APPENDIX

MATHEMATICAL APPENDICES

This appendix provides an overview of some of the mathematical concepts that you will find useful as you study microeconomics. In addition to introducing and summarizing the concepts, we will illustrate them by referring to selected examples from the textbook.

A.1 FUNCTIONAL RELATIONSHIPS

Economic analysis often requires that we understand how to relate economic variables to one another. There are three primary ways of expressing the relationships among variables: graphs, tables, and algebraic functions. For example, Figure A.1 contains information about the demand for paint in a market. The table at the bottom of the figure indicates how much paint consumers would purchase at various prices. For example, if the price of paint is $10 per liter, consumers in the market will buy 3 million liters per year. This information is also shown in the graph at point T. By convention, economists draw demand curves with price on the vertical axis and quantity on the horizontal axis. Since quantity is measured along the horizontal axis (in millions of liters), point T has the coordinates (3,10). Similarly, at a price of $8 per liter, consumers would buy 4 million liters [indicated on the graph at point U, with coordinates (4,8)]. Other points from data in the table are plotted at points S, V, and W. As the figure shows, tables and graphs can be very helpful in showing the relationships among variables.

We also often find it useful to express economic relationships with equations. We can express the relationship between price and quantity using functional notation:

$$Q = f(P) \tag{A.1}$$

where the function f tells us Q, the quantity of paint consumed (measured in millions of liters) when the price is P (measured in $ per liter). A specific function that describes the data in Figure A.1 is

$$Q = 8 - 0.5P \tag{A.2}$$

Equation (A.2) is therefore the demand function that contains all of the points shown in Figure A.1. We have written equations (A.1) and (A.2) with Q on the left-hand side and P on the right-hand side. This is the natural way to write a demand function if we want to ask the following question: "How does the number of units sold depend on the price?" The variable on the left-hand side (Q) is the *dependent* variable, and the variable on the right-hand side (P) is the *independent* variable.

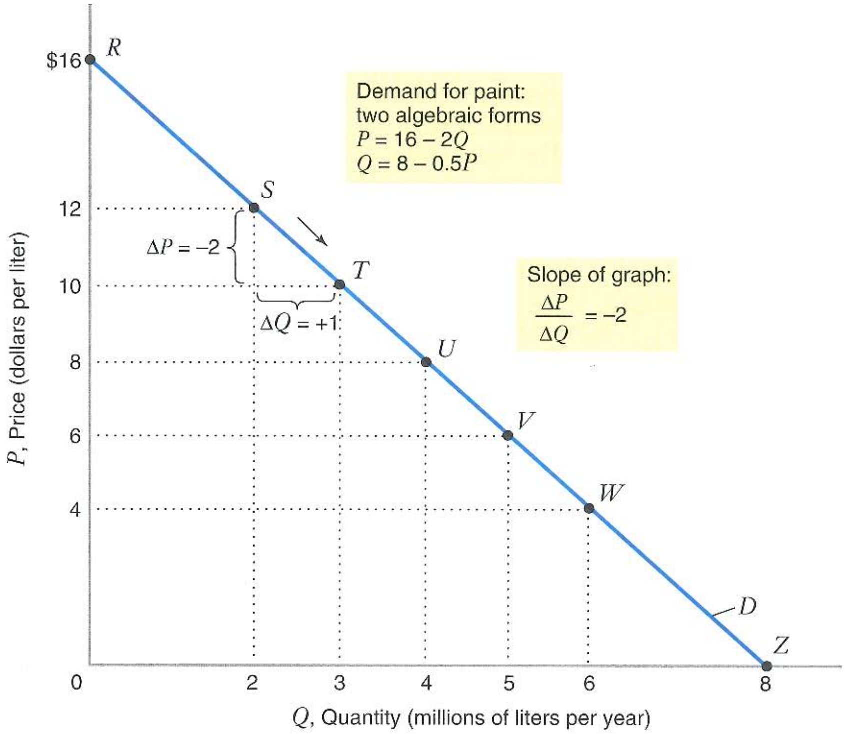
Let's use equation (A.2) to find out how much consumers will buy when the price is $8 per liter. When $P = 8$, then $Q = 8 - 0.5(8) = 4$. Thus, consumers will buy 4 million liters per year. To emphasize that Q is a function of P, equation (A.2) might also be written as $Q(P) = 8 - 0.5P$.

We might also use a demand function to answer a different question: "What price will induce consumers to demand any specified quantity?" Now we are asking how the price depends on the quantity we wish to sell. In other words, how does P depend on Q? We can let P take the role of the dependent variable and Q the independent variable. To see how P depends on Q, we can "invert" equation (A.2) by solving it for P in terms of Q. When we do so, we find that the inverse demand function can be expressed as equation (A.3):

$$P = 16 - 2Q \tag{A.3}$$

All of the combinations of price and quantity in the table in Figure A.1 also satisfy this equation. Let's use equation (A.3) to find out what price will make consumers demand 4 million liters per year. When we substitute $Q = 4$ into the equation, we find that $P = 16 - 2(4) = 8$. Thus, if we want consumers to demand 4 million liters per year, we should set the price at $8 per liter. To emphasize that P is a function of Q, we might also write equation (A.3) as $P(Q) = 16 - 2Q$.

When we draw a demand curve with P on the vertical axis and Q on the horizontal axis, the slope of the graph is just the "rise over the run," that is, the change in price (the vertical distance) divided by the change in quantity (the horizontal distance) as we move along the curve. For example, as we move from point S to point T, the change in price is $\Delta P = -2$, and the change in quantity is $\Delta Q = +1$. Thus, the slope is $\Delta P/\Delta Q = -2$. Since the demand curve in the example is a straight line, the slope is a constant everywhere on the curve. The vertical intercept of the



Point on Graph	Price of Paint ($ per liter)	Millions of Liters Purchased per Year
S	12	2
T	10	3
U	8	4
V	6	5
W	4	6

FIGURE A.1 Functional Relationships: Example with Demand Curve
The graph and table show the relationship between the quantity of paint purchased in a market (Q) and the price of paint (P). For example, the first row of the table indicates that when the price is $12 per liter, 2 million liters would be purchased each year. This corresponds to point S. The functional relationship between quantity and price can be represented algebraically in two ways. If we write price as a function of quantity, the form of the demand curve is $P = 16 - 2Q$. Equivalently, we may write quantity as a function of price, with $Q = 8 - 0.5P$.

demand curve occurs at point R, at a price of $16 per liter. This means that no paint would be sold at that price or any higher price.[1] If the price of paint were zero, then people would demand 8 million liters. This is the horizontal intercept in the graph, at point Z.

[1]You may recall from a course in algebra that the equation of a straight line is $y = mx + b$, where y is plotted on the vertical axis and x is measured on the horizontal axis. With such a graph m is the slope of the graph and b is the vertical intercept. In Figure A.1 the "y" variable is P because it is plotted on the vertical axis and the "x" variable (the one on the horizontal axis) is Q. Thus, instead of having the equation $y = -2x + 16$, with the example we have $P = -2Q + 16$. The slope is -2 and the vertical intercept is 16.

For practice drawing supply and demand curves from an equation, you might review Learning-By-Doing Exercises 2.1 and 2.2.

LEARNING-BY-DOING EXERCISE A.1

Graphing Total Cost

This example will help you see how to draw a graph and construct a table for a total cost function. Suppose that the function representing the relationship between the total costs of production (C) and the quantity produced (Q) is as follows:

$$C(Q) = Q^3 - 10Q^2 + 40Q \quad \text{(A.4)}$$

Problem In a table, show the total cost of producing each of the amounts of output: $Q = 0$, $Q = 1$, $Q = 2$, $Q = 3$, $Q = 4$, $Q = 5$, $Q = 6$, $Q = 7$. Draw the total cost function on a graph with total cost on the vertical axis and quantity on the horizontal axis.

Solution The first two columns of Table A.1 show the total cost for each level of output. For example, to produce three units, we evaluate $C(Q)$ when $Q = 3$. We find that $C(3) = (3)^3 - 10(3)^2 + 40(3) = 57$. (Do not worry about the other columns in the table. We will refer to them later.)

The total cost curve is plotted in panel (a) in Figure A.2. [Do not worry about panel (b). We will refer to it later.]

A.2 WHAT IS A "MARGIN"?

Decision makers are often interested in the **marginal value** of a dependent variable. The marginal value measures the *change* in a dependent variable associated with a one-unit *change* in an independent variable. The marginal cost therefore measures the rate of change of cost, that is, $\Delta C/\Delta Q$. A decision maker may be interested in the marginal cost because it tells her how much *more* it will cost to produce one *more* unit.

Consider once again Table A.1, which shows the total cost based on equation (A.4). The dependent variable is total cost, and the independent variable is the quantity produced. The table shows two ways of measuring the marginal cost. Column three illustrates the first way by showing how the total cost changes when one more unit is produced. The column is labeled "Arc" Marginal Cost because it measures the change in total cost over an *arc*, or region, over which the quantity increases by one unit. For example, when the quantity increases from $Q = 2$ to $Q = 3$, total cost increases from $C(2) = 48$ to $C(3) = 57$. Thus, the marginal cost over this region of the cost curve is $C(3) - C(2) = 9$. Similarly, the marginal cost over the arc from $Q = 5$ to $Q = 6$ is $C(6) - C(5) = 21$.

We can also represent the marginal cost on a graph. Consider Figure A.2(a). The vertical axis measures total cost, and the horizontal axis indicates the quantity produced. We can show that the arc marginal cost approximates the slope of the total cost curve over a region of interest. For example, let's determine the marginal cost when we increase quantity from $Q = 5$ (at point A) to $Q = 6$ (at point B). We can construct a straight line segment connecting points A and B. The slope of this segment is the change in cost (the "rise"), which is 21, divided by the change in the quantity (the "run"), which is 1. Thus, the slope of the

TABLE A.1 Relating Total, Average, and Marginal Cost with a Table*

(1) Quantity Produced (units) Q	(2) Total Cost (\$) C	(3) "Arc" Marginal Cost (\$/unit) $C(Q) - C(Q-1)$	(4) "Point" Marginal Cost (\$/unit) dC/dQ	(5) Average Cost (\$/unit) C/Q
0	0		40	
		$C(1) - C(0) = 31$		
1	31		23	31
		$C(2) - C(1) = 17$		
2	48		12	24
		$C(3) - C(2) = 9$		
3	57		7	19
		$C(4) - C(3) = 7$		
4	64		8	16
		$C(5) - C(4) = 11$		
5	75		15	15
		$C(6) - C(5) = 21$		
6	96		28	16
		$C(7) - C(6) = 37$		
7	133		47	19

*The table shows the values of total cost, marginal cost, and average cost curves when the cost function is $C(Q) = Q^3 - 10Q^2 + 40Q$.

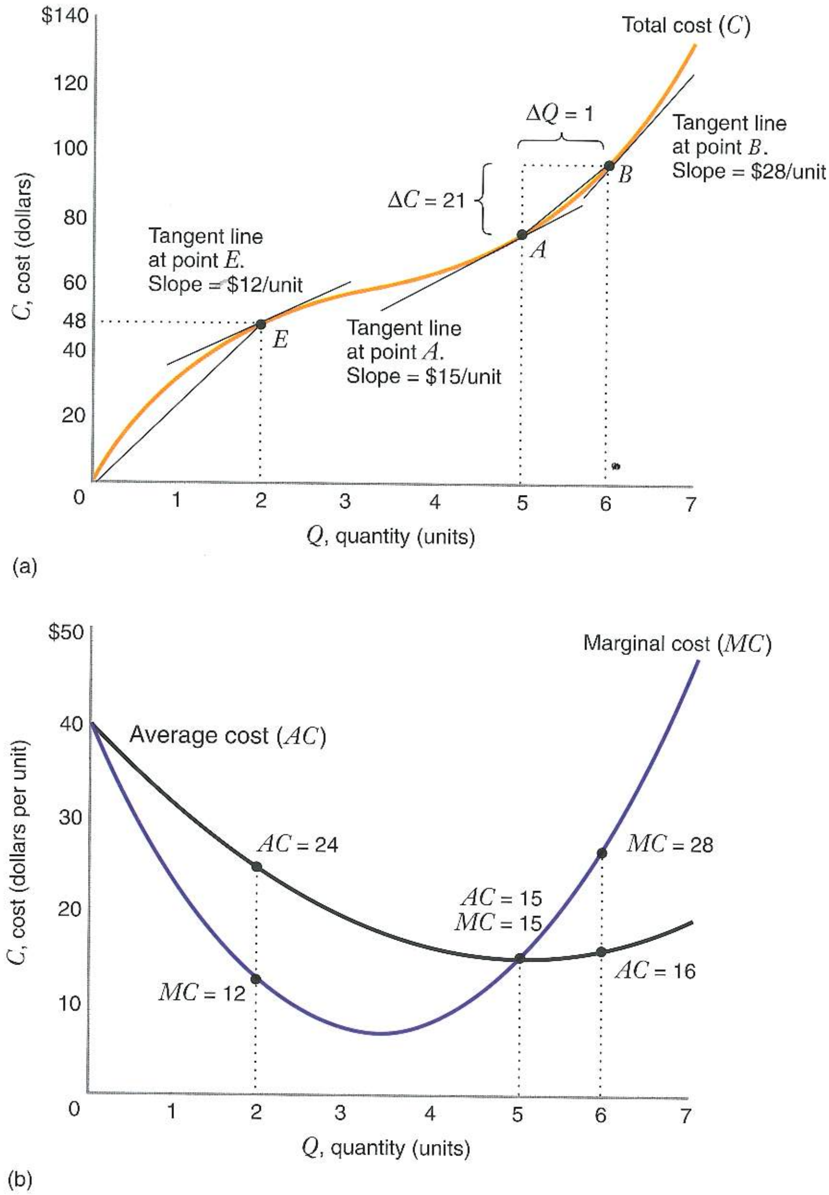


FIGURE A.2 Relating Total, Average, and Marginal Cost Graphically
Panel (a) shows the total cost of producing any specified amount of output. The units on the vertical axis of the top graph are monetary (dollars). The bottom graph shows the marginal and average cost curves corresponding to the total cost curve in the top graph. The units on the vertical axis of the bottom graph are dollars per unit. In panel (b), the *value of the marginal cost* at each quantity is the same as the *slope of the total cost* in panel (a).

segment connecting points A and B is the arc measure of the marginal cost. Note that over the region the slope of the total cost function changes. The arc marginal cost provides us with an *approximate* value of the slope of the graph over the region of interest.

Instead of approximating the marginal cost by measuring it over an *arc*, we could measure the marginal cost at any specified *point* (i.e., at a particular quantity). For example, *at* point A, the slope of the total cost curve is the slope of a line tangent to the total cost curve at A. The slope of this tangent line measures the rate of change of total cost at point A. Thus, *the slope of the line tangent to the total cost curve at point* A *measures the marginal cost at point* A. Similarly, the slope of the line tangent to the total cost curve at point B measures the marginal cost there.

How can we determine the value of the marginal cost at a point? One way to do this would be to construct a carefully drawn graph, and then measure the slope of the line tangent to the graph at the point of interest. For example, the slope of the total cost curve at point B (when $Q = 6$) is

\$28 per unit. Thus, the marginal cost when $Q = 6$ is \$28 per unit. Similarly, the marginal cost when $Q = 2$ is \$12 per unit because that is the slope of the line tangent to the total cost curve at point E. Column 4 in Table A.1 shows the exact "point" value of the marginal cost at each quantity.

As we will show later, instead of drawing and carefully measuring the slope of the graph, we can also use calculus to find the marginal cost at a point. (See Learning-By-Doing Exercise A.5.)

Relating Average and Marginal Values

The **average value** is the total value of the dependent variable divided by the value of the independent variable. Table A.1 also shows the average cost, that is, total cost divided by output, C/Q. The average cost is calculated in column 5.

We can also show the average cost curve on a graph. Consider the top graph in Figure A.2. We can show that the average cost at any quantity is the slope of a segment connecting the origin with the total cost curve. For example, let's determine the average cost when the quantity is $Q = 2$ (at point E). We can construct a line segment $0E$ connecting the origin to point E. The slope of this segment is the total cost (the *rise*), which is 48, divided by the quantity (the *run*), which is 2. Thus, the slope of the segment is the average cost, 24.

The value of the average cost is generally different from the value of the marginal cost. For example, the average cost at $Q = 2$ (again, the slope of the segment connecting the origin to point E) is 24, while the marginal cost (the slope of the line tangent to the total cost curve) is 12. We have plotted the values of the marginal and average cost on Figure A.2(b).

We need one graph to plot the value of the total cost and another to show the values of the average and marginal cost curves. The units of total cost are monetary, for example, dollars. Thus, the units along the vertical axis in the top graph are measured in *dollars*. However, the units of marginal cost, $\Delta C/\Delta Q$, and average cost, C/Q, are *dollars per unit*. The dimensions of total cost differ from the dimensions of average and marginal cost.

It is important to understand the relationship between marginal and average values. Since the marginal value represents the rate of change in the total value, the following statements must be true:

- The average value must *increase* if the marginal value is *greater* than the average value.
- The average value must *decrease* if the marginal value is *less than* the average value.
- The average value will be *constant* if the marginal value *equals* the average value.

These relationships hold for the marginal and average values of *any* measure. For example, suppose the average height of the students in your class is 180 centimeters. Now a new student, Mr. Margin, whose height is 190 centimeters, enters the class. What happens to the average height in the class? Since Mr. Margin's height exceeds the average height, the average height must increase.

Similarly, if Mr. Margin's height is 160 centimeters, the average height in the class must decrease. Finally, if Mr. Margin's height is exactly 180 centimeters, the average height in the class will remain unchanged.

This basic arithmetic insight helps us to understand the relationship between average and marginal product (see Figures 6.3 and 6.4), average and marginal cost (see Figures 8.7, 8.8, 8.9, and 8.10), average and marginal revenues for a monopolist (see Figures 11.2 and 11.4), and average and marginal expenditures for a monopsonist (see Figure 11.18).

LEARNING-BY-DOING EXERCISE A.2

Relating Average and Marginal Cost

This example will reinforce your understanding of the relationship between marginal and average values. Consider the average and marginal cost curves in Figure A.2(b).

Problem Use the relationship between marginal and average cost to explain why the average cost curve is rising, falling, or constant at each of the following quantities:

(a) $Q = 2$

(b) $Q = 5$

(c) $Q = 6$

Solution

(a) When $Q = 2$, the marginal cost curve lies *below* the average cost curve. Thus the average cost curve must be falling (have a negative slope).

(b) When $Q = 5$, the marginal cost curve is *equal* to the average cost curve (they intersect). Thus the average cost curve must be neither increasing nor decreasing (have a slope of zero) at that level of output. In this case, we see that this means we are at the minimum point on the average cost curve. (We will discuss minimum and maximum points of functions below.)

(c) When $Q = 6$, the marginal cost curve lies *above* the average cost curve. Thus the average cost curve must be rising (have a positive slope).

A.3 DERIVATIVES

In Figure A.2, we showed that one way to find the marginal cost is to plot the total cost curve and carefully measure the slope at each quantity. This is a tedious process, and it is not always easy to draw a precise tangent line and measure its slope accurately. Instead, we can use the powerful techniques of differential calculus to find the marginal cost or other marginal values we might want to know about.

Let's suppose that y is the dependent variable and x the independent variable in a function:

$$y = f(x)$$

Consider Figure A.3, which depicts the value of the dependent variable on the vertical axis and the value of the independent variable on the horizontal axis.

As we have already discussed, if y measures the *total* value, then the slope of the graph at any point measures the marginal value. (For example, if y measures total cost and x the quantity, then the slope of the cost function is the marginal cost at any quantity.) We can use a concept called a **derivative** to help us find the slope of a function at any point, such as point A in the figure.

We illustrate how a derivative works using Figure A.3. Let's begin with an algebraic approximation of the slope of the graph. The function $y = f(x)$ is curved, so we know that its slope will change as we move along the curve. We might approximate the slope of the graph at F by selecting two points on the curve, E and F. Let's draw a segment connecting these two points and call the segment EF. The slope of the segment is just the rise ($\Delta y = y_3 - y_1$) over the run ($\Delta x = x_3 - x_1$). Thus, the slope of EF is $\Delta y/\Delta x = (y_3 - y_1)/(x_3 - x_1)$. The graph indicates that the slope of EF will not exactly measure the slope of the tangent line at E, but it does give us an approximation of the slope. As the graph is drawn, the slope of EF will be less than the slope of the line tangent to the function at point E.

We can get a better approximation to the slope at E if we choose another point on the graph closer to E, such as point B. Let's draw a segment connecting these two points and call the segment EB. The slope of the segment EB is $\Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1)$. Once again, the graph tells us that the slope of EB will not exactly measure the slope of the tangent line at E (it still underestimates the slope at E), but it does give us a better approximation of the slope at E.

If we choose a point very close to E, the approximate calculation of the slope will approach the actual slope at point E. When the two points become very close to each other, Δx approaches zero. The value of the approximation as Δx approaches zero is the derivative, written dy/dx. We express the idea of the derivative mathematically as follows:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \qquad \text{(A.5)}$$

where the expression "$\lim_{\Delta x \to 0}$" tells us to evaluate the slope $\Delta y/\Delta x$ "in the limit" as Δx approaches zero. The value of the derivative dy/dx at point E is the slope of the graph at that point.

A.4 HOW TO FIND A DERIVATIVE

In this section we will show you how to find a derivative for a few of the functional forms commonly encountered in economic models. You can refer to any standard calculus book to learn more about derivatives, including derivatives of other types of functions not included here.

Derivative of a Constant

If the dependent variable y is a constant, its derivative with respect to x is zero. In other words, suppose $y = k$, where k is a constant. Then $dy/dx = 0$.

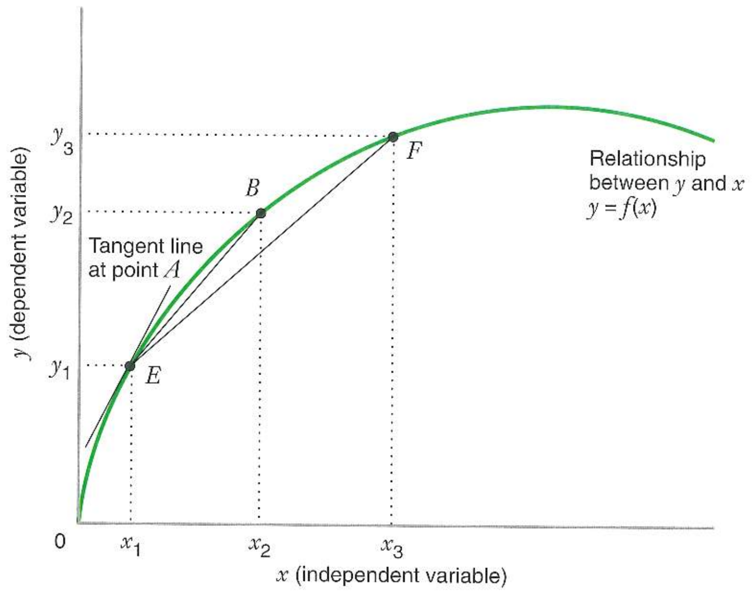


FIGURE A.3 The Meaning of a Derivative
When $x = x_1$, the derivative of y with respect to x (i.e., dy/dx) is the slope of the line tangent to point E.

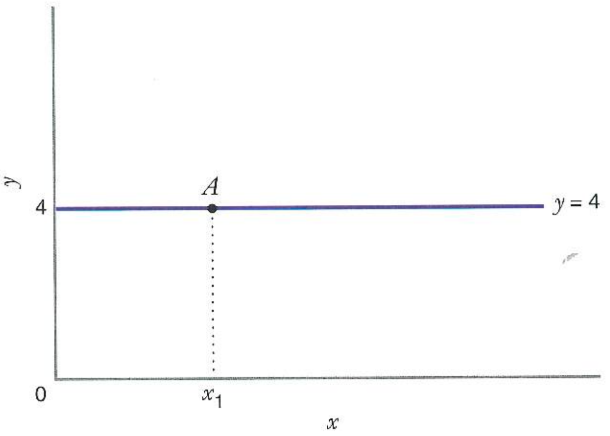


FIGURE A.4 Derivative of a Constant
The graph shows the function $y = 4$. Since the value of y does not vary as x changes, the graph is a horizontal line. The slope of the graph is always 0. The derivative $(dy/dx) = 0$ confirms the fact that the slope of the function is always 0.

Consider, for example, the function $y = 4$. Figure A.4 graphs this function. We can find the slope of this function in two ways. First, because the graph is flat, we know that the value of y does not vary as x changes. Thus, by inspection we observe that the slope of the graph is zero.

The second way to find the slope is to take the derivative. Since the derivative of a constant is zero, then $dy/dx = 0$. Since the derivative is always zero, the slope of the graph of the function $y = 4$ is always zero.

Derivative of a Power Function

A power function has the form:

$$y = ax^b \tag{A.6}$$

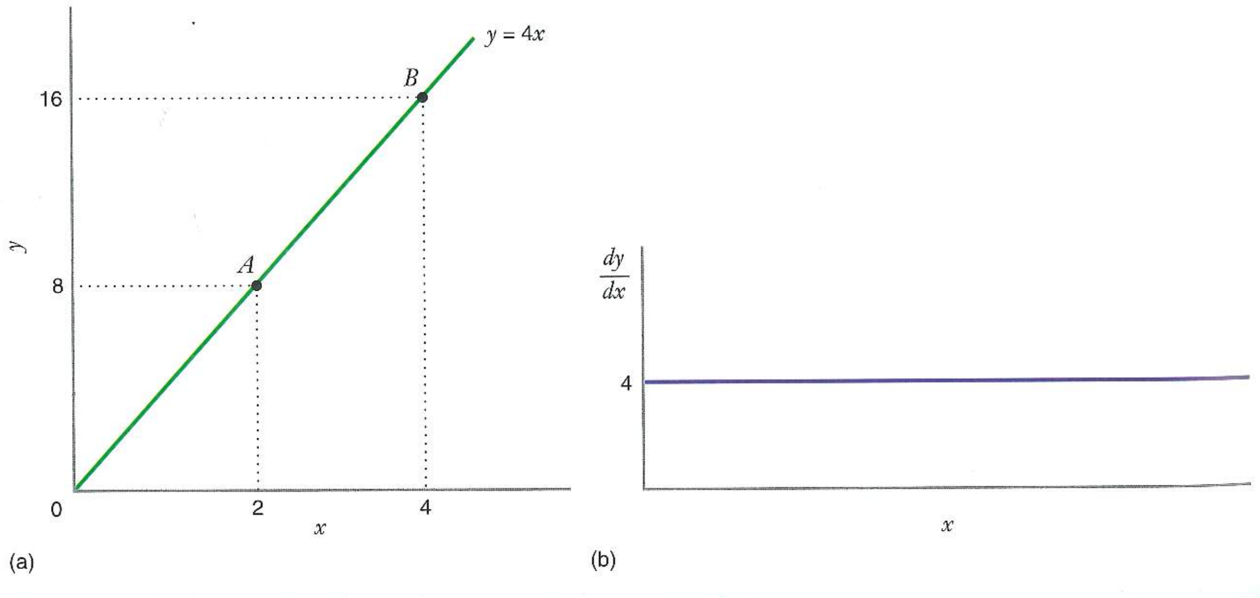


FIGURE A.5 Derivative of $y = 4x$
Panel (a) shows the function $y = 4x$. The slope of this graph is 4. Using the rule for the derivative of a power function, we find that the derivative $(dy/dx) = 4$, and plot the derivative in panel (b). The fact that the derivative is always 4 means that that slope of the function in panel (a) is always 4.

where a and b are constants. For such a function the derivative is

$$\frac{dy}{dx} = bax^{b-1} \tag{A.7}$$

Let's consider an example. Suppose $y = 4x$. The left graph of Figure A.5 shows this function. Since the function is a straight line, it has a constant slope. We can find the slope in two ways. First, take any two points on the graph, such as A and B. We find that the slope $\Delta y/\Delta x = (16-8)/(4-2) = 4$.

The second way to find the slope is to take the derivative. We recognize that $y = 4x$ is a power function like the one in equation (A.6), with $a = 4$ and $b = 1$. As equation (A.5) shows, the derivative is $dy/dx = bax^{b-1} = 4x^0 = 4$. Since the derivative dy/dx is always 4, the slope of the graph of the function $y = 4x$ is always 4.

LEARNING-BY-DOING EXERCISE A.3

Derivative of a Power Function

Consider the function $y = 3x^2$, shown in Figure A.6(a).

Problem Find the slope of this function when
(a) $x = -1$ (b) $x = 0$ (c) $x = +2$

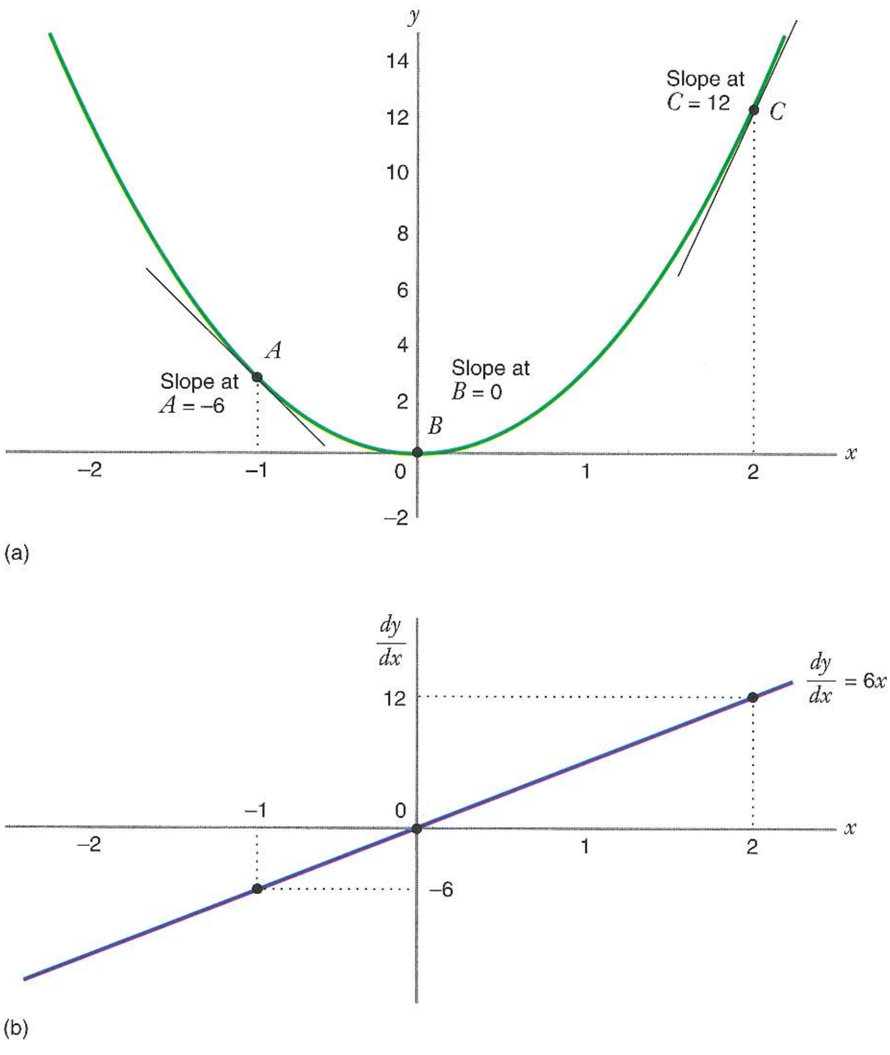


FIGURE A.6 Derivative of $y = 3x^2$
Panel (a) shows the function $y = 3x^2$. The slope of this graph varies as x changes. Using the rule for the derivative of a power function, we find that the derivative $(dy/dx) = 6x$, and plot the derivative in panel (b). When $x = -1$, the value of the derivative is -6. Thus the *slope* in panel (a) is -6 when $x = -1$. Similarly, the derivative tells us that the slope in panel (a) is zero when $x = 0$ and 12 when $x = 2$.

Solution

(a) We recognize that $y = 3x^2$ is a power function like the one in equation (A.6), with $a = 3$ and $b = 2$. As equation (A.7) shows, the derivative is $dy/dx = bax^{b-1} = 6x$. [The graph of the derivative is shown in Figure A.6(b).] Thus, the slope of the function $y = 3x^2$ will be $6x$. When $x = -1$, the value of the derivative is $dy/dx = 6(-1) = -6$. This tells us that the slope of the function $y = 3x^2$ [at point A in panel(a)] is -6.

(b) When $x = 0$, the value of the derivative is $dy/dx = 6(0) = 0$. Thus, the slope of the function $y = 3x^2$ at point B is 0.

(c) When $x = 2$, the value of the derivative is $dy/dx = 6(2) = 12$. Therefore, the slope of the function $y = 3x^2$ at point C is 12.

To summarize one of the uses of derivatives, consider Figure A.6. We could determine the slope of the curve in panel (a) at any point in two ways. First, we could graph the curve carefully, and construct a line segment tangent to the curve. For example, if we want to determine the slope at point A, we could draw a line tangent to A, and then measure the slope of the tangent line. If we did this properly, we would find that the slope at A is -6. However, this is a cumbersome approach and could easily lead to error, especially because the slope of the curve varies as x changes. An easier and more reliable way to find the slope is to find the derivative, and then calculate the value of the derivative for any point at which we want to know the slope.

LEARNING-BY-DOING EXERCISE A.4

Utility and Marginal Utility

In Chapter 3 (see Figure 3.2), we examined the utility function $U(y) = \sqrt{y}$. Here U is the dependent variable and y the independent variable. We observed that the corresponding marginal utility function is $MU(y) = 0.5\sqrt{y}$.

Problem Show that this marginal utility is correct.

Solution The marginal utility $MU(y)$ is the slope of the utility function, that is, the derivative dU/dy. We can easily find this derivative because $U(y) = \sqrt{y}$ is a power function. It may help to rewrite the utility function as $U(y) = y^{(1/2)}$. This is a power function with $U = ay^b$, where $a = 1$ and $b = 1/2$. The derivative is then $dU/dy = bay^{b-1} = (1/2)y^{(1/2)^{-1}} = 0.5y^{-1/2} = 0.5\sqrt{y}$.

Derivatives of a Natural Logarithm

A logarithmic function has the form:

$$y = \ln x \tag{A.8}$$

where "ln" denotes the natural logarithm of a number. The derivative of the natural logarithm is

$$\frac{dy}{dx} = \frac{1}{x} \tag{A.9}$$

Derivatives of Sums and Differences

Suppose $f(x)$ and $g(x)$ are two different functions of x. Suppose further that y is the sum of f and g, that is,

$$y = f(x) + g(x)$$

Then the derivative of y with respect to x is the *sum* of the derivatives of f and g. Thus,

$$\frac{dy}{dx} = \frac{df}{dx} + \frac{dg}{dx}$$

As an example, assume that $f(x) = 5x^2$ and that $g(x) = 2x$. Both f and g are power functions, with the derivatives $df/dx = 10x$ and $dg/dx = 2$. If $y = f(x) + g(x) = 5x^2 + 2x$, then $dy/dx = (df/dx) + (dg/dx) = 10x + 2$.

Similarly, if y is the difference between f and g, that is,

$$y = f(x) - g(x)$$

then the derivative of y with respect to x is the *difference* of the derivatives of f and g:

$$\frac{dy}{dx} = \frac{df}{dx} - \frac{dg}{dx}$$

LEARNING-BY-DOING EXERCISE A.5

Derivatives of Sums and Differences

Consider the cost function from Learning-By-Doing Exercise A.1:

$$C(Q) = Q^3 - 10Q^2 + 40Q$$

Problem Find the marginal cost when

(a) $Q = 2$

(b) $Q = 5$

(c) $Q = 6$

Solution The marginal cost $MC(Q)$ is the derivative of the total cost function dC/dQ. The total cost function is made up of three terms involving the sums and differences of power functions. Thus, $MC(Q) = 3Q^2 - 20Q + 40$.

(a) When $Q = 2$, the marginal cost is $MC(2) = 3(2)^2 - 20(2) + 40 = 12$. This marginal cost is the slope in panel (a) (the total cost curve) in Figure A.2 when the quantity is 2. The numerical value of the marginal cost is plotted in panel (b) of the same figure.

(b) When $Q = 5$, the marginal cost is $MC(5) = 3(5)^2 - 20(5) + 40 = 15$.

(c) When $Q = 6$, the marginal cost is $MC(6) = 3(6)^2 - 20(6) + 40 = 28$.

Note that the marginal costs calculated in this problem are the ones in column 4 of Table A.1.

Derivatives of Products

Suppose y is the product of $f(x)$ and $g(x)$, that is,

$$y = f(x)g(x)$$

Then the derivative of y with respect to x is

$$\frac{dy}{dx} = f\frac{dg}{dx} + g\frac{df}{dx}$$

As an example, assume that $f(x) = x^2$ and that $g(x) = (6 - x)$. The function f is a power function, while the function g is the sum of power functions. Their derivatives are thus $df/dx = 2x$ and $dg/dx = -1$. If $y = f(x)g(x) = x^2(6 - x)$, then $dy/dx = f(dg/dx) + g(df/dx) = x^2(-1) + (6 - x)(2x) = -3x^2 + 12x$.

As a check on this answer, we could first expand the function $y = x^2(6 - x) = 6x^2 - x^3$, and then take the derivative of this difference of power functions to get $dy/dx = 12x - 3x^2$.

Derivatives of Quotients

Suppose y is the quotient of $f(x)$ and $g(x)$, that is,

$$y = \frac{f(x)}{g(x)}$$

Then the derivative of y with respect to x is

$$\frac{dy}{dx} = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

As an example, assume once again that $f(x) = x^2$ and that $g(x) = (6 - x)$. As before, both f and g are power functions, with the derivatives $df/dx = 2x$ and $dg/dx = -1$. If

$$y = \frac{f(x)}{g(x)} = \frac{x^2}{(6 - x)}$$

then

$$\frac{dy}{dx} = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} = \frac{(6 - x)(2x) - (x^2)(-1)}{(6 - x)^2}$$

$$= \frac{12x - x^2}{(6 - x)^2}$$

There are other rules for finding derivatives for many other types of functions. However, the rules we have discussed in this section are the only ones you need to analyze the material covered in this book using calculus.

To sum up, derivatives are useful in helping us to understand and calculate many of the "marginal" concepts in economics. Three of the most commonly encountered marginal concepts are marginal utility, marginal cost, and marginal revenue.

- Suppose the function measuring total utility is $U(Q)$. Then the value of the derivative dU/dQ at any particular Q is the *slope* of the total utility curve *and* the marginal utility at that quantity. (See Learning-By-Doing Exercise A.4 and Figure 3.2.)
- Suppose the function measuring total cost is $C(Q)$. Then the value of the derivative dC/dQ at any particular Q is the *slope* of the total cost curve *and* the marginal cost at that quantity. (See Learning-By-Doing Exercise A.5, Table A.1, and Figure A.2.)
- Suppose the function measuring total revenue is $R(Q)$. Then the value of the derivative dR/dQ at any particular Q is the *slope* of the total revenue curve *and* the marginal revenue at that quantity.

A.5 MAXIMIZATION AND MINIMIZATION PROBLEMS

We can use derivatives to find where a function reaches a maximum or minimum. Suppose y, the dependent variable, is plotted on the vertical axis of a graph and x, the independent variable, is measured along the horizontal axis. The main idea is this: *A maximum or a minimum can only occur if the slope of the graph is zero.* In other words, at a maximum or a minimum, the derivative dy/dx must equal zero.

Let's consider an example of a maximum. Figure A.7 shows a graph of the function $y = -x^2 + 6x + 1$. We know that at a maximum of the function, the slope will be zero. Since the slope is just the derivative, we look for the value of x that makes the derivative equal to zero. We observe that y is a sum of power terms, with the derivative $dy/dx = -2x + 6$. At the maximum, the derivative is zero (i.e., $dy/dx = -2x + 6 = 0$). The derivative becomes zero when $x = 3$. Thus, the maximum value of y will then be $y = -3^2 + 6(3) + 1 = 10$.

Now let's consider a function that has a minimum. Consider again Figure A.6, showing a graph of the function $y = 3x^2$. We can use a derivative to verify that the function has its minimum at $x = 0$. We know that at the minimum of the function, the slope will be zero. Since the slope is just the derivative, we need to find the value of x that makes the

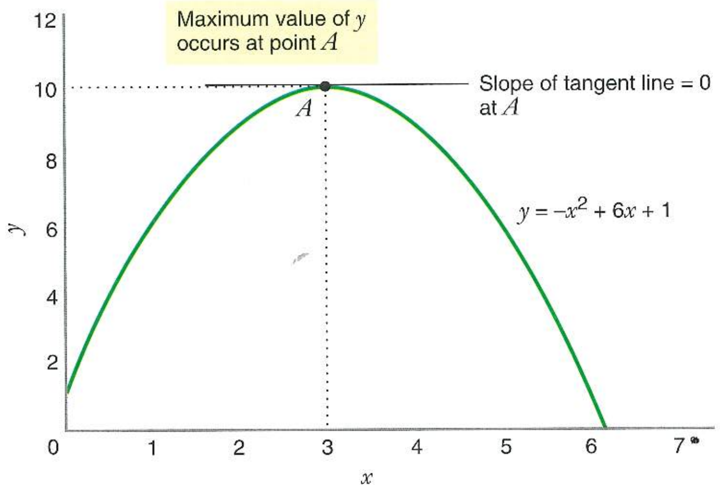


FIGURE A.7 Maximizing a Function
The graph illustrates that a function reaches its maximum when the slope is 0. At point A, when $x = 3$, y achieves its maximum value ($y = 10$). The slope of the curve—and, equivalently, the value of the derivative (dy/dx)—is 0 at point A.

derivative equal to zero. As we showed above, the derivative is $dy/dx = 6x$. At the minimum, the derivative is zero (i.e., $dy/dx = 6x = 0$). The derivative therefore becomes zero when $x = 0$. Thus, the minimum value of y will occur when $x = 0$.

As the two examples show, when the derivative is zero, we may have either a maximum or a minimum. If we observe that $dy/dx = 0$, from that information alone we cannot distinguish between a maximum and a minimum. To determine whether we have found a maximum or a minimum, we need to examine the *second derivative* of y with respect to x, denoted by d^2y/dx^2. The second derivative is just the derivative of the first derivative dy/dx. In other words, the first derivative (dy/dx) tells us the slope of the graph. The second derivative tells us whether the *slope* is increasing or decreasing as x increases. If the second derivative is negative, the slope is becoming less positive (or more negative) as x increases. If the second derivative is positive, the slope is becoming more positive (or less negative) as x increases.

- If we are at a point at which $dy/dx = 0$ and $d^2y/dx^2 < 0$, then that point is a maximum point on the function.
- If we are at a point at which $dy/dx = 0$ and $d^2y/dx^2 > 0$, then that point is a minimum point on the function.

To use the second derivative to see if we have found a maximum or a minimum, consider once again the function $y = -x^2 + 6x + 1$, shown in Figure A.7. We have already found that the slope of the graph is zero when $x = 3$, the value of x that made the derivative $dy/dx = -2x + 6$ equal to zero. We can verify that the graph reaches a maximum (and not a minimum) by examining the second derivative. The derivative of $-2x + 6$ with respect to x is the second derivative; thus $d^2y/dx^2 = -2$. Since the second derivative is negative, the slope of the graph is becoming less positive as we approach $x = 3$ from the left, and becomes more negative as we move to the right of $x = 3$. This verifies that the graph does achieve a maximum when $x = 3$.

Similarly, we can use a second derivative to show that the graph in Figure A.6 achieves a minimum (not a maximum) when $x = 0$. We have already found that the slope of the graph is zero when $x = 0$, the value of x that made the derivative $dy/dx = 6x$ equal to zero. The derivative of $6x$ with respect to x is the second derivative; thus $d^2y/dx^2 = 6$. Since the second derivative is positive, the slope of the graph is becoming less negative as we approach $x = 0$ from the left, and becomes ever more positive as we move to the right of $x = 0$. This verifies that the graph does achieve a minimum when $x = 0$.[2]

[2]The analysis in this appendix shows how to apply derivatives to find a *local* maximum or a *local* minimum. However, many functions will have more than one maximum or minimum. To find the *global* maximum for a function, you would have to compare the values of all of the local maxima, and then choose the one for which the function attains the highest value. Similarly, to find the *global* minimum for a function, you would have to compare the values of all of the local minima, and then choose the one for which the function attains the lowest value.

LEARNING-BY-DOING EXERCISE A.6

Using Derivatives to Find a Minimum

Consider once again the total cost function:

$$C(Q) = Q^3 - 10Q^2 + 40Q$$

The average cost function $AC(Q)$ is then $C(Q)/Q$:

$$AC(Q) = Q^2 - 10Q + 40$$

Panel (b) in Figure A.2 shows this average cost curve.

Problem

(a) Using a derivative, verify that the minimum of the average cost curve occurs when $Q = 5$. Also show that the value of the average cost is 15 at its minimum.

(b) Using the second derivative, verify that the average cost is minimized (and not maximized) when $Q = 5$.

Solution

(a) The average cost curve reaches its minimum when its slope (and, equivalently, the derivative dAC/dQ) is zero. Observe that $AC(Q)$ is a sum of power functions. Therefore, its derivative is $dAC/dQ = 2Q - 10$. When we set the derivative equal to zero we find that $Q = 5$. This is the quantity that minimizes AC. The value of the average cost at this quantity is $AC(5) = 5^2 - 10(5) + 40 = 15$.

(b) The second derivative of the average cost function is $d^2AC/dQ^2 = 2$. Since the second derivative is positive, the slope of the graph is becoming less negative as we approach $Q = 5$ from the left, and becomes ever more positive as we move to the right of $Q = 5$. This verifies that the graph does achieve a minimum when $Q = 5$.

Optimal Quantity Choice Rules

Once you understand how to use calculus to find a maximum or a minimum, it is easy to see how to apply the technique to economic problems. Let's first develop the optimal quantity choice rule for a profit-maximizing firm that takes all prices as given. We show in Chapter 9 [see equation (9.1)] that a price-taking firm maximizes profit when it chooses its output so that price equals marginal cost. The dependent variable is economic profit, denoted by π. Economic profit is the difference between the firm's total revenue (the market price, P, times the quantity it produces, Q) and the firm's total cost, $C(Q)$. Thus,

$$\pi = PQ - C(Q)$$

Because the firm has only a small share of the market, it takes the market price P as given (a constant). To maximize profit, the firm chooses Q so that the slope of the profit curve is zero (see Figure 9.1). In terms of calculus, the firm chooses Q so that $d\pi/dQ = 0$. The derivative of π is

$$\frac{d\pi}{dQ} = P - \frac{dC}{dQ}$$

where dC/dQ is just the marginal cost. Thus, the firm must choose Q so that price equals marginal cost to maximize profits (producing so that $d\pi/dQ = 0$).

Similarly, we show in Chapter 11 [see equation (11.1)] that a profit-maximizing monopolist chooses its output so that marginal revenue equals marginal cost. The dependent variable is economic profit, denoted by π. Economic profit is the difference between the firm's total revenue, $R(Q)$, and the firm's total cost, $C(Q)$. Thus,

$$\pi = R(Q) - C(Q)$$

To maximize profit, the firm chooses Q so that the slope of the profit curve is zero (see Figure 11.2). In terms of calculus, the firm chooses Q so that $d\pi/dQ = 0$. The derivative of π is

$$\frac{d\pi}{dQ} = \frac{dR}{dQ} - \frac{dC}{dQ}$$

where dR/dQ is the marginal revenue and dC/dQ is the marginal cost. Thus, the firm must choose Q so that marginal revenue equals marginal cost to maximize profits (again, producing so that $d\pi/dQ = 0$).

A.6 MULTIVARIABLE FUNCTIONS

Until now we have been dealing with functions that depend on only one variable. However, in many situations a dependent variable will be related to two or more independent variables. For example, the profit for a firm, π, may depend on the amounts of two outputs, Q_1 being the amount of the first good it produces and Q_2 the amount of the second good. Suppose the profit function for the firm is

$$\pi = 13Q_1 - 2(Q_1)^2 + Q_1Q_2 + 8Q_2 - 2(Q_2)^2 \quad \text{(A.10)}$$

Figure A.8 shows a graph of the profit function. The graph has three dimensions because there are three variables. The dependent variable, profit, is on the vertical axis. The graph shows the two independent variables, Q_1 and Q_2, on the other two axes. As the graph shows, the profit function is a "hill." The firm can maximize its profits at point A, producing $Q_1 = 4$ and $Q_2 = 3$, and then earning profits $\pi = 38$.

Let's see how we might use calculus to find the values of the independent variables (Q_1 and Q_2 in this example) that maximize a dependent variable (π in the example). To do so, we need to understand how a change in *each* of the

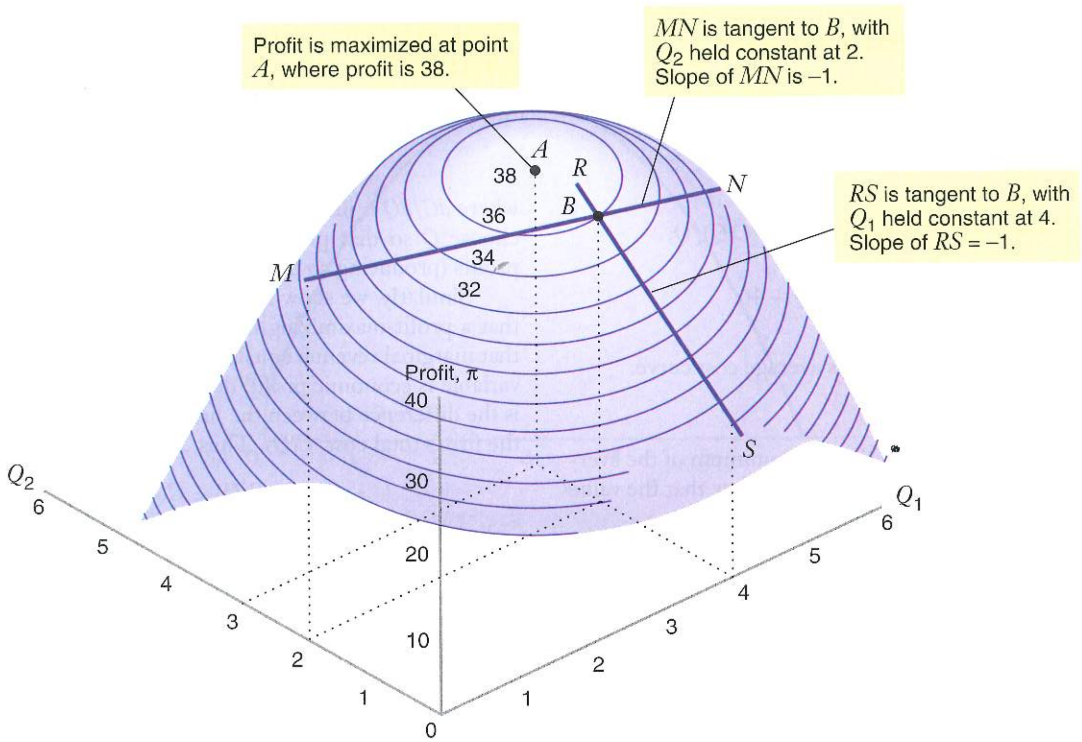


FIGURE A.8 Maximizing a Function of Two Variables
A function reaches its maximum when the slope is 0. At point A, when $Q_1 = 4$ and $Q_2 = 3$, the profit function achieves its maximum value of 38. The slope of the profit hill is 0 in all directions (and, equivalently, the values of the partial derivatives $\partial\pi/\partial Q_1$ and $\partial\pi/\partial Q_2$ are zero at point A).

At point B, when $Q_1 = 4$ and $Q_2 = 2$, the profit function achieves a lower value (36). The slope of the profit hill is not 0 in all directions. At B the value of the partial derivative $\partial\pi/\partial Q_2 = +4$. This means that the slope of the profit hill as we increase Q_2 (but hold $Q_1 = 4$) is 4. This is also the slope of the tangent line RS.

At B, the value of the partial derivative $\partial\pi/\partial Q_1 = -1$. This means that the slope of the profit hill as we increase Q_1 (but hold $Q_2 = 2$) is −1. This is also the slope of the tangent line MN.

independent variables affects the dependent variable, *holding constant the levels of all other independent variables*.

Consider point B in the graph, where $Q_1 = 4$, $Q_2 = 2$, and $\pi = 36$. As the graph shows, this is *not* the combination of outputs that maximizes profit.

The firm might ask how an increase in Q_2 affects π, holding constant the other independent variable Q_1. To find this information, we find the *partial derivative of π with respect to Q_2*, denoted by $\partial\pi/\partial Q_2$. To obtain this partial derivative, we take the derivative of equation (A.10), but treat the level of Q_1 as a constant. When we do this, the first two terms in equation (A.10) will be a constant because they depend only on Q_1; therefore the partial derivative of these terms with respect to Q_2 is zero. The partial derivative of the third term (Q_1Q_2) with respect to Q_2 is just Q_1. The partial derivative of the last two terms $[8Q_2 - 2(Q_2)^2]$ with respect to Q_2 will be $8 - 4Q_2$. When we put all of this information together, we learn that

$$\frac{\partial\pi}{\partial Q_2} = Q_1 + 8 - 4Q_2 \qquad \text{(A.11)}$$

Equation (A.11) measures the marginal profit (sometimes called marginal profitability) of Q_2. This marginal profit is the rate of change of profit (and the slope of the profit hill) as we vary Q_2, but hold Q_1 constant.

We illustrate what this partial derivative measures in Figure A.8. At point B we have drawn a line tangent to the profit hill (line RS). Along RS we are holding Q_1 constant ($Q_1 = 4$). We can find the slope of RS by evaluating the

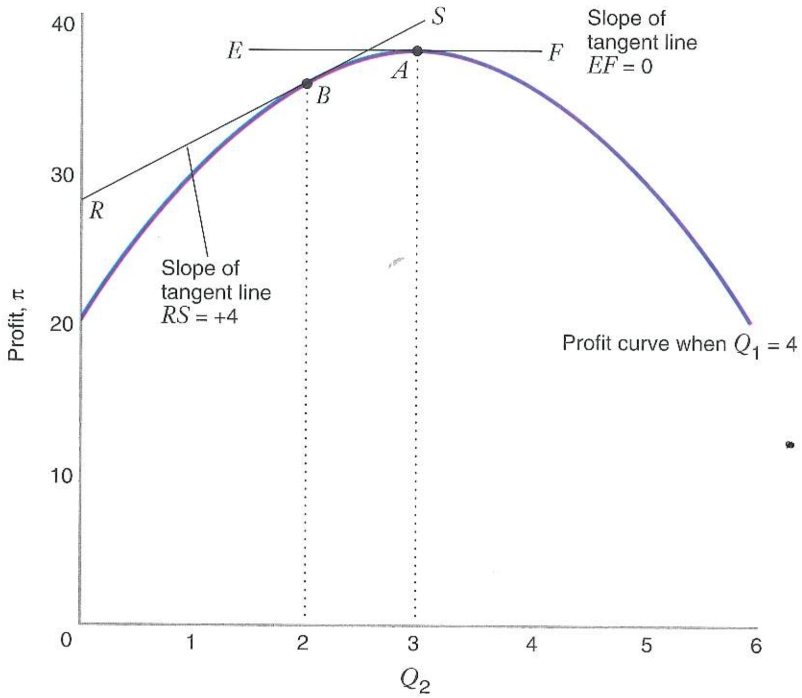


FIGURE A.9 Illustration of Partial Derivative
The graph shows a cross section of the profit hill in Figure A.8. We have drawn the cross section to show what the profit hill looks like when we vary Q_2, but hold Q_1 constant, with $Q_1 = 4$. Point *B* in this figure is therefore the same as point *B* in Figure A.8. We have also drawn the line tangent to the profit hill at point *B*. The value of the partial derivative of profit with respect to Q_2 (denoted by $\partial\pi/\partial Q_2$) measures the slope of this tangent line.

At point *A*, $Q_1 = 4$ and $Q_2 = 3$, the outputs that maximize profits. Point *A* is therefore the same as point *A* in Figure A.8. Since we have reached the top of the profit curve, the slope of the profit hill in Figure A.9 is 0. This means that the partial derivative $\partial\pi/\partial Q_2 = 0$.

partial derivative $\partial\pi/\partial Q_2 = Q_1 + 8 - 4Q_2$ when $Q_1 = 4$ and $Q_2 = 2$. The value of the derivative is therefore $\partial\pi/\partial Q_2 = (4) + 8 - 4(2) = 4$. The slope of *RS* (and therefore the slope of the profit hill at *B* in the direction of increasing Q_2) is 4.

To help you understand the meaning of a partial derivative, we have provided another view of the profit hill in Figure A.9. This graph shows a cross-sectional picture of the profit hill, showing what the profit hill looks like when we vary Q_2, but holds Q_1 constant, with $Q_1 = 4$. Point *B* in this figure is therefore the same as point *B* in Figure A.8. We have also drawn *RS*, the line tangent to the profit hill at point *B*. (The tangent line *RS* is the same in Figures A.8 and A.9.) The partial derivative of profit with respect to Q_2 (denoted by $\partial\pi/\partial Q_2$) measures the slope of this tangent line.[3] At point *B* the slope is 4.

Similarly, we could ask how an increase in Q_1 affects π, holding constant the other independent variable Q_2. To find this information, we find the partial derivative of π with respect to Q_1, denoted by $\partial\pi/\partial Q_1$. We take the derivative of equation (A.10), but treat the level of Q_2 as a constant. When we do this, the last two terms in equation (A.10) will be a constant because they depend only on Q_2; therefore the partial derivative of these terms with respect to Q_1 is zero. The partial derivative of the third term (Q_1Q_2) with respect to Q_1 is just Q_2. The partial derivative of the first two terms with

[3]Another way to see the meaning of the partial derivative illustrated in Figure A.9 is to substitute $Q_1 = 4$ into the profit function $\pi = 13Q_1 - 2(Q_1)^2 + Q_1Q_2 + 8Q_2 - 2(Q_2)^2$. Profits then become $\pi = 20 + 12Q - 2(Q_2)^2$. This is the equation of the profit hill in Figure A.9, because we have assumed Q_1 is held constant at 4. The slope of the profit hill in Figure A.9 is therefore $d\pi/dQ_2 = 12 - 4Q_2$. At point *B*, where $Q_2 = 2$, we find that $d\pi/dQ_2 = 4$, which is the slope of the tangent line *RS*.

respect to Q_1 will be $13 - 4Q_1$. When we put all of this information together, we learn that

$$\frac{\partial \pi}{\partial Q_1} = 13 - 4Q_1 + Q_2 \tag{A.12}$$

Equation (A.12) measures the marginal profit of Q_1, that is, the rate of change of profit as we vary Q_1, but hold Q_2 constant. Let's evaluate this partial derivative at point B in Figure A.8. When $Q_1 = 4$, and $Q_2 = 2$, we find that $\partial\pi/\partial Q_1 = 13 - 4(4) + 2 = -1$. Let's draw the line tangent to the profit hill at point B, holding Q_2 constant ($Q_2 = 2$), and label this line MN. The tangent line will have a slope of -1.

Finding a Maximum or a Minimum

How can we find the top of the profit hill in Figure A.8? At a maximum, the slope of the profit hill will be zero in all directions. This means that at a maximum *the partial derivatives* $\partial\pi/\partial Q_1$ *and* $\partial\pi/\partial Q_2$ *must both be zero*. Thus, in the example,

$$\frac{\partial \pi}{\partial Q_1} = 13 - 4Q_1 + Q_2 = 0$$

$$\frac{\partial \pi}{\partial Q_2} = Q_1 + 8 - 4Q_2 = 0$$

When we solve these two equations, we find that $Q_1 = 4$ and $Q_2 = 3$. These are the quantities that lead us to the top of the profit hill, point A in Figure A.8.[4] Let's also consider point A in Figure A.9, where $Q_1 = 4$ and $Q_2 = 3$, the outputs that maximize profits. Point A in this figure is therefore the same as point A in Figure A.8. Since we have reached the top of the profit curve, the slope of the profit hill at A in Figure A.9 is zero; this means that the partial derivative $\partial\pi/\partial Q_2$ is zero.

To practice taking partial derivatives, you might try the following exercises.

LEARNING-BY-DOING EXERCISE A.7

Marginal Utility with Two Independent Variables

In Chapter 3 (Learning-By-Doing Exercise 3.1), we introduced the utility function $U = \sqrt{xy}$. Here U is the dependent variable and x and y are the independent variables. The corresponding marginal utilities function are $MU_x = \sqrt{y}/(2\sqrt{x})$, and $MU_y = \sqrt{x}/(2\sqrt{y})$.

Problem Use partial derivatives to verify that these expressions for marginal utilities are correct.

Solution It may help to rewrite the utility function as $U = x^{1/2}y^{1/2}$. The marginal utility of x is just the partial derivative of U with respect to x, that is, $\partial U/\partial x$. To find this derivative, we treat y as a constant. Therefore, we only need to find the derivative of the term in brackets: $U = [x^{1/2}]\, y^{1/2}$. (The $y^{1/2}$ is just a multiplicative constant.) We observe that $x^{1/2}$ is a power function, with the derivative $(1/2)x^{-1/2}$, which can be rewritten as $1/(2\sqrt{x})$. The marginal utility is then $MU_x = \sqrt{y}/(2\sqrt{x})$.

Similarly, the marginal utility of y is just the partial derivative of U with respect to y, that is, $\partial U/\partial y$. To find this derivative, we treat x as a constant. Therefore, we only need to find the derivative of the term in brackets: $U = x^{1/2}\, [y^{1/2}]$. We observe that $y^{1/2}$ is a power function, with the derivative $(1/2)y^{-1/2}$, which can be rewritten as $1/(2\sqrt{y})$. The marginal utility is then $MU_y = \sqrt{x}/(2\sqrt{y})$.

LEARNING-BY-DOING EXERCISE A.8

Marginal Cost with Two Independent Variables

Problem Suppose the total cost C of producing two products is $C = Q_1 + \sqrt{Q_1 Q_2} + Q_2$, where Q_1 measures the number of units of the first product and Q_2 the number of units of the second. When $Q_1 = 16$ and $Q_2 = 1$, find the marginal cost of the first product, MC_1.

Solution It may help to rewrite the total cost function as $C = Q_1 + (Q_1)^{1/2}(Q_2)^{1/2} + Q_2$. The marginal cost of Q_1 is just the partial derivative of C with respect to Q_1, that is, $\partial C/\partial Q_1$. To find this derivative, we treat Q_2 as a constant. Let's consider each of the three terms in the cost function:

1. For the first term, the derivative of Q_1 with respect to Q_1 is 1.
2. For the second term, we only need to find the derivative of the term in brackets: $[(Q_1)^{1/2}]\,(Q_2)^{1/2}$. (The $(Q_2)^{1/2}$ is just a multiplicative constant.) We observe that $(Q_1)^{1/2}$ is a power function, with the derivative $(1/2)\,(Q_1)^{-1/2}$, which can be rewritten as $1/(2\sqrt{Q_1})$. The derivative of the second term is therefore $\sqrt{Q_2}/(2\sqrt{Q_1})$.
3. For the third term, Q_2 is being held constant. Since the derivative of a constant is zero, the derivative of the third term is zero.

Thus, the marginal cost of the first product is $MC_1 = 1 + \sqrt{Q_2}/(2\sqrt{Q_1})$. We can evaluate the marginal cost at any level of the outputs. For example, when $Q_1 = 16$

[4] To ensure that we have a maximum, or to distinguish a maximum from a minimum, we would also have to examine the second-order conditions for an optimum. In this appendix we do not present these conditions for a function with more than one independent variable and refer you to any standard calculus text. Also, the techniques we have discussed in this appendix may show you where a local maximum or minimum exists, but you may need to check further to see if the local maximum or minimum is a global optimum (see footnote 2).

and $Q_2 = 1$, we find that $MC_1 = 1 + \sqrt{1}/(2\sqrt{16}) = 9/8$. In words, when the firm is producing 16 units of the first output and 1 unit of the second, the marginal cost of the first product is 9/8.

A.7 CONSTRAINED OPTIMIZATION

As explained in Chapter 1, economic decision makers often want to extremize (maximize or minimize) the value of an economic variable such as profit, utility, or total production cost. However, they typically face constraints that limit the choices they can make. That is why economics is often described as a science of constrained choice.

Constrained optimization problems can be very large, often involving many decision variables and several constraints. In the next two sections, we present two approaches for solving constrained optimization problems. To facilitate the discussion, we focus here on a problem with two decision variables, x and y, and one constraint, although the principles are easily generalized to more complicated problems.

Let's represent the *objective function* (the function the decision maker wants to maximize or minimize) with the function $F(x, y)$. Let's describe the constraint she must satisfy by the function $G(x, y) = 0$.

For a maximization problem, we write the constrained optimization problem as follows:

$$\max_{(x,y)} F(x, y)$$
$$\text{subject to: } G(x, y) = 0$$

where the first line identifies the objective function to be maximized. (If the objective function were to be minimized, then the "max" would instead be a "min".) Underneath the "max" is a list of the endogenous variables that the decision maker controls (x and y). The second line represents the constraint the decision maker must satisfy. The decision maker can only choose values of x and y that satisfy $G(x, y) = 0$.

In Chapters 3 and 4 we explore one example of a constrained optimization problem, the consumer choice problem. A consumer may want to maximize his or her satisfaction, but must live within the constraints on available income. For that problem, F would be the utility function and G the budget constraint the consumer faces. In Chapter 7 we examine the cost-minimizing choice of inputs by a producer. A manager wants to minimize production costs, but may be required to supply a specified amount of output. The objective function is total cost, and the constraint is the amount of production required from the firm. In other settings managers often have budgetary constraints that limit the amount of money they can spend on an activity such as advertising.

In this section we show that it may be possible to solve a constrained optimization problem by substituting the constraint into the objective function, and then using calculus to find the maximum or minimum we seek. We illustrate how this might be done with two Learning-By-Doing Exercises.

LEARNING-BY-DOING EXERCISE A.9

Radio and Beer Advertising

Chapter 1 describes the problem facing a product manager for a small beer company that produces a high-quality microbrewed ale. The manager has a \$1 million advertising budget, and could spend the money on ads for TV or for radio. Table 1.1 illustrates new beer sales resulting from advertising. In Chapter 1 we did not give you the function that relates new beer sales to the amount of advertising, instead working with the values given in the table.

Now suppose you know that new beer sales (B, measured in barrels) depend on the amount of advertising on television (T, measured in hundreds of thousands of dollars) and radio (R, measured in hundreds of thousands of dollars) as follows:[5]

$$B(T, R) = 5000T - 250T^2 + 1000R - 50R^2$$

The function $B(T, R)$ is the objective function because this is the function that the decision maker wants to maximize. However, the manager can spend only \$1 million in total advertising. This means that the manager faces a constraint, namely, that $T + R = 10$. We write the maximization problem here as

$$\max_{(T,R)} B(T, R) \tag{A.13}$$
$$\text{subject to: } T + R = 10$$

where T and R are measured in hundreds of thousands of dollars.

Problem Solve this problem for the optimal amounts of radio and television advertising.

Solution The constraint has a simple form in this problem $(T + R = 10)$. From the constraint we know that $R = 10 - T$. We can just substitute this expression for R into the objective function as follows:

$$\begin{aligned} B &= 5000T - 250T^2 + 1000R - 50R^2 \\ &= 5000T - 250T^2 + 1000(10 - T) - 50(10 - T)^2 \\ &= 5000T - 300T^2 + 5000 \end{aligned}$$

The key point is the following: The new objective function $(B = 5000T - 300T^2 + 5000)$ already has the constraint "built in" because we have substituted the constraint into the original objective function $(B = 5000T - 250T^2 + 1000R - 50R^2)$. Now we can

[5]As an independent exercise, you may verify that the function $B(T, R) = 5000T - 250T^2 + 1000R - 50R^2$ gives the values of new beer sales in Table 1.1 for various combinations of television and radio advertising.

choose the optimal amount of TV advertising by setting the first derivative with respect to the amount of television advertising equal to zero:

$$\frac{dB}{dT} = 5000 - 600T = 0$$

This tells us that $T = 8.33$, that is, the manager should spend about \$833,333 on television advertising. We can then use the relationship $R = 10 - T$ to determine the optimal amount of radio advertising, so that $R = 1.67$. The manager should spend about \$166,667 on radio advertising. This "exact" solution is very close to the approximate solution developed in Chapter 1, using only the values displayed in the table.

LEARNING-BY-DOING EXERCISE A.10

The Farmer's Fencing Problem

Chapter 1 describes a constrained optimization involving the design of a fence for a farm. A farmer wishes to build a rectangular fence for his sheep. He has F feet of fence and cannot afford to purchase more. However, he can choose the dimensions of the pen, which will have a length of L feet and a width of W feet. He wishes to choose L and W to maximize the area of the pen; thus, the objective function is the area LW. He also faces a constraint; he must also make sure that the total amount of fencing he uses (the perimeter of the pen) not exceed F feet. In Chapter 1 we describe the farmer's decision as follows:

$$\max_{(L,W)} LW \qquad \text{(A.14)}$$
$$\text{subject to } 2L + 2W \le F$$

We know that the farmer will use all of the fence available if he wants to maximize the area of the pen. Therefore, we know that the constraint will be an equality, and the problem is simplified as follows:

$$\max_{(L,W)} LW \qquad \text{(A.15)}$$
$$\text{subject to } 2L + 2W = F$$

Problem Solve this problem to determine the optimal dimensions of the pen.

Solution The constraint has a simple form in this problem $(2L + 2W = F)$. The constraint tells us that $W = (F/2) - L$. We can just substitute this into the original objective function (LW) to find a new form of the objective function that already has the constraint built in:

$$\text{Area} = LW = L\left(\frac{F}{2} - L\right) = \frac{FL}{2} - L^2$$

Now we can choose the optimal length of the pen, L, by setting the first derivative equal to zero:

$$\frac{d\text{Area}}{dL} = \frac{F}{2} - 2L = 0$$

This tells us that $L = F/4$. We can then use the relationship $W = (F/2) - L$ to determine the optimal width, so that $W = F/4$. The solution tells us that the rectangle that maximizes the area of the pen will be a square, with sides $F/4$.

Before leaving this example, it is worth observing that we can use the results of the solution to perform *comparative statics* exercises, as described in Chapter 1. The exogenous variable in this problem (the one the farmer takes as given) is F, the amount of fence available to the farmer. The endogenous variables (the ones chosen by the farmer) are the length, L, the width W, and the area (Area $= LW$). We can use derivatives to answer the following questions:

1. How much will the length change when the amount of fence varies? We know that $L = F/4$. Therefore, $dL/dF = 1/4$. The length will increase by one-fourth foot when the perimeter is increased by one foot.
2. How much will the width change when the amount of fence varies? We know that $W = F/4$. Therefore, $dW/dF = 1/4$. The width will increase by one-fourth foot when the perimeter is increased by one foot.
3. How much will the area change when the amount of fence varies? We know that the Area $= LW = F^2/16$. Therefore, $d\text{Area}/dF = F/8$. The area will increase by about $F/8$ square feet when the perimeter is increased by one foot.

A.8 LAGRANGE MULTIPLIERS

In the previous section we showed how to solve a constrained optimization problem by solving the constraint for one of the variables and then substituting the constraint into the objective function. This technique is most likely to work when the constraint (or set of constraints) has a simple form. However, it may not be possible to use this approach in more complicated problems.

We now show how to solve constrained optimization problems by constructing an equation, called the *Lagrangian function*, that is a combination of the objective function and the constraint. We begin with a general description of the method, and then illustrate how to use it with two Learning-By-Doing Exercises.

We first construct the Lagrangian function as follows: $\Lambda(x, y, \lambda) = F(x, y) + \lambda G(x, y)$. This function is the sum of two terms: (1) the objective function, and (2) the constraint, multiplied by an unknown factor, λ, which is called the *Lagrange multiplier*. We then set the partial derivatives of the Lagrangian function with respect to the three unknowns (x, y, and λ) equal to zero.

$$\frac{\partial \Lambda}{\partial x} = 0 \longrightarrow \frac{\partial F(x, y)}{\partial x} - \lambda \frac{\partial G(x, y)}{\partial x} = 0 \quad \text{(A.16)}$$

$$\frac{\partial \Lambda}{\partial y} = 0 \longrightarrow \frac{\partial F(x, y)}{\partial y} - \lambda \frac{\partial G(x, y)}{\partial y} = 0 \quad \text{(A.17)}$$

$$\frac{\partial \Lambda}{\partial \lambda} = 0 \longrightarrow G(x, y) = 0 \quad \text{(A.18)}$$

We can then use the three equations (A.16, A.17, and A.18) to solve for the three unknowns. To see how to apply this method, consider the following two exercises.

LEARNING-BY-DOING EXERCISE A.11

Radio and Beer Advertising Revisited

Problem The problem here is the same as in Learning-By-Doing Exercise A.9. Now let's solve the problem using the method of Lagrange multipliers.

Solution We define the Lagrangian function

$$\Lambda(T, R, \lambda) = B(T, R) + \lambda(10 - T - R)$$

where λ is the Lagrange multiplier. Note that we have rewritten the constraint so that the right-hand side is zero (i.e., $10 - T - R = 0$). We then place the left-hand side of the constraint in the Lagrangian function.

The conditions for an interior optimum (with $T > 0$ and $R > 0$) are

$$\frac{\partial \Lambda}{\partial T} = 0 \longrightarrow \frac{\partial B(T, R)}{\partial T} - \lambda = 0 \quad \text{(A.19)}$$

$$\frac{\partial \Lambda}{\partial R} = 0 \longrightarrow \frac{\partial B(T, R)}{\partial R} - \lambda = 0 \quad \text{(A.20)}$$

$$\frac{\partial \Lambda}{\partial \lambda} = 0 \longrightarrow 10 - T - R = 0 \quad \text{(A.21)}$$

The partial derivatives in this problem are $\partial B(T, R)/\partial T = 5000 - 500T$ and $\partial B(T, R)/\partial R = 1000 - 100T$. Thus, we can write (A.18) as

$$5000 - 500T = \lambda, \text{ and} \quad \text{(A.22)}$$

$$1000 - 100R = \lambda \quad \text{(A.23)}$$

Since the right-hand sides of equations (A.22) and (A.23) are the same (λ), we know that at an optimum $5000 - 500T = 1000 - 100R$. This is equation (A.24). Equation (A.25) is the same as equation (A.21). Together, equations (A.24) and (A.25) give us two equations in two unknowns, T and R. We now know that the optimal amounts of radio and television advertising are determined by two equations:

$$5000 - 500T = 1000 - 100R \quad \text{(A.24)}$$

$$T + R = 10 \quad \text{(A.25)}$$

We then find that $T = \$8.33$ (hundred thousand) and $R = \$1.67$ (hundred thousand), the same solution we found in Learning-By-Doing Exercise A.9.

It is also possible to calculate the value of the Lagrange multiplier λ at the optimum, and this value has an important economic interpretation. We observe that $\lambda = 5000 - 500T = 5000 - 500(25/3) = 833.33$. (Alternatively, $\lambda = 1000 - 100R = 1000 - 100(5/3) = 833.33$.) The value of λ tells us (approximately) how much beer sales (the objective function) could be increased if the advertising budget were increased by one "unit" (in this problem a unit of advertising is \$100,000). The manager could expect sales to increase by about 833 barrels for every \$100,000 in extra advertising, or by about 0.00833 barrels for each additional dollar of advertising.

LEARNING-BY-DOING EXERCISE A.12

The Farmer's Fencing Problem Revisited

Problem The problem here is the same as in Learning-By-Doing Exercise A.10. Now let's solve the problem using the method of Lagrange.

Solution We define the Lagrangian

$$\Lambda(L, W, \lambda) = LW + \lambda(F - 2L - 2W)$$

where λ is the Lagrange multiplier. Note that we have rewritten the constraint so that the right-hand side is zero (i.e., $F - 2L - 2W = 0$). We then place the left-hand side of the constraint in the Lagrangian function.

The first-order necessary conditions for an interior optimum (with $L > 0$ and $W > 0$) are

$$\frac{\partial \Lambda}{\partial L} = 0 \longrightarrow \frac{\partial (LW)}{\partial L} - 2\lambda = 0 \quad \text{(A.26)}$$

$$\frac{\partial \Lambda}{\partial W} = 0 \longrightarrow \frac{\partial (LW)}{\partial W} - 2\lambda = 0 \quad \text{(A.27)}$$

$$\frac{\partial \Lambda}{\partial \lambda} = 0 \longrightarrow F - 2L - 2W = 0 \quad \text{(A.28)}$$

The partial derivatives in this problem are $[\partial(LW)/\partial L = W]$ and $[\partial(LW)/\partial W] = L$. Thus, we can write the first-order conditions (A.26) and (A.27) as

$$W = 2\lambda, \text{ and}$$

$$L = 2\lambda$$

Since the right-hand sides of equations (A.26) and (A.27) are the same (2λ), we know that at an optimum $W = L$. This is equation (A.29). Equation (A.30) is the same as equation (A.28). We now know that the optimal dimensions are determined by two equations:

$$W = L, \text{ and} \quad (A.29)$$

$$2L + 2W = F \quad (A.30)$$

We then find that $L = W = F/4$.

It is also possible to calculate the value of the Lagrange multiplier λ at the optimum. We know that $\lambda = L/2$ and that $L = F/4$. Therefore we know that $\lambda = F/8$. The value of λ tells us how much the area (measured in square feet) could be increased if the perimeter is increased by one unit (i.e., one foot). The farmer could expect the area to increase by about $F/8$ square feet for every extra foot of fence.

To see how to use the Lagrange multiplier, let's suppose that the amount of fence were increased from $F = 40$ feet to $F = 41$ feet. The Lagrange multiplier tells us that the area (the objective function) could then be increased by about $F/8$ square feet, or about 5 square feet.

Let's see how good this approximation is. With 40 feet of fence, the optimal dimensions are $L = W = 10$, and the area is $(10)(10) = 100$ square feet. With 41 feet of fence, the optimal dimensions are $L = W = 10.25$, and the area is $(10.25)(10.25) = 105.06$ square feet. Note that the approximation of the increase in the area using the Lagrange multiplier is very close to the actual increase in the area. The smaller the increase in the perimeter, the smaller will be the difference between the approximated and actual increase in the area.

In the text we have shown how Lagrange multipliers can be used to solve selected economic problems involving constrained optimization. In the appendix to Chapter 4, we use this method to solve the problem of consumer choice, where a consumer maximizes utility subject to a budget constraint. Also, in the appendix to Chapter 7 we apply this method to find the combination of inputs that will minimize the costs of producing any required level of output.

SUMMARY

- Economic analysis often requires that we understand how to relate economic variables to one another. There are three primary ways of expressing the relationships among variables: graphs, tables, and algebraic functions. **(LBD Exercise A.1)**

- The *marginal value* of a function measures the *change* in a dependent variable associated with a one-unit *change* in an independent variable. It also measures the slope of the graph of a function with the total value of the dependent variable on the vertical axis and the independent variable on the horizontal axis. The *average value* of a dependent variable is the total value of the dependent variable divided by the value of the independent variable. It is important to understand the relationship between marginal and average values:

- The average value must *increase* if the marginal value is *greater* than the average value.

- The average value must *decrease* if the marginal value is *less than* the average value.

- The average value will be *constant* if the marginal value *equals* the average value.

- Derivatives are useful in helping us to understand and calculate many of the "marginal" values in economics. Three of the most commonly encountered marginal values are marginal utility, marginal cost, and marginal revenue. The derivative of the total utility function is the *slope* of the total utility curve *and* the marginal utility. The derivative of the total cost function is the slope of the total cost curve *and* the marginal cost. The derivative of the total revenue function is the slope of the total revenue curve *and* the marginal revenue. **(LBD Exercises A.3, A.4, and A.5)**

- We can use derivatives to find where a function reaches a maximum or minimum. The function the decision maker wants to maximize or minimize is called the *objective function.* When there is only one dependent variable, the first derivative of the objective function with respect to the decision variable (the endogenous variable) will be zero at a maximum or a minimum. Equivalently, the slope of a graph of the objective function is zero at a maximum or minimum. We must check the second derivative to see if the function is maximized or minimized. **(LBD Exercise A.6)**

- We may also use derivatives to find marginal values (such as marginal cost, marginal revenue, and marginal utility) for a dependent variable that has more than one independent variable. To do so, we take the *partial* derivative of the dependent variable with respect to the independent variable of interest. To maximize or minimize an objective function with more than one dependent variable, we set all of the *partial* derivatives of the function equal to zero. **(LBD Exercises A.7 and A.8)**

- A constrained optimization problem is one in which a decision maker maximizes or minimizes an objective function subject to a set of constraints. There are two techniques for solving constrained optimization problems. Sometimes it may be possible to substitute constraints directly into the objective function, and then use derivatives to find an optimum. In more complex problems, it may not be possible to substitute the constraints into the objective function. One can then use the method of Lagrange multipliers to solve for a constrained optimum. **(LBD Exercises A.9, A.10, A.11, and A.12)**