Stochastic Navier-Stokes Equations perturbed by cylindrical Lévy noise on 2D rotating sphere

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Research rationales

Why study stochastic Navier-Stokes equations with noise, and stable Lévy noise?

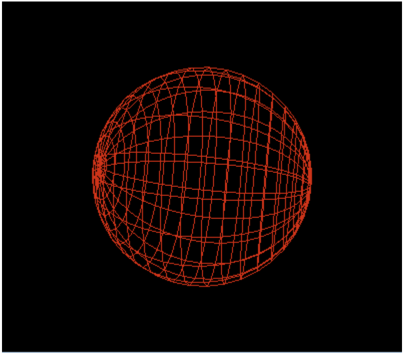
- More informative than deterministic equations.
- To prove in a “Cheaper” way for unsolved problems.
  - Clay millenium problem No. 3: Uniqueness in 3D is missing.
  - At atomic scale, fluid are not continuous fields
  - Lévy processes are perfect candidates to model discontinuity in infinite dimensions.

Why stable-type Lévy noise ?

- Has a 'heavy tail' that decays polynomially. Useful for modelling extreme events - earthquakes, stock market crashes.
- Allows one to take into account at the same time noise with a large numer of small random impulses and occasionally large random disturbance with infinite moments.

Research rationales: Rotating spheres?

- Modelling large scale interactions between Ocean and atmosphere requires taking into account that we are on the surface of rotating Earth.
- In cosmology, models of rotating fluids are used to describe many cosmological objects including black holes. (In this case, relativistic effects have to be taken into account which are not included in my thesis)



The Navier-Stokes equations

We consider the stochastic Navier-stokes equations on the 2D unit sphere $\mathbb{S}^2 \in (\theta, \phi)$ with rotation. Which is a system of 3 equations:

$$(1) \begin{cases} \partial_t u + (u \cdot \nabla) u - \nu L u + \omega \times u + \nabla p = f + \eta(x,t), \quad \text{on } (t,x) \in (0,T) \times \mathbb{S}^2 \\ \nabla \cdot u = 0, \qquad \text{on} \quad (t,x) \in [0,T) \times \mathbb{S}^2, \quad \text{Incompressible condition} \\ u(0) = u_0, \qquad \text{Initial conditions} \end{cases}$$

- u, p, ν, f are respectively velocity, pressure , viscosity and external forcing.
- $L = \Delta + 2\mathrm{Ric}$ is the stress tensor, where Δ is the Laplace-de Rham operator and Ric is the Ricci tensor on spheres.
- ω is the Coriolis acceleration which can be formally represented as $\omega(\cdot) = 2\Omega \cos\theta(\cdot)$.
- The noise process $\eta(x,t)$ can be viewed as some generalised derivative of H-valued Lévy process.

The sphere can be viewed as a surface embedded into $\mathbb{R}^3$, hence, given any two vector fields u, v on $\mathbb{S}^2$, we can find vector fields $\tilde{u}$ and $\tilde{v}$ defined on some nbhd of the surface $\mathbb{S}^2$ such that their restriction to $\mathbb{S}^2$ are equal to, resp. u and v, namely,

$$\tilde{u}|_{\mathbb{S}^2} = u \in T\mathbb{S}^2 \quad \text{and} \quad \tilde{v}|_{\mathbb{S}^2} = v \in T\mathbb{S}^2$$

So, define orthogonal projection $\pi_x : \mathbb{R}^3 \to T_x\mathbb{S}^2$

Then usual spherical calculus can be used to calculate standard curl and div operators.

Weak formulation of Stochastic Navier Stokes Equations

- Orthogonal-projection $P_{\mathcal{L}} : H \to H_{\mathcal{L}}$

$$P_{\mathcal{L}} = \mathrm{Id} - \Delta^{-1}(\nabla \otimes \nabla).$$

$P_{\mathcal{L}}$ decomposes the velocity vector into its divergence free part and the gradient of the scalar part, that is,

$$u = P_{\mathcal{L}}[(u \cdot \nabla)u] + \nabla\phi$$

- Define $A : D(A) \to H$ by $Au = -\nu P_{\mathcal{L}} L u$, $D(A) = (H^2(\mathbb{S}^2))^2 \cap V$.
- Define $B : V \times V \to V^*$ by $B(u, v) = P_{\mathcal{L}}[(u \cdot \nabla)v]$.
- $L(t) = P_{\mathcal{L}}\tilde{L}(t)$.
- $B(u) = P_{\mathcal{L}}[\,\mathrm{div}(u \otimes u)\,] = P_{\mathcal{L}}[\mathrm{div}\, uu^T] = B(u, u)$

Then, projecting (1) onto H yields

$$\begin{cases} u_t = -\nu A u - B(v+z, v+z) - Cv + \alpha z + f, \ (t,x) \in (0,T) \times \mathbb{S}^2 \\ u(0) = u_0 \in H \end{cases}$$

Suppose we have solved the projection problem (2) by finding the mild solution $u = P_{\mathcal{L}} u$, how do we recover the pressure from its projection?

Answer: By Hodge decomposition, the pressure satisfies

$$\nabla p = \nu \Delta u + (u \cdot \nabla) u - \partial_t \tilde{L}(t) - \mathcal{P}[-\nu \Delta u + (u \cdot \nabla) u - \partial_t \tilde{L}(t)]$$

Solve for u, Substitute u back to (1), we can recover ∇p in term of forcing term and the solution u.

My problem

Projecting (1) onto H leads to the abstract evolutionary equation in $u = u(t) = u(t,x)$:

$$(2) \quad \begin{cases} du(t) + (Au(t) + B(u(t)))dt + Cu = fdt + GdL(t), & t > 0 \\ u(0) = u_0 \in V \end{cases}$$

A **solution** to (2) is a process $\{u(t) \in X, t \geq 0\}$ which can be represented in the form $u(t) = v(t) + z_\alpha(t)$, where $\{z_\alpha(t), t \in \mathbb{R}\}$ is a stationary Ornstein-Uhlenbeck process z_α with drift $-\nu A - C - \alpha I$ and $v(t)$ is the solution to the problem (with $v_0 = u_0 - z_\alpha$):

$$(3) \qquad \begin{cases} \partial_t v = -\nu Av - B(v + z_\alpha, v + z_\alpha) - Cv + \alpha z_\alpha + f \\ v(0) = v_0 \end{cases}$$

This transformed system is interpreted as an integral equation

$$(4) \qquad v(t) = S(t)v_0 + \int_0^t S(t-s)B(v(s) + z_\alpha(s) + \alpha z_\alpha(s))ds$$

Weak formulation of Stochastic Navier Stokes Equations

To study the initial value proble (2), one needs to carefully study the properties of stochastic convolution

$$z_\alpha(t) = \int_0^t e^{-\hat{A}(t-s)} G dL(s)$$

which satisfies

$$(3) \qquad \begin{cases} dz_\alpha(t) + (\nu A + C + \alpha) z_\alpha(t) = G dL(t), t \geq 0 \\ z(0) = 0 \end{cases}$$

Research questions

- Well-posedness
  - Existence, uniqueness of **weak** and continuous dependence of initial data, forcing and driving noise.
  - Existence and Uniqueness of a **strong** solution.
- Invariant measures (Existence)
- Random Dynamical System: Existence of random attractors

Concepts of solutions: Weak solutions

Definition

Suppose that $z \in L^4_{loc}([0,T); \mathbb{L}^4(\mathbb{S}^2) \cap H)$, $v_0 \in H$, $f \in V'$. *A weak solution to (2) is a function* $v \in C([0,T);H) \cap L^2_{loc}([0,T);V)$ *which satisfies (3) in a weak sense for any* $\phi \in V$, $T > 0$, *and*

$$\partial_t (v, \phi) = (v_0, \phi) - \nu (v, A\phi) - b(v+z, v+z, \phi) - (\mathbf{C} v, \phi) + (\alpha z + f, \phi).$$

Equivalently, (3) holds as an equality in V' *for a.e.* $t \in [0,T]$.

Now if $f \in H$, and the following regularity is satisfied,

$$v \in L^\infty(0,T;V) \cap L^2(0,T;D(A)),$$

then the solution becomes strong. More precisely,

Concepts of solutions: Strong solutions

Definition (Strong solution)

Suppose that $z \in L^4_{loc}([0,T); \mathbb{L}^4(\mathbb{S}^2) \cap H)$, $v_0 \in V$, $f \in H$. *We say that* u *is a strong solution of the stochastic Navier-Stokes equations (2) on the time interval* $[0,T]$ *if* u *is a weak solution of (2) and in addition*

$$u \in L^\infty(0,T;V) \cap L^2(0,T;D(A)).$$

Problem 1: Well-posedness (Weak sense)

Existence follows from the Galerkin approximation on spheres. (See [3] and reference therein) More precisely, this is established in three steps.

1. Construct approximate solutions (Galerkin approximation)
2. Derive a-priori energy estimates for approximate solutions
   - Here I found some uniform apriori estimate on the solution to the transformed equation, and I show that the estimates exist globally in time.
3. Show approximate solutions converges.
   - Extract convergence subsequence, pass to the limit in the equation.

Uniqueness follows from classical arguments in the spirit of Lion & Prodi.

Existence and Uniqueness of weak solutions

Theorem

Assume that $\alpha \geq 0$, $z \in L^4_{loc}([0,\infty); \mathbb{L}^4(\mathbb{S}^2) \cap H)$, $f \in V'$ and $v_0 \in H$. Then, there exists a unique solution of (4) in the space $D(0,T;H) \cap L^2(0,T;V)$ which belongs to $D(h,T;V) \cap L^2_{loc}(h,T;D(A))$ for all $h > 0$ and $T > 0$. Moreover, if $v_0 \in V$, then $v \in D(0,T;V) \cap L^2_{loc}(0,T;D(A))$ for all $T > 0$. In particular, $v(T,z_n)u^0_n \to v(T,z_n)u_0$ in H. Moreover, if

$$\sum_{l=1}^{\infty} |\sigma_l|^{\beta} \lambda_l^{\beta/2} < \infty\,,$$

then the theorem holds.

Existence and Uniqueness of strong solutions

Theorem

Assume that $\alpha \geq 0$, $z \in L^4_{loc}([0, \infty); \mathbb{L}^4(\mathbb{S}^2) \cap H)$, $f \in H$ and $v_0 \in H$. Then, there exists $\mathbb{P}$-a.s. unique solution of (4) in the space $D(0, T; H) \cap L^2(0, T; V)$, which belongs to $D(\epsilon, T; V) \cap L^2_{loc}(\epsilon, T; D(A))$ for all $\epsilon > 0$, and $T > 0$. Moreover, if $v_0 \in V$, then $u \in D(0, T; V) \cap L^2_{loc}(0, T; D(A))$ for all $T > 0$, $\omega \in \Omega$. Moreover, if

$$\sum_{l=1}^{\infty} |\sigma_l|^{\beta} \lambda_l^{\beta/2} < \infty ,$$

then the theorem holds.

Proof.

We follow a classical fixed point typed argument of Lion& Prodi, also in [2]. This consists of proving Local existence and uniqueness and Uniform apriori estimates. □

What is invariant measures?

Roughly speaking, it is a stationary solution represents the long time behaviour of a given dynamical system.

Markov semigroup $$(P_t\phi)(u_0) = \mathbb{E}_{u_0}\phi(u_t)$$

Invariant measure $$\int P_t(u_0, A)d\mu(u_0) = \mu(A)$$
$$P_t^*\mu = \mu$$

Birkoff Ergodic Theorem $$\frac{1}{T}\int_0^T \phi(u_t)\,dt = \int \phi(u)\,d\mu(u)$$

How to prove invariant measures?

1. It is worth mentioning that the existence of invariant measures was not possible for the deterministic Navier-Stokes Equations. This motivates us to study the same question in "stochastic" sense.
2. In standard case of equations with **Gaussian** noise, existence of invariant measures can be established via three criterions.
   - Markov property
   - Feller property
   - Tightness of probability law

   The first two properties follow from well-posedness. Tightness follows from Krylov-Bogoliubov argument for Markov process.

Fir the SNSE with stable typed noise, existence of invariant measures is a more difficult problem compared to the Gaussian counterpart. This is because the energy estimates for 2nd moments of the solution are not available, which fails the standard argument (SLLN)

Random Dynamical Systems: preliminary

Definition

Given a metric DS $\mathfrak{T}$ and a Polish space (X, d), a map $\varphi : \mathbb{R} \times \Omega \times X \ni (t, \omega, x) \mapsto \varphi(t, \omega)x \in X$ is called a measurable random dynamical system (on X over ϑ), iff

- *φ is $(\mathcal{B}(\mathbb{R}) \times \mathcal{F} \times \mathcal{B}, \mathcal{B})$-measurable.*
- *The trajectories $\varphi(\cdot, \omega)x : \mathbb{R} \to X$ are càdlàg $\forall\, (\omega, x) \in \Omega \times \mathbb{R}$;*
- *φ is ϑ-cocycle:*

$$\varphi(t+s, \omega) = \varphi(t, \vartheta_s \omega) \circ \varphi(s, \omega) \quad \forall \quad s, t \in \mathbb{R}, \ \varphi(0, \omega) = id, \ \forall\, \omega \in \Omega$$

The concept of RDS is a relative new development combining ideas and methods from probability theory and dynamical systems.

Random Dynamical Systems: preliminary

1. **Attractor** is a central notion from mathematical physics, which conveys crucial geometric information about the asymptotic regime of a dynamical system as $t \to \infty$.
2. Now, given a probability space, a **Random Attractor** is a compact random set, invariant for the associated RDS and attracting every bounded random set in its basis of attraction.

More precisely,

Existence of random attractors

Definition

A random set $A : \Omega \to \mathfrak{C}(X)$ *is a random attractor iff*

- *A is a compact random set;*
- *A is* φ*-invariant, i.e.* $\mathbb{P}$*-a.s.*

$$\varphi(t, \omega)A(\omega) = A(\vartheta_t \omega), \tag{1}$$

- *A is attracting, in the sense that, for all* $B \in X$ *it holds*

$$\lim_{t\to\infty} \rho(\varphi(t, \vartheta_{-t}\omega)B(\varphi_{-t}\omega), A(\omega)) = 0.$$

Existence of random attractors: Proof

I proved the existence of random attractor using a simple argument inspired from Flandoli and Crauel [1]. Namely in three steps,

1. Using an apriori estimate for the strong solution. (i.e. a strong solution bounded in V and compact in H)
2. Using these estimates, we prove, respectively the existence of an absorbing ball in H at $t = -1$ and V at $t = 0$.
3. Using compact embedding of Sobolev spaces, we identified a compact absorbing set and consequently deduce the existence of a random attractor.

Fubini theorem for stochastic integral w.r.t. large and small jumps

We proved a new version of stochastic Fubini Theorem for the subordinated process $L(t) = W(Z(t))$, where W is a cylindrical Wiener process on Hilbert space and Z is a subordinator process. Our stochastic Fubini theorem capture both large and small jumps that states the following.

Theorem

$$\int_0^T \left(\int_0^s \Phi(s,\sigma) dL(\sigma) \right) ds = \int_0^T \left(\int_\sigma^T \Phi(s,\sigma) ds \right) dL(\sigma). \tag{5}$$

Some technical conditions required are

1. U and E be separable Hilbert spaces.
2. $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $T > 0$ is fixed
3. The mapping $[0, T] \times [0, T] \times \Omega \ni (s, \sigma, \omega) \mapsto \Phi(s, \sigma, \omega) \in L(U, E)$ is a strongly measurable with respect to the σ-algebra $\mathcal{B}([0, T]) \otimes \mathcal{P}_T$, where $\mathcal{P}_T$ stands for the predictable σ-algebra in $[0, T] \times \Omega$. More precisely, we assume that for every $y \in U$ the mapping $[0, T] \times [0, T] \times \Omega \ni (s, \sigma, \omega) \mapsto \Phi(s, \sigma, \omega)y \in E$ is measurable with respect to the σ-algebra $\mathcal{B}([0, T]) \otimes \mathcal{P}_T$.
4. Furthermore, assume that L is a U-valued Lévy process defined as $L(t) := W(Z(t))$, $Z(t)$ is a subordinator process belonging to $\mathrm{Sub}(p)$, i.e. $Z(t)$ has intensity measure satisfying

$$\rho(\{0\}) = 0, \quad \int_1^\infty \rho(d\xi) + \int_0^1 \xi \rho(d\xi) < \infty \quad \int_0^1 \xi^{\frac{p}{2}} \rho(d\xi) < \infty,$$

where ρ and ν are respectively the intensity measure on $\mathbb{R}$ and Lévy measure on U_0. One relates ρ and ν as

$$\nu(\Gamma) = \int_0^\infty \zeta_s(\Gamma)\rho(ds), \quad \Gamma \in \mathcal{B}(Y).$$

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Thank You!