6 Game Theory

The central assumption behind the analysis in this text is that people make the best choices they can given their objectives. For example, in the theory of choice in Chapter 2, a consumer chooses the affordable bundle maximizing his or her utility. The setting was made fairly simple by considering a single consumer in isolation, justified by the assumption that consumers are price takers, small enough relative to the market that their actions do not measurably impact others. Many situations are more complicated in that they involve strategic interaction. The best one person can do may often depend on what another does. For example, how loud a student plays his or her music may depend on how loud the student in the next dorm room plays his or hers, and vice versa. A gas station's profit-maximizing price may depend on what the competitor across the street charges. In this chapter, we will learn the tools economists use to deal with these strategic situations. The tools are quite general, applying to problems anywhere from the interaction between students in a dorm or players in a card game, all the way up to wars between countries. The tools are also particularly useful for analyzing the interaction among oligopoly firms, and we will draw on them extensively for this purpose later in the book.

Background

Game theory was originally developed during the 1920s and grew rapidly during World War II in response to the need to develop formal ways of thinking about military strategy.[1] One branch of game theory, called cooperative game theory, assumes the group of players reaches an outcome that is best for the group as a whole, producing the largest "pie" to be shared among them; the theory focuses

[1] Much of the pioneering work in game theory was done by the mathematician John von Newmann. The main reference is J. von Neumann and O. Morgenstern, *The Theory of Games and Economic Behavior* (Princeton, NJ: Princeton University Press, 1944).

on rules for how the "pie" should be divided. We will focus mostly on the second branch, called noncooperative game theory, in which players are guided instead by self-interest. We focus on noncooperative game theory for several reasons. Self-interested behavior does not always lead to an outcome that is best for the players as a group, and such outcomes are interesting (as we will see from the Prisoners' Dilemma to follow) and practically relevant. Second, the assumption of self-interested behavior is the natural extension of our analysis of single-player decision problems in earlier chapters to a strategic setting. Third, one can analyze attempts to cooperate using noncooperative game theory. Perhaps most importantly, noncooperative game theory is more widely used by economists. Still, cooperative game theory has proved useful to model bargaining games and political processes.

Basic Concepts

Game theory models seek to portray complex strategic situations in a simplified setting. Much like the previous models in this book, a game theory model abstracts the details of a problem to arrive at its mathematical representation. The greatest strength of this type of modeling is that it enables us to get to the heart of the problem.

Any situation in which individuals must make strategic choices and in which the final outcome depends on what each person chooses to do can be viewed as a game. All games have three basic elements: (1) players, (2) strategies, and (3) payoffs.

Players

Each decision maker in a game is called a *player*. The players may be individuals (as in card games), firms (as in an oligopoly), or entire nations (as in military conflicts). Players are characterized as having the ability to choose among a set of possible actions. The number of players, usually fixed throughout the "play" of the game, varies from game to game, with two-player, three-player, or n-player games being possible. In this chapter, we primarily study two-player games since many of the important concepts can be illustrated in this simple setting. We usually denote these players by A and B.

Strategies

A player's choice in a game is called a *strategy*. A strategy may simply be one of the set of possible actions available to the player, leading to the use of the terms *strategy* and *action* interchangeably in informal discourse. But a strategy can be more complicated than an action. A strategy can be a contingent plan of action based on what another player does first (as will be important when we get to sequential games). A strategy can involve probabilities of playing several actions (as will be important when we get to mixed strategies). The actions underlying the strategies can range from the very simple (taking another card in blackjack) to the very complex (building an anti-missile defense system). Although some games offer the players a choice among many different actions, most of the important concepts in this chapter can be illustrated for situations in which each player has only two actions available. Even when the player has only two actions available, the set of strategies may be much larger once we allow for contingent plans or for a range of probabilities of playing the actions.

Payoffs

The final returns to the players of a game at its conclusion are called *payoffs*. Payoffs include the utilities players obtain from explicit monetary payments plus any implicit feelings they have about the outcome, such as whether they are embarrassed or gain self-esteem. It is sometimes convenient to ignore these complications and take payoffs simply to be the explicit monetary payments involved in the game. This is sometimes a reasonable assumption (for example, in the case of profit for a profit-maximizing firm), but it should be recognized as a simplification. Players prefer to earn the highest payoffs possible.

EQUILIBRIUM

Best response
A strategy that produces the highest payoff among all possible strategies for a player given what the other player is doing.

Nash equilibrium
A set of strategies, one for each player, that are each best responses against one another.

Students who have taken a basic microeconomics course are familiar with the concept of market equilibrium (we will study this in detail in Chapter 10), defined as the point where supply equals demand. Both suppliers and demanders are content with the market equilibrium: given the equilibrium price and quantity, no market participant has an incentive to change his or her behavior. The question arises whether there are similar concepts in game theory models. Are there strategic choices that, once made, provide no incentives for the players to alter their behavior given what others are doing?

The most widely used approach to defining equilibrium in games is that proposed by Cournot (see Chapter 14) and generalized in the 1950s by John Nash (see Application 6.1: A Beautiful Mind for a discussion of the movie that increased his fame). An integral part of this definition of equilibrium is the notion of a best response. Player A's strategy a is a **best response** against player B's strategy b if A cannot earn more from any other possible strategy given that B is playing b. A **Nash equilibrium** is a set of strategies, one for each player, that are each best responses against one another. In a two-player game, a set of strategies (a^*, b^*) is a Nash equilibrium if a^* is a best response for A against b^* and b^* is a best response for B against a^*. A Nash equilibrium is stable in the sense that no single player has an incentive to deviate unilaterally to some other strategy. Put another way, outcomes that are not Nash equilibria are unstable because at least one player can switch to a strategy that would increase his or her payoffs given what the other players are doing.

Nash equilibrium is so widely used by economists as an equilibrium definition because, in addition to selecting an outcome that is stable, a Nash equilibrium

Application 6.1 A Beautiful Mind

In 1994, John Nash received the Nobel Prize in economics for the development of the equilibrium concept now known as Nash equilibrium. The publication of the best-selling biography *A Beautiful Mind* and the Oscar-award-winning movie of the same title has made Nash world famous.[1]

A Beautiful Blond

The movie dramatizes the development of Nash equilibrium in a single scene in which Nash is in a bar talking with his male classmates. They notice several women at the bar, one blond and the rest brunette, and it is posited that the blond is more desirable than the brunettes. Nash conceives of the situation as a game among the male classmates. If they all go for the blond, they will block each other and fail to get her, and indeed fail to get the brunettes because the brunettes will be annoyed at being second choice. He proposes that they all go for the brunettes. (The assumption is that there are enough brunettes that they do not have to compete for them, so the males will be successful in getting dates with them.) While they will not get the more desirable blond, each will at least end up with a date.

Confusion About Nash Equilibrium?

If it is thought that the Nash character was trying to solve for the Nash equilibrium of the game, he is guilty of making an elementary mistake! The outcome in which all male graduate students go for brunettes is not a Nash equilibrium. In a Nash equilibrium, no player can have a strictly profitable deviation given what the others are doing. But if all the other male graduate students went for brunettes, it would be strictly profitable for one of them to deviate and go for the blond because the deviator would have no competition for the blond, and she is assumed to provide a higher payoff. There are many Nash equilibria of this game, involving various subsets of males competing for the blond, but the outcome in which all males avoid the blond is not one of them.[2]

Nash Versus the Invisible Hand

Some sense can be made of the scene if we view the Nash character's suggested outcome not as what he thought was the Nash equilibrium of the game but as a suggestion for how they might cooperate to move to a different outcome and increase their payoffs. One of the central lessons of game theory is that equilibrium does not necessarily lead to an outcome that is best for all. In this chapter, we study the Prisoners' Dilemma, in which the Nash equilibrium is for both players to Confess when they could both benefit if they could agree to be Silent. In this chapter we also study the Battle of the Sexes, in which there is a Nash equilibrium where the players sometimes show up at different events, and this failure to coordinate ends up harming them both. The payoffs in the Beautiful Blond game can be specified in such a way that players do better if they all agree to ignore the blond than in the equilibrium in which all compete for the blond with some probability.[3] Adam Smith's famous "invisible hand," which directs the economy toward an efficient outcome under perfect competition, does not necessarily operate when players interact strategically in a game. Game theory opens up the possibility of conflict, miscoordination, and waste, just as observed in the real world.

To Think About

1. How would you write down the game corresponding to the bar scene from *A Beautiful Mind*? What are the Nash equilibria of your game? Should the females be included as players in the setup along with the males?
2. One of Nash's classmates suggested that Nash was trying to convince the others to go after the brunettes so that Nash could have the blond for himself. Is this a Nash equilibrium? Are there others like it? How can one decide how a game will be played if there are multiple Nash equilibria?

[1] The book is S. Nasar, *A Beautiful Mind* (New York: Simon and Schuster, 1997) and the movie is *A Beautiful Mind* (Universal Pictures, 2001).

[2] S. P. Anderson and M. Engers, "A Beautiful Blond: A Nash Coordination Game," University of Virginia working paper (February 2004).

[3] For example, the payoff to getting the blond can be set to 3, getting no date to 0, getting a brunette when no one else has gotten the blond to 2, and getting a brunette when someone else has gotten the blond to 1. Thus there is a loss due to envy if one gets the brunette when another has gotten the blond.

exists for all games. (As we will see, some games that at first appear not to have a Nash equilibrium will end up having one in mixed strategies.) The Nash equilibrium concept does have some problems. Some games have several Nash equilibria, some of which may be more plausible than others. In some applications, other equilibrium concepts may be more plausible than Nash equilibrium. The definition of Nash equilibrium leaves out the process by which players arrive at strategies they are prescribed to play. Economists have devoted a great deal of recent research to these issues, and the picture is far from settled. Still, Nash's concept provides an initial working definition of equilibrium that we can use to start our study of game theory.

Illustrating Basic Concepts

We can illustrate the basic components of a game and the concept of Nash equilibrium in perhaps the most famous of all noncooperative games, the Prisoners' Dilemma.

The Prisoners' Dilemma

First introduced by A. Tucker in the 1940s, its name stems from the following situation. Two suspects, *A* and *B*, are arrested for a crime. The district attorney has little evidence in the case and is anxious to extract a confession. She separates the suspects and privately tells each, "If you Confess and your partner doesn't, I can promise you a reduced (one-year) sentence, and on the basis of your confession, your partner will get 10 years. If you both Confess, you will each get a three-year sentence." Each suspect also knows that if neither of them confesses, the lack of evidence will cause them to be tried for a lesser crime for which they will receive two-year sentences.

The Game in Normal Form

Normal form Representation of a game using a payoff matrix.

The players in the game are the two suspects, *A* and *B*. (Though a third person, the district attorney, plays a role in the story, once she sets up the payoffs from confessing she does not make strategic decisions, so she does not need to be included in the game.) The players can choose one of two possible actions, Confess or Silent. The payoffs, as well as the players and actions, can be conveniently summarized, as shown in the matrix in Table 6-1. The representation of a game in a matrix like this is called the **normal form**. In the table, player *A*'s strategies, Confess or Silent, head the rows and *B*'s strategies head the columns. Payoffs corresponding to the

TABLE 6-1 Prisoners' Dilemma in Normal Form

		B Confess	*B* Silent
A	Confess	−3, −3	−1, −10
A	Silent	−10, −1	−2, −2

various combinations of strategies are shown in the body of the table. Since more prison time causes disutility, the prison terms for various outcomes enter with negative signs. We will adopt the convention that the first payoff in each box corresponds to the row player (player *A*) and the second corresponds to the column player (player *B*). To make this convention even clearer, we will make all of player *A*'s strategies and payoffs red and all of player *B*'s blue. For an example of how to read the table, if *A* Confesses and *B* is Silent, *A* earns −1 (for one year of prison) and *B* earns −10 (for 10 years of prison). The fact that the district attorney approaches each separately indicates that the game is simultaneous: a player cannot observe the other's action before choosing his or her own action.

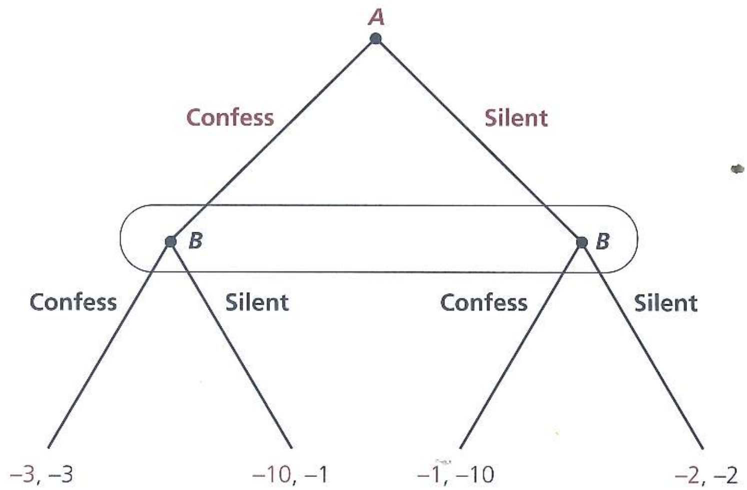
The Game in Extensive Form

The Prisoners' Dilemma game can also be represented as a game tree as in Figure 6-1, called the **extensive form**. Action proceeds from top to bottom. Each dark circle is a decision point for the player indicated there. The first move belongs to *A*, who can choose to Confess or be Silent. The next move belongs to *B*, who can also choose to Confess or be Silent. Payoffs are given at the bottom of the tree.

Extensive form Representation of a game as a tree.

To reflect the fact that the Prisoners' Dilemma is a simultaneous game, we would like to put the two players' moves at the same level in the tree, but the structure of a tree prevents us from doing that. To avoid this problem, we can arbitrarily choose one player (here *A*) to be at the top of the tree as the first mover and the other to be lower as the second mover, but then we draw an oval around

FIGURE 6-1 Prisoners' Dilemma in Extensive Form *A* chooses to Confess or be Silent, and *B* makes a similar choice. The oval surrounding *B*'s decision points indicates that *B* cannot observe *A*'s choice when *B* moves, since the game is simultaneous. Payoffs are listed at the bottom.



B's decision points to reflect the fact that B does not observe which action A has chosen and so does not observe which decision point has been reached when he or she makes his or her decision.

The choice to put A above B in the extensive form was arbitrary: we would have obtained the same representation if we put B above A and then had drawn an oval around A's decision points. As we will see when we discuss sequential games, having an order to the moves only matters if the second mover can observe the first mover's action. It usually is easier to use the extensive form to analyze sequential games and the normal form to analyze simultaneous games. Therefore, we will return to the normal-form representation of the Prisoners' Dilemma to solve for its Nash equilibrium.

Solving for the Nash Equilibrium

Return to the normal form of the Prisoners' Dilemma in Table 6-1. Consider each box in turn to see if any of the corresponding pairs of strategies constitute a Nash equilibrium. First consider the lower right box, corresponding to both players choosing Silent. There is reason to think this is the equilibrium of the game since the sum of the payoffs, -4, is greater than the sum of the payoffs in any of the other three outcomes (since all sums are negative, by "the greatest sum" we mean the one closest to 0). However, both playing Silent is in fact not a Nash equilibrium. To be a Nash equilibrium, both players' strategies must be best responses to each other. But given that B plays Silent, A can increase his or her payoff from -2 in the proposed equilibrium to -1 by deviating from Silent to Confess. Therefore, Silent is not A's best response to B's playing Silent. It is also true that B's playing Silent is not a best response to A's playing Silent (although demonstrating that at least one of the two players was not playing his or her best response was enough to rule out an outcome as being a Nash equilibrium). Next consider the top right box, where A plays Confess and B plays Silent. This is not a Nash equilibrium either. Given that A plays Confess, B can increase his or her payoff from -10 in the proposed equilibrium to -3 by deviating from Silent to Confess. Similarly, the bottom left box, in which A plays Silent and B plays Confess, can be shown not to be a Nash equilibrium since A is not playing a best response.

The remaining upper left box corresponds to both playing Confess. This is a Nash equilibrium. Given B plays Confess, A's best response is Confess since this leads A to earn -3 rather than -10. By the same logic, Confess is B's best response to A's playing Confess.

Rather than going through each outcome one by one, there is a shortcut to finding the Nash equilibrium directly by underlining payoffs corresponding to best responses. This method is useful in games having only two actions having small payoff matrices, but it becomes extremely useful when the number of actions increases and the payoff matrix grows. The method is outlined in Table 6-2. The first step is to compute A's best response to B's playing Confess. A compares his or her payoff in the first column from playing Confess, -3, to playing Silent, -10. The payoff -3 is higher than -10, so Confess is A's best response, and we underline -3. In step 2, we underline -1, corresponding to A's best

TABLE 6-2 Solving for Nash Equilibrium in Prisoners' Dilemma Using the Underlining Method

Step 1: Underline payoff for *A*'s best response to *B*'s playing Confess.

		B Confess	*B* Silent
A	**Confess**	$\underline{-3,}$ −3	−1, −10
	Silent	−10, −1	−2, −2

Step 2: Underline payoff for *A*'s best response to *B*'s playing Silent.

		B Confess	*B* Silent
A	**Confess**	$\underline{-3,}$ −3	$\underline{-1,}$ −10
	Silent	−10, −1	−2, −2

Step 3: Underline payoff for *B*'s best response to *A*'s playing Confess.

		B Confess	*B* Silent
A	**Confess**	$\underline{-3,}$ $\underline{-3}$	$\underline{-1,}$ −10
	Silent	−10, −1	−2, −2

Step 4: Underline payoff for *B*'s best response to *A*'s playing Silent.

		B Confess	*B* Silent
A	**Confess**	$\underline{-3,}$ $\underline{-3}$	$\underline{-1,}$ −10
	Silent	−10, $\underline{-1}$	−2, −2

TABLE 6-2 Solving for Nash Equilibrium in Prisoners' Dilemma Using the Underlining Method *(continued)*

Step 5: Nash equilibrium in box with both payoffs underlined.

		B	
		Confess	**Silent**
A	**Confess**	$\underline{-3}, \underline{-3}$	$\underline{-1}, -10$
	Silent	$-10, \underline{-1}$	$-2, -2$

response, Confess, to *B*'s playing Silent. In step 3, we underline −3, corresponding to *B*'s best response to *A*'s playing Confess. In step 4, we underline −1, corresponding to *B*'s best response to *A*'s playing Silent.

For an outcome to be a Nash equilibrium, both players must be playing a best response to each other. Therefore, both payoffs in the box must be underlined. As seen in step 5, the only box in which both payoffs are underlined is the upper left, with both players choosing Confess. In the other boxes, either one or no payoffs are underlined, meaning that one or both of the players are not playing a best response in these boxes, so they cannot be Nash equilibria.

The temptation is to say that the Nash equilibrium is −3,−3. This is not technically correct. Recall that the definition of Nash equilibrium involves a set of strategies, so it is proper to refer to the Nash equilibrium in the Prisoners' Dilemma as "both players choose Confess." True, each outcome corresponds to unique payoffs in this game, so there is little confusion in referring to an equilibrium by the associated payoffs rather than strategies. However, we will come across games later in the chapter in which different outcomes have the same payoffs, so referring to equilibria by payoffs leads to ambiguity.

Dominant Strategies

Dominant strategy
Best response to all of the other player's strategies.

Referring to step 5 in Table 6-2, not only is Confess a best response to the other players' equilibrium strategy (all that is required for Nash equilibrium), Confess is a best response to all strategies the other player might choose, called a **dominant strategy**. When a player has a dominant strategy in a game, there is good reason to predict that this is how the player will play the game. The player does not need to make a strategic calculation, imagining what the other might do in equilibrium. The player has one strategy that is best, regardless of what the other does. In most games, players do not have dominant strategies, so dominant strategies would not be a generally useful equilibrium definition (while Nash equilibrium is, since it exists for all games).

The Dilemma

The game is called the Prisoners' "Dilemma" because there is a better outcome for both players than the equilibrium. If both were Silent, they would each only get two years rather than three. But both being Silent is not stable; each would prefer to deviate to Confess. If the suspects could sign binding contracts, they would sign a contract that would have them both choose Silent. But such contracts would be difficult to write in the game because the district attorney approaches each suspect privately, so they cannot communicate; and even if they could sign a contract, the court would refuse to enforce it.

Situations resembling the Prisoners' Dilemma arise in many real world settings. The best outcome for students working on a group project together might be for all to work hard and earn a high grade on the project, but the individual incentive to shirk, each relying on the efforts of others, may prevent them from attaining such an outcome. A cartel agreement among dairy farmers to restrict output would lead to higher prices and profits if it could be sustained, but may be unstable because it may be too tempting for an individual farmer to try to sell more milk at the high price. We will study the stability of business cartels more formally in Chapter 14.

MIXED STRATEGIES

To analyze some games, we need to allow for more complicated strategies than simply choosing an action. We will consider **mixed strategies**, which have the player randomly select one of several possible actions. In contrast, the strategies we have considered so far that have each player choose one action or another with certainty, are called **pure strategies**. We will illustrate mixed strategies in another classic game, Matching Pennies.

Mixed strategy Randomly selecting from several possible actions.

Pure strategy A single action played with certainty.

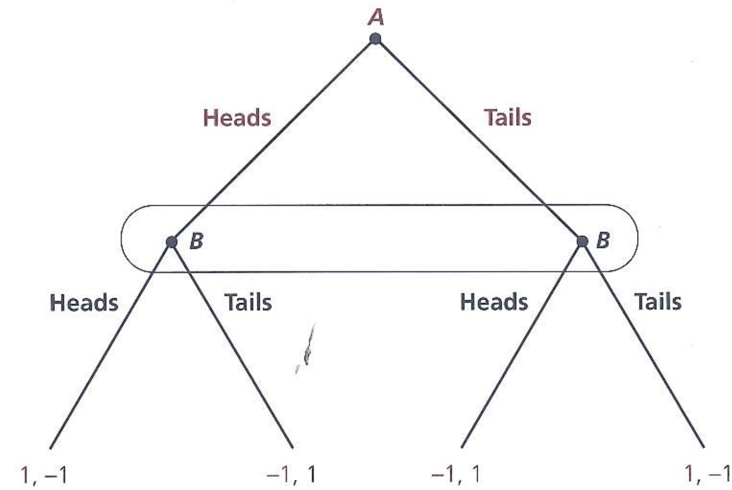
Matching Pennies

Matching Pennies is based on a children's game in which two players, *A* and *B*, each secretly choose whether to leave a penny with its head or tail facing up. The players then reveal their choices simultaneously. *A* wins *B*'s penny if the coins match (both Heads or both Tails), and *B* wins *A*'s penny if they do not. The normal form for the game is given in Table 6-3 and the extensive form in Figure 6-2. The game has the special property that the two players' payoffs in each box add to zero, called a zero-sum game. By contrast, the reader can check that the Prisoner's Dilemma is not a zero-sum game because the sum of players' payoffs varies across the different boxes.

TABLE 6-3 Matching Pennies Game in Normal Form

		B	
		Heads	Tails
A	Heads	1, −1	−1, 1
	Tails	−1, 1	1, −1

FIGURE 6-2 Matching Pennies Game in Extensive Form



To solve for the Nash equilibrium, we will use the method of underlining payoffs for best responses introduced previously for the Prisoners' Dilemma. Table 6-4 presents the results from this method. *A* always prefers to play the same action as *B*. *B* prefers to play a different action from *A*. There is no box with both payoffs underlined, so we have not managed to find a Nash equilibrium. One might be tempted to say that no Nash equilibrium exists for this game. But this contradicts our earlier claim that all games have Nash equilibria. The contradiction can be resolved by noting that the Matching Pennies game does have a Nash equilibrium, not in pure strategies, as would be found by our underlining method, but in mixed strategies.

TABLE 6-4 Solving for Pure-Strategy Nash Equilibrium in Matching Pennies Game

		B: Heads	B: Tails
A	Heads	$\underline{1}, -1$	$-1, \underline{1}$
	Tails	$-1, \underline{1}$	$\underline{1}, -1$

Solving for a Mixed-Strategy Nash Equilibrium

Rather than choosing Heads or Tails, suppose players secretly flip the penny and play whatever side turns up. The result of this strategy is a random choice of Heads with probability ½ and Tails with probability ½. This set of strategies, with both playing Heads or Tails with equal chance, is the mixed-strategy Nash equilibrium of the game. To verify this, we need to show that both players' strategies are best responses to each other.

In the proposed equilibrium, all four outcomes corresponding to the four boxes in the normal form in Table 6-3 are equally likely to occur, each occurring with probability $1/4$. Using the formula for expected payoffs from the previous chapter, A's expected payoff equals the probability-weighted sum of the payoffs in each outcome:

$$(1/4)(1) + (1/4)(-1) + (1/4)(-1) + (1/4)(1) = 0.$$

Similarly, B's expected payoff is also 0. The mixed strategies in the proposed equilibrium are best responses to each other if neither player can deviate to a strategy that produces a strictly higher payoff than 0. But there is no such profitable deviation. Given that B plays Heads and Tails with equal probabilities, the players' coins will match exactly half the time, whether A chooses Heads or Tails (or indeed even some random combination of the two actions); so A's payoff is 0 no matter what strategy it chooses. A cannot earn more than the 0 it earns in equilibrium. Similarly, given A is playing Heads and Tails with equal probabilities, B's expected payoff is 0 no matter what strategy it uses. So neither player has a strictly profitable deviation. (It should be emphasized here that if a deviation produces a tie with the player's equilibrium payoff, this is not sufficient to rule out the equilibrium; to rule out an equilibrium, one must demonstrate a deviation produces a strictly higher payoff than in equilibrium.)

Both players playing Heads and Tails with equal probabilities is the only mixed-strategy Nash equilibrium in this game. No other probabilities would work. For example, suppose B were to play Heads with probability $1/3$ and Tails with probability $2/3$. Then A would earn an expected payoff of $(1/3)(1)+(2/3)(-1) = -1/3$ from playing Heads and $(1/3)(-1)+(2/3)(1) = 1/3$ from playing Tails. Therefore, A would strictly prefer to play Tails as a pure strategy rather than playing a mixed strategy involving both Heads and Tails, and so B's playing Heads with probability $1/3$ and Tails with probability $2/3$ cannot be a mixed-strategy Nash equilibrium.

It should be emphasized that in mixed-strategy Nash equilibrium of Matching Pennies or indeed any game, players must be indifferent between the actions that are played with positive probability. If a player strictly preferred one action over another, the player would want to put all of the probability on the preferred action and none on the other action.

The Nash equilibrium in Matching Pennies involved equal probabilities of the two actions. In other games, the mixed-strategy Nash equilibrium can involve different probabilities of playing the two actions. We will see such a case in the Battle of the Sexes game below. In games with more than two actions, the mixed-strategy

MicroQuiz 6.1

In Matching Pennies, suppose B plays the equilibrium mixed strategy of Heads with probability $1/2$ and Tails with probability $1/2$. Use the formula for expected values to verify that A's expected payoff equals 0 from using any of the following strategies.

1. The pure strategy of Heads.
2. The pure strategy of Tails.
3. The mixed strategy of Heads with probability $1/2$ and Tails with probability $1/2$.
4. The mixed strategy of Heads with probability $1/3$ and Tails with probability $2/3$.

Nash equilibrium can sometimes involve playing more than two actions with positive probability.

Interpretation of Random Strategies

While at first glance it may seem bizarre to have players flipping coins or rolling dice in secret to determine their strategies, it may not be so unnatural in children's games such as Matching Pennies. Mixed strategies are also natural and common in sports, as discussed in Application 6.2: Mixed Strategies in Sports. Mixed strategies are important in law enforcement. Consider the enforcement of traffic laws such as speeding. Having police stationed continuously on every street may be prohibitively expensive. But having police stationed on only some of the streets only some of the time may deter speeding everywhere, especially if the punishment for speeding is high enough. Of course, if drivers knew which streets the police patrolled, they would drive within the limit on those and speed on the others. So, in practice, police are stationed at random places and times. Perhaps most familiar to students is the role of mixed strategies in class exams. Class time is usually too limited for the professor to examine students on every topic taught in class. But it may be sufficient to test students on a subset of topics to get them to study all of the material. If students knew which topics are on the test, they may be inclined to study only those and not the others, so the professor must choose which topics to include at random to get the students to study everything.

MULTIPLE EQUILIBRIA

Nash equilibrium is a useful solution concept because it exists for all games. A drawback is that some games have several or even many Nash equilibria. The possibility of multiple equilibria causes a problem for economists who would like to use game theory to make predictions, since it is unclear which of the Nash equilibria one should predict will happen. The possibility of multiple equilibria is illustrated in yet another classic game, the Battle of the Sexes.

TABLE 6-5 Battle of the Sexes in Normal Form

		B (Husband)	
		Ballet	Boxing
A (Wife)	Ballet	2, 1	0, 0
	Boxing	0, 0	1, 2

Battle of the Sexes

The game involves two players, a wife (A) and a husband (B) who are planning an evening out. Both prefer to be together rather than apart. Conditional on being together, the wife would prefer to go to a Ballet performance and the husband to a Boxing match. The normal form for the game is given in Table 6-5 and the extensive form in Figure 6-3.

Application 6.2 Mixed Strategies in Sports

Sports provide a setting in which mixed strategies arise quite naturally, and in a simple enough setting that we can see game theory in operation.

Soccer Penalty Kicks

In soccer, if a team commits certain offenses near its own goal, the other team is awarded a penalty kick, effectively setting up a game between the kicker and the goalie. Table 1 is based on a study of penalty kicks in elite European soccer leagues.[1] The first entry in each box is the frequency the penalty kick scores (taken to be the kicker's payoff), and the second entry is the frequency it does not score (taken to be the goalie's payoff). Kickers are assumed to have two actions: aim toward the "natural" side of the goal (left for right-footed kickers and right for left-footed players) or aim toward the other side. Kickers can typically kick harder and more accurately to their natural side. Goalies can try to jump one way or the other to try to block the kick. The ball travels too fast for the goalie to react to its direction, so the game is effectively simultaneous. Goalies know from scouting reports what side is natural for each kicker, so they can condition their actions on this information.

Do Mixed Strategies Predict Actual Outcomes?

Using the method of underlining payoffs corresponding to best responses, as shown in Table 1, we see that no box has both payoffs underlined, so there is no pure-strategy Nash equilibrium.

TABLE 1 Soccer Penalty Kick Game

		Goalie: Natural side for kicker	Goalie: Other side
Kicker	Natural side for kicker	.64, <u>.36</u>	<u>.94</u>, .06
	Other side	<u>.89</u>, .11	.44, <u>.56</u>

Using the same steps we will use to compute the mixed-strategy Nash equilibrium in the Battle of the Sexes below, one can show that the kicker kicks to his natural side 3/5 of the time and 2/5 of the time to his other side; the goalie jumps to the side that is natural for the kicker 2/3 of the time and the other side 1/3 of the time. Several important implications emerge from the theoretical analysis of the mixed-strategy Nash equilibrium that can be checked in the data. First, both actions have at least some chance of being played. This is borne out in the Chiappori et al. data: almost all of the kickers and goalies who are involved in three or more penalty kicks in the data choose each action at least once. Second, players obtain the same expected payoff in equilibrium regardless of the action taken. This is again borne out in the data, with kickers scoring about 75 percent of the time, whether they kick to their natural side or the opposite, and goalies being scored on about 75 percent of the time, whether they jump to the kicker's natural side or the opposite. Third, the goalie should jump to the side that is natural for the kicker more often. Otherwise, the higher speed and accuracy going to his natural side would lead the kicker to play the pure strategy of always kicking that way. Again, this conclusion is borne out in the data, with the goalie jumping to the kicker's natural side 60 percent of the time (note how close this is to the prediction of 2/3 we made above).

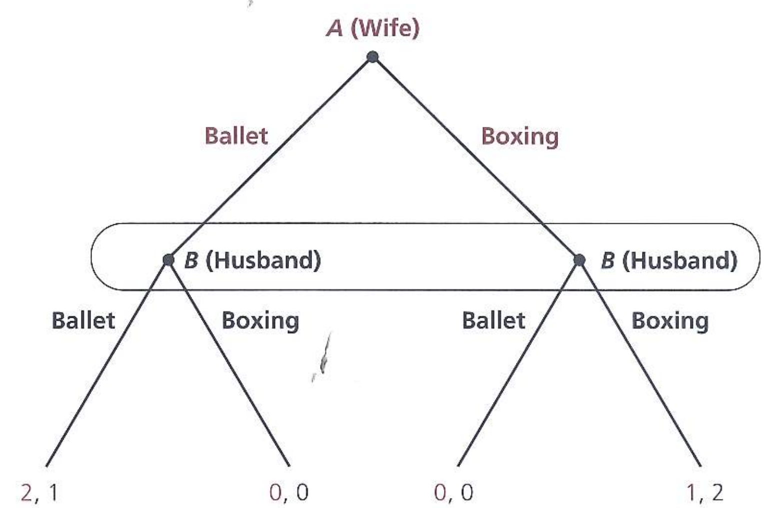
To Think About

1. Verify the mixed-strategy Nash equilibrium computed above for the penalty-kick game using the methods we will use for the Battle of the Sexes game.
2. Economists have studied mixed strategies in other sports, for example whether a tennis serve is aimed to the returner's backhand or forehand.[2] Can you think of other sports settings in which mixed strategies may be involved? Can you think of settings involving mixed strategies outside of sports and games and besides the ones noted in the text?

[1] P.-A. Chiappori, S. Levitt, and T. Groseclose, "Testing Mixed-Strategy Equilibria When Players Are Heterogeneous: The Case of Penalty Kicks in Soccer," *American Economic Review* (September 2002): 1138–1151.

[2] M. Walker and J. Wooders, "Minimax Play at Wimbledon," *American Economic Review* (December 2001), 1521–1538.

FIGURE 6-3 **Battle of the Sexes in Extensive Form**



To solve for the Nash equilibria, we will use the method of underlining payoffs for best responses introduced previously. Table 6-6 presents the results from this method. A player's best response is to play the same action as the other. Both payoffs are underlined in two boxes: the box in which both play Ballet and also in the box in which both play Boxing. Therefore, there are two pure-strategy Nash equilibria: (1) both play Ballet and (2) both play Boxing.

The problem of multiple equilibria is even worse than at first appears. Besides the two pure-strategy Nash equilibria, there is a mixed-strategy one. How does one know this? It is impossible to know exactly unless one performs all the calculations necessary to find a mixed-strategy Nash equilibrium. However, one could guess that there would be a mixed-strategy Nash equilibrium based on a famous but peculiar result that Nash equilibria tend to come in odd numbers. Therefore, finding an even number of pure-strategy Nash equilibria (two in this game, zero in Matching Pennies) should lead one to suspect that the game also has another Nash equilibrium, in mixed strategies.

TABLE 6-6 Solving for Pure-Strategy Nash Equilibria in the Battle of the Sexes

		B (Husband)	
		Ballet	Boxing
A (Wife)	Ballet	2, 1	0, 0
	Boxing	0, 0	1, 2

Computing Mixed Strategies in the Battle of the Sexes

It is instructive to go through the calculation of the mixed-strategy Nash equilibrium in the Battle of the Sexes since, unlike in Matching Pennies, the equilibrium probabilities do not end up being equal ($\frac{1}{2}$) for each action. Let w be the probability the wife plays Ballet and h the probability the husband plays Ballet. Remembering that the probabilities of exclusive and exhaustive events must add to one, the probability of playing Boxing is $1 - w$ for the wife and $1 - h$ for the husband; so once we know the probability each plays Ballet, we automatically know the probability each plays Boxing. Our task then is to compute the equilibrium values of w and h. The difficulty now is that w and h may potentially be any one of a continuum of values between 0 and 1, so we cannot set up a payoff matrix and use our underlining method to find best responses. Instead, we will graph players' **best-response functions.**

Best-response function
Function giving the payoff-maximizing choice for one player for each of a continuum of actions of the other player.

Let us start by computing the wife's best-response function. The wife's best-response function gives the w that maximizes her payoff for each of the husband's possible strategies, h. For a given h, there are three possibilities: she may strictly prefer to play Ballet; she may strictly prefer to play Boxing; or she may be indifferent between Ballet and Boxing. In terms of w, if she strictly prefers to play Ballet, her best response is $w = 1$. If she strictly prefers to play Boxing, her best response is $w = 0$. If she is indifferent about Ballet and Boxing, her best response is a tie between $w = 1$ and $w = 0$; in fact, it is a tie among $w = 0$, $w = 1$, and all values of w between 0 and 1!

To see this last point, suppose her expected payoff from playing both Ballet and Boxing is, say, $\frac{2}{3}$, and suppose she randomly plays Ballet and Boxing with probabilities w and $1 - w$. Her expected payoff (this should be reviewed, if necessary, from Chapter 5) would equal the probability she plays Ballet times her expected payoff if she plays Ballet plus the probability she plays Boxing times her expected payoff if she plays Boxing:

$$(w)(2/3) + (1 - w)(2/3) = 2/3.$$

This shows that she gets the same payoff, $2/3$, whether she plays Ballet for sure, Boxing for sure, or a mixed strategy involving any probabilities w, $1 - w$ of playing Ballet and Boxing. So her best response would be a tie among $w = 0$, $w = 1$, and all values in between.

Returning to the computation of the wife's best-response function, suppose the husband plays a mixed strategy of Ballet probability h and Boxing with probability $1 - h$. Referring to Table 6-7, her expected payoff from playing Ballet equals h (the probability the husband plays Ballet, and so they end up in Box 1) times 2 (her payoff in Box 1) plus $1 - h$ (the probability he plays Boxing, and so they end up in Box 2) times 0 (her payoff in Box 2), for a total expected payoff, after simplifying, of $2h$. Her expected payoff from playing Boxing equals h (the probability the husband plays Ballet, and so they end up in Box 3) times 0 (her payoff in Box 3) plus $1 - h$ (the probability he plays Boxing, and so they end up in Box 4) times 1 (her payoff in Box 4) for a total expected payoff, after simplifying, of $1 - h$.

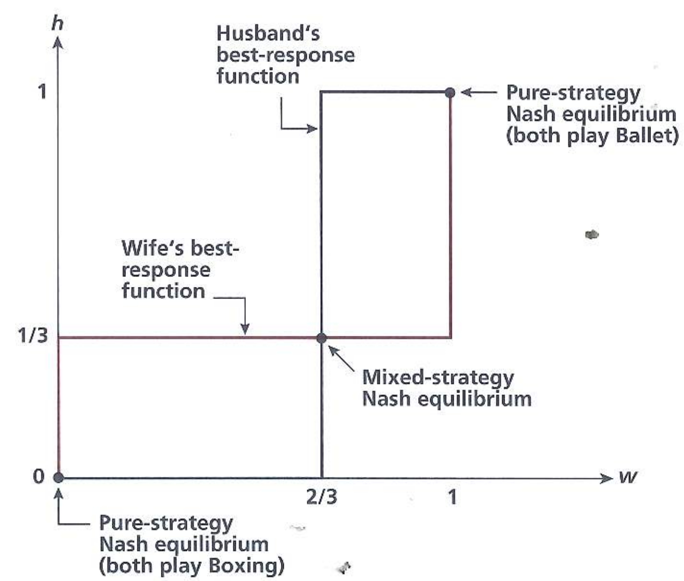
TABLE 6-7 Computing the Wife's Best Response to the Husband's Mixed Strategy

		B (Husband)		
		Ballet h	**Boxing $1-h$**	
A (Wife)	**Ballet**	**Box 1** 2, 1	**Box 2** 0, 0	$(h)(2) + (1-h)(0) = 2h$
	Boxing	**Box 3** 0, 0	**Box 4** 1, 2	$(h)(0) + (1-h)(1) = 1-h$

Based on the calculations from the previous paragraph, we can see that she prefers Ballet if $2h > 1 - h$ or, rearranging, $h > 1/3$. Therefore, in other words, her best response to any h greater than $1/3$ is $w = 1$. She prefers Boxing if $2h < 1 - h$ or, rearranging, $h < 1/3$. Therefore, her best response to any h less than $1/3$ is $w = 0$. She is indifferent between Ballet and Boxing if $h = 1/3$. Therefore, her best response to $h = 1/3$ includes $w = 0$, $w = 1$, and all values in between.

Figure 6-4 graphs her best-response function as the thick red line. Similar calculations can be used to derive the husband's best-response function, the thick

FIGURE 6-4 Best-Response Functions Allowing for Mixed Strategies in the Battle of the Sexes



blue line. The best-response functions intersect in three places, the three Nash equilibria. The figure allows us to recover the two pure-strategy Nash equilibria found before: the one in which $w = h = 1$ (that is, both play Ballet for sure) and the one in which $w = h = 0$ (that is, both play Boxing for sure). We also obtain the mixed-strategy Nash equilibrium $w = 2/3$ and $h = 1/3$. In words, the mixed-strategy Nash equilibrium involves the wife's playing Ballet with probability $2/3$ and Boxing with probability $1/3$ and the husband's playing Ballet with probability $1/3$ and Boxing with probability $2/3$.

At first glance, it seems that the wife puts more probability on Ballet because she prefers Ballet conditional on coordinating and the husband puts more probability on Boxing because he prefers Boxing conditional on coordinating. This intuition is misleading. The wife, for example, is indifferent between Ballet and Boxing in the mixed-strategy Nash equilibrium given her husband's strategy. She does not care what probabilities she plays Ballet and Boxing. What pins down her equilibrium probabilities is not her payoffs but her husband's. She has to put less probability on the action he prefers conditional on coordinating (Boxing) than on the other action (Ballet) or else he would not be indifferent between Ballet and Boxing and the probabilities would not form a Nash equilibrium.

MicroQuiz 6.2

1. In the Battle of the Sexes, does either player have a dominant strategy?

2. In general, can a game have a mixed-strategy Nash equilibrium if a player has a dominant strategy? Why or why not?

The Problem of Multiple Equilibria

Given that there are multiple equilibria, it is difficult to make a unique prediction about the outcome of the game. To solve this problem, game theorists have devoted a considerable amount of research to refining the Nash equilibrium concept, that is, coming up with good reasons for picking out one Nash equilibrium as being more "reasonable" than others. One suggestion would be to select the outcome with the highest total payoffs for the two players. This rule would eliminate the mixed-strategy Nash equilibrium in favor of one of the two pure-strategy equilibria. In the mixed-strategy equilibrium, we showed that each player's expected payoff is $2/3$ no matter which action is chosen, implying that the total expected payoff for the two players is $2/3 + 2/3 = 4/3$. In the two pure-strategy equilibria, total payoffs, equal to 3, exceed the total expected payoff in the mixed-strategy equilibrium.

A rule that selects the highest total payoff would not distinguish between the two pure-strategy equilibria. To select between these, one might follow T. Schelling's suggestion and look for a **focal point**.[2] For example, the equilibrium in which both play Ballet might be a logical focal point if the couple had a history of deferring to the wife's wishes on previous occasions. Without access to this

Focal point
Logical outcome on which to coordinate, based on information outside of the game.

[2] T. Schelling, *The Strategy of Conflict* (Cambridge, MA: Harvard University Press, 1960).

external information on previous interactions, it would be difficult for a game theorist to make predictions about focal points, however. Another suggestion would be that, since we have no reason to think one player is favored over another, to select the symmetric equilibrium. This rule might select the mixed-strategy Nash equilibrium because it is the only one that has equal payoffs (the wife's and husband's expected payoffs are both $2/3$). In sum, the basis for any of these selection rules is fairly weak. The Battle of the Sexes is one of those games for which there is simply no good way to solve the problem of multiple equilibria.

Coming up with a suitable refinement of Nash equilibrium allowing the selection of one from a set of multiple Nash equilibria is an issue at the forefront of current research in game theory. Perhaps an even more basic question, also at the forefront of current research, is why players are led to play Nash equilibria to begin with. One approach, described in Application 6.3: Evolutionary Game Theory, is to suppose that players are guided more by instinct than reason, instinct that has been honed by generations of evolution.

SEQUENTIAL GAMES

In some games, the order of moves matters. For example, in a bicycle race with a staggered start, the last racer has the advantage of knowing the time to beat. With new consumer technologies, for example HDTV sets, it may help to wait to buy until a critical mass of others have and so there are a sufficiently large number of program channels available.

Sequential games differ from the simultaneous games we have considered so far in that a player that moves after another can learn information about the play of the game up to that point, including what actions other players have chosen. The player can use this information to form more sophisticated strategies than simply choosing an action; the player's strategy can be a contingent plan, with the action played depending on what the other players do.

To illustrate the new concepts raised by sequential games, and in particular to make a stark contrast between sequential and simultaneous games, we will take a simultaneous game we have discussed already, the Battle of the Sexes, and turn it into a sequential game.

The Sequential Battle of the Sexes

Consider the Battle of the Sexes game analyzed previously with all the same actions and payoffs, but change the order of moves. Rather than the wife and husband making a simultaneous choice, the wife moves first, choosing Ballet or Boxing, the husband observes this choice (say the wife calls him from her chosen location), and then the husband makes his choice. The wife's possible strategies have not changed: she can choose the simple actions Ballet or Boxing (or perhaps a mixed strategy involving both actions, although this will not be a relevant consideration in the sequential game). The husband's set of possible strategies has expanded. For each of the wife's two actions, he can choose one of two actions,

Application 6.3 Evolutionary Game Theory

The development of the theory of evolution had been so successful in describing how living things' physical structure came to be that biologists began wondering if the theory could also explain animal behavior. Could the theory of evolution explain how hard two dogs were willing to fight over a bone? Could it explain why a bee might sacrifice itself to defend the hive?

A Biologist's Conception

John Maynard Smith pioneered the use of evolutionary game theory to understand animal behavior.[1] The theory assumes that animals behave according to instinctual strategies passed down from generation to generation through one's genes, just as physical characteristics are passed down. The more successful a particular strategy, the more "fit" is an animal in terms of the ability to survive and pass down its genes to subsequent generations. New strategies arise through mutations. Mutations quickly disappear in the population if they are not successful against the strategies that are predominant in the population. But if the mutations are successful, they can come to dominate in the population.

The Hawk-Dove Game

To see these ideas more concretely, consider the famous Hawk-Dove game shown in Table 1. Animals *A* and *B* play one of two possible actions when confronting each other over a piece of food. An animal may fight aggressively for the food (the Hawk strategy), or it may be more passive (the Dove strategy). If an animal playing Dove meets one playing Hawk, the Hawk gets the food without a fight. If two Hawks meet, they divide the benefit from the food, v, less the cost of fighting, c, in half. If two Doves meet, they divide v in half and do not fight. Animals from a large population are matched at random to play the game. The animal that gets the higher payoff in their meeting passes the genes for its strategy on to the next generation.

A strategy is said to be evolutionarily stable if, given it is pervasive in the population, any other strategy is introduced as a small mutation, this mutation will not spread in the population. Dove is not an evolutionarily stable strategy. If the whole population is playing Dove and a few Hawks are introduced, random matching will lead the Hawks nearly always to be matched against Doves, and Hawks will get a higher payoff in these matches (v compared to 0). On the other hand, Hawk is evolutionarily stable as long as the cost of fighting c does not exceed the benefit v.

TABLE 1 Hawk-Dove Game

		B	
		Hawk	**Dove**
A	**Hawk**	$(v - c)/2, (v - c)/2$	$v, 0$
	Dove	$0, v$	$v/2, v/2$

Evolutionary Game Theory in Economics

Evolutionary game theory has received considerable recent attention in economics. First, it provides a way of selecting among multiple Nash equilibria. It can be shown that evolutionarily stable strategies are Nash equilibria, but not all Nash equilibria are evolutionarily stable.

Second, and more importantly, evolutionary game theory provides an equilibrium concept that does not depend on players being hyperrational as does Nash equilibrium. In a two-player game, for example, Nash equilibrium requires player *A* to be rational in the sense that *A* plays a best response to *B*'s equilibrium action, but *A* must also know that *B* is rational, since *A* expects *B* to play a best response, and *A* must know that *B* knows that *A* is rational, and so on, a mind-boggling chain of reasoning. There is thus appeal in a concept such as evolutionarily stable strategies, which require no rationality and can explain regularities in even animal behavior.

To Think About

1. Researchers have applied evolutionary arguments to explain norms regarding sexual behavior (monogamy, polygamy, adultery).[2] How might one construct such arguments?
2. Are humans evolving faster than other animals? What factors might affect human evolution, given that nature plays less of a role in preventing survival to maturity as civilization advances?

[1] J. Maynard Smith, *Evolution and the Theory of Games* (Cambridge: Cambridge University Press, 1982).

[2] T. Burnham and J. Phelan, *Mean Genes: From Sex to Money to Food: Taming Our Primal Instincts* (New York: Penguin Books, 2000).

TABLE 6-8 Husband's Contingent Strategies

Contingent strategy	Strategy written equivalently in conditional format
Always go to Ballet	Ballet | Ballet, Ballet | Boxing
Follow his wife	Ballet | Ballet, Boxing | Boxing
Do the opposite	Boxing | Ballet, Ballet | Boxing
Always go to Boxing	Boxing | Ballet, Boxing | Boxing

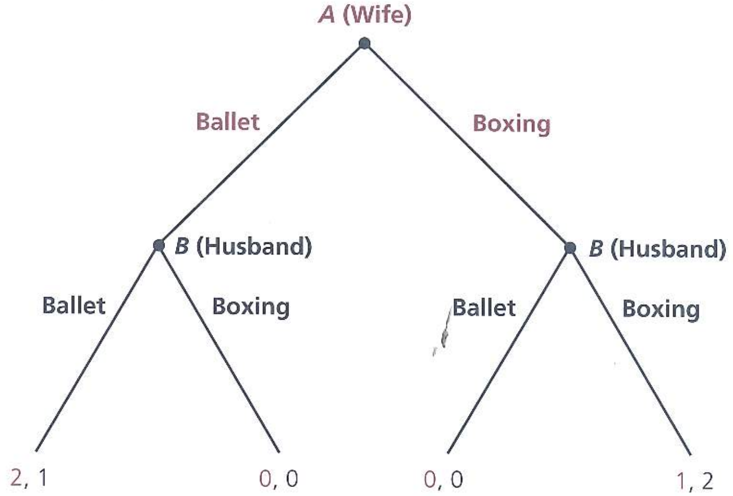
so he has four possible strategies, which are listed in Table 6-8. The vertical bar in the second equivalent way of writing the strategies means "conditional on," so, for example, "Boxing | Ballet" should be read "the husband goes to Boxing conditional on the wife's going to Ballet." The husband still can choose a simple action, with "Ballet" now interpreted as "always go to Ballet" and "Boxing" as "always go to Boxing," but he can also follow her or do the opposite.

Given that the husband has four pure strategies rather than just two, the normal form, given in Table 6-9, must now be expanded to have eight boxes. Roughly speaking, the normal form is twice as complicated as that for the simultaneous version of the game in Table 6-5. By contrast, the extensive form, given in Figure 6-5, is no more complicated than the extensive form for the simultaneous version of the game in Figure 6-3. The only difference between the extensive forms is that the oval around the husband's decision points has been removed. In

TABLE 6-9 Sequential Version of the Battle of the Sexes in Normal Form

		B (Husband)			
		Ballet | Ballet Ballet | Boxing	Ballet | Ballet Boxing | Boxing	Boxing | Ballet Ballet | Boxing	Boxing | Ballet Boxing | Boxing
A (Wife)	Ballet	2, 1	2, 1	0, 0	0, 0
	Boxing	0, 0	1, 2	0, 0	1, 2

FIGURE 6-5 Sequential Version of the Battle of the Sexes in Extensive Form



the sequential version of the game, the husband's decision points are not gathered together in a dotted oval because the husband observes his wife's action and so knows which one he is on before moving. We can begin to see why the extensive form becomes more useful than the normal form for sequential games, especially in games with many rounds of moves.

To solve for the Nash equilibria, we will return to the normal form and use the method of underlining payoffs for best responses introduced previously. Table 6-10 presents the results from this method. One complication that arises

TABLE 6-10 Solving for Nash Equilibria in the Sequential Version of the Battle of the Sexes

		B (Husband)			
		Ballet | Ballet Ballet | Boxing	Ballet | Ballet Boxing | Boxing	Boxing | Ballet Ballet | Boxing	Boxing | Ballet Boxing | Boxing
A (Wife)	Ballet	Nash equilibrium 1 2, 1	Nash equilibrium 2 2, 1	0, 0	0, 0
	Boxing	0, 0	1, 2	0, 0	Nash equilibrium 3 1, 2

in the method of underlining payoffs is that there are ties for best responses in this game. For example, if the husband plays the strategy "Boxing | Ballet, Ballet | Boxing," that is, if he does the opposite of his wife, then she earns zero no matter what action she chooses. To apply the underlining method properly, we need to underline both zeroes in the third column. There are also ties between the husband's best responses to his wife's playing Ballet (his payoff is 1 if he plays either "Ballet | Ballet, Ballet | Boxing" or "Ballet | Ballet, Boxing | Boxing") and to his wife's playing Boxing (his payoff is 2 if he plays either "Ballet | Ballet, Boxing | Boxing" or "Boxing | Ballet, Boxing | Boxing"). Again, as shown in the table, we need to underline the payoffs for all the strategies that tie for the best response. There are three pure-strategy Nash equilibria:

1. Wife plays Ballet, husband plays "Ballet | Ballet, Ballet | Boxing."
2. Wife plays Ballet, husband plays "Ballet | Ballet, Boxing | Boxing."
3. Wife plays Boxing, husband plays "Boxing | Ballet, Boxing | Boxing."

As we saw with the simultaneous version of the Battle of the Sexes, here again with the sequential version we have multiple equilibria. Here, however, game theory offers a good way to select among the equilibria. Consider the third Nash equilibrium. The husband's strategy, "Boxing | Ballet, Boxing | Boxing," involves an implicit threat that he will choose Boxing even if his wife chooses Ballet. This threat is sufficient to deter her from choosing Ballet. Given she chooses Boxing in equilibrium, his strategy earns him 2, which is the best he can do in any outcome. So the outcome is a Nash equilibrium. But the husband's threat is not credible, that is, it is an empty threat. If the wife really were to choose Ballet first, he would be giving up a payoff of 1 by choosing Boxing rather than Ballet. It is clear why he would want to threaten to choose Boxing, but it is not clear that such a threat should be believed. Similarly, the husband's strategy, "Ballet | Ballet, Ballet | Boxing," in the first Nash equilibrium also involves an empty threat, the threat that he will choose Ballet if his wife chooses Boxing. (This is an odd threat to make since he does not gain from making it, but it is an empty threat nonetheless.)

MICROQUIZ 6.3

Refer to the normal form of the sequential Battle of the Sexes.

1. Provide examples in which referring to equilibria using payoffs is ambiguous but with strategies is unambiguous.

2. Explain why "Boxing" or "Ballet" is not a complete description of the second-mover's strategy.

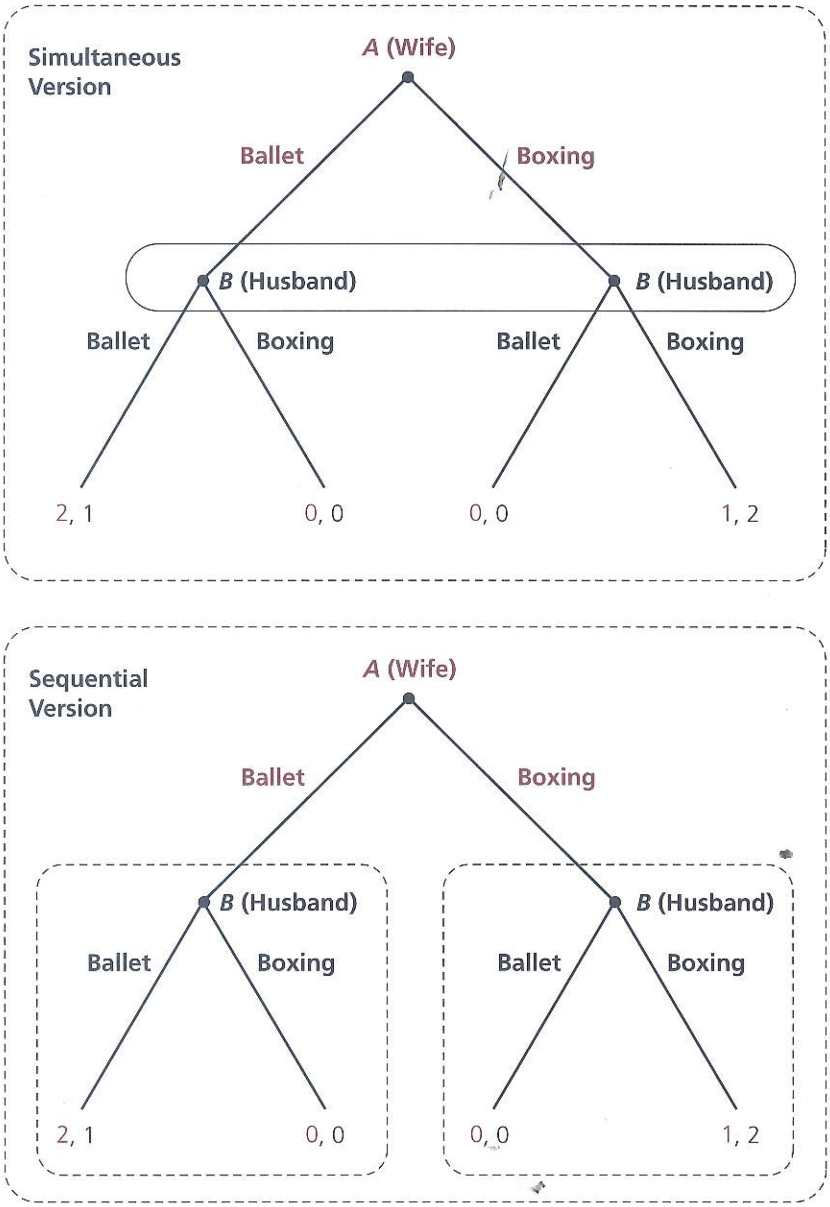
Subgame-Perfect Equilibrium

Proper subgame
Part of the game tree including an initial decision not connected to another in an oval and everything branching out below it.

Game theory offers a formal way of selecting the reasonable Nash equilibria in sequential games using the concept of subgame-perfect equilibrium. Subgame-perfect equilibrium is a refinement that rules out empty threats by requiring strategies to be rational even for contingencies that do not arise in equilibrium.

Before defining subgame-perfect equilibrium formally, we need to say what a proper subgame is. A subgame is a part of the extensive form beginning with a decision point and including everything that branches out below it. The decision point at the top of a **proper subgame** is not connected to another in the same oval. Conceptually, this means that the player who moves first in a proper subgame knows the actions played by others that have led up to that point. It is easier to see what a proper subgame is than to define it in words. Figure 6-6 shows the extensive forms from the simultaneous and sequential versions of the Battle of the Sexes, with dotted lines drawn around the proper subgames in each. In the

FIGURE 6-6 Proper Subgames in the Battle of the Sexes



simultaneous Battle of the Sexes, there is only one decision point that is not connected to another in an oval, the topmost one. Therefore, there is only one proper subgame, the game itself. In the sequential Battle of the Sexes, there are three proper subgames: the game itself, and two lower subgames starting with decision points where the husband gets to move.

Subgame-perfect equilibrium Strategies that form a Nash equilibrium on every proper subgame.

A **subgame-perfect equilibrium** is a set of strategies, one for each player, that form a Nash equilibrium on every proper subgame. A subgame-perfect equilibrium is always a Nash equilibrium. This is true since the whole game is a proper subgame of itself, so a subgame-perfect equilibrium must be a Nash equilibrium on the whole game. In the simultaneous version of the Battle of the Sexes, there is nothing more to say since there are no other subgames besides the whole game itself.

In the sequential version of the Battle of the Sexes, the concept of subgame-perfect equilibrium has more bite. In addition to strategies having to form a Nash equilibrium on the whole game itself, they must form Nash equilibria on the two other proper subgames, starting with the decision points at which the husband moves. These subgames are simple decision problems, and so it is easy to compute the corresponding Nash equilibria. For the left-hand subgame, beginning with the husband's decision point following his wife's choosing Ballet, he has a simple decision between Ballet, which earns him a payoff of 1, and Boxing, which earns him a payoff of 0. The Nash equilibrium in this simple decision subgame is for the husband to choose Ballet. For the right-hand subgame, beginning with the husband's decision point following from his wife's choosing Boxing, he has a simple decision between Ballet, which earns him 0, and Boxing, which earns him 2. The Nash equilibrium in this simple decision subgame is for him to choose Boxing. Thus we see that the husband has only one strategy that can be part of a subgame-perfect equilibrium: "Ballet | Ballet, Boxing | Boxing." Any other strategy has him playing something that is not a Nash equilibrium on some proper subgame. Returning to the three enumerated Nash equilibria, only the second one is subgame-perfect. The first and the third are not. For example, the third equilibrium, in which the husband always goes to Boxing, is ruled out as a subgame-perfect equilibrium because the husband would not go to Boxing if the wife indeed went to Ballet; he would go to Ballet as well. Subgame-perfect equilibrium thus rules out the empty threat of always going to Boxing that we were uncomfortable with in the previous section.

More generally, subgame-perfect equilibrium rules out any sort of empty threat in any sequential game. In effect, Nash equilibrium only requires behavior to be rational on the part of the game tree that is reached in equilibrium. Players can choose potentially irrational actions on other parts of the game tree. In particular, a player can threaten to damage both of them in order to "scare" the other from choosing certain actions. Subgame-perfect equilibrium requires rational behavior on all parts of the game tree. Threats to play irrationally, that is, threats to choose something other than one's best response, are ruled out as being empty.

Subgame-perfect equilibrium is not a useful refinement in a simultaneous game because a simultaneous game has no proper subgames besides the game itself, and so subgame-perfect equilibrium would not reduce the set of Nash equilibria.

Backward Induction

Our approach to solving for the equilibrium in the sequential Battle of the Sexes was to find all the Nash equilibria using the normal form, and then to sort through them for the subgame-perfect equilibrium. A shortcut to find the subgame-perfect equilibrium directly is to use **backward induction**. Backward induction works as follows: identify all of the subgames at the bottom of the extensive form; find the Nash equilibria on these subgames; replace the (potentially complicated) subgames with the actions and payoffs resulting from Nash equilibrium play on these subgames; then move up to the next level of subgames and repeat the procedure.

Backward induction Solving for equilibrium by working backwards from the end of the game to the beginning.

Figure 6-7 illustrates the use of backward induction to solve for the subgame-perfect equilibrium of the sequential Battle of the Sexes. First compute the Nash equilibria of the bottom-most subgames, in this case the subgames corresponding to the husband's decision problems. In the subgame following his wife's choosing Ballet, he would choose Ballet, giving payoffs 2 for her and 1 for him. In the subgame following his wife's choosing Boxing, he would choose Boxing, giving payoffs 1 for her and 2 for him. Next, substitute the husband's equilibrium strategies for the subgames themselves. The resulting game is a simple decision problem for the wife, drawn in the lower panel of the figure, a choice between Ballet, which would give her a payoff of 2 and Boxing, which would give her a payoff of 1. The Nash equilibrium of this game is for her to choose the action with the higher payoff, Ballet. In sum, backward induction allows us to jump straight to the subgame-perfect equilibrium, in which the wife chooses Ballet and the husband chooses "Ballet | Ballet, Boxing | Boxing," and bypass the other Nash equilibria.

Backward induction is particularly useful in games in which there are multiple rounds of sequential play. As rounds are added, it quickly becomes too hard to solve for all the Nash equilibria and then to sort through which are subgame-perfect. With backward induction, an additional round is simply accommodated by adding another iteration of the procedure.

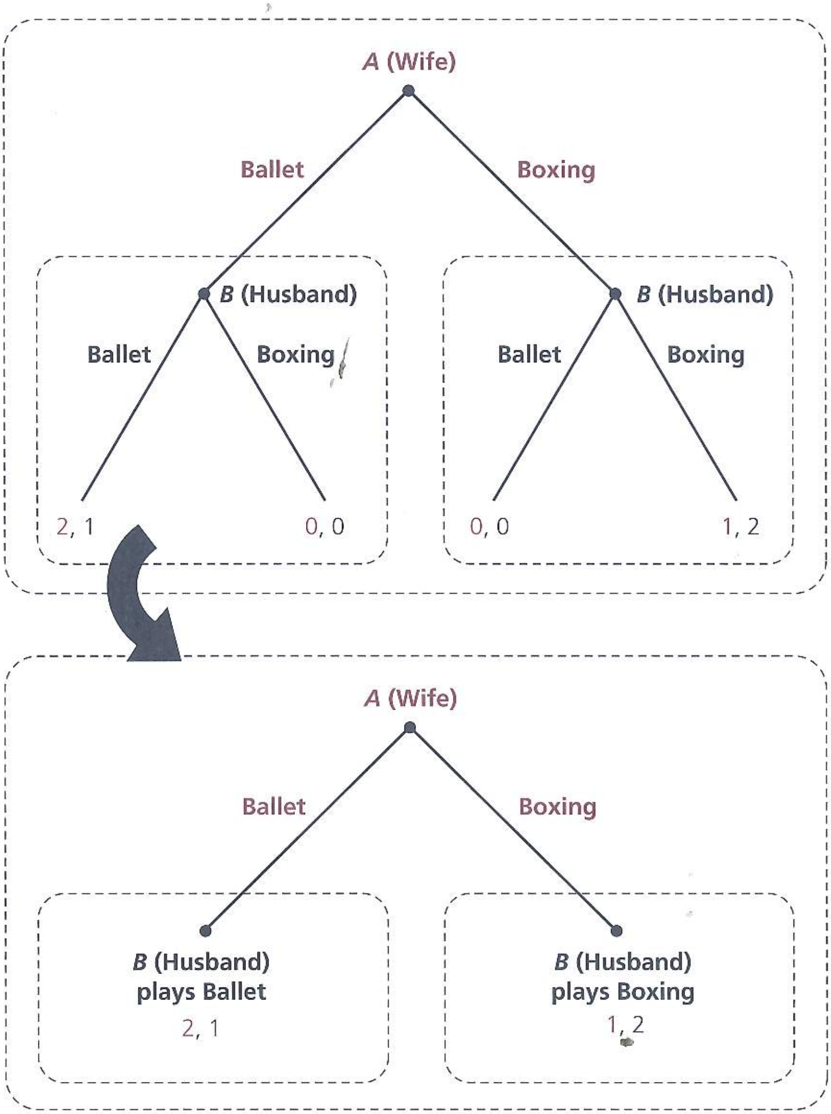
Application 6.4: Laboratory Experiments discusses whether human subjects play games the way theory predicts in experimental settings, including whether subjects play the subgame-perfect equilibrium in sequential games.

Repeated Games

So far, we have examined one-shot games in which each player is given one choice and the game ends. In many real-world settings, the same players play the same **stage game** several or even many times. For example, the players in the Prisoners' Dilemma may anticipate committing future crimes together and thus playing future Prisoners' Dilemmas together. Gas stations located across the street from each other, when they set their prices each morning, effectively play a new pricing game every day. As we saw with the Prisoners' Dilemma, when such games are played once, the equilibrium outcome may be worse for all players

Stage game Simple game that is played repeatedly.

FIGURE 6-7 Backward Induction in the Sequential Battle of the Sexes



Trigger strategy
Strategy in a repeated game where the player stops cooperating in order to punish another player's break with cooperation.

than some other, more cooperative, outcome. Repetition opens up the possibility of the cooperative outcome being played in equilibrium. Players can adopt **trigger strategies**, whereby they play the cooperative outcome as long as all have cooperated up to that point, but revert to playing the Nash equilibrium if anyone breaks with cooperation. We will investigate the conditions under which trigger strategies work to increase players' payoffs. We will focus on subgame-perfect equilibria of the repeated games.

Application 6.4 Laboratory Experiments

Experimental economics tests how well economic theory matches the behavior of experimental subjects in laboratory settings. The methods are similar to those used in experimental psychology—often conducted on campus using undergraduates as subjects—the main difference being that experiments in economics tend to involve incentives in the form of explicit monetary payments paid to subjects. The importance of experimental economics was highlighted in 2002, when Vernon Smith received the Nobel prize in economics for his pioneering work in the field.

Experiments with the Prisoners' Dilemma

There have been hundreds of tests of whether players Confess in the Prisoners' Dilemma, as predicted by Nash equilibrium, or whether they play the cooperative outcome of Silent. In the experiments of Cooper et al.,[1] subjects played the game 20 times, against different, anonymous opponents. Play converged to the Nash equilibrium as subjects gained experience with the game. Players played the cooperative action 43 percent of the time in the first five rounds, falling to only 20 percent of the time in the last five rounds.

Experiments with the Ultimatum Game

Experimental economics has also tested to see whether subgame-perfect equilibrium is a good predictor of behavior in sequential games. In one widely studied sequential game, the Ultimatum Game, the experimenter provides a pot of money to two players. The first mover (Proposer) proposes a split of this pot to the second mover. The second mover (Responder) then decides whether to accept the offer, in which case players are given the amount of money indicated, or reject the offer, in which case both players get nothing. As one can see by using backward induction, in the subgame-perfect equilibrium, the Proposer should offer a minimal share of the pot and this should be accepted by the Responder.

In experiments, the division tends to be much more even than in the subgame-perfect equilibrium.[2] The most common offer is a 50-50 split. Responders tend to reject offers giving them less than 30 percent of the pot. This result is observed even when the pot is as high as $100, so that rejecting a 30 percent offer means turning down $30. Some economists have suggested that money may not be a true measure of players' payoffs, which may include other factors such as how fairly the pot is divided.[3] Even if a Proposer does not care directly about fairness, the fear that the Responder may care about fairness and thus might reject an uneven offer out of spite may lead the Proposer to propose an even split.

Experiments with the Dictator Game

To test whether players care directly about fairness or act out of fear of the other player's spite, researchers experimented with a related game, the Dictator Game. In the Dictator Game, the Proposer chooses a split of the pot, and this split is implemented without input from the Responder. Proposers tend to offer a less-even split than in the Ultimatum Game, but still offer the Responder some of the pot, suggesting Responders had some residual concern for fairness. The details of the experimental design are crucial, however, as one ingenious experiment showed.[4] The experiment was designed so that the experimenter would never learn which Proposers had made which offers. With this element of anonymity, Proposers almost never gave an equal split to Responders and, indeed, took the whole pot for themselves two-thirds of the time. The results suggest that Proposers care more about being thought of as fair rather than truly being fair.

To Think About

1. As an experimenter, how would you choose the following aspects of experimental design? Are there any tradeoffs involved?
   a. Size of the payoffs.
   b. Ability of subjects to see opponents.
   c. Playing the same game against the same opponent repeatedly.
   d. Informing subjects fully about the experimental design.
2. How would you construct an experiment involving the Battle of the Sexes? What theoretical issues might it be interesting to test with your experiment?

[1] R. Cooper, D. V. DeJong, R. Forsythe, and T. W. Ross, "Cooperation Without Reputation: Experimental Evidence from Prisoner's Dilemma Games," *Games and Economic Behavior* (February 1996): 187–218.

[2] For a review of Ultimatum Game experiments and a textbook treatment of experimental economics more generally, see D. D. Davis and C. A. Holt, *Experimental Economics* (Princeton: Princeton University Press, 1993).

[3] See, for example, M. Rabin, "Incorporating Fairness into Game Theory and Economics," *American Economic Review* (December 1993): 1281–1302.

[4] E. Hoffman, K. McCabe, K. Shachat, and V. Smith, "Preferences, Property Rights, and Anonymity in Bargaining Games," *Games and Economic Behavior* (November 1994): 346–380.

Definite Time Horizon

For many stage games, repeating them a known, finite number of times does not increase the possibility for cooperation. To see this point concretely, suppose the Prisoners' Dilemma were repeated for 10 periods. Use backward induction to solve for the subgame-perfect equilibrium. The lowest subgame is the one-shot Prisoners' Dilemma played in the 10th period. Regardless of what happened before, the Nash equilibrium on this subgame is for both to play Confess. Folding the game back to the ninth period, trigger strategies that condition play in the 10th period on what happens in the ninth are ruled out. Nothing that happens in the ninth period affects what happens subsequently because, as we just argued, the players both Confess in the 10th period no matter what. It is as if the ninth period is the last, and again the Nash equilibrium on this subgame is again for both to play Confess. Working backward in this way, we see that players will Confess each period; that is, players will simply repeat the Nash equilibrium of the stage game 10 times. The same argument would apply for any definite number of repetitions.

Indefinite Time Horizon

If the number of times the stage game is repeated is indefinite, matters change significantly. The number of repetitions is indefinite if players know the stage game will be repeated but are uncertain of exactly how many times. For example, the partners in crime in the Prisoners' Dilemma may know that they will participate in many future crimes together, sometimes be caught, and thus have to play the Prisoners' Dilemma game against each other, but may not know exactly how many opportunities for crime they will have or how often they will be caught, and so will not know exactly how many times they will play the stage game. With an indefinite number of repetitions, there is no final period from which to start applying backward induction, and thus no final period for trigger strategies to begin unraveling. Under certain conditions, more cooperation can be sustained than in the stage game.

Suppose the two players play the following repeated version of the Prisoners' Dilemma. The game is played in the first period for certain, but for how many more periods after that the game is played is uncertain. Let r be the probability the game is repeated for another period and $1 - r$ the probability the repetitions stop for good. Thus, the probability the game lasts at least one period is 1, at least two periods is r, at least three periods is r^2, and so forth.

Suppose players use the trigger strategies of playing the cooperative action, Silent, as long a no one cheats by playing Confess, but that players both play Confess forever afterward if either of them had ever cheated. To show that such strategies form a subgame-perfect equilibrium, we need to check that a player cannot gain by cheating. In equilibrium, both players play Silent and each earns -2 each period the game is played, implying a player's expected payoff over the course of the entire game is

$$(-2)(1 + r + r^2 + r^3 + \cdots) \tag{6.1}$$

If a player cheats and plays Confess, given the other is playing Silent, the cheater earns -1 in that period, but then both play Confess every period, from then on, each earning -3 each period, for a total expected payoff of

$$-1 \quad + \quad (-3)(r + r^2 + r^3 + \cdots) \tag{6.2}$$

For cooperation to be a subgame-perfect equilibrium, (6.1) must exceed (6.2). Adding 2 to both expressions, and then adding $3(r + r^2 + r^3 + \cdots)$ to both expressions, (6.1) exceeds (6.2) if

$$r + r^2 + r^3 + \cdots > 1 \tag{6.3}$$

To proceed further, we need to find a simple expression for the series $r + r^2 + r^3 + \cdots$. A standard mathematical result is that the series $r + r^2 + r^3 + \cdots$ equals $r/(1 - r)$.[3] Substituting this result in (6.3), we see that (6.3) holds, and so cooperation on Silent can be sustained, if r is greater than ½.[4]

This result means that players can cooperate in the repeated Prisoners' Dilemma only if the probability of repetition r is high enough. Players are deterred from getting the short-run gain that cheating and playing Confess provides (a payoff of -1 rather than -2) with the threat of the loss of future payoffs as cheating leads them to revert to both playing Confess thereafter. This threat only works if the probability the game is repeated, and so future payoffs are actually realized, is high enough.

> **MICROQUIZ 6.4**
>
> Consider the indefinitely repeated Prisoners' Dilemma.
>
> **1.** For what value of r does the repeated game become simply the stage game?
>
> **2.** Suppose at some point while playing the grim strategy, players relent and go back to the cooperative outcome (Silent). If this relenting were anticipated, how would it affect the ability to sustain the cooperative outcome using trigger strategies to begin with?

Other strategies can be used to try to elicit cooperation in the repeated game. We considered strategies that had players revert to the Nash equilibrium of Confess each period forever. This strategy, which involves the harshest possible punishment for deviation, is called the grim strategy. Less harsh punishments include the so-called tit-for-tat strategy, which involves only one round of punishment for cheating. Since it involves the harshest punishment possible, the grim strategy elicits cooperation for the largest range of cases (the lowest value of r) of any strategy. Harsh punishments work well because, if players succeed in cooperating, they never experience the losses from the punishment in equilibrium. If there were uncertainty about the

[3] Let $S = r + r^2 + r^3 + \cdots$. Multiplying both sides by r, $rS = r^2 + r^3 + r^4 + \cdots$. Subtracting rS from S, we have $S - rS = (r + r^2 + r^3 + \cdots) - (r^2 + r^3 + r^4 + \cdots) = r$ since all of the terms on the right-hand side cancel except for the leading r. Thus $(1 - r)S = r$, or, rearranging, $S = r/(1 - r)$.

[4] The mathematics are the same in an alternative version of the game in which the stage game is repeated with certainty each period for an infinite number of periods, but in which future payoffs are discounted according to a per-period interest rate. One can show that cooperation is possible if the per-period interest rate is less than 100 percent.

economic environment, or about the rationality of the other player, the grim strategy may not lead to as high payoffs as less-harsh strategies.

One might ask whether the threat to punish the other player (whether forever as in the grim strategy or for one round with tit-for-tat) is an empty threat since punishment harms both players. The answer is no. The punishment involves reverting to the Nash equilibrium, in which both players choose best responses, and so it is a credible threat and is consistent with subgame-perfect equilibrium.

Continuous Actions

Most of the insight from economic situations can often be gained by distilling the situation down to a game with two actions, as with all the games studied so far. Other times, additional insight can be gained by allowing more actions, sometimes even a continuum. Firms' pricing, output or investment decisions, bids in auctions, and so forth are often modeled by allowing players a continuum of actions. Such games can no longer be represented in the normal form we are used to seeing in this chapter, and the underlining method cannot be used to solve for Nash equilibrium. Still, the new techniques for solving for Nash equilibria will have the same logic as those seen so far. We will illustrate the new techniques in a game called the Tragedy of the Commons.

Tragedy of the Commons

The game involves two shepherds, A and B, who graze their sheep on a common (land that can be freely used by community members). Let s_A and s_B be the number of sheep each grazes, chosen simultaneously. Because the common only has a limited amount of space, if more sheep graze, there is less grass for each one, and they grow less quickly. To be concrete, suppose the benefit A gets from each sheep (in terms of mutton and wool) equals

$$120 - s_A - s_B \tag{6.4}$$

The total benefit A gets from a flock of s_A sheep is therefore

$$s_A(120 - s_A - s_B) \tag{6.5}$$

While we cannot use the method of underlining payoffs for best responses, we can compute A's best-response function. Recall the use of best-response functions in computing the mixed-strategy Nash equilibrium in the Battle of the Sexes game. We resorted to best-response functions because, although the Battle of the Sexes game has only two actions, there is a continuum of possible mixed strategies over those two actions. In the Tragedy of the Commons here, we need to resort to best-response functions because we start off with a continuum of actions.

A's best-response function gives the s_A that maximizes A's payoff for any s_B. A's best response will be the number of sheep such that the marginal benefit of an additional sheep equals the marginal cost. His marginal benefit of an additional sheep is[5]

$$120 - 2s_A - s_B \tag{6.6}$$

The total cost of grazing sheep is 0 since they graze freely on the common, and so the marginal cost of an additional sheep is also 0. Equating the marginal benefit in (6.6) with the marginal cost of 0 and solving for s_A, A's best-response function equals

$$s_A = 60 - \frac{s_B}{2} \tag{6.7}$$

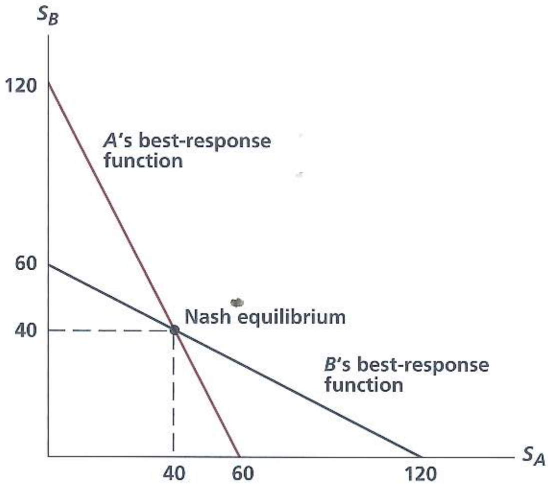
By symmetry, B's best-response function is

$$s_B = 60 - \frac{s_A}{2} \tag{6.8}$$

For actions to form a Nash equilibrium, they must be best responses to each other; in other words, they must be the simultaneous solution to (6.7) and (6.8). The simultaneous solution is shown graphically in Figure 6-8. The best-response functions are graphed with s_A on the horizontal axis and s_B on the vertical (the inverse of A's best-response function is actually what is graphed). The Nash equilibrium, which lies at the intersection of the two functions, involves each grazing 40 sheep.

The game is called a tragedy because the shepherds end up overgrazing in equilibrium. They overgraze because they do not take into account the reduction in the value of other's sheep when they choose the size of their flocks. If each grazed 30 rather than 40 sheep, one can show that each would earn a total payoff of 1,800 rather than the 1,600 they each earn in equilibrium. Over-consumption is a typical finding in settings where multiple parties have free access to a common resource, such as

FIGURE 6-8 Best-Response Functions in the Tragedy of the Commons



[5] One can take the formula for the marginal benefit in (6.6) as given or can use calculus to verify it. Differentiating the benefit function (6.5), which can be rewritten $120s_A - s_A^2 - s_A s_B$, term by term with respect to s_A (treating s_B as a constant) yields the marginal benefit (6.6).

multiple wells pumping oil from a common underground pool or multiple fishing boats fishing in the same ocean area, and is often a reason given for restricting access to such common resources through licensing and other government interventions.

Shifting Equilibria

One reason it is useful to allow players to have continuous actions is that it is easier in this setting to analyze how a small change in one of the game's parameters shifts the equilibrium. For example, suppose A's benefit per sheep rises from (6.4) to

$$132 - s_A - s_B \tag{6.9}$$

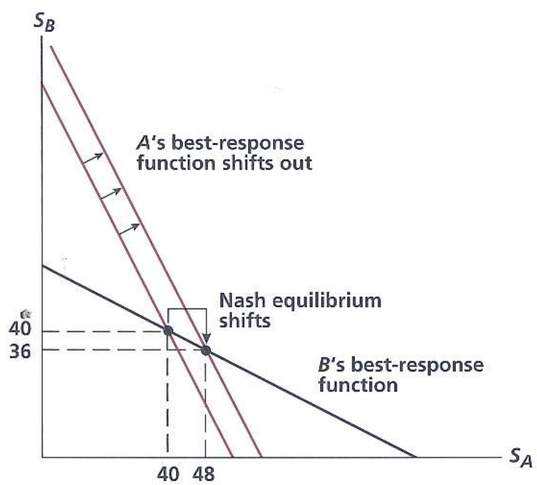
A's best-response function becomes

$$s_A = 66 - \frac{s_B}{2} \tag{6.10}$$

B's stays the same as in (6.8). As shown in Figure 6-9, in the new Nash equilibrium, A increases his flock to 48 sheep and B decreases his to 36. It is clear why the size of A's flock increases: the increase in A's benefit shifts his best-response function out. The interesting strategic effect is that—while nothing about B's benefit has changed, and so B's best-response function remains the same as before—having observed A's benefit increasing from (6.4) to (6.9), B anticipates that it must choose a best response to a higher quantity by A, and so ends up reducing the size of his flock.

Games with continuous actions offer additional insights in other contexts, as shown in Application 6.5: Terrorism.

FIGURE 6-9 Shift in Equilibrium When *A*'s Benefit Increases An increase in *A*'s benefit per sheep shifts his best-response function out. Though *B*'s best-response function remains the same, his equilibrium number of sheep falls in the new Nash equilibrium.



N-PLAYER GAMES

Just as we can often capture the essence of a situation using a game with two actions, as we have seen with all the games studied so far, we can often distill the number of players down to two as well. However in some cases, it is useful to study games with more than two players. This is particularly useful to answer the question of how a change in the number of players would affect

Application 6.5 Terrorism

Few issues raise as much public-policy concern as terrorism, given the continued attacks in the Middle East and Europe and the devastating attack on the World Trade Center and Pentagon in the United States on September 11, 2001. In this application, we will see that game theory can be usefully applied to analyze terrorism and the best defensive measures against it.

Defending Targets against Terrorism

Consider a sequential game between a government and a terrorist. The players have the opposite objectives: the government wants to minimize the expected damage from terrorism, and the terrorist wants to maximize expected damage. For simplicity, assume the terrorist can attack one of two targets: target 1 (say a nuclear power plant) leads to considerable damage if successfully attacked; target 2 (say a restaurant) leads to less damage. The government moves first, choosing s_1, the proportion of its security force guarding target 1. The remainder of the security force, $1 - s_1$, guards target 2. (Note that the government's action is a continuous variable between 0 and 1, so this is an application of our general discussion of games with continuous actions in the text.) The terrorist moves second, choosing which target to attack. Assume the probability of successful attack on target 1 is $1 - s_1$ and on target 2 is s_1, implying that the larger the security force guarding a particular target, the lower the probability of a successful attack.

To solve for the subgame-perfect equilibrium, we will apply backward induction, meaning in this context that we will consider the terrorist's (the second-mover's) decision first. The terrorist will compute the expected damage from attacking each target, equal to the probability of a successful attack multiplied by the damage caused if the attack is successful. The terrorist will attack the target with the highest expected damage. Moving backward to the first mover's (the government's) decision, the way for the government to minimize the expected damage from terrorism is to divide the security force between the two targets so that the expected damage is equalized. (Suppose the expected damage from attacking target 1 were strictly higher than target 2. Then the terrorist would definitely attack target 1, and the government could reduce expected damage from this attack by shifting some of the security force from target 2 to target 1.) Using some numbers, if the damage from a successful attack on target 1 is 10 times that on target 2, the government should put 10 times the security force on target 1. The terrorist ends up playing a mixed strategy in equilibrium, with each target having a positive probability of being attacked.

Bargaining with Terrorists

Terrorism raises many more issues than those analyzed above. Suppose terrorists have taken hostages and demand the release of prisoners in return for the hostages' freedom. Should a country bargain with the terrorists?[1] The official policy of countries, including the United States and Israel, is no. Using backward induction, it is easy to see why countries would like to commit to not bargaining, since this would preclude any benefit from taking hostages and deter the terrorists from taking hostages in the first place. But a country's commitment to not bargain may not be credible, especially if the hostages are "important" enough, as was the case when the Israeli parliament voted to bargain for the release of 21 students taken hostage in a high school in Maalot, Israel, in 1974. (The vote came after the deadline set by the terrorists, and the students ended up being killed.) The country's commitment may still be credible in some scenarios. If hostage incidents are expected to arise over time repeatedly, the country may refuse to bargain as part of a long-term strategy to establish a reputation for not bargaining. Another possibility is that the country may not trust the terrorists to free the hostages after the prisoners are released, in which case there would be little benefit from bargaining with them.

To Think About

1. The U.S. government has considered analyzing banking transactions to look for large, suspicious movements of cash as a screen for terrorists. What are the pros and cons of such a screen? How would the terrorists respond in equilibrium if they learned of this screen? Would it still be a useful tool?
2. Is it sensible to model the terrorist as wanting to maximize expected damage? Instead, the terrorist may prefer to attack "high-visibility" targets, even if this means lower expected damage, or may prefer to maximize the sum of damage plus defense/deterrence expenditures. Which alternative is most plausible? How would these alternatives affect the game?

[1] See H. E. Lapan and T. Sandler, "To Bargain or not to Bargain: That Is the Question," *American Economic Review* (May 1988): 16–20.

MICROQUIZ 6.5

Suppose the Tragedy of the Commons involved three shepherds (A, B, and C). Suppose the benefit per sheep is $120 - s_A - s_B - s_C$, implying that, for example, A's total benefit is $s_A(120 - s_A - s_B - s_C)$ and marginal benefit is $120 - 2s_A - s_B - s_C$.

1. Solve the three equations that come from equating each of the three shepherds' marginal benefit of a sheep to the marginal cost (zero) to find the Nash equilibrium.

2. Compare the total number of sheep on the common with three shepherds to that with two.

the equilibrium (see, for example, MicroQuiz 6.5). The problems at the end of the chapter will provide some examples of how to draw the normal form in games with more than two players.

INCOMPLETE INFORMATION

In all the games studied so far, there was no private information. All players knew everything there was to know about each others' payoffs, available actions, and so forth. Matters become more complicated, and potentially more interesting, if players know something about themselves that others do not know. For example, one's bidding strategy in a sealed-bid auction for a painting would be quite different if one knew the valuation of everyone else at the auction compared to the (more realistic) case in which one did not. Card games would be quite different, certainly not as fun, if all hands were played face up. Games in which players do not share all relevant information in common are called games of **incomplete information.**

Incomplete information Some players have information about the game that others do not.

We will devote most of Chapter 17 to studying games of incomplete information. We will study signaling games, which include students choosing how much education to obtain in order to signal their underlying aptitude, which might be difficult to observe directly, to prospective employers. We will study screening games, which include the design of deductible policies by insurance companies in order to deter high-risk consumers from purchasing. As mentioned, auctions and card games also fall in the realm of games of incomplete information. Such games are at the forefront of current research in game theory.

SUMMARY

This chapter provided a brief overview of game theory. Game theory provides an organized way of understanding decision making in strategic environments. We introduced the following broad ideas:

- The basic building blocks of all games are players, actions, and payoffs.
- Nash equilibrium is the most widely used equilibrium concept. Strategies form a Nash equilibrium if all players' strategies are best responses to each other. All games have at least one Nash equilibrium. Sometimes the Nash equilibrium is in mixed strategies, which we learned how to compute. Some games have multiple Nash equilibria, and it may be difficult in these cases to make predictions about which one will end up being played.
- We studied several classic games, including the Prisoners' Dilemma, Matching Pennies, and Battle

of the Sexes. These games each demonstrated important principles. Many strategic situations can be distilled down to one of these games.

- Sequential games introduce the possibility of contingent strategies for the second mover and often expand the set of Nash equilibria. Subgame-perfect equilibrium rules out outcomes involving noncredible threats. One can easily solve for subgame-perfect equilibrium using backward induction.
- In some games such as the Prisoners' Dilemma, all players are worse off in the Nash equilibrium than in some other outcome. If the game is repeated an indefinite number of times, players can use trigger strategies to try to enforce the better outcome.

REVIEW QUESTIONS

1. Why is the Prisoners' Dilemma a "dilemma" for the players involved? How might they solve this dilemma through pre-game discussions or post-game threats? If you were arrested and the D.A. tried this ploy, what would you do? Would it matter whether you were very close friends with your criminal accomplice?
2. In the Tragedy of the Commons, we saw how a small change in *A*'s benefit resulted in a shift in *A*'s best-response function and a movement along *B*'s best-response function. Can you think of other factors that might shift *A*'s best-response function? Relate this discussion to shifts in an individual's demand curve versus movements along it.
3. In game theory, players maximize payoffs. Is this assumption different from the one we used in Chapter 2 through Chapter 4?
4. The Battle of the Sexes is a coordination game. What coordination games arise in your experience? How do you go about solving coordination problems?
5. Choose a setting from student life. Try to model it as a game, with a set number of players, payoffs, and actions. Is it like any of the classic games studied in this chapter?
6. In the sequential games such as the sequential Battle of the Sexes, why does Nash equilibrium allow for outcomes with non-credible threats? Why does subgame-perfect equilibrium rule them out?
7. What is the difference between an action and a strategy?
8. Why are Nash equilibria identified by the strategies rather than the payoffs involved?
9. Which of the following activities might be represented as a zero-sum game? Which are clearly not zero-sum?
   a. Flipping a coin for $1.
   b. Playing blackjack.
   c. Choosing which candy bar to buy from a vendor.
   d. Reducing taxes through various "creative accounting" methods and seeking to avoid detection by the IRS.
   e. Deciding when to rob a particular house, knowing that the residents may adopt various counter-theft strategies.
10. Which of these relationships would be better modeled as involving repetitions and which not, or does it depend? For those that are repeated, which are more realistically seen as involving a definite number of repetitions and which an indefinite number?
    a. Two nearby gas stations posting their prices each morning.
    b. A professor testing students in a course.
    c. Students entering a dorm room lottery together.
    d. Accomplices committing a crime.
    e. Two lions fighting for a mate.

PROBLEMS

6.1 Consider a simultaneous game in which player *A* chooses one of two actions (Up or Down), and *B* chooses one of two actions (Left or Right). The game has the following payoff matrix, where the first payoff in each entry is for *A* and the second for *B*.

		B Left	*B* Right
A	Up	10, *X*	41, 30
	Down	20, 30	50, *Y*

a. Give conditions on *X* and *Y* that ensure that player *B* has a dominant strategy of playing Left.
b. Give conditions on *X* and *Y* that ensure that player *B* has a dominant strategy of playing Right.
c. Given your answer to part a, is there a Nash equilibrium in that setting? If so, what is it?
d. Given your answer to part b, is there a Nash equilibrium in that setting? If so, what is it?
e. Can you come up with values for *X* and *Y* such that there are two Nash equilibria?

6.2 The Chicken Game is played by two macho teens who speed toward each other on a single-lane road. The first to veer off is branded a coward or "chicken," whereas the one who doesn't turn gains peer group esteem. Of course, if neither veers, both die in the resulting crash. Payoffs to the Chicken Game are provided in the following table.

		Teen *B* Chicken	Teen *B* Not Chicken
Teen *A*	Chicken	2, 2	1, 3
	Not Chicken	3, 1	0, 0

a. Find the pure-strategy Nash equilibrium or equilibria.
b. Compute the mixed-strategy Nash equilibrium. As part of your answer, draw the best-response function diagram for the mixed strategies.
c. Suppose the game is played sequentially, with teen *A* moving first and committing to this action by throwing away the steering wheel. What are teen *B*'s contingent strategies? Write down the normal and extensive forms for the sequential version of the game.
d. Use the normal form for the sequential version of the game to solve for the Nash equilibria.
e. Identify the proper subgames in the extensive form for the sequential version of the game. Use backward induction to solve for the subgame-perfect equilibrium. Explain why the other Nash equilibria of the sequential game are "unreasonable."

6.3 Return to the Battle of the Sexes in Table 6-5. Compute the mixed-strategy Nash equilibrium under the following modifications and compare it to the one computed in the text. Draw the corresponding best-response-function diagram for the mixed strategies.

a. Double all of the payoffs.
b. Double the payoff from coordinating on one's preferred activity from 2 to 4 but leave all other payoffs the same.
c. Change the payoff from choosing one's preferred activity alone (that is, not coordinating with one's spouse) from 0 to ½ for each but leave all the other payoffs the same.

6.4 Two classmates *A* and *B* are assigned an extra group project. Each student can choose to Shirk or Work. If one or more players chooses Work, the project is completed and provides each with extra credit valued at 4 payoff units each. The cost of completing the project is that 6 total units of effort (measured in payoff units) is divided equally among all players who choose to Work and this is subtracted from their payoff. If both Shirk, they do not have to expend any effort but the project is not completed, giving each a payoff of 0. The teacher can only tell whether the project is completed and not which students contributed to it.

a. Write down the normal form for this game, assuming students choose to Shirk or Work simultaneously.
b. Find the Nash equilibrium or equilibria.
c. Does either player have a dominant strategy? What game from the chapter does this resemble?

6.5 Consider the Tragedy of the Commons game from the chapter with two shepherds, A and B, where s_A and s_B denote the number of sheep each grazes on the common pasture. Assume that the benefit per sheep (in terms of mutton and wool) equals

$$300 - s_A - s_B$$

implying that the total benefit from a flock of s_A sheep is

$$s_A(300 - s_A - s_B)$$

and that the marginal benefit of an additional sheep (as one can use calculus to show or can take for granted) is

$$300 - 2s_A - s_B$$

Assume the (total and marginal) cost of grazing sheep is zero since the common can be freely used.

a. Compute the flock sizes and shepherds' total benefits in the Nash equilibrium.
b. Draw the best-response-function diagram corresponding to your solution.
c. Suppose A's benefit per sheep rises to $330 - s_A - s_B$. Compute the new Nash equilibrium flock sizes. Show the change from the original to the new Nash equilibrium in your best-response-function diagram.

6.6 Find the pure-strategy Nash equilibrium or equilibria of the following game with three actions for each player.

		B: Left	B: Center	B: Right
A	Up	4, 3	5, −1	6, 2
A	Middle	2, 1	7, 4	3, 6
A	Down	3, 0	9, 6	0, 8

6.7 Consider a simultaneous game in which player A chooses one of two actions (Up or Down), and B chooses one of two actions (Left or Right). The game has the following payoff matrix, where the first payoff in each entry is for A and the second for B.

		B: Left	B: Right
A	Up	100, 99	41, 30
A	Down	90, 80	45, 82

a. Find the Nash equilibrium or equilibria.
b. Which player, if any, has a dominant strategy?

6.8 Suppose A can somehow change the game in problem 6.7 to a new one in which his or her payoffs from Up are increased by 5.

a. Does this change your answer to part a from problem 6.7? If so, how?
b. Does this change your answer to part b from problem 6.7? If so, how?
c. Does A benefit from changing the game by increasing his or her payoff in this way?

6.9 Return to the game given by the payoff matrix in problem 6.7.

a. Write down the extensive form for the simultaneous move game.
b. Suppose the game is now sequential move, with A moving first and then B. Write down the extensive form for this sequential-move game.
c. Write down the normal form for the sequential-move game. Find all the Nash equilibria. Which Nash equilibrium is subgame-perfect?

6.10 In this problem higher payoffs are worse than lower payoffs. The following game is a version of the Prisoners' Dilemma, but the payoffs are slightly different than in Table 6-1.

		B	
		Silent	Confess
A	Silent	10, 10	20, −5
	Confess	−5, 20	15, 15

a. Verify that the Nash equilibrium is the usual one for the Prisoners' Dilemma and that both players have dominant strategies.
b. Suppose the stage game is played infinitely often with a probability p that the game is continued to the next stage and $1 - p$ that the game ends for good. Compute the level of p that is required for a subgame-perfect equilibrium in which both players play a trigger strategy where both are Silent if no one deviates but resort to a grim strategy (that is, both play Confess forever after) if anyone deviates to Confess.
c. Continue to suppose the stage game is played infinitely often, as in b. Is there a value of p for which there exists a subgame-perfect equilibrium in which both players play a trigger strategy where both are Silent if no one deviates but resort to tit-far-tat (that is, both play Confess for one period and go back to Silent forever after that) if anyone deviates to Confess. Remember that p is a probability so it must be between 0 and 1.