BLUEPRINT FOR THE LIQUID TENSOR EXPERIMENT

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Remark 0.1. This text is based on the lecture notes on Analytic Geometry [Sch20], by Peter Scholze. The final section is copy-pasted from those lecture notes almost verbatim. This text is meant as a blueprint for the Liquid Tensor Experiment.

Remark 0.2. In this text $\mathbf{N}$ denotes the natural numbers *including* 0.

1. Breen–Deligne data

The goal of this section is to a give a precise statement of a variant of the Breen–Deligne resolution. This variant is not actually a resolution, but it is sufficient for our purposes, and is much easier to state and prove.

We first recall the original statement of the Breen–Deligne resolution.

Theorem 1.1 (Breen–Deligne)**.** *For an abelian group A, there is a resolution, functorial in A, of the form*

$$\dots \to \bigoplus_{i=1}^{n_i} \mathbb{Z}[A^{r_{ij}}] \to \dots \to \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \to \mathbb{Z}[A^2] \to \mathbb{Z}[A] \to A \to 0.$$

What does a homomorphism $f\colon \mathbb{Z}[A^m] \to \mathbb{Z}[A^n]$ that is functorial in A look like? We should perhaps say more precisely what we mean by this. The idea is that m and n are fixed, and for each abelian group A we have a group homomorphism $f_A\colon \mathbb{Z}[A^m] \to \mathbb{Z}[A^n]$ uch that if $\phi\colon A \to B$ is a group homomorphism inducing $\phi_i\colon \mathbf{Z}[A^i] \to \mathbf{Z}[B^i]$ for each natural number i then the obvious square commutes: $\phi_n \circ f_A = f_B \circ \phi_m$.

The map f_A is specified by what it does to the generators $(a_1, a_2, a_3, \dots, a_m) \in A^m$. It can send such an element to an arbitrary element of $\mathbb{Z}[A^n]$, but one can check that universality implies that f_A will be a $\mathbb{Z}$-linear combination of "basic universal maps", where a "basic universal map" is one that sends $(a_1, a_2, \dots, a_m)$ to $(t_1, \dots, t_n)$, where t_i is a $\mathbb{Z}$-linear combination $c_{i,1} \cdot a_1 + \dots + c_{i,m} \cdot a_m$. So a "basic universal map" is specified by the $n \times m$-matrix c.

Definition 1.2. A *basic universal map* from exponent m to n, is an $n \times m$-matrix with coefficients in $\mathbb{Z}$.

Definition 1.3. A *universal map* from exponent m to n, is a formal $\mathbb{Z}$-linear combination of basic universal maps from exponent m to n.

If f is a basic universal map, then we write $[f]$ for the corresponding universal map.

Definition 1.4. Let $f = \sum_g n_g[g]$ be a universal map. We say that f is *bound by* a natural number N if $\sum_g |n_g| \le N$.

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We point out that basic universal maps can be composed by matrix multiplication, and this formally induces a composition of universal maps. As mentioned above, one can also check (this has been formalised in Lean) that this construction gives a bijection between universal maps from exponent m to n and functorial collections $f_A \colon \mathbf{Z}[A^m] \to \mathbf{Z}[A^n]$.

Definition 1.5. In other words, we are considering the following two categories:

- the category whose objects are natural numbers, and whose morphisms are matrices;
- the category with the same objects, but with Hom-sets replaced by the free abelian groups generated by the sets of matrices. We denote this latter category FreeMat.

Both categories naturally come with a monoidal structure: for the first it is given by the Kronecker product of matrices (a.k.a. tensor product of linear maps) which induces a monoidal structure on FreeMat. As usual, we will denote this monoidal structure $\_ \otimes \_$. For example, if f is a basic universal map, then $2 \otimes f$ denotes the block matrix

$$\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$$

Definition 1.6. Let N be a natural number, and $i < N$. Then $\pi'_{N,i}$ denotes the basic universal map from exponent N to 1

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = (a_0, a_1, \dots, a_{N-1})^t$$

where $a_j = \delta_{ij}$.

Definition 1.7. Let N and n be natural numbers. Then π^N_n denotes the universal map from exponent $N \cdot n$ to n given by $\sum_{i<N} [\pi'_{N,i} \otimes n]$.

(On $\mathbf{Z}[A^{N \cdot n}] \to \mathbf{Z}[A^n]$ this map is the formal sum of the maps $\mathbf{Z}[A^{N \cdot n}] \to \mathbf{Z}[A^n]$ induced by the projection maps $A^{N \cdot n} = (A^n)^N \to A^n$.)

Definition 1.8. Let N and n be natural numbers. Then σ^N_n denotes the universal map from exponent $N \cdot n$ to n given by $[\sum_{i<N} \pi'_{N,i} \otimes n]$.

(On $\mathbf{Z}[A^{N \cdot n}] \to \mathbf{Z}[A^n]$ this map is induced by the summation map $A^{N \cdot n} = (A^n)^N \to A^n$.)

Definition 1.9. A *Breen–Deligne data* is a chain complex in FreeMat.

Concretely, this means that it consists of a sequence of exponents $n_0, n_1, n_2, \dots \in \mathbb{N}$, and universal maps f_i from exponent n_{i+1} to n_i, such that for all i we have $f_i \circ f_{i+1} = 0$.

A morphism of Breen–Deligne data is a morphism of chain complexes.

Definition 1.10. For every natural numbers N, the endofunctor $N \otimes \_$ on FreeMat induces an endofunctor of Breen–Deligne data.

Concretely, it maps a pair (n, f) of Breen–Deligne data, to the pair $N \otimes (n, f)$ consisting of exponents $N \cdot n_i$ and universal maps $N \otimes f_i$.

Let BD be Breen–Deligne data. The universal maps σ^N and π^N defined above, induce morphisms $\sigma^N_{\mathsf{BD}}, \pi^N_{\mathsf{BD}} \colon N \otimes \mathsf{BD} \to \mathsf{BD}$.

Definition 1.11. A *Breen–Deligne* package consists of Breen–Deligne data BD together with a homotopy h between π^2_{BD} and σ^2_{BD}.

Definition 1.12. Let BD be a Breen–Deligne package and N a power of 2. Then the homotopy h induces a homotopy between π^N_{BD} and σ^N_{BD} by iterative composition of the homotopy packaged in BD.

Definition 1.13. We will now construct an example of a Breen–Deligne package. In some sense, it is the "easiest" solution to the conditions posed above. The exponents will be $n_i = 2^i$, and the homotopies h_i will be the identity. Under these constraints, we recursively construct the universal maps f_i:

$$f_0 = \pi^2_1 - \sigma^2_1, \quad f_{i+1} = (\pi^2_{2^{i+1}} - \sigma^2_{2^{i+1}}) - (2 \otimes f_i).$$

We leave it as exercise for the reader, to verify that with these definitions (n, f, h) forms a Breen–Deligne package.

We now make definitions that will make precise some conditions between constants that will be needed when we construct Breen–Deligne complexes of normed abelian groups.

Definition 1.14. Let f be a basic universal map from exponent m to n. Let $c_1, c_2 \in \mathbb{R}_{\geq 0}$. We say that (c_1, c_2) is *f-suitable*, if for all i

$$\sum_j c_1 |f_{ij}| \leq c_2.$$

To orient the reader: later on we will be considering maps on normed abelian groups induced from universal maps, and this inequality will guarantee that if $\|m\| \leq c_1$ then $\|f(m)\| \leq c_2$.

Definition 1.15. Let f be a universal map from exponent m to n. Let $c_1, c_2 \in \mathbb{R}_{\geq 0}$. We say that (c_1, c_2) is *f-suitable*, if for all basic universal maps g that occur in the formal sum f, the pair of nonnegative reals (c_1, c_2) is g-suitable.

Definition 1.16. Let f be a universal map and let $r, r', c_1, c_2 \in \mathbb{R}_{\geq 0}$. We say that (c_1, c_2) is *very suitable* for (f, r, r') if there exist $N, b \in \mathbf{N}$ and $c' \in \mathbf{R}_{\geq 0}$ such that:

- f is bound by N (see Definition 1.4)
- (c_1, c') is f-suitable
- $r^b N \leq 1$
- $c' \leq (r')^b c_2$

Definition 1.17. Let $\mathsf{BD} = (n, f)$ be Breen–Deligne data, let $r, r' \in \mathbf{R}_{\geq 0}$, and let $\kappa = (\kappa_0, \kappa_1, \dots)$ be a sequence of nonnegative real numbers. We say that κ is BD-*suitable* (resp. *very suitable* for (BD, r, r')), if for all i, the pair (κ_{i+1}, κ_i) is f_i-suitable (resp. *very suitable* for (f_i, r, r')).

(Note! The order (κ_{i+1}, κ_i) is contravariant compared to Definition 1.15. This is because of the contravariance of $\widehat{V}(\_)$; see Definition 5.9.)

Definition 1.18. Let BD be a Breen–Deligne package with data (n, f) and homotopy h. Let κ, κ' be sequences of nonnegative real numbers. (In applications κ is a (n, f)-suitable sequence.)

Then κ' is *adept* to (BD, κ) if for all i the pair $(\kappa_i/2, \kappa'_{i+1}\kappa_{i+1})$ is h_i-suitable. (Recall that h_i is the homotopy map $n_i \to n_{i+1}$.)

Lemma 1.19. *Let BD be a Breen–Deligne package, N a power of 2, and let κ, κ' be sequences of nonnegative real numbers. Assume that κ' is adept to (BD, κ). Let h^N be the homotopy between π^N_{BD} and σ^N_{BD} defined in Def 1.12.*

For all i, the pair $(\kappa_i/N, \kappa'_{i+1}\kappa_{i+1})$ is h^N_i-suitable.

Proof. Omitted. (But done in Lean.) □

Lemma 1.20. *Let BD be a Breen–Deligne package, and let r, r' be nonnegative reals, such that $r < 1$ and $r' > 0$.*

There exists a sequence κ of positive real numbers such that κ is very suitable for (BD, r, r').

Proof. The sequence can be constructed recursively, which we leave as exercise for the reader. (It has been done in Lean.) □

Lemma 1.21. *Let BD be a Breen–Deligne package, and let r, r' be nonnegative reals, such that $0 < r < 1$ and $0 < r' \leq 1$. Let κ be any sequence of positive reals.*

There exists a sequence κ' of nonnegative real numbers κ' is adept to (BD, κ).

Proof. The sequence can be constructed recursively, which we leave as exercise for the reader. (It has been done in Lean.) □

2. Variants of normed groups

Remark 2.1. Normed groups are well-studied objects. In this text it will be helpful to work with the more general notion of *semi-normed group*. This drops the separation axiom $\|x\| = 0 \iff x = 0$ but is otherwise the same as a normed group.

The main difference is that this includes "uglier" objects, but creates a "nicer" category: semi-normed groups need not be Hausdorff, but quotients by arbitrary (possibly non-closed) subgroups are naturally semi-normed groups.

Nevertheless, there is the occasional use for the more restrictive notion of normed group, when we come to polyhedral lattices below (see Section 6.

In this text, a morphism of (semi)-normed groups will always be bouned. If the morphism is supposed to be norm-nonincreasing, this will be mentioned explicitly.

Definition 2.2. Let $r > 0$ be a real number. An *r-normed $\mathbb{Z}[T^{\pm 1}]$-module* is a semi-normed group V endowed with an automorphism $T \colon V \to V$ such that for all $v \in V$ we have $\|T(v)\| = r\|v\|$.

The remainder of this text sets up some algebraic variants of semi-normed groups.

Definition 2.3. A *pseudo-normed group* is an abelian group $(M, +)$, together with an increasing filtration $M_c \subseteq M$ of subsets M_c indexed by $\mathbb{R}_{\geq 0}$, such that each M_c contains 0, is closed under negation, and $M_{c_1} + M_{c_2} \subseteq M_{c_1+c_2}$. An example would be $M = \mathbb{R}$ or $M = \mathbb{Q}_p$ with $M_c := \{x : |x| \leq c\}$.

A pseudo-normed group M is *profinitely filtered* if each of the sets M_c is endowed with a topological space structure making it a profinite set, such that following maps are all continuous:

- the inclusion $M_{c_1} \to M_{c_2}$ (for $c_1 \leq c_2$);
- the negation $M_c \to M_c$;
- the addition $M_{c_1} \times M_{c_2} \to M_{c_1+c_2}$.

A *morphism* of profinitely filtered pseudo-normed groups $M \to N$ is a group homomorphism f that is

- *bounded*: there is a constant C such that $x \in M_c$ implies $f(x) \in N_{Cc}$;
- *continuous*: for one (or equivalently all) constants C as above, the induced map $M_c \to N_{Cc}$ is a morphism of profinite sets, i.e. continuous.

The reason the two definitions are equivalent is that a continuous injection between profinite sets must be a topological embedding.

Definition 2.4. Let r' be a positive real number. A profinitely filtered pseudo-normed group M has an *r'-action of T^{-1}* if it comes endowed with a distinguished morphism of profinitely filtered pseudo-normed groups $T^{-1}\colon M \to M$ that is bounded by r'^{-1}: if $x \in M_c$ then $T^{-1}x \in M_{c/r'}$.

A morphism $M \to N$ of profinitely filtered pseudo-normed groups with r'-action of T^{-1} is a morphism of profinitely filtered pseudo-normed groups f that commutes with the action of T^{-1} and is *strict*: if $x \in M_c$ then $f(x) \in N_c$.

3. Spaces of convergent power series

We will now construct the central example of profinitely filtered pseudo-normed groups with r'-action of T^{-1}.

Definition 3.1. Let $r' > 0$ be a real number, and let S be a finite set. Denote by $\overline{\mathcal{M}}_{r'}(S)$ the set

$$\left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \in T\mathbf{Z}[[T]] \right)_{s \in S} \;\middle|\; \sum_{n \geq 1, s \in S} |a_{n,s}| (r')^n < \infty \right\}.$$

Note that $\overline{\mathcal{M}}_{r'}(S)$ is naturally a pseudo-normed group with filtration given by

$$\overline{\mathcal{M}}_{r'}(S)_{\leq c} = \left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} \;\middle|\; \sum_{n \geq 1, s \in S} |a_{n,s}| (r')^n \leq c \right\}.$$

Lemma 3.2. *Let $r' > 0$ and $c \geq 0$ be real numbers, and let S be a finite set. The space $\overline{\mathcal{M}}_{r'}(S)_{\leq c}$ is the profinite limit of the finite sets*

$$\overline{\mathcal{M}}_{r'}(S)_{\leq c, \leq N} = \left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} \;\middle|\; \sum_{1 \leq n \leq N, s \in S} |a_{n,s}| (r')^n \leq c \right\}$$

This endows $\overline{\mathcal{M}}_{r'}(S)_{\leq c}$ with the profinite topology. In particular, it is a profinitely filtered pseudo-normed group.

Proof. Formalised, but omitted from this text. □

For the remainder of this section, let $r' > 0, c \geq 0$ be real numbers, and let S be a finite set.

Definition 3.3. There is a natural action of T^{-1} on $\overline{\mathcal{M}}_{r'}(S)$, via

$$T^{-1} \cdot \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} = \left(\sum_{n \geq 1} a_{n+1,s} T^n \right)_{s \in S}.$$

Lemma 3.4. *The natural action of T^{-1} on $\overline{\mathcal{M}}_{r'}(S)$ restricts to continuous maps*
$$T^{-1} \cdot \_ : \overline{\mathcal{M}}_r(S)_{\leq c} \to \overline{\mathcal{M}}_r(S)_{\leq c/r'}.$$
In particular, $\overline{\mathcal{M}}_{r'}(S)$ has an r'-action of T^{-1}.

Proof. Formalised, but omitted from this text. □

4. Some normed homological algebra

It will be convenient to use the following definition generalizing the notion of a bound on the norm of a right inverse of a normed group morphism (note that the morphisms we consider have no reason to have a right inverse, even when they are surjective).

Definition 4.1. Let G and H be semi-normed groups, let K be a subgroup of H and C be a positive real number. A morphism $f : G \to H$ is C-surjective onto K if, for all x in K, there exists some g in G such that $f(g) = x$ and $\|g\| \leq C\|x\|$. If $K = H$ we simply say f is C-surjective.

The following controlled surjectivity lemma will be used to prove Lemma 4.3 and Lemma 5.8.

Lemma 4.2. *Let G and H be normed groups. Let K be a subgroup of H and f a morphism from G to H. Assume that G is complete and f is C-surjective onto K. Then f is $(C+\varepsilon)$-surjective onto the topological closure of K for every positive ε.*

Proof. Let x be any element of the closure of K. First note the conclusion is trivial when $x = 0$, so we can assume $x \neq 0$. Then write x as a sum $\sum_{i\geq 0} x_i$ with all $x_i \in K$, $\|x - x_0\| \leq \varepsilon_0$ and $\|x_i\| \leq \epsilon_i$ for $i > 0$ for some sequence of positive numbers ϵ_i to be chosen later. By assumption, we can then lift each x_i to g_i such that $f(g_i) = x_i$ and $\|g_i\| \leq C\|x_i\|$, and then set $g = \sum g_i$. Because G is complete, this sum converges provided the ε_i sequence converges fast enough to zero. We then have $f(g) = x$ and
$$\|g\| \leq C \sum_{i\geq 0} \|x_i\| \leq C(\|x\| + \varepsilon_0) + C \sum_{i>0} \varepsilon_i \leq (C+\varepsilon)\|x\|$$
where the last inequality holds provided the ε_i sequence converges fast enough to zero. For instance $\varepsilon_i = \varepsilon \parallel x \parallel /(2^{i+1}C)$ satisfies all our constraints on the ε_i sequence (in particular they are positive because $x \neq 0$). □

The first application of the above lemma is a completion result for a quantitative version of being a complex.

Lemma 4.3. *Let $f : M_0 \to M_1$ and $g : M_1 \to M_2$ be bounded maps between normed groups. Assume there are positive constants C and D such that:*

- *f is C-surjective onto* $\ker g$.
- *g is D-surjective onto its image.*

Then for every positive ε, $\hat{f}$ is $(C+\varepsilon)$-surjective onto $\ker \hat{g}$.

Proof. Since f is C-surjective onto $\ker g$, $\hat{f}$ is C-surjective onto $\ker g$ seen as a subset of $\widehat{M_1}$. Hence this lemma will follow directly from Lemma 4.2 once we'll have proven that $\ker g$ is dense in $\ker \hat{g}$. Let $\hat{y}$ be an element of $\ker \hat{g}$. Pick any $\delta > 0$ and take $y \in M_1$ such that $\|\hat{y} - y\| \leq \delta$. Let $z = g(y) \in M_2$, which has norm $\|z\| = \|g(y)\| = \|g(y - \hat{y})\|$ bounded by $C_g\delta$, where C_g is the norm of g. We can thus find some $y' \in M_1$ with $\|y'\| \leq DC_g\delta$ and $g(y') = z$. Replacing y by $y - y'$, we can thus find $y \in \ker(g : M_1 \to M_2)$ such that still $\|\hat{y} - y\| \leq (1 + DC_g)\delta$; as δ was arbitrary, this gives the desired density. $\square$

Definition 4.4. A *system of complexes* of normed abelian groups is for each $c \in \mathbb{R}_{\geq 0}$ a complex

$$C_c^\bullet : C_c^0 \to C_c^1 \to \ldots$$

of normed abelian groups together with maps of complexes $\mathrm{res}_{c',c} : C_{c'}^\bullet \to C_c^\bullet$, for $c' \geq c$, satisfying $\mathrm{res}_{c,c} = \mathrm{id}$ and the obvious associativity condition. In other words, a functor from $(\mathbb{R}_{\geq 0})^{\mathrm{op}}$ to cochain complexes of semi-normed groups.

By convention, for every system of complexes $C_\bullet^\bullet$, we will set $C_c^{-1} = 0$ for all c. This will come up each time we write C_c^{i-1} and i could be 0.

In this section, given $x \in C_{c'}^\bullet$ and $c_0 \leq c \leq c'$ we will use the notation $x_{|c} := \mathrm{res}_{c',c}(x)$.

Definition 4.5. A system of complexes is *admissible* if all differentials and maps $\mathrm{res}_{c',c}^i$ are norm-nonincreasing.

Throughout the rest of this section, k (and k', k'') will denote reals at least 1, m will be a non-negative integer, and K, K', K'' will denote non-negative reals.

Definition 4.6. A cochain complex C of semi-normed groups is *normed exact* if for all $i \geq 0$, all $\varepsilon > 0$, and all $x \in C^i$ with $d(x) = 0$ there exists a $y \in C^{i-1}$ such that $d(y) = x$ and $\|y\| \leq (1+\varepsilon)\|x\|$.

Definition 4.7. Let $C_\bullet^\bullet$ be a system of complexes. For an integer $m \geq 0$ and reals $k \geq 1$, $K \geq 0$ and $c_0 \geq 0$, we say the datum $C_\bullet^\bullet$ is *$\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound K* if the following condition is satisfied. For all $c \geq c_0$ and all $x \in C_{kc}^i$ with $i \leq m$ there is some $y \in C_c^{i-1}$ such that

$$\|x_{|c} - dy\| \leq K\|dx\|.$$

We will also need a version where the inequality is relaxed by some arbitrary small additive constant.

Definition 4.8. Let $C_\bullet^\bullet$ be a system of complexes. For an integer $m \geq 0$ and reals $k \geq 1$, $K \geq 0$ and $c_0 \geq 0$, the datum $(C_c^\bullet)_c$ is *weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound K* if the following condition is satisfied. For all $c \geq c_0$, all $x \in C_{kc}^i$ with $i \leq m$ and any $\varepsilon > 0$ there is some $y \in C_c^{i-1}$ such that

$$\|x_{|c} - dy\| \leq K\|dx\| + \varepsilon.$$

We first note that the difference between those two definitions is only about cocyles if we are ready to lose a tiny something on the norm bound K.

Lemma 4.9. *Let $C_\bullet^\bullet$ be a system of complexes. If $C_\bullet^\bullet$ is weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound K and if, for all $c \geq c_0$ and all $x \in C_{kc}^i$ with $i \leq m$ such that $dx = 0$ there is some $y \in C_c^{i-1}$ such that $x_{|c} = dy$ then, for every positive δ, $C_\bullet^\bullet$ is $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound $K + \delta$.*

Proof. Let δ be some positive real number. Let x be an element of C_{kc}^i for some $c \geq c_0$ and $i \leq m$. If $dx = 0$ then the assumption we made about exact elements is exactly what we want.

Assume now that $dx \neq 0$. The weak exactness assumption applied to $\varepsilon = \delta\|dx\|$ gives some $y \in C_c^{i-1}$ such that

$$\begin{aligned}\|x_{|c} - dy\| &\leq K\|dx\| + \delta\|dx\| \\ &= (K+\delta)\|dx\|\end{aligned}$$

□

Lemma 4.10. *Let $k \geq 1$, $c_0 \geq 0$ be real numbers, and $m \in \mathbb{N}$. Let $C_\bullet^\bullet$ be a system of complexes, and for each $c \geq 0$ let D_c be a cochain complex of semi-normed groups. Let $f_c \colon C_{kc}^\bullet \to D_c^\bullet$ and $g_c \colon D_c^\bullet \to C_c^\bullet$ be norm-nonincreasing morphisms of cochain complexes of semi-normed groups such that $g_c \circ f_c$ is the restriction map $C_{kc}^\bullet \to C_c^\bullet$. Assume that for all $c \geq c_0$ the cochain complex D_c is normed exact. Then $C_\bullet^\bullet$ is weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound 1.*

Proof. Fix $c \geq c_0$, $i \leq m$, $x \in C_{kc}^i$, and $\varepsilon > 0$. Denote by δ the positive real number $\frac{\varepsilon}{\|x\|+1}$.

Clearly $f(d(x))$ is killed by d, so by normed exactness of D_c we find $x' \in D_c^i$ such that $d(x') = f(d(x))$ and $\|x'\| \leq (1+\delta)\|f(d(x))\|$. Similarly $d(f(x) - x') = 0$, so by exactness of D_c we find $y \in D_c^{i-1}$ such that $d(y) = f(x) - x'$.

We are done if we show that $\|x_{|c} - d(g(y))\| \leq \|d(x)\| + \varepsilon$. Observe that $x_{|c} - d(g(y)) = g(f(x)) - g(d(y)) = g(x')$, and therefore we shall show $\|g(x')\| \leq \|d(x)\| + \varepsilon$.

Now we use that f and g are norm-nonincreasing to calculate

$$\|g(x')\| \leq \|x'\| \leq (1+\delta)\|f(d(x))\| \leq (1+\delta)\|d(x)\|.$$

Finally, we have $(1+\delta)\|d(x)\| \leq \|d(x)\| + \varepsilon$ by our choice of δ. □

Lemma 4.11. *Let $M_\bullet^\bullet$ be an admissible collection of complexes of complete normed abelian groups.*

Assume that $M_c^\bullet$ is weakly $\leq k$-exact in degrees $\leq m$ for $c \geq c_0$ with bound K. Then $M_c^\bullet$, for every $\delta > 0$, it is $\leq k^2$-exact in degrees $\leq m$ for $c \geq c_0$ with bound $K + \delta$.

Proof. Lemma 4.9 ensures we only need to care about cocycles of M. More precisely, let x be a cocycle in $M_{k^2c}^i$ for some $i \leq m$ and $c \geq c_0$. We need to find $y \in M_c^{i-1}$ such that $dy = x_{|c}$.

By weak $\leq k$-exactness applied to x and a sequence ε_j to be chosen later, we can find a sequence $w^j \in M_{kc}^{i-1}$ such that

$$\|x_{kc} - dw^j\| \leq \varepsilon_j.$$

Then, by weak $\leq k$-exactness applied to each $w^{j+1} - w^j$ and a sequence δ_j to be chosen later, we can find a sequence $z^j \in M_c^{i-2}$ such that

$$\|(w^{j+1} - w^j)_{|c} - dz^j\| \leq K\|dw^{j+1} - dw^j\| + \delta_j.$$

We set $y^j := w^j_{|c} - \sum_{l=0}^{j-1} dz^l \in M_c^{i-1}$.

We have

$$\begin{aligned}\|y^{j+1} - y^j\| &= \left\|(w^{j+1} - w^j)_{|c} - dz^j\right\| \\ &\leq K\|dw^{j+1} - dw^j\| + \delta_j \\ &\leq 2K\varepsilon_j + \delta_j.\end{aligned}$$

So y^j is a Cauchy sequence as long as we make sure $2K\varepsilon_j + \delta_j \leq 2^{-j}$ for instance. Since M_c^{i-1} is complete, this sequence converges to some y. Because $dy^j = dw^j_{|c}$, we get that $\|x_{|c} - dy^j\| \leq \varepsilon_j$ and in the limit $x_{|c} = dy$. □

Proposition 4.12. *Let $M_\bullet^\bullet$ and $M_\bullet'^\bullet$ be two admissible collections of complexes of complete normed abelian groups. For each $c \geq c_0$ let $f_c^\bullet : M_c^\bullet \to M_c'^\bullet$ be a collection of maps between these collections of complexes that are norm-nonincreasing and which all commute with all restriction maps, and assume that there exists these maps satisfy*

$$\|x_{|c}\| \leq K''\|f(x)\|$$

for all $i \leq m+1$ and all $x \in M^i_{k''c}$. Let $N_c^\bullet = M_c'^\bullet/M_c^\bullet$ be the collection of quotient complexes, with the quotient norm; this is again an admissible collection of complexes.

Assume that $M_c^\bullet$ (resp. $M_c'^\bullet$) is weakly $\leq k$-exact (resp. $\leq k'$-exact) in degrees $\leq m$ for $c \geq c_0$ with bound K (resp. K'). Then $N_c^\bullet$ is weakly $\leq kk'k''$-exact in degrees $\leq m-1$ for $c \geq c_0$ with bound $K'(KK''+1)$.

Proof. Let $n \in N^i_{kk'k''c}$ for $i \leq m-1$. We fix $\varepsilon > 0$. We need to find an element $y \in N_c^{i-1}$ such that

$$\|n_{|c} - dy\| \leq K'(KK''+1)\|dn\| + \epsilon.$$

Pick any preimage $m' \in M'^i_{kk'k''c}$ of n. In particular dm' is a preimage of dn. By definition of the quotient norm, we can find $m_1 \in M^{i+1}_{kk'k''c}$ and $m_1'' \in (M')^{i+1}_{kk'k''c}$ such that

$$dm' = f(m_1) + m_1''$$

with $\|m_1''\| \leq \|dn\| + \varepsilon_1$, for some positive ε_1 to be chosen later.

Applying the differential to the last displayed equation, and using that this kills the image of d, and that f is a map of complexes, we see that

$$f(dm_1) = -dm_1''.$$

Using the norm bound on f, we get

$$\begin{aligned}\|dm_{1|kk'c}\| &\leq K''\|f(dm_1)\| = K''\|dm_1''\| \\ &\leq K''\|m_1''\| \leq K''\|dn\| + K''\varepsilon_1.\end{aligned}$$

On the other hand, weak exactness of M applied to $m_{1|kk'c}$ gives $m_0 \in M^i_{k'c}$ such that

$$\|m_{1|kk'c|k'c} - dm_0\| \leq K\|dm_{1|kk'c}\| + \varepsilon_1$$

which combines with the previous estimate to give:

$$\|m_{1|k'c} - dm_0\| \leq KK''\,\|dn\| + (KK''+1)\varepsilon_1.$$

Now let $m'_{\mathrm{new}} = m'_{|k'c} - f(m_0) \in M'^i_{k'c}$; this is a lift of $n_{|k'c}$. Then

$$dm'_{\mathrm{new}} = dm'_{|k'c} - f(m_{1|k'c}) + f(m_{1|k'c} - dm_0) = m''_{1|k'c} + f(m_{1|k'c} - dm_0).$$

In particular,

$$\|dm'_{\mathrm{new}}\| \leq (KK''+1)\,\|dn\| + (KK''+2)\varepsilon_1.$$

Now weak exactness of M' gives $x \in M_c'^{i-1}$ such that

$$\|m'_{\mathrm{new}|c} - dx\| \leq K'\|dm'_{\mathrm{new}}\| + \varepsilon_1 \leq K'((KK''+1)\,\|dn\| + (KK''+2)\varepsilon_1) + \varepsilon_1.$$

In particular, letting $y \in N_c^{i-1}$ be the image of x, we get

$$\|n_{|c} - dy\| \leq K'(KK''+1)\,\|dn\| + (K'(KK''+2)+1)\varepsilon_1,$$

which is exactly what we wanted if we choose $\varepsilon_1 = \varepsilon/(K'(KK''+2)+1)$. ∎

We also need the 'dual' version of 4.12, for kernels instead of cokernels. Note that this is actually a bit easier to prove. (Should we put it first in the presentation?)

Proposition 4.13. *Let $M_\bullet^\bullet$ and $M_\bullet'^\bullet$ be two admissible collections of complexes of complete normed abelian groups. For each $c \geq c_0$ let $f_c^\bullet : M_c^\bullet \to M_c'^\bullet$ be a collection of maps between these collections of complexes which all commute with all restriction maps. Assume moreover that we are given two constants $r_1, r_2 \geq 0$ such that:*

- *for all $i, c \geq c_0$ and all $x \in M_c^i$*
$$\|f(x)\| \leq r_1 \|x\|;$$
- *for all $i \leq m+1, c \geq c_0$ and all $y \in M_c'^i$, there exists $x \in M_c^i$ such that*
$$f(x) = y \text{ and } \|x\| \leq r_2 \|y\|.$$

Let $N_c^\bullet$ be the collection of kernel complexes, with the induced norm; this is again an admissible collection of complexes.

Assume that $M_c^\bullet$ (resp. $M_c'^\bullet$) is weakly $\leq k$-exact (resp. $\leq k'$-exact) in degrees $\leq m$ for $c \geq c_0$ with bound K (resp. K'). Then $N_c^\bullet$ is weakly $\leq kk'$-exact in degrees $\leq m-1$ for $c \geq c_0$ with bound $K + r_1 r_2 K K'$.

Proof. Let $n \in N_{kk'c}^i \subseteq M_{kk'c}^i$ for $i \leq m-1$ and let $\varepsilon > 0$. We need to find an element $y \in N_c^{i-1}$ such that
$$\|n_{|c} - dy\| \leq K + r_1 r_2 K K' \|dn\| + \epsilon.$$
By weak exactness of $M_\bullet^\bullet$, we can find $m \in M_{k'c}^{i-i}$ such that
$$\|n_{|k'c} - dm\| \leq K \|dn\| + \epsilon_1,$$
where $\epsilon_1 > 0$ to be chosen later. By weak exactness of $M_\bullet'^\bullet$, we can find $m' \in M_c'^{i-2}$ such that
$$\|f(m)_{|c} - dm'\| \leq K' \|df(m)\| + \epsilon_2,$$
where $\epsilon_2 > 0$ to be chosen later. Let $m_1 \in M_c^{i-2}$ be a lift of m' and let $m_2 \in M_c^{i-1}$ be such that
$$f(m_2) = f(m_{|c} - dm_1) \text{ and } \|m_2\| \leq r_2 \|f(m_{|c} - dm_1)\|.$$
Set $y = m_{|c} - dm_1 - m_2 \in M_c^{i-1}$. By construction $f(y) = 0$, so $y \in N_c^{i-1}$. We compute
$$\|n_{|c} - dy\| = \|n_{|c} - dm_{|c} + d^2 m_1 - dm_2\| = \|n_{|c} - dm_{|c} - dm_2\| \leq$$
$$\|n_{|c} - dm_{|c}\| + \|dm_2\| = \|(n_{|k'c} - dm)_{|c}\| + \|dm_2\| \leq \|(n_{|k'c} - dm)\| + \|dm_2\| \leq$$
$$K\|dn\| + \epsilon_1 + \|dm_2\|.$$
Where we have used the defining property of m and admissibility of $M_\bullet^\bullet$. By the same assumption and since $f(n_{|k'c}) = f(n)_{|k'c} = 0$, we have
$$\|dm_2\| \leq \|m_2\| \leq r_2 \|f(m_{|c} - dm_1)\| = r_2 \|f(m)_{|c} - df(m_1)\| = r_2 \|f(m)_{|c} - dm'\| \leq$$
$$r_2(K'\|df(m)\| + \epsilon_2) = r_2(K'\|f(dm)\| + \epsilon_2) = r_2(K'\|f(n_{|k'c}) - f(dm)\| + \epsilon_2) =$$
$$r_2(K'\|f(n_{|k'c} - dm)\| + \epsilon_2) \leq r_2(K' r_1 \|n_{|k'c} - dm\| + \epsilon_2) \leq r_2(K' r_1 (K\|dn\| + \epsilon_1) + \epsilon_2)$$
In particular we get
$$\|n_{|c} - dy\| \leq K\|dn\| + \epsilon_1 + r_2(K' r_1 (K \|dn\| + \epsilon_1) + \epsilon_2) =$$
$$(K + r_1 r_2 K K')\|dn\| + \epsilon_1 (1 + r_1 r_2 K') + r_2 \epsilon_2.$$

Now let

$$\epsilon_1 = \frac{\epsilon}{2(1+r_1r_2K')} \text{ and } \epsilon_2 = \begin{cases} \frac{\epsilon}{2r_2} \text{ if } r_2 \neq 0 \\ 1 \text{ if } r_2 = 0 \end{cases}$$

In any case $r_2\epsilon_2 \leq \frac{\epsilon}{2}$ and so

$$\|n_{|c} - dy\| \leq (K + r_1r_2KK')\|dn\| + \epsilon$$

as required.

If $i = 0$, then all m, m', m_1 and m_2 are 0, so $y = 0$ as required. $\square$

Consider a system of double complexes $M_c^{p,q}$, $p, q \geq 0$, $c \geq c_0$,

$$\begin{array}{ccccccc}
M_c^{0,0} & \longrightarrow & M_c^{0,1} & \longrightarrow & M_c^{0,2} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
M_c^{1,0} & \longrightarrow & M_c^{1,1} & \longrightarrow & M_c^{1,2} & \longrightarrow & \dots \\
\downarrow & & \downarrow & & \downarrow & & \\
M_c^{2,0} & \longrightarrow & M_c^{2,1} & \longrightarrow & \ddots & & \\
\downarrow & & \downarrow & & & & \\
\vdots & & \vdots & & & &
\end{array}$$

of complete normed abelian groups.

Definition 4.14. We say that the system of double complexes $M_c^{p,q}$ satisfies the *normed spectral homotopy condition* for $m \in \mathbf{N}$ and $H, c_0 \in \mathbf{R}_{\geq 0}$ if the following condition is satisfied:

For $q = 0, \dots, m$ and $c \geq c_0$, there is a map $h^q_{k'c} \colon M^{0,q+1}_{k'c} \to M^{1,q}_c$ with

$$\|h^q_{k'c}(x)\|_{M_c^{1,q}} \leq H\|x\|_{M^{0,q+1}_{k'c}}$$

for all $x \in M^{0,q+1}_{k'c}$, and such that for all $c \geq c_0$ and $q = 0, \dots, m$ the "homotopic" map

$$\mathrm{res}^{1,q}_{k'^2c,k'c} \circ d^{0,q} + h^q_{k'^2c} \circ d'^{0,q}_{k'^2c} + d'^{1,q-1}_{k'c} \circ h^{q-1}_{k'^2c} \colon M^{0,q}_{k'^2c} \to M^{1,q}_{k'c}$$

factors as a composite of the restriction $\mathrm{res}^{0,q}_{k'^2c,c}$ and a map

$$\delta^{0,q}_c \colon M^{0,q}_c \to M^{1,q}_{k'c}$$

that is a map of complexes (in degrees $\leq m$), and satisfies the estimate

$$\|\delta^{0,q}_c(x)\|_{M^{1,q}_{k'c}} \leq \epsilon\|x\|_{M_c^{0,q}} \tag{4.1}$$

for all $x \in M_c^{0,q}$.

Proposition 4.15. *Fix an integer $m \geq 0$ and constants k, K. Then there exists an $\epsilon > 0$ and constants k_0, K_0, depending (only) on k, K and m, with the following property.*

Let $M_c^{p,q}$ be a system of double complexes as above, and assume that it is admissible. Assume further that there is some $k' \geq k_0$ and some $H > 0$, such that

1. *for $i = 0, \dots, m+1$, the rows $M_c^{i,q}$ are weakly $\leq k$-exact in degrees $\leq m-1$ for $c \geq c_0$ with bound K;*

(2) *for $j = 0, \dots, m$, the columns $M_c^{p,j}$ are weakly $\leq k$-exact in degrees $\leq m$ for $c \geq c_0$ with bound K;*
(3) *it satisfies the normed spectral homotopy condition for m, H and c_0.*

Then the first row is weakly $\leq k'^2$ exact in degrees $\leq m$ for $c \geq c_0$ with bound $2K_0H$.

We note that the bound on the homotopy is of a peculiar nature, in that the bound only depends on a deep restriction of x.

Proof. First, we treat the case $m = 0$. If $m = 0$, we claim that one can take $\epsilon = \frac{1}{2k}$ and $k_0 = k$. We have to prove exactness at the first step. Let $x_{k'^2c} \in M^{0,0}_{k'^2c}$ and denote $x_{k'c} = \mathrm{res}^{0,0}_{k'^2c,k'c}(x)$ and $x_c = \mathrm{res}^{0,0}_{k'^2c,c}(x)$. Then by assumption (2) (and $k' \geq k$), we have

$$\|x_c\|_{M_c^{0,0}} \leq k\|d^{0,0}_{k'c}(x_{k'c})\|_{M^{1,0}_{k'c}}.$$

On the other hand, by (3),

$$\|\mathrm{res}^{1,0}_{k'^2c,k'c}(d^{0,0}_{k'^2c}(x)) \pm h^0_{k'^2c}(d'^{0,0}_{k'^2c}(x))\|_{M^{1,0}_{k'c}} \leq \epsilon\|x_c\|_{M_c^{0,0}}.$$

In particular, noting that $\mathrm{res}^{1,0}_{k'^2c,k'c}(d^{0,0}_{k'^2c}(x)) = d^{0,0}_{k'c}(x_{k'c})$, we get

$$\|x_c\|_{M_c^{0,0}} \leq k\|d^{0,0}_{k'c}(x_{k'c})\|_{M^{1,0}_{k'c}} \leq k\epsilon\|x_c\|_{M_c^{0,0}} + kH\|d'^{0,0}_{k'^2c}(x)\|_{M^{0,1}_{k'^2c}}.$$

Thus, taking $\epsilon = \frac{1}{2k}$ as promised, this implies

$$\|x_c\|_{M_c^{0,0}} \leq 2kH\|d'^{0,0}_{k'^2c}(x)\|_{M^{0,1}_{k'^2c}}.$$

This gives the desired $\leq \max(k'^2, 2k_0H)$-exactness in degrees $\leq m$ for $c \geq c_0$.

Now we argue by induction on m. Consider the complex $N^{p,q}$ given by $M^{p,q+1}$ for $q \geq 1$ and $N^{p,0} = M^{p,1}/\overline{M^{p,0}}$ (the quotient by the closure of the image, which is also the completion of $M^{p,1}/M^{p,0}$), equipped with the quotient norm. Using the normed version of the snake lemma, Proposition **??** in the appendix to this lecture, one checks that this satisfies the assumptions for $m-1$, with k replaced by $\max(k^4, k^3 + k + 1)$. □

5. Completions of locally constant functions

Definition 5.1. Let V be a semi-normed group, and X a compact topological space. We denote by $V(X)$ the normed abelian group of locally constant functions $X \to V$ with respect to the sup norm. With $\widehat{V}(X)$ we denote the completion of $V(X)$.

These constructions are functorial in bounded group homomorphisms $V \to V'$ and contravariantly functorial in continuous maps $f\colon X \to X'$.

Note in particular that $V(f)$ and $\widehat{V}(f)$ are norm-nonincreasing morphisms of semi-normed groups.

Lemma 5.2. *Let $r \in \mathbb{R}_{>0}$, and let V be an r-normed $\mathbb{Z}[T^{\pm 1}]$-module. Let X be a compact space. Then $\widehat{V}(X)$ is naturally an r-normed $\mathbb{Z}[T^{\pm 1}]$-module, with the action of T given by post-composition.*

Proof. Formalised, but omitted from this text. □

We continue to use the notation of before: let $r' > 0, c \geq 0$ be real numbers, and let M be a profinitely filtered pseudo-normed group with r'-action by T^{-1} (see Section 2).

Lemma 5.3. *Let f be a basic universal map from exponent m to n. We get an induced homomorphism of profinitely filtered pseudo-normed groups $M^m \to M^n$ bounded by the maximum (over all i) of $\sum_j |f_{ij}|$, where the f_{ij} are the coefficients of the $n \times m$-matrix representing f.*

This construction is functorial in f.

Proof. Omitted. □

Definition 5.4. Let f be a basic universal map from exponent m to n, and let (c_2, c_1) be f-suitable. We get an induced map

$$V(f) \colon V(M^n_{\leq c_1}) \to V(M^m_{\leq c_2})$$

induced by the morphism of profinitely filtered pseudo-normed groups $M^m \to M^n$.

This construction is functorial in f.

Definition 5.5. Let $f = \sum_g n_g g$ be a universal map from exponent m to n, and let (c_2, c_1) be f-suitable. We get an induced map

$$V(f) \colon V(M^n_{\leq c_1}) \to V(M^m_{\leq c_2})$$

that is the sum $\sum n_g V(g)$.

This construction is functorial in f.

Definition 5.6. Let f be a universal map from exponent m to n, and let (c_2, c_1) be f-suitable. We get an induced map

$$\widehat{V}(f) \colon \widehat{V}(M^n_{\leq c_1}) \to \widehat{V}(M^m_{\leq c_2})$$

that is the completion of $V(f)$.

This construction is functorial in f.

Let $r > 0$, and assume now that V is an r-normed $\mathbb{Z}[T^{\pm 1}]$-module. Assume $r' \leq 1$.

Definition 5.7. There are two natural actions of T^{-1} on $\widehat{V}(M_{\leq c})$. The first comes from the r'-action of T^{-1} on M which gives a continuous map

$$M_{\leq cr'} \to M_{\leq c}$$

and thus a normed group morphism $V(M_{\leq c}) \to V(M_{\leq cr'})$ which can be extended by completion to

$$(T^{-1})^* \colon \widehat{V}(M_{\leq c}) \to \widehat{V}(M_{\leq cr'}).$$

The other comes from Lemma 5.2, using the r-normed $\mathbb{Z}[T^{\pm 1}]$-module V. Again by extension to completion, we get a map

$$[T^{-1}] \colon \widehat{V}(M_{\leq c}) \to \widehat{V}(M_{\leq c}),$$

that we can compose with the map $\widehat{V}(M_{\leq c}) \to \widehat{V}(M_{\leq cr'})$, obtained from the natural inclusion $M_{\leq cr'} \to M_{\leq c}$. We thus end up with two maps

$$(T^{-1})^*, [T^{-1}] \colon \widehat{V}(M_{\leq c}) \to \widehat{V}(M_{\leq cr'}).$$

and we define $\widehat{V}(M_{\leq c})^{T^{-1}}$ to be the equalizer of $(T^{-1})^*$ and $[T^{-1}]$. In other words, the kernel of $(T^{-1})^* - [T^{-1}]$.

We will also need to understand the image of $(T^{-1})^* - [T^{-1}]$. The next lemma ensures it is surjective with controlled preimages, see Definition 4.1.

Lemma 5.8. *Let M be a profinitely filtered pseudo-normed group with action of T^{-1}. For any $r \in (0,1)$, any r-normed $\mathbb{Z}[T^{\pm 1}]$-module V, any $c > 0$ and any a, the map*

$$\widehat{V}(M^a_{\leq c}) \xrightarrow{T^{-1}-[T^{-1}]^*} \widehat{V}(M^a_{\leq r'c})$$

has norm bounded by $r^{-1}+1$ and is $\frac{r}{1-r}(1+\epsilon)$-surjective.

Proof. The norm bound is clear because $[T^{-1}]^*$ is norm non-increasing and T^{-1} scales norm by r^{-1}. Quantitative surjectivity will follow from Lemma 4.2 once we'll have proven that $T^{-1} - [T^{-1}]^* : \widehat{V}(M^a_{\leq c}) \to \widehat{V}(M^a_{\leq r'c})$ is $r/(1-r)$-surjective onto $V(M^a_{\leq r'c})$.

We first note that any locally constant function $\varphi \in V(M^a_{\leq r'c})$ can be extended to a locally constant function $\bar{\varphi} \in V(M^a_{\leq c})$ with the same norm (recall f takes finitely many values and its norm is the maximum of norms of these values).

Let f be any element of $V(M^a_{\leq r'c})$. We inductively define a sequence of locally constant functions $h_n \in V(M^a_{\leq c})$ with $h_0 = T \circ \bar{f}$ and $h_{n+1} = T \circ \overline{[T^{-1}]^* h_n}$. Here we use the composition symbol to emphasize this is indeed the naive post-composition with T, there is no extra precomposition with a the inclusion map $\iota : M^a_{\leq r'c} \hookrightarrow M^a_{\leq c}$ as in the definition of T^{-1} seen as a map from $V(M^a_{\leq c})$ to $V(M^a_{\leq r'c})$.

Since $[T^{-1}]^*$ is norm non-increasing, extension is norm preserving and T scales norm by r, we get that $\|h_n\| \leq r^{n+1}\|f\|$. We then set $g_n = \sum_{i=0}^n h_i$. The norm estimate on h_n ensures g is a Cauchy sequence in $V(M^a_{\leq c})$ hence it converges to some g in $\widehat{V}(M^a_{\leq c})$. We compute:

$$\begin{aligned}
(T^{-1} - [T^{-1}]^*)g_n &= \sum_{k=0}^{n} \left(T^{-1}h_k - [T^{-1}]^* h_k\right) \\
&= T^{-1}h_0 + \sum_{k=0}^{n-1} \left(T^{-1}h_{k+1} - [T^{-1}]^* h_k\right) - [T^{-1}]^* h_n \\
&= \bar{f} \circ \iota + \sum_{k=0}^{n-1} \left(T^{-1} \circ T \circ \overline{[T^{-1}]^* h_k} \circ \iota - [T^{-1}]^* h_k\right) - [T^{-1}]^* h_n \\
&= f - [T^{-1}]^* h_n
\end{aligned}$$

which converges to f hence $(T^{-1} - [T^{-1}]^*)g = f$. In addition $\|g\| \leq \sum_n r^{n+1}\|f\| = r/(1-r)\|f\|$. ☐

Definition 5.9. Let f be a universal map from exponent m to n, and let (c_2, c_1) be f-suitable.

The natural map from Definition 5.6 restricts to a map

$$\widehat{V}(f)^{T^{-1}} : \widehat{V}(M^n_{\leq c_1})^{T^{-1}} \to \widehat{V}(M^m_{\leq c_2})^{T^{-1}}$$

Lemma 5.10. *Let $0 < r$ and $0 < r' \leq 1$ be real numbers. Let f be a universal map from exponent m to n, and let (c_2, c_1) be very suitable for (f, r, r'). Then*

$$\widehat{V}(f)^{T^{-1}} : \widehat{V}(M^n_{\leq c_1})^{T^{-1}} \to \widehat{V}(M^m_{\leq c_2})^{T^{-1}}$$

is norm-nonincreasing.

Proof. Use the assumption that (c_2, c_1) is very suitable for (f, r, r') in order to find $N, b \in \mathbf{N}$ and $c' \in \mathbf{R}_{\geq 0}$ such that:

- f is bound by N (see Definition 1.4)

- (c_2, c') is f-suitable
- $r^b N \leq 1$
- $c' \leq (r')^b c_1$

Now, notice that the norm of $\widehat{V}(f)$ is at most N, and $\widehat{V}(f)$ can be factored as

$$\widehat{V}(M^n_{\leq c_1})^{T^{-1}} \xrightarrow{\text{res}} \widehat{V}(M^n_{\leq c'})^{T^{-1}} \xrightarrow{\widehat{V}(f)} \widehat{V}(M^m_{\leq c_2})^{T^{-1}}$$

Now use the defining property of the equalizer to conclude that the restriction map has norm less than $1/N$, and therefore the composition is norm-nonincreasing. □

Definition 5.11. Let $0 < r$ and $0 < r' \leq 1$ be real numbers, and let V be an r-normed $\mathbb{Z}[T^{\pm 1}]$-module. Let $\mathsf{BD} = (n, f)$ be Breen–Deligne data, and let $\kappa = (\kappa_0, \kappa_1, \kappa_2, \dots)$ be a sequence of constants in $\mathbb{R}_{\geq 0}$ that is very suitable for (BD, r, r'). Let M be a profinitely filtered pseudo-normed group with r'-action of T^{-1}.

For every $c \in \mathbb{R}_{\geq 0}$, the maps from Definition 5.9 induced by the universal maps f_i from the Breen–Deligne $\mathsf{BD} = (n, f)$ assemble into a complex of normed abelian groups

$$C^{\mathrm{BD}}_\kappa(M)^\bullet_c \colon 0 \to \dots \to \widehat{V}(M^{n_i}_{\leq \kappa_i})^{T^{-1}} \to \widehat{V}(M^{n_{i+1}}_{\leq \kappa_{i+1}})^{T^{-1}} \to \dots .$$

Together, these complexes fit into a system of complexes with the natural restriction maps.

By Lemma 5.10 the differentials are norm-nonincreasing. It is clear that the restriction maps are also norm-nonincreasing, and therefore the system is admissible.

6. Polyhedral lattices

Definition 6.1. A *polyhedral lattice* is a finite free abelian group Λ equipped with a norm $\|\cdot\|_\Lambda \colon \Lambda \otimes \mathbb{R} \to \mathbb{R}$ such that there exists a finite set $\{\lambda_1, \dots, \lambda_n\} \subset \Lambda$ that generate the norm: that is to say, for every $\lambda \in \Lambda$ there exist $c_1, \dots, c_n \in \mathbb{Q}$ such that $\lambda = \sum c_i \lambda_i$ and $\|\lambda\| = c_i \|\lambda_i\|$.

Equivalently (but not verified in Lean): the norm is given by the supremum of finitely many linear functions on Λ; or once more, equivalently, the "unit ball" $\{\lambda \in \Lambda \otimes \mathbb{R} \mid \|\lambda\|_\Lambda \leq 1\}$ is a polyhedron.

Finally, we can prove the key combinatorial lemma, ensuring that any element of $\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))$ can be decomposed into N elements whose norm is roughly $\frac{1}{N}$ of the original element.

Definition 6.2. Let M be a pseudo-normed group, $N \in \mathbb{N}$, and $d \in \mathbb{R}_{\geq 0}$. We say that M is *N-splittable* with error term d, if for all c and $x \in M_c$, there exists a decomposition

$$x = x_1 + x_2 + \dots + x_N,$$

with $x_i \in M_{c/N+d}$.

Lemma 6.3. *Let Λ be a polyhedral lattice. Then for all positive integers N there is a constant d such that for all $c > 0$ one can write any $x \in \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c}$ as*

$$x = x_1 + \dots + x_N$$

where all $x_i \in \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c/N+d}$.

In other words, for all N, there exists a d such that $\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))$ is N-splittable with error term d.

As preparation for the proof, we have the following results.

Lemma 6.4 (Gordan's lemma)**.** *Let Λ be a finite free abelian group, and let $\lambda_1, \dots, \lambda_m \in \Lambda$ be elements. Let $M \subset \mathrm{Hom}(\Lambda, \mathbb{Z})$ be the submonoid $\{x \mid x(\lambda_i) \geq 0 \text{ for all } i = 1, \dots, m\}$. Then M is finitely generated as monoid.*

Proof. This is a standard result. We omit the proof here. It is done in Lean. □

Lemma 6.5. *Let Λ be a finite free abelian group, let N be a positive integer, and let $\lambda_1, \dots, \lambda_m \in \Lambda$ be elements. Then there is a finite subset $A \subset \Lambda^\vee$ such that for all $x \in \Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z})$ there is some $x' \in A$ such that $x - x' \in N\Lambda^\vee$ and for all $i = 1, \dots, m$, the numbers $x'(\lambda_i)$ and $(x - x')(\lambda_i)$ have the same sign, i.e. are both nonnegative or both nonpositive.*

Proof. It suffices to prove the statement for all x such that $\lambda_i(x) \geq 0$ for all i; indeed, applying this variant to all $\pm\lambda_i$, one gets the full statement.

Thus, consider the submonoid $\Lambda^\vee_+ \subset \Lambda^\vee$ of all x that pair nonnegatively with all λ_i. This is a finitely generated monoid by Lemma 6.4; let $y_1, \dots, y_M$ be a set of generators. Then we can take for A all sums $n_1 y_1 + \dots + n_M y_M$ where all $n_j \in \{0, \dots, N-1\}$. □

Lemma 6.6. *Let $x_0, x_1, \dots$ be a sequence of reals, and assume that $\sum_{i=0}^{\infty} x_i$ converges absolutely. For every natural number $N > 0$, there exists a partition $\mathbb{N} = A_1 \sqcup A_2 \sqcup \dots \sqcup A_N$ such that for each $j = 1, \dots, N$ we have $\sum_{i \in A_j} x_i \leq (\sum_{i=0}^{\infty} x_i)/N + 1$*

Proof. Define the A_j recursively: assume that the natural numbers $0, \dots, n$ have been placed into the sets $A_1, \dots, A_N$. Then add the number $n+1$ to the set A_j for which

$$\sum_{i=0, i \in A_j}^{n} x_i$$

is minimal. □

Lemma 6.7. *For all natural numbers $N > 0$, and for all $x \in \overline{\mathcal{M}}_{r'}(S)_{\leq c}$ one can decompose x as a sum*

$$x = x_1 + \dots + x_N$$

with all $x_i \in \overline{\mathcal{M}}_{r'}(S)_{\leq c/N+1}$.

Proof. Choose a bijection $S \times \mathbb{N} \cong \mathbb{N}$, and transport the result from Lemma 6.6. □

Proof of Lemma 6.3. Pick $\lambda_1, \dots, \lambda_m \in \Lambda$ generating the norm. We fix a finite subset $A \subset \Lambda^\vee$ satisfying the conclusion of the previous lemma. Write

$$x = \sum_{n \geq 1, s \in S} x_{n,s} T^n[s]$$

with $x_{n,s} \in \Lambda^\vee$. Then we can decompose

$$x_{n,s} = N x^0_{n,s} + x^1_{n,s}$$

where $x^1_{n,s} \in A$ and we have the same-sign property of the last lemma. Letting $x^0 = \sum_{n \geq 1, s \in S} x^0_{n,s} T^n[s]$, we get a decomposition

$$x = N x^0 + \sum_{a \in A} a x_a$$

with $x_a \in \overline{\mathcal{M}}_{r'}(S)$ (with the property that in the basis given by the $T^n[s]$, all coefficients are 0 or 1). Crucially, we know that for all $i = 1, \dots, m$, we have

$$\|x(\lambda_i)\| = N\|x^0(\lambda_i)\| + \sum_{a \in A} |a(\lambda_i)| \|x_a\|$$

by using the same sign property of the decomposition.

Using this decomposition of x, we decompose each term into N summands. This is trivial for the first term Nx^0, and each summand of the second term decomposes with $d = 1$ by Lemma 6.7. (It follows that in general one can take for d the supremum over all i of $\sum_{a \in A} |a(\lambda_i)|$.) □

Definition 6.8. Let Λ be a polyhedral lattice, and let $N > 0$ be a natural number. (We think of N as being fixed once and for all, and thus it does not show up in the notation below.)

By Λ' we denote Λ^N endowed with the norm

$$\|(\lambda_1, \dots, \lambda_N)\|_{\Lambda'} = \tfrac{1}{N}(\|\lambda_1\|_\Lambda + \dots + \|\lambda_N\|_\Lambda).$$

This is a polyhedral lattice.

Lemma 6.9. *For any $m \geq 1$, let $\Lambda'^{(m)}$ be given by $\Lambda'^m / \Lambda \otimes (\mathbb{Z}^m)_{\sum = 0}$; for $m = 0$, we set $\Lambda'^{(0)} = \Lambda$. Then $\Lambda'^{(m)}$ is a polyhedral lattice.*

Proof. The proof is done in Lean. TODO: write down a proof here. □

Definition 6.10. For any $m \geq 1$, let $\Lambda'^{(m)}$ be given by $\Lambda'^m / \Lambda \otimes (\mathbb{Z}^m)_{\sum=0}$; for $m = 0$, we set $\Lambda'^{(0)} = \Lambda$. Then $\Lambda'^{(\bullet)}$ is a cosimplicial polyhedral lattice, the Čech conerve of $\Lambda \to \Lambda'$.

In particular, $\Lambda'^{(0)} = \Lambda \to \Lambda' = \Lambda'^{(1)}$ is the diagonal embedding.

Definition 6.11. Let Λ be a polyhedral lattice, and M a profinitely filtered pseudo-normed group.

Endow $\mathrm{Hom}(\Lambda, M)$ with the subspaces

$$\mathrm{Hom}(\Lambda, M)_{\leq c} = \{f \colon \Lambda \to M \mid \forall x \in \Lambda, f(x) \in M_{\leq c\|x\|}\}.$$

As Λ is polyhedral, it is enough to check the given condition on f for a finite collection of x that generate the norm.

These subspaces are profinite subspaces of M^Λ, and thus they make $\mathrm{Hom}(\Lambda, M)$ ito a profinitely filtered pseudo-normed group.

If M has an action of T^{-1}, then so does $\mathrm{Hom}(\Lambda, M)$.

7. End of proof

Now we state the following result, which is our main goal.

N.b.: It differs from Theorem 9.4 of [Sch20] only in one aspect: we assume that the sets S are finite, rather than profinite.

Theorem 7.1. *Let $\mathsf{BD} = (n, f, h)$ be a Breen–Deligne package, and let $\kappa = (\kappa_0, \kappa_1, \kappa_2, \dots)$ be a sequence of constants in $\mathbb{R}_{\geq 0}$ that is BD-suitable. Fix radii $1 > r' > r > 0$. For any m there is some k and c_0 such that for all finite sets S and all r-normed $\mathbb{Z}[T^{\pm 1}]$-modules V, the system of complexes*

$$C_\kappa^{\mathsf{BD}}(\overline{\mathcal{M}}_{r'}(S))_c^\bullet \colon \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c})^{T^{-1}} \to \widehat{V}(\overline{\mathcal{M}}_{r'}(S)^2_{\leq \kappa_1 c})^{T^{-1}} \to \dots$$

is $\leq k$-exact in degrees $\leq m$ for $c \geq c_0$.

We will prove Theorem 7.1 by induction on m. Unfortunately, the induction requires us to prove a stronger statement.

Theorem 7.2. *Fix radii $1 > r' > r > 0$. For any m there is some k such that for all polyhedral lattices Λ there is a constant $c_0(\Lambda) > 0$ such that for all finite sets S and all r-normed $\mathbb{Z}[T^{\pm 1}]$-modules V, the system of complexes*

$$C^\bullet_{\Lambda,c}\colon \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^2_{\leq \kappa_1 c})^{T^{-1}} \to \dots$$

is $\leq k$-exact in degrees $\leq m$ for $c \geq c_0(\Lambda)$.

Proof. Use $\Lambda = \mathbb{Z}$, and the isomorphism $\mathrm{Hom}(\mathbb{Z}, A) \cong A$. □

A word on universal constants: We fix once and for all, the constants $0 < r < r' \leq 1$ a Breen–Deligne package BD, and a sequence of positive constants κ that is very suitable for (BD, r, r'). Once the full proof is formalized, we can come back to this place and write a bit more about the other constants.

The global strategy of the proof is to construct a system of double complexes such that its first row is the system $C^\bullet_{\Lambda,\bullet}$ occuring in Theorem 7.2. We can then verify the conditions to Proposition 4.15 and conclude from there. For the time being, we will let M denote an arbitrary profinitely filtered pseudo-normed group with action of T^{-1}, and whenever needed we can specialize to $M = \overline{\mathcal{M}}_{r'}(S)$.

Further choices of constants: We will argue by induction on m, so assume the result for $m-1$ (this is no assumption for $m = 0$, so we do not need an induction start). This gives us some $k > 1$ for which the statement of Theorem 7.2 holds true for $m-1$; if $m = 0$, simply take any $k > 1$. In the proof below, we will increase k further in a way that depends only on m and r. After this modified choice of k, we fix ϵ and k_0 as provided by Proposition 4.15. Fix a sequence $(\kappa'_i)_i$ of nonnegative reals that is adept to (BD, κ). (Such a sequence exists by Lemma 1.21.) Moreover, we let k' be the supremum of k_0 and the c'_i for $i = 0, \dots, m+1$. Finally, choose a positive integer b so that $2k'(\frac{r}{r'})^b \leq \epsilon$, and let N be the minimal power of 2 that satisfies

$$k'/N \leq (r')^b.$$

Then in particular $r^b N \leq 2k'(\frac{r}{r'})^b \leq \epsilon$.

Definition 7.3. Let $\Lambda^{(\bullet)}$ be the cosimplicial polyhedral lattice of Definition 6.10, and recall from 6.11 that $\mathrm{Hom}(\Lambda^{(m)}, M)$ is a profinitely filtered pseudo-normed group with action of T^{-1}.

Hence $\mathrm{Hom}(\Lambda^{(\bullet)}, M)$ is a simplicial profinitely filtered pseudo-normed group with action of T^{-1}.

Now apply the construction of the system of complexes from Definition 5.11 to obtain a cosimplicial system of complexes

$$C^{\mathsf{BD}}_\kappa(\mathrm{Hom}(\Lambda^{(\bullet)}, M))^\bullet_\bullet.$$

Now take the alternating face map cochain complex to obtain a system of double complexes, whose objects are

$$\widehat{V}(\mathrm{Hom}(\Lambda^{(m)}, M)^{n_i}_{\leq \kappa_i c})^{T^{-1}}.$$

As final step, rescale the norm on the object in row m by $m!$, so that all columns become admissible: the vertical differential from row m to row $m+1$ is an alternating sum of $m+1$ maps that are all norm-nonincreasing.

Lemma 7.4. *In particular, for any $c > 0$, we have*

$$\mathrm{Hom}(\Lambda', M)_{\leq c} = \mathrm{Hom}(\Lambda, M)^N_{\leq c/N},$$

with the map to $\mathrm{Hom}(\Lambda, M)_{\leq c}$ *given by the sum map.*

Proof. Omitted (but done in Lean). □

Lemma 7.5. *Similarly, for any $c > 0$, we have*

$$\mathrm{Hom}(\Lambda'^{(m)}, M)_{\leq c} = \mathrm{Hom}(\Lambda', M)^{m/\,\mathrm{Hom}(\Lambda, M)_{\leq c}}_{\leq c},$$

the m-fold fibre product of $\mathrm{Hom}(\Lambda', M)_{\leq c}$ *over* $\mathrm{Hom}(\Lambda, M)_{\leq c}$.

Proof. Omitted (but done in Lean). □

Lemma 7.6. *There is a canonical isomorphism between the first row of the double complex*

$$C^{\mathsf{BD}}_{\kappa}(\mathrm{Hom}(\Lambda^{(1)}, M))^{\bullet}$$

and

$$C^{N \otimes \mathsf{BD}}_{\kappa/N}(\mathrm{Hom}(\Lambda, M))^{\bullet}$$

which identifies the map induced by the diagonal embedding $\Lambda \to \Lambda' = \Lambda^{(1)}$ with the map induced by $\sigma^N : N \otimes \mathsf{BD} \to \mathsf{BD}$.

Proof. Omitted (but done in Lean). □

Proposition 7.7. *Let $S' \to S$ be a surjective morphism of profinite sets, and let $S_\bullet \to S$ be its Cech nerve. Let V be a semi-normed group. Then the complex*

$$0 \to \widehat{V}(S) \to \widehat{V}(S_0) \to \widehat{V}(S_1) \to \ldots$$

is exact, and whenever $f \in \ker(\widehat{V}(S_m) \to \widehat{V}(S_{m+1}))$ with $\|f\| \leq c$, then for any $\epsilon > 0$ there is some $g \in \widehat{V}(S_{m-1})$ with $\|g\| \leq (1+\epsilon)c$ such that $d(g) = f$. In other words, the complex is normed exact in the sense of Definition 4.6.

Proof. Follow the proof of [Sch19, Theorem 3.3]: When S and all S_i are finite, the Cech cover splits, so a contracting homotopy gives the result with constant 1. In general, write the Cech cover as a cofiltered limit of Cech covers of finite sets by finite sets, pass to the filtered colimit, and complete, using Lemma 4.3. □

Lemma 7.8. *Let M be a profinitely filtered pseudo-normed group with T^{-1}-action that is N-splittable with error term $d \geq 0$. Let $k \geq 1$ be a real number, and let $c_0 > 0$ satisfy $d \leq \frac{(k-1)c_0}{N}$. For every c, consider the Cech nerve of the summation map $M^N_{c/N} \to M_c$. By applying the functor $\widehat{V}(\_)$ and taking the alternating face map complex, we obtain a system of complexes*

$$\widehat{V}(M_{\leq c}) \to \widehat{V}(M^N_{\leq c/N}) \to \ldots$$

This system of complexes is weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound 1.

Proof. For every constant c, consider the pullback

$$\begin{tikzcd}
 & M_c \arrow[r] & M_{kc} \\
 & X_c \arrow[u] \arrow[r] & M^N_{kc/N} \arrow[u] \\
M^N_{c/N} \arrow[uur] \arrow[ur, dotted] \arrow[urr] & &
\end{tikzcd}$$

We therefore get morphisms of cochain complexes

$$\begin{tikzcd}
\widehat{V}(M_{kc}) \arrow[r] \arrow[d] & \widehat{V}(M_c) \arrow[r] \arrow[d] & \widehat{V}(M_c) \arrow[d] \\
\widehat{V}(M^N_{kc/N}) \arrow[r] \arrow[d] & \widehat{V}(X_c) \arrow[r] \arrow[d] & \widehat{V}(M^N_{c/N}) \arrow[d] \\
\vdots \arrow[r] & \vdots \arrow[r] & \vdots
\end{tikzcd}$$

where all the columns are of the form "alternating face map complex of $\widehat{V}(\_)$ applied to a Cech nerve". Note that the horizontal maps are norm-nonincreasing and their compositions are restriction maps.

Claim: for $c \geq c_0$, the map $X_c \to M_c$ is surjective.

Indeed, by assumption every $x \in M_c$ can be decomposed into a sum $x = x_1 + \cdots + x_N$ with $x_i \in M_{c/N+d} \subset M_{kc/N}$, since $c \geq c_0$ and $d \leq \frac{(k-1)c_0}{N}$.

By Proposition 7.7, the middle column is normed exact (in the sence of Definition 4.6). The result follows from Lemma 4.10. $\square$

Proposition 7.9. *Let d be the constant from Proposition 6.3. Let $k > 1$ and $c_0 > 0$ be real numbers such that*

$$(k-1) * c_0/N \geq d.$$

Let m be any natural number, and put

$$K = (m+2) + \frac{r+1}{r(1-r)}(m+2)^2$$

Finally, let c_0' be $\frac{c_0}{r' \cdot n_i}$, where n_i is the i-th index in our fixed Breen–Deligne data.

Then i-th column in the double complex are (k^2, K)-weak bounded exact in degrees $\leq m$ for $c \geq c_0'$.

Proof. Let $M^{(m)}$ denote $\mathrm{Hom}(\Lambda^{(m)}, \overline{\mathcal{M}}_{r'}(S))^{n_i}$. We also write M for $M^{(0)} = \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))^{n_i}$ and M' for $M^{(1)}$. By Proposition 6.3, M is N-splittable with error term d.

Consider the diagram of morphisms of systems of complexes

$$\begin{tikzcd}
\widehat{V}(M_c)^{T^{-1}} \arrow[r] \arrow[d] & \widehat{V}(M_c) \arrow[r, "T^{-1}-[T^{-1}]^*"] \arrow[d] & \widehat{V}(M_c) \arrow[d] \\
\widehat{V}(M'_c)^{T^{-1}} \arrow[r] \arrow[d] & \widehat{V}(M'_c) \arrow[r, "T^{-1}-[T^{-1}]^*"] \arrow[d] & \widehat{V}(M'_c) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\widehat{V}(M_c^{(m)})^{T^{-1}} \arrow[r] & \widehat{V}(M_c^{(m)}) \arrow[r, "T^{-1}-[T^{-1}]^*"] & \widehat{V}(M_c^{(m)})
\end{tikzcd}$$

By Lemmas 7.8 and 7.5, we know that the two columns on the right are weakly $\leq k$-exact in degrees $\leq m$ and for $c \geq c_0$ with bound 1.

The result now follows from Lemma 5.8, and Proposition 4.13. □

Proposition 7.10. *Let h be the homotopy packaged with BD, and let h^N denote the n-th iterated composition of h (see Def 1.12) which is a homotopy between π^N and $\sigma^N : N \otimes \mathsf{BD} \to \mathsf{BD}$.*

Let $H \in \mathbf{R}_{\geq 0}$ be such that for $i = 0, \dots, m$ the universal map h_i^N is bound by H (see Def 1.4).

Then the double complex satisfies the normed homotopy homotopy condition (Def 4.14) for m, H, and c_0.

Proof. By Lemma 7.6 we may replace the first row by

$$C_{\kappa/N}^{N\otimes\mathsf{BD}}(\mathrm{Hom}(\Lambda, M))^\bullet.$$

Now it is important to recall that we have chosen $k' \geq \kappa'_i$ for all $i = 0, \dots, m+1$.

Our goal is to find, in degrees $\leq m$, a homotopy between the two maps from the first row

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)_{\leq c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^2_{\leq \kappa_1 c})^{T^{-1}} \to \dots$$

to the second row

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)^N_{\leq c/N})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^{2N}_{\leq \kappa_1 c/N})^{T^{-1}} \to \dots$$

respectively induced by σ^N and π^N (which are maps $N \otimes \mathsf{BD}$

By Definition 1.12 and Lemma 1.19 we can find this homotopy between the complex for $k'c$ and the complex for c. (Here we use $k' \geq c'_i$ for $i = 0, \dots, m$.) By assumption, the norm of these maps is bounded by H.

Finally, it remains to establish the estimate (4.1) on the homotopic map. We note that this takes $x \in \widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq k'^2\kappa_i c})^{T^{-1}}$ (with $i = q$ in the notation of (4.1)) to the element

$$y \in \widehat{V}(\mathrm{Hom}(\Lambda, M)^{Na_i}_{\leq k'\kappa_i c/N})^{T^{-1}}$$

that is the sum of the N pullbacks along the N projection maps $\mathrm{Hom}(\Lambda, M)^{Na_i}_{\leq k'\kappa_i c/N} \to \mathrm{Hom}(\Lambda, M)^{a_i}_{\leq k'^2\kappa_i c}$. We note that these actually take image in $\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq \kappa_i c}$ as $N \geq k'$, so this actually gives a well-defined map

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq \kappa_i c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^{Na_i}_{\leq k'\kappa_i c/N})^{T^{-1}}.$$

We need to see that this map is of norm $\leq \epsilon$. Now note that by our choice of N, we actually have $k'\kappa_i c/N \leq (r')^b\kappa_i c$, so this can be written as the composite of the restriction map

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq \kappa_i c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq (r')^b\kappa_i c})^{T^{-1}}$$

and

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)^{a_i}_{\leq (r')^b\kappa_i c})^{T^{-1}} \to \widehat{V}(\mathrm{Hom}(\Lambda, M)^{Na_i}_{\leq k'\kappa_i c/N})^{T^{-1}}.$$

The first map has norm exactly r^b, by T^{-1}-invariance, and as multiplication by T scales the norm with a factor of r on $\widehat{V}$. (Here is where we use $r' > r$, ensuring different scaling behaviour of the norm on source and target.) The second map has norm at most N (as it is a sum of N maps of norm ≤ 1). Thus, the total map has norm $\leq r^b N$. But by our choice of N, we have $r^b N \leq \epsilon$, giving the result. □

Proof of Theorem 7.2. By induction, the first condition of Proposition 4.15 is satisfied for all $c \geq c_0$ with c_0 large enough (depending on Λ but not V or S).

The second condition is Proposition 7.9, and the third condition has been checked in Proposition 7.10.

Thus, we can apply Proposition 4.15, and get the desired $\leq \max(k'^2, 2k_0H)$-exactness in degrees $\leq m$ for $c \geq c_0$, where k', k_0 and H were defined only in terms of k, m, r' and r, while c_0 depends on Λ (but not on V or S). This proves the inductive step. □

Question 7.11. Can one make the constants explicit, and how large are they? [1] Modulo the Breen-Deligne resolution, all the arguments give in principle explicit constants; and actually the proof of the existence of the Breen-Deligne resolution should be explicit enough to ensure the existence of bounds on the c_i and c_i'.

Answer: yes, we can do this. And we should write up a clean account asap, when we have cleaned up the proof in Lean.

References

[Sch19] P. Scholze. Lectures on Condensed Mathematics. 2019.
[Sch20] P. Scholze. Lectures on Analytic Geometry. 2020.

[1] A back of the envelope calculation seems to suggest that k is roughly doubly exponential in m, and that N has to be taken of roughly the same magnitude.