Stochastic Navier-Stokes Equations perturbed by cylindrical Lévy noise on 2D rotating sphere

Leanne Dong

Supervised by: Prof. Ben Goldys

School of Mathematics and Statistics
University of Sydney

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Research rationales

Why study stochastic Navier-Stokes equations with noise, and stable Lévy noise?

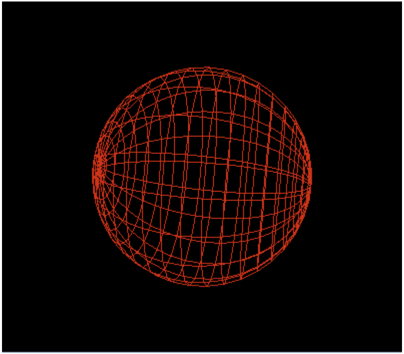
- More informative than deterministic equations.
- To prove in a "Cheaper" way for unsolved problems.
  - Clay millenium problem No. 3: Uniqueness in 3D is missing.
  - At atomic scale, fluid are not continuous fields
  - Lévy processes are perfect candidates to model discontinuity in infinite dimensions.

Why stable-type Lévy noise ?

- Has a 'heavy tail' that decays polynomially. Useful for modelling extreme events - earthquakes, stock market crashes.
- Allows one to take into account at the same time noise with a large numer of small random impulses and occasionally large random disturbance with infinite moments.

Research rationales: Rotating spheres?

- Modelling large scale interactions between Ocean and atmosphere requires taking into account that we are on the surface of rotating Earth.
- In cosmology, models of rotating fluids are used to describe many cosmological objects including black holes. (In this case, relativistic effects have to be taken into account which are not included in my thesis)



The Navier-Stokes equations

We consider the stochastic Navier-stokes equations on the 2D unit sphere $\mathbb{S}^2 \in (\theta, \phi)$ with rotation. Which is a system of 3 equations:

$$(1) \begin{cases} \partial_t u + (u \cdot \nabla) u - \nu L u + \omega \times u + \nabla p = f + \eta(x,t), \quad \text{on } (t,x) \in (0,T) \times \mathbb{S}^2 \\ \nabla \cdot u = 0, \qquad \text{on} \quad (t,x) \in [0,T) \times \mathbb{S}^2, \quad \text{Incompressible condition} \\ u(0) = u_0, \qquad \text{Initial conditions} \end{cases}$$

- u, p, ν, f are respectively velocity, pressure , viscosity and external forcing.
- $L = \Delta + 2\mathrm{Ric}$ is the stress tensor, where Δ is the Laplace-de Rham operator and Ric is the Ricci tensor on spheres.
- ω is the Coriolis acceleration which can be formally represented as $\omega(\cdot) = 2\Omega \cos\theta(\cdot)$.
- The noise process $\eta(x,t)$ can be viewed as some generalised derivative of H-valued Lévy process.

The sphere can be viewed as a surface embedded into $\mathbb{R}^3$, hence, given any two vector fields u, v on $\mathbb{S}^2$, we can find vector fields $\tilde{u}$ and $\tilde{v}$ defined on some nbhd of the surface $\mathbb{S}^2$ such that their restriction to $\mathbb{S}^2$ are equal to, resp. u and v, namely,

$$\tilde{u}|_{\mathbb{S}^2} = u \in T\mathbb{S}^2 \quad \text{and} \quad \tilde{v}|_{\mathbb{S}^2} = v \in T\mathbb{S}^2$$

So, define orthogonal projection $\pi_x : \mathbb{R}^3 \to T_x\mathbb{S}^2$
Then usual spherical calculus can be used to calculate standard curl and div operators.

Weak formulation of Stochastic Navier Stokes Equations

- Orthogonal-projection $P_{\mathcal{L}} : H \to H_{\mathcal{L}}$

$$P_{\mathcal{L}} = \mathrm{Id} - \Delta^{-1}(\nabla \otimes \nabla).$$

  $P_{\mathcal{L}}$ decomposes the velocity vector into its divergence free part and the gradient of the scalar part, that is,

$$u = P_{\mathcal{L}}[(u \cdot \nabla)u] + \nabla\phi$$

- Define $A : D(A) \to H$ by $Au = -\nu P_{\mathcal{L}} L u$, $D(A) = (H^2(\mathbb{S}^2))^2 \cap V$.
- Define $B : V \times V \to V^*$ by $B(u, v) = P_{\mathcal{L}}[(u \cdot \nabla)v]$.
- $L(t) = P_{\mathcal{L}} \tilde{L}(t)$.
- $B(u) = P_{\mathcal{L}}[\,\mathrm{div}(u \otimes u)\,] = P_{\mathcal{L}}[\mathrm{div}\, uu^T] = B(u, u)$

Then, projecting (1) onto H yields

$$\begin{cases} u_t = -\nu Au - B(v+z, v+z) - Cv + \alpha z + f, \ (t,x) \in (0,T) \times \mathbb{S}^2 \\ u(0) = u_0 \in H \end{cases}$$

Suppose we have solved the projection problem (2) by finding the mild solution $u = P_{\mathcal{L}} u$, how do we recover the pressure from its projection?

Answer: By Hodge decomposition, the pressure satisfies

$$\nabla p = \nu \Delta u + (u \cdot \nabla) u - \partial_t \tilde{L}(t) - \mathcal{P}[-\nu \Delta u + (u \cdot \nabla) u - \partial_t \tilde{L}(t)]$$

Solve for u, Substitute u back to (1), we can recover ∇p in term of forcing term and the solution u.

My problem

Projecting (1) onto H leads to a version of the two-dimensional Navier-Stokes equation of our interest, which is the abstract equation in $u = u(t) = u(t,x)$:

$$(2) \quad \begin{cases} du(t) + (Au(t) + B(u(t)))dt + Cu = fdt + GdL(t), & t > 0 \\ u(0) = u_0 \in V \end{cases}$$

We assume A generate a strongly continuous semigroup $\{S(t) : t \geq 0\}$ on X. $\{u(t) \in X, t \geq 0\}$ is said to be a mild solution of (1) if

$$(3) \quad u(t) = S(t)u_0 - \int_0^t S(t-s)B(u(s))ds + \int_0^t [S(t-s)]GdL(s).$$

Weak formulation of Stochastic Navier Stokes Equations

Define $z(t) = L_A(t) = \int_0^t S(t-s)GdL(s)$ as the solution of

$$\begin{cases} dz(t) + \hat{A}z(t) = dL(t), t \geq 0 \\ z(0) = 0 \end{cases}$$

With this definition, (3) can be written as

$$u(t) = S(t)u(0) + \int_0^t S(t-s)B(u(s) + z(s))ds$$