# Graph theoretic approach for automating electric circuit

Leanne Dong

Gina Cody School of Engineering and Computer Science
Concordia University Montréal

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# Overview

# Graph vocabulary

- Graph $G = (V, E)$ is a finite set of points ($V$: vertices or nodes) and lines joining some of these points ($E$ edges, arcs, links); Graphs can be directed and undirected;
- Cycle/Circuit/Loop: a path such that the final node of the last link coincides with the initial node of the first link;
- Tree: a graph that is connected but has no cycles;
- Spanning tree $T$: For a connected graph $G$, this is just a subgraph that contains all nodes of $G$;
- Branches: The edges of a spanning tree $T$;
- Incidence matrix $A_{inc} = \{a_{ij}\}$ is

$$a_{ij} = \begin{cases} 1 & \text{if } j\text{th edge is incident on the } i\text{th node} \\ 0 & \text{if } j\text{th edge is not incident incident on the } i\text{th node} \end{cases}$$

# Graph vocabulary

- Complete graph: this is a graph where each node is joined by an arc to every other node.
- Partial graph: a graph that contains all the nodes of the original graph but only some edges.
- subgraph: a graph that contains only a subset of the nodes of the original and all of the edges that join them.
- connected graph: a graph is connected if there is at least one chain from any nodes to every other node.

# Why graph theory for electric circuits?

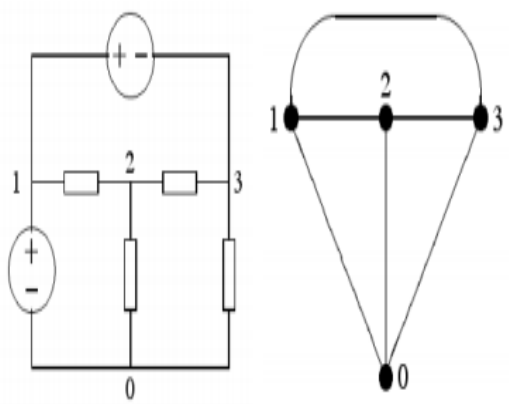
- Powerful, scalable methods to model complex networks arising many applications
- A Simple method which allows us to derive the differential equations for even very complicated circuits

# Graph for electric circuit: The idea

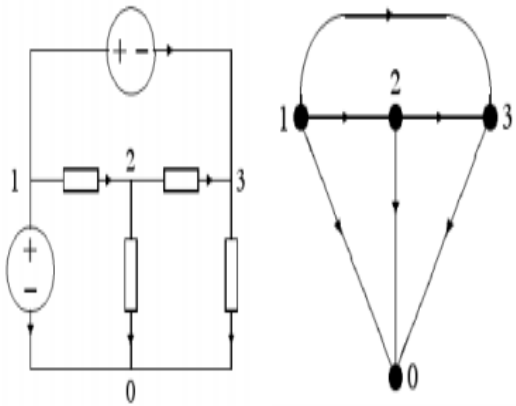
- Algebraic graph theory for electronic network simulation first due to Poincaré and Veblen
- ▸ Gotlieb and Corneil (1967)'s idea of generating spanning trees by finding fundamental set of cycles (cycle base)
- The above was further improved by ▸ Paton (1969). The DFS search was used to construct cycle base: Computational very efficient!
- Further improved by ▸ Tiernan (1970). Efficient but complicated. Fortunately this has been implemented in the Boost C++ libraries. See in particular the graph library . Boost::graph provide a comprehensive range of headers including ▸ Tiernan's efficient search algorithm in finding the fundamental cycles of the graph.

# Type of graphs in electrical theory

- Intuitively, circuits are graphs that describe the interconnection of electrical elements
  1. passive elements such as resistances, capacitances, inductances
  2. active elements
  3. sources (or excitation)
- Two variables: voltage (V) and currents (I).




Graph




Digraph– arrows indicate directions of currents and voltages' polarities

# Loop

- A loop is a set of branches of graph forming a closed path.
- Example: branches $\{a, c, d\}$, $\{b, d, e\}$ and $\{a, b, e, c\}$.

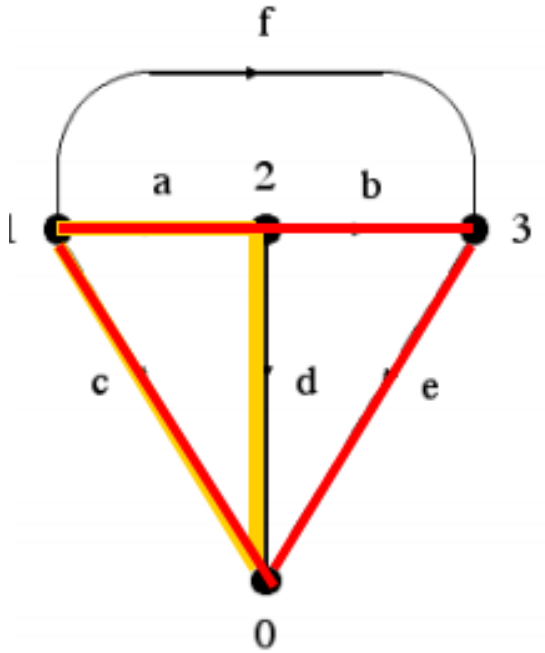



Figure: Braches and loops

# Tree and co-tree

- A tree is a set of branches of a graph which contains no loop.
- Thus, a tree is a maximal set of branches can be connected without forming a loop.
- After a tree is chosen, the remaining branches form a co-tree

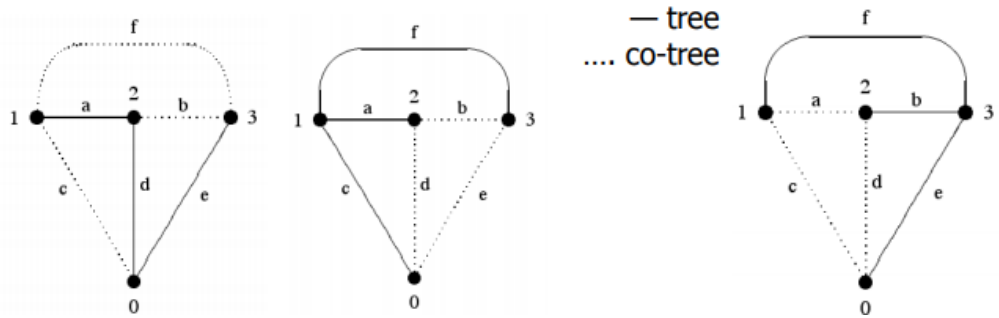



Figure: Tree and co-tree

# Graph visualisation

We are currently developing an undirected graph extension of ▸ Simple-graph-tool, which is a free tool visualizing graph theory and supporting weighted directed graph. Our **Main technology** are Qt (a framework for making GUI for applications in C++ language) and CMake (native build environment across platform)

# A very simple example

Table: PSPICE netlist.

| | | | |
|---|---|---|---|
| I1 | 0 | 1 | 1Amp |
| R2 | 1 | 0 | 1Ohm |
| R3 | 1 | 2 | 1Ohm |
| R4 | 2 | 0 | 1Ohm |

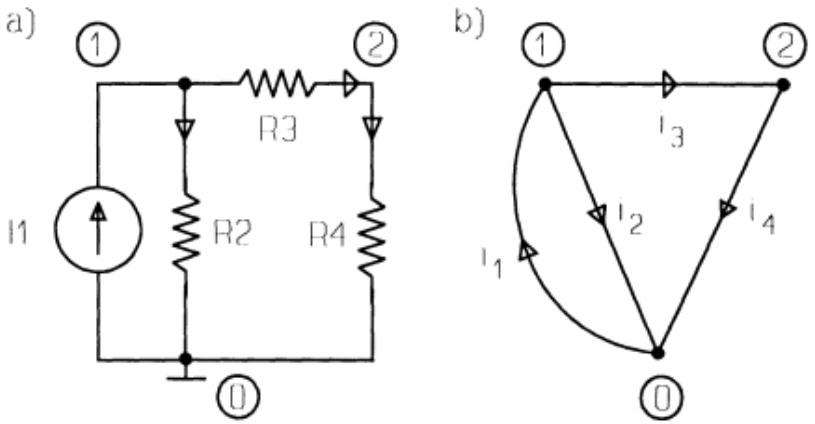



Figure: A simple circuit and its topology

# Kirchhoff's law again

**KVL**: At each loop or mesh,

$$\sum_{loop} V_k = 0$$

**KCL**: At each node,

$$\sum I_{inflow} = \sum I_{outflow}$$

# A very simple example: KCL

The KCL equations w.r.t. node 1, 2 and 0 are

$$\begin{aligned} -I_1 + I_2 + I_3 &= 0 \\ -I_3 + I_4 + \ &= 0 \\ I_1 - I_2 - I_4 &= 0 \end{aligned}$$

Linear dependency! $\Longrightarrow$ , Reference node at which the KCL equations may be omitted.

# A very simple example: KCL

Suppose we take node 0 as the reference node, so eliminate the 3rd equation, hence

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = 0$$

or simply

$$AI = 0$$

where $I$ is the vector of branch currents and $A$ is known as the incidence matrix containing only coefficients 0 or 1, $-1$.

# A very simple example: KVL

- Certain assembly of branches form loops (i.e. closed paths)
- In our last example, the loops are $\{I_1, R_2\}$, $\{I_1, R_3, R_4\}$ and $\{R_2, R_3, R_4\}$
- Conservation of energy: work done around a loop should be 0. Hence, the sum of voltages of branches constituting the loop must be 0. In the similar vein, the KVL equations can be written as a compact matrix form as

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = 0$$

or simply

$$BV = 0$$

# A very simple example: summary

By the superposition principle, the topological matrices $A$ and $B$ both contain the same full information on the assembly of branches.

# Analysis circuit networks

- Mesh Current Analysis: See tutorial
- Nodal Voltage Analysis

We choose Mesh current method over Nodal approach as we can exploit the independence of the loops. In this way, we obtain minimum set of equations needed to solve the circuit, without redundancies.

**Assumption of Mesh current method (verso.c)**:

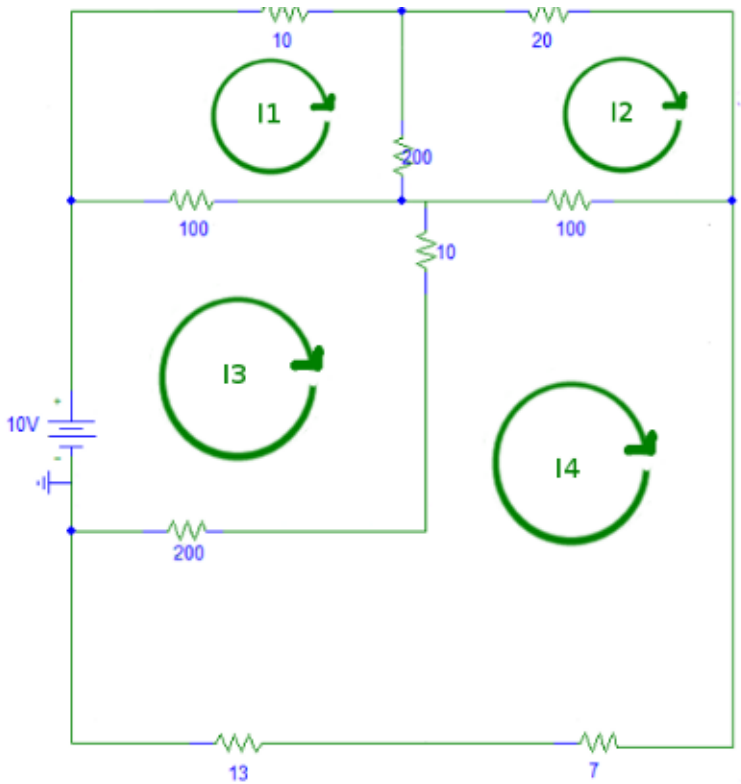
- Circuit is built up of several meshes with branches which contain resistance;
- By the superposition principle, one can split the analysis of the full circuit as the analysis of independent meshes;
- Set specific direction of the current in each mesh and so the original circuit is splitted into many meshes with given current direction (directed meshes).
- Superpose the folution to find the global solution of the circuit equations.

# The formal procedure

1. Describe the circuit in a computer-readable format (how data was stored and read);
2. Extract a set of independent loops;
3. For each loop, we associate a loop current;
4. Use KVL to set up a system of $n$ equations, one for each independent loop;
5. Solve the system of equations and thus find the values of the loop currents ($I$);
6. Use the superposition principle to find the total current in each circuit branch.

# A concrete example

- We show a set of independent loops with associated loop currents
- Resistance are conventionally assumed to be ohms.
- Voltage values are in volt.



# A concrete example: setting up the Kirchhoff equations

At each branch $b$, the total voltage drop $V^{(b)}$ across the branch is the sum of the voltage drop due to the loop currents $I_{l'}$ that traverse that branch:

$$V^{(b)} = \sum_{b \in l} \sum_{l' \in L_b} R^{(b)} I_{l'}$$

So basically the inflow outflow relations,

$$\begin{aligned}
(10 + 200 + 100) I_1 - 200 I_2 - 100 I_3 + 0 I_4 &= 0 \qquad (1) \\
-200 I_1 + (20 + 100 + 200) I_2 + 0 I_3 - 100 I_4 &= 0 \\
-100 I_1 + 0 I_2 + (100 + 10 + 200) I_3 - (200 + 10) I_4 &= 10 \\
0 I_1 - 100 I_2 - (10 + 200) I_3 + (100 + 7 + 13 + 200 + 10) I_4 &= 0
\end{aligned}$$

# Why find spanning trees

The construction of structures as spanning trees and efficient traversal methods to access the information is crucial to solve the electrical or hydraulic network problem.

# Traverse information in a Graph: DFS vs BFS

| Importance | Key | BFS | DFS |
|---|---|---|---|
| 1 | Definition | Breadth First Search | Depth First Search |
| 2 | Data structure | Use **queue** to find shortest path | Use **stack** to find shortest path |
| 3 | Source | BFS is better when target is closer | DFS is better when target is far |
| 4 | Speed | Slower | Faster |
| 5 | Time complexity | $O(V+E)$ | $O(V+E)$ |

Please check our simple-graph-tool for demonstration of DFS and BFS traversal

# Methods

- ▸ Gotlieb and Corneil (1967): Implementated by a Physicist Prof. Milotti's group, see ▸ CircuitMath.
- The above was further improved by ▸ Paton (1969). The DFS search was used to construct cycle base: Computational very efficient!
- Further improved by ▸ Tiernan (1970).

In term of storage and speed, Paton and Tiernan are better. Moreover, Tiernan's algorithm has been written into the modern C++ numerics libraries like ▸ Boost

# Method: Gotlieb and Corneil (1967)

**Assumptions**

- Undirected graph is represented by its adjacency matrix $A = \{a_{ij}\}$ is

$$a_{ij} = \begin{cases} 1 & \text{if } j\text{th node is connected with the } i\text{th node} \\ 0 & \text{if } j\text{th node is not connected with the } i\text{th node} \end{cases}$$

- Graph is connected
- Each connected component can be separated

# Method: Gotlieb and Corneil (1967)

**Key steps**

- Construct a set of disjoint trees by choosing some edges.
  1. The set of $n$ node is partitioned w.r.t. some chsen edges to form a set of disjoijnt trees with adjacency matrix $B$.
  2. For rach row $i$ of A, set $b_{ji} = b_{ij} = 1$ if $a_{ij}$ is the first superdiagonal element of the $i$th row of $A$ to equal 1; otherwise set $b_{ji} = b_{ij} = 0$.
- Divide the set of nodes of the graph into connected components w r.t. the edges chosen in step 1.
- Amalgamate the componets by adding appropriate edges to yield a spanning tree.
- Produce a set of fundamental cycles

# Method: Paton (1969)

This has been implemented by a computer scientist Philipp Sch. He demonstrated how in principle one can enumerate all cycles of a graph. However the method does not scale up well.

# Method: Tiernan (1970)

A theoretically very eddifcient search algorithm that uses an exhaustive search to find all elementary circuits of a directed graph. For more detail, one shall consult with

- The official Boost Graph Library site : ▸ boost::Graph
- Textbook: ▸ The Boost Graph library
- Uni-Resource: ▸ Brown
- Tutorial: ▸ technical-recipes

# Circuit equations

Having found the independent loops, we can set up the system of equations like the one we show before and solve them. To fix idea, we consider mainly simple linear circuits. The solutions can be trivially attained using the LU decomposition method via some scientific software. One choice is the routines **ludcmp** and **lubksb** in the Numerical Recipes C library

# Branch currents and voltages

- The solution of linear equation yields all the independent loop currents;
- From them, we can find the branch currents by summing over all the independent loop currents that traverse a given branch.
- By looping over all branches, one can construct a new matrix $I = \{i_{ik}\}$ where the element $i_{jk}$ stores the current in the branch that links the i-th node with the k-th node.
- Similarly, we calculate a voltage matrix $V = \{v_{jk}\}$, where the matrix elemet $v_{jk} = r_{jk}$, $i_{jk}$ stores the voltage drop in the branch that links the $j$th node with the $k$th node.
- Voltage are set by the power supplies w.r..t. a ground terminal: one can find the voltage of any node w.r.t. ground by following any path along the circuit that joins the node with the ground node and summing all the voltage drops.
- Finaly, save the results on a file and save for further analysis

# Potential improvements

- A comparison of graph-based method with the nodal method used in established programs such as SPICE
- Optimising circuit representation in the description file
- Deal with sparse matrix case, the program efficiency could be boosted with the use of sparse matrix code
- code could be cleaned up with C++, performance could be boosted via better data structures, such as linked lists (represent loops) and other graph structures
- Can be extended to handle time-dependent voltage and currents, instead of the linear system we have seen, we can manipulate the corresponding DEs and use a DE solver
- Could be extended to include non-linear elements like semiconductor diodes, transistors, etc, this would require the intro of a nonlinear equation solver

# Potential improvements

- Can be implemented in Object-Oriented sense, utilise the efficient C++ library like Boost.
- Could work on a graphical interface both for input - by placing components in a special window - and for output - by automatically plotting output functions
- Could be used in the analysis of fluid flow - using the analogy between hydraulic and circuit networks.

**Rule of Sum for High performance code**: Efficiency with algorithms and performance through data structure.

# Proposal: object oriented algorithms in C++

- OOP features allow handling data more efficiently.
- C++ is the most ideal language for building either an academic or for a live simulation system given **Performance** is the key.
- In case any security issue arise from accepting data over a network, we can consider switching to Rust. From C++ to Rust is not a long learning curve as they are relative.
- Java, C# are relatively memory safe. But low in performance compared to C++.
- We would encounter some PDEs in the CFD component later on, C++ allows us to integrate GPU in obtainling massively-parallel PDE solutions. Specifically, for solving large, spare liner or nonlinear system.

# Numerical Algorithms

- Power system is represented by thousands of differential or algebraic equations (DAEs).
- DAEs are discretized using an integration algorithm.
- These equations are usually solved through Newton iteration.
- The core of the solution of the nonlinear system of equations is the solution of the linear system of equations;.

# Review: Algorithms for loop extraction

- DFS: Ivetić, Vasilić, Stanić and Prodanović (2016) identify the set of basis loops from which minimum loops are obtained from transformations of spanning tree. Cycles extracted in their work may not be minimal ones.
- Gotlieb and Corneil (1967)
- Tiernan (1970)

# Algorithms for equations solver

- Gaussian elimination (Only for linear equations)
- Iterative methods: Newton-type algorithms;
- Hardy-cross method: suitable for large network

# Solving the previous concrete circuit problem

Simplify (1), write in matrix form, we have

$$\left(\begin{array}{cccc|c} 310 & -200 & -100 & 0 & 0 \\ -200 & 320 & 0 & -100 & 0 \\ -100 & 0 & 310 & -210 & 10 \\ 0 & -100 & -210 & 330 & 0 \end{array}\right)$$

**Solution**:
We invoke C++ to solve the problem for us.
`https://godbolt.org/z/eTu_xC`
**Exercise**: Solve the above using nodal analysis.

# Solution: implementation in C++ with **Eigen**

```
#include <iostream>
#include </usr/include/eigen3/Eigen/Dense>

using namespace std;
using namespace Eigen;

int main()
{
   Matrix4f A;
   Vector4f b;
   A << 310,-200,-100,0, -200,320,0,-100, -100,0,310,-210, 0,-100,-210,330;
   b << 0, 0, 10,0;
   cout << "Here is the matrix A:\n" << A << endl;
   cout << "Here is the vector b:\n" << b << endl;
   Vector4f x = A.colPivHouseholderQr().solve(b);
   cout << "The solution is:\n" << x << endl;
}
```

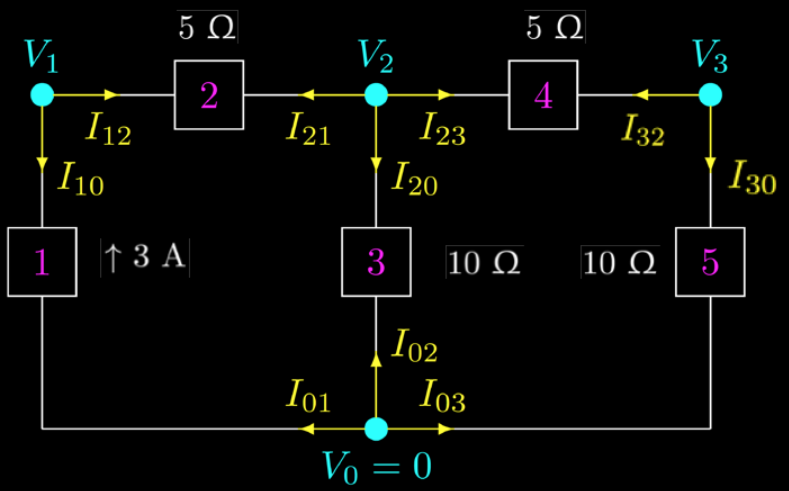
The solution is found to be 0.199872, 0.190431, 0.238743, 0.209634.

# Create adjacency and incidence matrice

It is easy to draw in C++.

- Adjacency: `https://godbolt.org/z/zfxn8i`
- Incidence: `https://godbolt.org/z/2ZaTPp`

# Generic Example of a Network model (Linear case, Juergen, 2019)




Reference Node (0)

$$(0) \quad I_{01} + I_{02} + I_{03} = 0$$

$$(1) \quad I_{10} + I_{12} = 0$$

$$(2) \quad I_{20} + I_{21} + I_{23} = 0$$

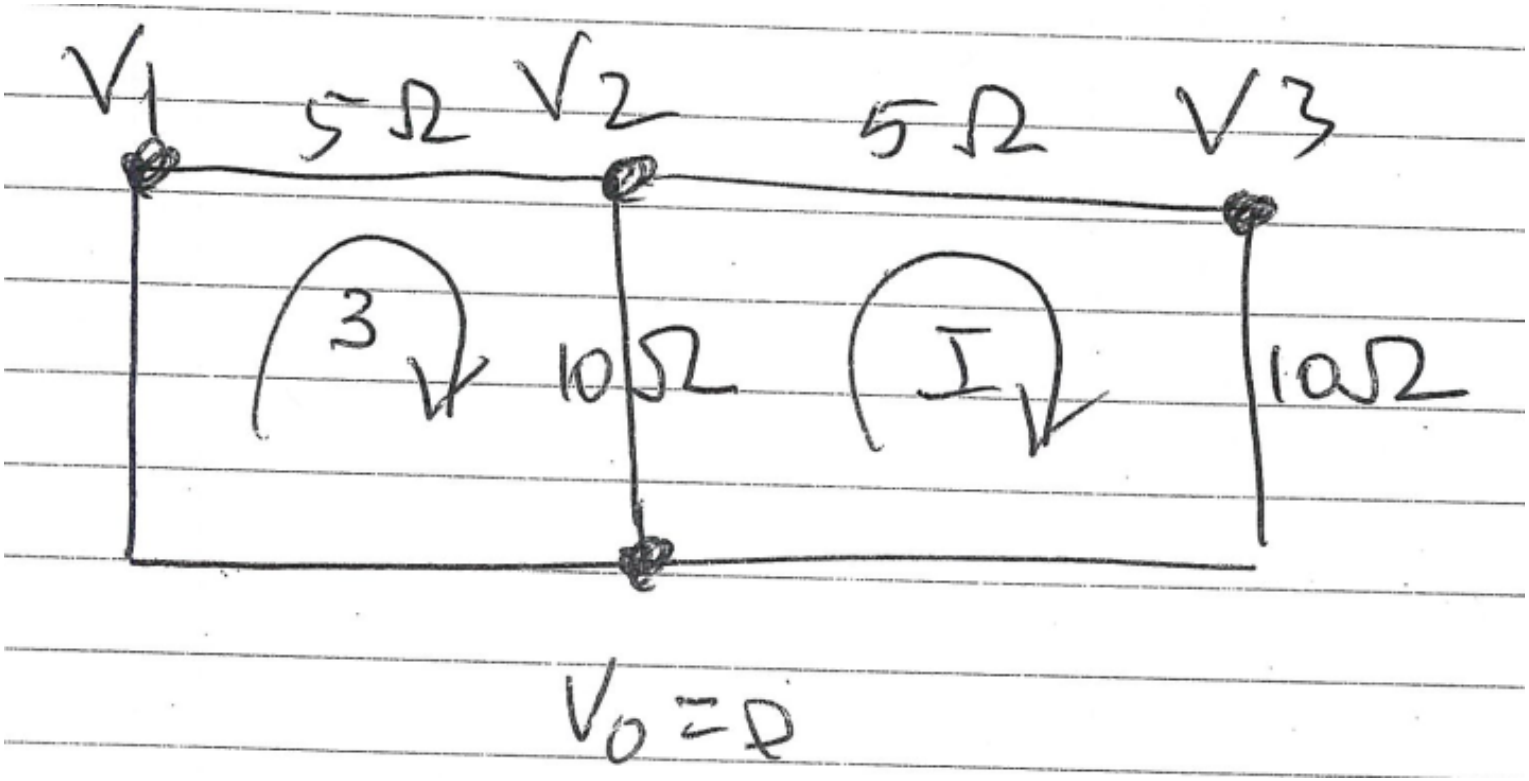
$$(3) \quad I_{30} + I_{32} = 0$$

$$(1) \quad -3\,\mathrm{A} + \frac{V_1 - V_2}{5\,\Omega} = 0$$

$$(2) \quad \frac{V_2}{10\,\Omega} + \frac{V_2 - V_1}{5\,\Omega} + \frac{V_2 - V_3}{5\,\Omega} = 0$$

$$(3) \quad \frac{V_3}{10\,\Omega} + \frac{V_3 - V_2}{5\,\Omega} = 0$$

# Network graph




KVL: $\sum_{loop} V_k = 0$

# Solution

Assume the positrons approach. That is

$$V_a \xrightarrow[R]{I} V_b \qquad I > 0 \text{ implies } V_a > V_b \Longrightarrow IR = V_a - V_b$$

**Circuit A**:

$$V_1 = 3 \times 5 + 3 \times 10 - 10I = 0$$

**Circuit B**:

$$-5I - 10I - 10I + 3 \times 10 = 0$$

implies $I = \frac{30}{25} = \frac{6}{5}$, then

$$V_1 = 45 - 10I = 33.$$

We now find $V_2$, $V_3$ by

$$V_1 - V_2 = 3 \times 5 \rightarrow \ V_2 = V_1 - 15 = 18$$
$$V_2 - V_3 = 5I = 6 \rightarrow \ V_3 = 12$$

# Example: Linear case

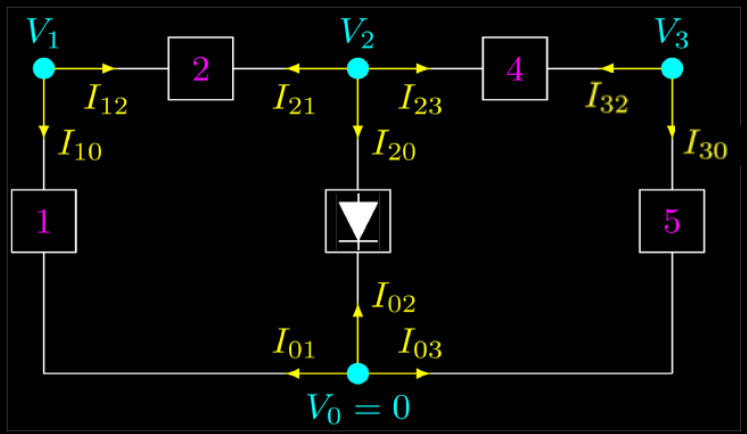
```
#include <iostream>
#include </usr/include/eigen3/Eigen/Dense>

using namespace std;
using namespace Eigen;

int main()
{
   Matrix3f A;
   Vector3f b;
   A << 0.2,-0.2,0,  -0.2,0.5,-0.2,  0,-0.2,0.3;
   b << 3, 0, 0;
   cout << "Here is the matrix A:\n" << A << endl;
   cout << "Here is the vector b:\n" << b << endl;
   Vector3f x = A.colPivHouseholderQr().solve(b);
   cout << "The solution is:\n" << x << endl;
}
```

# Generic Example of a Network model (Nonlinear case, Juergen, 2019)

## A Generic Example of a „Network Model“ Applied to a DC Network with a Diode




$$(1) \quad -3\,\mathrm{A} + \frac{V_1 - V_2}{5\,\Omega} = 0$$

$$(2) \quad I_\mathrm{s}\left(\exp\left(\frac{qV_2}{nkT}\right) - 1\right) + \frac{V_2 - V_1}{5\,\Omega} + \frac{V_2 - V_3}{5\,\Omega} = 0$$

$$(3) \quad \frac{V_3}{10\,\Omega} + \frac{V_3 - V_2}{5\,\Omega} = 0$$

# The Shockley Equation

$$I = I_s\left(\frac{qV}{n\kappa T} - 1\right) = \left(\frac{V}{nV_T} - 1\right)$$

where

1. $I_s$ - **saturation current**. $I_s \tilde{1}0^{-14}$ for small signal diodes at 300 Kelvin: .
2. $n$ - emission coefficient: $n = 1, 2$ for small signal diodes.
3. $\kappa = 1.38 \times 10^{-23} J/K$ - the Boltzman's constant.
4. $q = 1.60 \times 10^{-19} C$ - the charge of the electron;
5. $V_T$-thermal voltage- $V_T = \kappa T/q$

We will now implement this in C++. Assume $n = 2$, $T = 300$.. Along with other constants, these together yields $\frac{q}{n\kappa T}$ The complete example can be easily implemented in C++ with ▸ Eigen at `https://godbolt.org/z/p_LVgt`.

# Exercise

Implement the Diode example using mesh current method by hand.

# Creating adjacency matrix

```
#include <iostream>
#include <vector>
#include <utility>
void print_graph(const std::vector<std::vector<int> > &adj);
void addEdge(std::vector<std::vector<int> >& adj, int u, int v);

int main()
{
// Initialise array to hold adjacency matrix, vec<>s is already dynamic
    std::vector<std::vector<int> > adj(3, std::vector<int>(3));
    addEdge(adj,0,1);     // edge from node 0 to node 1
    addEdge(adj,0,2);
    addEdge(adj,1,2);
    print_graph(adj);
}

void print_graph(const std::vector<std::vector<int> >& adj)
{
    for(std::size_t i = 0; i < adj.size(); i++ )
    {
        for(std::size_t j = 0 ; j < adj[i].size(); j++ )
        {
            std::cout << adj[i][j]<< "  ";
        }
        std::cout << std::endl;
    }
}


void addEdge(std::vector<std::vector<int> >& adj, int u , int v)
{
    adj[u][v]=1;
    adj[v][u]=1;
}
```

# Creating incidence matrix

```
#include <iostream>
#include <vector>
#include <utility>
void displayMatrix(const std::vector<std::vector<int> >& inc);
void add_edge(std::vector<std::vector<int> >& inc, int u, int v);
int ed_no=0;

int main()
{
// Initialise array to hold incidence matrix, vec<>s is already dynamic
   std::vector <std::vector<int> > inc(3, std::vector<int>(4));
//there are 3 vertices and 4 edges in the graph pila has
   add_edge(inc, 0, 1);
   add_edge(inc, 0, 2);
   add_edge(inc, 1, 2);

   displayMatrix(inc);
}

void displayMatrix(const std::vector<std::vector<int> >& inc)
{
   for(std::size_t i = 0; i < inc.size(); i++)
   {
      for(std::size_t j = 0; j < inc[i].size(); j++)
      {
         std::cout << inc[i][j] << "  ";
      }
      std::cout << std::endl;
   }
}
void add_edge(std::vector<std::vector<int> >& inc, int u, int v)
{//function to add edge into the matrix with edge number
   inc[u][ed_no]=1;
   inc[v][ed_no]=1;
   ed_no++; //increase the edge number
}
```

# Project 1

- Learn thoroughly Milloti's source codes, re-implement Milotti's approach in C++ with a more efficient graph algorithms. (Hint: Try DFS to find spanning tree)
- Develop a simulator (Either Qt or OpenGL) to visualize the Electrical Distribution Network and displays results in a more intuitive manner for any specific configuration of the power flow problem. This visualizer also allows for easy interaction with the program.

# Project 2

- Implement a Diode example using Milotti's graph theoretic approach. (C or C++)
- Hint: Extend the current C project to nonlinear element and the construction of systems of differential equations to manipulate time-dependent signals, much like in the well-known program SPICE.