BLUEPRINT FOR THE LIQUID TENSOR EXPERIMENT

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Remark 0.1. This text is based on the lecture notes on Analytic Geometry [Sch20], by Peter Scholze. The final section is copy-pasted from those lecture notes almost verbatim. This text is meant as a blueprint for the Liquid Tensor Experiment.

1. Breen–Deligne data

The goal of this section is to a give a precise statement of the Breen–Deligne resolution. We first give the statement, and provide details later.

Theorem 1.1 (Breen–Deligne)**.** *For an abelian group A, there is a resolution, functorial in A, of the form*

$$\dots \to \bigoplus_{i=1}^{n_i} \mathbb{Z}[A^{r_{ij}}] \to \dots \to \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0.$$

I (Johan Commelin) have not figured out the details. But it seems to be possible to avoid the $\bigoplus_{i=1}^{n_i}$, so we will aim for something like the following statement.

Theorem 1.2. *For an abelian group A, there is a resolution, functorial in A, of the form*

$$\dots \to \mathbb{Z}[A^{n_i}] \to \dots \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0.$$

What does a homomorphism $f\colon \mathbb{Z}[A^m] \to \mathbb{Z}[A^n]$ that is functorial in A look like? We should perhaps say more precisely what we mean by this. The idea is that m and n are fixed, and for each abelian group A we have a group homomorphism $f_A\colon \mathbb{Z}[A^m] \to \mathbb{Z}[A^n]$ such that if $\phi : A \to B$ is a group homomorphism inducing $\phi_i : \mathbf{Z}[A^i] \to \mathbf{Z}[B^i]$ for each natural number i then the obvious square commutes: $\phi_n \circ f_A = f_B \circ \phi_m$.

The map f_A is specified by what it does to the generators $(a_1, a_2, a_3, \dots, a_m) \in A^m$. It can send such an element to an arbitrary element of $\mathbb{Z}[A^n]$, but one can check that universality implies that f_A will be a $\mathbb{Z}$-linear combination of "basic universal maps", where a "basic universal map" is one that sends $(a_1, a_2, \dots, a_m)$ to $(t_1, \dots, t_n)$, where t_i is a $\mathbb{Z}$-linear combination $c_{i,1} \cdot a_1 + \dots + c_{i,m} \cdot a_m$. So a "basic universal map" is specified by the $n \times m$-matrix c.

Definition 1.3. A *basic universal map* from exponent m to n, is an $n \times m$-matrix with coefficients in $\mathbb{Z}$.

Definition 1.4. A *universal map* from exponent m to n, is a formal $\mathbb{Z}$-linear combination of basic universal maps from exponent m to n.

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We point out that basic universal maps can be composed by matrix multiplication, and this formally induces a composition of universal maps. As mentioned above, one can also check (this has been formalised in Lean) that this construction gives a bijection between universal maps from exponent m to n and functorial collections $f_A : \mathbf{Z}[A^m] \to \mathbf{Z}[A^n]$.

Definition 1.5. The addition on A^n induces a universal map $\sigma_\alpha \colon \mathbb{Z}[(A^n)^2] \to \mathbb{Z}[A^n]$, namely the formal generator $(I_n I_n)$, where I_n denotes the $n \times n$ identity matrix. (Here α stands for "addition".)

Definition 1.6. The formal sum of the two projections $(A^n)^2 \to A^n$ induces a universal map $\sigma_\pi \colon \mathbb{Z}[(A^n)^2] \to \mathbb{Z}[A^n]$, namely the formal sum $(I_n 0) + (0 I_n)$, where I_n denotes the $n \times n$ identity matrix, and 0 the $n \times n$ zero matrix. (Here π stands for "projections".)

Definition 1.7. Let f be a universal map from exponent m to n. Then $f \oplus f$ denotes the universal map from exponent $2m$ to $2n$, that applies f componentwise. If f is a generator (i.e. a basic universal map) then $f \oplus f$ is

$$\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}.$$

Definition 1.8. A tuple (n, f) of *Breen–Deligne data* consists of a sequence of exponents $n_0, n_1, n_2, \dots \in \mathbb{N}$, and universal maps f_i from exponent n_{i+1} to n_i.

Such a tuples is a *complex* if for all i we have $f_i \circ f_{i+1} = 0$.

A *universal morphism* of Breen–Deligne data (or complexes) $(m, f) \to (n, g)$ is a collection of universal maps ϕ_i from exponent m_i to n_i such that $g_i \circ \phi_{i+1} = \phi_i \circ f_i$ as universal maps from exponent m_{i+1} to n_i (i.e., the squares commute).

Definition 1.9. If (n, f) is a tuple of Breen–Deligne data, then $(n, f) \oplus (n, f)$ is the tuple consisting of exponents $2n_i$ and universal maps $f_i \oplus f_i$.

The two universal map σ_α and σ_π explained in the examples above, can be checked to induce universal maps of complexes: $(n, f) \oplus (n, f) \to (n, f)$.

Definition 1.10. A *homotopy* for a tuple (n, f) of Breen–Deligne data is a homotopy between the maps of complexes

$$\sigma_\alpha, \sigma_\pi \colon (n, f) \oplus (n, f) \to (n, f)$$

In other words, it consists of universal maps h_i from exponent $2n_i$ to n_{i+1}, such that $f_0 \circ h_0 = \sigma_\alpha - \sigma_\pi$ as universal maps from exponent $2n_0$ to n_0, and for all $i \geq 0$ we have

$$f_{i+1} \circ h_{i+1} + h_i \circ (f_i \oplus f_i) = \sigma_\alpha - \sigma_\pi$$

as universal maps from exponent $2n_{i+1}$ to n_{i+1}. Note that the first condition is morally the $i = -1$ case of the displayed equation, if we set $h_{-1} = 0$.

A *Breen–Deligne package* is a triple (n, f, h), such that (n, f) is Breen–Deligne data that is a complex, and h is a homotopy for (n, f).

Definition 1.11. We will now construct an example of a Breen–Deligne package. In some sense, it is the "easiest" solution to the conditions posed above. The exponents will be $n_i = 2^i$, and the homotopies h_i will be the identity. Under these constraints, we recursively construct the universal maps f_i:

$$f_0 = \sigma_\alpha - \sigma_\pi, \quad f_{i+1} = (\sigma_\alpha - \sigma_\pi) - (f_i \oplus f_i).$$

We leave it as exercise for the reader, to verify that with these definitions (n, f, h) forms a Breen–Deligne package.

We now make three definitions that will make precise some conditions between constants that will be needed when we construct Breen–Deligne complexes of normed abelian groups.

Definition 1.12. Let f be a basic universal map from exponent m to n. Let $c_1, c_2 \in \mathbb{R}_{\geq 0}$. We say that (c_1, c_2) is *f-suitable*, if for all i

$$\sum_j c_1 |f_{ij}| \leq c_2.$$

To orient the reader: later on we will be considering maps on normed abelian groups induced from universal maps, and this inequality will guarantee that if $\|m\| \leq c_1$ then $\|f(m)\| \leq c_2$.

Definition 1.13. Let f be a universal map from exponent m to n. Let $c_1, c_2 \in \mathbb{R}_{\geq 0}$. We say that (c_1, c_2) is *f-suitable*, if for all basic universal maps g that occur in the formal sum f, the pair of nonnegative reals (c_1, c_2) is g-suitable.

Definition 1.14. Let (n, f) be Breen–Deligne data, and let $c = (c_0, c_1, \dots)$ be a sequence of nonnegative real numbers. We say that c is *(n, f, h)-suitable*, if for all i, the pair (c_{i+1}, c_i) is f_i-suitable.

(Note! The order (c_{i+1}, c_i) is contravariant compared to Definition 1.13. This is because of the contravariance of $\widehat{V}(\_)$; see Definition 3.6.)

2. Spaces of convergent power series

Definition 2.1. A *pseudo-normed group* is an abelian group $(M, +)$, together with an increasing filtration $M_c \subseteq M$ of subsets M_c indexed by $\mathbb{R}_{\geq 0}$, such that each M_c contains 0, is closed under negation, and $M_{c_1} + M_{c_2} \subseteq M_{c_1+c_2}$. An example would be $M = \mathbb{R}$ or $M = \mathbb{Q}_p$ with $M_c := \{x : |x| \leq c\}$.

A pseudo-normed group M is *profinitely filtered* if each of the sets M_c is endowed with a topological space structure making it a profinite set, such that following maps are all continuous:

- the inclusion $M_{c_1} \to M_{c_2}$ (for $c_1 \leq c_2$);
- the negation $M_c \to M_c$;
- the addition $M_{c_1} \times M_{c_2} \to M_{c_1+c_2}$.

A *morphism* of profinitely filtered pseudo-normed groups $M \to N$ is a group homomorphism f that is

- *bounded*: there is a constant C such that $x \in M_c$ implies $f(x) \in N_{Cc}$;
- *continuous*: for one (or equivalently all) constants C as above, the induced map $M_c \to N_{Cc}$ is a morphism of profinite sets, i.e. continuous.

The reason the two definitions are equivalent is that a continuous injection between profinite sets must be a topological embedding.

Definition 2.2. Let r' be a positive real number. A profinitely filtered pseudo-normed group M has an *r'-action of T^{-1}* if it comes endowed with a distinguished morphism of profinitely filtered pseudo-normed groups $T^{-1} \colon M \to M$ that is bounded by r'^{-1}: if $x \in M_c$ then $T^{-1}x \in M_{c/r'}$.

A morphism $M \to N$ of profinitely filtered pseudo-normed groups with r'-action of T^{-1} is a morphism of profinitely filtered pseudo-normed groups f that commutes with the action of T^{-1} and is *strict*: if $x \in M_c$ then $f(x) \in N_c$.

We will now construct the central example of profinitely filtered pseudo-normed groups with r'-action of T^{-1}.

Definition 2.3. Let $r' > 0$ be a real number, and let S be a finite set. Denote by $\overline{\mathcal{M}}_{r'}(S)$ the set

$$\left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \in T\mathbf{Z}[[T]] \right)_{s \in S} \middle| \sum_{n \geq 1, s \in S} |a_{n,s}|(r')^n < \infty \right\}.$$

Note that $\overline{\mathcal{M}}_{r'}(S)$ is naturally a pseudo-normed group with filtration given by

$$\overline{\mathcal{M}}_{r'}(S)_{\leq c} = \left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} \middle| \sum_{n \geq 1, s \in S} |a_{n,s}|(r')^n \leq c \right\}.$$

Lemma 2.4. *Let $r' > 0$ and $c \geq 0$ be real numbers, and let S be a finite set. The space $\overline{\mathcal{M}}_{r'}(S)_{\leq c}$ is the profinite limit of the finite sets*

$$\overline{\mathcal{M}}_{r'}(S)_{\leq c, \leq N} = \left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} \middle| \sum_{1 \leq n \leq N, s \in S} |a_{n,s}|(r')^n \leq c \right\}$$

This endows $\overline{\mathcal{M}}_{r'}(S)_{\leq c}$ with the profinite topology. In particular, it is a profinitely filtered pseudo-normed group.

Proof. Formalised, but omitted from this text. □

For the remainder of this section, let $r' > 0, c \geq 0$ be real numbers, and let S be a finite set.

Definition 2.5. There is a natural action of T^{-1} on $\overline{\mathcal{M}}_{r'}(S)$, via

$$T^{-1} \cdot \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} = \left(\sum_{n \geq 1} a_{n+1,s} T^n \right)_{s \in S}.$$

Lemma 2.6. *The natural action of T^{-1} on $\overline{\mathcal{M}}_{r'}(S)$ restricts to continuous maps*

$$T^{-1} \cdot \_ : \overline{\mathcal{M}}_{r}(S)_{\leq c} \to \overline{\mathcal{M}}_{r}(S)_{\leq c/r'}.$$

In particular, $\overline{\mathcal{M}}_{r'}(S)$ has an r'-action of T^{-1}.

Proof. Formalised, but omitted from this text. □

Lemma 2.7. *Let f be a basic universal map from exponent m to n. We get an induced homomorphism of profinitely filtered pseudo-normed groups*

$$\overline{\mathcal{M}}_{r'}(S)^m \to \overline{\mathcal{M}}_{r'}(S)^n$$

bounded by the maximum (over all i) of $\sum_j |f_{ij}|$, where the f_{ij} are the coefficients of the $n \times m$-matrix representing f.

Proof. Omitted. □

3. Completions of locally constant functions

Definition 3.1. Let V be a normed abelian group, and X a compact topological space. We denote by $V(X)$ the normed abelian group of locally constant functions $X \to V$ with respect to the sup norm. With $\widehat{V}(X)$ we denote the completion of $V(X)$.

These constructions are functorial in bounded group homomorphisms $V \to V'$ and contravariantly functorial in continuous maps $X \to X'$.

We continue to use the notation of before: let $r' > 0, c \geq 0$ be real numbers, and let M be a profinitely filtered pseudo-normed group with r'-action by T^{-1}.

Definition 3.2. Let f be a universal map from exponent m to n, and let (c_1, c_2) be f-suitable. We get an induced map

$$\widehat{V}(f)\colon \widehat{V}(M^m_{\leq c_1}) \to \widehat{V}(M^n_{\leq c_2})$$

that is the sum $\sum n_g V(g)$, if f is the formal sum $\sum n_g g$ of basic universal maps.

Definition 3.3. Let $r > 0$ be a real number. An *r-normed $\mathbb{Z}[T^{\pm 1}]$-module* is a normed abelian group V endowed with an automorphism $T\colon V \to V$ such that for all $v \in V$ we have $\|T(v)\| = r\|v\|$.

Lemma 3.4. *Let $r \in \mathbb{R}_{>0}$, and let V be an r-normed $\mathbb{Z}[T^{\pm 1}]$-module. Let X be a compact space. Then $\widehat{V}(X)$ is naturally an r-normed $\mathbb{Z}[T^{\pm 1}]$-module, with the action of T given by post-composition.*

Proof. Formalised, but omitted from this text. □

Let $r > 0$, and let V be an r-normed $\mathbb{Z}[T^{\pm 1}]$-module. Assume $r' \leq 1$.

Definition 3.5. There are two natural actions of T^{-1} on $\widehat{V}(M_{\leq c})$. The first comes from the r'-action of T^{-1} on M which gives a continuous map

$$M_{\leq cr'} \to M_{\leq c}$$

and thus a map

$$(T^{-1})^*\colon \widehat{V}(M_{\leq c}) \to \widehat{V}(M_{\leq cr'}).$$

The other comes from Lemma 3.4, using the r-normed $\mathbb{Z}[T^{\pm 1}]$-module V. We get a map

$$[T^{-1}]\colon \widehat{V}(M_{\leq c}) \to \widehat{V}(M_{\leq c}),$$

that we can compose with the map $\widehat{V}(M_{\leq c}) \to \widehat{V}(M_{\leq cr'})$, obtained from the natural inclusion $M_{\leq cr'} \to M_{\leq c}$. We thus end up with two maps

$$(T^{-1})^*, [T^{-1}]\colon \widehat{V}(M_{\leq c}) \to \widehat{V}(M_{\leq cr'}).$$

and we define $\widehat{V}(M_{\leq c})^{T^{-1}}$ to be the equalizer of $(T^{-1})^*$ and $[T^{-1}]$. In other words, the kernel of $(T^{-1})^* - [T^{-1}]$.

Definition 3.6. Let f be a universal map from exponent m to n, and let (c_1, c_2) be f-suitable.

The natural map from Definition 3.2 restricts to a map

$$\widehat{V}(f)^{T^{-1}}\colon \widehat{V}(M^n_{\leq c_2})^{T^{-1}} \to \widehat{V}(M^m_{\leq c_1})^{T^{-1}}$$

4. Some normed homological algebra

Definition 4.1. A *system of complexes* of normed abelian groups is for each sufficiently large c (i.e. all $c \geq c_0$ for some $c_0 > 0$), a complex

$$C_c^\bullet : C_c^0 \to C_c^1 \to \ldots$$

of normed abelian groups together with maps of complexes $\mathrm{res}_{c',c} : C_{c'}^\bullet \to C_c^\bullet$, for $c' \geq c \geq c_0$, satisfying $\mathrm{res}_{c,c} = \mathrm{id}$ and the obvious associativity condition. We use notation $(C_c^\bullet)_{c \geq c_0}$ for a system of complexes, although we will frequently omit any mention of the lower bound c_0 and just write $C_\bullet^\bullet$.

By convention, for every system of complexes $(C_c^\bullet)_{c \geq c_0}$, we will set $C_c^{-1} = 0$ for all $c \geq c_0$. This will come up each time we write C_c^{i-1} and i could be 0.

In this section, given $x \in C_{c'}^\bullet$ and $c_0 \leq c \leq c'$ we will use the notation $x_{|c} := \mathrm{res}_{c',c}(x)$.

Definition 4.2. A system of complexes is *admissible* if all differentials and maps $\mathrm{res}_{c',c}^i$ are norm-nonincreasing.

Throughout the rest of this section, k (and k', k'') will denote reals at least 1, m will be a non-negative integer, and K, K', K'' will denote non-negative reals.

Definition 4.3. Let $(C_c^\bullet)_{c \geq c_0}$ be a system of complexes. For an integer $m \geq 0$ and reals $k \geq 1$, $c_0' \geq c_0$ and $K \geq 0$, we say the datum $(C_c^\bullet)_{c \geq c_0}$ is *$\leq k$-exact in degrees $\leq m$ and for $c \geq c_0'$ with bound K* if the following condition is satisfied. For all $c \geq c_0'$ and all $x \in C_{kc}^i$ with $i \leq m$ there is some $y \in C_c^{i-1}$ such that

$$\|x_{|c} - dy\| \leq K\|dx\|.$$

We will also need a version where the inequality is relaxed by some arbitrary small additive constant.

Definition 4.4. Let $(C_c^\bullet)_{c \geq c_0}$ be a system of complexes. For an integer $m \geq 0$ and reals $k \geq 1$, $c_0' \geq c_0$ and $K \geq 0$, the datum $(C_c^\bullet)_c$ is *weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0'$ with bound K* if the following condition is satisfied. For all $c \geq c_0'$, all $x \in C_{kc}^i$ with $i \leq m$ and any $\varepsilon > 0$ there is some $y \in C_c^{i-1}$ such that

$$\|x_{|c} - dy\| \leq K\|dx\| + \varepsilon.$$

We first note that the difference between those two definitions is only about cocyles if we are ready to lose a tiny something on the norm bound K.

Lemma 4.5. *Let $C_\bullet^\bullet$ be a system of complexes. If $C_\bullet^\bullet$ is weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0'$ with bound K and if, for all $c \geq c_0'$ and all $x \in C_{kc}^i$ with $i \leq m$ such that $dx = 0$ there is some $y \in C_c^{i-1}$ such that $x_{|c} = dy$ then, for every positive δ, $C_\bullet^\bullet$ is $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0'$ with bound $K + \delta$.*

Proof. Let δ be some positive real number. Let x be an element of C_{kc}^i for some $c \geq c_0'$ and $i \leq m$. If $dx = 0$ then the assumption we made about exact elements is exactly what we want.

Assume now that $dx \neq 0$. The weak exactness assumption applied to $\varepsilon = \delta\|dx\|$ gives some $y \in C_c^{i-1}$ such that

$$\begin{aligned}\|x_{|c} - dy\| &\leq K\|dx\| + \delta\|dx\| \\ &= (K + \delta)\|dx\|\end{aligned}$$

□

A more important observation is that, in both definitions, we can also ask some control on the norm of y if we are ready to square the restriction depth factor k.

Lemma 4.6. *Let $C_\bullet^\bullet$ be a system of complexes which is weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0'$ with bound K. For all $c \geq c_0'$, all $x \in C_{k^2c}^i$ with $i \leq m$, all $\varepsilon > 0$ and all $\delta > 0$ there is some $y \in C_c^{i-1}$ such that*

$$\|x_{|c} - dy\| \leq K\|dx\| + \varepsilon \quad \textit{and} \quad \|y\| \leq K(K+1)\|x\| + \delta.$$

Proof. Fix x, ε and δ. The weak exactness assumption applied to x and some η to be chosen later gives us $w \in C_{kc}^{i-1}$ such that

$$\|x_{|kc} - dw\| \leq K\|dx\| + \eta.$$

Then the weak exactness assumption applied to w and some τ to be chosen later gives us $z \in C_c^{i-2}$ such that

$$\|w_{|c} - dz\| \leq K\|dw\| + \tau.$$

We set $y = w_{|c} - dz$. Since $dy = dw_{|c}$, we get the first required estimate as long as $\eta \leq \varepsilon$. And we have:

$$\begin{aligned}\|y\| &\leq K\|dw\| + \tau \\ &\leq K(\|x_{|kc}\| + K\|dx\| + \eta) + \tau \\ &\leq K(K+1)\|x\| + K\eta + \tau\end{aligned}$$

which is fine as long as $K\eta + \tau \leq \delta$. So we set $\eta = \min(\varepsilon, \delta/(2K))$ (interpreted as ε if $K = 0$) and $\tau = \delta/2$. □

Lemma 4.7. *Let $(M_c^\bullet)_{c \geq c_0}$ be an admissible collection of complexes of normed abelian groups.*

Assume that $M_c^\bullet$ is weakly $\leq k$-exact in degrees $\leq m$ for $c \geq c_0$ with bound K. Then the completion $\overline{M_c^\bullet}$ is weakly $\leq k^2$-exact in degrees $\leq m$ for $c \geq c_0$ with bound K.

Proof. Let $x \in \overline{M_{k^2c}^i}$, where $c \geq c_0$ and $i \leq m$ and let $\epsilon > 0$. We can write $x = \sum_j x^j$ where

- $x^j \in M_{k^2c}^i$ for all $j \geq 0$,
- $\|x - x^0\| \leq \varepsilon_0$ for some positive ε_0 to be chosen later. This implies that $\|dx - dx^0\| \leq \varepsilon_0$ and in particular $\|dx^0\| \leq \|dx\| + \varepsilon_0$,
- $\|x^j\| \leq \varepsilon_j$ if $j > 0$, for some positive ε_j to be chosen later. This implies $\|dx^j\| \leq \varepsilon_j$ for all $j > 0$.

Using Lemma 4.6, we get a sequence y^j in M_c^{i-1} such that

$$\|x^j_{|c} - dy^j\| \leq K\|dx^j\| + \delta_j \quad \text{and} \quad \|y^j\| \leq K(K+1)\|x^j\| + \tau_j.$$

for positive sequences δ and τ to be chosen later.

Since M_c^{i-1} is complete, the series $\sum y^j$ converges as soon as we can guarantee that $\sum \|y^j\|$ converges. Our estimates ensure this convergence as soon as the sum of the $K(K+1)\varepsilon_j + \tau_j$ converges so here we only need ε and τ to be summable.

We then set $y = \sum y^j$ and compute:

$$\begin{aligned}
\|x_{|c} - dy\| &= \left\| \sum_{j\geq 0} x^j_{|c} - dy^j \right\| \\
&\leq \sum_{j\geq 0} \left\| x^j_{|c} - dy^j \right\| \\
&\leq \sum_{j\geq 0} K\|dx^j\| + \delta_j \\
&\leq K\|dx\| + K\varepsilon_0 + \delta_0 + \sum_{j>0} (K\varepsilon_j + \delta_j)
\end{aligned}$$

So everything is fine as long as $\sum_{j\geq 0}(K\varepsilon_j + \delta_j) \leq \varepsilon$, say $\varepsilon_j = \varepsilon 2^{-j-2}/K$ and $\delta_j = \varepsilon 2^{-j-2}$. □

Lemma 4.8. *Let $(M_c^\bullet)_{c\geq c_0}$ be an admissible collection of complexes of complete normed abelian groups.*

Assume that $M_c^\bullet$ is weakly $\leq k$-exact in degrees $\leq m$ for $c \geq c_0$ with bound K. Then $M_c^\bullet$, for every $\delta > 0$, it is $\leq k^2$-exact in degrees $\leq m$ for $c \geq c_0$ with bound $K + \delta$.

Proof. Lemma 4.5 ensures we only need to care about cocycles of M. More precisely, let x be a cocycle in $M^i_{k^2 c}$ for some $i \leq m$ and $c \geq c_0$. We need to find $y \in M_c^{i-1}$ such that $dy = x_{|c}$.

By weak $\leq k$-exactness applied to x and a sequence ε_j to be chosen later, we can find a sequence $w^j \in M^{i-1}_{kc}$ such that

$$\|x_{kc} - dw^j\| \leq \varepsilon_j.$$

Then, by weak $\leq k$-exactness applied to each $w^{j+1} - w^j$ and a sequence δ_j to be chosen later, we can find a sequence $z^j \in M_c^{i-2}$ such that

$$\|(w^{j+1} - w^j)_{|c} - dz^j\| \leq K\|dw^{j+1} - dw^j\| + \delta_j.$$

We set $y^j := w^j_{|c} - \sum_{l=0}^{j-1} dz^l \in M_c^{i-1}$.

We have

$$\begin{aligned}
\|y^{j+1} - y^j\| &= \left\| (w^{j+1} - w^j)_{|c} - dz^j \right\| \\
&\leq K\|dw^{j+1} - dw^j\| + \delta_j \\
&\leq 2K\varepsilon_j + \delta_j.
\end{aligned}$$

So y^j is a Cauchy sequence as long as we make sure $2K\varepsilon_j + \delta_j \leq 2^{-j}$ for instance. Since M_c^{i-1} is complete, this sequence converges to some y. Because $dy^j = dw^j_{|c}$, we get that $\|x_{|c} - dy^j\| \leq \varepsilon_j$ and in the limit $x_{|c} = dy$. □

Proposition 4.9. *Let $(M_c^\bullet)_{c\geq c_0}$ and $(M_c'^\bullet)_{c\geq c_0}$ be two admissible collections of complexes of complete normed abelian groups. For each $c \geq c_0$ let $f_c^\bullet : M_c^\bullet \to M_c'^\bullet$ be a collection of maps between these collections of complexes that are norm-nonincreasing and which all commute with all restriction maps, and assume that there exists these maps satisfy*

$$\|x_{|c}\| \leq K''\|f(x)\|$$

for all $i \leq m+1$ and all $x \in M^i_{k''c}$. Let $N_c^\bullet = M_c'^\bullet / M_c^\bullet$ be the collection of quotient complexes, with the quotient norm; this is again an admissible collection of complexes.

Assume that $M_c^\bullet$ (resp. $M_c'^\bullet$) is weakly $\leq k$-exact (resp. $\leq k'$-exact) in degrees $\leq m$ for $c \geq c_0$ with bound K (resp. K'). Then $N_c^\bullet$ is weakly $\leq kk'k''$-exact in degrees $\leq m-1$ for $c \geq c_0$ with bound $K'(KK''+1)$.

Proof. Let $n \in N^i_{kk'k''c}$ for $i \leq m-1$. We fix $\varepsilon > 0$. We need to find an element $y \in N_c^{i-1}$ such that

$$\|n_{|c} - dy\| \leq K'(KK''+1)\|dn\| + \epsilon.$$

Pick any preimage $m' \in M'^i_{kk'k''c}$ of n. In particular dm' is a preimage of dn. By definition of the quotient norm, we can find $m_1 \in M^{i+1}_{kk'k''c}$ and $m_1'' \in (M')^{i+1}_{kk'k''c}$ such that

$$dm' = f(m_1) + m_1''$$

with $\|m_1''\| \leq \|dn\| + \varepsilon_1$, for some positive ε_1 to be chosen later.

Applying the differential to the last displayed equation, and using that this kills the image of d, and that f is a map of complexes, we see that

$$f(dm_1) = -dm_1''.$$

Using the norm bound on f, we get

$$\begin{aligned}\|dm_{1|kk'c}\| &\leq K''\|f(dm_1)\| = K''\|dm_1''\| \\ &\leq K''\|m_1''\| \leq K''\|dn\| + K''\varepsilon_1.\end{aligned}$$

On the other hand, weak exactness of M applied to $m_{1|kk'c}$ gives $m_0 \in M^i_{k'c}$ such that

$$\|m_{1|kk'c|k'c} - dm_0\| \leq K\|dm_{1|kk'c}\| + \varepsilon_1$$

which combines with the previous estimate to give:

$$\|m_{1|k'c} - dm_0\| \leq KK''\,\|dn\| + (KK''+1)\varepsilon_1.$$

Now let $m'_{\text{new}} = m'_{|k'c} - f(m_0) \in M'^i_{k'c}$; this is a lift of $n_{|k'c}$. Then

$$dm'_{\text{new}} = dm'_{|k'c} - f(m_{1|k'c}) + f(m_{1|k'c} - dm_0) = m''_{1|k'c} + f(m_{1|k'c} - dm_0).$$

In particular,

$$\|dm'_{\text{new}}\| \leq (KK''+1)\,\|dn\| + (KK''+2)\varepsilon_1.$$

Now weak exactness of M' gives $x \in M_c'^{i-1}$ such that

$$\|m'_{\text{new}|c} - dx\| \leq K'\|dm'_{\text{new}}\| + \varepsilon_1 \leq K'((KK''+1)\,\|dn\| + (KK''+2)\varepsilon_1) + \varepsilon_1.$$

In particular, letting $y \in N_c^{i-1}$ be the image of x, we get

$$\|n_{|c} - dy\| \leq K'(KK''+1)\,\|dn\| + (K'(KK''+2)+1)\varepsilon_1,$$

which is exactly what we wanted if we choose $\varepsilon_1 = \varepsilon/(K'(KK''+2)+1)$. □

Proposition 4.10. *Let $(M_c^\bullet)_{c\geq c_0}$ and $(M_c'^\bullet)_{c\geq c_0}$ be two admissible collections of complexes of complete normed abelian groups. For $c \geq c_0$ let $f_c^\bullet : M_c^\bullet \to M_c'^\bullet$ be a collection of maps between these collections of complexes that is strictly compatible with the norm and commutes with restriction maps, and assume that it satisfies*

$$\|x_{|c}\| \leq K''\|f(x)\|$$

for all $i \leq m+1$ and all $x \in M^i_{k''c}$. Let $N_c^\bullet = M_c'^\bullet/\overline{M_c^\bullet}$ be the collection of quotient complexes, with the quotient norm; this is again an admissible collection of complexes.

Assume that $M_c^\bullet$ (resp. $M_c'^\bullet$) is $\leq k$-exact (resp. $\leq k'$-exact) in degrees $\leq m$ for $c \geq c_0$ with bound K (resp. K'). Then, for every $\delta > 0$, $N_c^\bullet$ is $\leq (kk'k'')^2$-exact in degrees $\leq m-1$ for $c \geq c_0$ with bound $K'(KK''+1)+\delta$.

Proof. The exactness assumptions on M and M' give the corresponding weak exactness condition. Hence Proposition 4.9 ensures that $N_c^\bullet$ is weakly $\leq kk'k''$-exact in degrees $\leq m-1$ for $c \geq c_0$ with bound $K'(KK''+1)$. Since $N_c^\bullet$ is a complex of complete groups, Lemma 4.8 gives the required exactness. □

Proposition 4.11. *Fix an integer $m \geq 0$ and constants k, K. Then there exists an $\epsilon > 0$ and constants k_0, K_0, depending (only) on k, K and m, with the following property.*

Consider an admissible system of double complexes $M_c^{p,q}$, $p, q \geq 0$, $c \geq c_0$, of complete normed abelian groups as well as some $k' \geq k_0$ and some $H > 0$, such that

$$\begin{array}{ccccccc}
M_c^{0,0} & \longrightarrow & M_c^{0,1} & \longrightarrow & M_c^{0,2} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
M_c^{1,0} & \longrightarrow & M_c^{1,1} & \longrightarrow & M_c^{1,2} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
M_c^{2,0} & \longrightarrow & M_c^{2,1} & \longrightarrow & \ddots & & \\
\downarrow & & \downarrow & & & & \\
\vdots & & \vdots & & & &
\end{array}$$

(1) *for $i = 0, \dots, m+1$, the rows $M_c^{i,q}$ are weakly $\leq k$-exact in degrees $\leq m-1$ for $c \geq c_0$ with bound K;*
(2) *for $j = 0, \dots, m$, the columns $M_c^{p,j}$ are weakly $\leq k$-exact in degrees $\leq m$ for $c \geq c_0$ with bound K;*
(3) *for $q = 0, \dots, m$ and $c \geq c_0$, there is a map $h_{k'c}^q : M_{k'c}^{0,q+1} \to M_c^{1,q}$ with*

$$\|h_{k'c}^q(x)\|_{M_c^{1,q}} \leq H\|x\|_{M_{k'c}^{0,q+1}}$$

for all $x \in M_c^{0,q+1}$, and such that for all $c \geq c_0$ and $x \in M_{k'^2c}^{0,q}$, one has

$$\|\mathrm{res}_{k'^2c,k'c}^{1,q}(d^{0,q}(x)) \pm h_{k'^2c}^q(d_{k'^2c}'^{0,q}(x)) \pm d_{k'c}'^{1,q-1}(h_{k'^2c}^{q-1}(x))\|_{M_{k'c}^{1,q}} \leq \epsilon\|\mathrm{res}_{k'^2c,c}^{0,q}(x)\|_{M_c^{0,q}}. \tag{4.1}$$

Then the first row is weakly $\leq k'^2$ exact in degrees $\leq m$ for $c \geq c_0$ with bound $2K_0H$.

We note that the bound on the homotopy is of a peculiar nature, in that the bound only depends on a deep restriction of x.

Proof. First, we treat the case $m = 0$. If $m = 0$, we claim that one can take $\epsilon = \frac{1}{2k}$ and $k_0 = k$. We have to prove exactness at the first step. Let $x_{k'^2c} \in M_{k'^2c}^{0,0}$ and denote $x_{k'c} = \mathrm{res}_{k'^2c,k'c}^{0,0}(x)$ and $x_c = \mathrm{res}_{k'^2c,c}^{0,0}(x)$. Then by assumption (2) (and $k' \geq k$), we have

$$\|x_c\|_{M_c^{0,0}} \leq k\|d_{k'c}^{0,0}(x_{k'c})\|_{M_{k'c}^{1,0}}.$$

On the other hand, by (3),

$$\|\mathrm{res}_{k'^2c,k'c}^{1,0}(d_{k'^2c}^{0,0}(x)) \pm h_{k'^2c}^0(d_{k'^2c}'^{0,0}(x))\|_{M_{k'c}^{1,0}} \leq \epsilon\|x_c\|_{M_c^{0,0}}.$$

In particular, noting that $\mathrm{res}^{1,0}_{k'^2c,k'c}(d^{0,0}_{k'^2c}(x)) = d^{0,0}_{k'c}(x_{k'c})$, we get

$$\|x_c\|_{M^{0,0}_c} \leq k\|d^{0,0}_{k'c}(x_{k'c})\|_{M^{1,0}_{k'c}} \leq k\epsilon\|x_c\|_{M^{0,0}_c} + kH\|d'^{0,0}_{k'^2c}(x)\|_{M^{0,1}_{k'^2c}}.$$

Thus, taking $\epsilon = \frac{1}{2k}$ as promised, this implies

$$\|x_c\|_{M^{0,0}_c} \leq 2kH\|d'^{0,0}_{k'^2c}(x)\|_{M^{0,1}_{k'^2c}}.$$

This gives the desired $\leq \max(k'^2, 2k_0H)$-exactness in degrees $\leq m$ for $c \geq c_0$.

Now we argue by induction on m. Consider the complex $N^{p,q}$ given by $M^{p,q+1}$ for $q \geq 1$ and $N^{p,0} = M^{p,1}/\overline{M^{p,0}}$ (the quotient by the closure of the image, which is also the completion of $M^{p,1}/M^{p,0}$), equipped with the quotient norm. Using the normed version of the snake lemma, Proposition 4.10 in the appendix to this lecture, one checks that this satisfies the assumptions for $m-1$, with k replaced by $\max(k^4, k^3+k+1)$. $\square$

5. More Breen–Deligne

We need the following results about the Breen-Deligne resolution for normed abelian groups. Let us consider here abelian groups M (in any topos) equipped with an increasing filtration $M_{\leq c} \subset M$ by subobjects indexed by the positive real numbers, such that $0 \in M_{\leq c}$, $-M_{\leq c} = M_{\leq c}$ and $M_{\leq c} + M_{\leq c'} \subset M_{\leq c+c'}$; we need no further conditions. Let us call these pseudo-normed abelian groups.

Fix a choice of a functorial Breen-Deligne resolution

$$C(M) : ... \to \mathbb{Z}[M^{a_i}] \to ... \to \mathbb{Z}[M^{a_1}] \to \mathbb{Z}[M^{a_0}] \to M \to 0$$

of an abelian group M; purely for notational convenience, we can and do assume that each term is of the form $\mathbb{Z}[M^{a_i}]$ (as opposed to a finite direct sum of such). The possibility of doing this follows from the proof of [Sch19, Theorem 4.10], noting that a functor of the form $A \mapsto \mathbb{Z}[A^n] \oplus \mathbb{Z}[A^m]$ admits a surjection from the functor $A \mapsto \mathbb{Z}[A^{n+m}] \oplus \mathbb{Z}$; this gives a resolution where all terms are of the form $\mathbb{Z}[A^{a_i}] \oplus \mathbb{Z}^m$. Now pass to the quotient of these complexes corresponding to the map $0 \to A$; this gives a complex all of whose terms are of the form $\mathbb{Z}[A^{a_i}]/\mathbb{Z}$. Noting that $\mathbb{Z}[A^{a_i}]$ is functorially isomorphic to $\mathbb{Z}[A^{a_i}]/\mathbb{Z} \oplus \mathbb{Z}$ (via splitting $0 \to A^{a_i} \to 0$), we can then add an acyclic complex of $\mathbb{Z}$'s in each degree to get a resolution all of whose terms are of the form $\mathbb{Z}[A^{a_i}]$.

We also need some homotopies. More precisely, we start with the following homotopy.

Lemma 5.1. *For an abelian group M, the maps σ_1, σ_2 from*

$$C(M^2) : ... \to \mathbb{Z}[M^{2a_i}] \to ... \to \mathbb{Z}[M^{2a_1}] \to \mathbb{Z}[M^{2a_0}]$$

to

$$C(M) : ... \to \mathbb{Z}[M^{a_i}] \to ... \to \mathbb{Z}[M^{a_1}] \to \mathbb{Z}[M^{a_0}],$$

induced by addition $M^2 \to M$, respectively the sum of the two maps induced by two projections $M^2 \to M$, are homotopic, via some functorial homotopy

$$h_i : \mathbb{Z}[M^{2a_i}] \to \mathbb{Z}[M^{a_{i+1}}].$$

If M is a pseudo-normed abelian group object in any topos, then σ_1 and σ_2 are well-defined as maps of complexes from

$$C(M^2)_{\leq c/2} : ... \to \mathbb{Z}[M^{2a_i}_{\leq c_i c/2}] \to ... \to \mathbb{Z}[M^{2a_1}_{\leq c_1 c/2}] \to \mathbb{Z}[M^{2a_0}_{\leq c/2}]$$

to

$$C(M)_{\leq c} : \ldots \to \mathbb{Z}[M^{a_i}_{\leq c_i c}] \to \ldots \to \mathbb{Z}[M^{a_1}_{\leq c_1 c}] \to \mathbb{Z}[M^{a_0}_{\leq c}]$$

for all $c > 0$. In that case, for all $i \geq 0$ there are universal constants c'_i such that h_i defines well-defined maps

$$\mathbb{Z}[M^{2a_i}_{\leq c_i c/2}] \to \mathbb{Z}[M^{a_{i+1}}_{\leq c'_i c_{i+1} c}]$$

for all $c > 0$.

Proof. This is a consequence of the proof of the existence of the Breen-Deligne resolution, proved in the same way as [Sch19, Proposition 4.17]. The existence of the constants c'_i is again formal, as in the last lemma. □

Now we need the following generalization to adding N elements.

Lemma 5.2. *Let N be a power of 2. The maps of complexes σ_1, σ_2 from*

$$C(M^N) : \ldots \to \mathbb{Z}[M^{Na_i}] \to \ldots \to \mathbb{Z}[M^{Na_1}] \to \mathbb{Z}[M^{Na_0}]$$

to

$$C(M) : \ldots \to \mathbb{Z}[M^{a_i}] \to \ldots \to \mathbb{Z}[M^{a_1}] \to \mathbb{Z}[M^{a_0}],$$

induced by addition $M^N \to M$, respectively the sum of the N maps induced by the N projections $M^N \to M$, are homotopic, via some functorial homotopy

$$h_i^N : \mathbb{Z}[M^{Na_i}] \to \mathbb{Z}[M^{a_{i+1}}]$$

which moreover satisfies the following bound, with the same constants $c'_0, c'_1, \ldots$ as in the previous lemma:

If M is a pseudo-normed abelian group object in any topos, then σ_1 and σ_2 are well-defined as maps of complexes from

$$C(M^N)_{\leq c/N} : \ldots \to \mathbb{Z}[M^{Na_i}_{\leq c_i c/N}] \to \ldots \to \mathbb{Z}[M^{Na_1}_{\leq c_1 c/N}] \to \mathbb{Z}[M^{Na_0}_{\leq c/N}]$$

to

$$C(M)_{\leq c} : \ldots \to \mathbb{Z}[M^{a_i}_{\leq c_i c}] \to \ldots \to \mathbb{Z}[M^{a_1}_{\leq c_1 c}] \to \mathbb{Z}[M^{a_0}_{\leq c}]$$

for all $c > 0$. In that case, h_i^N defines well-defined maps

$$\mathbb{Z}[M^{Na_i}_{\leq c_i c/N}] \to \mathbb{Z}[M^{a_{i+1}}_{\leq c'_i c_{i+1} c}]$$

for all $c > 0$.

Proof. Let $N = 2^m$. For each $j = 0, \ldots, m-1$, the two maps from $C(M^{2^{j+1}})$ to $C(M^{2^j})$ from the previous lemma are homotopic, and we use the homotopy from that lemma. Composing homotopies (which amounts concretely to a certain sum) we get the desired homotopy from $C(M^{2^m})$ to $C(M)$. It follows directly from this construction that the constants c'_i are unchanged. □

6. Polyhedral lattices

Definition 6.1. A *polyhedral lattice* is a finite free abelian group Λ equipped with a norm $\|\cdot\|_\Lambda : \Lambda\otimes\mathbb{R}\to\mathbb{R}$ (so $\Lambda\otimes\mathbb{R}$ is a Banach space) that is given by the supremum of finitely many linear functions on Λ; equivalently, the "unit ball" $\{\lambda\in\Lambda\otimes\mathbb{R} \mid \|\lambda\|_\Lambda\leq 1\}$ is a polyhedron.

Finally, we can prove the key combinatorial lemma, ensuring that any element of $\mathrm{Hom}(\Lambda,\overline{\mathcal{M}}_{r'}(S))$ can be decomposed into N elements whose norm is roughly $\frac{1}{N}$ of the original element.

Lemma 6.2. *Let Λ be a polyhedral lattice. Then for all positive integers N there is a constant d such that for all $c>0$ one can write any $x\in\mathrm{Hom}(\Lambda,\overline{\mathcal{M}}_{r'}(S))_{\leq c}$ as*

$$x=x_1+\ldots+x_N$$

where all $x_i\in\mathrm{Hom}(\Lambda,\overline{\mathcal{M}}_{r'}(S))_{\leq c/N+d}$.

As preparation for the proof, we have the following results.

Lemma 6.3 (Gordan's lemma)**.** *Let Λ be a finite free abelian group, and let $\lambda_1,\ldots,\lambda_m\in\Lambda$ be elements. Let $M\subset\mathrm{Hom}(\Lambda,\mathbb{Z})$ be the submonoid $\{x\mid x(\lambda_i)\geq 0$ for all $i=1,\ldots,m\}$. Then M is finitely generated as monoid.*

Proof. This is a standard result. We have not formalised it yet, but it is work in progress. ☐

Lemma 6.4. *Let Λ be a finite free abelian group, let N be a positive integer, and let $\lambda_1,\ldots,\lambda_m\in\Lambda$ be elements. Then there is a finite subset $A\subset\Lambda^\vee$ such that for all $x\in\Lambda^\vee=\mathrm{Hom}(\Lambda,\mathbb{Z})$ there is some $x'\in A$ such that $x-x'\in N\Lambda^\vee$ and for all $i=1,\ldots,m$, the numbers $x'(\lambda_i)$ and $(x-x')(\lambda_i)$ have the same sign, i.e. are both nonnegative or both nonpositive.*

Proof. It suffices to prove the statement for all x such that $\lambda_i(x)\geq 0$ for all i; indeed, applying this variant to all $\pm\lambda_i$, one gets the full statement.

Thus, consider the submonoid $\Lambda^\vee_+\subset\Lambda^\vee$ of all x that pair nonnegatively with all λ_i. This is a finitely generated monoid by Lemma 6.3; let $y_1,\ldots,y_M$ be a set of generators. Then we can take for A all sums $n_1y_1+\ldots+n_My_M$ where all $n_j\in\{0,\ldots,N-1\}$. ☐

Lemma 6.5. *Let $x_0,x_1,\ldots$ be a sequence of reals, and assume that $\sum_{i=0}^\infty x_i$ converges absolutely. For every natural number $N>0$, there exists a partition $\mathbb{N}=A_1\sqcup A_2\sqcup\cdots\sqcup A_N$ such that for each $j=1,\ldots,N$ we have $\sum_{i\in A_j}x_i\leq(\sum_{i=0}^\infty x_i)/N+1$*

Proof. Define the A_j recursively: assume that the natural numbers $0,\ldots,n$ have been placed into the sets $A_1,\ldots,A_N$. Then add the number $n+1$ to the set A_j for which

$$\sum_{i=0,i\in A_j}^{n} x_i$$

is minimal. ☐

Lemma 6.6. *For all natural numbers $N>0$, and for all $x\in\overline{\mathcal{M}}_{r'}(S)_{\leq c}$ one can decompose x as a sum*

$$x=x_1+\ldots+x_N$$

with all $x_i\in\overline{\mathcal{M}}_{r'}(S)_{\leq c/N+1}$.

Proof. Choose a bijection $S \times \mathbb{N} \cong \mathbb{N}$, and transport the result from Lemma 6.5. □

Proof of Lemma 6.2. Pick $\lambda_1, \dots, \lambda_m \in \Lambda$ generating the norm. We fix a finite subset $A \subset \Lambda^\vee$ satisfying the conclusion of the previous lemma. Write

$$x = \sum_{n \geq 1, s \in S} x_{n,s} T^n[s]$$

with $x_{n,s} \in \Lambda^\vee$. Then we can decompose

$$x_{n,s} = N x^0_{n,s} + x^1_{n,s}$$

where $x^1_{n,s} \in A$ and we have the same-sign property of the last lemma. Letting $x^0 = \sum_{n \geq 1, s \in S} x^0_{n,s} T^n[s]$, we get a decomposition

$$x = N x^0 + \sum_{a \in A} a x_a$$

with $x_a \in \overline{\mathcal{M}}_{r'}(S)$ (with the property that in the basis given by the $T^n[s]$, all coefficients are 0 or 1). Crucially, we know that for all $i = 1, \dots, m$, we have

$$\|x(\lambda_i)\| = N \|x^0(\lambda_i)\| + \sum_{a \in A} |a(\lambda_i)| \|x_a\|$$

by using the same sign property of the decomposition.

Using this decomposition of x, we decompose each term into N summands. This is trivial for the first term Nx^0, and each summand of the second term decomposes with $d = 1$ by Lemma 6.6. (It follows that in general one can take for d the supremum over all i of $\sum_{a \in A} |a(\lambda_i)|$.) □

Definition 6.7. Let Λ be a polyhedral lattice, and let $N > 0$ be a natural number. (We think of N as being fixed once and for all, and thus it does not show up in the notation below.)

By Λ' we denote Λ^N endowed with the norm

$$\|(\lambda_1, \dots, \lambda_N)\|_{\Lambda'} = \tfrac{1}{N}(\|\lambda_1\|_\Lambda + \dots + \|\lambda_N\|_\Lambda).$$

This is a polyhedral lattice.

Definition 6.8. For any $m \geq 1$, let $\Lambda'^{(m)}$ be given by $\Lambda'^m / \Lambda \otimes (\mathbb{Z}^m)_{\sum = 0}$; for $m = 0$, we set $\Lambda'^{(0)} = \Lambda$. Then $\Lambda'^{(\bullet)}$ is a cosimplicial polyhedral lattice, the Čech conerve of $\Lambda \to \Lambda'$.

In particular, $\Lambda'^{(0)} = \Lambda \to \Lambda' = \Lambda'^{(1)}$ is the diagonal embedding.

7. End of proof

Definition 7.1. Let $r, r' \in \mathbb{R}_{>0}$, and let V be an r-normed $\mathbb{Z}[T^{\pm 1}]$-module. Assume $r' \leq 1$. Let M be a profinitely filtered pseudo-normed group with r'-action of T^{-1}. Let $\mathrm{BD} = (n, f, h)$ be a Breen–Deligne package, and let $c' = (c'_0, c'_1, c'_2, \dots)$ be a sequence of constants in $\mathbb{R}_{\geq 0}$ that is BD-suitable.

For every $c \in \mathbb{R}_{\geq 0}$, the maps from Definition 3.6 induced by the universal maps f_i from the Breen–Deligne package (n, f, h) assemble into a complex of normed abelian groups

$$C^{\mathrm{BD}}_{c'}(M)^\bullet_c \colon \dots \to \widehat{V}(M^{n_i}_{\leq c_i})^{T^{-1}} \to \widehat{V}(M^{n_{i+1}}_{\leq c_{i+1}})^{T^{-1}} \to \dots \to 0.$$

Together, these complexes fit into a system of complexes with the natural restriction maps.

Now we state the following result, which is our main goal.

N.b.: It differs from Theorem 9.4 of [Sch20] only in one aspect: we assume that the sets S are finite, rather than profinite.

Theorem 7.2. *Let $BD = (n, f, h)$ be a Breen–Deligne package, and let $c' = (c'_0, c'_1, c'_2, \dots)$ be a sequence of constants in $\mathbb{R}_{\geq 0}$ that is BD-suitable. Fix radii $1 > r' > r > 0$. For any m there is some k and c_0 such that for all finite sets S and all r-normed $\mathbb{Z}[T^{\pm 1}]$-modules V, the system of complexes*

$$C^{BD}_{c'}(\overline{\mathcal{M}}_{r'}(S))^{\bullet}_c \colon \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c})^{T^{-1}} \to \widehat{V}(\overline{\mathcal{M}}_{r'}(S)^2_{\leq c'_1 c})^{T^{-1}} \to \dots$$

is $\leq k$-exact in degrees $\leq m$ for $c \geq c_0$.

Remark 7.3. Note: the text below is copied almost verbatim from [Sch20]**. Small parts have been formalized. We expect that the text will be rewritten and expanded as the formalization project progresses.**

We will prove Theorem 7.2 by induction on m. Unfortunately, the induction requires us to prove a stronger statement.

Endow $\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))$ with the subspaces

$$\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c} = \{f : \Lambda \to \overline{\mathcal{M}}_{r'}(S) \mid \forall x \in \Lambda, f(x) \in \overline{\mathcal{M}}_{r'}(S)_{\leq c\|x\|}\}.$$

As Λ is polyhedral, it is enough to check the given condition for finitely many x.

Now we claim the following generalization of Theorem 7.2.

Theorem 7.4. *Fix radii $1 > r' > r > 0$. For any m there is some k such that for all polyhedral lattices Λ there is a constant $c_0(\Lambda) > 0$ such that for all finite sets S and all r-normed $\mathbb{Z}[T^{\pm 1}]$-modules V, the system of complexes*

$$C^{\bullet}_{\Lambda,c} \colon \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^2_{\leq c_1 c})^{T^{-1}} \to \dots$$

is $\leq k$-exact in degrees $\leq m$ for $c \geq c_0(\Lambda)$.

Proof. Use $\Lambda = \mathbb{Z}$, and the isomorphism $\mathrm{Hom}(\mathbb{Z}, A) \cong A$. □

We note that the constants $c_1, c_2, \dots$ implicit in the choice of the complex are chosen once and for all (after fixing r and r'), and it can be ensured that the transition maps in the complex are norm-nonincreasing. Indeed, if the c_i are suitable as in Definition 1.14, the maps

$$\widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{a_i}_{\leq c_i c}) \to \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{a_{i+1}}_{\leq c_{i+1} c})$$

will have bounded norm, independently of V (as they are a certain universal finite sum of maps induced by maps between the profinite sets in paranthesis, each of which induces a map of norm bounded by 1), so on the subspace of T^{-1}-invariants, one can shrink the norm down to 1 by shrinking c_{i+1}. We make and fix this choice of the c_i for the statement of Theorem 7.4, and the rest of the proof.

Proof of Theorem 7.4. We argue by induction on m, so assume the result for $m - 1$ (this is no assumption for $m = 0$, so we do not need an induction start). This gives us some $k > 1$ for which the statement of Theorem 7.4 holds true for $m - 1$; if $m = 0$, simply take any $k > 1$. In the proof below, we will increase k further in a way that depends only on m and r. After this modified choice of k, we fix ϵ and k_0 as provided by Proposition 4.11. Moreover, we let k' be the supremum of k_0

and the c_i' from Lemma 5.1 (and 5.2) for $i = 0, \ldots, m$. Finally, choose a positive integer b so that $2k'(\frac{r}{r'})^b \leq \epsilon$, and let N be the minimal power of 2 that satisfies

$$k'/N \leq (r')^b.$$

Then in particular $r^b N \leq \frac{2}{k'}(\frac{r}{r'})^b \leq \epsilon$.

Consider the cosimplicial polyhedral lattice from Definition 6.8. In particular, for any $c > 0$, we have

$$\mathrm{Hom}(\Lambda'^{(m)}, \overline{\mathcal{M}}_{r'}(S))_{\leq c} = \mathrm{Hom}(\Lambda', \overline{\mathcal{M}}_{r'}(S))_{\leq c}^{m/\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c}},$$

the m-fold fibre product of $\mathrm{Hom}(\Lambda', \overline{\mathcal{M}}_{r'}(S))_{\leq c}$ over $\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c}$; and

$$\mathrm{Hom}(\Lambda', \overline{\mathcal{M}}_{r'}(S))_{\leq c} = \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^N_{\leq c/N},$$

with the map to $\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c}$ given by the sum map.

Consider the collection of double complexes $C^\bullet_{\Lambda'^{(\bullet)},c}$ associated to this cosimplicial polyhedral lattice by Dold-Kan. Up to rescaling the norms in the complex for $\Lambda'^{(m)}$ by a universal constant (something like $(m+2)!$), the differentials are strictly compatible with norms (as they are an alternating sum of $m+1$ face maps, all of which are of norm ≤ 1), so this collection of normed double complexes is admissible. By induction, the first condition of Proposition 4.11 is satisfied for all $c \geq c_0$ with c_0 large enough (depending on Λ but not V or S). By Lemma 6.2, and noting that $\mathrm{Hom}(\Lambda'^{(\bullet)}, \overline{\mathcal{M}}_{r'}(S))_{\leq c}$ is the Čech nerve of

$$\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^N_{\leq c/N} \xrightarrow{\Sigma} \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c},$$

also the second condition is satisfied, with k the maximum of the previous k and some constant depending only on m and r, provided we take c_0 large enough so that $(k-1)r'c_i c_0/N$ is at least the d of Lemma 6.2 for all $i = 0, \ldots, m$ (so this choice of c_0 again depends on Λ). Indeed, then one can splice a surjection of profinite sets between the maps

$$\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{Na}_{\leq c_i c/N} \to \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^a_{\leq c_i c}$$

and

$$\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{Na}_{\leq k c_i c/N} \to \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^a_{\leq k c_i c},$$

and so the transition map between the columns of that double complex factors over a similar complex arising from a simplicial hypercover of profinite sets, so the constants are bounded by Proposition 8.4, Lemma 8.5, and Proposition 4.10 (plus probably some other results of which we need to work out the details). At this point, we have finalized our choice of k (and, as promised, this choice depended only on m and r), and so we also finalized the constants ϵ, k' and N from the first paragraph of the proof.

Finally, to check the third condition, we use Lemma 5.2 to find, in degrees $\leq m$, a homotopy between the two maps from the first row

$$\widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^2_{\leq c_1 c})^{T^{-1}} \to \ldots$$

to the second row

$$\widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^N_{\leq c/N})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{2N}_{\leq c_1 c/N})^{T^{-1}} \to \ldots$$

respectively induced by the addition $\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^N_{\leq c/N} \to \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c}$ (which is the map that forms part of the double complex), and the map that is the sum of the N maps induced by the N projection maps

$$\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^N_{\leq c/N} \to \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c/N} \subset \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c}.$$

By Lemma 5.2, we can find this homotopy between the complex for $k'c$ and the complex for c, by our choice of $k' \geq c'_i$ for $i = 0, \ldots, m$. As N is fixed, the homotopy is the universal homotopy from Lemma 5.2, and in particular its norm is bounded by some universal constant H.

Finally, it remains to establish the estimate (4.1) on the homotopic map. We note that this takes $x \in \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{a_i}_{\leq k'^2 c_i c})^{T^{-1}}$ (with $i = q$ in the notation of (4.1)) to the element

$$y \in \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{Na_i}_{\leq k' c_i c/N})^{T^{-1}}$$

that is the sum of the N pullbacks along the N projection maps $\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{Na_i}_{\leq k' c_i c/N} \to \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{a_i}_{\leq k'^2 c_i c}$. We note that these actually take image in $\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{a_i}_{\leq c_i c}$ as $N \geq k'$, so this actually gives a well-defined map

$$\widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{a_i}_{\leq c_i c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{Na_i}_{\leq k' c_i c/N})^{T^{-1}}.$$

We need to see that this map is of norm $\leq \epsilon$. Now note that by our choice of N, we actually have $k' c_i c/N \leq (r')^b c_i c$, so this can be written as the composite of the restriction map

$$\widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{a_i}_{\leq c_i c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{a_i}_{\leq (r')^b c_i c})^{T^{-1}}$$

and

$$\widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{a_i}_{\leq (r')^b c_i c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{Na_i}_{\leq k' c_i c/N})^{T^{-1}}.$$

The first map has norm exactly r^b, by T^{-1}-invariance, and as multiplication by T scales the norm with a factor of r on $\widehat{V}$.[1] The second map has norm at most N (as it is a sum of N maps of norm ≤ 1). Thus, the total map has norm $\leq r^b N$. But by our choice of N, we have $r^b N \leq \epsilon$, giving the result.

Thus, we can apply Proposition 4.11, and get the desired $\leq \max(k'^2, 2k_0 H)$-exactness in degrees $\leq m$ for $c \geq c_0$, where k', k_0 and H were defined only in terms of k, m, r' and r, while c_0 depends on Λ (but not on V or S). This proves the inductive step. □

Question 7.5. Can one make the constants explicit, and how large are they?[2] Modulo the Breen-Deligne resolution, all the arguments give in principle explicit constants; and actually the proof of the existence of the Breen-Deligne resolution should be explicit enough to ensure the existence of bounds on the c_i and c'_i.

This completes the proof of all results announced so far.

[1] Here is where we use $r' > r$, ensuring different scaling behaviour of the norm on source and target.

[2] A back of the envelope calculation seems to suggest that k is roughly doubly exponential in m, and that N has to be taken of roughly the same magnitude.

8. Relevant material that should move to a better place

We often use the following exactness property:

Proposition 8.1. *Let $M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_2 \xrightarrow{d_2} M_3$ be a four-term complex of bounded maps of normed abelian groups. Assume that, for some positive constants C and D, for all $y \in \ker(d_1 : M_1 \to M_2)$ there is some $x \in M_0$ with $d_0(x) = y$ and $\|x\| \leq C\|y\|$, and similarly for all $z \in \ker(d_2 : M_2 \to M_3)$, there is some $y \in M_1$ with $d_1(y) = z$ and $\|y\| \leq D\|z\|$.*

Then $\widehat{M}_0 \xrightarrow{\hat{d}_0} \widehat{M}_1 \xrightarrow{\hat{d}_1} \widehat{M}_2 \xrightarrow{\hat{d}_2} \widehat{M}_3$ is a complex, and for all $\hat{y} \in \widehat{M}_1$ and all $\epsilon > 0$ there is some $\hat{x} \in \widehat{M}_0$ with $\hat{d}_1(\hat{x}) = \hat{y}$ and $\|\hat{x}\| \leq (C + \epsilon)\|\hat{y}\|$.

Proof. First, we claim that $\ker(d_1 : M_1 \to M_2)$ is dense in $\ker(\hat{d}_1 : \widehat{M}_1 \to \widehat{M}_2)$. Pick any $\delta > 0$ and take $y \in M_1$ such that $\|\hat{y} - y\| \leq \delta$. Let $z = d_1(y) \in M_2$, which has norm $\|z\| = \|d_1(y)\| = \|d_1(y - \hat{y})\|$ bounded by $C_{d_1}\delta$, where C_{d_1} is the norm of d_1. We can thus find some $y' \in M_1$ with $\|y'\| \leq DC_{d_1}\delta$ and $d_1(y') = z$. Replacing y by $y - y'$, we can thus find $y \in \ker(d_1 : M_1 \to M_2)$ such that still $\|\hat{y} - y\| \leq (1 + DC_{d_1})\delta$; as δ was arbitrary, this gives the desired density.

This implies that one can write $\hat{y}$ as a sum $y_0 + y_1 + \ldots$ with $y_i \in \ker(d_1)$ and $\|y_i\| \leq \epsilon_i$ for $i > 0$ for any given sequence of positive numbers $\epsilon_1 \geq \epsilon_2 \geq \ldots$. Indeed, we can inductively choose the y_i so that $\|\hat{y} - y_0 - \ldots - y_i\| \leq \frac{1}{2}\epsilon_{i+1}$, in which case $\|y_i\| \leq \frac{1}{2}(\epsilon_i + \epsilon_{i+1}) \leq \epsilon_i$. Taking the sequence of ϵ_i's sufficiently small so that $\sum_{i>0} \epsilon_i \leq \frac{\|\hat{y}\|}{2C}\epsilon$, we can lift all y_i to x_i with $\|x_i\| \leq C\|y_i\|$, and then $\hat{x} = x_0 + x_1 + \ldots$ maps to $\hat{y}$ and satisfies

$$\|\hat{x}\| \leq \|x_0\| + C\sum_{i>0} \epsilon_i \leq C\|y_0\| + C\sum_{i>0} \epsilon_i \leq C\|\hat{y}\| + 2C\sum_{i>0} \epsilon_i \leq (C + \epsilon)\|\hat{y}\|. \qquad \square$$

Proposition 8.2. *Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be maps of normed abelian groups with $gf = 0$ and such that for all $m \in \ker(M \to M'')$ there is some $m' \in M'$ with $f(m') = m$ with $\|m'\| \leq C\|m\|$.*

Then $\widehat{M'} \xrightarrow{\hat{f}} \widehat{M} \xrightarrow{\hat{g}} \widehat{M''}$ satisfy $\hat{g}\hat{f} = 0$ and for all $m \in \ker(\widehat{M} \to \widehat{M''})$ and all $\epsilon > 0$ there is some $m' \in \widehat{M'}$ with $\hat{f}(\widehat{m'}) = \widehat{m}$ and $\|m'\| \leq (C + \epsilon)\|m\|$.

Proof. One can write any element $m \in \widehat{M}$ as a sum $m_0 + m_1 + \ldots$ with $m_i \in M$ and $\|m_i\| \leq \epsilon_i$ for $i > 0$ for any given sequence of positive numbers $\epsilon_1 \geq \epsilon_2 \geq \ldots$. Indeed, we can inductively choose the m_i so that $\|m - m_0 - \ldots - m_i\| \leq \frac{1}{2}\epsilon_{i+1}$, in which case $\|m_i\| \leq \frac{1}{2}(\epsilon_i + \epsilon_{i+1}) \leq \epsilon_i$. Taking the sequence of ϵ_i's sufficiently small so that $\sum_{i>0} \epsilon_i \leq \frac{\|m\|}{2C}\epsilon$, we can lift all m_i to m'_i with $\|m'_i\| \leq C\|m_i\|$, and then $m' = m'_0 + m'_1 + \ldots$ satisfies

$$\|m'\| \leq \|m'_0\| + C\sum_{i>0} \epsilon_i \leq C\|m_0\| + C\sum_{i>0} \epsilon_i \leq C\|m\| + 2C\sum_{i>0} \epsilon_i \leq (C + \epsilon)\|m\|.$$

$\square$

See Definition 3.1 for the definition of $\widehat{M}$.

Proposition 8.3. *The condensed abelian group $\widehat{M}$ is canonically identified with the condensed abelian group associated to the topological abelian group $\widehat{M}_{\mathrm{top}}$ given by the completion of M equipped with the topology induced by the norm. The norm defines a natural map of condensed sets*

$$\|\cdot\| : \widehat{M} \to \mathbb{R}_{\geq 0}.$$

Proof. Note that in the supremum norm any continuous function from S to $\widehat{M}_{\mathrm{top}}$ can be approximated by locally constant functions arbitrarily well, and that the space of continuous functions from S to $\widehat{M}_{\mathrm{top}}$ is complete with respect to the supremum norm. That $\|\cdot\|$ defines a map of condensed sets $\widehat{M} \to \mathbb{R}_{\geq 0}$ follows for example from this identification with $\underline{\widehat{M}_{\mathrm{top}}}$, as the norm is by definition a continuous map $\widehat{M}_{\mathrm{top}} \to \mathbb{R}_{\geq 0}$. □

Proposition 8.4. *For any hypercover $S_\bullet \to S$ of a profinite set S by profinite sets S_i, the complex*

$$0 \to \widehat{M}(S) \to \widehat{M}(S_0) \to \widehat{M}(S_1) \to \ldots$$

is exact, and whenever $f \in \ker(\widehat{M}(S_m) \to \widehat{M}(S_{m+1}))$ with $\|f\| \leq c$, then for any $\epsilon > 0$ there is some $g \in \widehat{M}(S_{m-1})$ with $\|g\| \leq (1+\epsilon)c$ such that $d(g) = f$.

Proof. Follow the proof of [Sch19, Theorem 3.3]: When S and all S_i are finite, the hypercover splits, so a contracting homotopy gives the result with constant 1. In general, write the hypercover as a cofiltered limit of hypercovers of finite sets by finite sets, pass to the filtered colimit, and complete, using Proposition 8.2. □

Lemma 8.5. *Let M be a profinitely filtered pseudo-normed group with action of T^{-1}. For any r-normed $\mathbb{Z}[T^{\pm 1}]$-module V, any $c > 0$ and any a, the map*

$$\widehat{V}(M^a_{\leq c}) \xrightarrow{T^{-1} - [T^{-1}]^*} \widehat{V}(M^a_{\leq r'c})$$

is surjective, has norm bounded by $r^{-1} + 1$, and for any $f \in \widehat{V}(M^a_{\leq r'c})$ and $\epsilon > 0$ there is some $g \in \widehat{V}(M^a_{\leq c})$ with $f(x) = T^{-1}g(x) - g(T^{-1}x)$ and $\|g\| \leq \frac{r}{1-r}(1+\epsilon)\|f\|$.

Proof. Given $f : M^a_{\leq r'c} \to \widehat{V}$, choose an extension to a map $\tilde{f} : M^a \to \widehat{V}$ with $\|\tilde{f}\| \leq (1+\epsilon)\|f\|$. Such an extension exists: By induction (and using a sequence of ϵ_n's with $\prod_n (1+\epsilon_n) \leq 1+\epsilon$), it suffices to see that for any closed immersion $A \subset B$ of profinite sets and a map $f_A : A \to \widehat{V}$, there is an extension $f_B : B \to \widehat{V}$ of f_A with $\|f_B\| \leq (1+\epsilon)\|f_A\|$. To see this, write f_A as a (fast) convergent sum of maps that factor over a finite quotient of A; for maps factoring over a finite quotient of A, the extension is clear (and can be done in a norm-preserving way), as any map from A to a finite set can be extended to a map from B to the same finite set.

Given $\tilde{f}$, we can now define $g : M^a_{\leq c}$ by

$$g(x) = T\tilde{f}(x) + T^2\tilde{f}(T^{-1}x) + \ldots + T^{n+1}\tilde{f}(T^{-n}x) + \ldots \in \widehat{V};$$

then $\|g\| \leq \frac{r}{1-r}\|\tilde{f}\| \leq \frac{r}{1-r}(1+\epsilon)\|f\|$. □

References

[Sch19] P. Scholze. Lectures on Condensed Mathematics. 2019.
[Sch20] P. Scholze. Lectures on Analytic Geometry. 2020.