Graph theoretic approach for automating electric circuit

Leanne Dong

June 3, 2020

Overview

Basic concepts

- Graph $G = (V, E)$ is a finite set of points (V: vertices or nodes) and lines joining some of these points (E edges, arcs, links); Graphs can be directed and undirected;
- Cycle/Circuit/Loop: a path such that the final node of the last link coincides with the initial node of the first link;
- Tree: a graph that is connected but has no cycles;
- Spanning tree T: For a connected graph G, this is just a subgraph that contains all nodes of G;
- Branches: The edges of a spanning tree T;
- Incidence matrix $A_{inc} = \{a_{ij}\}$ is

$$a_{ij} = \begin{cases} 1 & \text{if } j\text{th edge is incident out of the } i\text{th vertex} \\ -1 & \text{if } j\text{th edge is incident into the } i\text{th vertex} \\ 0 & \text{if } j\text{th edge is not incident incident on the } i\text{th vertex} \end{cases}$$

Why graph theory for electric circuits?

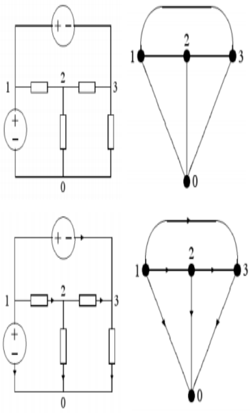
- Powerful, scalable methods to model complex networks arising many applications
- A Simple method which allows us to derive the differential equations for even very complicated circuits

Graph for electric circuit: The idea

- Algebraic graph theory for electronic network simulation first due to Poincaré and Veblen
- Gotlieb and Corneil (1967)'s idea of generating spanning trees by finding fundamental set of cycles (cycle base)
- The above was further improved by Paton (1969). The DFS search was used to construct cycle base: Computational very efficient!
- Further improved by Tiernan (1970). Efficient but complicated. Fortunately this has been implemented in the Boost C++ libraries. See in particular the graph library . Boost::graph provide a comprehensive range of headers including Tiernan's efficient search algorithm in finding the fundamental cycles of the graph.

Type of graphs in electrical theory

- Intuitively, circuits are graphs that describe the interconnection of electrical elements
  1. passive elements such as resistances, capacitances, inductances
  2. active elements
  3. sources (or excitation)
- Two variables: voltage (V) and currents (I).



Graph

Digraph— arrows indicate directions of currents and voltages' polarities

Loop

- A loop is a set of branches of graph forming a closed path.
- Example: branches $\{a, c, d\}$, $\{b, d, e\}$ and $\{a, b, e, c\}$.

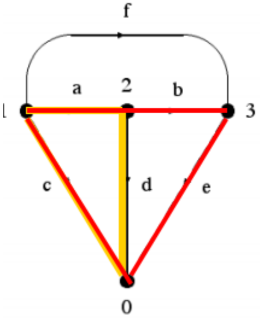


Figure: Braches and loops

Tree and co-tree

- A tree is a set of branches of a graph which contains no loop.
- Thus, a tree is a maximal set of branches can be connected without forming a loop.
- After a tree is chosen, the remaining branches form a co-tree

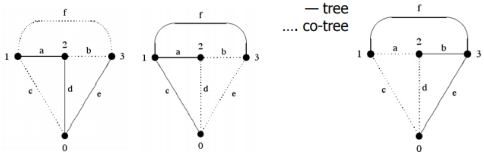


Figure: Tree and co-tree

A very simple example

Table: PSPICE netlist.

I1	0	1	1Amp
R2	1	0	1Ohm
R3	1	2	1Ohm
R4	2	0	1Ohm

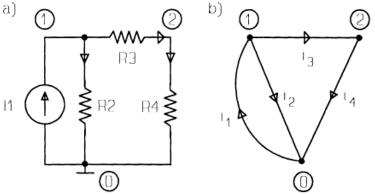


Figure 1.6 (a) A simple circuit and (b) a graph representing its topology

Figure: A simple circuit and its topology

Kirchhoff's law again

KVL: At each loop or mesh,

$$\sum_{loop} V_k = 0$$

KCL: At each node,

$$\sum I_{inflow} = \sum I_{flow}$$

A very simple example: KCL

The KCL equations w.r.t. node 1, 2 and 0 are

$$
\begin{aligned}
-I_1 + I_2 + I_3 &= 0 \\
-I_3 + I_4 + \quad &= 0 \\
I_1 - I_2 - I_4 &= 0
\end{aligned}
$$

Linear dependency! $\implies$, Reference node at which the KCL equations may be omitted.

A very simple example: KCL

Suppose we take node 0 as the reference node, so eliminate the 3rd equation, hence

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \end{pmatrix} = 0$$

or simply

$$AI = 0$$

where I is the vector of branch currents and A is known as the incidence matrix containing only coefficients 0 or 1, -1.

A very simple example: KVL

- Certain assembly of branches form loops (i.e. closed paths)
- In our last example, the loops are $\{I_1, R_2\}$, $\{I_1, R_3, R_4\}$ and $\{R_2, R_3, R_4\}$
- Conservation of energy: work done around a loop should be 0. Hence, the sum of voltages of branches constituting the loop must be 0. In the similar vein, the KVL equations can be written as a compact matrix form as

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = 0$$

or simply

$$BV = 0$$

A very simple example: summary

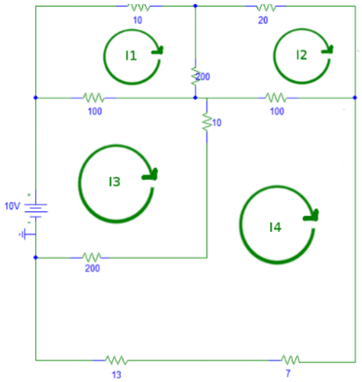
By the superposition principle, the topological matrices A and B both contain the same full information on the assembly of branches.

The formal procedure

1. Describe the circuit in a computer-readable format;
2. Find a set of independent loops;
3. For each loop, we associate a loop current;
4. Use KVL to set up a system of n equations, one for each independent loop;
5. Solve the system of equations and thus find the values of the loop currents (I);
6. Use the superposition principle to find the total current in each circuit branch.

A concrete example

- We show a set of independent loops with associated loop currents
- Resistance are conventionally assumed to be ohms.
- Voltage values are in volt.



A concrete example: setting up the Kirchhoff equations

At each branch b, the total voltage drop $V^{(b)}$ across the branch is the sum of the voltage drop due to the loop currents $I_{l'}$ that traverse that branch:

$$V^{(b)} = \sum_{b \in l} \sum_{l' \in L_b} R^{(b)} I_{l'}$$

So basically the inflow outflow relations,

$$\begin{aligned}
(10+200+100)I_1 - 200I_2 - 100I_3 + 0I_4 &= 0 \\
-200I_1 + (20+100+200)I_2 + 0I_3 - 100I_4 &= 0 \\
-100I_1 + 0I_2 + (100+10+200)I_3 - (200+10)I_4 &= 10 \\
0I_1 - 100I_2 - (10+200)I_3 + (100+7+13+200+10)I_4 &= 0
\end{aligned}$$

Circuit equations

Having found the independent loops, we can set up the system of equations like the one we show before and solve them. To fix idea, we consider mainly simple linear circuits. The solutions can be trivially attained using the LU decomposition method via some scientific software. One choice is the routines **ludcmp** and **lubksb** in the Numerical Recipes C library

Branch currents and voltages

Comparison with SPICE

Future plan

Develop existing approach in modern C++. Make use of algorithm library and Boost.

```
#ifndef BOOST_GRAPH_CYCLE_HPP
#define BOOST_GRAPH_CYCLE_HPP

#include <vector>

#include <boost/config.hpp>
#include <boost/graph/graph_concepts.hpp>
#include <boost/graph/graph_traits.hpp>
#include <boost/graph/properties.hpp>

#include <boost/concept/detail/concept_def.hpp>
namespace boost {
    namespace concepts {
        BOOST_concept(CycleVisitor,(Visitor)(Path)(Graph))
        {
            BOOST_CONCEPT_USAGE(CycleVisitor)
            {
                vis.cycle(p, g);
            }
        private:
            Visitor vis;
            Graph g;
            Path p;
        };
    } /* namespace concepts */
using concepts::CycleVisitorConcept;
} /* namespace boost */
#include <boost/concept/detail/concept_undef.hpp>
```