BLUEPRINT FOR THE LIQUID TENSOR EXPERIMENT

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Remark 0.1. This text is based on the lecture notes on Analytic Geometry [Sch20], by Peter Scholze. The final section is copy-pasted from those lecture notes almost verbatim. This text is meant as a blueprint for the Liquid Tensor Experiment.

Remark 0.2. In this text $\mathbf{N}$ denotes the natural numbers *including* 0.

1. Breen–Deligne data

The goal of this section is to a give a precise statement of a variant of the Breen–Deligne resolution. This variant is not actually a resolution, but it is sufficient for our purposes, and is much easier to state and prove.

We first recall the original statement of the Breen–Deligne resolution.

Theorem 1.1 (Breen–Deligne)**.** *For an abelian group A, there is a resolution, functorial in A, of the form*

$$\ldots \to \bigoplus_{i=1}^{n_i} \mathbb{Z}[A^{r_{ij}}] \to \ldots \to \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0.$$

What does a homomorphism $f\colon \mathbb{Z}[A^m] \to \mathbb{Z}[A^n]$ that is functorial in A look like? We should perhaps say more precisely what we mean by this. The idea is that m and n are fixed, and for each abelian group A we have a group homomorphism $f_A\colon \mathbb{Z}[A^m] \to \mathbb{Z}[A^n]$ uch that if $\phi\colon A \to B$ is a group homomorphism inducing $\phi_i\colon \mathbf{Z}[A^i] \to \mathbf{Z}[B^i]$ for each natural number i then the obvious square commutes: $\phi_n \circ f_A = f_B \circ \phi_m$.

The map f_A is specified by what it does to the generators $(a_1, a_2, a_3, \ldots, a_m) \in A^m$. It can send such an element to an arbitrary element of $\mathbb{Z}[A^n]$, but one can check that universality implies that f_A will be a $\mathbb{Z}$-linear combination of "basic universal maps", where a "basic universal map" is one that sends $(a_1, a_2, \ldots, a_m)$ to $(t_1, \ldots, t_n)$, where t_i is a $\mathbb{Z}$-linear combination $c_{i,1} \cdot a_1 + \cdots + c_{i,m} \cdot a_m$. So a "basic universal map" is specified by the $n \times m$-matrix c.

Definition 1.2. A *basic universal map* from exponent m to n, is an $n \times m$-matrix with coefficients in $\mathbb{Z}$.

Definition 1.3. A *universal map* from exponent m to n, is a formal $\mathbb{Z}$-linear combination of basic universal maps from exponent m to n.

If f is a basic universal map, then we write $[f]$ for the corresponding universal map.

Definition 1.4. Let $f = \sum_g n_g[g]$ be a universal map. We say that f is *bound by* a natural number N if $\sum_g |n_g| \leq N$.

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We point out that basic universal maps can be composed by matrix multiplication, and this formally induces a composition of universal maps. As mentioned above, one can also check (this has been formalised in Lean) that this construction gives a bijection between universal maps from exponent m to n and functorial collections $f_A\colon \mathbf{Z}[A^m] \to \mathbf{Z}[A^n]$.

Definition 1.5. In other words, we are considering the following two categories:

- the category whose objects are natural numbers, and whose morphisms are matrices;
- the category with the same objects, but with Hom-sets replaced by the free abelian groups generated by the sets of matrices. We denote this latter category FreeMat.

Both categories naturally come with a monoidal structure: for the first it is given by the Kronecker product of matrices (a.k.a. tensor product of linear maps) which induces a monoidal structure on FreeMat. As usual, we will denote this monoidal structure $\_ \otimes \_$. For example, if f is a basic universal map, then $2 \otimes f$ denotes the block matrix

$$\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$$

Definition 1.6. Let N be a natural number, and $i < N$. Then $\pi'_{N,i}$ denotes the basic universal map from exponent N to 1

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = (a_0, a_1, \dots, a_{N-1})^t$$

where $a_j = \delta_{ij}$.

Definition 1.7. Let N and n be natural numbers. Then π_n^N denotes the universal map from exponent $N \cdot n$ to n given by $\sum_{i<N} [\pi'_{N,i} \otimes n]$.

(On $\mathbf{Z}[A^{N\cdot n}] \to \mathbf{Z}[A^n]$ this map is the formal sum of the maps $\mathbf{Z}[A^{N\cdot n}] \to \mathbf{Z}[A^n]$ induced by the projection maps $A^{N\cdot n} = (A^n)^N \to A^n$.)

Definition 1.8. Let N and n be natural numbers. Then σ_n^N denotes the universal map from exponent $N \cdot n$ to n given by $[\sum_{i<N} \pi'_{N,i} \otimes n]$.

(On $\mathbf{Z}[A^{N\cdot n}] \to \mathbf{Z}[A^n]$ this map is induced by the summation map $A^{N\cdot n} = (A^n)^N \to A^n$.)

Definition 1.9. A *Breen–Deligne data* is a chain complex in FreeMat.

Concretely, this means that it consists of a sequence of exponents $n_0, n_1, n_2, \dots \in \mathbb{N}$, and universal maps f_i from exponent n_{i+1} to n_i, such that for all i we have $f_i \circ f_{i+1} = 0$.

A morphism of Breen–Deligne data is a morphism of chain complexes.

Definition 1.10. For every natural numbers N, the endofunctor $N \otimes \_$ on FreeMat induces an endofunctor of Breen–Deligne data.

Concretely, it maps a pair (n, f) of Breen–Deligne data, to the pair $N \otimes (n, f)$ consisting of exponents $N \cdot n_i$ and universal maps $N \otimes f_i$.

Let BD be Breen–Deligne data. The universal maps σ^N and π^N defined above, induce morphisms $\sigma^N_{\mathsf{BD}}, \pi^N_{\mathsf{BD}}\colon N \otimes \mathsf{BD} \to \mathsf{BD}$.

Definition 1.11. A *Breen–Deligne* package consists of Breen–Deligne data BD together with a homotopy h between π^2_{BD} and σ^2_{BD}.

Definition 1.12. Let BD be a Breen–Deligne package and N a power of 2. Then the homotopy h induces a homotopy between π^N_{BD} and σ^N_{BD} by iterative composition of the homotopy packaged in BD.

Definition 1.13. We will now construct an example of a Breen–Deligne package. In some sense, it is the "easiest" solution to the conditions posed above. The exponents will be $n_i = 2^i$, and the homotopies h_i will be the identity. Under these constraints, we recursively construct the universal maps f_i:

$$f_0 = \pi^2_1 - \sigma^2_1, \quad f_{i+1} = (\pi^2_{2^{i+1}} - \sigma^2_{2^{i+1}}) - (2 \otimes f_i).$$

We leave it as exercise for the reader, to verify that with these definitions (n, f, h) forms a Breen–Deligne package.

We now make definitions that will make precise some conditions between constants that will be needed when we construct Breen–Deligne complexes of normed abelian groups.

Definition 1.14. Let f be a basic universal map from exponent m to n. Let $c_1, c_2 \in \mathbb{R}_{\geq 0}$. We say that (c_1, c_2) is *f-suitable*, if for all i

$$\sum_j c_1 |f_{ij}| \leq c_2.$$

To orient the reader: later on we will be considering maps on normed abelian groups induced from universal maps, and this inequality will guarantee that if $\|m\| \leq c_1$ then $\|f(m)\| \leq c_2$.

Definition 1.15. Let f be a universal map from exponent m to n. Let $c_1, c_2 \in \mathbb{R}_{\geq 0}$. We say that (c_1, c_2) is *f-suitable*, if for all basic universal maps g that occur in the formal sum f, the pair of nonnegative reals (c_1, c_2) is g-suitable.

Definition 1.16. Let f be a universal map and let $r, r', c_1, c_2 \in \mathbb{R}_{\geq 0}$. We say that (c_1, c_2) is *very suitable* for (f, r, r') if there exist $N, b \in \mathbf{N}$ and $c' \in \mathbf{R}_{\geq 0}$ such that:

- f is bound by N (see Definition 1.4)
- (c_1, c') is f-suitable
- $r^b N \leq 1$
- $c' \leq (r')^b c_2$

Definition 1.17. Let $\mathsf{BD} = (n, f)$ be Breen–Deligne data, let $r, r' \in \mathbf{R}_{\geq 0}$, and let $\kappa = (\kappa_0, \kappa_1, \dots)$ be a sequence of nonnegative real numbers. We say that κ is BD-*suitable* (resp. *very suitable* for (BD, r, r')), if for all i, the pair (κ_{i+1}, κ_i) is f_i-suitable (resp. *very suitable* for (f_i, r, r')).

(Note! The order (κ_{i+1}, κ_i) is contravariant compared to Definition 1.15. This is because of the contravariance of $\widehat{V}(\_)$; see Definition 5.9.)

Definition 1.18. Let BD be a Breen–Deligne package with data (n, f) and homotopy h. Let κ, κ' be sequences of nonnegative real numbers. (In applications κ is a (n, f)-suitable sequence.)

Then κ' is *adept* to (BD, κ) if for all i the pair $(\kappa_i/2, \kappa'_{i+1}\kappa_{i+1})$ is h_i-suitable. (Recall that h_i is the homotopy map $n_i \to n_{i+1}$.)

Lemma 1.19. *Let* BD *be a Breen–Deligne package,* N *a power of* 2*, and let* κ, κ' *be sequences of nonnegative real numbers. Assume that* κ' *is adept to* (BD, κ)*. Let* h^N *be the homotopy between* π^N_{BD} *and* σ^N_{BD} *defined in Def 1.12.*

For all i*, the pair* $(\kappa_i/N, \kappa'_{i+1}\kappa_{i+1})$ *is* h^N_i*-suitable.*

Proof. Omitted. (But done in Lean.) □

Lemma 1.20. *Let* BD *be a Breen–Deligne package, and let* r, r' *be nonnegative reals, such that* $r < 1$ *and* $r' > 0$*.*

There exists a sequence κ *of positive real numbers such that* κ *is very suitable for* (BD, r, r')*.*

Proof. The sequence can be constructed recursively, which we leave as exercise for the reader. (It has been done in Lean.) □

Lemma 1.21. *Let* BD *be a Breen–Deligne package, and let* r, r' *be nonnegative reals, such that* $0 < r < 1$ *and* $0 < r' \leq 1$*. Let* κ *be any sequence of positive reals.*

There exists a sequence κ' *of nonnegative real numbers* κ' *is adept to* (BD, κ)*.*

Proof. The sequence can be constructed recursively, which we leave as exercise for the reader. (It has been done in Lean.) □

2. Variants of normed groups

Remark 2.1. Normed groups are well-studied objects. In this text it will be helpful to work with the more general notion of *semi-normed group*. This drops the separation axiom $\|x\| = 0 \iff x = 0$ but is otherwise the same as a normed group.

The main difference is that this includes "uglier" objects, but creates a "nicer" category: semi-normed groups need not be Hausdorff, but quotients by arbitrary (possibly non-closed) subgroups are naturally semi-normed groups.

Nevertheless, there is the occasional use for the more restrictive notion of normed group, when we come to polyhedral lattices below (see Section 6.

In this text, a morphism of (semi)-normed groups will always be bouned. If the morphism is supposed to be norm-nonincreasing, this will be mentioned explicitly.

Definition 2.2. Let $r > 0$ be a real number. An *r-normed* $\mathbb{Z}[T^{\pm 1}]$*-module* is a semi-normed group V endowed with an automorphism $T \colon V \to V$ such that for all $v \in V$ we have $\|T(v)\| = r\|v\|$.

The remainder of this text sets up some algebraic variants of semi-normed groups.

Definition 2.3. A *pseudo-normed group* is an abelian group $(M, +)$, together with an increasing filtration $M_c \subseteq M$ of subsets M_c indexed by $\mathbb{R}_{\geq 0}$, such that each M_c contains 0, is closed under negation, and $M_{c_1} + M_{c_2} \subseteq M_{c_1+c_2}$. An example would be $M = \mathbb{R}$ or $M = \mathbb{Q}_p$ with $M_c := \{x : |x| \leq c\}$.

A pseudo-normed group M is *profinitely filtered* if each of the sets M_c is endowed with a topological space structure making it a profinite set, such that following maps are all continuous:

- the inclusion $M_{c_1} \to M_{c_2}$ (for $c_1 \leq c_2$);
- the negation $M_c \to M_c$;
- the addition $M_{c_1} \times M_{c_2} \to M_{c_1+c_2}$.

A *morphism* of profinitely filtered pseudo-normed groups $M \to N$ is a group homomorphism f that is

- *bounded*: there is a constant C such that $x \in M_c$ implies $f(x) \in N_{Cc}$;
- *continuous*: for one (or equivalently all) constants C as above, the induced map $M_c \to N_{Cc}$ is a morphism of profinite sets, i.e. continuous.

The reason the two definitions are equivalent is that a continuous injection between profinite sets must be a topological embedding.

Definition 2.4. Let r' be a positive real number. A profinitely filtered pseudo-normed group M has an *r'-action of T^{-1}* if it comes endowed with a distinguished morphism of profinitely filtered pseudo-normed groups $T^{-1}\colon M \to M$ that is bounded by r'^{-1}: if $x \in M_c$ then $T^{-1}x \in M_{c/r'}$.

A morphism $M \to N$ of profinitely filtered pseudo-normed groups with r'-action of T^{-1} is a morphism of profinitely filtered pseudo-normed groups f that commutes with the action of T^{-1} and is *strict*: if $x \in M_c$ then $f(x) \in N_c$.

3. Spaces of convergent power series

We will now construct the central example of profinitely filtered pseudo-normed groups with r'-action of T^{-1}.

Definition 3.1. Let $r' > 0$ be a real number, and let S be a finite set. Denote by $\overline{\mathcal{M}}_{r'}(S)$ the set

$$\left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \in T\mathbf{Z}[[T]] \right)_{s \in S} \,\middle|\, \sum_{n \geq 1, s \in S} |a_{n,s}| (r')^n < \infty \right\}.$$

Note that $\overline{\mathcal{M}}_{r'}(S)$ is naturally a pseudo-normed group with filtration given by

$$\overline{\mathcal{M}}_{r'}(S)_{\leq c} = \left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} \,\middle|\, \sum_{n \geq 1, s \in S} |a_{n,s}| (r')^n \leq c \right\}.$$

Lemma 3.2. *Let $r' > 0$ and $c \geq 0$ be real numbers, and let S be a finite set. The space $\overline{\mathcal{M}}_{r'}(S)_{\leq c}$ is the profinite limit of the finite sets*

$$\overline{\mathcal{M}}_{r'}(S)_{\leq c, \leq N} = \left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} \,\middle|\, \sum_{1 \leq n \leq N, s \in S} |a_{n,s}| (r')^n \leq c \right\}$$

This endows $\overline{\mathcal{M}}_{r'}(S)_{\leq c}$ with the profinite topology. In particular, it is a profinitely filtered pseudo-normed group.

Proof. Formalised, but omitted from this text. □

For the remainder of this section, let $r' > 0, c \geq 0$ be real numbers, and let S be a finite set.

Definition 3.3. There is a natural action of T^{-1} on $\overline{\mathcal{M}}_{r'}(S)$, via

$$T^{-1} \cdot \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} = \left(\sum_{n \geq 1} a_{n+1,s} T^n \right)_{s \in S}.$$

Lemma 3.4. *The natural action of T^{-1} on $\overline{\mathcal{M}}_{r'}(S)$ restricts to continuous maps*

$$T^{-1} \cdot \_ : \overline{\mathcal{M}}_r(S)_{\leq c} \to \overline{\mathcal{M}}_r(S)_{\leq c/r'}.$$

In particular, $\overline{\mathcal{M}}_{r'}(S)$ has an r'-action of T^{-1}.

Proof. Formalised, but omitted from this text. □

4. Some normed homological algebra

It will be convenient to use the following definition generalizing the notion of a bound on the norm of a right inverse of a normed group morphism (note that the morphisms we consider have no reason to have a right inverse, even when they are surjective).

Definition 4.1. Let G and H be semi-normed groups, let K be a subgroup of H and C be a positive real number. A morphism $f : G \to H$ is C-surjective onto K if, for all x in K, there exists some g in G such that $f(g) = x$ and $\|g\| \leq C\|x\|$. If $K = H$ we simply say f is C-surjective.

The following controlled surjectivity lemma will be used to prove Lemma 4.3 and Lemma 5.8.

Lemma 4.2. *Let G and H be normed groups. Let K be a subgroup of H and f a morphism from G to H. Assume that G is complete and f is C-surjective onto K. Then f is $(C+\varepsilon)$-surjective onto the topological closure of K for every positive ε.*

Proof. Let x be any element of the closure of K. First note the conclusion is trivial when $x = 0$, so we can assume $x \neq 0$. Then write x as a sum $\sum_{i\geq 0} x_i$ with all $x_i \in K$, $\|x - x_0\| \leq \varepsilon_0$ and $\|x_i\| \leq \epsilon_i$ for $i > 0$ for some sequence of positive numbers ϵ_i to be chosen later. By assumption, we can then lift each x_i to g_i such that $f(g_i) = x_i$ and $\|g_i\| \leq C\|x_i\|$, and then set $g = \sum g_i$. Because G is complete, this sum converges provided the ε_i sequence converges fast enough to zero. We then have $f(g) = x$ and

$$\|g\| \leq C \sum_{i\geq 0} \|x_i\| \leq C(\|x\| + \varepsilon_0) + C \sum_{i>0} \varepsilon_i \leq (C+\varepsilon)\|x\|$$

where the last inequality holds provided the ε_i sequence converges fast enough to zero. For instance $\varepsilon_i = \varepsilon \parallel x \parallel / (2^{i+1}C)$ satisfies all our constraints on the ε_i sequence (in particular they are positive because $x \neq 0$). □

The first application of the above lemma is a completion result for a quantitative version of being a complex.

Lemma 4.3. *Let $f : M_0 \to M_1$ and $g : M_1 \to M_2$ be bounded maps between normed groups. Assume there are positive constants C and D such that:*

- *f is C-surjective onto* $\ker g$.
- *g is D-surjective onto its image.*

Then for every positive ε, $\hat{f}$ is $(C+\varepsilon)$-surjective onto $\ker \hat{g}$.

Proof. Since f is C-surjective onto $\ker g$, $\hat{f}$ is C-surjective onto $\ker g$ seen as a subset of $\widehat{M_1}$. Hence this lemma will follow directly from Lemma 4.2 once we'll have proven that $\ker g$ is dense in $\ker \hat{g}$. Let $\hat{y}$ be an element of $\ker \hat{g}$. Pick any $\delta > 0$ and take $y \in M_1$ such that $\|\hat{y} - y\| \leq \delta$. Let $z = g(y) \in M_2$, which has norm $\|z\| = \|g(y)\| = \|g(y - \hat{y})\|$ bounded by $C_g\delta$, where C_g is the norm of g. We can thus find some $y' \in M_1$ with $\|y'\| \leq DC_g\delta$ and $g(y') = z$. Replacing y by $y - y'$, we can thus find $y \in \ker(g : M_1 \to M_2)$ such that still $\|\hat{y} - y\| \leq (1 + DC_g)\delta$; as δ was arbitrary, this gives the desired density. □

Definition 4.4. A *system of complexes* of normed abelian groups is for each sufficiently large c (i.e. all $c \geq c_0$ for some $c_0 > 0$), a complex

$$C_c^\bullet : C_c^0 \to C_c^1 \to \ldots$$

of normed abelian groups together with maps of complexes $\mathrm{res}_{c',c} : C_{c'}^\bullet \to C_c^\bullet$, for $c' \geq c \geq c_0$, satisfying $\mathrm{res}_{c,c} = \mathrm{id}$ and the obvious associativity condition. We use notation $(C_c^\bullet)_{c \geq c_0}$ for a system of complexes, although we will frequently omit any mention of the lower bound c_0 and just write $C_\bullet^\bullet$.

By convention, for every system of complexes $(C_c^\bullet)_{c \geq c_0}$, we will set $C_c^{-1} = 0$ for all $c \geq c_0$. This will come up each time we write C_c^{i-1} and i could be 0.

In this section, given $x \in C_{c'}^\bullet$ and $c_0 \leq c \leq c'$ we will use the notation $x_{|c} := \mathrm{res}_{c',c}(x)$.

Definition 4.5. A system of complexes is *admissible* if all differentials and maps $\mathrm{res}_{c',c}^i$ are norm-nonincreasing.

Throughout the rest of this section, k (and k', k'') will denote reals at least 1, m will be a non-negative integer, and K, K', K'' will denote non-negative reals.

Definition 4.6. Let $(C_c^\bullet)_{c \geq c_0}$ be a system of complexes. For an integer $m \geq 0$ and reals $k \geq 1$, $c_0' \geq c_0$ and $K \geq 0$, we say the datum $(C_c^\bullet)_{c \geq c_0}$ is *$\leq k$-exact in degrees $\leq m$ and for $c \geq c_0'$ with bound K* if the following condition is satisfied. For all $c \geq c_0'$ and all $x \in C_{kc}^i$ with $i \leq m$ there is some $y \in C_c^{i-1}$ such that

$$\|x_{|c} - dy\| \leq K\|dx\|.$$

We will also need a version where the inequality is relaxed by some arbitrary small additive constant.

Definition 4.7. Let $(C_c^\bullet)_{c \geq c_0}$ be a system of complexes. For an integer $m \geq 0$ and reals $k \geq 1$, $c_0' \geq c_0$ and $K \geq 0$, the datum $(C_c^\bullet)_c$ is *weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0'$ with bound K* if the following condition is satisfied. For all $c \geq c_0'$, all $x \in C_{kc}^i$ with $i \leq m$ and any $\varepsilon > 0$ there is some $y \in C_c^{i-1}$ such that

$$\|x_{|c} - dy\| \leq K\|dx\| + \varepsilon.$$

We first note that the difference between those two definitions is only about cocyles if we are ready to lose a tiny something on the norm bound K.

Lemma 4.8. *Let $C_\bullet^\bullet$ be a system of complexes. If $C_\bullet^\bullet$ is weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0'$ with bound K and if, for all $c \geq c_0'$ and all $x \in C_{kc}^i$ with $i \leq m$ such that $dx = 0$ there is some $y \in C_c^{i-1}$ such that $x_{|c} = dy$ then, for every positive δ, $C_\bullet^\bullet$ is $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0'$ with bound $K + \delta$.*

Proof. Let δ be some positive real number. Let x be an element of C_{kc}^i for some $c \geq c_0'$ and $i \leq m$. If $dx = 0$ then the assumption we made about exact elements is exactly what we want.

Assume now that $dx \neq 0$. The weak exactness assumption applied to $\varepsilon = \delta\|dx\|$ gives some $y \in C_c^{i-1}$ such that

$$\begin{aligned}\|x_{|c} - dy\| &\leq K\|dx\| + \delta\|dx\| \\ &= (K+\delta)\|dx\|\end{aligned}$$

□

Lemma 4.9. *Let $(M_c^\bullet)_{c \geq c_0}$ be an admissible collection of complexes of complete normed abelian groups.*

Assume that $M_c^\bullet$ is weakly $\leq k$-exact in degrees $\leq m$ for $c \geq c_0$ with bound K. Then $M_c^\bullet$, for every $\delta > 0$, it is $\leq k^2$-exact in degrees $\leq m$ for $c \geq c_0$ with bound $K + \delta$.

Proof. Lemma 4.8 ensures we only need to care about cocycles of M. More precisely, let x be a cocycle in $M_{k^2c}^i$ for some $i \leq m$ and $c \geq c_0$. We need to find $y \in M_c^{i-1}$ such that $dy = x_{|c}$.

By weak $\leq k$-exactness applied to x and a sequence ε_j to be chosen later, we can find a sequence $w^j \in M_{kc}^{i-1}$ such that

$$\|x_{kc} - dw^j\| \leq \varepsilon_j.$$

Then, by weak $\leq k$-exactness applied to each $w^{j+1} - w^j$ and a sequence δ_j to be chosen later, we can find a sequence $z^j \in M_c^{i-2}$ such that

$$\|(w^{j+1} - w^j)_{|c} - dz^j\| \leq K\|dw^{j+1} - dw^j\| + \delta_j.$$

We set $y^j := w^j_{|c} - \sum_{l=0}^{j-1} dz^l \in M_c^{i-1}$.

We have

$$\begin{aligned}\|y^{j+1} - y^j\| &= \left\|(w^{j+1} - w^j)_{|c} - dz^j\right\| \\ &\leq K\|dw^{j+1} - dw^j\| + \delta_j \\ &\leq 2K\varepsilon_j + \delta_j.\end{aligned}$$

So y^j is a Cauchy sequence as long as we make sure $2K\varepsilon_j + \delta_j \leq 2^{-j}$ for instance. Since M_c^{i-1} is complete, this sequence converges to some y. Because $dy^j = dw^j_{|c}$, we get that $\|x_{|c} - dy^j\| \leq \varepsilon_j$ and in the limit $x_{|c} = dy$. □

Proposition 4.10. *Let $(M_c^\bullet)_{c \geq c_0}$ and $(M_c'^\bullet)_{c \geq c_0}$ be two admissible collections of complexes of complete normed abelian groups. For each $c \geq c_0$ let $f_c^\bullet : M_c^\bullet \to M_c'^\bullet$ be a collection of maps between these collections of complexes that are norm-nonincreasing and which all commute with all restriction maps, and assume that there exists these maps satisfy*

$$\|x_{|c}\| \leq K''\|f(x)\|$$

for all $i \leq m+1$ and all $x \in M_{k''c}^i$. Let $N_c^\bullet = M_c'^\bullet / M_c^\bullet$ be the collection of quotient complexes, with the quotient norm; this is again an admissible collection of complexes.

Assume that $M_c^\bullet$ (resp. $M_c'^\bullet$) is weakly $\leq k$-exact (resp. $\leq k'$-exact) in degrees $\leq m$ for $c \geq c_0$ with bound K (resp. K'). Then $N_c^\bullet$ is weakly $\leq kk'k''$-exact in degrees $\leq m-1$ for $c \geq c_0$ with bound $K'(KK''+1)$.

Proof. Let $n \in N^i_{kk'k''c}$ for $i \leq m-1$. We fix $\varepsilon > 0$. We need to find an element $y \in N^{i-1}_c$ such that

$$\|n_{|c} - dy\| \leq K'(KK''+1)\|dn\| + \epsilon.$$

Pick any preimage $m' \in M'^i_{kk'k''c}$ of n. In particular dm' is a preimage of dn. By definition of the quotient norm, we can find $m_1 \in M^{i+1}_{kk'k''c}$ and $m''_1 \in (M')^{i+1}_{kk'k''c}$ such that

$$dm' = f(m_1) + m''_1$$

with $\|m''_1\| \leq \|dn\| + \varepsilon_1$, for some positive ε_1 to be chosen later.

Applying the differential to the last displayed equation, and using that this kills the image of d, and that f is a map of complexes, we see that

$$f(dm_1) = -dm''_1.$$

Using the norm bound on f, we get

$$\begin{aligned}\|dm_{1|kk'c}\| &\leq K''\|f(dm_1)\| = K''\|dm''_1\| \\ &\leq K''\|m''_1\| \leq K''\|dn\| + K''\varepsilon_1.\end{aligned}$$

On the other hand, weak exactness of M applied to $m_{1|kk'c}$ gives $m_0 \in M^i_{k'c}$ such that

$$\|m_{1|kk'c|k'c} - dm_0\| \leq K\|dm_{1|kk'c}\| + \varepsilon_1$$

which combines with the previous estimate to give:

$$\|m_{1|k'c} - dm_0\| \leq KK''\,\|dn\| + (KK''+1)\varepsilon_1.$$

Now let $m'_{\mathrm{new}} = m'_{|k'c} - f(m_0) \in M'^i_{k'c}$; this is a lift of $n_{|k'c}$. Then

$$dm'_{\mathrm{new}} = dm'_{|k'c} - f(m_{1|k'c}) + f(m_{1|k'c} - dm_0) = m''_{1|k'c} + f(m_{1|k'c} - dm_0).$$

In particular,

$$\|dm'_{\mathrm{new}}\| \leq (KK''+1)\,\|dn\| + (KK''+2)\varepsilon_1.$$

Now weak exactness of M' gives $x \in M'^{i-1}_c$ such that

$$\|m'_{\mathrm{new}|c} - dx\| \leq K'\|dm'_{\mathrm{new}}\| + \varepsilon_1 \leq K'((KK''+1)\,\|dn\| + (KK''+2)\varepsilon_1) + \varepsilon_1.$$

In particular, letting $y \in N^{i-1}_c$ be the image of x, we get

$$\|n_{|c} - dy\| \leq K'(KK''+1)\,\|dn\| + (K'(KK''+2)+1)\varepsilon_1,$$

which is exactly what we wanted if we choose $\varepsilon_1 = \varepsilon/(K'(KK''+2)+1)$. □

We also need the 'dual' version of 4.10, for kernels instead of cokernels. Note that this is actually a bit easier to prove. (Should we put it first in the presentation?)

Proposition 4.11. *Let $(M^\bullet_c)_{c\geq c_0}$ and $(M'^\bullet_c)_{c\geq c_0}$ be two admissible collections of complexes of complete normed abelian groups. For each $c \geq c_0$ let $f^\bullet_c : M^\bullet_c \to M'^\bullet_c$ be a collection of maps between these collections of complexes which all commute with all restriction maps. Assume moreover that we are given two constants $r_1, r_2 \geq 0$ such that:*

- *for all $i, c \geq c_0$ and all $x \in M^i_c$*

$$\|f(x)\| \leq r_1\|x\|;$$

- *for all $i \leq m+1, c \geq c_0$ and all $y \in M_c'^i$, there exists $x \in M_c^i$ such that*
$$f(x) = y \text{ and } \|x\| \leq r_2 \|y\|.$$

Let $N_c^\bullet$ be the collection of kernel complexes, with the induced norm; this is again an admissible collection of complexes.

Assume that $M_c^\bullet$ (resp. $M_c'^\bullet$) is weakly $\leq k$-exact (resp. $\leq k'$-exact) in degrees $\leq m$ for $c \geq c_0$ with bound K (resp. K'). Then $N_c^\bullet$ is weakly $\leq kk'$-exact in degrees $\leq m-1$ for $c \geq c_0$ with bound $K + r_1 r_2 K K'$.

Proof. Let $n \in N^i_{kk'c} \subseteq M^i_{kk'c}$ for $i \leq m-1$ and let $\varepsilon > 0$. We need to find an element $y \in N_c^{i-1}$ such that
$$\|n_{|c} - dy\| \leq K + r_1 r_2 K K' \|dn\| + \epsilon.$$
By weak exactness of $(M_c^\bullet)_{c \geq c_0}$, we can find $m \in M^{i-i}_{k'c}$ such that
$$\|n_{|k'c} - dm\| \leq K\|dn\| + \epsilon_1,$$
where $\epsilon_1 > 0$ to be chosen later. By weak exactness of $(M_c'^\bullet)_{c \geq c_0}$, we can find $m' \in M_c'^{i-2}$ such that
$$\|f(m)_{|c} - dm'\| \leq K'\|df(m)\| + \epsilon_2,$$
where $\epsilon_2 > 0$ to be chosen later. Let $m_1 \in M_c^{i-2}$ be a lift of m' and let $m_2 \in M_c^{i-1}$ be such that
$$f(m_2) = f(m_{|c} - dm_1) \text{ and } \|m_2\| \leq r_2 \|f(m_{|c} - dm_1)\|.$$
Set $y = m_{|c} - dm_1 - m_2 \in M_c^{i-1}$. By construction $f(y) = 0$, so $y \in N_c^{i-1}$. We compute
$$\begin{gathered}\|n_{|c} - dy\| = \|n_{|c} - dm_{|c} + d^2 m_1 - dm_2\| = \|n_{|c} - dm_{|c} - dm_2\| \leq \\ \|n_{|c} - dm_{|c}\| + \|dm_2\| = \|(n_{|k'c} - dm)_{|c}\| + \|dm_2\| \leq \|(n_{|k'c} - dm)\| + \|dm_2\| \leq \\ K\|dn\| + \epsilon_1 + \|dm_2\|.\end{gathered}$$
Where we have used the defining property of m and admissibility of $(M_c^\bullet)_{c \geq c_0}$. By the same assumption and since $f(n_{|k'c}) = f(n)_{|k'c} = 0$, we have
$$\begin{gathered}\|dm_2\| \leq \|m_2\| \leq r_2 \|f(m_{|c} - dm_1)\| = r_2 \|f(m)_{|c} - df(m_1)\| = r_2 \|f(m)_{|c} - dm'\| \leq \\ r_2(K'\|df(m)\| + \epsilon_2) = r_2(K'\|f(dm)\| + \epsilon_2) = r_2(K'\|f(n_{|k'c}) - f(dm)\| + \epsilon_2) = \\ r_2(K'\|f(n_{|k'c} - dm)\| + \epsilon_2) \leq r_2(K' r_1 \|n_{|k'c} - dm\| + \epsilon_2) \leq r_2(K' r_1 (K\|dn\| + \epsilon_1) + \epsilon_2)\end{gathered}$$
In particular we get
$$\begin{gathered}\|n_{|c} - dy\| \leq K\|dn\| + \epsilon_1 + r_2(K' r_1 (K\|dn\| + \epsilon_1) + \epsilon_2) = \\ (K + r_1 r_2 K K')\|dn\| + \epsilon_1(1 + r_1 r_2 K') + r_2 \epsilon_2.\end{gathered}$$
Now let
$$\epsilon_1 = \frac{\epsilon}{2(1 + r_1 r_2 K')} \text{ and } \epsilon_2 = \begin{cases} \frac{\epsilon}{2r_2} \text{ if } r_2 \neq 0 \\ 1 \text{ if } r_2 = 0 \end{cases}$$
In any case $r_2 \epsilon_2 \leq \frac{\epsilon}{2}$ and so
$$\|n_{|c} - dy\| \leq (K + r_1 r_2 K K')\|dn\| + \epsilon$$
as required.

If $i = 0$, then all m, m', m_1 and m_2 are 0, so $y = 0$ as required. $\square$

Consider a system of double complexes $M_c^{p,q}$, $p, q \geq 0$, $c \geq c_0$,

$$\begin{array}{ccccccc}
M_c^{0,0} & \longrightarrow & M_c^{0,1} & \longrightarrow & M_c^{0,2} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
M_c^{1,0} & \longrightarrow & M_c^{1,1} & \longrightarrow & M_c^{1,2} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
M_c^{2,0} & \longrightarrow & M_c^{2,1} & \longrightarrow & \ddots & & \\
\downarrow & & \downarrow & & & & \\
\vdots & & \vdots & & & &
\end{array}$$

of complete normed abelian groups.

Definition 4.12. We say that the system of double complexes $M_c^{p,q}$ satisfies the *normed spectral homotopy condition* for $m \in \mathbf{N}$ and $H, c_0 \in \mathbf{R}_{\geq 0}$ if the following condition is satisfied:

For $q = 0, \dots, m$ and $c \geq c_0$, there is a map $h^q_{k'c} \colon M^{0,q+1}_{k'c} \to M^{1,q}_c$ with

$$\|h^q_{k'c}(x)\|_{M_c^{1,q}} \leq H \|x\|_{M^{0,q+1}_{k'c}}$$

for all $x \in M^{0,q+1}_{k'c}$, and such that for all $c \geq c_0$ and $q = 0, \dots, m$ the "homotopic" map

$$\mathrm{res}^{1,q}_{k'^2c,k'c} \circ d^{0,q} + h^q_{k'^2c} \circ d'^{0,q}_{k'^2c} + d'^{1,q-1}_{k'c} \circ h^{q-1}_{k'^2c} \colon M^{0,q}_{k'^2c} \to M^{1,q}_{k'c}$$

factors as a composite of the restriction $\mathrm{res}^{0,q}_{k'^2c,c}$ and a map

$$\delta^{0,q}_c \colon M^{0,q}_c \to M^{1,q}_{k'c}$$

that is a map of complexes (in degrees $\leq m$), and satisfies the estimate

$$\|\delta^{0,q}_c(x)\|_{M^{1,q}_{k'c}} \leq \epsilon \|x\|_{M^{0,q}_c} \tag{4.1}$$

for all $x \in M^{0,q}_c$.

Proposition 4.13. *Fix an integer $m \geq 0$ and constants k, K. Then there exists an $\epsilon > 0$ and constants k_0, K_0, depending (only) on k, K and m, with the following property.*

Let $M_c^{p,q}$ be a system of double complexes as above, and assume that it is admissible. Assume further that there is some $k' \geq k_0$ and some $H > 0$, such that

1. *for $i = 0, \dots, m+1$, the rows $M_c^{i,q}$ are weakly $\leq k$-exact in degrees $\leq m-1$ for $c \geq c_0$ with bound K;*
2. *for $j = 0, \dots, m$, the columns $M_c^{p,j}$ are weakly $\leq k$-exact in degrees $\leq m$ for $c \geq c_0$ with bound K;*
3. *it satisfies the normed spectral homotopy condition for m, H and c_0.*

Then the first row is weakly $\leq k'^2$ exact in degrees $\leq m$ for $c \geq c_0$ with bound $2K_0H$.

We note that the bound on the homotopy is of a peculiar nature, in that the bound only depends on a deep restriction of x.

Proof. First, we treat the case $m = 0$. If $m = 0$, we claim that one can take $\epsilon = \frac{1}{2k}$ and $k_0 = k$. We have to prove exactness at the first step. Let $x_{k'^2c} \in M^{0,0}_{k'^2c}$ and denote $x_{k'c} = \mathrm{res}^{0,0}_{k'^2c,k'c}(x)$ and $x_c = \mathrm{res}^{0,0}_{k'^2c,c}(x)$. Then by assumption (2) (and $k' \geq k$), we have

$$\|x_c\|_{M^{0,0}_c} \leq k\|d^{0,0}_{k'c}(x_{k'c})\|_{M^{1,0}_{k'c}}.$$

On the other hand, by (3),

$$\|\mathrm{res}^{1,0}_{k'^2c,k'c}(d^{0,0}_{k'^2c}(x)) \pm h^0_{k'^2c}(d'^{0,0}_{k'^2c}(x))\|_{M^{1,0}_{k'c}} \leq \epsilon\|x_c\|_{M^{0,0}_c}.$$

In particular, noting that $\mathrm{res}^{1,0}_{k'^2c,k'c}(d^{0,0}_{k'^2c}(x)) = d^{0,0}_{k'c}(x_{k'c})$, we get

$$\|x_c\|_{M^{0,0}_c} \leq k\|d^{0,0}_{k'c}(x_{k'c})\|_{M^{1,0}_{k'c}} \leq k\epsilon\|x_c\|_{M^{0,0}_c} + kH\|d'^{0,0}_{k'^2c}(x)\|_{M^{0,1}_{k'^2c}}.$$

Thus, taking $\epsilon = \frac{1}{2k}$ as promised, this implies

$$\|x_c\|_{M^{0,0}_c} \leq 2kH\|d'^{0,0}_{k'^2c}(x)\|_{M^{0,1}_{k'^2c}}.$$

This gives the desired $\leq \max(k'^2, 2k_0H)$-exactness in degrees $\leq m$ for $c \geq c_0$.

Now we argue by induction on m. Consider the complex $N^{p,q}$ given by $M^{p,q+1}$ for $q \geq 1$ and $N^{p,0} = M^{p,1}/\overline{M^{p,0}}$ (the quotient by the closure of the image, which is also the completion of $M^{p,1}/M^{p,0}$), equipped with the quotient norm. Using the normed version of the snake lemma, Proposition **??** in the appendix to this lecture, one checks that this satisfies the assumptions for $m - 1$, with k replaced by $\max(k^4, k^3 + k + 1)$. □

5. Completions of locally constant functions

Definition 5.1. Let V be a semi-normed group, and X a compact topological space. We denote by $V(X)$ the normed abelian group of locally constant functions $X \to V$ with respect to the sup norm. With $\widehat{V}(X)$ we denote the completion of $V(X)$.

These constructions are functorial in bounded group homomorphisms $V \to V'$ and contravariantly functorial in continuous maps $f \colon X \to X'$.

Note in particular that $V(f)$ and $\widehat{V}(f)$ are norm-nonincreasing morphisms of semi-normed groups.

Lemma 5.2. *Let $r \in \mathbb{R}_{>0}$, and let V be an r-normed $\mathbb{Z}[T^{\pm 1}]$-module. Let X be a compact space. Then $\widehat{V}(X)$ is naturally an r-normed $\mathbb{Z}[T^{\pm 1}]$-module, with the action of T given by post-composition.*

Proof. Formalised, but omitted from this text. □

We continue to use the notation of before: let $r' > 0, c \geq 0$ be real numbers, and let M be a profinitely filtered pseudo-normed group with r'-action by T^{-1} (see Section 2).

Lemma 5.3. *Let f be a basic universal map from exponent m to n. We get an induced homomorphism of profinitely filtered pseudo-normed groups $M^m \to M^n$ bounded by the maximum (over all i) of $\sum_j |f_{ij}|$, where the f_{ij} are the coefficients of the $n \times m$-matrix representing f.*

This construction is functorial in f.

Proof. Omitted. □

Definition 5.4. Let f be a basic universal map from exponent m to n, and let (c_2, c_1) be f-suitable. We get an induced map

$$V(f)\colon V(M^n_{\le c_1}) \to V(M^m_{\le c_2})$$

induced by the morphism of profinitely filtered pseudo-normed groups $M^m \to M^n$.

This construction is functorial in f.

Definition 5.5. Let $f = \sum_g n_g g$ be a universal map from exponent m to n, and let (c_2, c_1) be f-suitable. We get an induced map

$$V(f)\colon V(M^n_{\le c_1}) \to V(M^m_{\le c_2})$$

that is the sum $\sum n_g V(g)$.

This construction is functorial in f.

Definition 5.6. Let f be a universal map from exponent m to n, and let (c_2, c_1) be f-suitable. We get an induced map

$$\widehat{V}(f)\colon \widehat{V}(M^n_{\le c_1}) \to \widehat{V}(M^m_{\le c_2})$$

that is the completion of $V(f)$.

This construction is functorial in f.

Let $r > 0$, and assume now that V is an r-normed $\mathbb{Z}[T^{\pm 1}]$-module. Assume $r' \le 1$.

Definition 5.7. There are two natural actions of T^{-1} on $\widehat{V}(M_{\le c})$. The first comes from the r'-action of T^{-1} on M which gives a continuous map

$$M_{\le cr'} \to M_{\le c}$$

and thus a normed group morphism $V(M_{\le c}) \to V(M_{\le cr'})$ which can be extended by completion to

$$(T^{-1})^*\colon \widehat{V}(M_{\le c}) \to \widehat{V}(M_{\le cr'}).$$

The other comes from Lemma 5.2, using the r-normed $\mathbb{Z}[T^{\pm 1}]$-module V. Again by extension to completion, we get a map

$$[T^{-1}]\colon \widehat{V}(M_{\le c}) \to \widehat{V}(M_{\le c}),$$

that we can compose with the map $\widehat{V}(M_{\le c}) \to \widehat{V}(M_{\le cr'})$, obtained from the natural inclusion $M_{\le cr'} \to M_{\le c}$. We thus end up with two maps

$$(T^{-1})^*, [T^{-1}]\colon \widehat{V}(M_{\le c}) \to \widehat{V}(M_{\le cr'}).$$

and we define $\widehat{V}(M_{\le c})^{T^{-1}}$ to be the equalizer of $(T^{-1})^*$ and $[T^{-1}]$. In other words, the kernel of $(T^{-1})^* - [T^{-1}]$.

We will also need to understand the image of $(T^{-1})^* - [T^{-1}]$. The next lemma ensures it is surjective with controlled preimages, see Definition 4.1.

Lemma 5.8. *Let M be a profinitely filtered pseudo-normed group with action of T^{-1}. For any $r \in (0,1)$, any r-normed $\mathbb{Z}[T^{\pm 1}]$-module V, any $c > 0$ and any a, the map*

$$\widehat{V}(M^a_{\le c}) \xrightarrow{T^{-1} - [T^{-1}]^*} \widehat{V}(M^a_{\le r'c})$$

has norm bounded by $r^{-1} + 1$ and is $\frac{r}{1-r}(1+\epsilon)$-surjective.

Proof. The norm bound is clear because $[T^{-1}]^*$ is norm non-increasing and T^{-1} scales norm by r^{-1}. Quantitative surjectivity will follow from Lemma 4.2 once we'll have proven that $T^{-1} - [T^{-1}]^* : \widehat{V}(M^a_{\le c}) \to \widehat{V}(M^a_{\le r'c})$ is $r/(1-r)$-surjective onto $V(M^a_{\le r'c})$.

We first note that any locally constant function $\varphi \in V(M^a_{\le r'c})$ can be extended to a locally constant function $\bar\varphi \in V(M^a_{\le c})$ with the same norm (recall f takes finitely many values and its norm is the maximum of norms of these values).

Let f be any element of $V(M^a_{\le r'c})$. We inductively define a sequence of locally constant functions $h_n \in V(M^a_{\le c})$ with $h_0 = T \circ \bar f$ and $h_{n+1} = T \circ \overline{[T^{-1}]^* h_n}$. Here we use the composition symbol to emphasize this is indeed the naive post-composition with T, there is no extra precomposition with a the inclusion map $\iota : M^a_{\le r'c} \hookrightarrow M^a_{\le c}$ as in the definition of T^{-1} seen as a map from $V(M^a_{\le c})$ to $V(M^a_{\le r'c})$.

Since $[T^{-1}]^*$ is norm non-increasing, extension is norm preserving and T scales norm by r, we get that $\|h_n\| \le r^{n+1}\|f\|$. We then set $g_n = \sum_{i=0}^n h_i$. The norm estimate on h_n ensures g is a Cauchy sequence in $V(M^a_{\le c})$ hence it converges to some g in $\widehat{V}(M^a_{\le c})$. We compute:

$$\begin{aligned}
(T^{-1} - [T^{-1}]^*)g_n &= \sum_{k=0}^{n} \Big(T^{-1}h_k - [T^{-1}]^* h_k\Big) \\
&= T^{-1}h_0 + \sum_{k=0}^{n-1} \Big(T^{-1}h_{k+1} - [T^{-1}]^* h_k\Big) - [T^{-1}]^* h_n \\
&= \bar f \circ \iota + \sum_{k=0}^{n-1} \Big(T^{-1} \circ T \circ \overline{[T^{-1}]^* h_k} \circ \iota - [T^{-1}]^* h_k\Big) - [T^{-1}]^* h_n \\
&= f - [T^{-1}]^* h_n
\end{aligned}$$

which converges to f hence $(T^{-1} - [T^{-1}]^*)g = f$. In addition $\|g\| \le \sum_n r^{n+1}\|f\| = r/(1-r)\|f\|$. ☐

Definition 5.9. Let f be a universal map from exponent m to n, and let (c_2, c_1) be f-suitable.

The natural map from Definition 5.6 restricts to a map

$$\widehat{V}(f)^{T^{-1}} : \widehat{V}(M^n_{\le c_1})^{T^{-1}} \to \widehat{V}(M^m_{\le c_2})^{T^{-1}}$$

Lemma 5.10. *Let $0 < r$ and $0 < r' \le 1$ be real numbers. Let f be a universal map from exponent m to n, and let (c_2, c_1) be very suitable for (f, r, r'). Then*

$$\widehat{V}(f)^{T^{-1}} : \widehat{V}(M^n_{\le c_1})^{T^{-1}} \to \widehat{V}(M^m_{\le c_2})^{T^{-1}}$$

is norm-nonincreasing.

Proof. Use the assumption that (c_2, c_1) is very suitable for (f, r, r') in order to find $N, b \in \mathbf{N}$ and $c' \in \mathbf{R}_{\ge 0}$ such that:

- f is bound by N (see Definition 1.4)
- (c_2, c') is f-suitable
- $r^b N \le 1$
- $c' \le (r')^b c_1$

Now, notice that the norm of $\widehat{V}(f)$ is at most N, and $\widehat{V}(f)$ can be factored as

$$\widehat{V}(M^n_{\le c_1})^{T^{-1}} \xrightarrow{\text{res}} \widehat{V}(M^n_{\le c'})^{T^{-1}} \xrightarrow{\widehat{V}(f)} \widehat{V}(M^m_{\le c_2})^{T^{-1}}$$

Now use the defining property of the equalizer to conclude that the restriction map has norm less than $1/N$, and therefore the composition is norm-nonincreasing. □

Definition 5.11. Let $0 < r$ and $0 < r' \le 1$ be real numbers, and let V be an r-normed $\mathbb{Z}[T^{\pm 1}]$-module. Let $\mathsf{BD} = (n, f)$ be Breen–Deligne data, and let $\kappa = (\kappa_0, \kappa_1, \kappa_2, \dots)$ be a sequence of constants in $\mathbb{R}_{\ge 0}$ that is very suitable for (BD, r, r'). Let M be a profinitely filtered pseudo-normed group with r'-action of T^{-1}.

For every $c \in \mathbb{R}_{\ge 0}$, the maps from Definition 5.9 induced by the universal maps f_i from the Breen–Deligne $\mathsf{BD} = (n, f)$ assemble into a complex of normed abelian groups

$$C^{\mathrm{BD}}_{\kappa}(M)^{\bullet}_{c} \colon 0 \to \dots \to \widehat{V}(M^{n_i}_{\le \kappa_i})^{T^{-1}} \to \widehat{V}(M^{n_{i+1}}_{\le \kappa_{i+1}})^{T^{-1}} \to \dots .$$

Together, these complexes fit into a system of complexes with the natural restriction maps.

By Lemma 5.10 the differentials are norm-nonincreasing. It is clear that the restriction maps are also norm-nonincreasing, and therefore the system is admissible.

6. Polyhedral lattices

Definition 6.1. A *polyhedral lattice* is a finite free abelian group Λ equipped with a norm $\|\cdot\|_\Lambda \colon \Lambda \otimes \mathbb{R} \to \mathbb{R}$ such that there exists a finite set $\{\lambda_1, \dots, \lambda_n\} \subset \Lambda$ that generate the norm: that is to say, for every $\lambda \in \Lambda$ there exist $c_1, \dots, c_n \in \mathbb{Q}$ such that $\lambda = \sum c_i \lambda_i$ and $\|\lambda\| = c_i \|\lambda_i\|$.

Equivalently (but not verified in Lean): the norm is given by the supremum of finitely many linear functions on Λ; or once more, equivalently, the "unit ball" $\{\lambda \in \Lambda \otimes \mathbb{R} \mid \|\lambda\|_\Lambda \le 1\}$ is a polyhedron.

Finally, we can prove the key combinatorial lemma, ensuring that any element of $\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))$ can be decomposed into N elements whose norm is roughly $\frac{1}{N}$ of the original element.

Lemma 6.2. *Let Λ be a polyhedral lattice. Then for all positive integers N there is a constant d such that for all $c > 0$ one can write any $x \in \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\le c}$ as*

$$x = x_1 + \dots + x_N$$

where all $x_i \in \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\le c/N + d}$.

As preparation for the proof, we have the following results.

Lemma 6.3 (Gordan's lemma)**.** *Let Λ be a finite free abelian group, and let $\lambda_1, \dots, \lambda_m \in \Lambda$ be elements. Let $M \subset \mathrm{Hom}(\Lambda, \mathbb{Z})$ be the submonoid $\{x \mid x(\lambda_i) \ge 0$ for all $i = 1, \dots, m\}$. Then M is finitely generated as monoid.*

Proof. This is a standard result. We omit the proof here. It is done in Lean. □

Lemma 6.4. *Let Λ be a finite free abelian group, let N be a positive integer, and let $\lambda_1, \dots, \lambda_m \in \Lambda$ be elements. Then there is a finite subset $A \subset \Lambda^\vee$ such that for all $x \in \Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z})$ there is some $x' \in A$ such that $x - x' \in N\Lambda^\vee$ and for all $i = 1, \dots, m$, the numbers $x'(\lambda_i)$ and $(x - x')(\lambda_i)$ have the same sign, i.e. are both nonnegative or both nonpositive.*

Proof. It suffices to prove the statement for all x such that $\lambda_i(x) \geq 0$ for all i; indeed, applying this variant to all $\pm\lambda_i$, one gets the full statement.

Thus, consider the submonoid $\Lambda^\vee_+ \subset \Lambda^\vee$ of all x that pair nonnegatively with all λ_i. This is a finitely generated monoid by Lemma 6.3; let $y_1, \ldots, y_M$ be a set of generators. Then we can take for A all sums $n_1 y_1 + \ldots + n_M y_M$ where all $n_j \in \{0, \ldots, N-1\}$. □

Lemma 6.5. *Let $x_0, x_1, \ldots$ be a sequence of reals, and assume that $\sum_{i=0}^\infty x_i$ converges absolutely. For every natural number $N > 0$, there exists a partition $\mathbb{N} = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_N$ such that for each $j = 1, \ldots, N$ we have $\sum_{i \in A_j} x_i \leq (\sum_{i=0}^\infty x_i)/N + 1$*

Proof. Define the A_j recursively: assume that the natural numbers $0, \ldots, n$ have been placed into the sets $A_1, \ldots, A_N$. Then add the number $n+1$ to the set A_j for which

$$\sum_{i=0, i \in A_j}^{n} x_i$$

is minimal. □

Lemma 6.6. *For all natural numbers $N > 0$, and for all $x \in \overline{\mathcal{M}}_{r'}(S)_{\leq c}$ one can decompose x as a sum*

$$x = x_1 + \ldots + x_N$$

with all $x_i \in \overline{\mathcal{M}}_{r'}(S)_{\leq c/N+1}$.

Proof. Choose a bijection $S \times \mathbb{N} \cong \mathbb{N}$, and transport the result from Lemma 6.5. □

Proof of Lemma 6.2. Pick $\lambda_1, \ldots, \lambda_m \in \Lambda$ generating the norm. We fix a finite subset $A \subset \Lambda^\vee$ satisfying the conclusion of the previous lemma. Write

$$x = \sum_{n \geq 1, s \in S} x_{n,s} T^n[s]$$

with $x_{n,s} \in \Lambda^\vee$. Then we can decompose

$$x_{n,s} = N x^0_{n,s} + x^1_{n,s}$$

where $x^1_{n,s} \in A$ and we have the same-sign property of the last lemma. Letting $x^0 = \sum_{n \geq 1, s \in S} x^0_{n,s} T^n[s]$, we get a decomposition

$$x = N x^0 + \sum_{a \in A} a x_a$$

with $x_a \in \overline{\mathcal{M}}_{r'}(S)$ (with the property that in the basis given by the $T^n[s]$, all coefficients are 0 or 1). Crucially, we know that for all $i = 1, \ldots, m$, we have

$$\|x(\lambda_i)\| = N\|x^0(\lambda_i)\| + \sum_{a \in A} |a(\lambda_i)| \|x_a\|$$

by using the same sign property of the decomposition.

Using this decomposition of x, we decompose each term into N summands. This is trivial for the first term Nx^0, and each summand of the second term decomposes with $d = 1$ by Lemma 6.6. (It follows that in general one can take for d the supremum over all i of $\sum_{a \in A} |a(\lambda_i)|$.) □

Definition 6.7. Let Λ be a polyhedral lattice, and let $N > 0$ be a natural number. (We think of N as being fixed once and for all, and thus it does not show up in the notation below.)

By Λ' we denote Λ^N endowed with the norm

$$\|(\lambda_1, \dots, \lambda_N)\|_{\Lambda'} = \tfrac{1}{N}(\|\lambda_1\|_\Lambda + \dots + \|\lambda_N\|_\Lambda).$$

This is a polyhedral lattice.

Lemma 6.8. *For any $m \geq 1$, let $\Lambda'^{(m)}$ be given by $\Lambda'^m / \Lambda \otimes (\mathbb{Z}^m)_{\sum=0}$; for $m = 0$, we set $\Lambda'^{(0)} = \Lambda$. Then $\Lambda'^{(m)}$ is a polyhedral lattice.*

Proof. The proof is done in Lean. TODO: write down a proof here. ☐

Definition 6.9. For any $m \geq 1$, let $\Lambda'^{(m)}$ be given by $\Lambda'^m / \Lambda \otimes (\mathbb{Z}^m)_{\sum=0}$; for $m = 0$, we set $\Lambda'^{(0)} = \Lambda$. Then $\Lambda'^{(\bullet)}$ is a cosimplicial polyhedral lattice, the Čech conerve of $\Lambda \to \Lambda'$.

In particular, $\Lambda'^{(0)} = \Lambda \to \Lambda' = \Lambda'^{(1)}$ is the diagonal embedding.

Definition 6.10. Let Λ be a polyhedral lattice, and M a profinitely filtered pseudo-normed group.

Endow $\mathrm{Hom}(\Lambda, M)$ with the subspaces

$$\mathrm{Hom}(\Lambda, M)_{\leq c} = \{f \colon \Lambda \to M \mid \forall x \in \Lambda, f(x) \in M_{\leq c\|x\|}\}.$$

As Λ is polyhedral, it is enough to check the given condition on f for a finite collection of x that generate the norm.

These subspaces are profinite subspaces of M^Λ, and thus they make $\mathrm{Hom}(\Lambda, M)$ ito a profinitely filtered pseudo-normed group.

If M has an action of T^{-1}, then so does $\mathrm{Hom}(\Lambda, M)$.

7. End of proof

Now we state the following result, which is our main goal.

N.b.: It differs from Theorem 9.4 of [Sch20] only in one aspect: we assume that the sets S are finite, rather than profinite.

Theorem 7.1. *Let $\mathsf{BD} = (n, f, h)$ be a Breen–Deligne package, and let $\kappa = (\kappa_0, \kappa_1, \kappa_2, \dots)$ be a sequence of constants in $\mathbb{R}_{\geq 0}$ that is BD-suitable. Fix radii $1 > r' > r > 0$. For any m there is some k and c_0 such that for all finite sets S and all r-normed $\mathbb{Z}[T^{\pm 1}]$-modules V, the system of complexes*

$$C_\kappa^{\mathsf{BD}}(\overline{\mathcal{M}}_{r'}(S))_c^\bullet \colon \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c})^{T^{-1}} \to \widehat{V}(\overline{\mathcal{M}}_{r'}(S)^2_{\leq \kappa_1 c})^{T^{-1}} \to \dots$$

is $\leq k$-exact in degrees $\leq m$ for $c \geq c_0$.

We will prove Theorem 7.1 by induction on m. Unfortunately, the induction requires us to prove a stronger statement.

Theorem 7.2. *Fix radii $1 > r' > r > 0$. For any m there is some k such that for all polyhedral lattices Λ there is a constant $c_0(\Lambda) > 0$ such that for all finite sets S and all r-normed $\mathbb{Z}[T^{\pm 1}]$-modules V, the system of complexes*

$$C_{\Lambda,c}^\bullet \colon \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^2_{\leq \kappa_1 c})^{T^{-1}} \to \dots$$

is $\leq k$-exact in degrees $\leq m$ for $c \geq c_0(\Lambda)$.

Proof. Use $\Lambda = \mathbb{Z}$, and the isomorphism $\mathrm{Hom}(\mathbb{Z}, A) \cong A$. ☐

A word on universal constants: We fix once and for all, the constants $0 < r < r' \leq 1$ a Breen–Deligne package BD, and a sequence of positive constants κ that is very suitable for (BD, r, r'). Once the full proof is formalized, we can come back to this place and write a bit more about the other constants.

The global strategy of the proof is to construct a system of double complexes such that its first row is the system $C^{\bullet}_{\Lambda,\bullet}$ occuring in Theorem 7.2. We can then verify the conditions to Proposition 4.13 and conclude from there. For the time being, we will let M denote an arbitrary profinitely filtered pseudo-normed group with action of T^{-1}, and whenever needed we can specialize to $M = \overline{\mathcal{M}}_{r'}(S)$.

Further choices of constants: We will argue by induction on m, so assume the result for $m-1$ (this is no assumption for $m = 0$, so we do not need an induction start). This gives us some $k > 1$ for which the statement of Theorem 7.2 holds true for $m-1$; if $m = 0$, simply take any $k > 1$. In the proof below, we will increase k further in a way that depends only on m and r. After this modified choice of k, we fix ϵ and k_0 as provided by Proposition 4.13. Fix a sequence $(\kappa'_i)_i$ of nonnegative reals that is adept to (BD, κ). (Such a sequence exists by Lemma 1.21.) Moreover, we let k' be the supremum of k_0 and the c'_i for $i = 0, \dots, m+1$. Finally, choose a positive integer b so that $2k'(\frac{r}{r'})^b \leq \epsilon$, and let N be the minimal power of 2 that satisfies

$$k'/N \leq (r')^b.$$

Then in particular $r^b N \leq 2k'(\frac{r}{r'})^b \leq \epsilon$.

Definition 7.3. Let $\Lambda^{(\bullet)}$ be the cosimplicial polyhedral lattice of Definition 6.9, and recall from 6.10 that $\mathrm{Hom}(\Lambda^{(m)}, M)$ is a profinitely filtered pseudo-normed group with action of T^{-1}.

Hence $\mathrm{Hom}(\Lambda^{(\bullet)}, M)$ is a simplicial profinitely filtered pseudo-normed group with action of T^{-1}.

Now apply the construction of the system of complexes from Definition 5.11 to obtain a cosimplicial system of complexes

$$C^{\mathsf{BD}}_{\kappa}(\mathrm{Hom}(\Lambda^{(\bullet)}, M))^{\bullet}_{\bullet}.$$

Now take the alternating face map cochain complex to obtain a system of double complexes, whose objects are

$$\widehat{V}(\mathrm{Hom}(\Lambda^{(m)}, M)^{n_i}_{\leq \kappa_i c})^{T^{-1}}.$$

As final step, rescale the norm on the object in row m by $m!$, so that all columns become admissible: the vertical differential from row m to row $m+1$ is an alternating sum of $m+1$ maps that are all norm-nonincreasing.

Lemma 7.4. *In particular, for any $c > 0$, we have*

$$\mathrm{Hom}(\Lambda', M)_{\leq c} = \mathrm{Hom}(\Lambda, M)^N_{\leq c/N},$$

with the map to $\mathrm{Hom}(\Lambda, M)_{\leq c}$ given by the sum map.

Proof. Omitted (but done in Lean). □

Lemma 7.5. *Similarly, for any $c > 0$, we have*

$$\mathrm{Hom}(\Lambda'^{(m)}, M)_{\leq c} = \mathrm{Hom}(\Lambda', M)^{m/\mathrm{Hom}(\Lambda, M)_{\leq c}}_{\leq c},$$

the m-fold fibre product of $\mathrm{Hom}(\Lambda', M)_{\leq c}$ over $\mathrm{Hom}(\Lambda, M)_{\leq c}$.

Lemma 7.6. *There is a canonical isomorphism between the first row of the double complex*

$$C^{\mathsf{BD}}_{\kappa}(\mathrm{Hom}(\Lambda^{(1)}, M))^{\bullet}$$

and

$$C^{N\otimes\mathsf{BD}}_{\kappa/N}(\mathrm{Hom}(\Lambda, M))^{\bullet}$$

which identifies the map induced by the diagonal embedding $\Lambda \to \Lambda' = \Lambda^{(1)}$ with the map induced by $\sigma^N \colon N \otimes \mathsf{BD} \to \mathsf{BD}$.

Proof. Omitted (but done in Lean). ☐

Proposition 7.7. *Let $S' \to S$ be a surjective morphism of profinite sets, and let $S_\bullet \to S$ be its Cech nerve. Then the complex*

$$0 \to \widehat{M}(S) \to \widehat{M}(S_0) \to \widehat{M}(S_1) \to \dots$$

is exact, and whenever $f \in \ker(\widehat{M}(S_m) \to \widehat{M}(S_{m+1}))$ with $\|f\| \le c$, then for any $\epsilon > 0$ there is some $g \in \widehat{M}(S_{m-1})$ with $\|g\| \le (1+\epsilon)c$ such that $d(g) = f$.

Proof. Follow the proof of [Sch19, Theorem 3.3]: When S and all S_i are finite, the cechcover splits, so a contracting homotopy gives the result with constant 1. In general, write the cechcover as a cofiltered limit of cechcovers of finite sets by finite sets, pass to the filtered colimit, and complete, using Lemma 4.3. ☐

Proposition 7.8. *Let d be the constant from Proposition 6.2. Let $k > 1$ and $c_0 > 0$ be real numbers such that*

$$(k-1) * c_0/N \ge d.$$

Let m be any natural number, and put

$$K = (m+2) + \frac{r+1}{r(1-r)}(m+2)^2$$

Finally, let c_0' be $\frac{c_0}{r' \cdot n_i}$, where n_i is the i-th index in our fixed Breen–Deligne data.

Then i-th column in the double complex are (k^2, K)-weak bounded exact in degrees $\le m$ for $c \ge c_0'$.

Proof. This is almost completely done in Lean. The proof given below has to be expanded and rewritten.

By Lemma 6.2, and noting that $\mathrm{Hom}(\Lambda'^{(\bullet)}, \overline{\mathcal{M}}_{r'}(S))_{\le c}$ is the Čech nerve of

$$\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^N_{\le c/N} \xrightarrow{\Sigma} \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\le c},$$

also the second condition is satisfied, with k the maximum of the previous k and some constant depending only on m and r, provided we take c_0 large enough so that $(k-1)r'c_ic_0/N$ is at least the d of Lemma 6.2 for all $i = 0, \dots, m$ (so this choice of c_0 again depends on Λ). Indeed, then one can splice a surjection of profinite sets between the maps

$$\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{Na}_{\le c_i c/N} \to \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{a}_{\le c_i c}$$

and

$$\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{Na}_{\le k c_i c/N} \to \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{a}_{\le k c_i c},$$

and so the transition map between the columns of that double complex factors over a similar complex arising from a simplicial cechcover of profinite sets, so the constants are bounded as claimed in the statement, by Proposition 7.7, Lemma 5.8, and Proposition 4.11. $\square$

Proposition 7.9. *Let h be the homotopy packaged with BD, and let h^N denote the n-th iterated composition of h (see Def 1.12) which is a homotopy between π^N and $\sigma^N : N \otimes \mathsf{BD} \to \mathsf{BD}$.*

Let $H \in \mathbf{R}_{\geq 0}$ be such that for $i = 0, \dots, m$ the universal map h_i^N is bound by H (see Def 1.4).

Then the double complex satisfies the normed homotopy homotopy condition (Def 4.12) for m, H, and c_0.

Proof. By Lemma 7.6 we may replace the first row by

$$C_{\kappa/N}^{N\otimes\mathsf{BD}}(\mathrm{Hom}(\Lambda, M))^{\bullet}.$$

Now it is important to recall that we have chosen $k' \geq \kappa_i'$ for all $i = 0, \dots, m+1$.

Our goal is to find, in degrees $\leq m$, a homotopy between the two maps from the first row

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)_{\leq c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^2_{\leq \kappa_1 c})^{T^{-1}} \to \dots$$

to the second row

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)^N_{\leq c/N})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^{2N}_{\leq \kappa_1 c/N})^{T^{-1}} \to \dots$$

respectively induced by σ^N and π^N (which are maps $N \otimes \mathsf{BD}$

By Definition 1.12 and Lemma 1.19 we can find this homotopy between the complex for $k'c$ and the complex for c. (Here we use $k' \geq c_i'$ for $i = 0, \dots, m$.) By assumption, the norm of these maps is bounded by H.

Finally, it remains to establish the estimate (4.1) on the homotopic map. We note that this takes $x \in \widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq k'^2 \kappa_i c})^{T^{-1}}$ (with $i = q$ in the notation of (4.1)) to the element

$$y \in \widehat{V}(\mathrm{Hom}(\Lambda, M)^{Na_i}_{\leq k' \kappa_i c/N})^{T^{-1}}$$

that is the sum of the N pullbacks along the N projection maps $\mathrm{Hom}(\Lambda, M)^{Na_i}_{\leq k'\kappa_i c/N} \to \mathrm{Hom}(\Lambda, M)^{a_i}_{\leq k'^2 \kappa_i c}$. We note that these actually take image in $\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq \kappa_i c}$ as $N \geq k'$, so this actually gives a well-defined map

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq \kappa_i c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^{Na_i}_{\leq k' \kappa_i c/N})^{T^{-1}}.$$

We need to see that this map is of norm $\leq \epsilon$. Now note that by our choice of N, we actually have $k'\kappa_i c/N \leq (r')^b \kappa_i c$, so this can be written as the composite of the restriction map

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq \kappa_i c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq (r')^b \kappa_i c})^{T^{-1}}$$

and

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq (r')^b \kappa_i c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^{Na_i}_{\leq k' \kappa_i c/N})^{T^{-1}}.$$

The first map has norm exactly r^b, by T^{-1}-invariance, and as multiplication by T scales the norm with a factor of r on $\widehat{V}$. (Here is where we use $r' > r$, ensuring different scaling behaviour of the norm on source and target.) The second map has norm at most N (as it is a sum of N maps of norm ≤ 1). Thus, the total map has norm $\leq r^b N$. But by our choice of N, we have $r^b N \leq \epsilon$, giving the result. $\square$

Proof of Theorem 7.2. By induction, the first condition of Proposition 4.13 is satisfied for all $c \geq c_0$ with c_0 large enough (depending on Λ but not V or S).

The second condition is Proposition 7.8, and the third condition has been checked in Proposition 7.9.

Thus, we can apply Proposition 4.13, and get the desired $\leq \max(k'^2, 2k_0H)$-exactness in degrees $\leq m$ for $c \geq c_0$, where k', k_0 and H were defined only in terms of k, m, r' and r, while c_0 depends on Λ (but not on V or S). This proves the inductive step. □

Question 7.10. Can one make the constants explicit, and how large are they? [1] Modulo the Breen-Deligne resolution, all the arguments give in principle explicit constants; and actually the proof of the existence of the Breen-Deligne resolution should be explicit enough to ensure the existence of bounds on the c_i and c_i'.

Answer: yes, we can do this. And we should write up a clean account asap, when we have cleaned up the proof in Lean.

References

[Sch19] P. Scholze. Lectures on Condensed Mathematics. 2019.
[Sch20] P. Scholze. Lectures on Analytic Geometry. 2020.

[1]A back of the envelope calculation seems to suggest that k is roughly doubly exponential in m, and that N has to be taken of roughly the same magnitude.