BLUEPRINT FOR THE LIQUID TENSOR EXPERIMENT

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Remark 0.0.1. This text is based on the lecture notes on Analytic Geometry [Sch20], by Peter Scholze. The final subsection is copy-pasted from those lecture notes almost verbatim. This text is meant as a blueprint for the Liquid Tensor Experiment.

Remark 0.0.2. In this text $\mathbf{N}$ denotes the natural numbers *including* 0.

Introduction. The goal of this document is to provide a detailed account of the proof of the following theorem, along side a computer verification in the Lean theorem prover (see Section 0.0.1).

Theorem (Clausen–Scholze)**.** *Let $0 < p' < p \leq 1$ be real numbers, let S be a profinite set, and let V be a p-Banach space. Let $\mathcal{M}_{p'}(S)$ be the space of p'-measures on S. Then*

$$\mathrm{Ext}^i_{\mathrm{Cond(Ab)}}(\mathcal{M}_{p'}(S), V) = 0$$

for $i \geq 1$.

We will explain this statement in more detail in Section 0.0.1 below.

Remark. *Status report of the project as of 10-06-2021. The main ingredient in the proof of the above Theorem is the highly technical Theorem 1.7.1.*

We have completed a computer verification of this first target in Lean, and this document contains an account of the proof. We are in the process of cleaning up and documenting this first step.

Once that is done, we will work on the second step: deducing Theorem from Theorem 1.7.1.

0.0.1. *On the statement of the main goal.* For the definition of condensed sets and condensed abelian groups, we refer to [Sch19].

A p-Banach space V is a complete topological $\mathbb{R}$-vector space whose topology is induced by a p-norm; that is, a norm satisfying $\|rv\| = |r|^p\|v\|$.

We will now explain the space $\mathcal{M}_{p'}(S)$. TODO (For now, see Definition 6.3 of [Sch20].)

On the Lean theorem prover and computer verification of proofs. The Lean theorem prover is developed by Leonardo de Moura at Microsoft Research.

TODO: write a short paragraph about what computer verification means, add pointers to further tutorials/material

1. First part

1.1. **Breen–Deligne data.** The goal of this subsection is to a give a precise statement of a variant of the Breen–Deligne resolution. This variant is not actually a resolution, but it is sufficient for our purposes, and is much easier to state and prove.

We first recall the original statement of the Breen–Deligne resolution.

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Theorem (Breen–Deligne)**.** *For an abelian group A, there is a resolution, functorial in A, of the form*

$$\ldots \to \bigoplus_{i=1}^{n_i} \mathbb{Z}[A^{r_{ij}}] \to \ldots \to \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0.$$

What does a homomorphism $f\colon \mathbb{Z}[A^m] \to \mathbb{Z}[A^n]$ that is functorial in A look like? We should perhaps say more precisely what we mean by this. The idea is that m and n are fixed, and for each abelian group A we have a group homomorphism $f_A\colon \mathbb{Z}[A^m] \to \mathbb{Z}[A^n]$ such that if $\phi\colon A \to B$ is a group homomorphism inducing $\phi_i\colon \mathbf{Z}[A^i] \to \mathbf{Z}[B^i]$ for each natural number i then the obvious square commutes: $\phi_n \circ f_A = f_B \circ \phi_m$.

The map f_A is specified by what it does to the generators $(a_1, a_2, a_3, \ldots, a_m) \in A^m$. It can send such an element to an arbitrary element of $\mathbb{Z}[A^n]$, but one can check that universality implies that f_A will be a $\mathbb{Z}$-linear combination of "basic universal maps", where a "basic universal map" is one that sends $(a_1, a_2, \ldots, a_m)$ to $(t_1, \ldots, t_n)$, where t_i is a $\mathbb{Z}$-linear combination $c_{i,1} \cdot a_1 + \cdots + c_{i,m} \cdot a_m$. So a "basic universal map" is specified by the $n \times m$-matrix c.

Definition 1.1.1. A *basic universal map* from exponent m to n, is an $n \times m$-matrix with coefficients in $\mathbb{Z}$.

Definition 1.1.2. A *universal map* from exponent m to n, is a formal $\mathbb{Z}$-linear combination of basic universal maps from exponent m to n.

If f is a basic universal map, then we write $[f]$ for the corresponding universal map.

Definition 1.1.3. Let $f = \sum_g n_g [g]$ be a universal map. We say that f is *bound by* a natural number N if $\sum_g |n_g| \leq N$.

We point out that basic universal maps can be composed by matrix multiplication, and this formally induces a composition of universal maps. As mentioned above, one can also check (this has been formalised in Lean) that this construction gives a bijection between universal maps from exponent m to n and functorial collections $f_A\colon \mathbf{Z}[A^m] \to \mathbf{Z}[A^n]$.

Definition 1.1.4. In other words, we are considering the following two categories:

- the category whose objects are natural numbers, and whose morphisms are matrices;
- the category with the same objects, but with Hom-sets replaced by the free abelian groups generated by the sets of matrices. We denote this latter category FreeMat.

Both categories naturally come with a monoidal structure: for the first it is given by the Kronecker product of matrices (a.k.a. tensor product of linear maps) which induces a monoidal structure on FreeMat. As usual, we will denote this monoidal structure $\_ \otimes \_$. For example, if f is a basic universal map, then $2 \otimes f$ denotes the block matrix

$$\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$$

Definition 1.1.5. Let N be a natural number, and $i < N$. Then $\pi'_{N,i}$ denotes the basic universal map from exponent N to 1

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = (a_0, a_1, \dots, a_{N-1})^t$$

where $a_j = \delta_{ij}$.

Definition 1.1.6. Let N and n be natural numbers. Then π_n^N denotes the universal map from exponent $N \cdot n$ to n given by $\sum_{i<N} [\pi'_{N,i} \otimes n]$.

(On $\mathbf{Z}[A^{N \cdot n}] \to \mathbf{Z}[A^n]$ this map is the formal sum of the maps $\mathbf{Z}[A^{N \cdot n}] \to \mathbf{Z}[A^n]$ induced by the projection maps $A^{N \cdot n} = (A^n)^N \to A^n$.)

Definition 1.1.7. Let N and n be natural numbers. Then σ_n^N denotes the universal map from exponent $N \cdot n$ to n given by $[\sum_{i<N} \pi'_{N,i} \otimes n]$.

(On $\mathbf{Z}[A^{N \cdot n}] \to \mathbf{Z}[A^n]$ this map is induced by the summation map $A^{N \cdot n} = (A^n)^N \to A^n$.)

Definition 1.1.8. A *Breen–Deligne data* is a chain complex in FreeMat.

Concretely, this means that it consists of a sequence of exponents $n_0, n_1, n_2, \dots \in \mathbb{N}$, and universal maps f_i from exponent n_{i+1} to n_i, such that for all i we have $f_i \circ f_{i+1} = 0$.

A morphism of Breen–Deligne data is a morphism of chain complexes.

Definition 1.1.9. For every natural numbers N, the endofunctor $N \otimes \_$ on FreeMat induces an endofunctor of Breen–Deligne data.

Concretely, it maps a pair (n, f) of Breen–Deligne data, to the pair $N \otimes (n, f)$ consisting of exponents $N \cdot n_i$ and universal maps $N \otimes f_i$.

Let BD be Breen–Deligne data. The universal maps σ^N and π^N defined above, induce morphisms $\sigma_{\mathsf{BD}}^N, \pi_{\mathsf{BD}}^N \colon N \otimes \mathsf{BD} \to \mathsf{BD}$.

Definition 1.1.10. A *Breen–Deligne* package consists of Breen–Deligne data BD together with a homotopy h between π_{BD}^2 and σ_{BD}^2.

Definition 1.1.11. Let BD be a Breen–Deligne package and N a power of 2. Then the homotopy h induces a homotopy between π_{BD}^N and σ_{BD}^N by iterative composition of the homotopy packaged in BD.

Definition 1.1.12. We will now construct an example of a Breen–Deligne package. In some sense, it is the "easiest" solution to the conditions posed above. The exponents will be $n_i = 2^i$, and the homotopies h_i will be the identity. Under these constraints, we recursively construct the universal maps f_i:

$$f_0 = \pi_1^2 - \sigma_1^2, \quad f_{i+1} = (\pi_{2^{i+1}}^2 - \sigma_{2^{i+1}}^2) - (2 \otimes f_i).$$

We leave it as exercise for the reader, to verify that with these definitions (n, f, h) forms a Breen–Deligne package.

We now make definitions that will make precise some conditions between constants that will be needed when we construct Breen–Deligne complexes of normed abelian groups.

Definition 1.1.13. Let f be a basic universal map from exponent m to n. Let $c_1, c_2 \in \mathbb{R}_{\geq 0}$. We say that (c_1, c_2) is *f-suitable*, if for all i

$$\sum_j c_1 |f_{ij}| \leq c_2.$$

To orient the reader: later on we will be considering maps on normed abelian groups induced from universal maps, and this inequality will guarantee that if $\|m\| \leq c_1$ then $\|f(m)\| \leq c_2$.

Definition 1.1.14. Let f be a universal map from exponent m to n. Let $c_1, c_2 \in \mathbb{R}_{\geq 0}$. We say that (c_1, c_2) is *f-suitable*, if for all basic universal maps g that occur in the formal sum f, the pair of nonnegative reals (c_1, c_2) is g-suitable.

Definition 1.1.15. Let f be a universal map and let $r, r', c_1, c_2 \in \mathbb{R}_{\geq 0}$. We say that (c_1, c_2) is *very suitable* for (f, r, r') if there exist $N, b \in \mathbf{N}$ and $c' \in \mathbf{R}_{\geq 0}$ such that:

- f is bound by N (see Definition 1.1.3)
- (c_1, c') is f-suitable
- $r^b N \leq 1$
- $c' \leq (r')^b c_2$

Definition 1.1.16. Let $\mathsf{BD} = (n, f)$ be Breen–Deligne data, let $r, r' \in \mathbf{R}_{\geq 0}$, and let $\kappa = (\kappa_0, \kappa_1, \dots)$ be a sequence of nonnegative real numbers. We say that κ is BD-*suitable* (resp. *very suitable* for (BD, r, r')), if for all i, the pair (κ_{i+1}, κ_i) is f_i-suitable (resp. *very suitable* for (f_i, r, r')).

(Note! The order (κ_{i+1}, κ_i) is contravariant compared to Definition 1.1.14. This is because of the contravariance of $\widehat{V}(\_)$; see Definition 1.5.9.)

Definition 1.1.17. Let BD be a Breen–Deligne package with data (n, f) and homotopy h. Let κ, κ' be sequences of nonnegative real numbers. (In applications κ is a (n, f)-suitable sequence.)

Then κ' is *adept* to (BD, κ) if for all i the pair $(\kappa_i/2, \kappa'_{i+1}\kappa_{i+1})$ is h_i-suitable. (Recall that h_i is the homotopy map $n_i \to n_{i+1}$.)

Lemma 1.1.18. *Let BD be a Breen–Deligne package, N a power of 2, and let κ, κ' be sequences of nonnegative real numbers. Assume that κ' is adept to (BD, κ). Let h^N be the homotopy between π^N_{BD} and σ^N_{BD} defined in Def 1.1.11.*

For all i, the pair $(\kappa_i/N, \kappa'_{i+1}\kappa_{i+1})$ is h^N_i-suitable.

Proof. Omitted. (But done in Lean.) □

Lemma 1.1.19. *Let BD be a Breen–Deligne package, and let r, r' be nonnegative reals, such that $r < 1$ and $r' > 0$.*

There exists a sequence κ of positive real numbers such that κ is very suitable for (BD, r, r').

Proof. The sequence can be constructed recursively, which we leave as exercise for the reader. (It has been done in Lean.) □

Lemma 1.1.20. *Let BD be a Breen–Deligne package, and let r, r' be nonnegative reals, such that $0 < r < 1$ and $0 < r' \leq 1$. Let κ be any sequence of positive reals.*

There exists a sequence κ' of nonnegative real numbers that is adept to (BD, κ).

Proof. The sequence can be constructed recursively, which we leave as exercise for the reader. (It has been done in Lean.) □

1.2. **Variants of normed groups.** Normed groups are well-studied objects. In this text it will be helpful to work with the more general notion of *semi-normed group*. This drops the separation axiom $\|x\| = 0 \iff x = 0$ but is otherwise the same as a normed group.

The main difference is that this includes "uglier" objects, but creates a "nicer" category: semi-normed groups need not be Hausdorff, but quotients by arbitrary (possibly non-closed) subgroups are naturally semi-normed groups.

Nevertheless, there is the occasional use for the more restrictive notion of normed group, when we come to polyhedral lattices below (see Section 1.6).

In this text, a morphism of (semi)-normed groups will always be bouned. If the morphism is supposed to be norm-nonincreasing, this will be mentioned explicitly.

Definition 1.2.1. Let $r > 0$ be a real number. An *r-normed $\mathbb{Z}[T^{\pm 1}]$-module* is a semi-normed group V endowed with an automorphism $T\colon V \to V$ such that for all $v \in V$ we have $\|T(v)\| = r\|v\|$.

The remainder of this subsection sets up some algebraic variants of semi-normed groups.

Definition 1.2.2. A *pseudo-normed group* is an abelian group $(M, +)$, together with an increasing filtration $M_c \subseteq M$ of subsets M_c indexed by $\mathbb{R}_{\geq 0}$, such that each M_c contains 0, is closed under negation, and $M_{c_1} + M_{c_2} \subseteq M_{c_1+c_2}$. An example would be $M = \mathbb{R}$ or $M = \mathbb{Q}_p$ with $M_c := \{x : |x| \leq c\}$.

A pseudo-normed group M is *exhaustive* if $\bigcup_c M_c = M$.

All pseudo-normed groups that we consider will have a topology on the filtration sets M_c. The most general variant is the following notion.

Definition 1.2.3. A pseudo-normed group M is *CH-filtered* if each of the sets M_c is endowed with a topological space structure making it a compact Hausdorff space, such that following maps are all continuous:

- the inclusion $M_{c_1} \to M_{c_2}$ (for $c_1 \leq c_2$);
- the negation $M_c \to M_c$;
- the addition $M_{c_1} \times M_{c_2} \to M_{c_1+c_2}$.

The pseudo-normed group M is *profinitely filtered* if moreover the filtration sets M_c are totally disconnected, making them profinite sets.

Remark 1.2.4. The topologies on the filtration sets M_c will induce a topology on M: the colimit topology. If M is some sort of normed group, then this topology is typically genuinely different from the norm topology.

Definition 1.2.5. A *morphism* of CH-filtered pseudo-normed groups $M \to N$ is a group homomorphism $f\colon M \to N$ that is

- *bounded*: there is a constant C such that $x \in M_c$ implies $f(x) \in N_{Cc}$;
- *continuous*: for one (or equivalently all) constants C as above, the induced map $M_c \to N_{Cc}$ is a morphism of profinite sets, i.e. continuous.

The reason the two definitions of continuity are equivalent is that a continuous injection from a compact space to a Hausdorff space must be a topological embedding.

A morphism $f\colon M \to N$ is *strict* if $x \in M_c$ implies $f(x) \in N_c$ (in other words, if we can take $C = 1$ in the boundedness condition above).

We will also consider the analogue of an r-normed $\mathbb{Z}[T^{-1}]$-module in the pseudo-normed setting.

Definition 1.2.6. Let r' be a positive real number. A CH-filtered pseudo-normed group M has an *r'-action of T^{-1}* if it comes endowed with a distinguished morphism of CH-filtered pseudo-normed groups $T^{-1}\colon M \to M$ that is bounded by r'^{-1}: if $x \in M_c$ then $T^{-1}x \in M_{c/r'}$.

A morphism of CH-filtered pseudo-normed groups with r'-action of T^{-1} is a morphism $f\colon M \to N$ of CH-filtered pseudo-normed groups that commutes with the action of T^{-1}.

1.3. **Spaces of convergent power series.** We will now construct the central example of profinitely filtered pseudo-normed groups with r'-action of T^{-1}.

Definition 1.3.1. Let $r' > 0$ be a real number, and let S be a finite set. Denote by $\overline{\mathcal{L}}_{r'}(S)$ the set

$$\left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \in T\mathbf{Z}[[T]] \right)_{s \in S} \middle| \sum_{n \geq 1, s \in S} |a_{n,s}|(r')^n < \infty \right\}.$$

Note that $\overline{\mathcal{L}}_{r'}(S)$ is naturally a pseudo-normed group with filtration given by

$$\overline{\mathcal{L}}_{r'}(S)_{\leq c} = \left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} \middle| \sum_{n \geq 1, s \in S} |a_{n,s}|(r')^n \leq c \right\}.$$

Lemma 1.3.2. *Let $r' > 0$ and $c \geq 0$ be real numbers, and let S be a finite set. The space $\overline{\mathcal{L}}_{r'}(S)_{\leq c}$ is the profinite limit of the finite sets*

$$\overline{\mathcal{L}}_{r'}(S)_{\leq c, \leq N} = \left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} \middle| \sum_{1 \leq n \leq N, s \in S} |a_{n,s}|(r')^n \leq c \right\}$$

This endows $\overline{\mathcal{L}}_{r'}(S)_{\leq c}$ with the profinite topology. In particular, it is a profinitely filtered pseudo-normed group.

Proof. Formalised, but omitted from this text. □

For the remainder of this subsection, let $r' > 0, c \geq 0$ be real numbers, and let S be a finite set.

Definition 1.3.3. There is a natural action of T^{-1} on $\overline{\mathcal{L}}_{r'}(S)$, via

$$T^{-1} \cdot \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} = \left(\sum_{n \geq 1} a_{n+1,s} T^n \right)_{s \in S}.$$

Lemma 1.3.4. *The natural action of T^{-1} on $\overline{\mathcal{L}}_{r'}(S)$ restricts to continuous maps*

$$T^{-1} \cdot \_ \colon \overline{\mathcal{L}}_r(S)_{\leq c} \to \overline{\mathcal{L}}_r(S)_{\leq c/r'}.$$

In particular, $\overline{\mathcal{L}}_{r'}(S)$ has an r'-action of T^{-1}.

Proof. Formalised, but omitted from this text. □

1.4. **Some normed homological algebra.** It will be convenient to use the following definition generalizing the notion of a bound on the norm of a right inverse of a normed group morphism (note that the morphisms we consider have no reason to have a right inverse, even when they are surjective).

Definition 1.4.1. Let G and H be semi-normed groups, let K be a subgroup of H and C be a positive real number. A morphism $f : G \to H$ is C-surjective onto K if, for all x in K, there exists some g in G such that $f(g) = x$ and $\|g\| \le C\|x\|$. If $K = H$ we simply say f is C-surjective.

The following controlled surjectivity lemma will be used to prove Lemma 1.4.3 and Lemma 1.5.8.

Lemma 1.4.2. *Let G and H be normed groups. Let K be a subgroup of H and f a morphism from G to H. Assume that G is complete and f is C-surjective onto K. Then f is $(C+\varepsilon)$-surjective onto the topological closure of K for every positive ε.*

Proof. Let x be any element of the closure of K. First note the conclusion is trivial when $x = 0$, so we can assume $x \neq 0$. Then write x as a sum $\sum_{i\ge 0} x_i$ with all $x_i \in K$, $\|x - x_0\| \le \varepsilon_0$ and $\|x_i\| \le \epsilon_i$ for $i > 0$ for some sequence of positive numbers ϵ_i to be chosen later. By assumption, we can then lift each x_i to g_i such that $f(g_i) = x_i$ and $\|g_i\| \le C\|x_i\|$, and then set $g = \sum g_i$. Because G is complete, this sum converges provided the ε_i sequence converges fast enough to zero. We then have $f(g) = x$ and

$$\|g\| \le C \sum_{i\ge 0} \|x_i\| \le C(\|x\| + \varepsilon_0) + C \sum_{i>0} \varepsilon_i \le (C+\varepsilon)\|x\|$$

where the last inequality holds provided the ε_i sequence converges fast enough to zero. For instance $\varepsilon_i = \varepsilon \parallel x \parallel / (2^{i+1}C)$ satisfies all our constraints on the ε_i sequence (in particular they are positive because $x \neq 0$). □

The first application of the above lemma is a completion result for a quantitative version of being a complex.

Lemma 1.4.3. *Let $f : M_0 \to M_1$ and $g : M_1 \to M_2$ be bounded maps between normed groups. Assume there are positive constants C and D such that:*

- *f is C-surjective onto $\ker g$.*
- *g is D-surjective onto its image.*

Then for every positive ε, $\hat{f}$ is $(C+\varepsilon)$-surjective onto $\ker \hat{g}$.

Proof. Since f is C-surjective onto $\ker g$, $\hat{f}$ is C-surjective onto $\ker g$ seen as a subset of $\widehat{M_1}$. Hence this lemma will follow directly from Lemma 1.4.2 once we'll have proven that $\ker g$ is dense in $\ker \hat{g}$. Let $\hat{y}$ be an element of $\ker \hat{g}$. Pick any $\delta > 0$ and take $y \in M_1$ such that $\|\hat{y} - y\| \le \delta$. Let $z = g(y) \in M_2$, which has norm $\|z\| = \|g(y)\| = \|g(y - \hat{y})\|$ bounded by $C_g\delta$, where C_g is the norm of g. We can thus find some $y' \in M_1$ with $\|y'\| \le DC_g\delta$ and $g(y') = z$. Replacing y by $y - y'$, we can thus find $y \in \ker(g : M_1 \to M_2)$ such that still $\|\hat{y} - y\| \le (1 + DC_g)\delta$; as δ was arbitrary, this gives the desired density. □

Definition 1.4.4. A *system of complexes* of normed abelian groups is for each $c \in \mathbb{R}_{\ge 0}$ a complex

$$C_c^\bullet : C_c^0 \to C_c^1 \to \dots$$

of normed abelian groups together with maps of complexes $\mathrm{res}_{c',c} : C^\bullet_{c'} \to C^\bullet_c$, for $c' \geq c$, satisfying $\mathrm{res}_{c,c} = \mathrm{id}$ and the obvious associativity condition. In other words, a functor from $(\mathbb{R}_{\geq 0})^{\mathrm{op}}$ to cochain complexes of semi-normed groups.

By convention, for every system of complexes $C^\bullet_\bullet$, we will set $C^{-1}_c = 0$ for all c. This will come up each time we write C^{i-1}_c and i could be 0.

In this subsection, given $x \in C^\bullet_{c'}$ and $c_0 \leq c \leq c'$ we will use the notation $x_{|c} := \mathrm{res}_{c',c}(x)$.

Definition 1.4.5. A system of complexes is *admissible* if all differentials and maps $\mathrm{res}^i_{c',c}$ are norm-nonincreasing.

Throughout the rest of this subsection, k (and k', k'') will denote reals at least 1, m will be a non-negative integer, and K, K', K'' will denote non-negative reals.

Definition 1.4.6. A cochain complex C of semi-normed groups is *normed exact* if for all $i \geq 0$, all $\varepsilon > 0$, and all $x \in C^i$ with $d(x) = 0$ there exists a $y \in C^{i-1}$ such that $d(y) = x$ and $\|y\| \leq (1+\varepsilon)\|x\|$.

Definition 1.4.7. Let $C^\bullet_\bullet$ be a system of complexes. For an integer $m \geq 0$ and reals $k \geq 1$, $K \geq 0$ and $c_0 \geq 0$, we say the datum $C^\bullet_\bullet$ is *$\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound K* if the following condition is satisfied. For all $c \geq c_0$ and all $x \in C^i_{kc}$ with $i \leq m$ there is some $y \in C^{i-1}_c$ such that

$$\|x_{|c} - dy\| \leq K\|dx\|.$$

We will also need a version where the inequality is relaxed by some arbitrary small additive constant.

Definition 1.4.8. Let $C^\bullet_\bullet$ be a system of complexes. For an integer $m \geq 0$ and reals $k \geq 1$, $K \geq 0$ and $c_0 \geq 0$, the datum $(C^\bullet_c)_c$ is *weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound K* if the following condition is satisfied. For all $c \geq c_0$, all $x \in C^i_{kc}$ with $i \leq m$ and any $\varepsilon > 0$ there is some $y \in C^{i-1}_c$ such that

$$\|x_{|c} - dy\| \leq K\|dx\| + \varepsilon.$$

We first note that the difference between those two definitions is only about cocyles if we are ready to lose a tiny something on the norm bound K.

Lemma 1.4.9. *Let $C^\bullet_\bullet$ be a system of complexes. If $C^\bullet_\bullet$ is weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound K and if, for all $c \geq c_0$ and all $x \in C^i_{kc}$ with $i \leq m$ such that $dx = 0$ there is some $y \in C^{i-1}_c$ such that $x_{|c} = dy$ then, for every positive δ, $C^\bullet_\bullet$ is $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound $K + \delta$.*

Proof. Let δ be some positive real number. Let x be an element of C^i_{kc} for some $c \geq c_0$ and $i \leq m$. If $dx = 0$ then the assumption we made about exact elements is exactly what we want.

Assume now that $dx \neq 0$. The weak exactness assumption applied to $\varepsilon = \delta\|dx\|$ gives some $y \in C^{i-1}_c$ such that

$$\begin{aligned}\|x_{|c} - dy\| &\leq K\|dx\| + \delta\|dx\| \\ &= (K + \delta)\|dx\|\end{aligned}$$

□

Lemma 1.4.10. *Let $k \geq 1$, $c_0 \geq 0$ be real numbers, and $m \in \mathbb{N}$. Let $C^\bullet_\bullet$ be a system of complexes, and for each $c \geq 0$ let D_c be a cochain complex of semi-normed groups. Let $f_c \colon C^\bullet_{kc} \to D^\bullet_c$ and $g_c \colon D^\bullet_c \to C^\bullet_c$ be norm-nonincreasing morphisms of cochain complexes of semi-normed groups such that $g_c \circ f_c$ is the restriction map $C^\bullet_{kc} \to C^\bullet_c$. Assume that for all $c \geq c_0$ the cochain complex D_c is normed exact. Then $C^\bullet_\bullet$ is weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound 1.*

Proof. Fix $c \geq c_0$, $i \leq m$, $x \in C^i_{kc}$, and $\varepsilon > 0$. Denote by δ the positive real number $\frac{\varepsilon}{\|x\|+1}$.

Clearly $f(d(x))$ is killed by d, so by normed exactness of D_c we find $x' \in D^i_c$ such that $d(x') = f(d(x))$ and $\|x'\| \leq (1+\delta)\|f(d(x))\|$. Similarly $d(f(x) - x') = 0$, so by exactness of D_c we find $y \in D^{i-1}_c$ such that $d(y) = f(x) - x'$.

We are done if we show that $\|x_{|c} - d(g(y))\| \leq \|d(x)\| + \varepsilon$. Observe that $x_{|c} - d(g(y)) = g(f(x)) - g(d(y)) = g(x')$, and therefore we shall show $\|g(x')\| \leq \|d(x)\| + \varepsilon$.

Now we use that f and g are norm-nonincreasing to calculate

$$\|g(x')\| \leq \|x'\| \leq (1+\delta)\|f(d(x))\| \leq (1+\delta)\|d(x)\|.$$

Finally, we have $(1+\delta)\|d(x)\| \leq \|d(x)\| + \varepsilon$ by our choice of δ. □

Lemma 1.4.11. *Let $M^\bullet_\bullet$ be an admissible collection of complexes of complete normed abelian groups.*

Assume that $M^\bullet_c$ is weakly $\leq k$-exact in degrees $\leq m$ for $c \geq c_0$ with bound K. Then $M^\bullet_c$, for every $\delta > 0$, it is $\leq k^2$-exact in degrees $\leq m$ for $c \geq c_0$ with bound $K + \delta$.

Proof. Lemma 1.4.9 ensures we only need to care about cocycles of M. More precisely, let x be a cocycle in $M^i_{k^2 c}$ for some $i \leq m$ and $c \geq c_0$. We need to find $y \in M^{i-1}_c$ such that $dy = x_{|c}$.

By weak $\leq k$-exactness applied to x and a sequence ε_j to be chosen later, we can find a sequence $w^j \in M^{i-1}_{kc}$ such that

$$\|x_{kc} - dw^j\| \leq \varepsilon_j.$$

Then, by weak $\leq k$-exactness applied to each $w^{j+1} - w^j$ and a sequence δ_j to be chosen later, we can find a sequence $z^j \in M^{i-2}_c$ such that

$$\|(w^{j+1} - w^j)_{|c} - dz^j\| \leq K\|dw^{j+1} - dw^j\| + \delta_j.$$

We set $y^j := w^j_{|c} - \sum_{l=0}^{j-1} dz^l \in M^{i-1}_c$.

We have

$$\begin{aligned} \|y^{j+1} - y^j\| &= \left\|(w^{j+1} - w^j)_{|c} - dz^j\right\| \\ &\leq K\|dw^{j+1} - dw^j\| + \delta_j \\ &\leq 2K\varepsilon_j + \delta_j. \end{aligned}$$

So y^j is a Cauchy sequence as long as we make sure $2K\varepsilon_j + \delta_j \leq 2^{-j}$ for instance. Since M^{i-1}_c is complete, this sequence converges to some y. Because $dy^j = dw^j_{|c}$, we get that $\|x_{|c} - dy^j\| \leq \varepsilon_j$ and in the limit $x_{|c} = dy$. □

Proposition 1.4.12. *Let $M^\bullet_\bullet$ and $M'^\bullet_\bullet$ be two admissible collections of complexes of complete normed abelian groups. For each $c \geq c_0$ let $f^\bullet_c : M^\bullet_c \to M'^\bullet_c$ be a collection of maps between these* *collections of complexes that are norm-nonincreasing and which all commute with all restriction maps, and assume that there exists these maps satisfy*

$$\|x_{|c}\| \leq K''\|f(x)\|$$

for all $i \leq m+1$ and all $x \in M^i_{k''c}$. Let $N^\bullet_c = M'^\bullet_c / M^\bullet_c$ be the collection of quotient complexes, with the quotient norm; this is again an admissible collection of complexes.

Assume that $M^\bullet_c$ (resp. $M'^\bullet_c$) is weakly $\leq k$-exact (resp. $\leq k'$-exact) in degrees $\leq m$ for $c \geq c_0$ with bound K (resp. K'). Then $N^\bullet_c$ is weakly $\leq kk'k''$-exact in degrees $\leq m-1$ for $c \geq c_0$ with bound $K'(KK''+1)$.

Proof. Let $n \in N^i_{kk'k''c}$ for $i \leq m-1$. We fix $\varepsilon > 0$. We need to find an element $y \in N^{i-1}_c$ such that

$$\|n_{|c} - dy\| \leq K'(KK''+1)\|dn\| + \epsilon.$$

Pick any preimage $m' \in M'^i_{kk'k''c}$ of n. In particular dm' is a preimage of dn. By definition of the quotient norm, we can find $m_1 \in M^{i+1}_{kk'k''c}$ and $m''_1 \in (M')^{i+1}_{kk'k''c}$ such that

$$dm' = f(m_1) + m''_1$$

with $\|m''_1\| \leq \|dn\| + \varepsilon_1$, for some positive ε_1 to be chosen later.

Applying the differential to the last displayed equation, and using that this kills the image of d, and that f is a map of complexes, we see that

$$f(dm_1) = -dm''_1.$$

Using the norm bound on f, we get

$$\begin{aligned}\|dm_{1|kk'c}\| &\leq K''\|f(dm_1)\| = K''\|dm''_1\| \\ &\leq K''\|m''_1\| \leq K''\|dn\| + K''\varepsilon_1.\end{aligned}$$

On the other hand, weak exactness of M applied to $m_{1|kk'c}$ gives $m_0 \in M^i_{k'c}$ such that

$$\|m_{1|kk'c|k'c} - dm_0\| \leq K\|dm_{1|kk'c}\| + \varepsilon_1$$

which combines with the previous estimate to give:

$$\|m_{1|k'c} - dm_0\| \leq KK''\,\|dn\| + (KK''+1)\varepsilon_1.$$

Now let $m'_{\mathrm{new}} = m'_{|k'c} - f(m_0) \in M'^i_{k'c}$; this is a lift of $n_{|k'c}$. Then

$$dm'_{\mathrm{new}} = dm'_{|k'c} - f(m_{1|k'c}) + f(m_{1|k'c} - dm_0) = m''_{1|k'c} + f(m_{1|k'c} - dm_0).$$

In particular,

$$\|dm'_{\mathrm{new}}\| \leq (KK''+1)\,\|dn\| + (KK''+2)\varepsilon_1.$$

Now weak exactness of M' gives $x \in M'^{i-1}_c$ such that

$$\|m'_{\mathrm{new}|c} - dx\| \leq K'\|dm'_{\mathrm{new}}\| + \varepsilon_1 \leq K'((KK''+1)\,\|dn\| + (KK''+2)\varepsilon_1) + \varepsilon_1.$$

In particular, letting $y \in N^{i-1}_c$ be the image of x, we get

$$\|n_{|c} - dy\| \leq K'(KK''+1)\,\|dn\| + (K'(KK''+2)+1)\varepsilon_1,$$

which is exactly what we wanted if we choose $\varepsilon_1 = \varepsilon/(K'(KK''+2)+1)$. □

We also need the 'dual' version of 1.4.12, for kernels instead of cokernels. Note that this is actually a bit easier to prove. (Should we put it first in the presentation?)

Proposition 1.4.13. *Let $M_\bullet^\bullet$ and $M_\bullet'^\bullet$ be two admissible collections of complexes of complete normed abelian groups. For each $c \geq c_0$ let $f_c^\bullet : M_c^\bullet \to M_c'^\bullet$ be a collection of maps between these collections of complexes which all commute with all restriction maps. Assume moreover that we are given two constants $r_1, r_2 \geq 0$ such that:*

- *for all $i, c \geq c_0$ and all $x \in M_c^i$*
$$\|f(x)\| \leq r_1\|x\|;$$
- *for all $i \leq m+1, c \geq c_0$ and all $y \in M_c'^i$, there exists $x \in M_c^i$ such that*
$$f(x) = y \text{ and } \|x\| \leq r_2\|y\|.$$

Let $N_c^\bullet$ be the collection of kernel complexes, with the induced norm; this is again an admissible collection of complexes.

Assume that $M_c^\bullet$ (resp. $M_c'^\bullet$) is weakly $\leq k$-exact (resp. $\leq k'$-exact) in degrees $\leq m$ for $c \geq c_0$ with bound K (resp. K'). Then $N_c^\bullet$ is weakly $\leq kk'$-exact in degrees $\leq m-1$ for $c \geq c_0$ with bound $K + r_1r_2KK'$.

Proof. Let $n \in N_{kk'c}^i \subseteq M_{kk'c}^i$ for $i \leq m-1$ and let $\varepsilon > 0$. We need to find an element $y \in N_c^{i-1}$ such that
$$\|n_{|c} - dy\| \leq K + r_1r_2KK'\|dn\| + \epsilon.$$
By weak exactness of $M_\bullet^\bullet$, we can find $m \in M_{k'c}^{i-i}$ such that
$$\|n_{|k'c} - dm\| \leq K\|dn\| + \epsilon_1,$$
where $\epsilon_1 > 0$ to be chosen later. By weak exactness of $M_\bullet'^\bullet$, we can find $m' \in M_c'^{i-2}$ such that
$$\|f(m)_{|c} - dm'\| \leq K'\|df(m)\| + \epsilon_2,$$
where $\epsilon_2 > 0$ to be chosen later. Let $m_1 \in M_c^{i-2}$ be a lift of m' and let $m_2 \in M_c^{i-1}$ be such that
$$f(m_2) = f(m_{|c} - dm_1) \text{ and } \|m_2\| \leq r_2\|f(m_{|c} - dm_1)\|.$$
Set $y = m_{|c} - dm_1 - m_2 \in M_c^{i-1}$. By construction $f(y) = 0$, so $y \in N_c^{i-1}$. We compute
$$\|n_{|c} - dy\| = \|n_{|c} - dm_{|c} + d^2m_1 - dm_2\| = \|n_{|c} - dm_{|c} - dm_2\| \leq$$
$$\|n_{|c} - dm_{|c}\| + \|dm_2\| = \|(n_{|k'c} - dm)_{|c}\| + \|dm_2\| \leq \|(n_{|k'c} - dm)\| + \|dm_2\| \leq$$
$$K\|dn\| + \epsilon_1 + \|dm_2\|.$$
Where we have used the defining property of m and admissibility of $M_\bullet^\bullet$. By the same assumption and since $f(n_{|k'c}) = f(n)_{|k'c} = 0$, we have
$$\|dm_2\| \leq \|m_2\| \leq r_2\|f(m_{|c} - dm_1)\| = r_2\|f(m)_{|c} - df(m_1)\| = r_2\|f(m)_{|c} - dm'\| \leq$$
$$r_2(K'\|df(m)\| + \epsilon_2) = r_2(K'\|f(dm)\| + \epsilon_2) = r_2(K'\|f(n_{|k'c}) - f(dm)\| + \epsilon_2) =$$
$$r_2(K'\|f(n_{|k'c} - dm)\| + \epsilon_2) \leq r_2(K'r_1\|n_{|k'c} - dm\| + \epsilon_2) \leq r_2(K'r_1(K\|dn\| + \epsilon_1) + \epsilon_2)$$
In particular we get
$$\|n_{|c} - dy\| \leq K\|dn\| + \epsilon_1 + r_2(K'r_1(K\|dn\| + \epsilon_1) + \epsilon_2) =$$
$$(K + r_1r_2KK')\|dn\| + \epsilon_1(1 + r_1r_2K') + r_2\epsilon_2.$$

Now let

$$\epsilon_1 = \frac{\epsilon}{2(1 + r_1 r_2 K')} \text{ and } \epsilon_2 = \begin{cases} \frac{\epsilon}{2r_2} \text{ if } r_2 \neq 0 \\ 1 \text{ if } r_2 = 0 \end{cases}$$

In any case $r_2\epsilon_2 \leq \frac{\epsilon}{2}$ and so

$$\|n_{|c} - dy\| \leq (K + r_1 r_2 K K')\|dn\| + \epsilon$$

as required.

If $i = 0$, then all m, m', m_1 and m_2 are 0, so $y = 0$ as required. $\square$

Consider a system of double complexes $M_c^{p,q}$, $p, q \geq 0$, $c \geq c_0$,

$$\begin{tikzcd}
M_c^{0,0} \arrow[r] \arrow[d] & M_c^{0,1} \arrow[r] \arrow[d] & M_c^{0,2} \arrow[r] \arrow[d] & \dots \\
M_c^{1,0} \arrow[r] \arrow[d] & M_c^{1,1} \arrow[r] \arrow[d] & M_c^{1,2} \arrow[r] \arrow[d] & \dots \\
M_c^{2,0} \arrow[r] \arrow[d] & M_c^{2,1} \arrow[r] \arrow[d] & \ddots & \\
\vdots & \vdots & &
\end{tikzcd}$$

of complete normed abelian groups.

Definition 1.4.14. We say that the system of double complexes $M_c^{p,q}$ satisfies the *normed spectral homotopy condition* for $m \in \mathbf{N}$ and $H, c_0 \in \mathbf{R}_{\geq 0}$ if the following condition is satisfied:

For $q = 0, \dots, m$ and $c \geq c_0$, there is a map $h^q_{k'c} \colon M^{0,q+1}_{k'c} \to M^{1,q}_c$ with

$$\|h^q_{k'c}(x)\|_{M_c^{1,q}} \leq H\|x\|_{M^{0,q+1}_{k'c}}$$

for all $x \in M^{0,q+1}_{k'c}$, and such that for all $c \geq c_0$ and $q = 0, \dots, m$ the "homotopic" map

$$\mathrm{res}^{1,q}_{k'^2c,k'c} \circ d^{0,q} + h^q_{k'^2c} \circ d'^{0,q}_{k'^2c} + d'^{1,q-1}_{k'c} \circ h^{q-1}_{k'^2c} \colon M^{0,q}_{k'^2c} \to M^{1,q}_{k'c}$$

factors as a composite of the restriction $\mathrm{res}^{0,q}_{k'^2c,c}$ and a map

$$\delta^{0,q}_c \colon M^{0,q}_c \to M^{1,q}_{k'c}$$

that is a map of complexes (in degrees $\leq m$), and satisfies the estimate

$$\|\delta^{0,q}_c(x)\|_{M^{1,q}_{k'c}} \leq \epsilon\|x\|_{M^{0,q}_c} \tag{1.4.1}$$

for all $x \in M^{0,q}_c$.

Proposition 1.4.15. *Fix an integer $m \geq 0$ and constants k, K. Then there exists an $\epsilon > 0$ and constants k_0, K_0, depending (only) on k, K and m, with the following property.*

Let $M_c^{p,q}$ be a system of double complexes as above, and assume that it is admissible. Assume further that there is some $k' \geq k_0$ and some $H > 0$, such that

1. *for $i = 0, \dots, m+1$, the rows $M_c^{i,\cdot}$ are weakly $\leq k$-exact in degrees $\leq m-1$ for $c \geq c_0$ with bound K;*

(2) *for $j = 0, \dots, m$, the columns $M_c^{p,j}$ are weakly $\leq k$-exact in degrees $\leq m$ for $c \geq c_0$ with bound K;*
(3) *it satisfies the normed spectral homotopy condition for m, H and c_0.*

Then the first row is weakly $\leq k'^2$ exact in degrees $\leq m$ for $c \geq c_0$ with bound $2K_0H$.

We note that the bound on the homotopy is of a peculiar nature, in that the bound only depends on a deep restriction of x.

Proof of Proposition 1.4.15. First, we treat the case $m = 0$. If $m = 0$, we claim that one can take $\epsilon = \frac{1}{2k}$ and $k_0 = k$. We have to prove exactness at the first step. Let $x_{k'^2c} \in M^{0,0}_{k'^2c}$ and denote $x_{k'c} = \mathrm{res}^{0,0}_{k'^2c,k'c}(x)$ and $x_c = \mathrm{res}^{0,0}_{k'^2c,c}(x)$. Then by assumption (2) (and $k' \geq k$), we have

$$\|x_c\|_{M_c^{0,0}} \leq k\|d^{0,0}_{k'c}(x_{k'c})\|_{M^{1,0}_{k'c}}.$$

On the other hand, by (3),

$$\|\mathrm{res}^{1,0}_{k'^2c,k'c}(d^{0,0}_{k'^2c}(x)) \pm h^0_{k'^2c}(d'^{0,0}_{k'^2c}(x))\|_{M^{1,0}_{k'c}} \leq \epsilon\|x_c\|_{M_c^{0,0}}.$$

In particular, noting that $\mathrm{res}^{1,0}_{k'^2c,k'c}(d^{0,0}_{k'^2c}(x)) = d^{0,0}_{k'c}(x_{k'c})$, we get

$$\|x_c\|_{M_c^{0,0}} \leq k\|d^{0,0}_{k'c}(x_{k'c})\|_{M^{1,0}_{k'c}} \leq k\epsilon\|x_c\|_{M_c^{0,0}} + kH\|d'^{0,0}_{k'^2c}(x)\|_{M^{0,1}_{k'^2c}}.$$

Thus, taking $\epsilon = \frac{1}{2k}$ as promised, this implies

$$\|x_c\|_{M_c^{0,0}} \leq 2kH\|d'^{0,0}_{k'^2c}(x)\|_{M^{0,1}_{k'^2c}}.$$

This gives the desired $\leq \max(k'^2, 2k_0H)$-exactness in degrees $\leq m$ for $c \geq c_0$.

Now we argue by induction on m. Consider the complex $N^{p,q}$ given by $M^{p,q+1}$ for $q \geq 1$ and $N^{p,0} = M^{p,1}/\overline{M^{p,0}}$ (the quotient by the closure of the image, which is also the completion of $M^{p,1}/M^{p,0}$), equipped with the quotient norm. Using the normed version of the snake lemma, Proposition 1.4.12, one checks that this satisfies the assumptions for $m - 1$, with k replaced by $\max(k^4, k^3 + k + 1)$. □

1.5. **Completions of locally constant functions.**

Definition 1.5.1. Let V be a semi-normed group, and X a compact topological space. We denote by $V(X)$ the normed abelian group of locally constant functions $X \to V$ with respect to the sup norm. With $\widehat{V}(X)$ we denote the completion of $V(X)$.

These constructions are functorial in bounded group homomorphisms $V \to V'$ and contravariantly functorial in continuous maps $f\colon X \to X'$.

Note in particular that $V(f)$ and $\widehat{V}(f)$ are norm-nonincreasing morphisms of semi-normed groups.

Lemma 1.5.2. *Let $r \in \mathbb{R}_{>0}$, and let V be an r-normed $\mathbb{Z}[T^{\pm 1}]$-module. Let X be a compact space. Then $\widehat{V}(X)$ is naturally an r-normed $\mathbb{Z}[T^{\pm 1}]$-module, with the action of T given by post-composition.*

Proof. Formalised, but omitted from this text. □

We continue to use the notation of before: let $r' > 0, c \geq 0$ be real numbers, and let M be a profinitely filtered pseudo-normed group with r'-action by T^{-1} (see Section 1.2).

Lemma 1.5.3. *Let f be a basic universal map from exponent m to n. We get an induced homomorphism of profinitely filtered pseudo-normed groups $M^m \to M^n$ bounded by the maximum (over all i) of $\sum_j |f_{ij}|$, where the f_{ij} are the coefficients of the $n \times m$-matrix representing f.*

This construction is functorial in f.

Proof. Omitted. □

Definition 1.5.4. Let f be a basic universal map from exponent m to n, and let (c_2, c_1) be f-suitable. We get an induced map

$$V(f)\colon V(M^n_{\leq c_1}) \to V(M^m_{\leq c_2})$$

induced by the morphism of profinitely filtered pseudo-normed groups $M^m \to M^n$.

This construction is functorial in f.

Definition 1.5.5. Let $f = \sum_g n_g g$ be a universal map from exponent m to n, and let (c_2, c_1) be f-suitable. We get an induced map

$$V(f)\colon V(M^n_{\leq c_1}) \to V(M^m_{\leq c_2})$$

that is the sum $\sum n_g V(g)$.

This construction is functorial in f.

Definition 1.5.6. Let f be a universal map from exponent m to n, and let (c_2, c_1) be f-suitable. We get an induced map

$$\widehat{V}(f)\colon \widehat{V}(M^n_{\leq c_1}) \to \widehat{V}(M^m_{\leq c_2})$$

that is the completion of $V(f)$.

This construction is functorial in f.

Let $r > 0$, and assume now that V is an r-normed $\mathbb{Z}[T^{\pm 1}]$-module. Assume $r' \leq 1$.

Definition 1.5.7. There are two natural actions of T^{-1} on $\widehat{V}(M_{\leq c})$. The first comes from the r'-action of T^{-1} on M which gives a continuous map

$$M_{\leq cr'} \to M_{\leq c}$$

and thus a normed group morphism $V(M_{\leq c}) \to V(M_{\leq cr'})$ which can be extended by completion to

$$(T^{-1})^*\colon \widehat{V}(M_{\leq c}) \to \widehat{V}(M_{\leq cr'}).$$

The other comes from Lemma 1.5.2, using the r-normed $\mathbb{Z}[T^{\pm 1}]$-module V. Again by extension to completion, we get a map

$$[T^{-1}]\colon \widehat{V}(M_{\leq c}) \to \widehat{V}(M_{\leq c}),$$

that we can compose with the map $\widehat{V}(M_{\leq c}) \to \widehat{V}(M_{\leq cr'})$, obtained from the natural inclusion $M_{\leq cr'} \to M_{\leq c}$. We thus end up with two maps

$$(T^{-1})^*, [T^{-1}]\colon \widehat{V}(M_{\leq c}) \to \widehat{V}(M_{\leq cr'}).$$

and we define $\widehat{V}(M_{\leq c})^{T^{-1}}$ to be the equalizer of $(T^{-1})^*$ and $[T^{-1}]$. In other words, the kernel of $(T^{-1})^* - [T^{-1}]$.

We will also need to understand the image of $(T^{-1})^* - [T^{-1}]$. The next lemma ensures it is surjective with controlled preimages, see Definition 1.4.1.

Lemma 1.5.8. *Let M be a profinitely filtered pseudo-normed group with action of T^{-1}. For any $r \in (0,1)$, any r-normed $\mathbb{Z}[T^{\pm 1}]$-module V, any $c > 0$ and any a, the map*

$$\widehat{V}(M^a_{\le c}) \xrightarrow{T^{-1}-[T^{-1}]^*} \widehat{V}(M^a_{\le r'c})$$

has norm bounded by $r^{-1}+1$ and is $\frac{r}{1-r}(1+\epsilon)$-surjective.

Proof. The norm bound is clear because $[T^{-1}]^*$ is norm non-increasing and T^{-1} scales norm by r^{-1}. Quantitative surjectivity will follow from Lemma 1.4.2 once we'll have proven that $T^{-1}-[T^{-1}]^* : \widehat{V}(M^a_{\le c}) \to \widehat{V}(M^a_{\le r'c})$ is $r/(1-r)$-surjective onto $V(M^a_{\le r'c})$.

We first note that any locally constant function $\varphi \in V(M^a_{\le r'c})$ can be extended to a locally constant function $\bar{\varphi} \in V(M^a_{\le c})$ with the same norm (recall f takes finitely many values and its norm is the maximum of norms of these values).

Let f be any element of $V(M^a_{\le r'c})$. We inductively define a sequence of locally constant functions $h_n \in V(M^a_{\le c})$ with $h_0 = T \circ \bar{f}$ and $h_{n+1} = T \circ \overline{[T^{-1}]^* h_n}$. Here we use the composition symbol to emphasize this is indeed the naive post-composition with T, there is no extra precomposition with a the inclusion map $\iota : M^a_{\le r'c} \hookrightarrow M^a_{\le c}$ as in the definition of T^{-1} seen as a map from $V(M^a_{\le c})$ to $V(M^a_{\le r'c})$.

Since $[T^{-1}]^*$ is norm non-increasing, extension is norm preserving and T scales norm by r, we get that $\|h_n\| \le r^{n+1}\|f\|$. We then set $g_n = \sum_{i=0}^n h_i$. The norm estimate on h_n ensures g is a Cauchy sequence in $V(M^a_{\le c})$ hence it converges to some g in $\widehat{V}(M^a_{\le c})$. We compute:

$$\begin{aligned}
(T^{-1}-[T^{-1}]^*)g_n &= \sum_{k=0}^{n}\Big(T^{-1}h_k - [T^{-1}]^* h_k\Big) \\
&= T^{-1}h_0 + \sum_{k=0}^{n-1}\Big(T^{-1}h_{k+1} - [T^{-1}]^* h_k\Big) - [T^{-1}]^* h_n \\
&= \bar{f}\circ\iota + \sum_{k=0}^{n-1}\Big(T^{-1}\circ T \circ \overline{[T^{-1}]^* h_k}\circ\iota - [T^{-1}]^* h_k\Big) - [T^{-1}]^* h_n \\
&= f - [T^{-1}]^* h_n
\end{aligned}$$

which converges to f hence $(T^{-1}-[T^{-1}]^*)g = f$. In addition $\|g\| \le \sum_n r^{n+1}\|f\| = r/(1-r)\|f\|$. ☐

Definition 1.5.9. Let f be a universal map from exponent m to n, and let (c_2, c_1) be f-suitable.

The natural map from Definition 1.5.6 restricts to a map

$$\widehat{V}(f)^{T^{-1}} : \widehat{V}(M^n_{\le c_1})^{T^{-1}} \to \widehat{V}(M^m_{\le c_2})^{T^{-1}}$$

Lemma 1.5.10. *Let $0 < r$ and $0 < r' \le 1$ be real numbers. Let f be a universal map from exponent m to n, and let (c_2, c_1) be very suitable for (f, r, r'). Then*

$$\widehat{V}(f)^{T^{-1}} : \widehat{V}(M^n_{\le c_1})^{T^{-1}} \to \widehat{V}(M^m_{\le c_2})^{T^{-1}}$$

is norm-nonincreasing.

Proof. Use the assumption that (c_2, c_1) is very suitable for (f, r, r') in order to find $N, b \in \mathbf{N}$ and $c' \in \mathbf{R}_{\ge 0}$ such that:

- f is bound by N (see Definition 1.1.3)

- (c_2, c') is f-suitable
- $r^b N \leq 1$
- $c' \leq (r')^b c_1$

Now, notice that the norm of $\widehat{V}(f)$ is at most N, and $\widehat{V}(f)$ can be factored as

$$\widehat{V}(M^n_{\leq c_1})^{T^{-1}} \xrightarrow{\text{res}} \widehat{V}(M^n_{\leq c'})^{T^{-1}} \xrightarrow{\widehat{V}(f)} \widehat{V}(M^m_{\leq c_2})^{T^{-1}}$$

Now use the defining property of the equalizer to conclude that the restriction map has norm less than $1/N$, and therefore the composition is norm-nonincreasing. $\square$

Definition 1.5.11. Let $0 < r$ and $0 < r' \leq 1$ be real numbers, and let V be an r-normed $\mathbb{Z}[T^{\pm 1}]$-module. Let $\mathsf{BD} = (n, f)$ be Breen–Deligne data, and let $\kappa = (\kappa_0, \kappa_1, \kappa_2, \dots)$ be a sequence of constants in $\mathbb{R}_{\geq 0}$ that is very suitable for (BD, r, r'). Let M be a profinitely filtered pseudo-normed group with r'-action of T^{-1}.

For every $c \in \mathbb{R}_{\geq 0}$, the maps from Definition 1.5.9 induced by the universal maps f_i from the Breen–Deligne $\mathsf{BD} = (n, f)$ assemble into a complex of normed abelian groups

$$C^{\mathrm{BD}}_{\kappa}(M)^{\bullet}_{c} \colon 0 \to \dots \to \widehat{V}(M^{n_i}_{\leq \kappa_i})^{T^{-1}} \to \widehat{V}(M^{n_{i+1}}_{\leq \kappa_{i+1}})^{T^{-1}} \to \dots .$$

Together, these complexes fit into a system of complexes with the natural restriction maps.

By Lemma 1.5.10 the differentials are norm-nonincreasing. It is clear that the restriction maps are also norm-nonincreasing, and therefore the system is admissible.

1.6. **Polyhedral lattices.**

Definition 1.6.1. A *polyhedral lattice* is a finite free abelian group Λ equipped with a norm $\|\cdot\|_\Lambda \colon \Lambda \otimes \mathbb{R} \to \mathbb{R}$ such that there exists a finite set $\{\lambda_1, \dots, \lambda_n\} \subset \Lambda$ that generate the norm: that is to say, for every $\lambda \in \Lambda$ there exist $c_1, \dots, c_n \in \mathbb{Q}$ such that $\lambda = \sum c_i \lambda_i$ and $\|\lambda\| = c_i \|\lambda_i\|$.

Equivalently (but not verified in Lean): the norm is given by the supremum of finitely many linear functions on Λ; or once more, equivalently, the "unit ball" $\{\lambda \in \Lambda \otimes \mathbb{R} \mid \|\lambda\|_\Lambda \leq 1\}$ is a polyhedron.

Finally, we can prove the key combinatorial lemma, ensuring that any element of $\mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))$ can be decomposed into N elements whose norm is roughly $\frac{1}{N}$ of the original element.

Definition 1.6.2. Let M be a pseudo-normed group, $N \in \mathbb{N}$, and $d \in \mathbb{R}_{\geq 0}$. We say that M is *N-splittable* with error term d, if for all c and $x \in M_c$, there exists a decomposition

$$x = x_1 + x_2 + \cdots + x_N,$$

with $x_i \in M_{c/N+d}$.

Proposition 1.6.3. *Let Λ be a polyhedral lattice, and S a profinite set. Then for all positive integers N there is a constant d such that for all $c > 0$ one can write any $x \in \mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c}$ as*

$$x = x_1 + \dots + x_N$$

where all $x_i \in \mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c/N+d}$.

In other words, for all N, there exists a d such that $\mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))$ is N-splittable with error term d.

Proof. The desired statement is equivalent to the surjectivity of the map of profinite sets

$$\mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))^N_{\leq c/N+d} \times_{\mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c+Nd}} \mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c} \to \mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c}.$$

Note that, as a functor of S, both sides commute with cofiltered limits, so it is enough to handle finite S, by Tychonoff. But that is exactly the following Lemma 1.6.4. □

Lemma 1.6.4. *Let Λ be a polyhedral lattice, and S a finite set. Then for all positive integers N there is a constant d such that for all $c > 0$ one can write any $x \in \mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c}$ as*

$$x = x_1 + \ldots + x_N$$

where all $x_i \in \mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c/N+d}$.

In other words, for all N, there exists a d such that $\mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))$ is N-splittable with error term d.

As preparation for the proof, we have the following results.

Lemma 1.6.5 (Gordan's lemma)**.** *Let Λ be a finite free abelian group, and let $\lambda_1, \ldots, \lambda_m \in \Lambda$ be elements. Let $M \subset \mathrm{Hom}(\Lambda, \mathbb{Z})$ be the submonoid $\{x \mid x(\lambda_i) \geq 0$ for all $i = 1, \ldots, m\}$. Then M is finitely generated as monoid.*

Proof. This is a standard result. We omit the proof here. It is done in Lean. □

Lemma 1.6.6. *Let Λ be a finite free abelian group, let N be a positive integer, and let $\lambda_1, \ldots, \lambda_m \in \Lambda$ be elements. Then there is a finite subset $A \subset \Lambda^\vee$ such that for all $x \in \Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z})$ there is some $x' \in A$ such that $x - x' \in N\Lambda^\vee$ and for all $i = 1, \ldots, m$, the numbers $x'(\lambda_i)$ and $(x - x')(\lambda_i)$ have the same sign, i.e. are both nonnegative or both nonpositive.*

Proof. It suffices to prove the statement for all x such that $\lambda_i(x) \geq 0$ for all i; indeed, applying this variant to all $\pm\lambda_i$, one gets the full statement.

Thus, consider the submonoid $\Lambda^\vee_+ \subset \Lambda^\vee$ of all x that pair nonnegatively with all λ_i. This is a finitely generated monoid by Lemma 1.6.5; let $y_1, \ldots, y_M$ be a set of generators. Then we can take for A all sums $n_1 y_1 + \ldots + n_M y_M$ where all $n_j \in \{0, \ldots, N-1\}$. □

Lemma 1.6.7. *Let $x_0, x_1, \ldots$ be a sequence of reals, and assume that $\sum_{i=0}^\infty x_i$ converges absolutely. For every natural number $N > 0$, there exists a partition $\mathbb{N} = A_1 \sqcup A_2 \sqcup \cdots \sqcup A_N$ such that for each $j = 1, \ldots, N$ we have $\sum_{i \in A_j} x_i \leq (\sum_{i=0}^\infty x_i)/N + 1$*

Proof. Define the A_j recursively: assume that the natural numbers $0, \ldots, n$ have been placed into the sets $A_1, \ldots, A_N$. Then add the number $n+1$ to the set A_j for which

$$\sum_{i=0, i \in A_j}^{n} x_i$$

is minimal. □

Lemma 1.6.8. *For all natural numbers $N > 0$, and for all $x \in \overline{\mathcal{L}}_{r'}(S)_{\leq c}$ one can decompose x as a sum*

$$x = x_1 + \ldots + x_N$$

with all $x_i \in \overline{\mathcal{L}}_{r'}(S)_{\leq c/N+1}$.

Proof. Choose a bijection $S \times \mathbb{N} \cong \mathbb{N}$, and transport the result from Lemma 1.6.7. □

Proof of Lemma 1.6.4. Pick $\lambda_1, \dots, \lambda_m \in \Lambda$ generating the norm. We fix a finite subset $A \subset \Lambda^\vee$ satisfying the conclusion of the previous lemma. Write

$$x = \sum_{n \geq 1, s \in S} x_{n,s} T^n[s]$$

with $x_{n,s} \in \Lambda^\vee$. Then we can decompose

$$x_{n,s} = N x^0_{n,s} + x^1_{n,s}$$

where $x^1_{n,s} \in A$ and we have the same-sign property of the last lemma. Letting $x^0 = \sum_{n \geq 1, s \in S} x^0_{n,s} T^n[s]$, we get a decomposition

$$x = N x^0 + \sum_{a \in A} a x_a$$

with $x_a \in \overline{\mathcal{L}}_{r'}(S)$ (with the property that in the basis given by the $T^n[s]$, all coefficients are 0 or 1). Crucially, we know that for all $i = 1, \dots, m$, we have

$$\|x(\lambda_i)\| = N \|x^0(\lambda_i)\| + \sum_{a \in A} |a(\lambda_i)| \|x_a\|$$

by using the same sign property of the decomposition.

Using this decomposition of x, we decompose each term into N summands. This is trivial for the first term $N x^0$, and each summand of the second term decomposes with $d = 1$ by Lemma 1.6.8. (It follows that in general one can take for d the supremum over all i of $\sum_{a \in A} |a(\lambda_i)|$.) □

Definition 1.6.9. Let Λ be a polyhedral lattice, and let $N > 0$ be a natural number. (We think of N as being fixed once and for all, and thus it does not show up in the notation below.)

By Λ' we denote Λ^N endowed with the norm

$$\|(\lambda_1, \dots, \lambda_N)\|_{\Lambda'} = \tfrac{1}{N}(\|\lambda_1\|_\Lambda + \dots + \|\lambda_N\|_\Lambda).$$

This is a polyhedral lattice.

Lemma 1.6.10. *For any $m \geq 1$, let $\Lambda'^{(m)}$ be given by $\Lambda'^m / \Lambda \otimes (\mathbb{Z}^m)_{\sum = 0}$; for $m = 0$, we set $\Lambda'^{(0)} = \Lambda$. Then $\Lambda'^{(m)}$ is a polyhedral lattice.*

Proof. The proof is done in Lean. TODO: write down a proof here. □

Definition 1.6.11. For any $m \geq 1$, let $\Lambda'^{(m)}$ be given by $\Lambda'^m / \Lambda \otimes (\mathbb{Z}^m)_{\sum = 0}$; for $m = 0$, we set $\Lambda'^{(0)} = \Lambda$. Then $\Lambda'^{(\bullet)}$ is a cosimplicial polyhedral lattice, the Čech conerve of $\Lambda \to \Lambda'$.

In particular, $\Lambda'^{(0)} = \Lambda \to \Lambda' = \Lambda'^{(1)}$ is the diagonal embedding.

Definition 1.6.12. Let Λ be a polyhedral lattice, and M a profinitely filtered pseudo-normed group.

Endow $\mathrm{Hom}(\Lambda, M)$ with the subspaces

$$\mathrm{Hom}(\Lambda, M)_{\leq c} = \{f \colon \Lambda \to M \mid \forall x \in \Lambda, f(x) \in M_{\leq c\|x\|}\}.$$

As Λ is polyhedral, it is enough to check the given condition on f for a finite collection of x that generate the norm.

These subspaces are profinite subspaces of M^Λ, and thus they make $\mathrm{Hom}(\Lambda, M)$ ito a profinitely filtered pseudo-normed group.

If M has an action of T^{-1}, then so does $\mathrm{Hom}(\Lambda, M)$.

1.7. **Key technical result.** Now we state the following result, which is the key technical result on our to the main goal.

Theorem 1.7.1. *Let* $\mathsf{BD} = (n, f, h)$ *be a Breen–Deligne package, and let* $\kappa = (\kappa_0, \kappa_1, \kappa_2, \dots)$ *be a sequence of constants in* $\mathbb{R}_{\geq 0}$ *that is* BD*-suitable. Fix radii* $1 > r' > r > 0$*. For any* m *there is some* k *and* c_0 *such that for all profinite sets* S *and all* r*-normed* $\mathbb{Z}[T^{\pm 1}]$*-modules* V*, the system of complexes*

$$C_\kappa^{\mathsf{BD}}(\overline{\mathcal{L}}_{r'}(S))_c^\bullet \colon \widehat{V}(\overline{\mathcal{L}}_{r'}(S)_{\leq c})^{T^{-1}} \to \widehat{V}(\overline{\mathcal{L}}_{r'}(S)^2_{\leq \kappa_1 c})^{T^{-1}} \to \dots$$

is $\leq k$*-exact in degrees* $\leq m$ *for* $c \geq c_0$*.*

We will prove Theorem 1.7.1 by induction on m. Unfortunately, the induction requires us to prove a stronger statement.

Theorem 1.7.2. *Fix radii* $1 > r' > r > 0$*. For any* m *there is some* k *such that for all polyhedral lattices* Λ *there is a constant* $c_0(\Lambda) > 0$ *such that for all profinite sets* S *and all* r*-normed* $\mathbb{Z}[T^{\pm 1}]$*-modules* V*, the system of complexes*

$$C_{\Lambda,c}^\bullet \colon \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))^2_{\leq \kappa_1 c})^{T^{-1}} \to \dots$$

is $\leq k$*-exact in degrees* $\leq m$ *for* $c \geq c_0(\Lambda)$*.*

Proof of Theorem 1.7.1. Use $\Lambda = \mathbb{Z}$, and the isomorphism $\mathrm{Hom}(\mathbb{Z}, A) \cong A$. □

A word on universal constants: We fix once and for all, the constants $0 < r < r' \leq 1$ a Breen–Deligne package BD, and a sequence of positive constants κ that is very suitable for (BD, r, r'). Once the full proof is formalized, we can come back to this place and write a bit more about the other constants.

The global strategy of the proof is to construct a system of double complexes such that its first row is the system $C^\bullet_{\Lambda,\bullet}$ occuring in Theorem 1.7.2. We can then verify the conditions to Proposition 1.4.15 and conclude from there. For the time being, we will let M denote an arbitrary profinitely filtered pseudo-normed group with action of T^{-1}, and whenever needed we can specialize to $M = \overline{\mathcal{L}}_{r'}(S)$.

Further choices of constants: We will argue by induction on m, so assume the result for $m-1$ (this is no assumption for $m = 0$, so we do not need an induction start). This gives us some $k > 1$ for which the statement of Theorem 1.7.2 holds true for $m-1$; if $m = 0$, simply take any $k > 1$. In the proof below, we will increase k further in a way that depends only on m and r. After this modified choice of k, we fix ϵ and k_0 as provided by Proposition 1.4.15. Fix a sequence $(\kappa_i')_i$ of nonnegative reals that is adept to (BD, κ). (Such a sequence exists by Lemma 1.1.20.) Moreover, we let k' be the supremum of k_0 and the c_i' for $i = 0, \dots, m+1$. Finally, choose a positive integer b so that $2k'(\frac{r}{r'})^b \leq \epsilon$, and let N be the minimal power of 2 that satisfies

$$k'/N \leq (r')^b.$$

Then in particular $r^b N \leq 2k'(\frac{r}{r'})^b \leq \epsilon$.

Definition 1.7.3. Let $\Lambda^{(\bullet)}$ be the cosimplicial polyhedral lattice of Definition 1.6.11, and recall from 1.6.12 that $\mathrm{Hom}(\Lambda^{(m)}, M)$ is a profinitely filtered pseudo-normed group with action of T^{-1}.

Hence $\mathrm{Hom}(\Lambda^{(\bullet)}, M)$ is a simplicial profinitely filtered pseudo-normed group with action of T^{-1}.

Now apply the construction of the system of complexes from Definition 1.5.11 to obtain a cosimplicial system of complexes

$$C_\kappa^{\mathsf{BD}}(\mathrm{Hom}(\Lambda^{(\bullet)}, M))_\bullet^\bullet.$$

Now take the alternating face map cochain complex to obtain a system of double complexes, whose objects are

$$\widehat{V}(\mathrm{Hom}(\Lambda^{(m)}, M)^{n_i}_{\leq \kappa_i c})^{T^{-1}}.$$

As final step, rescale the norm on the object in row m by $m!$, so that all columns become admissible: the vertical differential from row m to row $m+1$ is an alternating sum of $m+1$ maps that are all norm-nonincreasing.

Lemma 1.7.4. *In particular, for any $c > 0$, we have*

$$\mathrm{Hom}(\Lambda', M)_{\leq c} = \mathrm{Hom}(\Lambda, M)^N_{\leq c/N},$$

with the map to $\mathrm{Hom}(\Lambda, M)_{\leq c}$ given by the sum map.

Proof. Omitted (but done in Lean). □

Lemma 1.7.5. *Similarly, for any $c > 0$, we have*

$$\mathrm{Hom}(\Lambda'^{(m)}, M)_{\leq c} = \mathrm{Hom}(\Lambda', M)^{m/\mathrm{Hom}(\Lambda,M)_{\leq c}}_{\leq c},$$

the m-fold fibre product of $\mathrm{Hom}(\Lambda', M)_{\leq c}$ over $\mathrm{Hom}(\Lambda, M)_{\leq c}$.

Proof. Omitted (but done in Lean). □

Lemma 1.7.6. *There is a canonical isomorphism between the first row of the double complex*

$$C_\kappa^{\mathsf{BD}}(\mathrm{Hom}(\Lambda^{(1)}, M))^\bullet$$

and

$$C_{\kappa/N}^{N\otimes\mathsf{BD}}(\mathrm{Hom}(\Lambda, M))^\bullet$$

which identifies the map induced by the diagonal embedding $\Lambda \to \Lambda' = \Lambda^{(1)}$ with the map induced by $\sigma^N : N \otimes \mathsf{BD} \to \mathsf{BD}$.

Proof. Omitted (but done in Lean). □

Proposition 1.7.7. *Let $\pi : X \to B$ be a surjective morphism of profinite sets, and let $S_\bullet \to S_{-1}$, $S_{-1} := B$, be its augmented Čech nerve. Let V be a semi-normed group. Then the complex*

$$0 \to \widehat{V}(S_{-1}) \to \widehat{V}(S_0) \to \widehat{V}(S_1) \to \cdots$$

is exact. Furthermore, for all $\epsilon > 0$ and $f \in \ker(\widehat{V}(S_m) \to \widehat{V}(S_{m+1}))$, there exists some $g \in \widehat{V}(S_{m-1})$ such that $d(g) = f$ and $\|g\| \leq (1+\epsilon)\cdot\|f\|$. In other words, the complex is normed exact in the sense of Definition 1.4.6.

Proof. We argue similarly to [Sch19, Theorem 3.3], as follows. By applying Lemma 1.4.3, we first reduce to a statement which does not involve ϵ or completions. Explicitly, we must show that

$$0 \to V(S_1) \to V(S_0) \to V(S_1) \to \cdots$$

is exact, and that whenever $f \in \ker(V(S_m) \to V(S_{m+1}))$, there exists $g \in V(S_{m-1})$ such that $\|g\| \leq \|f\|$ and $d(g) = f$. The map $V(S_{-1}) \to V(S_0)$ is the one induced by $S_0 \to S_{-1}$ which agrees with $X \to B$. Since $X \to B$ is surjective, we easily see that $V(S_{-1}) \to V(S_0)$ is injective.

If X and B are finite, then the remaining assertions follow from the existence of a splitting $\sigma : B \to X$ of $\pi : X \to B$, as follows. The map σ provides maps $S_m \to S_{m+1}$ for all $m \geq -1$, defined explicitly as

$$(a_0, \dots, a_m) \mapsto (\sigma(\pi(a_0)), a_0, \dots, a_m)$$

if $m \geq 0$ and simply as σ if $m = -1$. Here, for $m \geq 0$, we have identified S_m with the $m+1$-fold fibered product $X \times_B \cdots \times_B X$. Applying $V(-)$, these maps induce $h_m : V(S_{m+1}) \to V(S_m)$, which form a contracting homotopy for the complex in question, and which are norm nonincreasing by the definition of $V(-)$. If $f \in \ker(V_m \to V_{m+1})$ is as above, then $g := h_m(f)$ satisfies $d(g) = f$ and $\|g\| \leq \|f\|$, as required.

In the general case, write $X = \varprojlim_i X_i$ where X_i vary over the discrete (hence finite) quotients of X. Since $X \to B$ is surjective, for each i there exists a unique maximal discrete quotient B_i of B such that $X \to B$ descends to $X_i \to B_i$. The maps $X_i \to B_i$ are again surjective, and one has

$$(X \to B) = \varprojlim_i (X_i \to B_i).$$

Let $S_{i,\bullet} \to S_{i,-1}$, $S_{i,-1} := B_i$, denote the augmented Čech nerve of $X_i \to B_i$.

The terms in the Čech nerve are themselves limits, hence we have $S_m = \varprojlim_i S_{i,m}$, with each $S_{i,m}$ finite. The functor $V(-)$, when considered as taking values in abelian groups, sends cofiltered limits to filtered colimits. Also, if $f \in V(S_m)$ is the pullback of $f_i \in V(S_{i,m})$, then for a sufficiently small index $j \leq i$, the image of $f : S_m \to V$ agrees with the image of $f_j : S_{j,m} \to V$, where f_j is the image of f_i under the map $V(S_{i,m}) \to V(S_{j,m})$ induced by the transition map $S_{j,m} \to S_{i,m}$.

Now suppose that $f \in \ker(V(S_m) \to V(S_{m+1}))$ is given. By the discussion above, there exists some i and some $f_i \in V(S_{i,m})$ such that f is the pullback of f_i with respect to the morphism $S_m \to S_{i,m}$ and such that the following additional conditions hold:

(1) One has $\|f_i\| = \|f\|$.
(2) One has $f_i \in \ker(V(S_{i,m}) \to V(S_{i,m+1}))$.

Let $h_m : V(S_{i,m}) \to V(S_{i,m-1})$ be the map constructed in the argument for the finite case $X_i \to B_i$. Put $g_i := h_m(f_i)$ and g the image of g_i in $V(S_{m-1})$. Since the maps $V(S_{i,\bullet}) \to V(S_\bullet)$ commute with the differentials, we have $d(g) = f$. Finally, the map $V(S_{i,m-1}) \to V(S_{m-1})$ is norm nonincreasing as it is induced from $S_{m-1} \to S_{i,m-1}$, so that

$$\|g\| \leq \|g_i\| \leq \|f_i\| = \|f\|,$$

as contended. □

Lemma 1.7.8. *Let M be a profinitely filtered pseudo-normed group with T^{-1}-action that is N-splittable with error term $d \geq 0$. Let $k \geq 1$ be a real number, and let $c_0 > 0$ satisfy $d \leq \frac{(k-1)c_0}{N}$. For every c, consider the Cech nerve of the summation map $M_{c/N}^N \to M_c$. By applying the functor $\widehat{V}(\_)$ and taking the alternating face map complex, we obtain a system of complexes*

$$\widehat{V}(M_{\leq c}) \to \widehat{V}(M_{\leq c/N}^N) \to \dots$$

This system of complexes is weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound 1.

Proof. For every constant c, consider the pullback

$$\begin{tikzcd}
 & M_c \arrow[r] & M_{kc} \\
 & X_c \arrow[u] \arrow[r] & M^N_{kc/N} \arrow[u] \\
M^N_{c/N} \arrow[uur] \arrow[ur, dotted] \arrow[urr] & &
\end{tikzcd}$$

We therefore get morphisms of cochain complexes

$$\begin{tikzcd}
\widehat{V}(M_{kc}) \arrow[r] \arrow[d] & \widehat{V}(M_c) \arrow[r] \arrow[d] & \widehat{V}(M_c) \arrow[d] \\
\widehat{V}(M^N_{kc/N}) \arrow[r] \arrow[d] & \widehat{V}(X_c) \arrow[r] \arrow[d] & \widehat{V}(M^N_{c/N}) \arrow[d] \\
\vdots \arrow[r] & \vdots \arrow[r] & \vdots
\end{tikzcd}$$

where all the columns are of the form "alternating face map complex of $\widehat{V}(\_)$ applied to a Cech nerve". Note that the horizontal maps are norm-nonincreasing and their compositions are restriction maps.

Claim: for $c \geq c_0$, the map $X_c \to M_c$ is surjective.

Indeed, by assumption every $x \in M_c$ can be decomposed into a sum $x = x_1 + \cdots + x_N$ with $x_i \in M_{c/N+d} \subset M_{kc/N}$, since $c \geq c_0$ and $d \leq \frac{(k-1)c_0}{N}$.

By Proposition 1.7.7, the middle column is normed exact (in the sence of Definition 1.4.6). The result follows from Lemma 1.4.10. $\square$

Proposition 1.7.9. *Let d be the constant from Proposition 1.6.3. Let $k > 1$ and $c_0 > 0$ be real numbers such that*

$$(k-1) * c_0/N \geq d.$$

Let m be any natural number, and put

$$K = (m+2) + \frac{r+1}{r(1-r)}(m+2)^2$$

Finally, let c_0' be $\frac{c_0}{r' \cdot n_i}$, where n_i is the i-th index in our fixed Breen–Deligne data.

Then i-th column in the double complex is (k^2, K)-weak bounded exact in degrees $\leq m$ for $c \geq c_0'$.

Proof. Let $M^{(m)}$ denote $\mathrm{Hom}(\Lambda^{(m)}, \overline{\mathcal{L}}_{r'}(S))^{n_i}$. We also write M for $M^{(0)} = \mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))^{n_i}$ and M' for $M^{(1)}$. By Proposition 1.6.4, M is N-splittable with error term d.

Consider the diagram of morphisms of systems of complexes

$$\begin{array}{ccccc}
\widehat{V}(M_c)^{T^{-1}} & \longrightarrow & \widehat{V}(M_c) & \xrightarrow{T^{-1}-[T^{-1}]^*} & \widehat{V}(M_c) \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{V}(M'_c)^{T^{-1}} & \longrightarrow & \widehat{V}(M'_c) & \xrightarrow{T^{-1}-[T^{-1}]^*} & \widehat{V}(M'_c) \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{V}(M_c^{(m)})^{T^{-1}} & \longrightarrow & \widehat{V}(M_c^{(m)}) & \xrightarrow{T^{-1}-[T^{-1}]^*} & \widehat{V}(M_c^{(m)})
\end{array}$$

By Lemmas 1.7.8 and 1.7.5, we know that the two columns on the right are weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound 1.

The result now follows from Lemma 1.5.8, and Proposition 1.4.13. □

Proposition 1.7.10. *Let h be the homotopy packaged with BD, and let h^N denote the n-th iterated composition of h (see Def 1.1.11) which is a homotopy between π^N and $\sigma^N : N \otimes \mathsf{BD} \to \mathsf{BD}$.*

Let $H \in \mathbf{R}_{\geq 0}$ be such that for $i = 0, \dots, m$ the universal map h_i^N is bound by H (see Def 1.1.3).

Then the double complex satisfies the normed homotopy homotopy condition (Def 1.4.14) for m, H, and c_0.

Proof. By Lemma 1.7.6 we may replace the first row by

$$C_{\kappa/N}^{N\otimes\mathsf{BD}}(\mathrm{Hom}(\Lambda, M))^{\bullet}.$$

Now it is important to recall that we have chosen $k' \geq \kappa'_i$ for all $i = 0, \dots, m+1$.

Our goal is to find, in degrees $\leq m$, a homotopy between the two maps from the first row

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)_{\leq c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^2_{\leq \kappa_1 c})^{T^{-1}} \to \dots$$

to the second row

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)^N_{\leq c/N})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^{2N}_{\leq \kappa_1 c/N})^{T^{-1}} \to \dots$$

respectively induced by σ^N and π^N (which are maps $N \otimes \mathsf{BD}$

By Definition 1.1.11 and Lemma 1.1.18 we can find this homotopy between the complex for $k'c$ and the complex for c. (Here we use $k' \geq c'_i$ for $i = 0, \dots, m$.) By assumption, the norm of these maps is bounded by H.

Finally, it remains to establish the estimate (eq. 1.4.1) on the homotopic map. We note that this takes $x \in \widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq k'^2 \kappa_i c})^{T^{-1}}$ (with $i = q$ in the notation of (eq. 1.4.1)) to the element

$$y \in \widehat{V}(\mathrm{Hom}(\Lambda, M)^{Na_i}_{\leq k' \kappa_i c/N})^{T^{-1}}$$

that is the sum of the N pullbacks along the N projection maps $\mathrm{Hom}(\Lambda, M)^{Na_i}_{\leq k' \kappa_i c/N} \to \mathrm{Hom}(\Lambda, M)^{a_i}_{\leq k'^2 \kappa_i c}$. We note that these actually take image in $\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq \kappa_i c}$ as $N \geq k'$, so this actually gives a well-defined map

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq \kappa_i c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^{Na_i}_{\leq k' \kappa_i c/N})^{T^{-1}}.$$

We need to see that this map is of norm $\leq \epsilon$. Now note that by our choice of N, we actually have $k'\kappa_i c/N \leq (r')^b \kappa_i c$, so this can be written as the composite of the restriction map

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq \kappa_i c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq (r')^b \kappa_i c})^{T^{-1}}$$

and

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq (r')^b \kappa_i c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^{N a_i}_{\leq k'\kappa_i c/N})^{T^{-1}}.$$

The first map has norm exactly r^b, by T^{-1}-invariance, and as multiplication by T scales the norm with a factor of r on $\widehat{V}$. (Here is where we use $r' > r$, ensuring different scaling behaviour of the norm on source and target.) The second map has norm at most N (as it is a sum of N maps of norm ≤ 1). Thus, the total map has norm $\leq r^b N$. But by our choice of N, we have $r^b N \leq \epsilon$, giving the result. □

Proof of Theorem 1.7.2. By induction, the first condition of Proposition 1.4.15 is satisfied for all $c \geq c_0$ with c_0 large enough (depending on Λ but not V or S).

The second condition is Proposition 1.7.9, and the third condition has been checked in Proposition 1.7.10.

Thus, we can apply Proposition 1.4.15, and get the desired $\leq \max(k'^2, 2k_0 H)$-exactness in degrees $\leq m$ for $c \geq c_0$, where k', k_0 and H were defined only in terms of k, m, r' and r, while c_0 depends on Λ (but not on V or S). This proves the inductive step. □

Question 1.7.11. Can one make the constants explicit, and how large are they? [1] Modulo the Breen-Deligne resolution, all the arguments give in principle explicit constants; and actually the proof of the existence of the Breen-Deligne resolution should be explicit enough to ensure the existence of bounds on the c_i and c_i'.

Answer: yes, we can do this. And we should write up a clean account asap, when we have cleaned up the proof in Lean.

2. Second part

2.1. **Variants of normed groups.** This subsection continues some of the theory of (pseudo)-normed groups, started in Subsection 1.2.

Definition 2.1.1. A p-Banach space V is a complete topological $\mathbb{R}$-vector space whose topology is induced by a p-norm; that is, a norm satisfying $\|rv\| = |r|^p \|v\|$.

Lemma 2.1.2. *A p-Banach V is (up to non-canonical choice) an r-Banach $\mathbb{Z}[T^{\pm 1}]$-module, where $r = 2^{-p}$. (See Definition 1.2.1 for the definition of r-normed $\mathbb{Z}[T^{\pm 1}]$-modules.)*

Proof. Obvious. □

Definition 2.1.3. We will consider the following categories:

- CHPNG the category of CH-filtered pseudo-normed groups with bounded morphisms.
- CHPNG_1 the category of exhaustive CH-filtered pseudo-normed groups with strict morphisms.
- ProfinPNG the category of profinitely filtered pseudo-normed groups with bounded morphisms.

[1] A back of the envelope calculation seems to suggest that k is roughly doubly exponential in m, and that N has to be taken of roughly the same magnitude.

- $\mathrm{ProfinPNG}_1$ the category of exhaustive profinitely filtered pseudo-normed groups with strict morphisms.
- $\mathrm{ProfinPNGTinv}_1^{r'}$ the category of exhaustive profinitely filtered pseudo-normed groups with r'-action of T^{-1} and strict morphisms.

Proposition 2.1.4. *Consider an inverse system $(X_i)_i$ of compact-Hausdorffly-filtered-pseudonormed abelian groups where all transition maps $X_i \to X_j$ send $X_{i,\leq c}$ to $X_{j,\leq c}$. Then*

$$X_{\leq c} := \varprojlim_i X_{i,\leq c}$$

is compact Hausdorff, and

$$X = \bigcup_c X_{\leq c}$$

is naturally a compact-Hausdorffly-filtered pseudonormed abelian group which is the limit of $(X_i)_i$ in the strict category structure.

Proof. One can define negation and addition on X as continuous maps $- : X_{\leq c} \to X_{\leq c}$ and $+ : X_{\leq c} \times X_{\leq c'} \to X_{\leq c+c'}$, and these pass to the unions. It should then be straightforward to check the axioms. □

2.2. **Laurent Measures.**

Definition 2.2.1. Let $0 < r' < 1$ be a real number, and let S be a profinite set. Then $\mathcal{L}_{r'}(S)$ denotes TODO

For every $\alpha \in \mathbf{R}$ let $\lfloor \alpha \rfloor \in \mathbf{Z}$ be the largest integer that satisfies $\lfloor \alpha \rfloor \leq \alpha$: in particular,

$$\alpha - 1 < \lfloor \alpha \rfloor \leq \alpha \qquad \text{and} \qquad \alpha \geq 0 \Longrightarrow \lfloor \alpha \rfloor \geq 0.$$

Now fix $0 < \xi < 1$ and let $x \in \mathbf{R}_{\geq 0}$.

Definition 2.2.2. For all $n \in \mathbf{N}$, set

$$y_n(x) = y_n = \begin{cases} x & \text{if } n = 0 \\ y_{n-1} - \left\lfloor \frac{y_{n-1}}{\xi^{n-1}} \right\rfloor \xi^{n-1} & \text{if } n \geq 1 \end{cases}$$

Lemma 2.2.3. *For all $n \in \mathbf{N}$, we have*

$$y_{n+1} = x - \sum_{i=0}^{n} \left\lfloor \frac{y_i}{\xi^i} \right\rfloor \xi^i.$$

Proof. By induction: when $n = 0$ this boils down to

$$y_1 = x - \left\lfloor \frac{y_0}{\xi^0} \right\rfloor \xi^0$$

which is the definition of y_1 since $x = y_0$.

The step $n \to (n+1)$ goes as follows:

$$y_{n+1} \overset{\text{def}}{=} y_n - \left\lfloor \frac{y_n}{\xi^n} \right\rfloor \xi^n \overset{\text{Ind.}}{=} \left(x - \sum_{i=0}^{n-1} \left\lfloor \frac{y_i}{\xi^i} \right\rfloor \xi^i \right) - \left\lfloor \frac{y_n}{\xi^n} \right\rfloor \xi^n = x - \sum_{i=0}^{n} \left\lfloor \frac{y_i}{\xi^i} \right\rfloor \xi^i. \qquad \square$$

Lemma 2.2.4. *The sequence*

$$n \mapsto \left\lfloor \frac{y_n}{\xi^n} \right\rfloor$$

is bounded.

Proof. By definition, for all $n \geq 1$,

$$\frac{y_n}{\xi^n} = \frac{1}{\xi^n}\left(y_{n-1} - \left\lfloor \frac{y_{n-1}}{\xi^{n-1}} \right\rfloor \xi^{n-1}\right) = \frac{1}{\xi}\left(\frac{y_{n-1}}{\xi^{n-1}} - \left\lfloor \frac{y_{n-1}}{\xi^{n-1}} \right\rfloor\right) \leq \frac{1}{\xi} \cdot 1 = \frac{1}{\xi}$$

Thus, the sequence $n \mapsto \frac{y_n}{\xi^n}$ is bounded by $\max\{x = y_0, \xi^{-1}\}$ and the same holds, *a fortiori*, for the sequence $n \mapsto \lfloor \frac{y_n}{\xi^n} \rfloor$. □

Lemma 2.2.5. *We have*

$$\lim_{n \to +\infty} y_n = 0.$$

Proof. Indeed, for every $n \in \mathbf{N}$,

$$\frac{y_n}{\xi^n} - 1 < \left\lfloor \frac{y_n}{\xi^n} \right\rfloor \leq \frac{y_n}{\xi^n}$$

so, multiplying everything by $\xi^n > 0$,

$$y_n - \xi^n < \left\lfloor \frac{y_n}{\xi^n} \right\rfloor \xi^n \leq y_n.$$

Taking limits,

$$\lim_{n \to +\infty} y_n - \lim_{n \to +\infty} \xi^n = \lim_{n \to +\infty} y_n \leq \lim_{n \to +\infty} \left\lfloor \frac{y_n}{\xi^n} \right\rfloor \xi^n = 0 \leq \lim_{n \to +\infty} y_n.$$

By Lemma 2.2.4, the term $\left\lfloor \frac{y_n}{\xi^n} \right\rfloor \xi^n$ tends to 0 as $n \to +\infty$ because $0 < |\xi| < 1$. The lemma follows. □

Proposition 2.2.6. (1) *For all* $0 < r < 1$,

$$\sum_{i=0}^{+\infty} \left\lfloor \frac{y_i}{\xi^i} \right\rfloor r^i \qquad \textit{converges.}$$

(2)

$$\sum_{i=0}^{+\infty} \left\lfloor \frac{y_i}{\xi^i} \right\rfloor \xi^i = x;$$

Proof. The first point follows from Lemma 2.2.4, using that $0 < r < 1$.

For the second point, taking limits in Lemma 2.2.3, we obtain

$$\lim_{n \to +\infty} y_{n+1} = x - \sum_{i=0}^{+\infty} a_i \xi^i.$$

The result follows from Lemma 2.2.5. □

The next theorem is the first statement of Theorem 6.9 in [Sch20].

Definition 2.2.7. Fix $0 < \xi < 1$, and let $\theta_\xi \colon \mathbf{Z}((T))_r \to \mathbf{R}$ be the evaluation map

$$\sum a_n T^n \longmapsto \sum a_n(\xi) T^n.$$

Theorem 2.2.8. *The map θ_ξ is surjective.*

Proof. Pick $x \in \mathbf{R}$, and consider the power series

$$F(T) = \sum_{n \geq 0} \left\lfloor \frac{y_n(x)}{\xi^m} \right\rfloor T^n \in \mathbf{Z}[\![T]\!]$$

where $\{y_n(x)\}$ is the sequence defined in Definition 2.2.2. By Proposition 2.2.6, the series converges on the open unit disks, so it belongs to $\mathbf{Z}((T))_r$ and it specializes to x when evaluated at $T = \xi$. □

2.3. **Real Measures.**

Definition 2.3.1. Let $0 < p' < 1$ be a real number, and let S be a profinite set. Then $\mathcal{M}_{p'}(S)$ denotes TODO

2.4. **The MacLane Q'-construction.** In this subsection we will focus on the functorial complex induced by the Breen–Deligne package described in Definition 1.1.12. This complex is also known as MacLane's Q'-construction. (TODO: Rewrite the subsection on Breen–Deligne packages to reflect this.)

Proposition 2.4.1. *For any $i \geq 0$, the functor $A \mapsto H_i(Q'(A))$ has the following properties:*

(1) *It is additive, i.e.*

$$H_i(Q'(A \oplus B)) \cong H_i(Q'(A)) \oplus H_i(Q'(B)).$$

(2) *It commutes with filtered colimits, i.e. for a filtered inductive system A_i,*

$$\varinjlim_i H_i(Q'(A)) \cong H_i(Q'(\varinjlim_i A_i)).$$

In particular, for torsion-free abelian groups A, there is a functorial isomorphism

$$H_i(Q'(A)) \cong H_i(Q'(\mathbb{Z})) \otimes A.$$

As the proof shows, we do not really need the Q'-construction here: Any Breen–Deligne package will do.

Proof. Let us do the easy things first. Part (2) is clear as everything in sight commutes with filtered colimits. Assuming (1), we note that there is a natural map

$$H_i(Q'(\mathbb{Z})) \times A \to H_i(Q'(A))$$

induced by functoriality of $H_i(Q'(-))$. To check that this is bilinear and induces an isomorphism

$$H_i(Q'(\mathbb{Z})) \otimes A \cong H_i(Q'(A)),$$

we can reduce to the case that A is finitely generated by (2). In that case A is finite free, and the result follows from (1).

Thus, it remains to prove part (1), which has already been formalized. We recall that the direct sum of two abelian groups M and N is characterized as the abelian group P with maps $i_M : M \to P$, $i_N : N \to P$, $p_M : P \to M$, $p_N : P \to N$, satisfying $p_M i_M = \mathrm{id}_M$, $p_N i_N = \mathrm{id}_N$, $p_M i_N = 0$, $p_N i_M = 0$, $\mathrm{id}_P = i_M p_M + i_N p_N$. Apply this to $M = H_i(Q'(A))$, $N = H_i(Q'(B))$ and $P = H_i(Q'(A \oplus B))$, with all maps induced by applying $H_i(Q'(-))$ to the similar maps for A, B and $A \oplus B$. The fact that $H_i(Q'(-))$ is a functor already gives all identities except $\mathrm{id}_P = i_M p_M + i_N p_N$, and the only issue is the question whether $H_i(Q'(-))$ induces additive maps on morphism spaces.

But if $f, g : C \to D$ are any two maps of abelian groups, then $H_i(Q'(f+g)) = H_i(Q'(f)) + H_i(Q'(g))$, by reducing to the universal case of the two projections $D^2 \to D$ and using the homotopy baked into Definition 1.1.12. □

We note that by functoriality of the Q'-construction, it can also be applied to condensed abelian groups.

Corollary 2.4.2. *For torsion-free condensed abelian groups A, there is a natural isomorphism*

$$H_i(Q'(A)) \cong H_i(Q'(\mathbb{Z})) \otimes A$$

of condensed abelian groups.

Here, we only need to be able to tensor condensed abelian groups with (abstract) abelian groups. (With more effort, one could prove that $H_i(Q'(\mathbb{Z}))$ is even finitely generated.) In that case, the tensor product functor can be defined very naively by tensoring the values at any S with the given abstract abelian group.

Proof. Evaluating at $S \in \mathrm{ExtrDisc}$, we note that $S \mapsto H_i(Q'(A(S)))$ is already a condensed abelian group, and agrees with $H_i(Q'(\mathbb{Z})) \otimes A(S)$. Thus, the same is true after sheafification. □

If A is a torsion-free condensed abelian group equipped with an endomorphism f, then $Q'(A)$ is also equipped with the endomorphism f induced by functoriality, and by functoriality all previous assertions upgrade to $\mathbb{Z}[f]$-modules.

Proposition 2.4.3. *Let M and N be condensed abelian groups with endomorphisms f_M, f_N. Assume that M is torsion-free (over $\mathbb{Z}$). Then*

$$\mathrm{Ext}^i_{\mathbb{Z}[f]}(M, N) = 0$$

for all $i \geq 0$ if and only if

$$\mathrm{Ext}^i_{\mathbb{Z}[f]}(Q'(M), N) = 0$$

for all $i \geq 0$. More precisely, the first vanishes for $0 \leq i \leq j$ if and only if the second vanishes for $0 \leq i \leq j$.

At this point, we need to be able to talk about Ext-groups of (bounded to the right) complexes of condensed abelian groups (against condensed abelian groups).

The statement is also true without the torsion-freeness assumption on M, but slightly more nasty to prove then (and not required for the application).

Proof. We induct on j. Consider first the case $j = 0$; then any map $Q'(M) \to N$ factors uniquely over $H_0 Q'(M)[0] = M[0]$, yielding the result. Now assume that both sides vanish for $0 \leq i < j$; we need to see that the vanishing of the Ext^i's is equivalent. Consider the triangle

$$\tau_{\geq 1} Q'(M) \to Q'(M) \to M[0] \to .$$

Taking the corresponding long exact sequence of Ext-groups against N, we see that it suffices to see that

$$\mathrm{Ext}^i_{\mathbb{Z}[f]}(\tau_{\geq 1} Q'(M), N) = 0$$

for $0 \leq i \leq j$. But we can prove by descending induction on t that

$$\mathrm{Ext}^i_{\mathbb{Z}[f]}(\tau_{\geq t} Q'(M), N) = 0.$$

This is trivially true for $t > i$. Now look at the triangle

$$\tau_{\geq t+1} Q'(M) \to \tau_{\geq t} Q'(M) \to H_t(Q'(M))[t] \to$$

and the corresponding long exact sequence. It becomes sufficient to prove that

$$\mathrm{Ext}^i_{\mathbb{Z}[f]}(H_t(Q'(M))[t], N) = 0$$

for $0 \leq i \leq j$. Trivially,

$$\mathrm{Ext}^i_{\mathbb{Z}[f]}(H_t(Q'(M))[t], N) = \mathrm{Ext}^{i-t}_{\mathbb{Z}[f]}(H_t(Q'(M)), N).$$

Note that $t \geq 1$ here, so $i - t < j$ (and can be assumed ≥ 0). Also $H_t(Q'(M)) \cong H_t(Q'(\mathbb{Z})) \otimes M$. Thus, it suffices to show that for every abelian group A and every $0 \leq i < j$,

$$\mathrm{Ext}^i_{\mathbb{Z}[f]}(A \otimes M, N) = 0.$$

If A is free, then $A \otimes M$ is a direct sum of copies of M, and the result follows as Ext turns direct sums into products (and we assumed the vanishing of $\mathrm{Ext}^i_{\mathbb{Z}[f]}(M, N)$ for $0 \leq i < j$). In general, one can pick a two-term free resolution of A and use the long exact sequence. □

2.5. **Condensed abelian groups.**

Remark 2.5.1. For the time being, the following facts will be used without proof in this text. (They have or will be formalized in Lean though.)

- There is a natural functor Top $\to$ Cond(Sets).
- The category of condensed abelian groups (resp. condensed R-modules) is an abelian category with enough projectives. For S an extremally disconnected set, the objects $\mathbb{Z}[S]$ (resp. $R[S]$) is projective.
- We write $H^i(S, M)$ for $\mathrm{Ext}^i(\mathbb{Z}[S], M)$.

Definition 2.5.2. Consider an exact sequence of abelian groups

$$X' \xrightarrow{f} X \xrightarrow{g} X''$$

such that all of X', X and X'' carry the structure of compact-Hausdorffly-filtered-pseudonormed abelian groups. Assume that the maps are strict, i.e., $f(X'_{\leq c}) \subset X_{\leq c}$ and $g(X_{\leq c}) \subset X''_{\leq c}$. We say that the sequence is *exact with constant* c_f if $\ker(g) \cap X_{\leq c} \subset f(X'_{\leq c_f c})$.

Proposition 2.5.3. *Consider an inverse system*

$$(X'_i \xrightarrow{f_i} X_i \xrightarrow{g_i} X''_i)_i$$

of exact sequences that are exact with constant c_f *(independent of* i*). Moreover, assume that the transition maps* $X'_i \to X'_j$*,* $X_i \to X_j$ *and* $X''_i \to X''_j$ *are strict, and let* X'*,* X *and* X'' *be their limits. Then*

$$X' \xrightarrow{f} X \xrightarrow{g} X''$$

is exact with the same constant c_f*.*

Proof. Pass to cofiltered limits of compact Hausdorff spaces in the statements $\ker(g) \cap X_{\leq c} \subset f(X'_{\leq c_f c})$, noting that cofiltered limits of surjections of compact Hausdorff spaces are still surjective (by an application of Tychonoff). □

Definition 2.5.4. There is a natural functor

$$\mathrm{CHPNG} \to \mathrm{Cond}(\mathrm{Ab})$$
$$M \mapsto \underline{M}$$

where $\underline{M}(S)$ is defined to be collection of functions $f\colon S \to M$ that factor as continuous through M_c, for some c. In symbols:

$$M(S) = \{f\colon S \to M \mid \exists c, f(S) \subset M_c \text{ and } f\colon S \to M_c \text{ is continuous}\}.$$

Proposition 2.5.5. *Consider an exact sequence of abelian groups*

$$X' \xrightarrow{f} X \xrightarrow{g} X''$$

such that all of X', X and X'' carry the structure of compact-Hausdorffly-filtered-pseudonormed abelian groups. Assume that $f(X'_{\leq c}) \subset X_{\leq c}$ and $g(X_{\leq c}) \subset X''_{\leq c}$. If the sequence is exact with constant c_f, then the sequence

$$\underline{X'} \to \underline{X} \to \underline{X''}$$

of condensed abelian groups is exact.

Proof. TODO: Rewrite this proof. It is written for the case $\bullet \xrightarrow{g} \bullet \xrightarrow{0} \bullet$.

We evaluate at $S \in \mathrm{ExtrDisc}$. Any map $S \to X''$ factors over some $X''_{\leq c}$, and then $g : X_{\leq c_g c} \times_{X''} X''_{\leq c} \to X''_{\leq c}$ is a surjection of compact Hausdorff spaces; as S is extremally disconnected, the map $S \to X''_{\leq c}$ can be lifted, showing that $g : X(S) \to X''(S)$ is surjective. A similar argument shows that the kernel of $g : X(S) \to X''(S)$ is in the image of $f : X'(S) \to X(S)$, the latter clearly being injective. □

Lemma 2.5.6. *Let $S = \varprojlim S_i$ be a profinite set. Then $\mathbb{Z}[S]$ is naturally a profinitely filtered pseudo-normed group, via $\mathbb{Z}[S]_{\leq c} = \varprojlim \mathbb{Z}[S_i]_{\leq c}$, where $\mathbb{Z}[S_i]_{\leq c}$ is the set $\{\sum_{s \in S_i} n_s[s] \mid \sum_s |n_s| \leq c\}$.*

There is a natural isomorphism between the free condensed abelian group $\mathbb{Z}[S]$ and the colimit $\bigcup_c \mathbb{Z}[S]_{\leq c}$ of condensed sets.

Proof. For now, see Lemma 2.1 of [Sch20]. □

Proposition 2.5.7. *Let*

$$M\colon \mathrm{ProFin}^{\mathrm{op}} \to \mathrm{Ab}$$

be a functor, i.e. a presheaf of abelian groups on ProFin*. Assume that M preserves finite products, and that for any surjective map $f : T \to S$, the complex*

$$0 \to M(S) \to M(T) \to M(T \times_S T) \to M(T \times_S T \times_S T) \to \ldots$$

is exact.

Then M is a condensed abelian group, and for all profinite sets S and $i > 0$, one has $H^i(S, M) = 0$ for $i > 0$.

Proof. We prove by induction on $i > 0$ that $H^i(S, M) = 0$ for all profinite sets S, so assume the vanishing of $\mathrm{Ext}^1, \ldots, \mathrm{Ext}^i$ for some $i \geq 0$. (This is vacuous for $i = 0$.) We aim to prove that $H^{i+1}(S, M) = 0$ for all profinite sets S. Pick any profinite set S and a cover $T \to S$ with $T \in \mathrm{ExtrDisc}$. We get a long exact sequence of condensed abelian groups

$$\ldots \to \mathbb{Z}[T \times_S T \times_S T] \to \mathbb{Z}[T \times_S T] \to \mathbb{Z}[T] \to \mathbb{Z}[S] \to 0 :$$

Indeed, taken as presheaves on ExtrDisc, this is already true on the level of presheaves, where it reduces to the case of surjections of sets in which case one can write down a contracting homotopy. (Actually, the similar result is true in any topos, where one has to maybe argue a bit more carefully.)

The following argument is making explicit something usually seen through a spectral sequence. Define inductively

$$K_1 = \ker(\mathbb{Z}[T] \to \mathbb{Z}[S]),$$
$$K_2 = \ker(\mathbb{Z}[T \times_S T] \to \mathbb{Z}[T])$$

etc. One gets exact sequences

$$0 \to K_n \to \mathbb{Z}[T^{n/S}] \to K_{n-1} \to 0$$

for $n \geq 2$. From the long exact sequence

$$... \to H^i(T, M) \to \mathrm{Ext}^i(K_1, M) \to H^{i+1}(S, M) \to H^{i+1}(T, M) = 0$$

we see that we have to prove that $\mathrm{Ext}^i(K_1, M) = 0$ (if $i > 0$, otherwise that $M(T)$ surjects onto $\mathrm{Hom}(K_1, M)$). Assuming $i > 0$, we can go on, and using the inductive hypothesis applied to the fibre products $T^{*/S}$, we inductively see that

$$H^{i+1}(S, M) = \mathrm{Ext}^i(K_1, M) = \mathrm{Ext}^{i-1}(K_2, M) = ... = \mathrm{Ext}^1(K_i, M)$$

and eventually that this is the same as the cokernel of

$$M(T^{i/S}) \to \mathrm{Hom}(K_{i+1}, M).$$

But there is an exact sequence

$$0 \to \mathrm{Hom}(K_{i+1}, M) \to M(T^{(i+1)/S}) \to \mathrm{Hom}(K_{i+2}, M)$$

and $\mathrm{Hom}(K_{i+2}, M)$ injects into $M(T^{(i+2)/S})$. We see that

$$\mathrm{Hom}(K_{i+1}, M) = \ker(M(T^{(i+1)/S}) \to M(T^{(i+2)/S}))$$

and we need to see that

$$M(T^{i/S}) \to \mathrm{Hom}(K_{i+1}, M) = \ker(M(T^{(i+1)/S}) \to M(T^{(i+2)/S}))$$

is surjective, which is precisely the exactness of the Čech complex. □

Proposition 2.5.8. *Let $(M, \|\cdot\|)$ be a complete normed group, regarded as a topological group. Then the corresponding condensed abelian group $\underline{M}$ sends any profinite set S to the completion of normed group of locally constant maps $S \to M$ (with the supremum norm).*

Proposition 2.5.9. *Let $(M, \|\cdot\|)$ be a complete normed group, regarded as a topological group. Then for any profinite set S, one has $H^i(S, \underline{M}) = 0$ for $i > 0$.*

Proof. This follows Proposition 2.5.7 and the part of [Sch20, Proposition 8.19] that is already formalized. □

Lemma 2.5.10. *Let $0 < r < r' < 1$ be real numbers. Let S be a profinite set, and let V be a r-normed (Banach?) $\mathbb{Z}[T^{\pm 1}]$-module. Then $\mathrm{Ext}^i_{\mathrm{Mod}^{\mathrm{cond}}_{\mathbb{Z}[T^{-1}]}}(\overline{\mathcal{L}}_{r'}(S), V) = 0$ for all $i \geq 0$. In other words,*

$$\mathrm{Ext}^i_{\mathbb{Z}}(\overline{\mathcal{L}}_{r'}(S), V) \xrightarrow{[T^{-1}]_L - [T^{-1}]_V} \mathrm{Ext}^i_{\mathbb{Z}}(\overline{\mathcal{L}}_{r'}(S), V)$$

is a bijection for all i.

Proof. With Proposition 2.4.3, it suffices to prove the following assertion. Pick $1 > r' > r > 0$, a profinite S, and some r-Banach $\mathbb{Z}[T^{\pm 1}]$-module V as before. Then we want to prove that

$$\mathrm{Ext}^i_{\mathbb{Z}[T^{-1}]}(Q'(\overline{\mathcal{M}}_{r'}(S)), V) = 0$$

for all $i \geq 0$.

At this point, it is profitable to rewrite this again as the bijectivity of

$$(T^{-1})_V - (T^{-1})_{\mathcal{M}} : \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S)), V) \to \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S)), V).$$

Now these Ext-groups can be computed! More precisely, recall that $Q'(\overline{\mathcal{M}}_{r'}(S))$ is a complex of the form

$$\ldots \to \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)^2] \to \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)] \to 0.$$

Termwise, the Ext-groups turn into cohomology groups

$$H^i(\overline{\mathcal{M}}_{r'}(S)^{2^j}, V).$$

Unfortunately, $\overline{\mathcal{M}}_{r'}(S)$ itself is not profinite, so we cannot directly apply Proposition 2.5.9. To get around this last cliff, we write $Q'(\overline{\mathcal{M}}_{r'}(S))$ as a filtered colimit of complexes

$$Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq c} : \ldots \to \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)^2_{\leq \kappa_1 c}] \to \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)_{\leq \kappa_0 c}] \to 0$$

where the constants $\kappa_0 = 1, \kappa_1, \ldots$ are positive and chosen so that all differentials are well-defined. (The possibility of choosing such constants has already been formalized; TODO include pointer.) It suffices to prove that

$$(T^{-1})_V - (T^{-1})_{\mathcal{M}} : \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq c}, V) \to \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq r'c}, V)$$

is a pro-isomorphism in c, as then the final result follows by passing to a derived limit over c, see Lemma 2.5.11 below. This final pro-isomorphism assertion can finally be written out, and it unravels to the statement of Theorem 1.7.1.

In passing to the derived limit over c, we use the following lemma. □

Lemma 2.5.11. *Assume that in each degree i, the map*

$$(T^{-1})_V - (T^{-1})_{\mathcal{M}} : \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq c}, V) \to \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq r'c}, V)$$

is a pro-isomorphism in c (i.e., pro-systems of kernels, and of cokernels, are pro-zero). Then

$$(T^{-1})_V - (T^{-1})_{\mathcal{M}} : \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S)), V) \to \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S)), V).$$

is an isomorphism.

Proof. We have

$$Q'(\overline{\mathcal{M}}_{r'}(S)) = \bigcup_n Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n},$$

inducing a resolution

$$0 \to \bigoplus_n Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n} \to \bigoplus_n Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n} \to Q'(\overline{\mathcal{M}}_{r'}(S)) \to 0.$$

Passing to a corresponding long exact sequence reduces one to checking that the squares

$$\begin{array}{ccc} \prod_n \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n}, V) & \longrightarrow & \prod_n \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n}, V) \\ \downarrow & & \downarrow \\ \prod_n \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n}, V) & \longrightarrow & \prod_n \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n}, V) \end{array}$$

are bicartesian (here, horizontal maps are shift minus identity, and vertical maps are $(T^{-1})_V - (T^{-1})_{\mathcal{M}}$). Equivalently, the horizontal maps become isomorphisms on vertical kernels, and vertical cokernels. But the vertical kernels and vertical cokernels induce pro-zero systems of abelian groups, and then the horizontal kernels and cokernels compute $\varprojlim_n$ and $\varprojlim_n^1$ of their systems, which vanish. □

Proposition 2.5.12. *Let $0 < r < 1$ be a real number, and let S be a profinite set. Decomposing $\mathbb{Z}((T))_r$ into positive and nonpositive coefficients yields a direct sum decomposition*

$$\mathbb{Z}((T))_r = T\mathbb{Z}[[T]]_r \oplus \mathbb{Z}[T^{-1}].$$

This extends to a decomposition of spaces of measures

$$\mathcal{L}_r(S) = \mathcal{L}(S, T\mathbb{Z}((T))_r) = \mathcal{L}(S, T\mathbb{Z}[[T]]_r) \oplus \mathcal{L}(S, \mathbb{Z}[T^{-1}])$$

where $\mathcal{M}(S, \mathbb{Z}[T^{-1}]) = \mathbb{Z}[T^{-1}][S]$ is the free condensed $\mathbb{Z}[T^{-1}]$-module on S. Letting $\overline{\mathcal{L}}_r(S) = \mathcal{L}(S, T\mathbb{Z}[[T]]_r)$, we get a short exact sequence of condensed $\mathbb{Z}[T^{-1}]$-modules

$$0 \to \mathbb{Z}[T^{-1}][S] \to \mathcal{L}_r(S) \to \overline{\mathcal{L}}_r(S) \to 0.$$

Proof. On $\mathbb{Z}((T))_{r,\leq c}$, only finitely many nonpositive coefficients can possibly be nonzero, and each of them is bounded. This shows that the nonpositive summand of $\mathbb{Z}((T))_r$ is given by $\mathbb{Z}[T^{-1}]$. To pass to profinite S, use Proposition 2.5.6. □

Lemma 2.5.13. *Let $0 < r < r' < 1$ be real numbers. Let S be a profinite set, and let V be an r-normed $\mathbb{Z}[T^{\pm 1}]$-module. Then $\mathrm{Ext}^i_{\mathrm{Mod}^{\mathrm{cond}}_{\mathbb{Z}[T^{-1}]}}(\mathcal{L}_{r'}(S), V) = 0$ for all $i > 0$. In other words,*

$$\mathrm{Ext}^i_{\mathbb{Z}}(\mathcal{L}_{r'}(S), V) \xrightarrow{[T^{-1}]_L - [T^{-1}]_V} \mathrm{Ext}^i_{\mathbb{Z}}(\mathcal{L}_{r'}(S), V)$$

is a bijection for all $i > 0$ and a surjection for $i = 0$.

Proof. Consider the long exact sequence of Ext-groups arising form the short exact sequence (Lemma 2.5.12)

$$0 \to \mathbb{Z}[T^{-1}][S] \to \mathcal{L}_{r'}(S) \to \overline{\mathcal{L}}_{r'}(S) \to 0$$

by applying $\mathrm{Ext}^*(\_, V)$.

By Lemma 2.5.7 all groups $\mathrm{Ext}^i_{\mathrm{Cond(Ab)}}(\mathbf{Z}[S], V)$ vanish for $i > 0$. And by Lemma 2.5.10 all groups $\mathrm{Ext}^i_{\mathrm{Mod}^{\mathrm{cond}}_{\mathbb{Z}[T^{-1}]}}(\overline{\mathcal{L}}_{r'}(S), V)$ vanish for $i \geq 0$. The result follows.

The "In other words" version can be proved without mentioning $\mathbb{Z}[T^{-1}]$-linear Ext groups, by using the same ingredients and the five lemma. □

Lemma 2.5.14. *Let $0 < p' < 1$ be a real number, let S be a profinite set, and let r' denote $(\frac{1}{2})^{p'}$. There is a short exact sequence of condensed $\mathbb{Z}[T^{-1}]$-modules*

$$0 \to \mathcal{L}_{r'}(S) \to \mathcal{L}_{r'}(S) \to \mathcal{M}_{p'}(S) \to 0$$

where the first map is multiplication by $2T - 1$, and the second is evaluation at $T = \frac{1}{2}$.

Proof. TODO □

Theorem 2.5.15 (Clausen–Scholze)**.** *Let $0 < p' < p \leq 1$ be real numbers, let S be a profinite set, and let V be a p-Banach space. Let $\mathcal{M}_{p'}(S)$ be the space of p'-measures on S. Then*

$$\mathrm{Ext}^i_{\mathrm{Cond(Ab)}}(\mathcal{M}_{p'}(S), V) = 0$$

for $i \geq 1$.

Proof. Recall from Lemma 2.5.14 the short exact sequence

$$0 \to \mathcal{L}_{r'}(S) \to \mathcal{L}_{r'}(S) \to \mathcal{M}_{p'}(S) \to 0.$$

Apply to this $\mathrm{Ext}^*(\_, V)$ to obtain a long exact sequence. Note that T acts on V via multiplication by $\frac{1}{2}$ (by Lemma 2.1.2. Hence we can use LemmaExt-L to obtain isomorphisms between the Ext-groups involving $\mathcal{L}_{r'}(S)$, for $i > 0$, and a surjection for $i = 0$. The result follows. □

References

[Sch19] P. Scholze. Lectures on Condensed Mathematics. 2019.
[Sch20] P. Scholze. Lectures on Analytic Geometry. 2020.