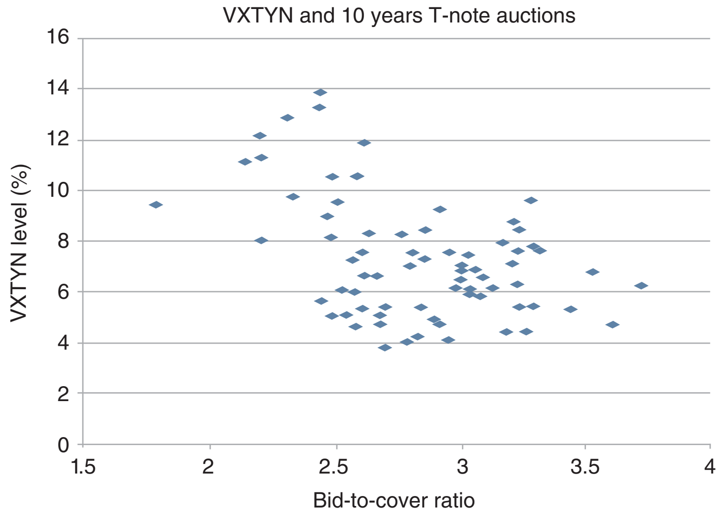

Figure 20.5 VXTYN and 10-year T-note auctions. Source: Chicago Board Options Exchange and Bloomberg.

In parallel with the discussion previously regarding the effect of swap rate volatility on corporate bond issuers, even the U.S. Treasury is not immune to IRV. Figure 20.5 illustrates how the bid-to-cover ratio in the 10-year Treasury Note auctions, a measure of demand for Treasury securities, is inversely related to IRV as measured by the CBOE/CBOT 10-year U.S. Treasury Note Volatility Index, VXTYN.[1]

The other successful category of exchange-listed IRDs includes futures and options on time deposits such as Eurodollar and Fed Funds at CME and Euribor and Sterling at NYSE LIFFE. A full listing of IRDs, their contract specifications, and volume statistics are maintained and publicly accessible on the various exchanges' websites.

20.2.4 Recent Developments in the IRD Market

The 2007–2008 financial crisis set in motion a tidal wave of reforms over the OTC derivative markets, and IRDs are one of the focal points of it given its sheer size and importance to the orderly functioning of global financial markets. Among the mind-numbing set of issues, regulators in various domiciles are focusing their efforts around implementing centralized clearing, settling, and reporting for a growing set of OTC IRDs. In response, various market utilities, from well-established exchanges and IDBs to a slew of new entrants, are adapting their businesses and positioning themselves the best they can for what the postreform landscape may look like.

While a detailed description of OTC reforms is not salient to this chapter, it is worth noting that the resulting shift in the longstanding dynamics between the listed and OTC worlds has already led to some previously unthinkable product innovations that are relevant to the subject at hand. For instance, in June 2012, CBOE announced that it had obtained swaption data licenses from multiple top IDBs in the IRD space to create a real-time index, named SRVX (in Figure 20.4), for tracking 1Y–10Y swap rate volatility. It is a safe bet to assume that a derivative exchange looking to enter the swaption volatility space would have had little chance of collaborations with IDBs before the crisis. What is more, many of the major IRD dealers have even expressed interest in the idea of an exchange-owned tradable swap rate volatility index, presumably in part because it overcomes the challenge dealers faced when trying to instill a sense of objectivity and credibility in their proprietary swap rate volatility indexes based on prices coming from their own trading desks. Other examples of such innovations include CME's deliverable IRS futures, which have garnered public support from multiple IRD dealers.

20.3 DISSECTING THE INSTRUMENTS

Evaluating IRDs poses new challenges compared to the equity derivative space. First, interest rate risk can take on different meanings: for example, it can relate to possibly imminent rate changes or to longer-term developments. Concerns about the former risk give rise to contracts such as time deposits (see Section 20.3.2), and concerns about the latter can be mitigated through IRS (see Section 20.3.3). The notion of risk adjustment differs in these two examples as it depends on varied risks affecting each market.

[1]Note that TYVIX has replaced VXTYN as the new ticker symbol while this article was in press.

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A second issue is that the payoffs to IRDs are obviously interest rate dependent, and yet we need to discount them relying on the very same interest rates – or at least relying on other interest rates that presumably correlate with the payoffs (see Chapter 18 for discussions regarding "multiple yield curves"). Is there a way to express the value of IRDs in a compact manner that also incorporates the random character of the discounting factors?

Standard financial theory holds that absence of arbitrage in frictionless markets implies that there is indeed a unit of account such that the price of all securities denominated in this account is martingales under a given probability. We term such unit of account *numéraire* to honor Léon Walras (1874) that famously concluded that any market is in equilibrium, given the equilibrium in the remaining markets, such that we can take the price of any commodity as given and study the equilibrium price of any other commodity in terms of the chosen numéraire.

The evaluation of fixed-income derivatives is facilitated by the relevant notion of numéraire that applies to each fixed-income asset class. Evaluating equity derivatives assuming constant rates typically requires no more than making reference to the money market account and the associated and well-known risk-neutral probability – "discounted asset prices are martingales under Q" (Cox, Ross, and Rubinstein, 1979; Harrison and Kreps, 1979). In contrast, we have appropriate notions of numéraire applying to each of the fixed-income asset classes we consider. In terms of the previous introductory example, the numéraire applying to time deposits is not the same as that applying to IRS.

This section provides foundations regarding IRDs that hinge upon these appropriate notions of numéraire. It develops representations of IRD prices, which we shall use in later sections to pin down their value based on specific modeling assumptions and as benchmarks to deal with the fundamental scope of the chapter to explain how to set a standard for forward-looking measures of IRV. We cover three markets, one for bonds (Section 20.3.1), one for time deposits (Section 20.3.2), and one for forward rate agreements and IRS (Section 20.3.3), as well as the main derivatives based on the third market (Section 20.3.4).

20.3.1 Government Bonds

This section deals with derivatives written on bonds, referred to as "government bonds," that are not subject to default risk – credit risk is not taken into account in any juncture of this chapter. Our objective is to develop basic representations for the price of forwards, futures, and options written on these bonds expressed in terms of the numéraire for this market that facilitates their evaluation.

Throughout this and the following section, we shall make reference to the following basic notation. We denote the instantaneous rate process with r_t and the price of a zero-coupon bond at t and expiring at T with $P_t(T)$, assuming no default risk. Hereafter, we shall also refer to r_t as the *short-term rate* relying on a standard terminology. We let the risk-neutral probability be Q and $\mathbb{E}_t$ be the risk-neutral expectation taken conditional upon the information set at time t; Chapter 19 provides a survey on risk-neutral pricing. Let $B_t(\mathbb{T})$ be the price at t of a coupon-bearing bond expiring at $\mathbb{T} \geq t$.

We assume the bond pays off coupons $\frac{C_i}{n}$ over the sequence of dates T_i, $i = 1, \ldots, N$, where n is the frequency of coupon payments and $\mathbb{T} \equiv T_N$. For example, $n = 2$ denotes semiannual coupon payments, in which case $T_i - T_{i-1} = \frac{1}{2}$. In the absence of arbitrage,

$$B_t(\mathbb{T}) \equiv \sum_{i=i_t}^{N} \frac{C_i}{n} P_t(T_i) + P_t(\mathbb{T}) \tag{20.1}$$

where T_{i_t} is the first coupon payment date after t.

We now discuss two basic derivative instruments referencing nondefaultable coupon-bearing bonds, which help formulate views on developments in sovereign bond markets – futures and forwards on coupon-bearing bonds. While an important portion of trading activity is concentrated in exchanges such as CME and in regard to bond futures, OTC markets still host trading in bond forwards, and so both are interesting instruments to survey.

We have one additional source of motivation to deal with both instruments in this chapter. Markets for trading government bond volatility rely upon highly liquid American-style options on futures. Chicago Board Options Exchange publishes an index of Treasury volatility based on American options on the 10-year U.S. Treasury Note futures traded at Chicago Mercantile Exchange. At the same time, we shall explain that a "model-free" forward-looking volatility gauge should rely on European options on zero-coupon bond forwards (see Section 20.5). Understanding how American *futures* options are priced relative to European *forward* options is therefore an element of paramount interest in the context of this chapter.

We now proceed with technical details. Theoretically, futures and forwardsdiffer because futures are expectations taken under the risk-neutral probability, and forwards are expectations under a different probability measure called the forward probability. We explain in the following how the risk-neutral and the forward probability are the same when the short-term rate is constant or nonrandom. Therefore, futures and forwards differ in the context of this chapter for the obvious reason that interest rates are taken to be random.

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20.3.1.1 Futures Denote with $\tilde{F}_t(S,\mathbb{T})$ the value at t of a future expiring at S on a coupon-bearing bond expiring at $\mathbb{T}$. Intuitively, a future position entails a continuous marking to market such that the expected instantaneous changes in the future position are zero under the risk-neutral probability, $\mathbb{E}_t(d\tilde{F}_t(S,\mathbb{T}))=0$, and because of the boundary condition, $\tilde{F}_S(S,\mathbb{T})=B_S(\mathbb{T})$, the future value satisfies,

$$\tilde{F}_t(S,\mathbb{T})=\mathbb{E}_t(B_S(\mathbb{T})) \tag{20.2}$$

In Section 20.5, we shall rely on Equation 20.2 to explain how to gauge the error made by approximating a Treasury volatility index calculated through American options on bond futures, rather than European options on bond forwards.

20.3.1.2 Forwards Next, let$F_t(S,\mathbb{T})$ be the price at t of a forward expiring at $S\le\mathbb{T}$. The payoff for going long the forward is $B_S(\mathbb{T})-F_t(S,\mathbb{T})$, and the position is costless at inception such that, in the absence of arbitrage, the forward satisfies

$$F_t(S,\mathbb{T})=\frac{1}{P_t(S)}\mathbb{E}_t\left(e^{-\int_t^S r_\tau d\tau}B_S(\mathbb{T})\right)=\frac{B_t(\mathbb{T})}{P_t(S)}$$

where the second equality follows because the discounted coupon-bearing bond is a martingale under the risk-neutral probability. Alternatively, elaborating as follows on the first equality, heuristically,

$$F_t(S,\mathbb{T})=\frac{1}{P_t(S)}\mathbb{E}_t\left(e^{-\int_t^S r_\tau d\tau}B_S(\mathbb{T})\right)=\mathbb{E}_t(\xi_S^F B_S(\mathbb{T}))=\mathbb{E}_t^{Q_{F^S}}(B_S(\mathbb{T})) \tag{20.3}$$

where Q_{F^S} is a new probability, defined through the Radon–Nikodym derivative,

$$\xi_S^F\equiv\left.\frac{dQ_{F^S}}{dQ}\right|_{\mathcal{F}_S}=\frac{e^{-\int_t^S r_\tau d\tau}}{P_t(S)} \tag{20.4}$$

and $\mathcal{F}_S$ denotes the information set at time S. The probability Q_{F^S}, referred to as S-forward probability, was introduced by Geman (1989) and Jamshidian (1989) and further analyzed by Geman, El Karoui, and Rochet (1995).[2]

We assume $F_t(S,\mathbb{T})$ is a diffusion process. By Equation (20.3), it is a martingale under the S-forward probability, and it is also strictly positive, and the forward price satisfies

$$\frac{dF_\tau(S,\mathbb{T})}{F_\tau(S,\mathbb{T})}=\upsilon_\tau(S,\mathbb{T})\cdot dW_\tau^{F^S},\quad \tau\in(t,S) \tag{20.5}$$

where $\upsilon_\tau(S,\mathbb{T})$ is an instantaneous volatility process adapted to $W_\tau^{F^S}$, which is a multidimensional Brownian motion under the S-forward probability. Note that $\upsilon_\tau(S,\mathbb{T})=\sigma_\tau^B(\mathbb{T})-\sigma_\tau(S)$ where $\sigma_\tau^B(\mathbb{T})$ denotes the instantaneous volatility of the log-changes in the coupon-bearing bond price in Equation 20.1 and $\sigma_\tau(S)$ is the instantaneous volatility of the log-changes of a zero-coupon bond expiring at S.

20.3.1.3 Options While OTC options commonly reference forwards or cash bonds, exchanges such as CME host trading of American options on futures. Section 20.5 explains that model-free volatility indexes should rely on European options on forwards, rather than American options on futures. We shall also deal with the latter in Section 20.5. In this section, we provide some details regarding the pricing of European options on forwards – with the option expiration being set equal to that of the forward.

Let $C_t^b(T,S)$ denote the price of a European option expiring at T and referencing a zero-coupon bond expiring at time S:

$$\begin{aligned}C_t^b(T,S)&=\mathbb{E}_t\left[e^{-\int_t^T r_\tau d\tau}\left(P_T(S)-K\right)^+\right]\\&=P_t(S)\cdot\mathbb{Q}_{F^S}(P_T(S)\ge K)-KP_t(T)\cdot\mathbb{Q}_{F^T}(P_T(S)\ge K)\end{aligned} \tag{20.6}$$

where $(x)^+\equiv\max(x,0)$ and $\mathbb{Q}_{F^j}(\cdot)$ denotes the cumulative distribution function under the j-forward probability.

[2]To explain heuristically how to arrive to the Radon–Nikodym derivative in Equation 20.4, consider a simple three-period market, with the current date being date-0, n states of nature occurring at date-2, and finally, the coupon-bearing bond (CBB) expiring at some date beyond date-2. Let B_i be the date-2 price of the CBB in state i and z_i be the discounting factor applying to payments contingent on the state of nature being i at date-2 – the counterpart to the numerator in Equation 20.4. Clearly, the arbitrage-free price of a zero-coupon bond expiring at date-2 is $P=\sum_{i=1}^n z_i q_i$, where q_i denotes the risk-neutral probability that state i will occur at date-2. Then, the price of the forward is, analogously to Equation 20.3, $F=\sum_{i=1}^n \frac{z_i B_i}{P}q_i=\sum_{i=1}^n B_i q_{F,i}$, where $q_{F,i}$ is defined through $\frac{q_{F,i}}{q_i}=\frac{z_i}{P}$. The meaning of this ratio parallels that of the Radon–Nikodym derivative in Equation 20.4, and it is immediate to verify that the "weights" $q_{F,i}$ sum to 1: $q_{F,i}$ is a probability, the counterpart to the forward probability in the continuous-time model.

Equation 20.6, derived in the appendix, is the bond option counterpart to the Black and Scholes (1973) formula in that (i) the underlying asset is a zero-coupon bond expiring at S, the current price of which, $P_t(S)$, multiplies the probability $\mathbb{Q}_{FS}$, and (ii) the present value of the strike price, $KP_t(T)$, multiplies the probability for the market numéraire in this problem, $\mathbb{Q}_{FT}$. Jamshidian (1989) first obtained Equation 20.6 in the context of a specific model for the short-term rate, reviewed in Section 20.4.2.

20.3.2 Time Deposits

A large portion of interest rate transactions involves interbank trading of time deposits as mentioned in Section 20.2, with the Libor being the best-known indicative cost of these transactions. The Libor serves as the basis of many IRDs surveyed in this chapter, such as interest rate futures and options, FRA, and IRS, caps, and floors, to mention the most important few. The focus of this section is on interest rate futures and options traded on exchanges, such as the CME's Eurodollar futures and options based on 3-month Libor.

Time deposit derivatives have in common with government bond derivatives the feature that their pricing is facilitated by making reference to the forward probability. Let $l_t(\Delta)$ be the simply compounded interest rate on a deposit for the time period from t to $t+\Delta$ (e.g., the 3-month Libor),

$$P_t(t+\Delta) \equiv \frac{1}{1+\Delta l_t(\Delta)}$$

where $P_t(t+\Delta)$ still denotes the time t price of a default-free zero-coupon bond expiring at $t+\Delta$, assuming that there is no counterparty risk. To simplify the exposition, we shall refer to $l_t(\Delta)$ as the Libor.

20.3.2.1 Forwards Consider a forward contract on the Libor, originated at time t, in which one party promises to pay the counterparty $100\times(1-l_S(\Delta))-Z_t(S,S+\Delta)$ at time S. The forward Libor price, $Z_t(S,S+\Delta)$, is defined as the clearing price for this derivative, that is, such that its initial value is zero at inception,

$$Z_t(S,S+\Delta) = 100\times(1-\hat{f}_t(S,S+\Delta)) \tag{20.7}$$

where

$$\hat{f}_t(S,S+\Delta) \equiv \mathbb{E}_t^{Q_{FS}}(l_S(\Delta))$$

and $\mathbb{E}_t^{Q_{FS}}$ denotes the expectation under the S-forward probability introduced in Equation 20.4. We refer to $\hat{f}_t(S,S+\Delta)$ as the forward Libor. It is obviously a martingale under Q_{FS}, with $\hat{f}_S(S,S+\Delta)=l_S(\Delta)$. Therefore, assuming that information is driven by Brownian motions, the forward price in Equation 20.7 satisfies

$$\frac{dZ_\tau(S,S+\Delta)}{Z_\tau(S,S+\Delta)} = v_\tau^z(S,\Delta)dW_\tau^{F^S}, \quad \tau\in(t,S) \tag{20.8}$$

where $W_{FS}(\tau)$ is a multidimensional Brownian motion under Q_{FS} and $v_\tau^z(S,\Delta)$ is a vector of instantaneous volatilities adapted to $W_{FS}(\tau)$.

Note the timing of the forward contract. The payoff takes place at S, that is, at the beginning of the debt servicing period, the Libor refers to $[S,S+\Delta]$. Section 20.3.3 reviews FRA and IRS, which are contracts for which the timing of payments occurs at the end of the debt servicing period.

20.3.2.2 Futures and Options Futures and options are determined exactly as in the government bond case and are not dealt with in this chapter to simplify the presentation. However, for reference, the appendix provides evaluation formulas and discussion regarding option evaluation in this context (see Eqs 20.115 and 20.116).

20.3.3 Forwards Rate Agreements and Interest Rate Swaps

20.3.3.1 Forward Agreements FRAs (in the sequel) are contracts that freeze the cost of capital for institutions such that at the end of a given debt servicing period, say, $[T,S]$, a counterparty (i) pays a fixed interest rate, K, and (ii) receives the variable interest rate that had prevailed over $[T,S]$. Assuming the variable interest rate to be the Libor $l_t(\Delta)$ introduced in the previous section, such a swap in interest rate payments has a time S payoff equal to

$$\Delta_S\times(l_T(\Delta_S)-K) \qquad \Delta_S\equiv S-T$$

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such that its value for any K is

$$\begin{aligned} FRA_t(T,S;K) &\equiv \mathbb{E}_t\left[e^{-\int_t^S r(\tau)d\tau}\Delta_S\left(l_T\left(\Delta_S\right)-K\right)\right] \\ &= \Delta_S(f_t(T,S)-K)P_t(S) \end{aligned} \tag{20.9}$$

where

$$f_t(T,S) \equiv \mathbb{E}_t^{Q_{FS}}(l_T(\Delta_S)) = \frac{1}{\Delta_S}\left(\frac{P_t(T)}{P_t(S)}-1\right) \tag{20.10}$$

and the last equality in Equation 20.9 follows by a change of probability. We shall return to the second equality in Equation 20.10 in a moment.

We refer to $f_t(T,S)$ as the forward swap rate: it is the value of K that makes the FRA worthless at origination, that is, $FRA_t(T,S;K)=0$. Note that $f_t(T,S)$ can actually be cast in a model-free fashion, that is, independent of any pricing model. Indeed, because $l_T(\Delta_S)$ is known at T, the value at T of $1+\Delta_S l_T(\Delta_S)$ dollars to be delivered at S is simply \$1 and, obviously, $P_t(T)$ dollars at t, such that the value of $\Delta_S l_T(\Delta_S)$ to be delivered at time S is simply $P_t(T)-P_t(S)$ at time t. Hinging upon this reasoning leads to the following simplification of Equation 20.9:

$$FRA_t(T,S;K) = P_t(T) - (1+\Delta_S K)P_t(S) \tag{20.11}$$

Solving for the value of K that renders $FRA_t(T,S;K)$ equal to zero yields the second equality in Equation 20.10.[3]

We now proceed with generalizing this basic contract with only one payment at S, to one in which periodic IRS take place.

20.3.3.2 Interest Rate Swaps Consider a basket of interest rate forwards, that is, one in which a party receives from another a payoff equal to $\pi_{T_i} \equiv \delta_{i-1}(l_{T_{i-1}}(\delta_{i-1})-K)$ at the reset date T_i, where K is the constant interest rate and $\delta_{i-1} \equiv T_i - T_{i-1}$, and n times. It is a forward-starting IRS, and the period over which the IRS payments will take place $T_n - T_1$ is known as the tenor of the contract.

A forward-starting IRS is valued as the present value of these single payoffs occurring over the reset dates,

$$\begin{aligned} v_{irs}^p(t) &\equiv \sum_{i=1}^n FRA_t(T_{i-1},T_i;K) \\ &= \sum_{i=1}^n [P_t(T_{i-1})-P_t(T_i)] - K\sum_{i=1}^n \delta_{i-1}P_t(T_i) \end{aligned} \tag{20.12}$$

where $FRA_t(\cdot)$ is as in Equation 20.11. The superscript p in v_{irs}^p stands for payer – by convention, the party who pays the fixed interest rate enters a swap *payer*, and the counterparty enters a swap *receiver*.

The forward swap rate is defined as the value of K such that the value of the IRS is zero at inception, that is, $v_{irs}^p(t)=0$, and equals

$$R_t(T_1,\ \dots\ ,T_n) \equiv \frac{P_t(T_0)-P_t(T_n)}{PVBP_t(T_1,\ \dots\ ,T_n)} \tag{20.13}$$

$$PVBP_T(T_1,\ \dots\ ,T_n) \equiv \sum_{i=1}^n \delta_{i-1}P_t(T_i) \tag{20.14}$$

Note that $PVBP_T(T_1,\ \dots\ ,T_n)$ is the value of an annuity paying out \$1 at each reset date. It is also known as the "price value of the basis point," that is, the present value impact of 1 basis point moves in the forward swap rate at T. Replacing the expression for $R_t(T_1,\ \dots\ ,T_n)$ in Equations 20.13 and 20.14 into Equation 20.12 leaves the following intuitive expression for a forward swap payer:

$$v_{irs}^p(t) = PVBP_t(T_1,\ \dots\ ,T_n)(R_t(T_1,\ \dots\ ,T_n)-K) \tag{20.15}$$

[3] The standard assumption that allows for this reasoning is that settlement does indeed occur at S. Should settlement occur at $T<S$ (say), the value of $1+\Delta_S l_T(\Delta_S)$ at T would obviously be higher than 1, and the previous reasoning would not go through: a payoff of $1+\Delta_S l_T(\Delta_S)$ at T is the same as a payoff of $(1+\Delta_S l_T(\Delta_S))^2$ at S. Brigo and Mercurio (2006, Chapter 13) and Veronesi (2010, Chapter 21) explain standard market practice to deal with this "convexity" issue.

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The forward swap rate displays a property to consider while pricing products referencing it, which we shall discuss in the next section. Consider the following equalities:

$$\begin{aligned} R_t(T_1, \dots, T_n) &= \frac{P_t(T_0) - P_t(T_n)}{PVBP_t(T_1, \dots, T_n)} \\ &= \mathbb{E}_t\left(\frac{e^{-\int_t^T r_\tau d\tau}}{PVBP_t\left(T_1, \dots, T_n\right)}(P_T(T_0) - P_T(T_n))\right) \\ &= \mathbb{E}_t(\xi_T^{sw,n} R_T(T_1, \dots, T_n)) \\ &= \mathbb{E}_t^{sw,n}(R_T(T_1, \dots, T_n)) \end{aligned}$$

where the second equality holds by the martingale property of the discounted prices of the zero-coupon bonds expiring at T_0 and at T_n, the third by the definition of the forward swap rate in Equations 20.13 and 20.14 and the definition of $\xi_T^{sw,n}$ below:

$$\xi_T^{sw,n} \equiv \left.\frac{dQ_{sw,n}}{dQ}\right|_{\mathcal{F}_T} = e^{-\int_t^T r_\tau d\tau} \frac{PVBP_T(T_1, \dots, T_n)}{PVBP_t(T_1, \dots, T_n)} \tag{20.16}$$

and the fourth by a change of probability, with $\mathbb{E}_t^{sw,n}(\cdot)$ denoting the expectation under $Q_{sw,n}$. That is, the forward swap rate is a martingale under the new probability, $Q_{sw,n}$ and, in a diffusion setting, satisfies

$$dR_\tau(T_1, \dots, T_n) = R_\tau(T_1, \dots, T_n)\sigma_\tau(T_1, \dots, T_n)dW_\tau^{sw,n}, \quad \tau \in [t, T] \tag{20.17}$$

for some instantaneous volatility process $\sigma_\tau(\cdot)$ adapted to the Brownian motion $W_\tau^{sw,n}$ defined under $Q_{sw,n}$. The probability $Q_{sw,n}$ is known as *annuity probability* and was introduced by Jamshidian (1997).

20.3.4 Caps, Floors, and Swaptions

We review other major derivatives written on Libor besides those on time deposits considered in Section 20.3.2.

First consider a cap, which is the same as an IRS except that it gives the holder the option to proceed with the swap and reach reset date only if it is convenient to do so, that is, when interest rates turn out to be higher than a strike. Therefore, a cap protects the holder from an increase in interest rates. A cap is made up of caplets, each delivering a payoff at the reset date T_i equal to

$$\delta_{i-1}(l_{T_{i-1}}(T_i) - K)^+, \quad i = 1, \dots, n$$

where K is a strike.[4] Therefore, the price of a caplet at t equals

$$\begin{aligned} Cap_t &\equiv \sum_{i=1}^{n} \mathbb{E}_t\left[e^{-\int_t^{T_i} r_\tau d\tau} \delta_{i-1}\left(l_{T_{i-1}}\left(T_i\right) - K\right)^+\right] \\ &= \sum_{i=1}^{n} \delta_{i-1} P_t(T_i) \mathbb{E}_t^{Q_F{}^{T_i}}(l_{T_{i-1}}(T_i) - K)^+ \end{aligned} \tag{20.18}$$

The benefit to be long a cap is to be protected period-by-period so to speak. It is the same as a basket of options with different maturities. A swaption works differently, in that the optionality kicks in “bundled.” Suppose, for example, we anticipate that 1 year from now, we might want to enter a 5-year payer swap to hedge against variable interest rates. At the same time, we would like to be sure that we would only benefit from interest rates going up while fixing the downside. Swaptions allow for this optionality as they provide their holder the right to enter a swap contract on a future date and at a certain strike.

Consider a payer swaption expiring at time T_0 with tenor period T_n–T_1 and strike K. Its payoff is the maximum between zero and the value of a payer IRS at T_0, which by Equation 20.15 is

$$(v_{irs}(T_0))^+ = PVBP_{T_0}(T_1, \dots, T_n)(R_{T_0}(T_1, \dots, T_n) - K)^+ \tag{20.19}$$

[4]Floors are defined in a symmetrically opposite way. They protect against a downward movement in interest rates, in that they are baskets of single *floorlets* that pay off, $\delta_{i-1}(K - l_{T_{i-1}}(T_i))^+$, at time T_i, $i = 1, \dots, n$. Floors could be priced through the put–call parity for caps and floors.

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such that the value at t is

$$\begin{aligned} Swpn_t^{\mathrm{p}} &= \mathbb{E}_t\left[e^{-\int_t^{T_0} r(\tau)d\tau} PVBP_{T_0}\left(T_1, \dots, T_n\right)\left(R_{T_0}(T_1, \dots, T_n) - K\right)^+\right] \\ &= PVBP_t(T_1, \dots, T_n)\mathbb{E}_t^{Q_{sw,n}}(R_{T_0}(T_1, \dots, T_n) - K)^+ \end{aligned} \tag{20.20}$$

where $\mathbb{E}_t^{Q_{sw,n}}$ denotes the time t expectation taken under the annuity probability $Q_{sw,n}$ defined through Equation 20.16. Swaption receivers work in the exactly opposite way and can be evaluated through the swaption parity,

$$Swpn_t^p(K) = \upsilon_{irs}^p(t) + Swpn_t^{\mathrm{r}}(K)$$

where $\upsilon_{irs}(t)$ is the value of the forward IRS in Equation 20.12.

20.4 EVALUATION PARADIGMS

This section examines the main evaluation paradigms for the instruments reviewed in the previous section. The literature on IRD evaluation is vast, relying on dozens of models either addressing empirical regularities or providing a standard for a convenient analytical framework. We cannot even attempt to review this literature and refer the reader to classical textbooks such as Brigo and Mercurio (2006) or Veronesi (2010). Rather, our focus in this section is to provide a broad illustration of models that are commonly used in practice and related to our main theme in this chapter.

Section 20.4.1 provides an overview of one of the earliest approaches that emerged in the literature, which is based on models of the short-term rate that determine the entire yield curve, possibly in combination with additional factors. This approach suffers from a drawback: it cannot fit the entire yield curve without error. Suppose a financial institution is pricing a set of IRDs with a model built up through this approach. How can we rely on a model that is not even able to pin down the initial yield curve? A second class of "no-arbitrage models" aims to fix this issue and is succinctly reviewed in Section 20.4.2. Finally, Section 20.4.3 surveys models designed to fit the cross section of IRD relying on notions of "implied volatility."

20.4.1 Models of the Short-term Rate

20.4.1.1 The Seminal Work Vasicek (1977) derives a model of the yield curve assuming the short-term rate r_τ is a continuous-time mean-reverting process with constant basis point volatility,

$$dr_\tau = \kappa(\bar{r} - r_\tau)d\tau + \sigma d\tilde{W}_\tau, \quad \bar{r} \equiv \mu - \frac{\lambda\sigma}{\kappa} \tag{20.21}$$

where $\tilde{W}_\tau$ is a Brownian motion under the risk-neutral probability, μ is the unconditional expectation of r_τ, κ is the speed of mean reversion, and finally, $\bar{r}$ is the unconditional expectation of r_τ taken under the risk-neutral probability Q, with λ denoting a risk adjustment arising through Girsanov theorem: that is, μ is the unconditional expectation of r_τ under the physical probability.

The model's major drawback is that the transition density of r_τ in Equation 20.21 is normal, which can lead to negative values of r_τ. Cox, Ingersoll, and Ross (1985) propose an alternative model in which the short-term rate is the solution to the so-called "square-root" process

$$dr_\tau = \hat{\kappa}(\hat{r} - r_\tau)d\tau + \hat{\sigma}\sqrt{r_\tau}d\tilde{W}_\tau \tag{20.22}$$

where basis point volatility is now time varying (although locally deterministic) and notation is similar to that utilized for the Vasicek model. It is well known that under these dynamics, the transition density of r_τ is noncentral chi-square, and the short-term rate is always positive under regularity conditions.

Both models lead to a closed-form solution for the price of a zero-coupon bond, which has the form

$$P_t(r_t; T) \equiv e^{a(T-t)-b(T-t)r_t} \tag{20.23}$$

for two functions $a(\cdot)$ and $b(\cdot)$ given in the appendix (Eqs 20.106 and 20.107 for Vasicek and Eqs 20.108 and 20.109 for Cox, Ingersoll, and Ross). For example, in Vasicek's model, the bond price exposure to the short-term rate is

$$b(T-t) = \frac{1 - e^{-\kappa(T-t)}}{\kappa} \tag{20.24}$$

The higher the persistence of the short-term rate, the lower the κ, the higher the exposure of the bond price to the short-term rate, and in turn the higher the bond return volatility. In the limit where κ is very small, the volatility of the return on the bond maturing in n years is equal to $v(n) \equiv \sigma \cdot n$. Even if the basis point volatility of the short-term rate is as small as, say, 2 bps, the model predicts that for a 10-year zero-coupon bond, return volatility $v(n)$ reaches 20%. The high persistence of the short-term rate creates a "risk for the long run" so to speak. The bond exposure for the Cox, Ingersoll, and Ross model is obviously different from b in Equation 20.24, but one can reach a similar conclusion with this model.

The fact that both models predict the price of a zero-coupon bond with the form in Equation 20.23 is due to a precise mathematical property reviewed in the next section.

20.4.1.2 Stochastic Volatility and Multifactor Extensions The Cox, Ingersoll, and Ross (1985) model surveyed previously predicts the short-term rate to have random basis point volatility but driven by the level of the short-term rate. This is now understood to be somehow a counterfactual feature, which led to formulations of models in which short-term rate volatility is driven by other factors than the level of interest rates.

Fong and Vasicek (1991) consider the following model, generalizing the Vasicek (1977) model in 20.21:

$$\begin{cases} dr_\tau = \kappa_r(\bar{r} - r_\tau)d\tau + v_\tau d\tilde{W}_{1\tau} \\ dv_\tau^2 = \kappa_v(\omega - v_\tau^2)d\tau + \xi v_\tau d\tilde{W}_{2\tau} \end{cases} \tag{20.25}$$

where $\tilde{W}_{i\tau}$ are two standard Brownian motions under the risk-neutral probability. It is a natural extension of the Vasicek model. The short-term rate is still mean reverting, but its basis point variance is now random and also mean reverting.

Longstaff and Schwartz (1992) propose another model, grounded in general equilibrium, for which the short-term rate is a linear combination of two factors,

$$r_\tau = \beta_1 y_{1\tau} + \beta_2 y_{2\tau} \tag{20.26}$$

for two constants β_i and two uncorrelated processes $y_{1\tau}$ and $y_{2\tau}$ – leading to an "affine" model, as explained later (see, also, the appendix). The Longstaff and Schwartz model also exhibits stochastic volatility, and like Fong and Vasicek, it has a solution with the following form:

$$P_t(r_t, v_t^2; T) \equiv e^{a(T-t)-b(T-t)r_t+c(T-t)v_t^2} \tag{20.27}$$

for three functions $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$ (see Appendix) and where v_t^2 denotes the instantaneous basis point variance of the changes in the short-term rate in Equation 20.26, just as in Fong and Vasicek.[5]

Note that there are now two factors driving the variation of the bond price. Persistence of either can inflate volatility of bond returns. Due to stochastic volatility, bond return volatility itself fluctuates. While very old, these two models are trailblazers for the new area of IRV pricing surveyed in this chapter and will also serve as an inspiration in Section 20.5 for the design of an evaluation model for derivatives on government bond volatility indexes.

An issue regarding both models is that there are only two factors driving the entire yield curve, which is counterfactual: it is well known since at least Litterman and Scheinkman (1991) (see also Knez, Litterman, and Scheinkman, 1994), that most of the variation of the yield curve is driven by at least three factors. Can we build up an analytically tractable multifactor model that predicts a bond price generalizing Equations 20.23 and 20.27?

Duffie and Kan (1996) prove the following result. Suppose that the short-term rate is the following generalization of the Longstaff and Schwartz equation 20.26:

$$r_\tau = \alpha + \beta \cdot y_\tau \tag{20.28}$$

where α is a constant, β is a vector, and y_τ is a diffusion in $\mathbb{R}^n$ solution to

$$dy_\tau = \kappa(\mu - y_\tau)d\tau + \Sigma V(y_\tau)d\tilde{W}_\tau \tag{20.29}$$

where $\tilde{W}_\tau$ is an n-dimensional Brownian motion under the risk-neutral probability, Σ is a full rank $n \times n$ matrix, and V is a full rank $n \times n$ diagonal matrix with elements

$$V(y)_{(ii)} = \sqrt{s_{1i} + s_{2i} \cdot y}, \quad i = 1, \ldots, n$$

[5]Mele (2003) studies the mapping from the instantaneous volatility of the short-term rate and the yield curve in more general models, identifying conditions leading to sign the relation between volatility and the yield curve – for example, the sign of $c(\cdot)$ in Equation 20.27.

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for some scalars s_{1i} and vectors s_{2i}. Then, the price of a zero-coupon bond has the following form that generalizes both Equations 20.23 and 20.27:

$$P(y_t, T-t) = e^{a^o(T-t)+a^y(T-t)\cdot y_t} \tag{20.30}$$

for some functions $a^o(\cdot)$ and $a^y(\cdot)$ (see Appendix for further details on these two functions).

This class of *affine* models is thoroughly surveyed by Piazzesi (2010), and all of the models described in the current section fall within this class. Note that affine models do not constitute the only analytically tractable class of models; the so-called "quadratic" formulation of the term structure discussed by Ahn, Dittmar, and Gallant (2002) also leads to closed-form solutions for the price of zero-coupon bonds.

20.4.2 No-Arbitrage Models

The approach in the previous section is to take the short-term rate process as given and price the entire yield curve. Obviously, this approach cannot lead to a perfect fit of the current yield curve. We now succinctly review models that help achieve this goal. These models are known as "no-arbitrage" simply because it is as if they relied on (rather than determine) current prices to imply the dynamics of the future yield curve through no-arb restrictions – no arbitrage is essentially the only assumption made, on top of standard price dynamics such as prices driven by Brownian motions.

20.4.2.1 Early Formulations Ho and Lee (1986) consider the first no-arbitrage model of the yield curve. In continuous time, the model assumes that the short-term rate is the solution to

$$dr_\tau = \theta_\tau d\tau + \sigma d\tilde{W}_\tau, \quad \tau \geq t \tag{20.31}$$

where $\tilde{W}_\tau$ is a Brownian motion under Q, σ is a constant basis point volatility, and θ_τ is an "infinite-dimensional" parameter (i.e., a curve continuous in calendar time and known at time t), which allows fitting of the initial yield curve without error.

Note that the model is clearly affine, and the price at τ of a zero-coupon bond expiring at T can be expressed as

$$P_\tau(r_\tau, T) = e^{\int_\tau^T \theta_s(s-T)ds+\frac{1}{6}\sigma^2(T-\tau)^3-(T-\tau)\cdot r_\tau}$$

Matching the instantaneous forward rate for T predicted by the model, $\varphi_\tau(T) \equiv -\frac{\partial}{\partial T}\ln P_\tau(r_\tau, T)$, to its hypothetically observed market counterpart, $\varphi_\tau^\$(T)$, gives

$$\theta_\tau = \frac{\partial}{\partial \tau}\varphi_t^\$(\tau) + \sigma^2(\tau - t) \tag{20.32}$$

By construction, the model fits the entire yield curve without error because the price satisfies $P_t(T) = e^{-\int_t^T \varphi_t^\$(\tau)d\tau}$ and the parameter θ_τ in Equation 20.32 is indeed shown to guarantee that $\varphi_t(\tau) = \varphi_t^\$(\tau)$ for each τ.

Note that with this θ_τ, the model predicts that the short-term rate in Equation 20.31 satisfies

$$r_t = \varphi_0^\$(t) + \frac{1}{2}\sigma^2 t^2 + \sigma \tilde{W}_t$$

and the instantaneous forward rate

$$d\varphi_\tau(T) = \sigma^2(T-\tau)d\tau + \sigma d\tilde{W}_\tau$$

That is, the model does not impose any restrictions on the drift of the forward rates other than one arising from no-arbitrage, but it is still able to match any observed price. These are the reasons we refer to this model as "no-arbitrage."

Hull and White (1990) generalize the previous model to 1 in which the short-term rate has a mean-reverting component,

$$dr_\tau = (\theta_\tau - \kappa r_\tau)d\tau + \sigma d\tilde{W}_\tau \tag{20.33}$$

where $\tilde{W}_\tau$ is a Q-Brownian motion, θ_τ is the infinite-dimensional parameter, and κ, σ are constants.[6] They find that the price of a bond predicted by the model is

$$P_\tau(r_\tau, T) = e^{\frac{1}{2}\sigma^2\int_\tau^T b^2(T-s)ds - \int_\tau^T \theta_s b(T-s)ds - b(T-\tau)\cdot r_\tau}$$

[6]Hull and White do in fact consider a more complex model in which κ and σ are both time varying.

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where the function $b(\cdot)$ is the same as in Equation 20.24 and

$$\theta_\tau = \frac{\partial}{\partial \tau}\varphi_t^{\$}(\tau) + \kappa f_{\$}\varphi_t^{\$}(\tau) + \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa(\tau - t)}\right) \tag{20.34}$$

The reason one may prefer the Hull and White model over the Ho and Lee model is that while the latter is capable of fitting the entire yield curve, a more complex model might be better to price derivatives. In fact, we may take this reasoning one step further and consider even more complex models than Hull and White. Brigo and Mercurio (2006) survey a rich variety of such models and examine their behavior vis-à-vis market data in detail.

20.4.2.2 Pricing Coupon-Bearing Bonds The Hull and White (1990) model does, of course, generalize Vasicek (1977). The latter is obtained from the former when θ_τ in Equation 20.33 is constant. In this section, we develop a pricing example to emphasize the importance of having θ_τ determined to fit the yield curve, as in Equation 20.34. The pricing problem concerns an option written on a coupon-bearing bond. In Section 20.5, we shall utilize this example to illustrate how to address a number of issues regarding the implementation of government bond volatility indexes.

First, recall the price of a call option expiring at T, written on a zero-coupon bond expiring at S, and struck at K, is given in Equation 20.6. Jamshidian (1989) shows that the two probabilities, $\mathbb{Q}_{Fj}$, can be calculated in closed form, such that

$$\overline{Call}_t^b(T; P_t(r_t, S), K, v) = P_t(r_t, S)\cdot\Phi(d_1) - KP_t(r_t, T)\cdot\Phi(d_1 - v_{[T,S]}) \tag{20.35}$$

where

$$d_1 = \frac{\ln\frac{P_t(r_t,S)}{KP_t(r_t,T)} + \frac{1}{2}v_{[T,S]}^2}{v_{[T,S]}}, \quad v_{[T,S]}^2 = \sigma^2\frac{1 - e^{-2\kappa(T-t)}}{2\kappa}b^2(S - T)$$

and $b(\cdot)$ is as in Equation 20.24 and $\Phi(\cdot)$ denotes the cumulative normal distribution.

Note that we have been silent regarding whether we are using Equation 20.34 to determine the "underlying" price $P_t(r_t, S)$ and the "discounting" price $P_t(r_t, T)$ in Equation (20.35). It does not really matter for the purpose of the current pricing exercise. It does matter, though, when it comes to pricing more complex instruments like options written on a coupon-bearing bond. Consider the payoff of such an option, given by

$$\left[\sum_{i=1}^n \frac{C_i}{n}P_{T_0}\left(T_i\right) + P_{T_0}(r_{T_0}, T_n) - K\right]^+ = \sum_{i=1}^N \bar{C}_i\cdot[P_{T_0}(r_{T_0}, T_i) - \mathcal{K}_i^*(K)]^+ \tag{20.36}$$

where $\bar{C}_i \equiv \frac{C_i}{n}$, for $i = 1, \ldots, N-1$, $\bar{C}_n \equiv \frac{C_n}{n} + 1$, $\mathcal{K}_i^*(K) \equiv P_{T_0}(r^*(K), T_i)$ and $r^*(K)$ is the solution to the following equation:

$$P_{T_0}(r^*, T_n) + \sum_{i=1}^n \frac{C_i}{n}P_{T_0}(r^*, T_i) = K \tag{20.37}$$

A unique solution to Equation 20.37 exists under mild regularity conditions because the bond price is inversely related to the short-term rate, in which case Equation 20.36 should also hold. Therefore, a call on a coupon-bearing bond is the same as a basket of call options with strike $\mathcal{K}_i^*(K)$, and its value is

$$Call_t^b(T_0; B_t(\mathbb{T}), K) \equiv \sum_{i=1}^n \bar{C}_i\cdot\overline{Call}_t^b(T_0; P_t(r_t, T_i), \mathcal{K}_i^*(K), v_{[T_0,T_i]}) \tag{20.38}$$

where $\overline{Call}_t^b$ is the call price in Equation 20.35. The price of a put can be determined through put–call parity. This formula is due to Jamshidian (1989).

A critical point to note is that the value of $r^*(K)$ and, hence, that of $\mathcal{K}_i^*(K)$ and ultimately that of $Call_t^b(\cdot)$ depends on the entire yield curve whenever θ_τ is calibrated to market data through Equation (20.37). In other words, different values of the yield curve lead to different values of $Call_t^b(\cdot)$ through the fictitious strike $\mathcal{K}_i^*(K)$.

20.4.2.3 Heath, Jarrow, and Morton Heath, Jarrow, and Morton (1992) (HJM, henceforth) generalize the early approach to no-arbitrage pricing in the fixed-income space. Their arguments can be understood as follows. Consider the following

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representation of a bond price in terms of the instantaneous forward rates:

$$P_\tau(T) = \frac{P_t^{\$}(T)}{P_t^{\$}(\tau)} \cdot e^{-\int_\tau^T (\varphi_\tau(\ell)-\varphi_t(\ell))d\ell} \tag{20.39}$$

Given the current prices, $P_t^{\$}(\tau)$ and $P_t^{\$}(T)$, the goal is to model the future forward rate movements,

$$\varphi_\tau(\ell) - \varphi_t(\ell)$$

Note that by construction, the current yield curve is fitted without error: set $\tau = t$ in Equation 20.39 and obtain $P_t(T) = P_t^{\$}(T)$ for each T.

HJM assume that under the risk-neutral probability, $\varphi_\tau(T)$ satisfies; for fixed T,

$$d\varphi_\tau(T) = \alpha_\tau(T)d\tau + \sigma_\tau^f(T)d\tilde{W}_\tau, \quad \tau \in (t, T] \tag{20.40}$$

where $\tilde{W}_\tau$ is a multidimensional process and α_τ and σ_τ^f are some adapted processes with $\varphi_t(T)$ given. A restriction can be found on the drift $\alpha_\tau(T)$ once we require the no-arbitrage condition that the expected returns of the zero-coupon bond must be equal to the short-term rate under the risk-neutral probability, Q. The restriction is

$$\alpha_\tau(T) = \sigma_\tau^f(T) \int_\tau^T \sigma_\tau^f(\ell)^\top d\ell \tag{20.41}$$

The appendix contains a derivation of Equation 20.41. Replacing the previous restriction in Equation 20.40 leads to the following expression for the short-term rate, $r_\tau \equiv \varphi_\tau(\tau)$:

$$r_\tau \equiv \varphi_\tau(\tau) = \varphi_t(\tau) + \int_t^\tau \alpha_s(\tau)ds + \int_t^\tau \sigma_s^f(\tau)d\tilde{W}_s, \quad \tau \in (t, T] \tag{20.42}$$

where $\alpha_\cdot(\cdot)$ is as in Equation 20.41. In principle, we could use Equation 20.42 to price IRDs. We would be simply left with specifying the volatility of the forward rates. For example, it is straightforward to check that once $\sigma_s(\tau)$ is constant, one is back to the Ho and Lee model.

More generally, one could specify richer dynamics than Ho and Lee and price IRDs by simulations relying on Equation 20.42. Needless to mention, one can rely on models much richer than the single factor in Equation 20.40 and derive restrictions generalizing Equation 20.41 and price IRDs accordingly. We now describe a simple class of models used in practice.

20.4.2.4 Market Models The original HJM framework relies on the notion of continuously compounded forward rates; yet in practice one must deal with discretely compounded rates. Black's (1976)formula is often used in practice to price IRDs such as caps, floors, and swaptions. The question then arises as to how to make the HJM framework consistent with market practice. Brace, Gatarek, and Musiela (1997), Jamshidian (1997), and Miltersen, Sandmann, and Sondermann (1997) develop restrictions the HJM should satisfy to address these issues, which lead to the "market model."

Assume that the forward rate $f_{i\tau} \equiv f_\tau(T_i, T_{i+1})$ in Equation 20.10 under the risk-neutral probability Q is the solution to

$$\frac{df_{i\tau}}{f_{i\tau}} = m_{i\tau}d\tau + \gamma_{i\tau}d\tilde{W}_\tau, \quad \tau \in [t, T_i], \quad i = 0, \dots, n-1 \tag{20.43}$$

where $\tilde{W}_\tau$ is a vector of standard Brownian motions under Q, $m_{i\tau}$ is a function measurable with respect to $\tilde{W}_\tau$, and $\gamma_{i\tau}$ is a *deterministic* vector-valued function of time.

Note, also, that by Equation 20.10,

$$\ln\left(\frac{P_\tau(T_i)}{P_\tau(T_{i+1})}\right) = \ln(1 + \delta_i f_{i\tau})$$

The idea is to expand both sides of the previous equation and identify the diffusion terms. Relying on Itô's lemma,

$$\sigma_\tau^B(T_i) - \sigma_\tau^B(T_{i+1}) = \frac{\delta_i f_{i\tau}}{1 + \delta_i f_{i\tau}}\gamma_{i\tau}, \quad \tau \in [t, T_i] \tag{20.44}$$

where $\sigma_\tau^B(T_i)$ denotes the time τ instantaneous return volatility of a zero-coupon bond expiring at T_i, consistent with the notation after Equation 20.5.

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Note that this restriction arises because a market model is being used to price IRDs. One could use any other no-arbitrage model (e.g., multifactor extensions of Hull and White, 1990) to price IRDs. The market model is, however, commonly used in practice. In the following, we shall see how Equation 20.44 is used to determine the price of a number of IRDs consistently (see Eq. 20.51).

20.4.2.5 Applications to Derivative Pricing Consider evaluating a cap through the market model. By Equation 20.18 and the relation $f_{i-1,T_{i-1}} \equiv f_{T_{i-1}}(T_{i-1}, T_i) = l_{T_{i-1}}(T_i)$,

$$Cap_t = \sum_{i=1}^{n} \delta_{i-1} P_t(T_i) \cdot \mathbb{E}_t^{Q_{F^{T_i}}} (f_{i-1,T_{i-1}} - K)^+ \tag{20.45}$$

We know already from Equation 20.10 that the forward rate $f_{i-1,\tau}$ is a martingale under $Q_{F^{T_i}}$. Assuming that it has deterministic volatility as in Equation 20.43

$$\frac{df_{i-1,\tau}}{f_{i-1,\tau}} = \gamma_{i-1,\tau} dW_\tau^{F^{T_i}}, \quad \tau \in [t, T_{i-1}], \quad i = 1, \dots, n \tag{20.46}$$

under $Q_{F^{T_i}}$. The expectation in Equation 20.45 can be determined through Black's (1976) formula,

$$\mathbb{E}_t^{Q_{F^{T_i}}} (f_{i-1,T_{i-1}} - K)^+ = Bl76(f_{i-1,t}; T_{i-1} - t, K, s_{i-1}) \tag{20.47}$$

where

$$Bl76(f_{i-1,t}; T_{i-1} - t, K, s_{i-1}) \equiv f_{i-1,t}\Phi(d_{i-1,t}) - K\Phi(d_{i-1,t} - s_{i-1}),$$

$$d_{i-1,t} = \frac{\ln \frac{f_{i-1,t}}{K} + \frac{1}{2}s_{i-1}^2}{s_{i-1}}, \quad s_{i-1}^2 = \int_t^{T_{i-1}} \gamma_{i-1,\tau}^2 d\tau$$

and Φ denotes the cumulative normal distribution.

Swaptions are priced similarly. Consider Equation 20.17 and assume that $\sigma_\tau(\cdot) \equiv \gamma_{n,sw,\tau}$, for some deterministic $\gamma_{n,sw,\tau}$. Because the forward swap rate is a martingale under the annuity probability, Equation 20.20 tells us that we can use Black's formula and obtain the price of a swaption payer expiring at T_0 as

$$Swpn_t^{\mathrm{p}}(K, R_{n,t}) = PVBP_t(T_1, \dots, T_n) \cdot Bl76(R_{n,t}; T_0 - t, K, \sqrt{\bar{V}}) \tag{20.48}$$

where $R_{n,t} \equiv R_t(T_1, \dots, T_n)$ to simplify notation and

$$Bl76\left(R_{n,t}; T_0 - t, K, \sqrt{\bar{V}}\right) = R_{n,t}\Phi\left(d_t\right) - K\Phi\left(d_t - \sqrt{\bar{V}}\right)$$

$$d_t = \frac{\ln \frac{R_{n,t}}{K} + \frac{1}{2}\bar{V}}{\sqrt{\bar{V}}}, \quad \bar{V} = \int_t^{T_0} \gamma_{n,sw,\tau}^2 d\tau$$

In practice, the market convention is to price swaptions through an implied Black's volatility, defined as the value of $\sqrt{\bar{V}}$ in Equation 20.48 that makes the Black's pricer return the market price,

$$\sigma_{iv}(K, R_{n,t}, T_0 - t)$$

$$Bl76(R_{n,t}; T_0 - t, K, \sigma_{iv}(K, R_{n,t}, T_0)) = \frac{Swpn_t^{\mathrm{p}}(K)}{PVBP_t(T_1, \dots, T_n)} \tag{20.49}$$

where $Swpn_t^{\mathrm{p}}(K)$ denotes the market price of the swaption payer struck at K.

An issue that arises here is that we cannot assume that Equations 20.17 and 20.46 simultaneously hold through their deterministic instantaneous volatilities. According to the model, the forward swap rate cannot be lognormal if we assume the forward is lognormal, and vice versa. One approach would be to price caps and floors using the previous lognormal model and proceed to price the swaptions through simulation. For example, one could consider a slightly different representation of the swaption payoff than in Equation 20.19, obtained by using Equation 20.9 in the definition in 20.12,

$$(v_{irs}^p(t))^+ = \left[\sum_{i=1}^{n} FRA_t(T_{i-1}, T_i; K)\right]^+$$

and by taking expectation of the previous payoff and a change in probability:

$$Swpn_t^{\mathrm{p}} = P_t(T_0)\mathbb{E}_t^{Q_F T_0}\left[\sum_{i=1}^{n} \delta_{i-1}\left(f_{i-1,T_0} - K\right) P_{T_0}(T_i)\right]^+ \tag{20.50}$$

Then, it can be shown that by Girsanov's theorem, the "fundamental restriction of the market model," Equation 20.44, implies

$$\frac{df_{i-1,\tau}}{f_{i-1,\tau}} = \gamma_{i-1,\tau}\sum_{j=0}^{i-1}\frac{\delta_j f_{j,\tau}}{1+\delta_j f_{j,\tau}}\gamma_{j,\tau}d\tau + \gamma_{i-1,\tau}dW_\tau^{F^{T_0}}, \quad i = 1, \dots, n \tag{20.51}$$

The various forward rates in Equation 20.50 may be simulated through Equation 20.51, leading to Monte Carlo approximations to the price in Equation (20.50). There are more complex models and numerical procedures devised in the literature to deal with these issues (see, e.g., Brigo and Mercurio, 2006, for a survey), but the logic remains the same as that described in this section: to use the market model's restrictions to price IRDs consistently.

20.4.3 Volatility

20.4.3.1 Local Volatility The logic underlying the previous no-arbitrage models is to price IRDs while making sure that, at the same time, the initial yield curve is fitted without error. We now deal with a similar issue regarding the *volatility* of fixed-income securities. Consider the pricing of swaptions and assume that the forward swap rate has nonrandom volatility, such that the price of a swaption payer is just as in Equation 20.48.

There is ample empirical evidence suggesting that the assumption of a nonrandom volatility is counterfactual (see, e.g., Mele and Obayashi, 2014a). Moreover, it is well known that the implied volatilities in Equation 20.49 are decreasing over moneyness, $K/R_{n,t}$ (setting for simplicity $R_{n,t} \equiv R_t(T_1, \dots, T_n)$) – a well-known empirical phenomenon known as the "skew." The presence of a skew poses quite a few challenges, both theoretical and practical. Theoretically, it implies an internally inconsistent conclusion that a model for the forward swap rate is needed for each swaption price corresponding to a specific strike. Practically, it does not allow for a consistent IRD risk management framework.

Naturally, these issues are very well known in the equity derivatives literature. In fact, research undertaken in the equity derivatives space suggests that stochastic volatility could be responsible for the skew (see, e.g., the early survey of Renault, 1997). Dupire (1994) provides a first fundamental step for dealing with these issues by introducing the so-called "local volatility" model. While this model was initially conceived to deal with equity derivatives, it may well be used to price IRDs such as swaptions.

Local volatility addresses an inverse problem. Let us continue with the forward swap example and consider Equation 20.17, assuming that $W_\tau^{n,sw}$ is scalar and that the volatility $\sigma_\tau(\cdot)$ is a function of both the forward swap rate and calendar time, $\sigma(R_{n,\tau}, \tau)$, say, such that

$$\frac{dR_{n,\tau}}{R_{n,\tau}} = \sigma(R_{n,\tau}, \tau)dW_\tau^{n,sw} \tag{20.52}$$

A standard approach in financial economics is to determine asset price restrictions given the dynamics of fundamentals and other assumptions. The local volatility model does, instead, reverse this protocol and, in the case of Equation 20.52, searches for the volatility function $\sigma(R_{n,\tau}, \tau)$ such that the price predicted by the model matches the market for each strike K. Mele and Obayashi (2014a; Appendix B) show that the resulting volatility function, say, $\sigma_{loc}(R_{n,\tau}, \tau)$, can be expressed in terms of the available swaption prices, similar to Dupire (1994) on the equity side,

$$\sigma_{loc}(K, \tau) = \sqrt{\mathbb{E}_t^{sw,n}(\sigma^2(R_{n,\tau}, \tau)|R_{n,\tau} = K)} = \sqrt{2\frac{\frac{\partial Swpn_t^p(K;\tau)}{\partial \tau}}{K^2\frac{\partial^2 Swpn_t^p(K;\tau)}{\partial K^2}}} \tag{20.53}$$

where we have emphasized the maturity of the swaption in the second argument of the swaption payer, $Swpn_t^p(K;\tau)$.[7]

By construction, the model does "fit the skew" through a function $\sigma_{loc}(R_{n,\tau}, \tau)$ that allows the model to match every market swaption price.[8] This local volatility, $\sigma_{loc}(R_{n,\tau}, \tau)$, is nonparametric in nature, entirely relying on data, and could be used to price illiquid swaptions or any other products relying on Monte Carlo simulations, for example.

[7]In deriving Equation 20.53, we have neglected the impact of the change in $PVBP_t$ occurring after a change in T; see Mele and Obayashi (2014a; Appendix B) for additional details.

[8]As usual in this context, this fit is only theoretical as it relies on continuous dependence on the data.

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Finally, note that we can use the previous expression to find an expression for the expectation of the integrated local variance from the time period $[T_1, T_2]$

$$\begin{aligned}
&\int_{T_1}^{T_2} \mathbb{E}_t^{sw,n}(\sigma^2(R_{n,\tau},\tau))d\tau = \\
&= \int_{T_1}^{T_2} \int_0^\infty (\mathbb{E}_t^{sw,n}(\sigma^2(R_{n,\tau},\tau)|R_{n,\tau}=K)\phi_\tau(K)dK)d\tau \\
&= \frac{2}{PVBP_t}\int_0^\infty \frac{Swpn_t^p(K;T_2) - Swpn_t^p(K;T_1)}{K^2}dK
\end{aligned} \tag{20.54}$$

where we have used the expression for the marginal density of $R_{n,\tau}$ under the annuity probability, $\phi_\tau(K) = \frac{1}{PVBP_t}\frac{\partial^2 Swpn_t^p(K;\tau)}{\partial K^2}$.

Equation 20.54 provides a model-free expression for expected percentage realized variance in swap markets, which is reminiscent of the model-free expression derived by Britten-Jones and Neuberger (2000) for equities. A remarkable difference is that this formula is rescaled by the $PVBP_t$, and the expectation is taken under the annuity probability, which are two themes we shall return to in great detail in Section 20.5.

20.4.3.2 SABR Local volatility models such as that in Equation 20.52 do, however, suffer from a drawback as pointed out by Hagan et al. (2002). Consider, for example, Equation 20.52 and for simplicity assume that the local volatility is a function of only the forward swap rate, $\sigma_{loc}(R_{n,\tau})$. Consider a generic "forward risk," denoted as X_τ, which is a martingale under a certain pricing probability, just as the forward swap rate under the annuity probability. Hagan and Woodward (1999) use perturbation methods and show that for any maturity, the implied Black's volatility of a European-style option with the forward risk X as the underlying is

$$\sigma_{iv}(K,X) = \sigma_{loc}\left(\frac{1}{2}(K+X)\right)\left(1 + \frac{1}{24}\frac{\sigma''_{loc}\left(\frac{1}{2}(K+X)\right)}{\sigma_{loc}(12(K+X))}(X-K)^2 + \cdots\right.$$

where the omitted terms are likely to be numerically negligible for practical purposes.

What are the dynamics of the market smile implied by this local volatility model? To illustrate, consider what happens to the first term in the previous expansion, say, $\hat{\sigma}_{iv}(K,X)$, when the forward increases from X to $X+\Delta X$,

$$\hat{\sigma}_{iv}(K, X+\Delta X) \equiv \sigma_{loc}\left(\frac{1}{2}(K+X+\Delta X)\right) = \hat{\sigma}_{iv}(K+\Delta X, X)$$

In other words, provided σ_{loc} is decreasing, the local volatility model predicts that as the forward X increases, the skew moves to the left, which might contradict market behavior. For example, let us assume the local volatility function is $\sigma_{loc}(x) = 0.04 \cdot x^{-1/2}$. Panel A of Figure 20.6 plots the implied volatility $\hat{\sigma}_{iv}(K,X)$ for $X = 3\%$ (solid line) and $X = 4\%$ (dashed line).

Hagan et al. (2002) consider a richer model, which they call SABR for "stochastic $\alpha\beta\rho$," in which the forward X_t satisfies

$$\begin{cases} dX_\tau = v_\tau X_\tau^\beta dW_{1\tau} \\ dv_\tau = \xi v_\tau\left(\rho dW_{1\tau} + \sqrt{1-\rho^2}dW_{2\tau}\right), \quad v_t \equiv \alpha_t \end{cases} \tag{20.55}$$

where $W_{i\tau}$ are two standard Brownian motions under the market probability; β, ρ, and ξ are constants; and α_t is interpreted as the initial condition for the unobserved stochastic volatility component of the forward. Note that the model allows the forward and its volatility to be conditionally correlated with instantaneous correlation equal to ρ.

HKLW show that the implied volatility predicted by this model is

$$\sigma_{iv}(K, X_t; \alpha_t) = \frac{\alpha_t}{(X_tK)^{(1-\beta)/2}}\frac{1 + \left(\frac{(1-\beta)^2}{24}\frac{\alpha_t^2}{(X_tK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\xi}{(X_tK)^{(1-\beta)/2}}\alpha_t + \frac{2-3\rho^2}{24}\xi^2\right)T + \cdots}{\left(1 + \frac{(1-\beta)^2}{24}\ln^2\frac{X_t}{K} + \frac{(1-\beta)^4}{1920}\ln^4\frac{X_t}{K} + \cdots\right.}\frac{z_t}{u(z_t)} \tag{20.56}$$

where

$$z_t \equiv \frac{\xi}{\alpha_t}(X_tK)^{(1-\beta)/2}\ln\frac{X_t}{K}, \quad u(z_t) \equiv \ln\frac{\sqrt{1-2\rho z_t + z_t^2} + z_t - \rho}{1-\rho}$$

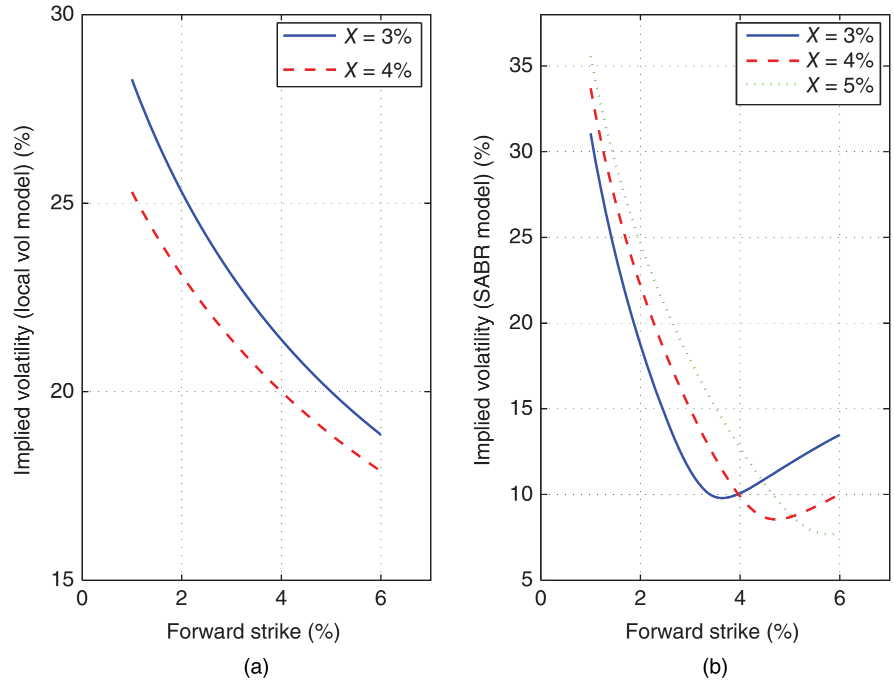

Figure 20.6 Panel A depicts the approximated implied volatility $\hat{\sigma}_{iv}(K, X)$ generated by a local volatility model with $\sigma_{loc}(x) = 0.04 \cdot x^{-1/2}$. Solid and dashed lines correspond to values of the forward X_t equal to 3% and 4%, respectively. Panel B depicts the approximated implied volatility $\sigma_{iv}(K, X_t; \alpha_t)$ in Equation 20.56 predicted by the stochastic alpha, beta, rho (SABR) model in Equation 20.55, with $\alpha_t = 0.02$, $\rho = -0.5$, $\beta = 0.5$, $\xi = 0.5$, and $T = 1$. Solid, dashed, and dotted lines correspond to values of the forward X_t equal to 3%, 4%, and 5%, respectively.

Panel B of Figure 20.6 depicts the behavior of the approximated implied volatility predicted by the SABR model obtained with hypothetical parameter values. The model can fix the counterfactual behavior of the skew predicted by a local volatility model: as the forward increases, the implied volatility shifts to the right while at the same time generating a downward-sloping "backbone," defined as the curve traced by the ATM volatility as the forward varies.[9]

The reason for a downward-sloping backbone is the coefficient $\beta < 1$ in Panel B of Figure 20.6. HKLW also show the origins of the asymmetric smile, that is, the skew, generated by their model, due to (i) a coefficient $\beta < 1$, which makes the instantaneous volatility in Equation 20.55, $\upsilon X^{\beta-1}$, decreasing in X, and (ii) a $\rho < 0$, which makes the transition density of the log-changes in X skewed toward the left, as in classical explanations of Heston (1993) given in related contexts. Finally, the volatility of volatility parameter ξ helps determine the curvature of the skew. The implied volatility shifts up as α_t increases – option prices increase with volatility in this model (e.g., Romano and Touzi, 1997) and so does implied volatility.

The SABR model is widely used in the market practice, especially while modeling the swaption skew. Note, however, that the model does not allow for a perfect matching of all available swaption prices, which by construction the local volatility can, at least theoretically. Finally, at the time of writing, it is not clear how this model could be incorporated in a market model of the kind surveyed in Section 20.4.2.4.

20.5 PRICING AND TRADING VOLATILITY

Volatility trading aims to produce P&L correlated with the volatility of a target asset class occurring over a given horizon. The portfolio design typically includes derivatives. Consider, for example, the swaption contract discussed in Section 20.3. Its value depends on the current forward swap rate and is mechanically determined through the Black's implied volatility as explained. The higher the Black's volatility, the higher the swaption value. One may attempt to create a certain portfolio of swaptions to bet on swap rate volatility.

[9]The comparative statics exercise in Figure 20.6 regards a change in the forward. Because the forward is correlated with volatility, an alternative comparative statics exercise is one in which both the forward changes and volatility change in accordance with their assumed correlation (see Bartlett, 2006). It is possible to show that in this case with negative correlation, an increase in the forward accompanied by a decrease in the volatility (consistent with the negative correlation) implies the skew shifts toward the left although the backbone is still downward sloping.

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The question arises as to whether such swaption-based portfolio returns correlate "enough" with volatility. Consider the previous swaption contract as a potential case to trade swap rate volatility. The price of a swaption payer, say, can increase both because volatility increases and the forward swap rate increases. To insulate the volatility component, one needs to consider a portfolio with more than just a swaption payer, aiming to hedge against the forward swap rate. A straddle, for example, is a portfolio including a payer and a receiver swaption, with the payer's delta roughly compensated by the receiver's. A position in this straddle does indeed succeed in insulating volatility in the short term, although we shall explain that for longer horizons, it could well deliver returns quite poorly correlated with realized volatility of the forward rate.

By construction, a portfolio with returns perfectly correlated with realized variance is valued the same as a contract delivering this variance at expiration against a threshold – a *variance swap*. We shall explain that the value of this contract is actually a portfolio that should theoretically include *many* derivatives written on the risk we are pricing the volatility of, in contrast with the previous simple two-option straddle.

While the literature on equity variance swaps is well known (see, e.g., the early work of Neuberger, 1994; Dumas, 1995; Demeterfi et al. 1999a,b; Bakshi and Madan, 2000; Britten-Jones and Neuberger, 2000; and Carr and Madan, 2001), that on IRV is still in its infancy and presents many new aspects to deal with, which were unknown in the equity case. This section surveys recent developments aiming to fill this void.

Section 20.5.1 develops an introductory example, which illustrates pitfalls arising while trading rate volatility through standard portfolio designs. Section 20.5.2 explains how to overcome these pitfalls through dedicated interest rate variance swap. Sections 20.5.3 and 20.5.4 discuss the pricing of interest rate variance swaps for three asset classes (government bonds, time deposits, and IRS) and ensuing indexes of forward-looking gauges of volatility. In practice, indexes cannot be traded directly, but one may trade derivatives referencing the index, such as futures and options, to gain exposure to movements in the index. Section 20.5.5 explains the main challenges encountered while modeling derivatives relating to IRV. It also presents a simple pricing model, which illustrates how the insights gained while constructing a government bond volatility index helps determine the value of these new products.

20.5.1 Standard Volatility Trading Practice

A standard practice to trade IRV relies on straddles. It is well known that equity option straddles might fail to lead to profits consistent with directional volatility views, due to "price dependency," that is, the circumstance that the straddle returns are affected by the direction of price movements rather than their absolute movements – that is, volatility. It is probably fair to say that this issue played a key role in the emergence of variance swap contracts, which aligns volatility views and payoffs by construction and also the redesign of the VIX index maintained by the CBOE since 2003.

Mele and Obayashi (2012, 2014a, Chapter 3) study the P&L of straddles in the important IRS segment, both theoretically and empirically. Theoretically, they show that ATM straddles' P&L inherit the same price dependency problem arising in the equity case, in that their daily P&L at the swaption expiry (say N days after entering the straddle) is, approximately,

$$P\&L_N \approx \sum_{n=1}^{N} \Gamma_n^{\$} \cdot [(\sigma_n^2 - \sigma_{iv}^2) PVBP_N] + \sum_{n=1}^{N} Straddle_n \cdot Vol_n(PVBP) \cdot \widetilde{\frac{\Delta R_n}{R_n}} \tag{20.57}$$

where $\Gamma_n^{\$}$ is the Dollar Gamma on day n (i.e., the swaption's Gamma times the square of the forward swap rate), $Straddle_n$ is the straddle value at n, σ_n is the instantaneous vol of the forward swap rate at n, σ_{iv} is the swaption Black's implied volatility on the first day the strategy is implemented, $Vol_n\,(PVBP)$ is the volatility of the annuity factor growth at n, and finally, $\widetilde{\frac{\Delta R_n}{R_n}}$ denotes the series of shocks affecting the forward swap rate.

That is, the P&L in 20.57 has two components: the first is an average of the "volatility views," $\sigma_n^2 - \sigma_{iv}^2$, weighted with Dollar Gamma, $\Gamma_n^{\$}$ (and the *PVBP* at the swaption expiry). The presence of Dollar Gamma generates a *price dependency*: even if the difference $\sigma_n^2 - \sigma_{iv}^2$ were positive for most of the time, inconvenient realizations of σ_n^2 could occur precisely when the Dollar Gamma is high, such that the first term in 20.57 could be negative even if $\sigma_n^2 - \sigma_{iv}^2$ is positive for most of the time. This price dependency is a feature common to equity option straddles. In the swaption straddle case, the annuity factor $PVBP_t$ in Equation 20.20 introduces an additional source of noise in the P&L (the second term in 20.57), making it even more unlikely that straddle returns would align with realized variance.

Empirically, Mele and Obayashi conclude that in a sample covering more than 10 years including the 2007–2009 financial crisis, straddle returns are "dispersed," with a P&L preserving the same sign of the changes in the realized variance for approximately 62% of the time for 1-month trades, as illustrated in Figure 20.7, and for approximately 65% of the time for 3-month trades. These findings motivate their security design of variance swaps on IRS, surveyed in Sections 20.5.2 and 20.5.3. In fact, the P&L labeled "volatility contract" originates from trades that have the design in Equation 20.69 of Section 20.5.2.4.

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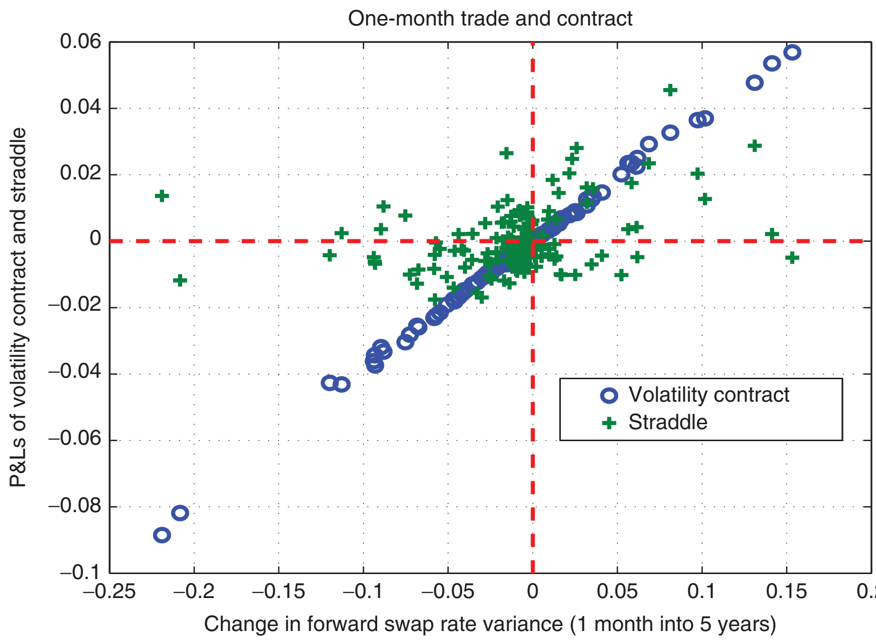

Figure 20.7 Empirical performance of directional volatility trades regarding interest rate swap markets. Source: Bloomberg.

20.5.2 An Introduction to Interest Rate Variance Swaps

One fundamental theme of this chapter relates to the evaluation of contracts referencing realized IRV. The idea underlying these contracts parallels that underlying equity variance swaps: the holder of the contract receives compensation from a counterparty, which is linked to the variance of a variable of interest realized in excess of a threshold determined in advance. This threshold is the "price of volatility," so to speak, and provides forward-looking information about volatility over the life of the variance swap. This section highlights general issues arising while pricing volatility in a model-free fashion and provides an evaluation framework leading to three concrete applications in Section 20.5.3.

20.5.2.1 From Equity to Rate Volatility Contracts The new definition of the CBOE VIX index in 2003 aims to incorporate advances made by financial theory over the previous decade on equity variance swaps. The new VIX index reflects the theoretically fair value of an equity variance swap cast in a "model-free" fashion – one that does not rely on any modeling assumptions beyond specification of standard price dynamics and absence of arbitrage. While variance swaps in the fixed-income space are still hypothetical at the time of writing, their importance as benchmarks for determining the fair value of volatility has already been actualized with the CBOE maintaining two indexes of IRV as discussed in Section 20.5.4.

A variance swap in the fixed-income space works similarly as in the equity case. Consider, for example, the instantaneous volatility of the forward bond price, $v_\tau(S, \mathbb{T})$ in Equation 20.5, and define the so-called realized integrated variance over the time interval $[t, T]$,

$$V(t, T, S, \mathbb{T}) \equiv \int_t^T \|v_\tau(S, \mathbb{T})\|^2 d\tau \tag{20.58}$$

Andersen, Bollerslev, and Diebold (2010) survey the literature of realized variance by highlighting estimation methods. Our emphasis in this chapter is different, motivated as we are to determine the economic value of this variance for a variety of fixed-income securities. Accordingly, assume that the risk we want to be protected against is that the realized variance of some fixed-income security is higher than a strike determined at the inception of the contract at time t. To illustrate, consider a swap linked to the realized variance of the forward bond price in Equation 20.58,

$$\pi(T, S, \mathbb{T}) \equiv V(t, T, S, \mathbb{T}) - \mathbb{P}_t(T, S, \mathbb{T}), \quad T \leq S \tag{20.59}$$

where T is the expiration of the variance swap and $\mathbb{P}_t(T, S, \mathbb{T})$ is the value of the strike, set such that the variance swap is worthless at origination,

$$\mathbb{P}_t(T, S, \mathbb{T}) = \frac{1}{P_t(T)} \mathbb{E}_t \left[e^{-\int_t^T r_\tau d\tau} V(t, T, S, \mathbb{T}) \right] = \mathbb{E}_t^{Q_{FT}} (V(t, T, S, \mathbb{T})) \tag{20.60}$$

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The second equality follows by a change of probability similar to that leading to the forward price in Equations 20.3 and 20.4. Therefore, the strike of a government bond variance swap is the expectation of the future variance under the *forward* probability. In contrast, the strike of an equity variance swap reflects the *risk-neutral* expectation of future variance, assuming interest rates are constant. Naturally, we cannot assume that interest rates are constant when evaluating fixed-income variance swaps, for the value of the latter would then be identically zero.

20.5.2.2 Model-Free Pricing The previous example suggests that while pricing IRV, reference needs to be made to the notion of numéraire and risk-adjusted probability applying to each asset class of interest. In particular, while the pricing probability of *equity* variance swaps is the risk-neutral probability assuming constant interest rates, Equation 20.60 reveals the pricing probability of government bond variance swaps to be the forward probability.

A desirable property of a variance swap is that it could be priced in a *model-free* fashion. “Model-free” means absence of reliance on assumptions rather than absence of arbitrage or standard assumptions on price dynamics and no-arbitrage. An example would suffice to further illustrate the meaning of model-free pricing in our context. Consider the SABR model in Section 20.4.3.2 (Eq. 20.55), which is a model that yields predictions about future expected volatility. For example, if $\beta = 1$, the model predicts that the expected integrated variance of the percentage changes in X_τ over an investment horizon equal to $T - t$ is

$$\mathbb{E}_t\left(\int_t^T v_\tau^2 d\tau\right) = v_t^2 \cdot \frac{e^{\xi^2(T-t)} - 1}{\xi^2} \tag{20.61}$$

which is unknown because the parameter ξ is unknown, even assuming we could observe the current variance level, v_t^2. The parameter ξ could be estimated, but uncertainty surrounding statistical inference translates to uncertainty regarding the estimate of the market expectations about future volatility.

A model-free approach to the evaluation of future volatility aims to infer the market’s expectations about volatility by looking into the entire spectrum of implied volatilities embedded in the price of out-of-the-money options. An implication of this approach is that the fair value of a variance swap should only link to the price of already tradable assets, such as (i) ATM interest rate options, (ii) the entire strip of out-of-the-money interest rate options, and (iii) zero-coupon bonds. It turns out this task is more involved than in the equity case due to the need to find the correct risk adjustment to be made while accounting for the numéraire in each market. For example, there is no guarantee that the appropriate probability to use while taking expectations is the risk-neutral probability, as in the hypothetical example of Equation 20.61.

To illustrate, consider the government bond variance swap in Equation 20.59. Does this payoff imply that the resulting variance swap could be priced in a model-free fashion? In the following, we shall see that in general, a model-free design of a variance contract is obtained by (i) rescaling the contract payoff by the market numéraire at the time of expiration of the variance swap and (ii) expressing the fair value of the contract as the expectation of future volatility under the market probability. In terms of the previous government bond example, the rescaling of item (i) is just $P_T(T) = 1$, such that we do not need to rescale by anything in the definition of the payoff $\pi(T, S, \mathbb{T})$ in Equation 20.59. Note, however, and importantly, that items (i) and (ii) are both required to price variance swaps in a model-free fashion.

For example, and consistently with item (ii), the expectation in Equation 20.60 is model-free whenever $T = S$. Intuitively, it is so because if $T = S$, the expectation in Equation 20.60 is proportional to the value of a portfolio of out-of-the-money options maturing at T and referencing a forward expiring at T. However, as soon as the expiration of the available options is shorter than that of the forward, $T < S$, the very same options cannot span risks generated by the volatility of the forwards. This situation is typical in Treasury markets, as we shall argue later, and a model-free expression for the variance swap is then typically only an approximation to its true value – a quite reasonable one in practice, as explained in the following.

We now proceed with a few technical details that formalize the previous conclusions. First, we provide a few additional albeit basic definitions regarding the notions of volatility we are interested in while modeling fixed-income securities.

20.5.2.3 Basis Point versus Percentage Volatility Unlike in equity markets, the appropriate gauge of IRV does not always rely on a “percentage notion,” as in the government bond volatility example of Equation 20.58. An alternative market standard is to quote volatility in “basis points.” While percentage volatility appears to be the right notion when reference is made to *price* movements, basis point volatility may be a more natural notion of uncertainty in *interest rate* movements. A rate increase from 10 bps to 15bps leads to the same percentage change as one from 100 bps to 150 bps, but, all else equal and accounting for convexity, the latter is a nearly 10-fold P&L and risk event. In this basic example, rate traders might find it more useful to speculate on whether a position might likely experience 5 bps moves or 50 bps moves over a given horizon.

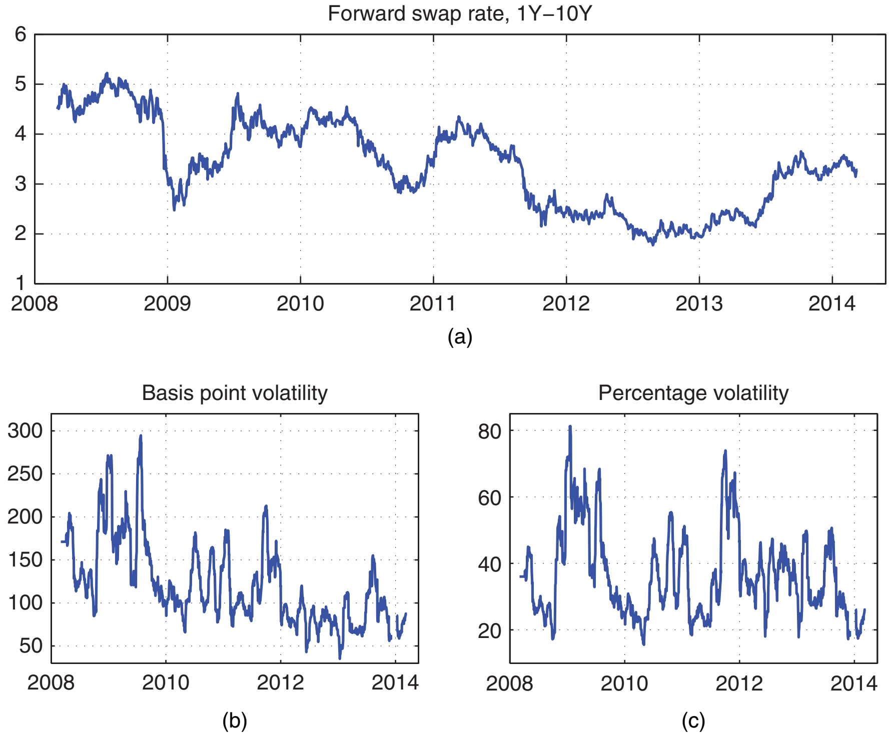

Figure 20.8 Panel A depicts the time series of the forward swap rate with 1-year maturity and 10-year tenor. Panels B and C depicts 1-month basis point realized volatility (Panel B) and 1-month percentage realized volatility (Panel C), both annualized. Source: Bloomberg.

To illustrate the notion of basis point volatility, consider the instantaneous volatility of the forward swap rate, $\sigma_\tau(\cdot)$, in Equation 20.17. The basis point and the percentage variance of the forward swap rate realized over the time interval $[t, T]$ are, respectively,

$$V_n^{\mathrm{bp}}(t, T) \equiv \int_t^T R_\tau^2(T_1, \dots, T_n) \, \|\sigma_\tau(T_1, \dots, T_n)\|^2 \, d\tau \qquad \text{and} \tag{20.62}$$

$$V_n(t, T) \equiv \int_t^T \|\sigma_\tau(T_1, \dots, T_n)\|^2 \, d\tau \tag{20.63}$$

While the concept of percentage variance is very well known and widely used in equity markets, our IRV contract design also needs to consider a basis point notion of volatility because absolute changes describe risk more effectively than relative changes in the context of yields and spreads.

Consider Figure 20.8, which depicts the 1Y–10Y forward swap rate, along with 1-month annualized realized volatilities, both basis point and percentage. During financial turmoil, basis point volatility appears to spike more when the forward swap rate is high. For example, basis point volatility over the second half of 2009 was more volatile than over the first half when interest rates were lower. The general downward trend in the forward swap rate over the sample period in Figure 20.8 is accompanied by a similar trend in basis point volatility. In contrast, percentage volatility seems to be less related to the general trend in the level of the forward swap rates.

Section 20.5.2.5 develops contract designs for variance swaps that are based on both basis point and percentage notions of realized variance. These contracts are cast in a context with random interest rates and numéraires and lead to technical issues unknown previously. For example, while in the standard equity case a percentage variance swap links to the so-called "log-contract," a basis point variance swap links to the "quadratic contract" under the numéraire probability. Furthermore, under standard conditions, the fair value can be decomposed as the product of a general fear factor and the level of interest rates consistently with the informal evidence in Figure 20.8 (see Eq. 20.81). We now explain these properties in detail.

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20.5.2.4 Numéraire Matching To illustrate the main issues arising while pricing IRV, consider an interest rate variance swap that delivers the percentage variance $V_n(t,T)$ in Equation 20.63,

$$\hat{\pi}_{irs}(T,n) \equiv V_n(t,T) - \hat{\mathbb{P}}_t(T,n) \tag{20.64}$$

where $\hat{\mathbb{P}}_t(T,n)$ is the fair value of the variance swap, determined at t, such that the contract has zero value. We argued earlier that a more compelling IRS variance contract should use the basis point notion of volatility. However, we start by pricing Equation 20.64 because we wish to emphasize how variance swaps are priced in connection with the notion of numéraire in this market – the annuity factor in Equation 20.15.

By standard arguments, and in analogy with the fair value of a government bond variance swap, $\mathbb{P}_t(T,S,\mathbb{T})$ in Equation 20.60, the fair value of the interest rate variance swap is determined as the expectation taken under the forward probability,

$$\hat{\mathbb{P}}_t(T,n) = \mathbb{E}_t^{Q_{FT}}(V_n(t,T)) \tag{20.65}$$

Next, let us attempt to determine the expectation on the right-hand side of Equation 20.65, by relying on standard "spanning arguments" developed in the equity case (see, e.g., Bakshi and Madan, 2000; Carr and Madan, 2001). First, consider a log-contract (Neuberger, 1994) on the forward swap rate, that is, a contract that promises to pay a payoff equal to $\ln \frac{R_T}{R_t}$ at expiration T. We are interested in the log-contract because of its linkage to realized variance. Applying Itô's lemma to Equation 20.17 leaves

$$\mathbb{E}_t^{sw,n}\left(\ln \frac{R_T}{R_t}\right) = -\frac{1}{2}\mathbb{E}_t^{sw,n}(V_n(t,T)) \tag{20.66}$$

where we have set the forward swap rate $R_\tau \equiv R_\tau(T_1, \dots, T_n)$ to simplify notation.

Moreover, the appendix reviews the arguments leading to the following expression for the log-contract on the forward swap rate:

$$\ln \frac{R_T}{R_t} = \frac{R_T - R_t}{R_t} - \left(\int_0^{R_t} \left(K - R_T\right)^+ \frac{1}{K^2} dK + \int_{R_t}^{\infty} (R_T - K)^+ \frac{1}{K^2} dK\right) \tag{20.67}$$

The first term on the right-hand side of the previous equation, once rescaled by the annuity factor $PVBP_T$, is the value at T of $\frac{1}{R_t}$ positions in a forward-starting IRS; the second term, rescaled by $PVBP_T$, is the time T payoff of a portfolio of out-of-the-money swaptions, with weights inversely proportional to the square of the strike.

In other words, the payoff of a log-contract can be linked to a portfolio comprising a forward and out-of-the-money derivatives, whence the "spanning" term. In particular, by taking expectations under the annuity probability and relying on results surveyed in Section 20.3,

$$\mathbb{E}_t^{sw,n}\left(\ln \frac{R_T}{R_t}\right) = -\frac{1}{PVBP_t(T_1, \dots, T_n)}\left(\int_0^{R_t} Swpn_t^{\mathrm{r}}(K)\frac{1}{K^2}dK + \int_{R_t}^{\infty} Swpn_t^{\mathrm{p}}(K)\frac{1}{K^2}dK\right) \tag{20.68}$$

with the same notation as in Section 20.3.

Note that there is no way to match Equations 20.66 and 20.68 to Equation 20.65: the expectation in Equation 20.65 is under the forward probability, whereas the expectations in Equations 20.66 and 20.68 are taken under the annuity probability. In other words, the forward swap rate R_τ is a martingale under the annuity probability, such that a model-free expression for the expected volatility (i.e., in terms of available swaption quotes) is only available under the annuity probability, not under the forward probability as required by Equation 20.65.

Now consider the following payoff design as an alternative to that in Equation (20.64)

$$\pi_{irs}(T,n) \equiv PVBP_T(T_1, \dots, T_n) \times (V_n(t,T) - \mathbb{P}_t(T,n)) \tag{20.69}$$

where the fair value is now the expectation of realized variance under the annuity probability

$$\mathbb{P}_t(T,n) = \mathbb{E}_t^{Q_{sw,n}}(V_n(t,T)) \tag{20.70}$$

We can now match Equations 20.66–20.68 to Equation 20.70, obtaining

$$\mathbb{P}_t(T,n) = \frac{2}{PVBP_t(T_1, \dots, T_n)}\left(\int_0^{R_t} Swpn_t^{\mathrm{r}}(K)\frac{1}{K^2}dK + \int_{R_t}^{\infty} Swpn_t^{p}(K)\frac{1}{K^2}dK\right) \tag{20.71}$$

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That is, we can price an interest rate variance swap in a model-free fashion once we rescale the payoff through the numéraire in this market, just as in Equation 20.69. In the next sections, we generalize to other markets and the notion of basis point volatility.

Before moving on to the theoretical generalization, we develop a numerical example to illustrate how Equation 20.71 can be implemented in practice, and an ensuing IRV index can be calculated.

20.5.2.5 A Numerical Example We provide an example regarding the implementation of the fair value of a variance swap on an IRS, $\mathbb{P}_t(T, n)$, in Equation 20.71, relying on hypothetical implied Black's volatilities for swaptions maturing in 1 month and tenor equal to 5 years (i.e., $\sigma_{iv}\left(K, R_{n,t}, T_0 - t\right)$ in Equation 20.49, with $T_0 - t = \frac{1}{12}$ and $T_n - T_0 = 5$) and regarding hypothetical market conditions on February 12, 2010. Panel A of Table 20.1 displays strike K (first column), implied vol (second), and basis point volatility (third), defined as

$$\sigma_{iv}^{bp}\left(K, R_{n,t}, T_0 - t\right) \equiv \sigma_{iv}(K, R_{n,t}, T_0 - t) \cdot R_{n,t} \tag{20.72}$$

TABLE 20.1 Calculation of the Fair Value of a Hypothetical Variance Swap on an IRS

Panel A

			Black's prices	
Strike rate (%)	Black's implied vol (%)	Basis point implied vol	Receiver swaption	Payer swaption
1.7352	36.1900	98.9869	≈ 0	$10.0000 \cdot 10^{-3}$
1.9852	36.1900	98.9869	$0.0007 \cdot 10^{-3}$	$7.5007 \cdot 10^{-3}$
2.2352	36.1200	98.7954	$0.0259 \cdot 10^{-3}$	$5.0259 \cdot 10^{-3}$
2.4352	35.9900	98.4398	$0.1773 \cdot 10^{-3}$	$3.1773 \cdot 10^{-3}$
2.5352	35.9300	98.2757	$0.3692 \cdot 10^{-3}$	$2.3692 \cdot 10^{-3}$
2.6352	35.8600	98.0843	$0.6793 \cdot 10^{-3}$	$1.6793 \cdot 10^{-3}$
2.6852	35.8300	98.0022	$0.8855 \cdot 10^{-3}$	$1.3855 \cdot 10^{-3}$
2.7352 (ATM)	35.8000	97.9202	$1.1272 \cdot 10^{-3}$	$1.1272 \cdot 10^{-3}$
2.7852	35.7600	97.8108	$1.4037 \cdot 10^{-3}$	$0.9037 \cdot 10^{-3}$
2.8352	35.7300	97.7287	$1.7142 \cdot 10^{-3}$	$0.7142 \cdot 10^{-3}$
2.9352	35.6700	97.5646	$2.4270 \cdot 10^{-3}$	$0.4270 \cdot 10^{-3}$
3.0352	35.6000	97.3731	$3.2406 \cdot 10^{-3}$	$0.2406 \cdot 10^{-3}$
3.2352	35.4700	97.0175	$5.0644 \cdot 10^{-3}$	$0.0644 \cdot 10^{-3}$
3.4852	35.3100	96.5799	$7.5092 \cdot 10^{-3}$	$0.0092 \cdot 10^{-3}$
3.7352	35.1400	96.1149	$10.0010 \cdot 10^{-3}$	$0.0010 \cdot 10^{-3}$

Panel B

			Weights		Contributions to strikes	
Strike rate (%)	Swaption type	Price	Basis point ΔK_i	Percentage $\Delta K_i / K_i^2$	Basis point contribution	Percentage contribution
1.7352	Receiver	≈ 0	0.0025	8.3031	≈ 0	≈ 0
1.9852	Receiver	$0.0007 \cdot 10^{-3}$	0.0025	6.3435	$0.0018 \cdot 10^{-6}$	$0.0046 \cdot 10^{-3}$
2.2352	Receiver	$0.0259 \cdot 10^{-3}$	0.0022	4.5035	$0.0583 \cdot 10^{-6}$	$0.1167 \cdot 10^{-3}$
2.4352	Receiver	$0.1773 \cdot 10^{-3}$	0.0015	2.5294	$0.2660 \cdot 10^{-6}$	$0.4485 \cdot 10^{-3}$
2.5352	Receiver	$0.3692 \cdot 10^{-3}$	0.0010	1.5559	$0.3692 \cdot 10^{-6}$	$0.5744 \cdot 10^{-3}$
2.6352	Receiver	$0.6793 \cdot 10^{-3}$	0.0008	1.0800	$0.5095 \cdot 10^{-6}$	$0.7337 \cdot 10^{-3}$
2.6852	Receiver	$0.8855 \cdot 10^{-3}$	0.0005	0.6935	$0.4428 \cdot 10^{-6}$	$0.6141 \cdot 10^{-3}$
2.7352	ATM	$1.1272 \cdot 10^{-3}$	0.0005	0.6683	$0.5636 \cdot 10^{-6}$	$0.7533 \cdot 10^{-3}$
2.7852	Payer	$0.9037 \cdot 10^{-3}$	0.0005	0.6446	$0.4518 \cdot 10^{-6}$	$0.5825 \cdot 10^{-3}$
2.8352	Payer	$0.7142 \cdot 10^{-3}$	0.0007	0.9330	$0.5357 \cdot 10^{-6}$	$0.6664 \cdot 10^{-3}$
2.9352	Payer	$0.4270 \cdot 10^{-3}$	0.0010	1.1607	$0.4270 \cdot 10^{-6}$	$0.4956 \cdot 10^{-3}$
3.0352	Payer	$0.2406 \cdot 10^{-3}$	0.0015	1.6282	$0.3609 \cdot 10^{-6}$	$0.3917 \cdot 10^{-3}$
3.2352	Payer	$0.0644 \cdot 10^{-3}$	0.0023	2.1497	$0.1448 \cdot 10^{-6}$	$0.1384 \cdot 10^{-3}$
3.4852	Payer	$0.0092 \cdot 10^{-3}$	0.0025	2.0582	$0.0229 \cdot 10^{-6}$	$0.0188 \cdot 10^{-3}$
3.7352	Payer	$0.0010 \cdot 10^{-3}$	0.0025	1.7919	$0.0024 \cdot 10^{-6}$	$0.0017 \cdot 10^{-3}$
				SUMS	$4.1567 \cdot 10^{-6}$	$5.5405 \cdot 10^{-3}$

Source: Bloomberg.

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The current forward swap rate is $R_{n,t} = 2.7352\%$ such that the ATM implied volatility is $\sigma_{iv}(K, R_{n,t}, T_0 - t)|_{K=R_{n,t}} = 35.80\%$ and $\sigma_{iv}^{bp}(K, R_{n,t}, T_0 - t)|_{K=R_{n,t}} = 2.7352\% \times 35.80\% = 0.979202\%$, that is, 97.9202 basis point volatility. Basis point volatilities are not needed to compute $\mathbb{P}_t(T, n)$ in 20.71 but are given for comparison purposes.

We plug the skew in the second column of Table 20.1 into the Black's formula, Equation 20.49, and calculate hypothetical swaption prices (normalized by the $PVBP_t$), reported in columns four and five of Table 20.1 (labeled "Black's prices"). We then approximate the integral in 20.71 using discretization steps equal to $\Delta K_i = \frac{1}{2}(K_{i+1} - K_{i-1})$ for $1 \le i < M$, $\Delta K_0 = (K_1 - K_0)$ and $\Delta K_M = (K_M - K_{M-1})$, where K_0 and K_M are the lowest and the highest available strikes and $M + 1$ is the total number of swaptions.

Panel B of Table 20.1 reports strikes (first column), swaption types used to estimate the fair value $\mathbb{P}_t(T, n)$ in 20.71 (second), and swaption prices (third). The fifth column reports the weight each price is given toward the estimation of $\mathbb{P}_t(T, n)$, before the final rescaling of 2; finally, the seventh column (labeled "percentage contribution") provides each swaption price multiplied by its weight (i.e., the third column multiplied by the fifth).

For later reference, the fourth column of Table 20.1 (Panel B) displays values for the weight ΔK_i, and the sixth column (labeled "basis point contribution") provides each swaption price rescaled by ΔK_i (i.e., the third column multiplied by the fourth). To anticipate, we shall utilize the values of column six while calculating the fair value of a basis point variance swap on an IRS (see Section 20.5.3.3, Eq. 20.95).

So we estimate the fair value of the variance swap $\mathbb{P}_t(T, n)$ to be $2 \times 5.5405 \cdot 10^{-3}$ in this example. We could, then, determine an index of percentage volatility based on this variance swap by annualizing this estimate of $\mathbb{P}_t(T, n)$ and expressing it in percentage terms, yielding the following forward-looking gauge of IRS volatility (1 month) on 5-year tenor forward swap rates:

$$\widehat{IRS\text{-}VI}_n(t, T) \equiv 100 \times \sqrt{\frac{2}{12^{-1}} \times 5.5405 \cdot 10^{-3}} = 36.4653$$

In comparison, the ATM implied volatility is $\sigma_{iv}(K, R_{n,t}, T_0 - t)|_{K=R_{n,t}} = 35.80\%$ as we mentioned earlier.

20.5.2.6 Contract Design Our goal is to design a contract that pays off an amount that depends on the realized variance of the risk of interest (say, the variance of the forward swap rate). However, a number of conceptual difficulties arise while attempting to price this contract in a model-free fashion, as interest rates for discounting are not independent of their volatility. How can we disentangle discounting from realized volatility of discounting, thereby pricing the pure volatility component? The answer is that we shall need to rescale the variance payoff with the market numéraire regarding the class of interest. This section provides introductory details.

Consider a forward-starting agreement originating at t, promising the following payoff at time T:

$$\Pi_T \equiv N_T \times (X_T - K) \tag{20.73}$$

where K is a constant; N_τ denotes the price of a tradable process, which is measurable with respect to the information at time τ

Concretely, N_τ is the value of the market numéraire and X_τ is a "forward risk." For example, X_τ can be the price of a forward expiring at T and written on a coupon-bearing bond, in which case $N_\tau \equiv P_\tau(T)$ and $N_T \equiv 1$. As another example, X_τ can be the forward swap rate, in which case $N_\tau \equiv PVBP_\tau$, such that the payoff Π_T in Equation 20.73 is that of an IRS payer. Even simpler, X_τ can be the price of a forward on a stock, and the numéraire is the price of a zero-coupon bond in a market with constant interest rates $\bar{r}$, say, $N_\tau \equiv e^{-\bar{r}(T-\tau)}$.

We are interested in pricing the realized volatility of X_τ to be formally defined later. We assume that X_τ is a diffusion process. Under standard regularity conditions, the clearing process X_τ is such that $X_t = K$ and is a martingale under the "market probability" Q_N, defined through the Radon–Nikodym derivative,

$$\xi_T^N \equiv \left.\frac{dQ_N}{dQ}\right|_{\mathcal{F}_T} = e^{-\int_t^T r_\tau d\tau} \frac{N_T}{N_t} \tag{20.74}$$

Therefore, X_τ satisfies

$$\frac{dX_\tau}{X_\tau} = \sigma_\tau \cdot dW_\tau \tag{20.75}$$

where W_τ is a vector Brownian motion under Q^N and σ_τ is the instantaneous volatility process adapted to W_τ.

As noted, market practice is to quote implied volatilities for fixed-income instruments both in percentage and basis point terms. Accordingly, we aim to price two notions of realized variance defined similarly as in the interest rate swap volatility case (see Eqs 20.62 and 20.63): one based on *arithmetic*, or *basis point*, changes of X_t in Equation 20.75 and another based on the

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logarithmic, or *percentage*, changes of X_t. Let $V^{bp}(t,T)$ and $V(t,T)$ denote the realized BP variance and percentage variance in the time interval $[t,T]$, namely,

$$V^{bp}(t,T) \equiv \int_t^T X_\tau^2 \|\sigma_\tau\|^2 d\tau \ \text{ and } \ V(t,T) \equiv \int_t^T \|\sigma_\tau\|^2 d\tau \tag{20.76}$$

We are searching for variance contracts linked to $V^{bp}(t,T)$ and $V(t,T)$ that can be priced in a model-free fashion.

Mele and Obayashi (2014a, Chapter 2) consider the following forward contracts with "stochastic multiplier," ones that are zero at the time of inception, t, and that at maturity T, pay off

$$\Phi_T^{bp} \equiv Y_T \times (V^{bp}(t,T) - K_Y^{bp}), \quad \Phi_T \equiv Y_T \times (V(t,T) - K_Y) \tag{20.77}$$

where Y_τ is measurable with respect to the Brownian motion W_τ in Equation 20.75 and K_Y^{bp} and K_Y are the fair values of the contracts to be determined. Naturally, we have that

$$K_Y^{bp} = \mathbb{E}_t^{Q_Y}(V^{bp}(t,T)), \quad K_Y = \mathbb{E}_t^{Q_Y}(V(t,T)) \tag{20.78}$$

where $\mathbb{E}_t^{Q_Y}$ denotes expectation at t under Q_Y and Q_Y is a probability defined formally through ξ_T^N in Equation 20.74, with $N \equiv Y$, and referred hereafter as *forward multiplier probability*.

Mele and Obayashi (2014a, Proposition 2.II) prove the following result:

Proposition 1 ***(model-free contracts.)*** *The fair values of K_Y^{bp} and K_Y in Equation 20.77 are model-free if and only if the Radon–Nikodym derivative of the forward multiplier probability Q_Y against the market numéraire probability Q_N is uncorrelated with $V^{bp}(t,T)$ and $V(t,T)$. They are given by*

$$K_Y^{bp} = \frac{2}{N_t}\left(\int_0^{X_t} Put_t(K)\,dK + \int_{X_t}^{\infty} Call_t(K)dK\right) \textit{(basis point pricing)} \tag{20.79}$$

and

$$K_Y = \frac{2}{N_t}\left(\int_0^{X_t} \frac{Put_t(K)}{K^2}dK + \int_{X_t}^{\infty} \frac{Call_t(K)}{K^2}dK\right) \qquad \textit{(percentage pricing)} \tag{20.80}$$

where

$$\frac{Call_t(K)}{N_t} = \mathbb{E}_t^{Q_N}\left(\frac{\max\{\Pi_T, 0\}}{N_T}\right), \quad \frac{Put_t(K)}{N_t} = \mathbb{E}_t^{Q_N}\left(\frac{\max\{-\Pi_T, 0\}}{N_T}\right)$$

and Π_T is the payoff in Equation 20.73.

Note that the previous proposition is not merely restating the famous conclusion that in the absence of arbitrage, there exists a numéraire N and a probability Q_N, such that security prices rescaled by the numéraire are martingales under Q_N. It states a stronger result about security design, namely, that *model-free* pricing of variance swaps is possible once we rescale the payoff of these contracts with the appropriate numéraire.

Indeed, Mele and Obayashi identify a host of possible variance swap "tilters" that make the fair value of the resulting contracts model-free, with the "numé raire" tilter being a specific case. However, given the economic appeal of the numéraire, and the familiarity with it among academics and practitioners, we will build upon this notion to develop our model-free security designs of interest rate variance swaps. Consider, for example, the previous interest rate variance swap with payoff rescaled by the annuity numéraire $PVBP_t$ in Equation 20.69. The ensuing strike, $\mathbb{P}_t(T,n)$ in Equation 20.71, is just the same as K_Y in Equation 20.80, once we account for the definition of the various derivative payoffs and risks.

Mele and Obayashi (2014a, Chapter 2) also show that if implied volatilities are homogeneous of degree zero in X and K ("sticky delta"),[10] then, K_Y in Equation 20.80 is independent of X_t, and there exists a function independent of X, $Sk(t,T)$, say, such that K_Y^{bp} in Equation 20.79 can be written as

$$K_Y^{bp} = X_t^2 \times Sk(t,T) \tag{20.81}$$

[10]A well-known example of models leading to "sticky delta" is the SABR model (when $\beta = 1$) in Section 20.4.3.2.

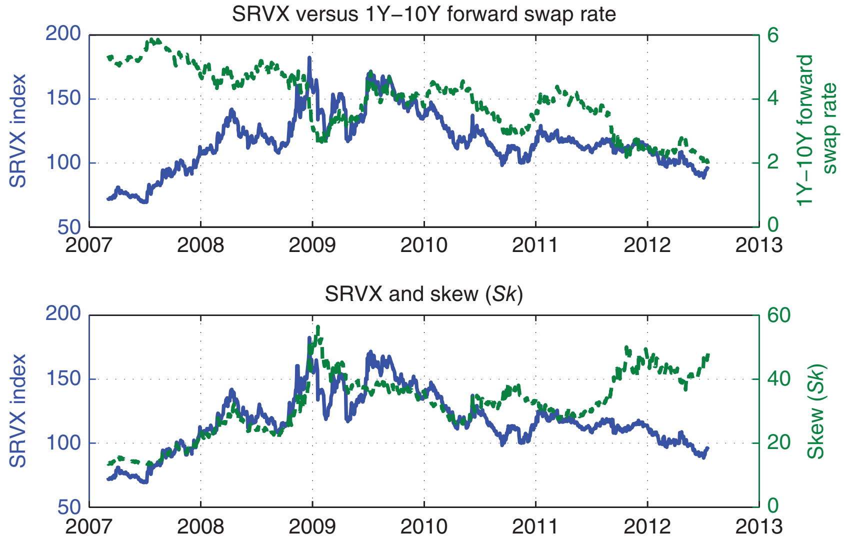

Figure 20.9 The CBOE SRVX interest rate swap volatility index (see Sections 20.5.4 and 20.5.5), along with the 1Y–10Y forward swap rate (Panel A) and the skew factor, $\sqrt{\frac{1}{T-t}Sk(t,T)}$, where $Sk(t,T)$ is as in Equation 20.81. Source: Chicago Board Options Exchange and Bloomberg.

The term $Sk(t,T)$ summarizes a pure "fear effect" or a "skew factor" and is shown to link to the entire skew of the options entering into the strike calculation.

Consider, for example, the SRVX index of swap rate volatility maintained by the CBOE (regarding volatility of 1Y–10Y forward swap rates), which is based on the strike K_Y in 20.79 (see Sections 20.5.4 and 20.5.5). Figure 20.9 depicts the behavior of the SRVX and both the forward swap rate and the annualized square root of the skew factor, $\sqrt{\frac{1}{T-t}Sk(t,T)}$. Mele, Obayashi, and Shalen (2015) analyze the empirical behavior of the CBOE SRVX and indeed show that this behavior can sometimes be determined by changes in the forward swap rate and at other times by the skew factor. For example, Figure 20.9 illustrates that the increase in the SRVX over the 2008 crisis is mostly driven by a fear factor; however, the SRVX decline in 2012 is explained by lower prevailing interest rates.

Section 20.5.3.3 explains that a basis point volatility index has additional properties, which could be utilized to assess uncertainty regarding developments in the *level* of interest rates and apply this property to swap markets (see Figure 20.11).

20.5.2.7 A Few Technical Details: Quadratic Contracts While the expression for the portfolio of out-of-the-money derivatives in Equation 20.80 may look familiar to those acquainted with the formula underlying the CBOE VIX index (see, e.g., Chicago Board Options Exchange, 2009), the expression in Equation 20.79 may be less familiar, and we therefore review the steps leading to it.

Suppose that the forward contract stochastic multiplier in Equation 20.77 is the same as the market numéraire, $Y_T = N_T$, and consider a "quadratic contract," one that delivers a payoff equal to X_T^2 at time T. The appendix shows that

$$X_T^2 - X_t^2 = 2X_t(X_T - X_t) + 2\left(\int_0^{X_t} \left(K - X_T\right)^+ dK + \int_{X_t}^{\infty} (X_T - K)^+ dK\right) \tag{20.82}$$

The first term on the right-hand side, once rescaled by the numéraire, N_T, is just the payoff Π_T in Equation 20.73 of $2X_t$ positions in a forward-starting agreement; the second term, rescaled by N_T, is the time T payoff of a portfolio of out-of-the-money options, with constant weights. By taking expectations under Q_N of both sides of the previous equation leaves

$$\mathbb{E}_t^{Q_N}(X_T^2) - X_t^2 = \frac{2}{N_t}\left(\int_0^{X_t} Put_t(K)\,dK + \int_{X_t}^{\infty} Call_t(K)dK\right) \tag{20.83}$$

Moreover, by applying Itô's lemma to Equation 20.75 leaves

$$\mathbb{E}_t^{Q_N}(X_T^2) - X_t^2 = \mathbb{E}_t^{Q_N}(V^{bp}(t,T)) \tag{20.84}$$

Matching Equations 20.83 and 20.84 to K_Y^{bp} in Equation 20.78 for $Y = N$ produces Equation 20.79. Note that in earlier work, Carr and Corso (2001) explain how to hedge the variance of price changes in markets with constant interest rates. In Mele and

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Obayashi (2014a, Chapter 2), we explain that their elegant replication arguments fail to work if interest rates are random. In subsequent chapters, we also show that the numéraire inherent in each market of interest can be incorporated into replicating portfolios that contain fixed-income securities.

The next subsection applies the general framework underlying Proposition I to analyze a number of cases that are relevant to market practice.

20.5.3 Pricing Volatility in Three Markets

20.5.3.1 Government Bonds Webegin with a case regarding the pricing of a government bond variance swap, which does not exactly conform with the previous framework. Consider the strike $\mathbb{P}_t(T, S, \mathbb{T})$ in Equation 20.60 and recall the variables it depends upon (i) the maturity of the contract, $T - t$; (ii) the maturity of the underlying forward price, $S - T$; and (iii) the expiration time of the coupon-bearing bond underlying the forward, $\mathbb{T}$.

Guided by intuition gained over the previous sections, we would like to find a model-free expression for $\mathbb{P}_t(T, S, \mathbb{T})$ based on options expiring at T and written on the forwards expiring at S. It turns out that this is impossible: the short-dated option cannot span the risks generated by the bond forward volatility. Let us explain the main issue. The dynamics of $F_\tau(S, \mathbb{T})$ under the "variance swap" pricing measure are those under Q_{F^T}, not under Q_{F^S} as in Equation 20.5, and by Girsanov theorem,

$$\frac{dF_\tau(S, \mathbb{T})}{F_\tau(S, \mathbb{T})} = v_\tau(S, \mathbb{T})(v_\tau(S, \mathbb{T}) - v_\tau(T, \mathbb{T}))d\tau + v_\tau(S, \mathbb{T}) \cdot dW_\tau^{F^T}, \quad \tau \in (t, T) \tag{20.85}$$

where $W_\tau^{F^T}$ is a multidimensional Brownian motion under Q_{F^T} and the nonzero drift reflects the adjustment made while moving from the probability Q_{F^S} to Q_{F^T}. Mele and Obayashi (2014a,b, Chapter 4) show that

$$\begin{aligned}\mathbb{P}_t(T, S, \mathbb{T}) &= 2(1 - \mathbb{E}_t^{Q_{F^T}}(e^{\tilde{\ell}(t,T,S,\mathbb{T})} - \tilde{\ell}(t, T, S, \mathbb{T}))) \\ &\quad + \frac{2}{P_t(T)}\left(\int_0^{F_t(S,\mathbb{T})} Put_t^b(K)\frac{1}{K^2}dK + \int_{F_t(S,\mathbb{T})}^{\infty} Call_t^b(K)\frac{1}{K^2}dK\right)\end{aligned} \tag{20.86}$$

where $Put_t^b(K)$ and $Call_t^b(K)$ denote the price of European out-of-the-money options written on the bond forward and struck at K and

$$\tilde{\ell}(t, T, S, \mathbb{T}) \equiv \int_t^T v_\tau(S, \mathbb{T})(v_\tau(S, \mathbb{T}) - v_\tau(T, \mathbb{T}))d\tau \tag{20.87}$$

Therefore, an index of forward-looking government bond volatility can be based on the annualized value of $\mathbb{P}(t, T, S, \mathbb{T})$,

$$GB - VI(t, T, S, \mathbb{T}) \equiv \sqrt{\frac{1}{T - t}\mathbb{P}_t(T, S, \mathbb{T})} \tag{20.88}$$

This index is model dependent as the term $\tilde{\ell}(t, T, S, \mathbb{T})$ in Equation 20.87 cannot be spanned through available fixed-income securities. Mele and Obayashi (2014a, Chapter 4) report that the effects of this maturity mismatch are limited whenever $T - S$ (the difference between the futures and option maturity) is as small as 2 months. For example, the CBOE/CBOT 10-year Treasury Note Volatility Index maintained by the CBOE relies on 1-month options and an underlying expiring in 3 months, such that the maturity mismatch is small and, at least in this case, so is the bias of not accounting for a model-dependent adjustment, that is, the first term on the right-hand side of Equation 20.86.[11]

Table 20.2 contains details regarding the implementation of the government bond volatility index in Equation 20.88 using hypothetical quotes reflecting market conditions on April 27, 2012, for 1-month options written on a 1-month forward on 10-year U.S. Treasury Notes (therefore, the term $\tilde{\ell}(t, T, S, \mathbb{T})$ in 20.87 is zero) and assuming these hypothetical data relate to *European* options on *forwards*, rather than *American* options on *futures*. The first column of Panel A reports strike K, with the ATM being equal to $K = 132$; the second column is given for reference and provides Black's implied volatilities for each strike[12]; finally, the third and fourth columns provide the option premiums.

Panel B of Table 20.2 contains details leading to an estimate of the index in Equation 20.88: the strikes (first column); the type of option used to implement the index (second); the option premiums (third); the weights used while approximating the integral

[11] Another complication is that the underlying of the future is actually the "cheapest to deliver" from a set of deliverable bonds into a futures contract. Throughout this chapter, we assume that the coupon-bearing bond underlying the index is one with fixed and known maturity date.

[12] Black's volatilities are defined similarly as in Equation 20.49 but with the price of 1-month zeros replacing the $PVBP_t$ and obviously the option premiums replacing the swaptions.

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TABLE 20.2 Calculation of the Fair Value of a Hypothetical Government Bond Variance Swap

Panel A

		Premiums	
Strike price (%)	Black's implied vol (%)	Put option	Call option
125.00	9.10	$0.2343 \cdot 10^{-3}$	$7.0234 \cdot 10^{-2}$
125.50	8.53	$0.2346 \cdot 10^{-3}$	$6.5234 \cdot 10^{-2}$
126.00	7.32	$0.1326 \cdot 10^{-3}$	$6.0132 \cdot 10^{-2}$
126.50	6.78	$0.1328 \cdot 10^{-3}$	$5.5132 \cdot 10^{-2}$
127.00	7.24	$0.3423 \cdot 10^{-3}$	$5.0342 \cdot 10^{-2}$
127.50	6.64	$0.3465 \cdot 10^{-3}$	$4.5346 \cdot 10^{-2}$
128.00	6.33	$0.4516 \cdot 10^{-3}$	$4.0451 \cdot 10^{-2}$
128.50	6.15	$0.6567 \cdot 10^{-3}$	$3.5656 \cdot 10^{-2}$
129.00	5.81	$0.8557 \cdot 10^{-3}$	$3.0855 \cdot 10^{-2}$
129.50	5.63	$1.2506 \cdot 10^{-3}$	$2.6250 \cdot 10^{-2}$
130.00	5.35	$1.7225 \cdot 10^{-3}$	$2.1722 \cdot 10^{-2}$
130.50	5.05	$2.3656 \cdot 10^{-3}$	$1.7365 \cdot 10^{-2}$
131.00	4.82	$3.3632 \cdot 10^{-3}$	$1.3363 \cdot 10^{-2}$
131.50	4.71	$4.9229 \cdot 10^{-3}$	$9.9229 \cdot 10^{-3}$
132.00 (ATM)	4.53	$6.8864 \cdot 10^{-3}$	$6.8864 \cdot 10^{-3}$
132.50	4.43	$9.5398 \cdot 10^{-3}$	$4.5398 \cdot 10^{-3}$
133.00	4.40	$1.2865 \cdot 10^{-2}$	$2.8655 \cdot 10^{-3}$
133.50	4.38	$1.6705 \cdot 10^{-2}$	$1.7053 \cdot 10^{-3}$
134.00	4.40	$2.0979 \cdot 10^{-2}$	$0.9793 \cdot 10^{-3}$
134.50	4.58	$2.5619 \cdot 10^{-2}$	$0.6192 \cdot 10^{-3}$
135.00	4.78	$3.0400 \cdot 10^{-2}$	$0.4000 \cdot 10^{-3}$
135.50	4.93	$3.5246 \cdot 10^{-2}$	$0.2462 \cdot 10^{-3}$
136.00	5.17	$4.0169 \cdot 10^{-2}$	$0.1696 \cdot 10^{-3}$
136.50	5.21	$4.5090 \cdot 10^{-2}$	$9.0837 \cdot 10^{-5}$

Panel B

			Weights		Contributions to strikes
Strike price (%)	Option type	Premiums	$\Delta K_i/K_i^2$		Premiums × weights
125.00	Put	$0.2343 \cdot 10^{-3}$	$3.2000 \cdot 10^{-3}$		$7.4976 \cdot 10^{-7}$
125.50	Put	$0.2346 \cdot 10^{-3}$	$3.1745 \cdot 10^{-3}$		$7.4494 \cdot 10^{-7}$
126.00	Put	$0.1326 \cdot 10^{-3}$	$3.1494 \cdot 10^{-3}$		$4.1762 \cdot 10^{-7}$
126.50	Put	$0.1328 \cdot 10^{-3}$	$3.1245 \cdot 10^{-3}$		$4.1512 \cdot 10^{-7}$
127.00	Put	$0.3423 \cdot 10^{-3}$	$3.1000 \cdot 10^{-3}$		$1.0613 \cdot 10^{-6}$
127.50	Put	$0.3465 \cdot 10^{-3}$	$3.0757 \cdot 10^{-3}$		$1.0658 \cdot 10^{-6}$
128.00	Put	$0.4516 \cdot 10^{-3}$	$3.0517 \cdot 10^{-3}$		$1.3781 \cdot 10^{-6}$
128.50	Put	$0.6567 \cdot 10^{-3}$	$3.0280 \cdot 10^{-3}$		$1.9887 \cdot 10^{-6}$
129.00	Put	$0.8557 \cdot 10^{-3}$	$3.0046 \cdot 10^{-3}$		$2.5710 \cdot 10^{-6}$
129.50	Put	$1.2506 \cdot 10^{-3}$	$2.9814 \cdot 10^{-3}$		$3.7289 \cdot 10^{-6}$
130.00	Put	$1.7225 \cdot 10^{-3}$	$2.9585 \cdot 10^{-3}$		$5.0963 \cdot 10^{-6}$
130.50	Put	$2.3656 \cdot 10^{-3}$	$2.9359 \cdot 10^{-3}$		$6.9454 \cdot 10^{-6}$
131.00	Put	$3.3632 \cdot 10^{-3}$	$2.9135 \cdot 10^{-3}$		$9.7990 \cdot 10^{-6}$
131.50	Put	$4.9229 \cdot 10^{-3}$	$2.8914 \cdot 10^{-3}$		$1.4234 \cdot 10^{-5}$
132.00	ATM	$6.8864 \cdot 10^{-3}$	$2.8696 \cdot 10^{-3}$		$1.9761 \cdot 10^{-5}$
132.50	Call	$4.5398 \cdot 10^{-3}$	$2.8479 \cdot 10^{-3}$		$1.2929 \cdot 10^{-5}$
133.00	Call	$2.8655 \cdot 10^{-3}$	$2.8266 \cdot 10^{-3}$		$8.0999 \cdot 10^{-6}$
133.50	Call	$1.7053 \cdot 10^{-3}$	$2.8054 \cdot 10^{-3}$		$4.7842 \cdot 10^{-6}$
134.00	Call	$0.9793 \cdot 10^{-3}$	$2.7845 \cdot 10^{-3}$		$2.7271 \cdot 10^{-6}$
134.50	Call	$0.6192 \cdot 10^{-3}$	$2.7632 \cdot 10^{-3}$		$1.7116 \cdot 10^{-6}$
135.00	Call	$0.4000 \cdot 10^{-3}$	$2.7434 \cdot 10^{-3}$		$1.0975 \cdot 10^{-6}$
135.50	Call	$0.2462 \cdot 10^{-3}$	$2.7232 \cdot 10^{-3}$		$6.7062 \cdot 10^{-7}$
136.00	Call	$0.1696 \cdot 10^{-3}$	$2.7032 \cdot 10^{-3}$		$4.5864 \cdot 10^{-7}$
136.50	Call	$9.0837 \cdot 10^{-5}$	$2.6835 \cdot 10^{-3}$		$2.4376 \cdot 10^{-7}$
				SUM	$1.0268 \cdot 10^{-4}$

Source: Bloomberg.

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of premiums in 20.86 (fourth), determined just as we did for the weights in Section 20.5.2.5 regarding variance swaps on IRS; and the contributions to strike (fifth). The fair value of the variance swap $\mathbb{P}_t(T, S, \mathbb{T})$ is estimated to equal $2 \times 1.0268 \cdot 10^{-4}$, such that the government bond volatility index can be estimated as

$$\widehat{GB\text{-}VI} = 100 \times \sqrt{\frac{1}{0.9980} \frac{2}{(1/12)} \times 1.0268 \cdot 10^{-4}} = 4.9692 \tag{20.89}$$

where 0.9980 is the price of a hypothetical 1-month zero on April 27, 2012. In comparison, the ATM implied volatility in Table 20.2 is 4.53%. Our estimate in 20.89, 4.9692, is well in line with the official value taken by the CBOE/CBOT VXTYN index (see Section 20.5.4) on that day, which was 4.87% (closing value).

The previous calculations rely on the assumption that European-style options on forwards are available for trading, which unfortunately does not hold in practice, as the options available to calculate a government bond volatility index are typically American options written on futures. This issue potentially affects the accuracy of the index as an indicator of the fair value of volatility due to the likely presence of an early exercise premium, and an index relying on American options likely overstates the true fair value of volatility.

Mele and Obayashi (2014a, Chapter 4) undertake a simple experiment aimed at gauging this bias. They consider a model without stochastic volatility, that is, the Vasicek model in Equation 20.21. While this model obviously predicts that the fair of volatility is constant, the authors' main concern is to assess the impact of the early exercise premium unconditionally, that is, independent of any particular value taken by the volatility of the short-term rate.

In a nutshell, the algorithm they use is the following. First, they determine the theoretical value of futures on the coupon-bearing bond based on Equation 20.2. Second, they estimate the parameter values of the Vasicek model under the physical probability. Third, they calibrate the risk premium coefficient, λ in Equation 20.21, to minimize a certain criterion of distance between the price of American call options on futures and the model-based price of American options on the futures. This step requires utilizing methods combining time discretizations and Monte Carlo simulations based on an algorithm proposed by Longstaff and Schwartz (2001); see Chapter 19 for a survey of methods regarding these approaches.

This procedure amounts to extract a pricing kernel from the cross section of American options on futures and parallels similar work attempted in the literature (see, e.g., Broadie, Chernov, and Johannes, 2007; Bikbov and Chernov, 2011). One then uses this risk premium coefficient to calculate the value of European options on forwards based on the Jamshidian (1989) formula (see Eq. 20.38). Alternative to a minimization criterion, one could extract a "risk premium skew," that is, a λ for each strike, but numerical experiments in Mele and Obayashi (2014a, Chapter 4) are only available for the first algorithm based on a minimization criterion. The experiments reveal that under realistic market conditions, a government bond volatility index based on American options written on futures overstates the true index value (i.e., based on European options on forwards) by approximately one relative percentage point.

20.5.3.2 Time Deposits Government bonds and time deposits share the same numéraire (see Section 20.3), and we can price a time deposit variance swap using the forward probability. In contrast to the government bond case, however, we translate the results into *basis point volatility of rates* to match market convention.

We convert forward Libor *prices* into forward Libor *rates* through Equation 20.7, which by Equation 20.8 satisfy

$$\frac{d\hat{f}_\tau(S, S+\Delta)}{\hat{f}_\tau(S, S+\Delta)} = v_\tau^f(S, \Delta) dW_\tau^{F^S}, \quad \tau \in (t, S) \tag{20.90}$$

where $v_\tau^f(S, \Delta) \equiv (1 - \hat{f}_\tau^{-1}(S, S+\Delta)) v_\tau^z(S, \Delta)$ by It ô's lemma.

The basis point Libor integrated rate variance is

$$V_t^{f,bp}(T, S, \Delta) \equiv \int_t^T \hat{f}_\tau^2(S, S+\Delta) \, \|v_\tau^f(S, \Delta)\|^2 d\tau$$

such that, by arguments similar to those leading to Equation 20.60, the fair value of the time deposit rate variance swap starting at t and paying off at T

$$V_t^{f,bp}(T, S, \Delta) - \mathbb{P}_t^{f,bp}(T, S, \Delta), \quad T \leq S,$$

is

$$\mathbb{P}_t^{f,bp}(T, S, \Delta) = \mathbb{E}_t^{Q_{F^T}}(V_t^{f,bp}(T, S, \Delta))$$

As in the government bond case, we face the complication that the maturity of the variance swap is T, while the forward Libor is a martingale under the S-forward probability as Equation 20.90 indicates. Moreover, we are dealing with a notion of basis point variance. Mele and Obayashi (2014a,b, Chapter 4) show that the expression for $\mathbb{P}_f^{bp}(t,T,S,\Delta)$ includes a model-dependent term

$$\mathbb{P}_t^{f,bp}(T,S,\Delta) = 2\left(\hat{f}_t^2(S,S+\Delta)\left(\mathbb{E}_t^{Q_{FT}}\left(e^{\tilde{\ell}_f(t,T,S)}\right)-1\right)-\mathbb{E}_t^{Q_{FT}}\left(\tilde{\ell}^{bp}(t,T,S)\right)\right)$$
$$+\frac{2}{P_t(T)}\left(\int_0^{\hat{f}_t(S,S+\Delta)} Put_t^f\left(K_f,T,S,\Delta\right)dK_f+\int_{\hat{f}_t(S,S+\Delta)}^{\infty} Call_t^f\left(K_f,T,S,\Delta\right)dK_f\right) \tag{20.91}$$

where $\tilde{\ell}_f(t,T,S)$ and $\tilde{\ell}^{bp}(t,T,S)$ are given in the appendix (see Eqs 20.113 and 20.114) and

$$Put_t^f(K_f,T,\Delta) = \frac{Call_t^z(100(1-K_f),T,S,\Delta)}{100}$$
$$Call_t^f(K_f,T,\Delta) = \frac{Put_t^z(100(1-K_f),T,S,\Delta)}{100}$$

with $Put_t^z(K_z,T,S,\Delta)$ and $Call_t^z(K_z,T,S,\Delta)$ denoting the prices of out-of-the-money puts and calls written on the forward Libor price $Z_t(S,S+\Delta)$ with strike price K_z and maturity T, and $Put_t^f(K_f,T,S,\Delta)$ and $Call_t^f(K_f,T,S,\Delta)$ are hypothetical out-of-the-money puts European-style options on the forward Libor rate (see Eqs 20.115 and 20.116 in the appendix).

Accordingly, an index of basis point time deposit rate volatility is

$$TD-VI_f^{bp}(t,T,S,\Delta) \equiv \sqrt{\frac{\mathbb{P}_t^{f,bp}(T,S,\Delta)}{T-t}} \tag{20.92}$$

where $\mathbb{P}_t^{f,bp}(T,S,\Delta)$ is as in Equation 20.91.

We estimate the fair value of the time deposit variance swap, $\mathbb{P}_t^{f,bp}(T,S,\Delta)$, and the index in Equation 20.92, based on option quotes on hypothetical market conditions as of June 26, 2012. Options relate to 3-month Eurodollars and are taken to be 3-month European options on forwards so that there is no maturity mismatch between options and forwards, that is, $\tilde{\ell}_f(t,T,S)=\tilde{\ell}^{bp}(t,T,S)=0$ in Equation 20.91. Table 20.3 reports strike K, implied volatilities, and option premiums, defined similarly as in Tables 20.1 and 20.2. The only difference is the way the implied volatilities in basis points are defined; they rely on a model in which the forward Libor price follows a Gaussian process with constant volatility as explained in the appendix.

Figure 20.10 plots the option premiums used for estimating $\mathbb{P}_t^{f,bp}(T,S,\Delta)$ (italicized in Table 20.3).[13] The weighted average of these premiums (with weights ΔK_i determined as in Tables 20.1 and 20.2 and equal to $0.1250 * 10^{-2}$) is $1.9210 \cdot 10^{-6}$, yielding an estimate of the basis point variance swap fair value equal to $\frac{1}{0.9996}\times 2\times 1.9210\cdot 10^{-6}$, where 0.9996 is the price as of June 26, 2012, of a hypothetical 3-month zero. An estimate of Equation 20.92 is

$$\widehat{TD\text{-}VI}^{bp} = 100\times 100\times\sqrt{\frac{1}{0.9996}\frac{2}{(3/12)}\times 1.9210\cdot 10^{-6}} = 39.2101 \tag{20.93}$$

In comparison, the ATM basis point volatility in Table 20.3 is 27.22 basis points, with the average basis point skew equal to 48.24 basis points.

20.5.3.3 Swaps In Section 20.5.2, we derived an expression for the fair value of an interest rate variance swap expressed in percentage terms (see Eq. 20.71) and argued that a model-free solution could be implemented by tilting the payoff of the contract by the annuity factor, $PVBP_t(T_1,\ldots,T_n)$. Now, how is a *basis point* variance contract valued? Consider a variance swap with the following payoff:

$$\pi_{irs}^{bp}(T,n) \equiv PVBP_T(T_1,\ldots,T_n)\times(V_n^{bp}(t,T)-\mathbb{P}_t^{bp}(T,n))$$

where $V_n^{bp}(t,T)$ is as in Equation 20.62 and $\mathbb{P}_t^{bp}(T,n)$ is its fair value. By an application of Proposition I, it equals

$$\mathbb{P}_t^{bp}(T,n) = \frac{2}{PVBP_t(T_1,\ldots,T_n)}\left(\int_0^{R_t} Swpn_t^r(K)\,dK+\int_{R_t}^{\infty} Swpn_t^p(K)\,dK\right)$$

[13]Note that the out-of-the-money puts (calls) on forward Libor prices in Table 20.3 correspond to the out-of-the-money calls (puts) on forward Libor rates in Equation 20.91.

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TABLE 20.3 Determination of Usable Options to Calculate the Fair Value of a Hypothetical Basis Point Time Deposit Variance Swap

	Implied volatility		Premiums	
Strike price (%)	Black's implied vol (%)	Basis point implied vol	Put option	Call option
98.750	0.91	89.75	*9.0197* $\cdot 10^{-5}$	7.5871 $\cdot 10^{-3}$
98.875	0.79	78.18	*9.2048* $\cdot 10^{-5}$	6.3395 $\cdot 10^{-5}$
99.000	0.67	66.27	*9.6557* $\cdot 10^{-5}$	5.0945 $\cdot 10^{-3}$
99.125	0.59	58.79	*1.3910* $\cdot 10^{-4}$	3.8876 $\cdot 10^{-3}$
99.250	0.49	48.66	*1.9261* $\cdot 10^{-4}$	2.6916 $\cdot 10^{-3}$
99.375	0.38	38.20	*2.8781* $\cdot 10^{-4}$	1.5373 $\cdot 10^{-3}$
99.500 (ATM)	0.27	27.22	*5.3566* $\cdot 10^{-4}$	*5.3566* $\cdot 10^{-4}$
99.625	0.24	23.66	1.3405 $\cdot 10^{-3}$	*9.1043* $\cdot 10^{-5}$
99.750	0.24	24.25	2.5069 $\cdot 10^{-3}$	*7.9202* $\cdot 10^{-6}$
99.875	0.31	27.50	3.7628 $\cdot 10^{-3}$	*3.8729* $\cdot 10^{-6}$

Source: Bloomberg.

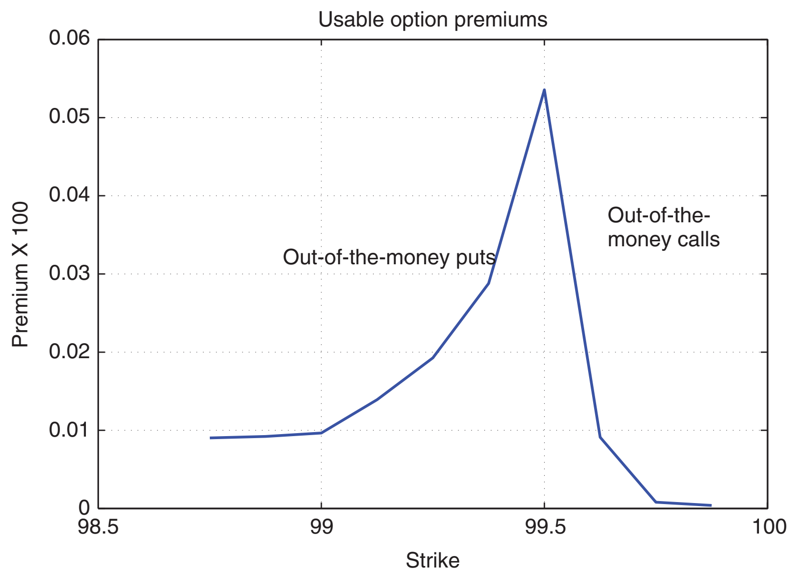

Figure 20.10 Option premiums used to determine the fair value of a time deposit basis point variance swap, $\mathbb{P}_t^{f,bp}(T,S,\Delta)$, in Equation 20.91, with $\tilde{\ell}_f(t,T,S) = \tilde{\ell}^{bp}(t,T,S) = 0$. The basis point time deposit volatility index based on these premiums is estimated to be $\widehat{TD\text{-}VI}^{\mathrm{bp}} = 39.2101$ (see Eq. 20.93). Source: Bloomberg.

Then, an index of IRS basis point volatility can be formulated as the square root of the annualized value of this strike:

$$IRS - VI_n^{bp}(t,T) \equiv \sqrt{\frac{1}{T-t}\mathbb{P}_t^{bp}(T,n)} \tag{20.94}$$

Using the same hypothetical market conditions on February 12, 2010, underlying Table 20.1 (Panel B, column six), we estimate the fair value $\mathbb{P}_t^{bp}(T,n)$ of the basis point variance swap on a 1-month into 5-year IRS to be $2 \times 4.1567 \cdot 10^{-6}$ and calculate an estimate of IRS-$\mathrm{VI}_n^{bp}(t,T)$ as

$$\widehat{IRS\text{-}VI}_n^{bp}(t,T) \equiv 100 \times 100 \times \sqrt{\frac{2}{12^{-1}} \times 4.1567 \cdot 10^{-6}} = 99.8803 \tag{20.95}$$

Note that this estimate is rescaled by 100^2 to make the index gauge be consistent with market practice to express basis point implied volatility as the product of rates times percentage volatility, where both rates and percentage volatilities are rescaled

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by 100.[14] In comparison, the ATM implied basis point volatility in Equation 20.72 is $\sigma_{iv}^{bp}(K, R_{n,t}, T_0 - t)|_{K=R_{n,t}} = 97.9202$ as determined in Section 20.5.2.5.

As anticipated, Mele and Obayashi (2012) first proposed this index of BP IRS volatility on the basis of the variance contract design in this section. Trolle and Schwartz (2013) arrive at the same expression in Equation 20.94 while solving for the conditional second moment of the swap rate distribution, which Mele, Obayashi, and Shalen (2015) label as "point-to-point" volatility, as opposed to the more familiar basis point "incremental" volatility $V_n^{bp}(t,T)$ in Equation 20.62,

$$V_{n,p-t-p}^{bp}(t,T) \equiv \sqrt{\frac{(R_T(T_1, \ldots, T_n) - R_t(T_1, \ldots, T_n))^2}{T-t}} \tag{20.96}$$

In contrast to $V_n^{bp}(t,T)$, point-to-point volatility tracks dispersion in changes of the forward swap rate over two distinct points in time. In other words, $V_{n,\text{p-t-p}}^{bp}(t,T)$ is a measure of distance between current and future forward swap rates and does not take into account movements that occur over the entire trading period, that is, $V_{n,\text{p-t-p}}^{bp}(t,T)$ could be small despite a period of market turbulence between t and T.

Incremental and point-to-point basis point volatilities can take on vastly different realizations, but they have the same expectation under the market probability. For simplicity, let $R_{n,\tau} \equiv R_\tau(T_1, \ldots, T_n)$ and $\sigma_{n,\tau} \equiv \sigma_\tau(T_1, \ldots, T_n)$, then

$$\begin{aligned}
\mathbb{E}_t^{sw,n}(R_{n,T} - R_{n,t})^2 &= \mathbb{E}_t^{sw,n}\left[\int_t^T R_{n,\tau}\sigma_{n,\tau}dW_\tau^{sw,n}\right]^2 \\
&= \mathbb{E}_t^{sw,n}\left[\int_t^T R_{n,\tau}^2\|\sigma_{n,\tau}\|^2 d\tau\right] \\
&= \mathbb{E}_t^{sw,n}[V_n^{bp}(t,T)]
\end{aligned}$$

where the first equality follows by Equation 20.17, the second by the so-called isometry property of Itô's integrals (e.g., Øksendal, 1998; p. 26), and the third by the definition of $V_n^{bp}(t,T)$ in Equation 20.62.

The fair value of a variance swap on an incremental and on a point-to-point basis are the same. Mele, Obayashi, and Shalen (2015) rely on this property and calculate approximate confidence bands for the forward swap rate forecasts based on the CBOE SRVX, which are reproduced in Figure 20.11.[15] Naturally, the longer the forecast horizon, the higher the dispersion of the forward swap rate forecasts.

20.5.4 Current Forward-Looking Indexes of IRV

There are many practical considerations when implementing the volatility indexes surveyed in this section, including the microstructure of each asset class considered, across both time and geographies. The indexes are made more robust by greater price discovery, breadth of option strikes, and depth of liquidity in the options driving the index. These characteristics also increase the viability of derivatives to be written on the indexes, such as volatility futures and options. Successful IRV index derivatives may in turn create a positive feedback loop by increasing the liquidity of the options used to calculate the indexes.

CBOE currently maintains two indexes of IRV that have the potential to satisfy the previous properties: (i) the SRVX index of IRS volatility, launched in June 2012, which relies on 1-year/10-year USD swaptions (based on Eq. 20.94); and (ii) the VXTYN, launched in May 2013, which is based on near-month CBOT options on 10-year T-note futures (based on Eq. 20.88). This section reviews some of the empirical properties of these indexes and explores their potential as possible benchmarks of volatility products designed to provide access to standardized IRV.

20.5.4.1 Market Disconnects How do current IRV indexes compare with the equity VIX? Figures 20.12 and 20.13 depict the behavior of the CBOE VIX vis-à-vis the CBOE SRVX and the CBOE/CBOT VXTYN. The two samples cover daily data from February 2007 (SRVX) and January 2008 (VXTYN) through February 2014, for a total of 1818 (SRVX) and 1525 (VXTYN) observations.

While equity volatility and IRV seem to respond similarly to pronounced global events, the timing and nature of their responses do differ. For example, the correlation between the daily changes of the VXTYN and VIX is less than 30%, yet the daily changes

[14]That is, consider the basis point variance in Equation 20.62, $V_n^{bp}(t,T)$, and suppose that the forward swap rate, $R_\tau(T_1, \ldots, T_n)$, and its instantaneous vol, $\sqrt{\|\sigma_\tau(T_1, \ldots, T_n)\|^2}$, are both expressed in decimals, as in Table 20.1. Then, reexpressing $R_\tau(T_1, \ldots, T_n)$ and $\sqrt{\|\sigma_\tau(T_1, \ldots, T_n)\|^2}$ in percentage terms implies rescaling $V_n^{bp}(t,T)$ by $100^2 \times 100^2$, leading to the scaling factor 100×100 in Equation 20.95.

[15]The bands in Figure 20.11 are only approximate because the forward swap rate is a martingale under the annuity probability but not necessarily Gaussian.

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Figure 20.11 Panel A: Time series behavior of the 3-month into 10-year forward swap rate along with approximate confidence bands constructed by adding and subtracting 1.96 times the CBOE SRVX with maturity 3 months and tenor equal to 10 years. Panel B: Times series behavior of forward swap rates and approximate confidence bands regarding 1 year into 10 years forward rates. Source: Chicago Board Options Exchange and Bloomberg.

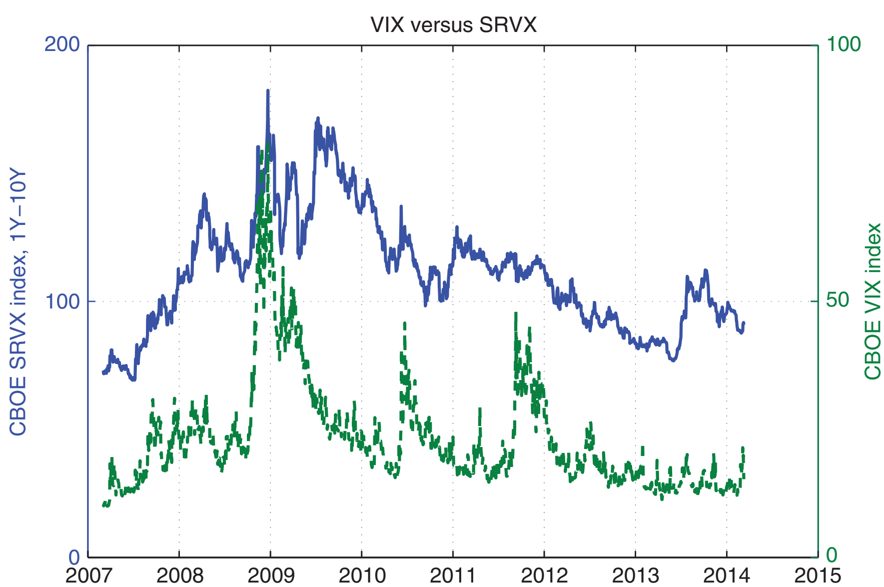

Figure 20.12 CBOE indexes of equity (VIX) and interest rate swap (SRVX) volatility. Source: Chicago Board Options Exchange.

of VXTYN and SRVX are correlated at more than 50%. There are at least five historical instances that vividly illustrate episodes of clear diversion between fixed-income and equity market volatilities.

First, right at the beginning of the crisis over summer 2007, the SRVX begins to trend upward for over 1 year, while over the same period, the VIX fluctuates without any apparent trend. Only in 2008 does the VIX experience a pronounced increase.

A second period of divergence occurs in 2009. Consider spring 2009 when positive economic news in the United States are accompanied by a bond sell-off and a stock market rally, with the VIX decreasing and both the SRVX and VXTYN increasing.

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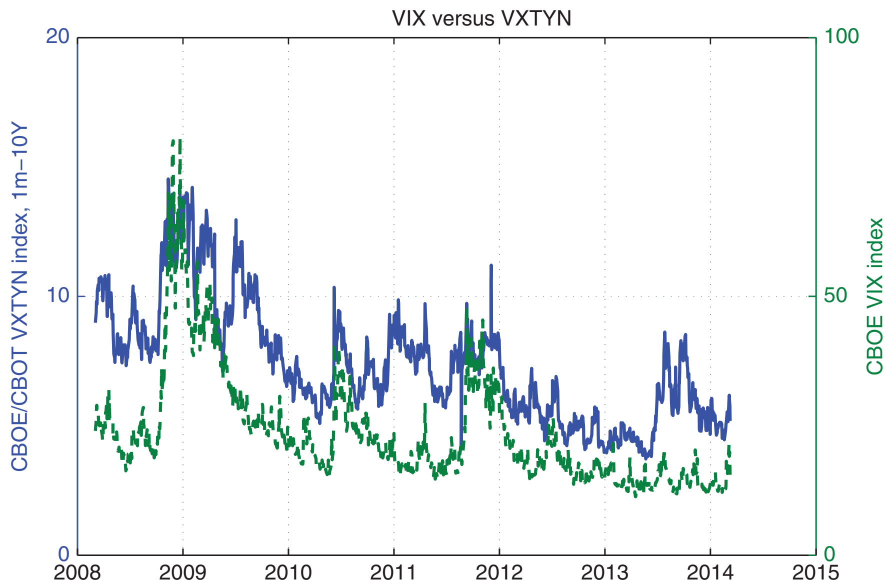

Figure 20.13 CBOE indexes of equity (VIX) and Treasury bond (VXTYN) volatility. Source: Chicago Board Options Exchange.

TABLE 20.4 Behavior of VXTYN, VIX, and S&P 500 over Days with Increasing Yields

10Y Yield up by:	No Obs.	VXTYN average Δ change (std error)	VIX average Δ change (std error)	S&P500 average change (std error)
> 0	685	5.06 bps (1.70 bps)	−63.49 bps (7.23 bps)	0.51% (0.05)
> 5 bps	297	14.17 bps (2.88 bps)	−90.07 bps (13.32 bps)	0.79% (0.10)
> 10 bps	112	18.41 bps (6.09 bps)	−113.48 bps (28.77 bps)	1.06% (0.23)
> 12 bps	73	29.01 bps (6.17 bps)	−101.95 bps (6.09 bps)	0.98% (0.27)
> 15 bps	25	34.48 bps (12.40 bps)	−114.72 bps (32.70 bps)	0.68% (0.65)

Source: CBOE and Bloomberg.

A rate sell-off is indeed good news for the aggregate stock market but may still increase fixed-income portfolio risk and "rate fear," a theme we shall return to later.

Third, the sovereign debt problems over 2011–2012 lead IRV to be suppressed through the use of monetary policy, while the equity VIX starts a downward trend only after 2012.

Fourth, both SRVX and VXTYN enter another high regime during the "FED Watch" episode over May–September 2013 – when QE tapering comes into focus. In this period, the VXTYN doubles in a matter of a few weeks, with the fluctuations of the equity VIX over these months, which seem to be independent of this episode.

Fifth, Yellen's "regime" over the last months of our sample seems to have stabilized IRV with virtually no apparent effects on the dynamics of the VIX.

20.5.4.2 The Index Tail Behavior How does expected IRV behave in times of extreme rate movements? Tables 20.4 and 20.5 document a few summary statistics regarding the VXTYN, with similar results reported by Mele, Obayashi, and Shalen (2015) regarding the SRVX.

Table 20.4 reveals a clear pattern – the VXTYN significantly increases over rate sell-offs, while the VIX decreases. An interpretation is that this negative dependence is a "reverse flight-to-quality" effect, in which investors flock back to the equity market as in the 2009 episode described previously. This interpretation is corroborated by the last column of Table 20.4, which shows that during days with increasing yields, the S&P 500 index increases, albeit with a nonmonotonic pattern. In particular, on days with moderate increases in the 10-year yield, the S&P increases, possibly driven by a reverse flight-to-quality effect; on days in which the 10-year yield moves up more drastically, a possible fear effect seems to curb equity market rallies.

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TABLE 20.5 Behavior of VXTYN, VIX, and S&P 500 over Days with Falling Yields

10Y yield down by:	No obs.	VXTYN average Δ change (std error)	VIX average Δ change (std error)	S&P500 average change (std error)
<0	723	−4.61 bps (1.45 bps)	63.39 bps (8.75 bps)	−0.48% (0.06)
>-2 bps	605	−4.52 bps (1.66 bps)	73.67 bps (10.21 bps)	−0.58% (0.07)
>-3 bps	468	−3.58 bps (2.04 bps)	91.93 bps (12.79 bps)	−0.76% (0.08)
>-5 bps	312	−0.19 bps (2.85 bps)	119.17 bps (17.38 bps)	−0.92% (0.11)
>-10 bps	100	13.71 bps (7.39 bps)	212.69 bps (42.41 bps)	−1.50% (0.29)
>-12 bps	55	23.63 bps (11.78 bps)	212.90 bps (64.60 bps)	−1.81% (0.40)
>-15 bps	28	44.03 bps (19.05 bps)	300.64 bps (113.99 bps)	−2.44% (0.69)

Source: CBOE and Bloomberg.

Table 20.5 reports the statistics regarding index behavior when the 10-year yield goes down. The VXTYN response is negative on days when yields are down by 5 bps, although this response is economically not important and not always statistically significant. When yields experience large falls, both the VXTYN and the VIX increase, possibly driven by a common fear that a fall in long-term yields might reflect bad times to come, which policy makers may have to handle in future rate cuts. This explanation seems to be corroborated by the fact that the S&P is indeed on average significantly negative during days on which the 10-year yield is down.

While forward-looking measures of volatility certainly bring valuable sources of information, they do not represent investable products. While a variety of investable products exist on the VIX, such as futures and options (see, e.g., Rhoads, 2011), only futures on VXTYN are available for trading since November 2014 at the time of writing. The next section provides an introduction to how to evaluate these products based on the variance swap pricing framework in the previous sections.

20.5.5 Products on IRV Indexes

The previous sections highlight that to price volatility in a model-free fashion, precise reference needs to be made to the appropriate market space – for example, the forward space for government bonds and time deposits and the annuity space for IRS.

We now illustrate how this explicit reference helps formulate models that jointly determine expected IRV and the price of derivatives thereon. For example, while pricing futures on an index of government bond volatility, we need to make sure that reference is made to risk-neutral expectations (needed to model future prices) of future expectations taken under the forward probability (needed to model government bond variance swaps). This added dimensionality defines a further difference of treatment between fixed-income and equity volatility – in the equity space, expected volatility and futures on it are both determined under the risk-neutral probability, Q, assuming as is standard that rates are constant.

Another key issue regards the very nature of IRV. A typical and pragmatic approach in the equity space is to model volatility in partial equilibrium, that is, separately from the stock price (see, e.g., Mencía and Sentana, 2013), by juxtaposing the dynamics of volatility to those of the asset returns as, for example, in the Heston (1993) model. This approach likely fails in the fixed-income space. Consider, for example, government bond volatility. Because the price of a zero-coupon bond converges to par, volatility cannot be taken to be exogenous, even ruling out stochastic volatility. Rather, it is determined endogenously, arising through no-arbitrage restrictions on bond price dynamics.

To illustrate, consider the following model with stochastic volatility,[16] which is an extension of the Ho and Lee (1986) model reviewed in Section 20.4, in which the short-term rate r_τ displays random and mean-reverting basis point volatility v_τ,

$$\begin{cases} dr_\tau = \theta_\tau d\tau + v_\tau dW_{1\tau} \\ dv_\tau^2 = k\left(m - v_\tau^2\right) d\tau + \xi v_\tau dW_{2\tau} \end{cases} \tag{20.97}$$

where $W_{i\tau}$ are Brownian motions under the risk-neutral probability; θ_τ is the infinite-dimensional parameter utilized to perfectly fit the initial yield curve at $\tau = t$ without error; k and m are the speed of mean reversion and the risk-neutral expectation of basis

[16]This model is introduced by Mele and Obayashi (2014a, Chapter 4, Appendix C) and is a special case of Mele and Obayashi (2014c).

point variance, v_t^2; and ξ is a "volatility of variance" parameter. The model is similar to the random volatility models encountered in Section 20.4.1, except it allows for a perfect fit of the yield curve through θ_τ.

The ability to fit the initial yield curve is a desirable property, as it allows feeding the model with information concerning the entire *current* yield curve and not only the current short-term rate as explained in Section 20.4. A particularly interesting feature in our context would be that the entire *current* yield curve can feed expected future *developments* in IRV, an aspect we discuss in the following. Fitting the current yield curve is also important for market making purposes as explained in Section 20.4: in our context, it offers a consistent framework for the risk management of IRD books that contain both IRV derivatives as well as more traditional IRDs and fixed-income assets.

The model predicts that the price of a zero-coupon bond at time $\tau \geq t$ and expiring at time $\mathcal{T} \geq \tau$ when the state is (r_τ, v_τ) is

$$P_\tau(r_\tau, v_\tau^2, \mathcal{T}) \equiv e^{\int_\tau^{\mathcal{T}}(s-\mathcal{T})\theta_s ds + km\int_\tau^{\mathcal{T}} C_{\mathcal{T}}(s)ds - (\mathcal{T}-\tau)r_\tau + C_{\mathcal{T}}(\tau)v_\tau^2} \tag{20.98}$$

where $C_{\mathcal{T}}(\tau)$ is the solution to the following Riccati's equation:

$$\dot{C}_{\mathcal{T}}(\tau) = kC_{\mathcal{T}}(\tau) - \left(\frac{1}{2}(\mathcal{T}-\tau)^2 + \frac{1}{2}\xi^2 C_{\mathcal{T}}^2(\tau)\right), \quad C_{\mathcal{T}}(\mathcal{T}) = 0 \tag{20.99}$$

and the dot denotes differentiation with respect to τ. Moreover, set the parameter θ_τ to

$$\theta_\tau = \frac{\partial f_\$(t,\tau)}{\partial \tau} + km\int_t^{\mathcal{T}} \frac{\partial^2 C_\tau(u)}{\partial \tau^2} du + \frac{\partial^2 C_\tau(t)}{\partial \tau^2} v_t^2$$

where $f_\$(t,\tau)$ is the forward rate at time t for maturity τ. Then, it can be shown that the model matches the yield curve initially observed (i.e., at t) without error, in that

$$P_t(r_t, v_t^2, \mathcal{T}) = e^{-\int_t^{\mathcal{T}} f_\$(t,\tau)d\tau} \equiv P_t(\mathcal{T}), \quad \text{for all } \mathcal{T}$$

The motivation leading to this model is to make predictions regarding the future yield curve and the price of government bond volatility products while feeding the model with all the bond prices observed at time t, not only the short-term rate. Methods and rationale underlying the model are then the same as those underlying the no-arbitrage approach reviewed in Section 20.4.

The attractive feature of this approach in our context is its potential to generate "price feedbacks," that is, the fact that a given shape of the yield curve today might convey information about the volatility to be expected within a given horizon. Mele and Obayashi (2014c) show this feature of the model does indeed arise once we consider the pricing of *coupon-bearing bonds*, although it does not, once we consider the volatility of *zero-coupon bonds*. However, in this introductory chapter, we illustrate the main features of government bond volatility pricing by making reference to the simplest zero-coupon bond case.

Consider, then, and again for simplicity, a forward expiring at S on a Zero-coupon bond expiring at $\mathbb{T}$, which by Equation 20.5 and a change in probability is shown to satisfy

$$\begin{cases} \frac{dF_\tau(S,\mathbb{T})}{F_\tau(S,\mathbb{T})} = v_\tau\left(-(\mathbb{T}-S)\,dW_{1,\tau}^{F^S} + \xi(C_{\mathbb{T}}(\tau) - C_S(\tau))dW_{2,\tau}^{F^S}\right) \\ dv_\tau^2 = (km - (k - \xi^2 C_S(\tau))v_\tau^2)d\tau + \xi v_\tau dW_{2,\tau}^{F^S} \end{cases} \tag{20.100}$$

where $W_{1,\tau}^{F^S}$ and $W_{2,\tau}^{F^S}$ are two independent Brownian motions under the forward probability Q_{F^S} and $C_S(\tau)$ is the bond price exposure to the basis point variance of the short-term rate in Equation 20.98, solution to Equation 20.99 for $\mathcal{T} \equiv S$. Therefore, the instantaneous percentage variance of the forward is

$$\|v_\tau(S,\mathbb{T})\|^2 \equiv \phi_\tau(S,\mathbb{T}) \cdot v_\tau^2 \tag{20.101}$$

where $\phi_\tau(S,\mathbb{T})$ is a time-varying, albeit nonrandom parameter, equal to

$$\phi_\tau(S,\mathbb{T}) \equiv (\mathbb{T}-S)^2 + \xi^2(C_{\mathbb{T}}(\tau) - C_S(\tau))^2$$

By taking the time t conditional expectation under Q_{F^S} of $\|v_\tau(S,\mathbb{T})\|^2$ in Equation 20.101, then annualizing and finally integrating the result over the time interval $[t,T]$, we have the following expression for the percentage volatility index predicted by the model for GB-VI$(t,T,S,\mathbb{T})$ in Equation 20.88:

$$GB\text{-}VI^z(v_t^2; t, T, S, \mathbb{T}) \equiv \sqrt{\Phi_1(t,T,S,\mathbb{T}) + \Phi_2(t,T,S,\mathbb{T}) \cdot v_t^2} \tag{20.102}$$

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where

$$\Phi_j(t,T,S,\mathbb{T}) \equiv \frac{1}{T-t}\int_t^T \phi_\tau(S,\mathbb{T})\bar{\phi}_{j\tau}(t,T)d\tau, \quad j=1,2$$

$$\bar{\phi}_{1\tau}(t,T) \equiv km \cdot \int_t^\tau e^{-\int_u^\tau (k-\xi^2 C_T(x))dx}du, \quad \bar{\phi}_{2\tau}(t,T) \equiv e^{-\int_t^\tau (k-\xi^2 C_T(x))dx} \tag{20.103}$$

and the superscript z on GB-VI emphasizes it is a volatility index regarding zero-coupon bonds.

The square of the government bond volatility index predicted by the model is affine in the basis point variance of the short-term rate, v_t^2, reflecting mean reversion under the forward probability: it is higher than its realized counterpart when v_t^2 is sufficiently small, and vice versa.[17] Naturally, the intercept Φ_1 and the variance loading Φ_2 in Equation 20.102 inflate the basis point variance of the short-term rate through the exposure of the bond price to volatility arising through the time-varying parameter $\phi_\tau(S,\mathbb{T})$ in Equation 20.103, just as the instantaneous bond volatility is magnified against v_t^2 by this very same parameter in Equation 20.101.

This case does not give rise to price feedbacks: the entire yield curve at t is indeed informative about developments in the entire yield curve through the parameter θ_τ; yet it does not inform us about expected volatility because Φ_1 and Φ_2 are independent of the yield curve at t. The situation differs in the case of coupon-bearing bonds as already mentioned.

Given Equation 20.102 and the assumptions made on the dynamics of v_t^2 in Equation 20.97, we can solve for the price of products written on the index. Consider, for example, evaluating a future on the index level in Equation 20.102, for delivery at time $t+\Delta$,

$$\begin{aligned}\mathcal{F}_t(v_t^2;\Delta,T,S,\mathbb{T}) &\equiv \mathbb{E}_t(GB-VI^z(v_{t+\Delta}^2;t+\Delta,T+\Delta,S+\Delta,\mathbb{T}+\Delta))\\ &= \int_0^\infty \sqrt{\Phi_1(t,T,S,\mathbb{T})+\Phi_2(t,T,S,\mathbb{T})\cdot x}\cdot f_\Delta(x|v_t^2)dx\end{aligned} \tag{20.104}$$

where $f_\Delta(x|v_t^2)$ is the transition density of the basis point variance at time $t+\Delta$ in Equation 20.97. Because v_t^2 is a square-root process just as the short-term rate in Equation 20.22, this density is noncentral chi-square by results reviewed in Section 20.4.

The model predicts market cycles between contango and backwardation. The market is in contango when the current basis point variance v_t^2 is low, such that due to mean reversion, the index level in Equation 20.102 is lower than its risk-neutral expectation of its future value, $\mathcal{F}_t(v_t^2;\Delta,T,S,\mathbb{T})$ in Equation 20.104. It then switches to backwardation as soon as the realized basis point variance increases to a sufficiently high level to make futures valued less than the current index levels.

The model could be equally used to evaluate options on the index levels. For example, the price of a call option expiring at time $t+\delta$ and struck at K is given by

$$\begin{aligned}&\mathcal{C}_t(v_t^2;\delta,T,S,\mathbb{T})\\ &\equiv \frac{1}{P_t(t+\delta)}\mathbb{E}_t^{Q_{F^{t+\delta}}}(\text{GB-VI}^z(v_{t+\delta}^2;t+\delta,T+\delta,S+\delta,\mathbb{T}+\delta)-K)^+\\ &= \frac{1}{P_t(t+\delta)}\int_0^\infty (\sqrt{\Phi_1(t,T,S,\mathbb{T})+\Phi_2(t,T,S,\mathbb{T})\cdot x}-K)^+\cdot f_\delta^F(x|v_t^2)dx\end{aligned} \tag{20.105}$$

where $f_\delta^F(x|v_t^2)$ denotes the transition density of basis point variance at time $t+\delta$ given v_t^2 under the forward probability $Q_{F^{t+\delta}}$, which is the same as that implied by the variance equation in 20.100 with $S\equiv t+\delta$.

Mele and Obayashi (2014,c) show how to generalize Equation 20.105 to the coupon-bearing bond case and note that the current yield curve does indeed affect option evaluation in this context.

20.6 CONCLUSIONS

IRV is a topic that is receiving a great deal of attention at the time of writing. Institutional and retail investors alike have become increasingly aware of IRV's potentially significant effects on their portfolio performance, especially with respect to their fixed-income positions. While volatility benchmarking and trading is now well established in equity markets, analogous

[17]Formally, the expected realized variance under Q_{F^T} is $\mathbb{E}_t^{Q_{F^T}}(\|v_\tau(S,\mathbb{T})\|^2)=\phi_\tau(S,\mathbb{T})(\bar{\phi}_{1\tau}(t,T)+\bar{\phi}_{2\tau}(t,T)v_t^2)$, and it can be higher or lower than the realized variance, $\|v_\tau(S,\mathbb{T})\|^2$ as described for ξ sufficiently small. Indeed, it is possible to show that for a fixed T, $\infty > C_T(x)\geq 0$ for all $x\leq T$ and $\xi\geq 0$, such that $\exists\xi^o : k-\xi^2C_T(x)>0$ in Equation 20.100, for all $\xi<\xi^o$ and $x\leq T$.

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initiatives are still in their infancy in the fixed-income space. This chapter surveys the main issues arising while dealing with this new topic and outlines solutions that have already been adopted. We describe both the institutional context and the pricing framework that is needed. This framework points to a new role IRDs might play while designing new tools to manage interest rate risks.

20.7 APPENDIX

Derivation of Equation 20.6. Let $\mathbb{I}_{\cdot}$ be the indicator function. We have

$$
\begin{aligned}
& C_t^b(T,S) \\
&= \mathbb{E}_t\left[e^{-\int_t^T r_\tau d\tau} P_T(S) \cdot \mathbb{I}_{P_T(S)\geq K}\right] - K \cdot \mathbb{E}_t\left[e^{-\int_t^T r_\tau d\tau} \cdot \mathbb{I}_{P_T(S)\geq K}\right] \\
&= \mathbb{E}_t\left[e^{-\int_t^S r_\tau d\tau} \cdot \mathbb{I}_{P_T(S)\geq K}\right] - K \cdot \mathbb{E}_t\left[e^{-\int_t^T r_\tau d\tau} \cdot \mathbb{I}_{P_T(S)\geq K}\right] \\
&= P_t(S) \cdot \mathbb{E}_t^{Q_{FS}}(\mathbb{I}_{P_T(S)\geq K}) - K P_t(T) \cdot \mathbb{E}_t^{Q_{FT}}(\mathbb{I}_{P_T(S)\geq K}) \\
&= P_t(S) \cdot \mathbb{Q}_{FS}(P_T(S) \geq K) - K P_t(T) \cdot \mathbb{Q}_{FT}(P_T(S) \geq K)
\end{aligned}
$$

where the second equality follows by the law of iterated expectations,

$$
\begin{aligned}
\mathbb{E}_t\left[e^{-\int_t^T r_\tau d\tau} P_T(S) \cdot \mathbb{I}_{P_T(S)\geq K}\right] &= \mathbb{E}_t\left[\mathbb{E}_T\left(e^{-\int_t^T r_\tau d\tau} P_T(S) \cdot \mathbb{I}_{P_T(S)\geq K}\right)\right] \\
&= \mathbb{E}_t\left[e^{-\int_t^T r_\tau d\tau} \cdot \mathbb{I}_{P_T(S)\geq K} \mathbb{E}_T\left(e^{-\int_T^S r_\tau d\tau}\right)\right] \\
&= \mathbb{E}_t\left[\mathbb{E}_T\left(e^{-\int_t^S r_\tau d\tau} \cdot \mathbb{I}_{P_T(S)\geq K}\right)\right] \\
&= \mathbb{E}_t\left[e^{-\int_t^S r_\tau d\tau} \cdot \mathbb{I}_{P_T(S)\geq K}\right]
\end{aligned}
$$

and the third equality by a change in probability to the S-forward (the first term) and the T-forward (the second term). Q.E.D.

Bond prices in Vasicek (1977) and Cox, Ingersoll, and Ross (1985). The bond price predicted by Vasicek and Cox, Ingersoll, and Ross model are as in Equation 20.23. For the Vasicek model, the two functions $a(\cdot)$ and $b(\cdot)$ are given by

$$
a(T-t) = \left(\frac{1-e^{-\kappa(T-t)}}{\kappa} - (T-t)\right)\left(\mu - \frac{1}{2}\left(\frac{\sigma}{\kappa}\right)^2\right) - \frac{\sigma^2}{4\kappa^3}(1-e^{-\kappa(T-t)})^2, \tag{20.106}
$$

$$
b(T-t) = \frac{1-e^{-\kappa(T-t)}}{\kappa} \tag{20.107}
$$

and for Cox, Ingersoll, and Ross,

$$
a(T-t) = \frac{2\hat{\kappa}\hat{r}}{\hat{\sigma}^2} \ln\left(\frac{2\gamma e^{\frac{1}{2}(\hat{\kappa}+\gamma)(T-t)}}{(\hat{\kappa}+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}\right), \tag{20.108}
$$

$$
b(T-t) = \frac{2(e^{\gamma(T-t)}-1)}{(\hat{\kappa}+\gamma)(e^{\gamma(T-t)}-1)+2\gamma}, \quad \gamma = \sqrt{\hat{\kappa}^2 + 2\hat{\sigma}^2}. \tag{20.109}
$$

Derivation of Equation 20.41. The price of a zero-coupon bond satisfies $P_t(T) = e^{Y_{t,T}}$, where $Y_{t,T} \equiv -\int_t^T \varphi_t(u) du$, such that by Itô's lemma,

$$
\frac{dP_\tau(T)}{P_\tau(T)} = dY_{\tau,T} + \frac{1}{2} dY_{\tau,T}^2, \tag{20.110}
$$

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where

$$\begin{aligned} dY_{\tau,T} &= \varphi_\tau(\tau)d\tau - \int_\tau^T d\varphi_\tau(u)du \\ &= r_\tau d\tau - \int_\tau^T (\alpha_\tau(u)d\tau + \sigma_\tau^f(u)d\tilde{W}_\tau)du \\ &= (r_\tau - \alpha_\tau^I(T))d\tau - \sigma_\tau^{If}(T)d\tilde{W}_\tau \end{aligned}$$

where we have used the definition of the short-term rate, $r_\tau \equiv \varphi_\tau(\tau)$, and Equation 20.40 and defined

$$\alpha_\tau^I(T) \equiv \int_\tau^T \alpha_\tau(u)du, \quad \sigma_\tau^{If}(T) \equiv \int_\tau^T \sigma_\tau^f(u)du$$

Equation 20.110 implies that in the absence of arbitrage,

$$\begin{aligned} \mathbb{E}_\tau\left(\frac{dP_\tau(T)}{P_\tau(T)}\right) &= r_\tau d\tau \\ &= \mathbb{E}_\tau(dY_{\tau,T}) + \frac{1}{2}\mathbb{E}_\tau\left(dY_{\tau,T}^2\right) \\ &= (r_\tau - \alpha_\tau^I(T))d\tau + \frac{1}{2}\|\sigma_\tau^{If}(T)\|^2 d\tau \end{aligned}$$

leaving

$$\alpha_\tau^I(T) = \frac{1}{2}\|\sigma_\tau^{If}(T)\|^2 \tag{20.111}$$

Differentiating Equation 20.111 with respect to T yields Equation 20.41. Q.E.D.

Derivation of Equation 20.67. We rely on standard arguments (see, e.g., Bakshi and Madan, 2000; Carr and Madan, 2001). Consider a twice-differentiable function f. By a Taylor's expansion with remainder, we have that

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \int_{x_0}^x (x-k)f''(k)dk \\ &= f(x_0) + f'(x_0)(x - x_0) + \int_0^{x_0} f''(k)(k-x)^+ dk \\ &+ \int_{x_0}^\infty f''(k)(x-k)^+ dk \end{aligned} \tag{20.112}$$

Equation 20.67 follows by taking $f \equiv \ln x$, $x \equiv R_T$, and $x_0 \equiv R_t$ in Equation 20.112. Q.E.D.

Derivation of Equation 20.82. This follows by Equation 20.112 after setting $f \equiv x^2$, $x \equiv X_T$, and $x_0 \equiv X_t$. Q.E.D.

The functions $\tilde{\ell}_f$ (t, T, S) and $\tilde{\ell}^{bp}$ (t, T, S) in Equation 20.91. The function summarizing the maturity mismatch $\tilde{\ell}_f(t, T, S)$ is the counterpart to that regarding the government volatility case in Equation 20.87, $\tilde{\ell}(t, T, S, \mathbb{T})$, and equals

$$\tilde{\ell}_f(t, T, S) \equiv -\int_t^T v_\tau^f(S, \Delta)(v_\tau^f(T, \Delta) - v_\tau^f(S, \Delta))d\tau \tag{20.113}$$

Regarding $\tilde{\ell}^{bp}(t, T, S)$, we have

$$\tilde{\ell}^{bp}(t, T, S) \equiv -\int_t^T f_\tau^2(S, S+\Delta)v_\tau^f(S, \Delta)(v_\tau^f(T, \Delta) - v_\tau^f(S, \Delta))d\tau \tag{20.114}$$

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Gaussian model with constant volatility and basis point implied volatility. Consider the price of a European call option maturing at time T on a forward Libor price maturing at T,

$$\begin{aligned} Call_t^z(K,T,\Delta) &= \mathbb{E}_t\left[e^{-\int_t^T r_\tau d\tau}\left(Z_T(T,T+\Delta)-K\right)^+\right] \\ &= P_t(T)\mathbb{E}_t^{Q_{FT}}(Z_T(T,T+\Delta)-K)^+ \end{aligned} \tag{20.115}$$

where the second equality follows by the usual change of probability. The price of a put option follows by the put–call parity,

$$Put_t^z(K,T,\Delta) = Call_t^z(K,T,\Delta) + P_t(T)(K - Z_t(T,T+\Delta)) \tag{20.116}$$

A standard benchmark to evaluate options relies on the assumption that the forward price in 20.8 is lognormal, with instantaneous volatility equal to a constant σ,

$$\frac{dZ_\tau(T,T+\Delta)}{Z_\tau(T,T+\Delta)} = \sigma dW_\tau^{F^T}, \quad \tau \in (t,T) \tag{20.117}$$

which leads to evaluate $Call_t^z(K,T,\Delta)$ through Black's formula,

$$\begin{aligned} \frac{Call_t^z(K,T,\Delta)}{P_t(T)} &= \mathbb{E}_t^{Q_{FT}}(Z_T(T,T+\Delta)-K)^+ \\ &= Bl76(Z_t(T,T+\Delta);T-t,K,\sigma) \end{aligned} \tag{20.118}$$

where

$$Bl76(Z_t;T-t,K,\sigma) \equiv Z_t\Phi(d_t) - K\Phi(d_t-\sigma), \quad d_t = \frac{\ln\frac{Z_t}{K}+\frac{1}{2}\sigma^2}{\sigma}$$

The Black's skew of implied volatilities in Table 20.3 is the mapping $K \mapsto \sigma(Z_t,K,T-t)$ where $\sigma(Z_t,K,T-t)$ denotes the value of σ such that Equation 20.118 holds when its left-hand side is replaced with market data.

An alternative model to 20.117 is one in which the instantaneous volatility $v_\tau^z(T,\Delta)$ in Equation 20.8 is such that the forward Libor price follows a Gaussian process with constant volatility σ_N,

$$dZ_\tau(T,T+\Delta) = \sigma_N dW_\tau^{F^T} \tag{20.119}$$

Assuming Equation 20.119 holds true, the expectation in Equation 20.115 is

$$\mathbb{E}_t^{Q_{FT}}(Z_T(T,T+\Delta)-K)^+ = (Z_t(T,T+\Delta)-K)\Phi(\delta_t) + \frac{\sigma_N\sqrt{T-t}}{\sqrt{2\pi}}e^{-\frac{1}{2}\delta_t^2} \tag{20.120}$$

where

$$\delta_t \equiv \frac{Z_t(T,T+\Delta)-K}{\sigma_N\sqrt{T-t}}$$

The basis point implied volatility skew in Table 20.3 is defined as the mapping $K \mapsto \sigma^{bp}(Z_t,K,T-t)$, where $\sigma^{bp}(Z_t,K,T-t)$ is the value of σ_N such that the model-based option price implied by Equation 20.120 equals the market price.

Details regarding affine models. We collect a few well-known results regarding affine models. We derive the two functions $a(\cdot)$ and $b(\cdot)$ in Equation 20.27 in the Longstaff and Schwartz (1992) model and, later, characterize them as solutions to ordinary differential equations in a more general context (see Eqs 20.128 and 20.129).

The basic assumption underlying the Longstaff and Schwartz model is that the two state processes $y_{1\tau}$ and $y_{2\tau}$ are uncorrelated and that under the risk-neutral probability, they are solutions to

$$dy_{i\tau} = \hat{\kappa}_i(\bar{y}_i - y_{i\tau})d\tau + \sqrt{y_{i\tau}}d\tilde{W}_{i\tau} \tag{20.121}$$

with obvious notation.

The bond price predicted by the model is

$$P_t(y_{1t}, y_{2t}; T) = \mathbb{E}_t\left(e^{-\int_t^T r_\tau d\tau}\right) = \mathbb{E}_t\left(e^{-\int_t^T \beta_1 y_{1\tau} d\tau}\right)\mathbb{E}_t\left(e^{-\int_t^T \beta_2 y_{2\tau} d\tau}\right)$$
$$= F_t(y_{1t}; T)F_t(y_{2t}; T) \tag{20.122}$$

where the second equality follows by Equation 20.26 and the assumption that $y_{1\tau}$ and $y_{2\tau}$ are uncorrelated, the two processes $y_{i\tau} \equiv \beta_i y_{i\tau}$, and are therefore solutions to the following square-root processes:

$$dy_{i\tau} = \hat{\kappa}_i(\hat{y}_i - y_{i\tau})d\tau + \hat{\sigma}_i\sqrt{y_{i\tau}}d\tilde{W}_{i\tau}, \qquad \hat{y}_i \equiv \beta_i\bar{y}_i, \quad \hat{\sigma}_i \equiv \sqrt{\beta_i}$$

such that by Equation 20.22, the two functions $F_t(\cdot\,; T)$ are the same pricing functions as in the Cox, Ingersoll, and Ross (1985) model,

$$F_t(y_{it}; T) = e^{a(T-t)-b(T-t)y_{it}} \tag{20.123}$$

with the two functions $a(\cdot)$ and $b(\cdot)$ being defined by Equations 20.108) and 20.109 and replacing $\hat{r} \equiv \hat{y}_i$, $\hat{\kappa} \equiv \hat{\kappa}_i$, and $\hat{\sigma} \equiv \hat{\sigma}_i$.

Finally, note that by Equations 20.26 and 20.121, the instantaneous variance of the short-term rate is

$$v_\tau^2 d\tau \equiv var(dr_\tau) = (\beta_1^2 y_{1\tau} + \beta_2^2 y_{2\tau})d\tau \tag{20.124}$$

Equations 20.26 and 20.124 can be used to express $y_{i\tau}$ in terms of the short-term rate r_τ and its instantaneous variance, v_τ^2, namely,

$$y_{1\tau} = \frac{\beta_2 r_\tau - v_\tau^2}{\beta_1(\beta_2 - \beta_1)}, \quad y_{2\tau} = \frac{v_\tau^2 - \beta_1 r_\tau}{\beta_2(\beta_2 - \beta_1)} \tag{20.125}$$

By plugging 20.125 into 20.123 and, then, 20.122 leaves Equation 20.27 for suitable functions $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$ and by abusing notation.

The model is a special case of the affine model in 20.29 introduced by Duffie and Kan (1996), yielding the pricing formula 20.30, where the two functions $a^o(\cdot)$ and $a^y(\cdot)$ are determined as follows. Note that in the absence of arbitrage, the expected return on a zero equals the short-term rate,

$$\mathbb{E}_\tau\left(\frac{dP_\tau}{P_\tau}\right) = r_\tau d\tau \tag{20.126}$$

By Itô's lemma, and the assumption that y_τ is as in Equation 20.29,

$$\mathbb{E}_\tau\left(\frac{dP}{P}\right) = \left(\frac{\partial P/\partial\tau}{P} + \frac{P_y}{P}\kappa(\mu - y) + \frac{1}{2}\mathrm{Tr}\left([\Sigma V(y)][\Sigma V(y)]^\top \frac{P_{yy}}{P}\right)\right)d\tau$$
$$= \left(-a^o + -a^y y + a^y\kappa(\mu - y) + \frac{1}{2}\sum_{i=1}^n (a^y\Sigma)_i^2 s_{1i} + \frac{1}{2}\sum_{i=1}^n (a^y\Sigma)_i^2 s_{2i} y\right)d\tau \tag{20.127}$$

where subscripts denote partial derivatives, dots denote differentiation with respect to time left to maturity, and the second equality follows by the assumption that the pricing function P_τ is as in Equation 20.30. Therefore, matching 20.127 and 20.28 through 20.126 produces that $a^o(\cdot)$ and $a^y(\cdot)$ are solutions to the following Riccati's equations:

$$\begin{cases} a^o(\tau) = -\alpha + a^y(\tau)\kappa\mu + \frac{1}{2}\sum_{i=1}^n (a^y(\tau)\Sigma)_i^2 s_{1i} \\ a^y(\tau) = -\beta - a^y(\tau)\kappa + \frac{1}{2}\sum_{i=1}^n (a^y(\tau)\Sigma)_i^2 s_{2i} \end{cases} \tag{20.128}$$

subject to the boundary conditions

$$a^o(0) = 0 \quad \text{and} \quad a^y(0) = 0$$

Consider, for example, the Fong and Vasicek (1991) model in 20.25. This model is still a special case of the general affine model 20.29, namely, after setting

$$r_\tau = y_{1\tau}, \quad v_\tau^2 = y_{2\tau}$$

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and

$$\begin{cases} dy_{1\tau} = \kappa_r(\bar{r} - y_{1\tau})d\tau + \sqrt{y_{2\tau}}d\tilde{W}_{1\tau} \\ dy_{2\tau} = \kappa_v(\omega - y_{2\tau})d\tau + \xi\sqrt{y_{2\tau}}d\tilde{W}_{2\tau} \end{cases}$$

Therefore, the price is as in Equation 20.27 as claimed in the main text and by Equation 20.128, with the three functions $a(\cdot)$, $b(\cdot)$, and $c(\cdot)$ satisfying

$$\begin{cases} a\,(\tau) = -b(\tau)\kappa_r\bar{r} + c(\tau)\kappa_v\omega \\ b(\tau) = 1 - b(\tau)\kappa_r \\ \dot{c}(\tau) = -c(\tau)\kappa_v + \frac{1}{2}(b^2(\tau) + c^2(\tau)\xi^2) \end{cases} \tag{20.129}$$

subject to the boundary conditions $a(0) = b(0) = c(0) = 0$.

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