

# BLUEPRINT FOR THE LIQUID TENSOR EXPERIMENT

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**Remark 0.0.1.** This text is based on the lecture notes on Analytic Geometry [Sch20], by Peter Scholze. The final subsection is copy-pasted from those lecture notes almost verbatim. This text is meant as a blueprint for the Liquid Tensor Experiment.

**Remark 0.0.2.** In this text  $\mathbf{N}$  denotes the natural numbers *including* 0.

**Introduction.** The goal of this document is to provide a detailed account of the proof of the following theorem, along side a computer verification in the Lean theorem prover (see Section 0.0.1).

**Theorem** (Clausen–Scholze). *Let  $0 < p' < p \leq 1$  be real numbers, let  $S$  be a profinite set, and let  $V$  be a  $p$ -Banach space. Let  $\mathcal{M}_{p'}(S)$  be the space of  $p'$ -measures on  $S$ . Then*

$$\mathrm{Ext}_{\mathrm{Cond}(\mathrm{Ab})}^i(\mathcal{M}_{p'}(S), V) = 0$$

for  $i \geq 1$ .

We will explain this statement in more detail in Section 0.0.1 below.

**Remark.** *Status report of the project as of 10-06-2021. The main ingredient in the proof of the above Theorem is the highly technical Theorem 1.7.1.*

*We have completed a computer verification of this first target in Lean, and this document contains an account of the proof. We are in the process of cleaning up and documenting this first step.*

*Once that is done, we will work on the second step: deducing Theorem from Theorem 1.7.1.*

0.0.1. *On the statement of the main goal.* For the definition of condensed sets and condensed abelian groups, we refer to [Sch19].

A  $p$ -Banach space  $V$  is a complete topological  $\mathbb{R}$ -vector space whose topology is induced by a  $p$ -norm; that is, a norm satisfying  $\|rv\| = |r|^p \|v\|$ .

We will now explain the space  $\mathcal{M}_{p'}(S)$ . TODO (For now, see Definition 6.3 of [Sch20].)

*On the Lean theorem prover and computer verification of proofs.* The Lean theorem prover is developed by Leonardo de Moura at Microsoft Research.

TODO: write a short paragraph about what computer verification means, add pointers to further tutorials/material

## 1. FIRST PART

1.1. **Breen–Deligne data.** The goal of this subsection is to give a precise statement of a variant of the Breen–Deligne resolution. This variant is not actually a resolution, but it is sufficient for our purposes, and is much easier to state and prove.

We first recall the original statement of the Breen–Deligne resolution.

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**Theorem** (Breen–Deligne). *For an abelian group  $A$ , there is a resolution, functorial in  $A$ , of the form*

$$\dots \rightarrow \bigoplus_{i=1}^{n_i} \mathbb{Z}[A^{r_{ij}}] \rightarrow \dots \rightarrow \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \rightarrow \mathbb{Z}[A^2] \rightarrow \mathbb{Z}[A] \rightarrow A \rightarrow 0.$$

What does a homomorphism  $f: \mathbb{Z}[A^m] \rightarrow \mathbb{Z}[A^n]$  that is functorial in  $A$  look like? We should perhaps say more precisely what we mean by this. The idea is that  $m$  and  $n$  are fixed, and for each abelian group  $A$  we have a group homomorphism  $f_A: \mathbb{Z}[A^m] \rightarrow \mathbb{Z}[A^n]$  such that if  $\phi: A \rightarrow B$  is a group homomorphism inducing  $\phi_i: \mathbb{Z}[A^i] \rightarrow \mathbb{Z}[B^i]$  for each natural number  $i$  then the obvious square commutes:  $\phi_n \circ f_A = f_B \circ \phi_m$ .

The map  $f_A$  is specified by what it does to the generators  $(a_1, a_2, a_3, \dots, a_m) \in A^m$ . It can send such an element to an arbitrary element of  $\mathbb{Z}[A^n]$ , but one can check that universality implies that  $f_A$  will be a  $\mathbb{Z}$ -linear combination of “basic universal maps”, where a “basic universal map” is one that sends  $(a_1, a_2, \dots, a_m)$  to  $(t_1, \dots, t_n)$ , where  $t_i$  is a  $\mathbb{Z}$ -linear combination  $c_{i,1} \cdot a_1 + \dots + c_{i,m} \cdot a_m$ . So a “basic universal map” is specified by the  $n \times m$ -matrix  $c$ .

**Definition 1.1.1.** A *basic universal map* from exponent  $m$  to  $n$ , is an  $n \times m$ -matrix with coefficients in  $\mathbb{Z}$ .

**Definition 1.1.2.** A *universal map* from exponent  $m$  to  $n$ , is a formal  $\mathbb{Z}$ -linear combination of basic universal maps from exponent  $m$  to  $n$ .

If  $f$  is a basic universal map, then we write  $[f]$  for the corresponding universal map.

**Definition 1.1.3.** Let  $f = \sum_g n_g [g]$  be a universal map. We say that  $f$  is *bound by* a natural number  $N$  if  $\sum_g |n_g| \leq N$ .

We point out that basic universal maps can be composed by matrix multiplication, and this formally induces a composition of universal maps. As mentioned above, one can also check (this has been formalised in Lean) that this construction gives a bijection between universal maps from exponent  $m$  to  $n$  and functorial collections  $f_A: \mathbb{Z}[A^m] \rightarrow \mathbb{Z}[A^n]$ .

**Definition 1.1.4.** In other words, we are considering the following two categories:

- the category whose objects are natural numbers, and whose morphisms are matrices;
- the category with the same objects, but with Hom-sets replaced by the free abelian groups generated by the sets of matrices. We denote this latter category  $\text{FreeMat}$ .

Both categories naturally come with a monoidal structure: for the first it is given by the Kronecker product of matrices (a.k.a. tensor product of linear maps) which induces a monoidal structure on  $\text{FreeMat}$ . As usual, we will denote this monoidal structure  $\_ \otimes \_$ . For example, if  $f$  is a basic universal map, then  $2 \otimes f$  denotes the block matrix

$$\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}$$

**Definition 1.1.5.** Let  $N$  be a natural number, and  $i < N$ . Then  $\pi'_{N,i}$  denotes the basic universal map from exponent  $N$  to 1

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = (a_0, a_1, \dots, a_{N-1})^t$$

where  $a_j = \delta_{ij}$ .

**Definition 1.1.6.** Let  $N$  and  $n$  be natural numbers. Then  $\pi_n^N$  denotes the universal map from exponent  $N \cdot n$  to  $n$  given by  $\sum_{i < N} [\pi'_{N,i} \otimes n]$ .

(On  $\mathbf{Z}[A^{N \cdot n}] \rightarrow \mathbf{Z}[A^n]$  this map is the formal sum of the maps  $\mathbf{Z}[A^{N \cdot n}] \rightarrow \mathbf{Z}[A^n]$  induced by the projection maps  $A^{N \cdot n} = (A^n)^N \rightarrow A^n$ .)

**Definition 1.1.7.** Let  $N$  and  $n$  be natural numbers. Then  $\sigma_n^N$  denotes the universal map from exponent  $N \cdot n$  to  $n$  given by  $[\sum_{i < N} \pi'_{N,i} \otimes n]$ .

(On  $\mathbf{Z}[A^{N \cdot n}] \rightarrow \mathbf{Z}[A^n]$  this map is induced by the summation map  $A^{N \cdot n} = (A^n)^N \rightarrow A^n$ .)

**Definition 1.1.8.** A *Breen–Deligne data* is a chain complex in  $\text{FreeMat}$ .

Concretely, this means that it consists of a sequence of exponents  $n_0, n_1, n_2, \dots \in \mathbb{N}$ , and universal maps  $f_i$  from exponent  $n_{i+1}$  to  $n_i$ , such that for all  $i$  we have  $f_i \circ f_{i+1} = 0$ .

A morphism of Breen–Deligne data is a morphism of chain complexes.

**Definition 1.1.9.** For every natural numbers  $N$ , the endofunctor  $N \otimes \_$  on  $\text{FreeMat}$  induces an endofunctor of Breen–Deligne data.

Concretely, it maps a pair  $(n, f)$  of Breen–Deligne data, to the pair  $N \otimes (n, f)$  consisting of exponents  $N \cdot n_i$  and universal maps  $N \otimes f_i$ .

Let  $\text{BD}$  be Breen–Deligne data. The universal maps  $\sigma^N$  and  $\pi^N$  defined above, induce morphisms  $\sigma_{\text{BD}}^N, \pi_{\text{BD}}^N: N \otimes \text{BD} \rightarrow \text{BD}$ .

**Definition 1.1.10.** A *Breen–Deligne package* consists of Breen–Deligne data  $\text{BD}$  together with a homotopy  $h$  between  $\pi_{\text{BD}}^2$  and  $\sigma_{\text{BD}}^2$ .

**Definition 1.1.11.** Let  $\text{BD}$  be a Breen–Deligne package and  $N$  a power of 2. Then the homotopy  $h$  induces a homotopy between  $\pi_{\text{BD}}^N$  and  $\sigma_{\text{BD}}^N$  by iterative composition of the homotopy packaged in  $\text{BD}$ .

**Definition 1.1.12.** We will now construct an example of a Breen–Deligne package. In some sense, it is the “easiest” solution to the conditions posed above. The exponents will be  $n_i = 2^i$ , and the homotopies  $h_i$  will be the identity. Under these constraints, we recursively construct the universal maps  $f_i$ :

$$f_0 = \pi_1^2 - \sigma_1^2, \quad f_{i+1} = (\pi_{2^{i+1}}^2 - \sigma_{2^{i+1}}^2) - (2 \otimes f_i).$$

We leave it as exercise for the reader, to verify that with these definitions  $(n, f, h)$  forms a Breen–Deligne package.

We now make definitions that will make precise some conditions between constants that will be needed when we construct Breen–Deligne complexes of normed abelian groups.

**Definition 1.1.13.** Let  $f$  be a basic universal map from exponent  $m$  to  $n$ . Let  $c_1, c_2 \in \mathbb{R}_{\geq 0}$ . We say that  $(c_1, c_2)$  is  $f$ -suitable, if for all  $i$

$$\sum_j c_1 |f_{ij}| \leq c_2.$$

To orient the reader: later on we will be considering maps on normed abelian groups induced from universal maps, and this inequality will guarantee that if  $\|m\| \leq c_1$  then  $\|f(m)\| \leq c_2$ .

**Definition 1.1.14.** Let  $f$  be a universal map from exponent  $m$  to  $n$ . Let  $c_1, c_2 \in \mathbb{R}_{\geq 0}$ . We say that  $(c_1, c_2)$  is  $f$ -suitable, if for all basic universal maps  $g$  that occur in the formal sum  $f$ , the pair of nonnegative reals  $(c_1, c_2)$  is  $g$ -suitable.

**Definition 1.1.15.** Let  $f$  be a universal map and let  $r, r', c_1, c_2 \in \mathbb{R}_{\geq 0}$ . We say that  $(c_1, c_2)$  is *very suitable* for  $(f, r, r')$  if there exist  $N, b \in \mathbb{N}$  and  $c' \in \mathbb{R}_{\geq 0}$  such that:

- $f$  is bound by  $N$  (see Definition 1.1.3)
- $(c_1, c')$  is  $f$ -suitable
- $r^b N \leq 1$
- $c' \leq (r')^b c_2$

**Definition 1.1.16.** Let  $\text{BD} = (n, f)$  be Breen–Deligne data, let  $r, r' \in \mathbb{R}_{\geq 0}$ , and let  $\kappa = (\kappa_0, \kappa_1, \dots)$  be a sequence of nonnegative real numbers. We say that  $\kappa$  is *BD-suitable* (resp. *very suitable* for  $(\text{BD}, r, r')$ ), if for all  $i$ , the pair  $(\kappa_{i+1}, \kappa_i)$  is  $f_i$ -suitable (resp. *very suitable* for  $(f_i, r, r')$ ).

(Note! The order  $(\kappa_{i+1}, \kappa_i)$  is contravariant compared to Definition 1.1.14. This is because of the contravariance of  $\widehat{V}(\_)$ ; see Definition 1.5.9.)

**Definition 1.1.17.** Let  $\text{BD}$  be a Breen–Deligne package with data  $(n, f)$  and homotopy  $h$ . Let  $\kappa, \kappa'$  be sequences of nonnegative real numbers. (In applications  $\kappa$  is a  $(n, f)$ -suitable sequence.)

Then  $\kappa'$  is *adept* to  $(\text{BD}, \kappa)$  if for all  $i$  the pair  $(\kappa_i/2, \kappa'_{i+1}\kappa_{i+1})$  is  $h_i$ -suitable. (Recall that  $h_i$  is the homotopy map  $n_i \rightarrow n_{i+1}$ .)

**Lemma 1.1.18.** Let  $\text{BD}$  be a Breen–Deligne package,  $N$  a power of 2, and let  $\kappa, \kappa'$  be sequences of nonnegative real numbers. Assume that  $\kappa'$  is adept to  $(\text{BD}, \kappa)$ . Let  $h^N$  be the homotopy between  $\pi_{\text{BD}}^N$  and  $\sigma_{\text{BD}}^N$  defined in Def 1.1.11.

For all  $i$ , the pair  $(\kappa_i/N, \kappa'_{i+1}\kappa_{i+1})$  is  $h_i^N$ -suitable.

*Proof.* Omitted. (But done in Lean.) □

**Lemma 1.1.19.** Let  $\text{BD}$  be a Breen–Deligne package, and let  $r, r'$  be nonnegative reals, such that  $r < 1$  and  $r' > 0$ .

There exists a sequence  $\kappa$  of positive real numbers such that  $\kappa$  is very suitable for  $(\text{BD}, r, r')$ .

*Proof.* The sequence can be constructed recursively, which we leave as exercise for the reader. (It has been done in Lean.) □

**Lemma 1.1.20.** Let  $\text{BD}$  be a Breen–Deligne package, and let  $r, r'$  be nonnegative reals, such that  $0 < r < 1$  and  $0 < r' \leq 1$ . Let  $\kappa$  be any sequence of positive reals.

There exists a sequence  $\kappa'$  of nonnegative real numbers that is adept to  $(\text{BD}, \kappa)$ .

*Proof.* The sequence can be constructed recursively, which we leave as exercise for the reader. (It has been done in Lean.) □

**1.2. Variants of normed groups.** Normed groups are well-studied objects. In this text it will be helpful to work with the more general notion of *semi-normed group*. This drops the separation axiom  $\|x\| = 0 \iff x = 0$  but is otherwise the same as a normed group.

The main difference is that this includes “uglier” objects, but creates a “nicer” category: semi-normed groups need not be Hausdorff, but quotients by arbitrary (possibly non-closed) subgroups are naturally semi-normed groups.

Nevertheless, there is the occasional use for the more restrictive notion of normed group, when we come to polyhedral lattices below (see Section 1.6).

In this text, a morphism of (semi)-normed groups will always be bounded. If the morphism is supposed to be norm-nonincreasing, this will be mentioned explicitly.

**Definition 1.2.1.** Let  $r > 0$  be a real number. An  $r$ -normed  $\mathbb{Z}[T^{\pm 1}]$ -module is a semi-normed group  $V$  endowed with an automorphism  $T: V \rightarrow V$  such that for all  $v \in V$  we have  $\|T(v)\| = r\|v\|$ .

The remainder of this subsection sets up some algebraic variants of semi-normed groups.

**Definition 1.2.2.** A *pseudo-normed group* is an abelian group  $(M, +)$ , together with an increasing filtration  $M_c \subseteq M$  of subsets  $M_c$  indexed by  $\mathbb{R}_{\geq 0}$ , such that each  $M_c$  contains 0, is closed under negation, and  $M_{c_1} + M_{c_2} \subseteq M_{c_1+c_2}$ . An example would be  $M = \mathbb{R}$  or  $M = \mathbb{Q}_p$  with  $M_c := \{x : |x| \leq c\}$ .

A pseudo-normed group  $M$  is *exhaustive* if  $\bigcup_c M_c = M$ .

All pseudo-normed groups that we consider will have a topology on the filtration sets  $M_c$ . The most general variant is the following notion.

**Definition 1.2.3.** A pseudo-normed group  $M$  is *CH-filtered* if each of the sets  $M_c$  is endowed with a topological space structure making it a compact Hausdorff space, such that following maps are all continuous:

- the inclusion  $M_{c_1} \rightarrow M_{c_2}$  (for  $c_1 \leq c_2$ );
- the negation  $M_c \rightarrow M_c$ ;
- the addition  $M_{c_1} \times M_{c_2} \rightarrow M_{c_1+c_2}$ .

The pseudo-normed group  $M$  is *profinutely filtered* if moreover the filtration sets  $M_c$  are totally disconnected, making them profinite sets.

**Remark 1.2.4.** The topologies on the filtration sets  $M_c$  will induce a topology on  $M$ : the colimit topology. If  $M$  is some sort of normed group, then this topology is typically genuinely different from the norm topology.

**Definition 1.2.5.** A *morphism* of CH-filtered pseudo-normed groups  $M \rightarrow N$  is a group homomorphism  $f: M \rightarrow N$  that is

- *bounded*: there is a constant  $C$  such that  $x \in M_c$  implies  $f(x) \in N_{Cc}$ ;
- *continuous*: for one (or equivalently all) constants  $C$  as above, the induced map  $M_c \rightarrow N_{Cc}$  is a morphism of profinite sets, i.e. continuous.

The reason the two definitions of continuity are equivalent is that a continuous injection from a compact space to a Hausdorff space must be a topological embedding.

A morphism  $f: M \rightarrow N$  is *strict* if  $x \in M_c$  implies  $f(x) \in N_c$  (in other words, if we can take  $C = 1$  in the boundedness condition above).

We will also consider the analogue of an  $r$ -normed  $\mathbb{Z}[T^{-1}]$ -module in the pseudo-normed setting.

**Definition 1.2.6.** Let  $r'$  be a positive real number. A CH-filtered pseudo-normed group  $M$  has an  $r'$ -action of  $T^{-1}$  if it comes endowed with a distinguished morphism of CH-filtered pseudo-normed groups  $T^{-1}: M \rightarrow M$  that is bounded by  $r'^{-1}$ : if  $x \in M_c$  then  $T^{-1}x \in M_{c/r'}$ .

A morphism of CH-filtered pseudo-normed groups with  $r'$ -action of  $T^{-1}$  is a morphism  $f: M \rightarrow N$  of CH-filtered pseudo-normed groups that commutes with the action of  $T^{-1}$ .

**1.3. Spaces of convergent power series.** We will now construct the central example of profinitely filtered pseudo-normed groups with  $r'$ -action of  $T^{-1}$ .

**Definition 1.3.1.** Let  $r' > 0$  be a real number, and let  $S$  be a finite set. Denote by  $\overline{\mathcal{L}}_{r'}(S)$  the set

$$\left\{ \left( \sum_{n \geq 1} a_{n,s} T^n \in T\mathbb{Z}[[T]] \right)_{s \in S} \left| \sum_{n \geq 1, s \in S} |a_{n,s}| (r')^n < \infty \right. \right\}.$$

Note that  $\overline{\mathcal{L}}_{r'}(S)$  is naturally a pseudo-normed group with filtration given by

$$\overline{\mathcal{L}}_{r'}(S)_{\leq c} = \left\{ \left( \sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} \left| \sum_{n \geq 1, s \in S} |a_{n,s}| (r')^n \leq c \right. \right\}.$$

**Lemma 1.3.2.** Let  $r' > 0$  and  $c \geq 0$  be real numbers, and let  $S$  be a finite set. The space  $\overline{\mathcal{L}}_{r'}(S)_{\leq c}$  is the profinite limit of the finite sets

$$\overline{\mathcal{L}}_{r'}(S)_{\leq c, \leq N} = \left\{ \left( \sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} \left| \sum_{1 \leq n \leq N, s \in S} |a_{n,s}| (r')^n \leq c \right. \right\}$$

This endows  $\overline{\mathcal{L}}_{r'}(S)_{\leq c}$  with the profinite topology. In particular, it is a profinitely filtered pseudo-normed group.

*Proof.* Formalised, but omitted from this text. □

For the remainder of this subsection, let  $r' > 0, c \geq 0$  be real numbers, and let  $S$  be a finite set.

**Definition 1.3.3.** There is a natural action of  $T^{-1}$  on  $\overline{\mathcal{L}}_{r'}(S)$ , via

$$T^{-1} \cdot \left( \sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} = \left( \sum_{n \geq 1} a_{n+1,s} T^n \right)_{s \in S}.$$

**Lemma 1.3.4.** The natural action of  $T^{-1}$  on  $\overline{\mathcal{L}}_{r'}(S)$  restricts to continuous maps

$$T^{-1} \cdot \_ : \overline{\mathcal{L}}_r(S)_{\leq c} \rightarrow \overline{\mathcal{L}}_r(S)_{\leq c/r'}.$$

In particular,  $\overline{\mathcal{L}}_{r'}(S)$  has an  $r'$ -action of  $T^{-1}$ .

*Proof.* Formalised, but omitted from this text. □

**1.4. Some normed homological algebra.** It will be convenient to use the following definition generalizing the notion of a bound on the norm of a right inverse of a normed group morphism (note that the morphisms we consider have no reason to have a right inverse, even when they are surjective).

**Definition 1.4.1.** Let  $G$  and  $H$  be semi-normed groups, let  $K$  be a subgroup of  $H$  and  $C$  be a positive real number. A morphism  $f : G \rightarrow H$  is  $C$ -surjective onto  $K$  if, for all  $x$  in  $K$ , there exists some  $g$  in  $G$  such that  $f(g) = x$  and  $\|g\| \leq C\|x\|$ . If  $K = H$  we simply say  $f$  is  $C$ -surjective.

The following controlled surjectivity lemma will be used to prove Lemma 1.4.3 and Lemma 1.5.8.

**Lemma 1.4.2.** *Let  $G$  and  $H$  be normed groups. Let  $K$  be a subgroup of  $H$  and  $f$  a morphism from  $G$  to  $H$ . Assume that  $G$  is complete and  $f$  is  $C$ -surjective onto  $K$ . Then  $f$  is  $(C + \varepsilon)$ -surjective onto the topological closure of  $K$  for every positive  $\varepsilon$ .*

*Proof.* Let  $x$  be any element of the closure of  $K$ . First note the conclusion is trivial when  $x = 0$ , so we can assume  $x \neq 0$ . Then write  $x$  as a sum  $\sum_{i \geq 0} x_i$  with all  $x_i \in K$ ,  $\|x - x_0\| \leq \varepsilon_0$  and  $\|x_i\| \leq \varepsilon_i$  for  $i > 0$  for some sequence of positive numbers  $\varepsilon_i$  to be chosen later. By assumption, we can then lift each  $x_i$  to  $g_i$  such that  $f(g_i) = x_i$  and  $\|g_i\| \leq C\|x_i\|$ , and then set  $g = \sum g_i$ . Because  $G$  is complete, this sum converges provided the  $\varepsilon_i$  sequence converges fast enough to zero. We then have  $f(g) = x$  and

$$\|g\| \leq C \sum_{i \geq 0} \|x_i\| \leq C(\|x\| + \varepsilon_0) + C \sum_{i > 0} \varepsilon_i \leq (C + \varepsilon)\|x\|$$

where the last inequality holds provided the  $\varepsilon_i$  sequence converges fast enough to zero. For instance  $\varepsilon_i = \varepsilon \|x\| / (2^{i+1}C)$  satisfies all our constraints on the  $\varepsilon_i$  sequence (in particular they are positive because  $x \neq 0$ ).  $\square$

The first application of the above lemma is a completion result for a quantitative version of being a complex.

**Lemma 1.4.3.** *Let  $f : M_0 \rightarrow M_1$  and  $g : M_1 \rightarrow M_2$  be bounded maps between normed groups. Assume there are positive constants  $C$  and  $D$  such that:*

- $f$  is  $C$ -surjective onto  $\ker g$ .
- $g$  is  $D$ -surjective onto its image.

*Then for every positive  $\varepsilon$ ,  $\hat{f}$  is  $(C + \varepsilon)$ -surjective onto  $\ker \hat{g}$ .*

*Proof.* Since  $f$  is  $C$ -surjective onto  $\ker g$ ,  $\hat{f}$  is  $C$ -surjective onto  $\ker g$  seen as a subset of  $\widehat{M_1}$ . Hence this lemma will follow directly from Lemma 1.4.2 once we'll have proven that  $\ker g$  is dense in  $\ker \hat{g}$ . Let  $\hat{y}$  be an element of  $\ker \hat{g}$ . Pick any  $\delta > 0$  and take  $y \in M_1$  such that  $\|\hat{y} - y\| \leq \delta$ . Let  $z = g(y) \in M_2$ , which has norm  $\|z\| = \|g(y)\| = \|g(y - \hat{y})\|$  bounded by  $C_g \delta$ , where  $C_g$  is the norm of  $g$ . We can thus find some  $y' \in M_1$  with  $\|y'\| \leq DC_g \delta$  and  $g(y') = z$ . Replacing  $y$  by  $y - y'$ , we can thus find  $y \in \ker(g : M_1 \rightarrow M_2)$  such that still  $\|\hat{y} - y\| \leq (1 + DC_g)\delta$ ; as  $\delta$  was arbitrary, this gives the desired density.  $\square$

**Definition 1.4.4.** A *system of complexes* of normed abelian groups is for each  $c \in \mathbb{R}_{\geq 0}$  a complex

$$C_c^\bullet : C_c^0 \rightarrow C_c^1 \rightarrow \dots$$

of normed abelian groups together with maps of complexes  $\text{res}_{c',c}^\bullet : C_{c'}^\bullet \rightarrow C_c^\bullet$ , for  $c' \geq c$ , satisfying  $\text{res}_{c,c}^\bullet = \text{id}$  and the obvious associativity condition. In other words, a functor from  $(\mathbb{R}_{\geq 0})^{\text{op}}$  to cochain complexes of semi-normed groups.

By convention, for every system of complexes  $C^\bullet$ , we will set  $C_c^{-1} = 0$  for all  $c$ . This will come up each time we write  $C_c^{i-1}$  and  $i$  could be 0.

In this subsection, given  $x \in C_{c'}^\bullet$  and  $c_0 \leq c \leq c'$  we will use the notation  $x|_c := \text{res}_{c',c}^\bullet(x)$ .

**Definition 1.4.5.** A system of complexes is *admissible* if all differentials and maps  $\text{res}_{c',c}^i$  are norm-nonincreasing.

Throughout the rest of this subsection,  $k$  (and  $k', k''$ ) will denote reals at least 1,  $m$  will be a non-negative integer, and  $K, K', K''$  will denote non-negative reals.

**Definition 1.4.6.** A cochain complex  $C$  of semi-normed groups is *normed exact* if for all  $i \geq 0$ , all  $\varepsilon > 0$ , and all  $x \in C^i$  with  $d(x) = 0$  there exists a  $y \in C^{i-1}$  such that  $d(y) = x$  and  $\|y\| \leq (1 + \varepsilon)\|x\|$ .

**Definition 1.4.7.** Let  $C^\bullet$  be a system of complexes. For an integer  $m \geq 0$  and reals  $k \geq 1, K \geq 0$  and  $c_0 \geq 0$ , we say the datum  $C^\bullet$  is  *$\leq k$ -exact in degrees  $\leq m$  and for  $c \geq c_0$  with bound  $K$*  if the following condition is satisfied. For all  $c \geq c_0$  and all  $x \in C_{kc}^i$  with  $i \leq m$  there is some  $y \in C_c^{i-1}$  such that

$$\|x|_c - dy\| \leq K\|dx\|.$$

We will also need a version where the inequality is relaxed by some arbitrary small additive constant.

**Definition 1.4.8.** Let  $C^\bullet$  be a system of complexes. For an integer  $m \geq 0$  and reals  $k \geq 1, K \geq 0$  and  $c_0 \geq 0$ , the datum  $(C_c^\bullet)_c$  is *weakly  $\leq k$ -exact in degrees  $\leq m$  and for  $c \geq c_0$  with bound  $K$*  if the following condition is satisfied. For all  $c \geq c_0$ , all  $x \in C_{kc}^i$  with  $i \leq m$  and any  $\varepsilon > 0$  there is some  $y \in C_c^{i-1}$  such that

$$\|x|_c - dy\| \leq K\|dx\| + \varepsilon.$$

We first note that the difference between those two definitions is only about cocycles if we are ready to lose a tiny something on the norm bound  $K$ .

**Lemma 1.4.9.** Let  $C^\bullet$  be a system of complexes. If  $C^\bullet$  is weakly  $\leq k$ -exact in degrees  $\leq m$  and for  $c \geq c_0$  with bound  $K$  and if, for all  $c \geq c_0$  and all  $x \in C_{kc}^i$  with  $i \leq m$  such that  $dx = 0$  there is some  $y \in C_c^{i-1}$  such that  $x|_c = dy$  then, for every positive  $\delta$ ,  $C^\bullet$  is  $\leq k$ -exact in degrees  $\leq m$  and for  $c \geq c_0$  with bound  $K + \delta$ .

*Proof.* Let  $\delta$  be some positive real number. Let  $x$  be an element of  $C_{kc}^i$  for some  $c \geq c_0$  and  $i \leq m$ . If  $dx = 0$  then the assumption we made about exact elements is exactly what we want.

Assume now that  $dx \neq 0$ . The weak exactness assumption applied to  $\varepsilon = \delta\|dx\|$  gives some  $y \in C_c^{i-1}$  such that

$$\begin{aligned} \|x|_c - dy\| &\leq K\|dx\| + \delta\|dx\| \\ &= (K + \delta)\|dx\| \end{aligned}$$

□



**Lemma 1.4.10.** *Let  $k \geq 1$ ,  $c_0 \geq 0$  be real numbers, and  $m \in \mathbb{N}$ . Let  $C^\bullet$  be a system of complexes, and for each  $c \geq 0$  let  $D_c$  be a cochain complex of semi-normed groups. Let  $f_c: C_{kc}^\bullet \rightarrow D_c^\bullet$  and  $g_c: D_c^\bullet \rightarrow C_c^\bullet$  be norm-nonincreasing morphisms of cochain complexes of semi-normed groups such that  $g_c \circ f_c$  is the restriction map  $C_{kc}^\bullet \rightarrow C_c^\bullet$ . Assume that for all  $c \geq c_0$  the cochain complex  $D_c$  is normed exact. Then  $C^\bullet$  is weakly  $\leq k$ -exact in degrees  $\leq m$  and for  $c \geq c_0$  with bound 1.*

*Proof.* Fix  $c \geq c_0$ ,  $i \leq m$ ,  $x \in C_{kc}^i$ , and  $\varepsilon > 0$ . Denote by  $\delta$  the positive real number  $\frac{\varepsilon}{\|x\|+1}$ .

Clearly  $f(d(x))$  is killed by  $d$ , so by normed exactness of  $D_c$  we find  $x' \in D_c^i$  such that  $d(x') = f(d(x))$  and  $\|x'\| \leq (1 + \delta)\|f(d(x))\|$ . Similarly  $d(f(x) - x') = 0$ , so by exactness of  $D_c$  we find  $y \in D_c^{i-1}$  such that  $d(y) = f(x) - x'$ .

We are done if we show that  $\|x|_c - d(g(y))\| \leq \|d(x)\| + \varepsilon$ . Observe that  $x|_c - d(g(y)) = g(f(x)) - g(d(y)) = g(x')$ , and therefore we shall show  $\|g(x')\| \leq \|d(x)\| + \varepsilon$ .

Now we use that  $f$  and  $g$  are norm-nonincreasing to calculate

$$\|g(x')\| \leq \|x'\| \leq (1 + \delta)\|f(d(x))\| \leq (1 + \delta)\|d(x)\|.$$

Finally, we have  $(1 + \delta)\|d(x)\| \leq \|d(x)\| + \varepsilon$  by our choice of  $\delta$ .  $\square$

**Lemma 1.4.11.** *Let  $M^\bullet$  be an admissible collection of complexes of complete normed abelian groups.*

*Assume that  $M_c^\bullet$  is weakly  $\leq k$ -exact in degrees  $\leq m$  for  $c \geq c_0$  with bound  $K$ . Then  $M_c^\bullet$ , for every  $\delta > 0$ , it is  $\leq k^2$ -exact in degrees  $\leq m$  for  $c \geq c_0$  with bound  $K + \delta$ .*

*Proof.* Lemma 1.4.9 ensures we only need to care about cocycles of  $M$ . More precisely, let  $x$  be a cocycle in  $M_{k^2c}^i$  for some  $i \leq m$  and  $c \geq c_0$ . We need to find  $y \in M_c^{i-1}$  such that  $dy = x|_c$ .

By weak  $\leq k$ -exactness applied to  $x$  and a sequence  $\varepsilon_j$  to be chosen later, we can find a sequence  $w^j \in M_{kc}^{i-1}$  such that

$$\|x_{kc} - dw^j\| \leq \varepsilon_j.$$

Then, by weak  $\leq k$ -exactness applied to each  $w^{j+1} - w^j$  and a sequence  $\delta_j$  to be chosen later, we can find a sequence  $z^j \in M_c^{i-2}$  such that

$$\|(w^{j+1} - w^j)|_c - dz^j\| \leq K\|dw^{j+1} - dw^j\| + \delta_j.$$

We set  $y^j := w^j|_c - \sum_{l=0}^{j-1} dz^l \in M_c^{i-1}$ .

We have

$$\begin{aligned} \|y^{j+1} - y^j\| &= \|(w^{j+1} - w^j)|_c - dz^j\| \\ &\leq K\|dw^{j+1} - dw^j\| + \delta_j \\ &\leq 2K\varepsilon_j + \delta_j. \end{aligned}$$

So  $y^j$  is a Cauchy sequence as long as we make sure  $2K\varepsilon_j + \delta_j \leq 2^{-j}$  for instance. Since  $M_c^{i-1}$  is complete, this sequence converges to some  $y$ . Because  $dy^j = dw^j|_c$ , we get that  $\|x|_c - dy^j\| \leq \varepsilon_j$  and in the limit  $x|_c = dy$ .  $\square$

**Proposition 1.4.12.** *Let  $M^\bullet$  and  $M'^\bullet$  be two admissible collections of complexes of complete normed abelian groups. For each  $c \geq c_0$  let  $f_c^\bullet: M_c^\bullet \rightarrow M'_c^\bullet$  be a collection of maps between these*

collections of complexes that are norm-nonincreasing and which all commute with all restriction maps, and assume that there exists these maps satisfy

$$\|x|_c\| \leq K'' \|f(x)\|$$

for all  $i \leq m+1$  and all  $x \in M_{kk''c}^i$ . Let  $N_c^\bullet = M_c'^\bullet / M_c^\bullet$  be the collection of quotient complexes, with the quotient norm; this is again an admissible collection of complexes.

Assume that  $M_c^\bullet$  (resp.  $M_c'^\bullet$ ) is weakly  $\leq k$ -exact (resp.  $\leq k'$ -exact) in degrees  $\leq m$  for  $c \geq c_0$  with bound  $K$  (resp.  $K'$ ). Then  $N_c^\bullet$  is weakly  $\leq kk'k''$ -exact in degrees  $\leq m-1$  for  $c \geq c_0$  with bound  $K'(KK''+1)$ .

*Proof.* Let  $n \in N_{kk'k''c}^i$  for  $i \leq m-1$ . We fix  $\varepsilon > 0$ . We need to find an element  $y \in N_c^{i-1}$  such that

$$\|n|_c - dy\| \leq K'(KK''+1)\|dn\| + \varepsilon.$$

Pick any preimage  $m' \in M_{kk'k''c}^i$  of  $n$ . In particular  $dm'$  is a preimage of  $dn$ . By definition of the quotient norm, we can find  $m_1 \in M_{kk'k''c}^{i+1}$  and  $m_1'' \in (M')_{kk'k''c}^{i+1}$  such that

$$dm' = f(m_1) + m_1''$$

with  $\|m_1''\| \leq \|dn\| + \varepsilon_1$ , for some positive  $\varepsilon_1$  to be chosen later.

Applying the differential to the last displayed equation, and using that this kills the image of  $d$ , and that  $f$  is a map of complexes, we see that

$$f(dm_1) = -dm_1''.$$

Using the norm bound on  $f$ , we get

$$\begin{aligned} \|dm_{1|kk'c}\| &\leq K'' \|f(dm_1)\| = K'' \|dm_1''\| \\ &\leq K'' \|m_1''\| \leq K'' \|dn\| + K'' \varepsilon_1. \end{aligned}$$

On the other hand, weak exactness of  $M$  applied to  $m_{1|kk'c}$  gives  $m_0 \in M_{k'c}^i$  such that

$$\|m_{1|kk'c|k'c} - dm_0\| \leq K \|dm_{1|kk'c}\| + \varepsilon_1$$

which combines with the previous estimate to give:

$$\|m_{1|k'c} - dm_0\| \leq KK'' \|dn\| + (KK''+1)\varepsilon_1.$$

Now let  $m'_{\text{new}} = m'_{|k'c} - f(m_0) \in M_{k'c}^i$ ; this is a lift of  $n_{|k'c}$ . Then

$$dm'_{\text{new}} = dm'_{|k'c} - f(m_{1|k'c} - dm_0) = m''_{1|k'c} + f(m_{1|k'c} - dm_0).$$

In particular,

$$\|dm'_{\text{new}}\| \leq (KK''+1)\|dn\| + (KK''+2)\varepsilon_1.$$

Now weak exactness of  $M'$  gives  $x \in M_c^{i-1}$  such that

$$\|m'_{\text{new}|c} - dx\| \leq K' \|dm'_{\text{new}}\| + \varepsilon_1 \leq K'((KK''+1)\|dn\| + (KK''+2)\varepsilon_1) + \varepsilon_1.$$

In particular, letting  $y \in N_c^{i-1}$  be the image of  $x$ , we get

$$\|n|_c - dy\| \leq K'(KK''+1)\|dn\| + (K'(KK''+2)+1)\varepsilon_1,$$

which is exactly what we wanted if we choose  $\varepsilon_1 = \varepsilon / (K'(KK''+2)+1)$ .  $\square$

We also need the ‘dual’ version of 1.4.12, for kernels instead of cokernels. Note that this is actually a bit easier to prove. (Should we put it first in the presentation?)

**Proposition 1.4.13.** *Let  $M_\bullet^\bullet$  and  $M'_\bullet^\bullet$  be two admissible collections of complexes of complete normed abelian groups. For each  $c \geq c_0$  let  $f_c^\bullet : M_c^\bullet \rightarrow M'_c{}^\bullet$  be a collection of maps between these collections of complexes which all commute with all restriction maps. Assume moreover that we are given two constants  $r_1, r_2 \geq 0$  such that:*

- *for all  $i, c \geq c_0$  and all  $x \in M_c^i$*

$$\|f(x)\| \leq r_1 \|x\|;$$

- *for all  $i \leq m+1, c \geq c_0$  and all  $y \in M_c^{i'}$ , there exists  $x \in M_c^i$  such that*

$$f(x) = y \text{ and } \|x\| \leq r_2 \|y\|.$$

*Let  $N_c^\bullet$  be the collection of kernel complexes, with the induced norm; this is again an admissible collection of complexes.*

*Assume that  $M_c^\bullet$  (resp.  $M'_c{}^\bullet$ ) is weakly  $\leq k$ -exact (resp.  $\leq k'$ -exact) in degrees  $\leq m$  for  $c \geq c_0$  with bound  $K$  (resp.  $K'$ ). Then  $N_c^\bullet$  is weakly  $\leq kk'$ -exact in degrees  $\leq m-1$  for  $c \geq c_0$  with bound  $K + r_1 r_2 K K'$ .*

*Proof.* Let  $n \in N_{kk'c}^i \subseteq M_{kk'c}^i$  for  $i \leq m-1$  and let  $\varepsilon > 0$ . We need to find an element  $y \in N_c^{i-1}$  such that

$$\|n_{|c} - dy\| \leq K + r_1 r_2 K K' \|dn\| + \varepsilon.$$

By weak exactness of  $M_\bullet^\bullet$ , we can find  $m \in M_{k'c}^{i-i}$  such that

$$\|n_{|k'c} - dm\| \leq K \|dn\| + \varepsilon_1,$$

where  $\varepsilon_1 > 0$  to be chosen later. By weak exactness of  $M'_\bullet{}^\bullet$ , we can find  $m' \in M_c^{i-2}$  such that

$$\|f(m)_{|c} - dm'\| \leq K' \|df(m)\| + \varepsilon_2,$$

where  $\varepsilon_2 > 0$  to be chosen later. Let  $m_1 \in M_c^{i-2}$  be a lift of  $m'$  and let  $m_2 \in M_c^{i-1}$  be such that

$$f(m_2) = f(m_{|c} - dm_1) \text{ and } \|m_2\| \leq r_2 \|f(m_{|c} - dm_1)\|.$$

Set  $y = m_{|c} - dm_1 - m_2 \in M_c^{i-1}$ . By construction  $f(y) = 0$ , so  $y \in N_c^{i-1}$ . We compute

$$\begin{aligned} \|n_{|c} - dy\| &= \|n_{|c} - dm_{|c} + d^2 m_1 - dm_2\| = \|n_{|c} - dm_{|c} - dm_2\| \leq \\ &\|n_{|c} - dm_{|c}\| + \|dm_2\| = \|(n_{|k'c} - dm)_{|c}\| + \|dm_2\| \leq \|(n_{|k'c} - dm)\| + \|dm_2\| \leq \\ &K \|dn\| + \varepsilon_1 + \|dm_2\|. \end{aligned}$$

Where we have used the defining property of  $m$  and admissibility of  $M_\bullet^\bullet$ . By the same assumption and since  $f(n_{|k'c}) = f(n)_{|k'c} = 0$ , we have

$$\begin{aligned} \|dm_2\| &\leq \|m_2\| \leq r_2 \|f(m_{|c} - dm_1)\| = r_2 \|f(m)_{|c} - df(m_1)\| = r_2 \|f(m)_{|c} - dm'\| \leq \\ &r_2 (K' \|df(m)\| + \varepsilon_2) = r_2 (K' \|f(dm)\| + \varepsilon_2) = r_2 (K' \|f(n_{|k'c}) - f(dm)\| + \varepsilon_2) = \\ &r_2 (K' \|f(n_{|k'c} - dm)\| + \varepsilon_2) \leq r_2 (K' r_1 \|n_{|k'c} - dm\| + \varepsilon_2) \leq r_2 (K' r_1 (K \|dn\| + \varepsilon_1) + \varepsilon_2) \end{aligned}$$

In particular we get

$$\begin{aligned} \|n_{|c} - dy\| &\leq K \|dn\| + \varepsilon_1 + r_2 (K' r_1 (K \|dn\| + \varepsilon_1) + \varepsilon_2) = \\ &(K + r_1 r_2 K K') \|dn\| + \varepsilon_1 (1 + r_1 r_2 K') + r_2 \varepsilon_2. \end{aligned}$$

Now let

$$\epsilon_1 = \frac{\epsilon}{2(1 + r_1 r_2 K')} \text{ and } \epsilon_2 = \begin{cases} \frac{\epsilon}{2r_2} & \text{if } r_2 \neq 0 \\ 1 & \text{if } r_2 = 0 \end{cases}$$

In any case  $r_2 \epsilon_2 \leq \frac{\epsilon}{2}$  and so

$$\|n|_c - dy\| \leq (K + r_1 r_2 K K') \|dn\| + \epsilon$$

as required.

If  $i = 0$ , then all  $m$ ,  $m'$ ,  $m_1$  and  $m_2$  are 0, so  $y = 0$  as required.  $\square$

Consider a system of double complexes  $M_c^{p,q}$ ,  $p, q \geq 0$ ,  $c \geq c_0$ ,

$$\begin{array}{ccccccc} M_c^{0,0} & \longrightarrow & M_c^{0,1} & \longrightarrow & M_c^{0,2} & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ M_c^{1,0} & \longrightarrow & M_c^{1,1} & \longrightarrow & M_c^{1,2} & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ M_c^{2,0} & \longrightarrow & M_c^{2,1} & \longrightarrow & \ddots & & \\ \downarrow & & \downarrow & & & & \\ \vdots & & \vdots & & & & \end{array}$$

of complete normed abelian groups.

**Definition 1.4.14.** We say that the system of double complexes  $M_c^{p,q}$  satisfies the *normed spectral homotopy condition* for  $m \in \mathbf{N}$  and  $H, c_0 \in \mathbf{R}_{\geq 0}$  if the following condition is satisfied:

For  $q = 0, \dots, m$  and  $c \geq c_0$ , there is a map  $h_{k'_c}^q: M_{k'_c}^{0,q+1} \rightarrow M_c^{1,q}$  with

$$\|h_{k'_c}^q(x)\|_{M_c^{1,q}} \leq H \|x\|_{M_{k'_c}^{0,q+1}}$$

for all  $x \in M_{k'_c}^{0,q+1}$ , and such that for all  $c \geq c_0$  and  $q = 0, \dots, m$  the “homotopic” map

$$\text{res}_{k'^2 c, k'_c}^{1,q} \circ d^{0,q} + h_{k'^2 c}^q \circ d_{k'^2 c}'^{0,q} + d_{k'_c}'^{1,q-1} \circ h_{k'^2 c}^{q-1}: M_{k'^2 c}^{0,q} \rightarrow M_{k'_c}^{1,q}$$

factors as a composite of the restriction  $\text{res}_{k'^2 c, c}^{0,q}$  and a map

$$\delta_c^{0,q}: M_c^{0,q} \rightarrow M_{k'_c}^{1,q}$$

that is a map of complexes (in degrees  $\leq m$ ), and satisfies the estimate

$$(1.4.1) \quad \|\delta_c^{0,q}(x)\|_{M_{k'_c}^{1,q}} \leq \epsilon \|x\|_{M_c^{0,q}}$$

for all  $x \in M_c^{0,q}$ .

**Proposition 1.4.15.** Fix an integer  $m \geq 0$  and constants  $k, K$ . Then there exists an  $\epsilon > 0$  and constants  $k_0, K_0$ , depending (only) on  $k, K$  and  $m$ , with the following property.

Let  $M_c^{p,q}$  be a system of double complexes as above, and assume that it is admissible. Assume further that there is some  $k' \geq k_0$  and some  $H > 0$ , such that

- (1) for  $i = 0, \dots, m+1$ , the rows  $M_c^{i,q}$  are weakly  $\leq k$ -exact in degrees  $\leq m-1$  for  $c \geq c_0$  with bound  $K$ ;

(2) for  $j = 0, \dots, m$ , the columns  $M_c^{p,j}$  are weakly  $\leq k$ -exact in degrees  $\leq m$  for  $c \geq c_0$  with bound  $K$ ;

(3) it satisfies the normed spectral homotopy condition for  $m$ ,  $H$  and  $c_0$ .

Then the first row is weakly  $\leq k'^2$  exact in degrees  $\leq m$  for  $c \geq c_0$  with bound  $2K_0H$ .

We note that the bound on the homotopy is of a peculiar nature, in that the bound only depends on a deep restriction of  $x$ .

*Proof of Proposition 1.4.15.* First, we treat the case  $m = 0$ . If  $m = 0$ , we claim that one can take  $\epsilon = \frac{1}{2k}$  and  $k_0 = k$ . We have to prove exactness at the first step. Let  $x_{k'^2c} \in M_{k'^2c}^{0,0}$  and denote  $x_{k'c} = \text{res}_{k'^2c, k'c}^{0,0}(x)$  and  $x_c = \text{res}_{k'^2c, c}^{0,0}(x)$ . Then by assumption (2) (and  $k' \geq k$ ), we have

$$\|x_c\|_{M_c^{0,0}} \leq k \|d_{k'c}^{0,0}(x_{k'c})\|_{M_{k'c}^{1,0}}.$$

On the other hand, by (3),

$$\|\text{res}_{k'^2c, k'c}^{1,0}(d_{k'^2c}^{0,0}(x)) \pm h_{k'^2c}^0(d_{k'^2c}^{0,0}(x))\|_{M_{k'c}^{1,0}} \leq \epsilon \|x_c\|_{M_c^{0,0}}.$$

In particular, noting that  $\text{res}_{k'^2c, k'c}^{1,0}(d_{k'^2c}^{0,0}(x)) = d_{k'c}^{0,0}(x_{k'c})$ , we get

$$\|x_c\|_{M_c^{0,0}} \leq k \|d_{k'c}^{0,0}(x_{k'c})\|_{M_{k'c}^{1,0}} \leq k\epsilon \|x_c\|_{M_c^{0,0}} + kH \|d_{k'^2c}^{0,0}(x)\|_{M_{k'^2c}^{0,1}}.$$

Thus, taking  $\epsilon = \frac{1}{2k}$  as promised, this implies

$$\|x_c\|_{M_c^{0,0}} \leq 2kH \|d_{k'^2c}^{0,0}(x)\|_{M_{k'^2c}^{0,1}}.$$

This gives the desired  $\leq \max(k'^2, 2k_0H)$ -exactness in degrees  $\leq m$  for  $c \geq c_0$ .

Now we argue by induction on  $m$ . Consider the complex  $N^{p,q}$  given by  $M^{p,q+1}$  for  $q \geq 1$  and  $N^{p,0} = M^{p,1}/\overline{M^{p,0}}$  (the quotient by the closure of the image, which is also the completion of  $M^{p,1}/M^{p,0}$ ), equipped with the quotient norm. Using the normed version of the snake lemma, Proposition 1.4.12, one checks that this satisfies the assumptions for  $m-1$ , with  $k$  replaced by  $\max(k^4, k^3 + k + 1)$ .  $\square$

### 1.5. Completions of locally constant functions.

**Definition 1.5.1.** Let  $V$  be a semi-normed group, and  $X$  a compact topological space. We denote by  $V(X)$  the normed abelian group of locally constant functions  $X \rightarrow V$  with respect to the sup norm. With  $\widehat{V}(X)$  we denote the completion of  $V(X)$ .

These constructions are functorial in bounded group homomorphisms  $V \rightarrow V'$  and contravariantly functorial in continuous maps  $f: X \rightarrow X'$ .

Note in particular that  $V(f)$  and  $\widehat{V}(f)$  are norm-nonincreasing morphisms of semi-normed groups.

**Lemma 1.5.2.** Let  $r \in \mathbb{R}_{>0}$ , and let  $V$  be an  $r$ -normed  $\mathbb{Z}[T^{\pm 1}]$ -module. Let  $X$  be a compact space. Then  $\widehat{V}(X)$  is naturally an  $r$ -normed  $\mathbb{Z}[T^{\pm 1}]$ -module, with the action of  $T$  given by post-composition.

*Proof.* Formalised, but omitted from this text.  $\square$

We continue to use the notation of before: let  $r' > 0, c \geq 0$  be real numbers, and let  $M$  be a profinitely filtered pseudo-normed group with  $r'$ -action by  $T^{-1}$  (see Section 1.2).

**Lemma 1.5.3.** *Let  $f$  be a basic universal map from exponent  $m$  to  $n$ . We get an induced homomorphism of profinitely filtered pseudo-normed groups  $M^m \rightarrow M^n$  bounded by the maximum (over all  $i$ ) of  $\sum_j |f_{ij}|$ , where the  $f_{ij}$  are the coefficients of the  $n \times m$ -matrix representing  $f$ .*

*This construction is functorial in  $f$ .*

*Proof.* Omitted. □

**Definition 1.5.4.** Let  $f$  be a basic universal map from exponent  $m$  to  $n$ , and let  $(c_2, c_1)$  be  $f$ -suitable. We get an induced map

$$V(f): V(M_{\leq c_1}^n) \rightarrow V(M_{\leq c_2}^m)$$

induced by the morphism of profinitely filtered pseudo-normed groups  $M^m \rightarrow M^n$ .

This construction is functorial in  $f$ .

**Definition 1.5.5.** Let  $f = \sum_g n_g g$  be a universal map from exponent  $m$  to  $n$ , and let  $(c_2, c_1)$  be  $f$ -suitable. We get an induced map

$$V(f): V(M_{\leq c_1}^n) \rightarrow V(M_{\leq c_2}^m)$$

that is the sum  $\sum n_g V(g)$ .

This construction is functorial in  $f$ .

**Definition 1.5.6.** Let  $f$  be a universal map from exponent  $m$  to  $n$ , and let  $(c_2, c_1)$  be  $f$ -suitable. We get an induced map

$$\widehat{V}(f): \widehat{V}(M_{\leq c_1}^n) \rightarrow \widehat{V}(M_{\leq c_2}^m)$$

that is the completion of  $V(f)$ .

This construction is functorial in  $f$ .

Let  $r > 0$ , and assume now that  $V$  is an  $r$ -normed  $\mathbb{Z}[T^{\pm 1}]$ -module. Assume  $r' \leq 1$ .

**Definition 1.5.7.** There are two natural actions of  $T^{-1}$  on  $\widehat{V}(M_{\leq c})$ . The first comes from the  $r'$ -action of  $T^{-1}$  on  $M$  which gives a continuous map

$$M_{\leq cr'} \rightarrow M_{\leq c}$$

and thus a normed group morphism  $V(M_{\leq c}) \rightarrow V(M_{\leq cr'})$  which can be extended by completion to

$$(T^{-1})^*: \widehat{V}(M_{\leq c}) \rightarrow \widehat{V}(M_{\leq cr'}).$$

The other comes from Lemma 1.5.2, using the  $r$ -normed  $\mathbb{Z}[T^{\pm 1}]$ -module  $V$ . Again by extension to completion, we get a map

$$[T^{-1}]: \widehat{V}(M_{\leq c}) \rightarrow \widehat{V}(M_{\leq c}),$$

that we can compose with the map  $\widehat{V}(M_{\leq c}) \rightarrow \widehat{V}(M_{\leq cr'})$ , obtained from the natural inclusion  $M_{\leq cr'} \rightarrow M_{\leq c}$ . We thus end up with two maps

$$(T^{-1})^*, [T^{-1}]: \widehat{V}(M_{\leq c}) \rightarrow \widehat{V}(M_{\leq cr'}).$$

and we define  $\widehat{V}(M_{\leq c})^{T^{-1}}$  to be the equalizer of  $(T^{-1})^*$  and  $[T^{-1}]$ . In other words, the kernel of  $(T^{-1})^* - [T^{-1}]$ .

We will also need to understand the image of  $(T^{-1})^* - [T^{-1}]$ . The next lemma ensures it is surjective with controlled preimages, see Definition 1.4.1.

**Lemma 1.5.8.** *Let  $M$  be a profinitely filtered pseudo-normed group with action of  $T^{-1}$ . For any  $r \in (0, 1)$ , any  $r$ -normed  $\mathbb{Z}[T^{\pm 1}]$ -module  $V$ , any  $c > 0$  and any  $a$ , the map*

$$\widehat{V}(M_{\leq c}^a) \xrightarrow{T^{-1} - [T^{-1}]^*} \widehat{V}(M_{\leq r'c}^a)$$

*has norm bounded by  $r^{-1} + 1$  and is  $\frac{r}{1-r}(1 + \epsilon)$ -surjective.*

*Proof.* The norm bound is clear because  $[T^{-1}]^*$  is norm non-increasing and  $T^{-1}$  scales norm by  $r^{-1}$ . Quantitative surjectivity will follow from Lemma 1.4.2 once we'll have proven that  $T^{-1} - [T^{-1}]^* : \widehat{V}(M_{\leq c}^a) \rightarrow \widehat{V}(M_{\leq r'c}^a)$  is  $r/(1-r)$ -surjective onto  $V(M_{\leq r'c}^a)$ .

We first note that any locally constant function  $\varphi \in V(M_{\leq r'c}^a)$  can be extended to a locally constant function  $\bar{\varphi} \in V(M_{\leq c}^a)$  with the same norm (recall  $f$  takes finitely many values and its norm is the maximum of norms of these values).

Let  $f$  be any element of  $V(M_{\leq r'c}^a)$ . We inductively define a sequence of locally constant functions  $h_n \in V(M_{\leq c}^a)$  with  $h_0 = T \circ \bar{f}$  and  $h_{n+1} = T \circ \overline{[T^{-1}]^* h_n}$ . Here we use the composition symbol to emphasize this is indeed the naive post-composition with  $T$ , there is no extra precomposition with  $\iota$  as in the definition of  $T^{-1}$  seen as a map from  $V(M_{\leq c}^a)$  to  $V(M_{\leq r'c}^a)$ .

Since  $[T^{-1}]^*$  is norm non-increasing, extension is norm preserving and  $T$  scales norm by  $r$ , we get that  $\|h_n\| \leq r^{n+1}\|f\|$ . We then set  $g_n = \sum_{i=0}^n h_i$ . The norm estimate on  $h_n$  ensures  $g$  is a Cauchy sequence in  $V(M_{\leq c}^a)$  hence it converges to some  $g$  in  $\widehat{V}(M_{\leq c}^a)$ . We compute:

$$\begin{aligned} (T^{-1} - [T^{-1}]^*)g_n &= \sum_{k=0}^n \left( T^{-1}h_k - [T^{-1}]^*h_k \right) \\ &= T^{-1}h_0 + \sum_{k=0}^{n-1} \left( T^{-1}h_{k+1} - [T^{-1}]^*h_k \right) - [T^{-1}]^*h_n \\ &= \bar{f} \circ \iota + \sum_{k=0}^{n-1} \left( T^{-1} \circ T \circ \overline{[T^{-1}]^*h_k} \circ \iota - [T^{-1}]^*h_k \right) - [T^{-1}]^*h_n \\ &= f - [T^{-1}]^*h_n \end{aligned}$$

which converges to  $f$  hence  $(T^{-1} - [T^{-1}]^*)g = f$ . In addition  $\|g\| \leq \sum_n r^{n+1}\|f\| = r/(1-r)\|f\|$ .  $\square$

**Definition 1.5.9.** Let  $f$  be a universal map from exponent  $m$  to  $n$ , and let  $(c_2, c_1)$  be  $f$ -suitable.

The natural map from Definition 1.5.6 restricts to a map

$$\widehat{V}(f)^{T^{-1}} : \widehat{V}(M_{\leq c_1}^n)^{T^{-1}} \rightarrow \widehat{V}(M_{\leq c_2}^m)^{T^{-1}}$$

**Lemma 1.5.10.** *Let  $0 < r$  and  $0 < r' \leq 1$  be real numbers. Let  $f$  be a universal map from exponent  $m$  to  $n$ , and let  $(c_2, c_1)$  be very suitable for  $(f, r, r')$ . Then*

$$\widehat{V}(f)^{T^{-1}} : \widehat{V}(M_{\leq c_1}^n)^{T^{-1}} \rightarrow \widehat{V}(M_{\leq c_2}^m)^{T^{-1}}$$

*is norm-nonincreasing.*

*Proof.* Use the assumption that  $(c_2, c_1)$  is very suitable for  $(f, r, r')$  in order to find  $N, b \in \mathbb{N}$  and  $c' \in \mathbb{R}_{\geq 0}$  such that:

- $f$  is bound by  $N$  (see Definition 1.1.3)

- $(c_2, c')$  is  $f$ -suitable
- $r^b N \leq 1$
- $c' \leq (r')^b c_1$

Now, notice that the norm of  $\widehat{V}(f)$  is at most  $N$ , and  $\widehat{V}(f)$  can be factored as

$$\widehat{V}(M_{\leq c_1}^n)^{T^{-1}} \xrightarrow{\text{res}} \widehat{V}(M_{\leq c'}^n)^{T^{-1}} \xrightarrow{\widehat{V}(f)} \widehat{V}(M_{\leq c_2}^n)^{T^{-1}}$$

Now use the defining property of the equalizer to conclude that the restriction map has norm less than  $1/N$ , and therefore the composition is norm-nonincreasing.  $\square$

**Definition 1.5.11.** Let  $0 < r$  and  $0 < r' \leq 1$  be real numbers, and let  $V$  be an  $r$ -normed  $\mathbb{Z}[T^{\pm 1}]$ -module. Let  $\text{BD} = (n, f)$  be Breen–Deligne data, and let  $\kappa = (\kappa_0, \kappa_1, \kappa_2, \dots)$  be a sequence of constants in  $\mathbb{R}_{\geq 0}$  that is very suitable for  $(\text{BD}, r, r')$ . Let  $M$  be a profinitely filtered pseudo-normed group with  $r'$ -action of  $T^{-1}$ .

For every  $c \in \mathbb{R}_{\geq 0}$ , the maps from Definition 1.5.9 induced by the universal maps  $f_i$  from the Breen–Deligne  $\text{BD} = (n, f)$  assemble into a complex of normed abelian groups

$$C_{\kappa}^{\text{BD}}(M)_c^{\bullet}: 0 \rightarrow \dots \rightarrow \widehat{V}(M_{\leq \kappa_i}^{n_i})^{T^{-1}} \rightarrow \widehat{V}(M_{\leq \kappa_{i+1}}^{n_{i+1}})^{T^{-1}} \rightarrow \dots$$

Together, these complexes fit into a system of complexes with the natural restriction maps.

By Lemma 1.5.10 the differentials are norm-nonincreasing. It is clear that the restriction maps are also norm-nonincreasing, and therefore the system is admissible.

## 1.6. Polyhedral lattices.

**Definition 1.6.1.** A *polyhedral lattice* is a finite free abelian group  $\Lambda$  equipped with a norm  $\|\cdot\|_{\Lambda}: \Lambda \otimes \mathbb{R} \rightarrow \mathbb{R}$  such that there exists a finite set  $\{\lambda_1, \dots, \lambda_n\} \subset \Lambda$  that generate the norm: that is to say, for every  $\lambda \in \Lambda$  there exist  $c_1, \dots, c_n \in \mathbb{Q}$  such that  $\lambda = \sum c_i \lambda_i$  and  $\|\lambda\| = c_i \|\lambda_i\|$ .

Equivalently (but not verified in Lean): the norm is given by the supremum of finitely many linear functions on  $\Lambda$ ; or once more, equivalently, the “unit ball”  $\{\lambda \in \Lambda \otimes \mathbb{R} \mid \|\lambda\|_{\Lambda} \leq 1\}$  is a polyhedron.

Finally, we can prove the key combinatorial lemma, ensuring that any element of  $\text{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))$  can be decomposed into  $N$  elements whose norm is roughly  $\frac{1}{N}$  of the original element.

**Definition 1.6.2.** Let  $M$  be a pseudo-normed group,  $N \in \mathbb{N}$ , and  $d \in \mathbb{R}_{\geq 0}$ . We say that  $M$  is  *$N$ -splittable* with error term  $d$ , if for all  $c$  and  $x \in M_c$ , there exists a decomposition

$$x = x_1 + x_2 + \dots + x_N,$$

with  $x_i \in M_{c/N+d}$ .

**Proposition 1.6.3.** Let  $\Lambda$  be a polyhedral lattice, and  $S$  a profinite set. Then for all positive integers  $N$  there is a constant  $d$  such that for all  $c > 0$  one can write any  $x \in \text{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c}$  as

$$x = x_1 + \dots + x_N$$

where all  $x_i \in \text{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c/N+d}$ .

In other words, for all  $N$ , there exists a  $d$  such that  $\text{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))$  is  $N$ -splittable with error term  $d$ .



*Proof.* The desired statement is equivalent to the surjectivity of the map of profinite sets

$$\mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c/N+d}^N \times_{\mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c+Nd}} \mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c} \rightarrow \mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c}.$$

Note that, as a functor of  $S$ , both sides commute with cofiltered limits, so it is enough to handle finite  $S$ , by Tychonoff. But that is exactly the following Lemma 1.6.4.  $\square$

**Lemma 1.6.4.** *Let  $\Lambda$  be a polyhedral lattice, and  $S$  a finite set. Then for all positive integers  $N$  there is a constant  $d$  such that for all  $c > 0$  one can write any  $x \in \mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c}$  as*

$$x = x_1 + \dots + x_N$$

where all  $x_i \in \mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c/N+d}$ .

In other words, for all  $N$ , there exists a  $d$  such that  $\mathrm{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))$  is  $N$ -splittable with error term  $d$ .

As preparation for the proof, we have the following results.

**Lemma 1.6.5** (Gordan's lemma). *Let  $\Lambda$  be a finite free abelian group, and let  $\lambda_1, \dots, \lambda_m \in \Lambda$  be elements. Let  $M \subset \mathrm{Hom}(\Lambda, \mathbb{Z})$  be the submonoid  $\{x \mid x(\lambda_i) \geq 0 \text{ for all } i = 1, \dots, m\}$ . Then  $M$  is finitely generated as monoid.*

*Proof.* This is a standard result. We omit the proof here. It is done in Lean.  $\square$

**Lemma 1.6.6.** *Let  $\Lambda$  be a finite free abelian group, let  $N$  be a positive integer, and let  $\lambda_1, \dots, \lambda_m \in \Lambda$  be elements. Then there is a finite subset  $A \subset \Lambda^\vee$  such that for all  $x \in \Lambda^\vee = \mathrm{Hom}(\Lambda, \mathbb{Z})$  there is some  $x' \in A$  such that  $x - x' \in N\Lambda^\vee$  and for all  $i = 1, \dots, m$ , the numbers  $x'(\lambda_i)$  and  $(x - x')(\lambda_i)$  have the same sign, i.e. are both nonnegative or both nonpositive.*

*Proof.* It suffices to prove the statement for all  $x$  such that  $\lambda_i(x) \geq 0$  for all  $i$ ; indeed, applying this variant to all  $\pm \lambda_i$ , one gets the full statement.

Thus, consider the submonoid  $\Lambda_+^\vee \subset \Lambda^\vee$  of all  $x$  that pair nonnegatively with all  $\lambda_i$ . This is a finitely generated monoid by Lemma 1.6.5; let  $y_1, \dots, y_M$  be a set of generators. Then we can take for  $A$  all sums  $n_1 y_1 + \dots + n_M y_M$  where all  $n_j \in \{0, \dots, N-1\}$ .  $\square$

**Lemma 1.6.7.** *Let  $x_0, x_1, \dots$  be a sequence of reals, and assume that  $\sum_{i=0}^\infty x_i$  converges absolutely. For every natural number  $N > 0$ , there exists a partition  $\mathbb{N} = A_1 \sqcup A_2 \sqcup \dots \sqcup A_N$  such that for each  $j = 1, \dots, N$  we have  $\sum_{i \in A_j} x_i \leq (\sum_{i=0}^\infty x_i)/N + 1$*

*Proof.* Define the  $A_j$  recursively: assume that the natural numbers  $0, \dots, n$  have been placed into the sets  $A_1, \dots, A_N$ . Then add the number  $n+1$  to the set  $A_j$  for which

$$\sum_{i=0, i \in A_j}^n x_i$$

is minimal.  $\square$

**Lemma 1.6.8.** *For all natural numbers  $N > 0$ , and for all  $x \in \overline{\mathcal{L}}_{r'}(S)_{\leq c}$  one can decompose  $x$  as a sum*

$$x = x_1 + \dots + x_N$$

with all  $x_i \in \overline{\mathcal{L}}_{r'}(S)_{\leq c/N+1}$ .

*Proof.* Choose a bijection  $S \times \mathbb{N} \cong \mathbb{N}$ , and transport the result from Lemma 1.6.7.  $\square$

*Proof of Lemma 1.6.4.* Pick  $\lambda_1, \dots, \lambda_m \in \Lambda$  generating the norm. We fix a finite subset  $A \subset \Lambda^\vee$  satisfying the conclusion of the previous lemma. Write

$$x = \sum_{n \geq 1, s \in S} x_{n,s} T^n[s]$$

with  $x_{n,s} \in \Lambda^\vee$ . Then we can decompose

$$x_{n,s} = Nx_{n,s}^0 + x_{n,s}^1$$

where  $x_{n,s}^1 \in A$  and we have the same-sign property of the last lemma. Letting  $x^0 = \sum_{n \geq 1, s \in S} x_{n,s}^0 T^n[s]$ , we get a decomposition

$$x = Nx^0 + \sum_{a \in A} ax_a$$

with  $x_a \in \overline{\mathcal{L}_{r'}}(S)$  (with the property that in the basis given by the  $T^n[s]$ , all coefficients are 0 or 1). Crucially, we know that for all  $i = 1, \dots, m$ , we have

$$\|x(\lambda_i)\| = N\|x^0(\lambda_i)\| + \sum_{a \in A} |a(\lambda_i)| \|x_a\|$$

by using the same sign property of the decomposition.

Using this decomposition of  $x$ , we decompose each term into  $N$  summands. This is trivial for the first term  $Nx^0$ , and each summand of the second term decomposes with  $d = 1$  by Lemma 1.6.8. (It follows that in general one can take for  $d$  the supremum over all  $i$  of  $\sum_{a \in A} |a(\lambda_i)|$ .)  $\square$

**Definition 1.6.9.** Let  $\Lambda$  be a polyhedral lattice, and let  $N > 0$  be a natural number. (We think of  $N$  as being fixed once and for all, and thus it does not show up in the notation below.)

By  $\Lambda'$  we denote  $\Lambda^N$  endowed with the norm

$$\|(\lambda_1, \dots, \lambda_N)\|_{\Lambda'} = \frac{1}{N}(\|\lambda_1\|_{\Lambda} + \dots + \|\lambda_N\|_{\Lambda}).$$

This is a polyhedral lattice.

**Lemma 1.6.10.** For any  $m \geq 1$ , let  $\Lambda'^{(m)}$  be given by  $\Lambda'^m/\Lambda \otimes (\mathbb{Z}^m)_{\Sigma=0}$ ; for  $m = 0$ , we set  $\Lambda'^{(0)} = \Lambda$ . Then  $\Lambda'^{(m)}$  is a polyhedral lattice.

*Proof.* The proof is done in Lean. TODO: write down a proof here.  $\square$

**Definition 1.6.11.** For any  $m \geq 1$ , let  $\Lambda'^{(m)}$  be given by  $\Lambda'^m/\Lambda \otimes (\mathbb{Z}^m)_{\Sigma=0}$ ; for  $m = 0$ , we set  $\Lambda'^{(0)} = \Lambda$ . Then  $\Lambda'^{(\bullet)}$  is a cosimplicial polyhedral lattice, the Čech conerve of  $\Lambda \rightarrow \Lambda'$ .

In particular,  $\Lambda'^{(0)} = \Lambda \rightarrow \Lambda' = \Lambda'^{(1)}$  is the diagonal embedding.

**Definition 1.6.12.** Let  $\Lambda$  be a polyhedral lattice, and  $M$  a profinitely filtered pseudo-normed group.

Endow  $\text{Hom}(\Lambda, M)$  with the subspaces

$$\text{Hom}(\Lambda, M)_{\leq c} = \{f: \Lambda \rightarrow M \mid \forall x \in \Lambda, f(x) \in M_{\leq c\|x\|}\}.$$

As  $\Lambda$  is polyhedral, it is enough to check the given condition on  $f$  for a finite collection of  $x$  that generate the norm.

These subspaces are profinite subspaces of  $M^\Lambda$ , and thus they make  $\text{Hom}(\Lambda, M)$  into a profinitely filtered pseudo-normed group.

If  $M$  has an action of  $T^{-1}$ , then so does  $\text{Hom}(\Lambda, M)$ .

**1.7. Key technical result.** Now we state the following result, which is the key technical result on our to the main goal.

**Theorem 1.7.1.** *Let  $\text{BD} = (n, f, h)$  be a Breen–Deligne package, and let  $\kappa = (\kappa_0, \kappa_1, \kappa_2, \dots)$  be a sequence of constants in  $\mathbb{R}_{\geq 0}$  that is BD-suitable. Fix radii  $1 > r' > r > 0$ . For any  $m$  there is some  $k$  and  $c_0$  such that for all profinite sets  $S$  and all  $r$ -normed  $\mathbb{Z}[T^{\pm 1}]$ -modules  $V$ , the system of complexes*

$$C_{\kappa}^{\text{BD}}(\overline{\mathcal{L}}_{r'}(S))_{\bullet}^{\bullet} : \widehat{V}(\overline{\mathcal{L}}_{r'}(S))_{\leq c}^{T^{-1}} \rightarrow \widehat{V}(\overline{\mathcal{L}}_{r'}(S))_{\leq \kappa_1 c}^2{}^{T^{-1}} \rightarrow \dots$$

*is  $\leq k$ -exact in degrees  $\leq m$  for  $c \geq c_0$ .*

We will prove Theorem 1.7.1 by induction on  $m$ . Unfortunately, the induction requires us to prove a stronger statement.

**Theorem 1.7.2.** *Fix radii  $1 > r' > r > 0$ . For any  $m$  there is some  $k$  such that for all polyhedral lattices  $\Lambda$  there is a constant  $c_0(\Lambda) > 0$  such that for all profinite sets  $S$  and all  $r$ -normed  $\mathbb{Z}[T^{\pm 1}]$ -modules  $V$ , the system of complexes*

$$C_{\Lambda, c}^{\bullet} : \widehat{V}(\text{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq c})^{T^{-1}} \rightarrow \widehat{V}(\text{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))_{\leq \kappa_1 c}^2)^{T^{-1}} \rightarrow \dots$$

*is  $\leq k$ -exact in degrees  $\leq m$  for  $c \geq c_0(\Lambda)$ .*

*Proof of Theorem 1.7.1.* Use  $\Lambda = \mathbb{Z}$ , and the isomorphism  $\text{Hom}(\mathbb{Z}, A) \cong A$ . □

**A word on universal constants:** We fix once and for all, the constants  $0 < r < r' \leq 1$  a Breen–Deligne package  $\text{BD}$ , and a sequence of positive constants  $\kappa$  that is very suitable for  $(\text{BD}, r, r')$ . Once the full proof is formalized, we can come back to this place and write a bit more about the other constants.

**The global strategy** of the proof is to construct a system of double complexes such that its first row is the system  $C_{\Lambda, \bullet}^{\bullet}$  occurring in Theorem 1.7.2. We can then verify the conditions to Proposition 1.4.15 and conclude from there. For the time being, we will let  $M$  denote an arbitrary profinitely filtered pseudo-normed group with action of  $T^{-1}$ , and whenever needed we can specialize to  $M = \overline{\mathcal{L}}_{r'}(S)$ .

**Further choices of constants:** We will argue by induction on  $m$ , so assume the result for  $m - 1$  (this is no assumption for  $m = 0$ , so we do not need an induction start). This gives us some  $k > 1$  for which the statement of Theorem 1.7.2 holds true for  $m - 1$ ; if  $m = 0$ , simply take any  $k > 1$ . In the proof below, we will increase  $k$  further in a way that depends only on  $m$  and  $r$ . After this modified choice of  $k$ , we fix  $\epsilon$  and  $k_0$  as provided by Proposition 1.4.15. Fix a sequence  $(\kappa'_i)_i$  of nonnegative reals that is adept to  $(\text{BD}, \kappa)$ . (Such a sequence exists by Lemma 1.1.20.) Moreover, we let  $k'$  be the supremum of  $k_0$  and the  $c'_i$  for  $i = 0, \dots, m + 1$ . Finally, choose a positive integer  $b$  so that  $2k'(\frac{r}{r'})^b \leq \epsilon$ , and let  $N$  be the minimal power of 2 that satisfies

$$k'/N \leq (r')^b.$$

Then in particular  $r^b N \leq 2k'(\frac{r}{r'})^b \leq \epsilon$ .

**Definition 1.7.3.** Let  $\Lambda^{(\bullet)}$  be the cosimplicial polyhedral lattice of Definition 1.6.11, and recall from 1.6.12 that  $\text{Hom}(\Lambda^{(m)}, M)$  is a profinitely filtered pseudo-normed group with action of  $T^{-1}$ .

Hence  $\text{Hom}(\Lambda^{(\bullet)}, M)$  is a simplicial profinitely filtered pseudo-normed group with action of  $T^{-1}$ .

Now apply the construction of the system of complexes from Definition 1.5.11 to obtain a cosimplicial system of complexes

$$C_{\kappa}^{\text{BD}}(\text{Hom}(\Lambda^{\bullet}, M))_{\bullet}.$$

Now take the alternating face map cochain complex to obtain a system of double complexes, whose objects are

$$\widehat{V}(\text{Hom}(\Lambda^{(m)}, M)_{\leq \kappa_i c}^{n_i})^{T^{-1}}.$$

As final step, rescale the norm on the object in row  $m$  by  $m!$ , so that all columns become admissible: the vertical differential from row  $m$  to row  $m+1$  is an alternating sum of  $m+1$  maps that are all norm-nonincreasing.

**Lemma 1.7.4.** *In particular, for any  $c > 0$ , we have*

$$\text{Hom}(\Lambda', M)_{\leq c} = \text{Hom}(\Lambda, M)_{\leq c/N}^N,$$

with the map to  $\text{Hom}(\Lambda, M)_{\leq c}$  given by the sum map.

*Proof.* Omitted (but done in Lean). □

**Lemma 1.7.5.** *Similarly, for any  $c > 0$ , we have*

$$\text{Hom}(\Lambda'^{(m)}, M)_{\leq c} = \text{Hom}(\Lambda', M)_{\leq c}^{m/\text{Hom}(\Lambda, M)_{\leq c}},$$

the  $m$ -fold fibre product of  $\text{Hom}(\Lambda', M)_{\leq c}$  over  $\text{Hom}(\Lambda, M)_{\leq c}$ .

*Proof.* Omitted (but done in Lean). □

**Lemma 1.7.6.** *There is a canonical isomorphism between the first row of the double complex*

$$C_{\kappa}^{\text{BD}}(\text{Hom}(\Lambda^{(1)}, M))^{\bullet}$$

and

$$C_{\kappa/N}^{N \otimes \text{BD}}(\text{Hom}(\Lambda, M))^{\bullet}$$

which identifies the map induced by the diagonal embedding  $\Lambda \rightarrow \Lambda' = \Lambda^{(1)}$  with the map induced by  $\sigma^N: N \otimes \text{BD} \rightarrow \text{BD}$ .

*Proof.* Omitted (but done in Lean). □

**Proposition 1.7.7.** *Let  $\pi: X \rightarrow B$  be a surjective morphism of profinite sets, and let  $S_{\bullet} \rightarrow S_{-1}$ ,  $S_{-1} := B$ , be its augmented Čech nerve. Let  $V$  be a semi-normed group. Then the complex*

$$0 \rightarrow \widehat{V}(S_{-1}) \rightarrow \widehat{V}(S_0) \rightarrow \widehat{V}(S_1) \rightarrow \dots$$

is exact. Furthermore, for all  $\epsilon > 0$  and  $f \in \ker(\widehat{V}(S_m) \rightarrow \widehat{V}(S_{m+1}))$ , there exists some  $g \in \widehat{V}(S_{m-1})$  such that  $d(g) = f$  and  $\|g\| \leq (1 + \epsilon) \cdot \|f\|$ . In other words, the complex is normed exact in the sense of Definition 1.4.6.

*Proof.* We argue similarly to [Sch19, Theorem 3.3], as follows. By applying Lemma 1.4.3, we first reduce to a statement which does not involve  $\epsilon$  or completions. Explicitly, we must show that

$$0 \rightarrow V(S_1) \rightarrow V(S_0) \rightarrow V(S_1) \rightarrow \dots$$

is exact, and that whenever  $f \in \ker(V(S_m) \rightarrow V(S_{m+1}))$ , there exists  $g \in V(S_{m-1})$  such that  $\|g\| \leq \|f\|$  and  $d(g) = f$ . The map  $V(S_{-1}) \rightarrow V(S_0)$  is the one induced by  $S_0 \rightarrow S_{-1}$  which agrees with  $X \rightarrow B$ . Since  $X \rightarrow B$  is surjective, we easily see that  $V(S_{-1}) \rightarrow V(S_0)$  is injective.

If  $X$  and  $B$  are finite, then the remaining assertions follow from the existence of a splitting  $\sigma : B \rightarrow X$  of  $\pi : X \rightarrow B$ , as follows. The map  $\sigma$  provides maps  $S_m \rightarrow S_{m+1}$  for all  $m \geq -1$ , defined explicitly as

$$(a_0, \dots, a_m) \mapsto (\sigma(\pi(a_0)), a_0, \dots, a_m)$$

if  $m \geq 0$  and simply as  $\sigma$  if  $m = -1$ . Here, for  $m \geq 0$ , we have identified  $S_m$  with the  $m+1$ -fold fibered product  $X \times_B \cdots \times_B X$ . Applying  $V(-)$ , these maps induce  $h_m : V(S_{m+1}) \rightarrow V(S_m)$ , which form a contracting homotopy for the complex in question, and which are norm nonincreasing by the definition of  $V(-)$ . If  $f \in \ker(V_m \rightarrow V_{m+1})$  is as above, then  $g := h_m(f)$  satisfies  $d(g) = f$  and  $\|g\| \leq \|f\|$ , as required.

In the general case, write  $X = \varprojlim_i X_i$  where  $X_i$  vary over the discrete (hence finite) quotients of  $X$ . Since  $X \rightarrow B$  is surjective, for each  $i$  there exists a unique maximal discrete quotient  $B_i$  of  $B$  such that  $X \rightarrow B$  descends to  $X_i \rightarrow B_i$ . The maps  $X_i \rightarrow B_i$  are again surjective, and one has

$$(X \rightarrow B) = \varprojlim_i (X_i \rightarrow B_i).$$

Let  $S_{i,\bullet} \rightarrow S_{i,-1}$ ,  $S_{i,-1} := B_i$ , denote the augmented Čech nerve of  $X_i \rightarrow B_i$ .

The terms in the Čech nerve are themselves limits, hence we have  $S_m = \varprojlim_i S_{i,m}$ , with each  $S_{i,m}$  finite. The functor  $V(-)$ , when considered as taking values in abelian groups, sends cofiltered limits to filtered colimits. Also, if  $f \in V(S_m)$  is the pullback of  $f_i \in V(S_{i,m})$ , then for a sufficiently small index  $j \leq i$ , the image of  $f : S_m \rightarrow V$  agrees with the image of  $f_j : S_{j,m} \rightarrow V$ , where  $f_j$  is the image of  $f_i$  under the map  $V(S_{i,m}) \rightarrow V(S_{j,m})$  induced by the transition map  $S_{j,m} \rightarrow S_{i,m}$ .

Now suppose that  $f \in \ker(V(S_m) \rightarrow V(S_{m+1}))$  is given. By the discussion above, there exists some  $i$  and some  $f_i \in V(S_{i,m})$  such that  $f$  is the pullback of  $f_i$  with respect to the morphism  $S_m \rightarrow S_{i,m}$  and such that the following additional conditions hold:

- (1) One has  $\|f_i\| = \|f\|$ .
- (2) One has  $f_i \in \ker(V(S_{i,m}) \rightarrow V(S_{i,m+1}))$ .

Let  $h_m : V(S_{i,m}) \rightarrow V(S_{i,m-1})$  be the map constructed in the argument for the finite case  $X_i \rightarrow B_i$ . Put  $g_i := h_m(f_i)$  and  $g$  the image of  $g_i$  in  $V(S_{m-1})$ . Since the maps  $V(S_{i,\bullet}) \rightarrow V(S_\bullet)$  commute with the differentials, we have  $d(g) = f$ . Finally, the map  $V(S_{i,m-1}) \rightarrow V(S_{m-1})$  is norm nonincreasing as it is induced from  $S_{m-1} \rightarrow S_{i,m-1}$ , so that

$$\|g\| \leq \|g_i\| \leq \|f_i\| = \|f\|,$$

as contended. □

**Lemma 1.7.8.** *Let  $M$  be a profinitely filtered pseudo-normed group with  $T^{-1}$ -action that is  $N$ -splittable with error term  $d \geq 0$ . Let  $k \geq 1$  be a real number, and let  $c_0 > 0$  satisfy  $d \leq \frac{(k-1)c_0}{N}$ . For every  $c$ , consider the Čech nerve of the summation map  $M_{c/N}^N \rightarrow M_c$ . By applying the functor  $\widehat{V}(\_)$  and taking the alternating face map complex, we obtain a system of complexes*

$$\widehat{V}(M_{\leq c}) \rightarrow \widehat{V}(M_{\leq c/N}^N) \rightarrow \dots$$

*This system of complexes is weakly  $\leq k$ -exact in degrees  $\leq m$  and for  $c \geq c_0$  with bound 1.*

*Proof.* For every constant  $c$ , consider the pullback

$$\begin{array}{ccccc}
 & & M_c & \longrightarrow & M_{kc} \\
 & & \uparrow & & \uparrow \\
 & & X_c & \longrightarrow & M_{kc/N}^N \\
 & \nearrow & & \nearrow & \\
 M_{c/N}^N & \xrightarrow{\quad} & & & 
 \end{array}$$

We therefore get morphisms of cochain complexes

$$\begin{array}{ccccc}
 \widehat{V}(M_{kc}) & \longrightarrow & \widehat{V}(M_c) & \longrightarrow & \widehat{V}(M_c) \\
 \downarrow & & \downarrow & & \downarrow \\
 \widehat{V}(M_{kc/N}^N) & \longrightarrow & \widehat{V}(X_c) & \longrightarrow & \widehat{V}(M_{c/N}^N) \\
 \downarrow & & \downarrow & & \downarrow \\
 \vdots & \longrightarrow & \vdots & \longrightarrow & \vdots
 \end{array}$$

where all the columns are of the form “alternating face map complex of  $\widehat{V}(\_)$  applied to a Čech nerve”. Note that the horizontal maps are norm-nonincreasing and their compositions are restriction maps.

Claim: for  $c \geq c_0$ , the map  $X_c \rightarrow M_c$  is surjective.

Indeed, by assumption every  $x \in M_c$  can be decomposed into a sum  $x = x_1 + \dots + x_N$  with  $x_i \in M_{c/N+d} \subset M_{kc/N}$ , since  $c \geq c_0$  and  $d \leq \frac{(k-1)c_0}{N}$ .

By Proposition 1.7.7, the middle column is normed exact (in the sense of Definition 1.4.6). The result follows from Lemma 1.4.10.  $\square$

**Proposition 1.7.9.** *Let  $d$  be the constant from Proposition 1.6.3. Let  $k > 1$  and  $c_0 > 0$  be real numbers such that*

$$(k-1) * c_0 / N \geq d.$$

*Let  $m$  be any natural number, and put*

$$K = (m+2) + \frac{r+1}{r(1-r)}(m+2)^2$$

*Finally, let  $c'_0$  be  $\frac{c_0}{r \cdot n_i}$ , where  $n_i$  is the  $i$ -th index in our fixed Breen–Deligne data.*

*Then  $i$ -th column in the double complex is  $(k^2, K)$ -weak bounded exact in degrees  $\leq m$  for  $c \geq c'_0$ .*

*Proof.* Let  $M^{(m)}$  denote  $\text{Hom}(\Lambda^{(m)}, \overline{\mathcal{L}}_{r'}(S))^{n_i}$ . We also write  $M$  for  $M^{(0)} = \text{Hom}(\Lambda, \overline{\mathcal{L}}_{r'}(S))^{n_i}$  and  $M'$  for  $M^{(1)}$ . By Proposition 1.6.4,  $M$  is  $N$ -splittable with error term  $d$ .

Consider the diagram of morphisms of systems of complexes

$$\begin{array}{ccccc}
\widehat{V}(M_c)^{T^{-1}} & \longrightarrow & \widehat{V}(M_c) & \xrightarrow{T^{-1}-[T^{-1}]^*} & \widehat{V}(M_c) \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{V}(M'_c)^{T^{-1}} & \longrightarrow & \widehat{V}(M'_c) & \xrightarrow{T^{-1}-[T^{-1}]^*} & \widehat{V}(M'_c) \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
\widehat{V}(M_c^{(m)})^{T^{-1}} & \longrightarrow & \widehat{V}(M_c^{(m)}) & \xrightarrow{T^{-1}-[T^{-1}]^*} & \widehat{V}(M_c^{(m)})
\end{array}$$

By Lemmas 1.7.8 and 1.7.5, we know that the two columns on the right are weakly  $\leq k$ -exact in degrees  $\leq m$  and for  $c \geq c_0$  with bound 1.

The result now follows from Lemma 1.5.8, and Proposition 1.4.13.  $\square$

**Proposition 1.7.10.** *Let  $h$  be the homotopy packaged with  $\text{BD}$ , and let  $h^N$  denote the  $n$ -th iterated composition of  $h$  (see Def 1.1.11) which is a homotopy between  $\pi^N$  and  $\sigma^N: N \otimes \text{BD} \rightarrow \text{BD}$ .*

*Let  $H \in \mathbf{R}_{\geq 0}$  be such that for  $i = 0, \dots, m$  the universal map  $h_i^N$  is bound by  $H$  (see Def 1.1.3).*

*Then the double complex satisfies the normed homotopy homotopy condition (Def 1.4.14) for  $m$ ,  $H$ , and  $c_0$ .*

*Proof.* By Lemma 1.7.6 we may replace the first row by

$$C_{\kappa/N}^{N \otimes \text{BD}}(\text{Hom}(\Lambda, M))^{\bullet}.$$

Now it is important to recall that we have chosen  $k' \geq \kappa'_i$  for all  $i = 0, \dots, m+1$ .

Our goal is to find, in degrees  $\leq m$ , a homotopy between the two maps from the first row

$$\widehat{V}(\text{Hom}(\Lambda, M)_{\leq c})^{T^{-1}} \rightarrow \widehat{V}(\text{Hom}(\Lambda, M)_{\leq \kappa_1 c}^2)^{T^{-1}} \rightarrow \dots$$

to the second row

$$\widehat{V}(\text{Hom}(\Lambda, M)_{\leq c/N}^N)^{T^{-1}} \rightarrow \widehat{V}(\text{Hom}(\Lambda, M)_{\leq \kappa_1 c/N}^{2N})^{T^{-1}} \rightarrow \dots$$

respectively induced by  $\sigma^N$  and  $\pi^N$  (which are maps  $N \otimes \text{BD}$

By Definition 1.1.11 and Lemma 1.1.18 we can find this homotopy between the complex for  $k'c$  and the complex for  $c$ . (Here we use  $k' \geq c'_i$  for  $i = 0, \dots, m$ .) By assumption, the norm of these maps is bounded by  $H$ .

Finally, it remains to establish the estimate (eq. 1.4.1) on the homotopic map. We note that this takes  $x \in \widehat{V}(\text{Hom}(\Lambda, M)_{\leq k'^2 \kappa_i c}^{a_i})^{T^{-1}}$  (with  $i = q$  in the notation of (eq. 1.4.1)) to the element

$$y \in \widehat{V}(\text{Hom}(\Lambda, M)_{\leq k' \kappa_i c/N}^{Na_i})^{T^{-1}}$$

that is the sum of the  $N$  pullbacks along the  $N$  projection maps  $\text{Hom}(\Lambda, M)_{\leq k' \kappa_i c/N}^{Na_i} \rightarrow \text{Hom}(\Lambda, M)_{\leq k'^2 \kappa_i c}^{a_i}$ .

We note that these actually take image in  $\text{Hom}(\Lambda, M)_{\leq \kappa_i c}^{a_i}$  as  $N \geq k'$ , so this actually gives a well-defined map

$$\widehat{V}(\text{Hom}(\Lambda, M)_{\leq \kappa_i c}^{a_i})^{T^{-1}} \rightarrow \widehat{V}(\text{Hom}(\Lambda, M)_{\leq k' \kappa_i c/N}^{Na_i})^{T^{-1}}.$$

We need to see that this map is of norm  $\leq \epsilon$ . Now note that by our choice of  $N$ , we actually have  $k' \kappa_i c / N \leq (r')^b \kappa_i c$ , so this can be written as the composite of the restriction map

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)_{\leq \kappa_i c}^{a_i})^{T^{-1}} \rightarrow \widehat{V}(\mathrm{Hom}(\Lambda, M)_{\leq (r')^b \kappa_i c}^{a_i})^{T^{-1}}$$

and

$$\widehat{V}(\mathrm{Hom}(\Lambda, M)_{\leq (r')^b \kappa_i c}^{a_i})^{T^{-1}} \rightarrow \widehat{V}(\mathrm{Hom}(\Lambda, M)_{\leq k' \kappa_i c / N}^{Na_i})^{T^{-1}}.$$

The first map has norm exactly  $r^b$ , by  $T^{-1}$ -invariance, and as multiplication by  $T$  scales the norm with a factor of  $r$  on  $\widehat{V}$ . (Here is where we use  $r' > r$ , ensuring different scaling behaviour of the norm on source and target.) The second map has norm at most  $N$  (as it is a sum of  $N$  maps of norm  $\leq 1$ ). Thus, the total map has norm  $\leq r^b N$ . But by our choice of  $N$ , we have  $r^b N \leq \epsilon$ , giving the result.  $\square$

*Proof of Theorem 1.7.2.* By induction, the first condition of Proposition 1.4.15 is satisfied for all  $c \geq c_0$  with  $c_0$  large enough (depending on  $\Lambda$  but not  $V$  or  $S$ ).

The second condition is Proposition 1.7.9, and the third condition has been checked in Proposition 1.7.10.

Thus, we can apply Proposition 1.4.15, and get the desired  $\leq \max(k'^2, 2k_0 H)$ -exactness in degrees  $\leq m$  for  $c \geq c_0$ , where  $k'$ ,  $k_0$  and  $H$  were defined only in terms of  $k$ ,  $m$ ,  $r'$  and  $r$ , while  $c_0$  depends on  $\Lambda$  (but not on  $V$  or  $S$ ). This proves the inductive step.  $\square$

**Question 1.7.11.** Can one make the constants explicit, and how large are they? <sup>1</sup> Modulo the Breen-Deligne resolution, all the arguments give in principle explicit constants; and actually the proof of the existence of the Breen-Deligne resolution should be explicit enough to ensure the existence of bounds on the  $c_i$  and  $c'_i$ .

Answer: yes, we can do this. And we should write up a clean account asap, when we have cleaned up the proof in Lean.

## 2. SECOND PART

**2.1. Variants of normed groups.** This subsection continues some of the theory of (pseudo)-normed groups, started in Subsection 1.2.

**Definition 2.1.1.** A  $p$ -Banach space  $V$  is a complete topological  $\mathbb{R}$ -vector space whose topology is induced by a  $p$ -norm; that is, a norm satisfying  $\|rv\| = |r|^p \|v\|$ .

**Lemma 2.1.2.** A  $p$ -Banach  $V$  is (up to non-canonical choice) an  $r$ -Banach  $\mathbb{Z}[T^{\pm 1}]$ -module, where  $r = 2^{-p}$ . (See Definition 1.2.1 for the definition of  $r$ -normed  $\mathbb{Z}[T^{\pm 1}]$ -modules.)

*Proof.* Obvious.  $\square$

**Definition 2.1.3.** We will consider the following categories:

- **CHPNG** the category of CH-filtered pseudo-normed groups with bounded morphisms.
- **CHPNG<sub>1</sub>** the category of exhaustive CH-filtered pseudo-normed groups with strict morphisms.
- **ProfinPNG** the category of profinitely filtered pseudo-normed groups with bounded morphisms.

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<sup>1</sup>A back of the envelope calculation seems to suggest that  $k$  is roughly doubly exponential in  $m$ , and that  $N$  has to be taken of roughly the same magnitude.



- $\text{ProfinPNG}_1$  the category of exhaustive profinitely filtered pseudo-normed groups with strict morphisms.
- $\text{ProfinPNGTinv}_1^{r'}$  the category of exhaustive profinitely filtered pseudo-normed groups with  $r'$ -action of  $T^{-1}$  and strict morphisms.

**Proposition 2.1.4.** *Consider an inverse system  $(X_i)_i$  of compact-Hausdorffly-filtered-pseudonormed abelian groups where all transition maps  $X_i \rightarrow X_j$  send  $X_{i,\leq c}$  to  $X_{j,\leq c}$ . Then*

$$X_{\leq c} := \varprojlim_i X_{i,\leq c}$$

*is compact Hausdorff, and*

$$X = \bigcup_c X_{\leq c}$$

*is naturally a compact-Hausdorffly-filtered pseudonormed abelian group which is the limit of  $(X_i)_i$  in the strict category structure.*

*Proof.* One can define negation and addition on  $X$  as continuous maps  $- : X_{\leq c} \rightarrow X_{\leq c}$  and  $+: X_{\leq c} \times X_{\leq c'} \rightarrow X_{\leq c+c'}$ , and these pass to the unions. It should then be straightforward to check the axioms.  $\square$

## 2.2. Real Measures.

**Definition 2.2.1.** Let  $0 < p' < 1$  be a real number, and let  $S$  be a finite set. Then  $\mathcal{M}_{p'}(S)$  denotes the real vector space

$$\mathcal{M}_{p'}(S) = \left\{ \sum_{s \in S} a_s \text{ such that } a_s \in \mathbf{R} \right\}$$

endowed with the  $\ell^{p'}$ -norm  $\|\sum_{s \in S} a_s\|_{\ell^{p'}} = \sum_{s \in S} |a_s|^{p'}$ .

For every finite set  $S$ , the space  $\mathcal{M}_{p'}(S)$  can be written as the filter colimit

$$\mathcal{M}_{p'}(S) = \varinjlim_{c>0} \mathcal{M}_{p'}(S)_{\leq c}$$

where  $\mathcal{M}_{p'}(S)_{\leq c} = \{F \in \mathcal{M}_{p'}(S) \text{ such that } \|F\|_{\ell^{p'}} \leq c\}$ . Now, given a profinite set  $S = \varprojlim S_i$ , where all  $S_i$ 's are finite groups, and a positive real number  $c > 0$ , define  $\mathcal{M}_{p'}(S)_{\leq c}$  as

$$\mathcal{M}_{p'}(S)_{\leq c} = \varprojlim \mathcal{M}_{p'}(S_i)_{\leq c}$$

endowed with the projective limit topology. Finally, set  $\mathcal{M}_{p'}(S) = \varinjlim_c \mathcal{M}_{p'}(S)_{\leq c}$ .

## 2.3. Laurent Measures.

**Definition 2.3.1.** Let  $0 < r' < 1$  be a real number, and let  $S$  be a finite set. Then  $\mathcal{L}_{r'}(S)$  denotes the group

$$\mathcal{L}_{r'}(S) = \left\{ \sum_{s \in S, n \in \mathbf{Z}} a_{n,s} T^n \mid a_{n,s} \in \mathbf{Z}, \sum_{n \in \mathbf{Z}, s \in S} |a_{n,s}| r'^n < +\infty \right\}.$$

The group  $\mathcal{L}_{r'}(S)$  is filtered by the subsets

$$\mathcal{L}_{r'}(S)_{\leq c} = \left\{ \sum_{s \in S, n \in \mathbf{Z}} a_{n,s} T^n \mid a_{n,s} \in \mathbf{Z}, \sum_{n \in \mathbf{Z}, s \in S} |a_{n,s}| r^n \leq c \right\}$$

for  $c > 0$ . Each subset  $\mathcal{L}_{r'}(S)_{\leq c}$  can be written as

$$\mathcal{L}_{r'}(S)_{\leq c} = \varprojlim_A \mathcal{L}_{r'}(S)_{A, \leq c}$$

where  $A$  runs through the finite subsets of  $\mathbf{Z}$  and  $\mathcal{L}_{r'}(S)_{A, \leq c}$  is the finite set

$$\mathcal{L}_{r'}(S)_{A, \leq c} = \left\{ \sum_{s \in S, n \in A} a_{n,s} T^n \mid a_{n,s} \in \mathbf{Z}, \sum_{n \in A, s \in S} |a_{n,s}| r^n \leq c \right\}.$$

This defines a profinite topology on  $\mathcal{L}_{r'}(S)_{\leq c}$ , for each  $c > 0$ .

**Definition 2.3.2.** Let  $0 < r' < 1$  be a real number, and let  $S$  be a profinite set with a presentation  $S = \varprojlim S_i$ , where all  $S_i$  are finite. For all  $c > 0$ , let  $\mathcal{L}_{r'}(S)_{\leq c}$  denote the projective limit

$$\mathcal{L}_{r'}(S)_{\leq c} = \varprojlim \mathcal{L}_{r'}(S_i)_{\leq c},$$

endowed with the projective limit topology, and set

$$\mathcal{L}_{r'}(S) = \varinjlim_{c>0} \mathcal{L}_{r'}(S)_{\leq c}$$

Now fix  $0 < \xi < 1$  and let  $x \in \mathbf{R}_{\geq 0}$ . For simplicity, denote by  $\mathbf{Z}((T))_{r'}$  the group  $\mathcal{L}_{r'}(*)$ . The next theorem is the first statement of Theorem 6.9 in [Sch20].

**Definition 2.3.3.** Fix  $0 < \xi < 1$ , and let  $\vartheta_\xi: \mathbf{Z}(! (T)!)_r \rightarrow \mathbf{R}$  be the evaluation map

$$\sum a_n T^n \mapsto \sum a_n \xi^n.$$

One of the key results proven in [Sch20, §6] is the surjectivity and continuity of the map  $\vartheta_\xi$ . In order to prove the surjectivity, we need the following

**Definition 2.3.4.** For all  $n \in \mathbf{N}$ , set

$$Y_\xi(x, n) = \begin{cases} x & \text{if } n = 0 \\ Y_\xi(x, n-1) - \left\lfloor \frac{Y_\xi(x, n-1)}{\xi^{n-1}} \right\rfloor \xi^{n-1} & \text{if } n \geq 1 \end{cases}$$

where  $\lfloor * \rfloor$  denotes the floor of  $*$ , namely the greatest integer that is less or equal than  $*$ .

**Lemma 2.3.5.** *The map  $\vartheta_\xi$  is surjective.*

*Proof.* Pick  $x \in \mathbf{R}$ , and consider the power series

$$F(T) = \sum_{n \geq 0} \left\lfloor \frac{Y_\xi(x, n)}{\xi^n} \right\rfloor T^n \in \mathbf{Z}[[T]]$$

where  $\{Y_\xi(x, n)\}_n$  is the sequence defined in Definition 2.3.4. The fact that  $0 < \xi < 1$  ensures that the series converges on the open unit disk, and it can be proven that  $F(\xi) = x$ .  $\square$

Given any *finite* set  $S$ , the above map  $\vartheta_\xi$  defines a map  $\vartheta_{\xi, S}: \mathcal{L}_{r'}(S) \rightarrow \mathcal{M}_{p'}(S)$  by extending  $\vartheta_\xi$  componentwise. For the purpose of this project, the special case where  $\xi = 1/2$  is enough, and we write  $\theta_S$  to denote  $\vartheta_{\xi, S}$ .

**Lemma 2.3.6.** *Let  $0 < p' < 1$  be a real number, let  $S$  be a finite set, and let  $r'$  denote  $(\frac{1}{2})^{p'}$ . The map*

$$\theta_S: \mathcal{L}_{r'}(S) \longrightarrow \mathcal{M}_{p'}(S)$$

*induced by evaluating  $T$  at  $1/2$  is continuous*

*Proof.* Omitted for now, but done in Lean. □

By taking projective limits, we obtain a map  $\vartheta_S$  for every profinite  $S$ .

**Theorem 2.3.7.** *Let  $0 < p' < 1$  be a real number, let  $S$  be a profinite set, and let  $r'$  denote  $(\frac{1}{2})^{p'}$ . The map*

$$\theta_S: \mathcal{L}_{r'}(S) \longrightarrow \mathcal{M}_{p'}(S)$$

*induced by evaluating  $T$  at  $1/2$  is continuous and surjective.*

*Proof.* The proof of surjectivity for finite  $S$  follows immediately from Lemma 2.3.5, and a general argument shows that the surjectivity is preserved by taking inverse limits. Similarly, the proof of continuity is a direct consequence of Lemma 2.3.6. □

With Theorem 2.3.7 at our disposal, we can move on and analyse the kernel of the map  $\theta_S$ .

**2.4. The MacLane  $Q'$ -construction.** In this subsection we will focus on the functorial complex induced by the Breen–Deligne package described in Definition 1.1.12. This complex is also known as MacLane’s  $Q'$ -construction. (TODO: Rewrite the subsection on Breen–Deligne packages to reflect this.)

**Proposition 2.4.1.** *For any  $i \geq 0$ , the functor  $A \mapsto H_i(Q'(A))$  has the following properties:*

(1) *It is additive, i.e.*

$$H_i(Q'(A \oplus B)) \cong H_i(Q'(A)) \oplus H_i(Q'(B)).$$

(2) *It commutes with filtered colimits, i.e. for a filtered inductive system  $A_i$ ,*

$$\lim_{\substack{\longrightarrow \\ i}} H_i(Q'(A)) \cong H_i(Q'(\lim_{\substack{\longrightarrow \\ i}} A_i)).$$

*In particular, for torsion-free abelian groups  $A$ , there is a functorial isomorphism*

$$H_i(Q'(A)) \cong H_i(Q'(\mathbb{Z})) \otimes A.$$

As the proof shows, we do not really need the  $Q'$ -construction here: Any Breen–Deligne package will do.

*Proof.* Let us do the easy things first. Part (2) is clear as everything in sight commutes with filtered colimits. Assuming (1), we note that there is a natural map

$$H_i(Q'(\mathbb{Z})) \times A \rightarrow H_i(Q'(A))$$

induced by functoriality of  $H_i(Q'(-))$ . To check that this is bilinear and induces an isomorphism

$$H_i(Q'(\mathbb{Z})) \otimes A \cong H_i(Q'(A)),$$

we can reduce to the case that  $A$  is finitely generated by (2). In that case  $A$  is finite free, and the result follows from (1).

Thus, it remains to prove part (1), which has already been formalized. We recall that the direct sum of two abelian groups  $M$  and  $N$  is characterized as the abelian group  $P$  with maps  $i_M : M \rightarrow P$ ,  $i_N : N \rightarrow P$ ,  $p_M : P \rightarrow M$ ,  $p_N : P \rightarrow N$ , satisfying  $p_M i_M = \text{id}_M$ ,  $p_N i_N = \text{id}_N$ ,  $p_M i_N = 0$ ,  $p_N i_M = 0$ ,  $\text{id}_P = i_M p_M + i_N p_N$ . Apply this to  $M = H_i(Q'(A))$ ,  $N = H_i(Q'(B))$  and  $P = H_i(Q'(A \oplus B))$ , with all maps induced by applying  $H_i(Q'(-))$  to the similar maps for  $A$ ,  $B$  and  $A \oplus B$ . The fact that  $H_i(Q'(-))$  is a functor already gives all identities except  $\text{id}_P = i_M p_M + i_N p_N$ , and the only issue is the question whether  $H_i(Q'(-))$  induces additive maps on morphism spaces. But if  $f, g : C \rightarrow D$  are any two maps of abelian groups, then  $H_i(Q'(f+g)) = H_i(Q'(f)) + H_i(Q'(g))$ , by reducing to the universal case of the two projections  $D^2 \rightarrow D$  and using the homotopy baked into Definition 1.1.12.  $\square$

We note that by functoriality of the  $Q'$ -construction, it can also be applied to condensed abelian groups.

**Corollary 2.4.2.** *For torsion-free condensed abelian groups  $A$ , there is a natural isomorphism*

$$H_i(Q'(A)) \cong H_i(Q'(\mathbb{Z})) \otimes A$$

*of condensed abelian groups.*

Here, we only need to be able to tensor condensed abelian groups with (abstract) abelian groups. (With more effort, one could prove that  $H_i(Q'(\mathbb{Z}))$  is even finitely generated.) In that case, the tensor product functor can be defined very naively by tensoring the values at any  $S$  with the given abstract abelian group.

*Proof.* Evaluating at  $S \in \text{ExtrDisc}$ , we note that  $S \mapsto H_i(Q'(A(S)))$  is already a condensed abelian group, and agrees with  $H_i(Q'(\mathbb{Z})) \otimes A(S)$ . Thus, the same is true after sheafification.  $\square$

If  $A$  is a torsion-free condensed abelian group equipped with an endomorphism  $f$ , then  $Q'(A)$  is also equipped with the endomorphism  $f$  induced by functoriality, and by functoriality all previous assertions upgrade to  $\mathbb{Z}[f]$ -modules.

**Proposition 2.4.3.** *Let  $M$  and  $N$  be condensed abelian groups with endomorphisms  $f_M$ ,  $f_N$ . Assume that  $M$  is torsion-free (over  $\mathbb{Z}$ ). Then*

$$\text{Ext}_{\mathbb{Z}[f]}^i(M, N) = 0$$

*for all  $i \geq 0$  if and only if*

$$\text{Ext}_{\mathbb{Z}[f]}^i(Q'(M), N) = 0$$

*for all  $i \geq 0$ . More precisely, the first vanishes for  $0 \leq i \leq j$  if and only if the second vanishes for  $0 \leq i \leq j$ .*

At this point, we need to be able to talk about Ext-groups of (bounded to the right) complexes of condensed abelian groups (against condensed abelian groups).

The statement is also true without the torsion-freeness assumption on  $M$ , but slightly more nasty to prove then (and not required for the application).

*Proof.* We induct on  $j$ . Consider first the case  $j = 0$ ; then any map  $Q'(M) \rightarrow N$  factors uniquely over  $H_0 Q'(M)[0] = M[0]$ , yielding the result. Now assume that both sides vanish for  $0 \leq i < j$ ; we need to see that the vanishing of the  $\text{Ext}^i$ 's is equivalent. Consider the triangle

$$\tau_{\geq 1} Q'(M) \rightarrow Q'(M) \rightarrow M[0] \rightarrow .$$

Taking the corresponding long exact sequence of Ext-groups against  $N$ , we see that it suffices to see that

$$\mathrm{Ext}_{\mathbb{Z}[f]}^i(\tau_{\geq 1}Q'(M), N) = 0$$

for  $0 \leq i \leq j$ . But we can prove by descending induction on  $t$  that

$$\mathrm{Ext}_{\mathbb{Z}[f]}^i(\tau_{\geq t}Q'(M), N) = 0.$$

This is trivially true for  $t > i$ . Now look at the triangle

$$\tau_{\geq t+1}Q'(M) \rightarrow \tau_{\geq t}Q'(M) \rightarrow H_t(Q'(M))[t] \rightarrow$$

and the corresponding long exact sequence. It becomes sufficient to prove that

$$\mathrm{Ext}_{\mathbb{Z}[f]}^i(H_t(Q'(M))[t], N) = 0$$

for  $0 \leq i \leq j$ . Trivially,

$$\mathrm{Ext}_{\mathbb{Z}[f]}^i(H_t(Q'(M))[t], N) = \mathrm{Ext}_{\mathbb{Z}[f]}^{i-t}(H_t(Q'(M)), N).$$

Note that  $t \geq 1$  here, so  $i - t < j$  (and can be assumed  $\geq 0$ ). Also  $H_t(Q'(M)) \cong H_t(Q'(\mathbb{Z})) \otimes M$ . Thus, it suffices to show that for every abelian group  $A$  and every  $0 \leq i < j$ ,

$$\mathrm{Ext}_{\mathbb{Z}[f]}^i(A \otimes M, N) = 0.$$

If  $A$  is free, then  $A \otimes M$  is a direct sum of copies of  $M$ , and the result follows as Ext turns direct sums into products (and we assumed the vanishing of  $\mathrm{Ext}_{\mathbb{Z}[f]}^i(M, N)$  for  $0 \leq i < j$ ). In general, one can pick a two-term free resolution of  $A$  and use the long exact sequence.  $\square$

## 2.5. Condensed abelian groups.

**Remark 2.5.1.** For the time being, the following facts will be used without proof in this text. (They have or will be formalized in Lean though.)

- There is a natural functor  $\mathrm{Top} \rightarrow \mathrm{Cond}(\mathrm{Sets})$ .
- The category of condensed abelian groups (resp. condensed  $R$ -modules) is an abelian category with enough projectives. For  $S$  an extremally disconnected set, the objects  $\mathbb{Z}[S]$  (resp.  $R[S]$ ) is projective.
- We write  $H^i(S, M)$  for  $\mathrm{Ext}^i(\mathbb{Z}[S], M)$ .

**Definition 2.5.2.** Consider an exact sequence of abelian groups

$$X' \xrightarrow{f} X \xrightarrow{g} X''$$

such that all of  $X'$ ,  $X$  and  $X''$  carry the structure of compact-Hausdorffly-filtered-pseudonormed abelian groups. Assume that the maps are strict, i.e.,  $f(X'_{\leq c}) \subset X_{\leq c}$  and  $g(X_{\leq c}) \subset X''_{\leq c}$ . We say that the sequence is *exact with constant  $c_f$*  if  $\ker(g) \cap X_{\leq c} \subset f(X'_{\leq c_f c})$ .

**Proposition 2.5.3.** Consider an inverse system

$$(X'_i \xrightarrow{f_i} X_i \xrightarrow{g_i} X''_i)_i$$

of exact sequences that are exact with constant  $c_f$  (independent of  $i$ ). Moreover, assume that the transition maps  $X'_i \rightarrow X'_j$ ,  $X_i \rightarrow X_j$  and  $X''_i \rightarrow X''_j$  are strict, and let  $X'$ ,  $X$  and  $X''$  be their limits. Then

$$X' \xrightarrow{f} X \xrightarrow{g} X''$$

is exact with the same constant  $c_f$ .

*Proof.* Pass to cofiltered limits of compact Hausdorff spaces in the statements  $\ker(g) \cap X_{\leq c} \subset f(X'_{\leq c_f c})$ , noting that cofiltered limits of surjections of compact Hausdorff spaces are still surjective (by an application of Tychonoff).  $\square$

**Definition 2.5.4.** There is a natural functor

$$\begin{aligned} \text{CHPNG} &\rightarrow \text{Cond}(\text{Ab}) \\ M &\mapsto \underline{M} \end{aligned}$$

where  $\underline{M}(S)$  is defined to be collection of functions  $f: S \rightarrow M$  that factor as continuous through  $M_c$ , for some  $c$ . In symbols:

$$M(S) = \{f: S \rightarrow M \mid \exists c, f(S) \subset M_c \text{ and } f: S \rightarrow M_c \text{ is continuous}\}.$$

**Proposition 2.5.5.** Consider an exact sequence of abelian groups

$$X' \xrightarrow{f} X \xrightarrow{g} X''$$

such that all of  $X'$ ,  $X$  and  $X''$  carry the structure of compact-Hausdorffly-filtered-pseudonormed abelian groups. Assume that  $f(X'_{\leq c}) \subset X_{\leq c}$  and  $g(X_{\leq c}) \subset X''_{\leq c}$ . If the sequence is exact with constant  $c_f$ , then the sequence

$$\underline{X'} \rightarrow \underline{X} \rightarrow \underline{X''}$$

of condensed abelian groups is exact.

*Proof.* TODO: Rewrite this proof. It is written for the case  $\bullet \xrightarrow{g} \bullet \xrightarrow{0} \bullet$ .

We evaluate at  $S \in \text{ExtrDisc}$ . Any map  $S \rightarrow X''$  factors over some  $X''_{\leq c}$ , and then  $g: X_{\leq c_g c} \times_{X''} X''_{\leq c} \rightarrow X''_{\leq c}$  is a surjection of compact Hausdorff spaces; as  $S$  is extremally disconnected, the map  $S \rightarrow X''_{\leq c}$  can be lifted, showing that  $g: X(S) \rightarrow X''(S)$  is surjective. A similar argument shows that the kernel of  $g: X(S) \rightarrow X''(S)$  is in the image of  $f: X'(S) \rightarrow X(S)$ , the latter clearly being injective.  $\square$

**Lemma 2.5.6.** Let  $S = \varprojlim S_i$  be a profinite set. Then  $\mathbb{Z}[S]$  is naturally a profinitely filtered pseudo-normed group, via  $\mathbb{Z}[S]_{\leq c} = \varprojlim \mathbb{Z}[S_i]_{\leq c}$ , where  $\mathbb{Z}[S_i]_{\leq c}$  is the set  $\{\sum_{s \in S_i} n_s[s] \mid \sum_s |n_s| \leq c\}$ .

There is a natural isomorphism between the free condensed abelian group  $\mathbb{Z}[S]$  and the colimit  $\bigcup_c \mathbb{Z}[S]_{\leq c}$  of condensed sets.

*Proof.* For now, see Lemma 2.1 of [Sch20].  $\square$

**Proposition 2.5.7.** Let

$$M: \text{ProFin}^{\text{op}} \rightarrow \text{Ab}$$

be a functor, i.e. a presheaf of abelian groups on  $\text{ProFin}$ . Assume that  $M$  preserves finite products, and that for any surjective map  $f: T \rightarrow S$ , the complex

$$0 \rightarrow M(S) \rightarrow M(T) \rightarrow M(T \times_S T) \rightarrow M(T \times_S T \times_S T) \rightarrow \dots$$

is exact.

Then  $M$  is a condensed abelian group, and for all profinite sets  $S$  and  $i > 0$ , one has  $H^i(S, M) = 0$  for  $i > 0$ .

*Proof.* We prove by induction on  $i > 0$  that  $H^i(S, M) = 0$  for all profinite sets  $S$ , so assume the vanishing of  $\text{Ext}^1, \dots, \text{Ext}^i$  for some  $i \geq 0$ . (This is vacuous for  $i = 0$ .) We aim to prove that  $H^{i+1}(S, M) = 0$  for all profinite sets  $S$ . Pick any profinite set  $S$  and a cover  $T \rightarrow S$  with  $T \in \text{ExtrDisc}$ . We get a long exact sequence of condensed abelian groups

$$\dots \rightarrow \mathbb{Z}[T \times_S T \times_S T] \rightarrow \mathbb{Z}[T \times_S T] \rightarrow \mathbb{Z}[T] \rightarrow \mathbb{Z}[S] \rightarrow 0 :$$

Indeed, taken as presheaves on  $\text{ExtrDisc}$ , this is already true on the level of presheaves, where it reduces to the case of surjections of sets in which case one can write down a contracting homotopy. (Actually, the similar result is true in any topos, where one has to maybe argue a bit more carefully.)

The following argument is making explicit something usually seen through a spectral sequence. Define inductively

$$\begin{aligned} K_1 &= \ker(\mathbb{Z}[T] \rightarrow \mathbb{Z}[S]), \\ K_2 &= \ker(\mathbb{Z}[T \times_S T] \rightarrow \mathbb{Z}[T]) \end{aligned}$$

etc. One gets exact sequences

$$0 \rightarrow K_n \rightarrow \mathbb{Z}[T^n/S] \rightarrow K_{n-1} \rightarrow 0$$

for  $n \geq 2$ . From the long exact sequence

$$\dots \rightarrow H^i(T, M) \rightarrow \text{Ext}^i(K_1, M) \rightarrow H^{i+1}(S, M) \rightarrow H^{i+1}(T, M) = 0$$

we see that we have to prove that  $\text{Ext}^i(K_1, M) = 0$  (if  $i > 0$ , otherwise that  $M(T)$  surjects onto  $\text{Hom}(K_1, M)$ ). Assuming  $i > 0$ , we can go on, and using the inductive hypothesis applied to the fibre products  $T^{*/S}$ , we inductively see that

$$H^{i+1}(S, M) = \text{Ext}^i(K_1, M) = \text{Ext}^{i-1}(K_2, M) = \dots = \text{Ext}^1(K_i, M)$$

and eventually that this is the same as the cokernel of

$$M(T^{i/S}) \rightarrow \text{Hom}(K_{i+1}, M).$$

But there is an exact sequence

$$0 \rightarrow \text{Hom}(K_{i+1}, M) \rightarrow M(T^{(i+1)/S}) \rightarrow \text{Hom}(K_{i+2}, M)$$

and  $\text{Hom}(K_{i+2}, M)$  injects into  $M(T^{(i+2)/S})$ . We see that

$$\text{Hom}(K_{i+1}, M) = \ker(M(T^{(i+1)/S}) \rightarrow M(T^{(i+2)/S}))$$

and we need to see that

$$M(T^{i/S}) \rightarrow \text{Hom}(K_{i+1}, M) = \ker(M(T^{(i+1)/S}) \rightarrow M(T^{(i+2)/S}))$$

is surjective, which is precisely the exactness of the Čech complex.  $\square$

**Proposition 2.5.8.** *Let  $(M, \|\cdot\|)$  be a complete normed group, regarded as a topological group. Then the corresponding condensed abelian group  $\underline{M}$  sends any profinite set  $S$  to the completion of normed group of locally constant maps  $S \rightarrow M$  (with the supremum norm).*

*Proof.* This is a standard result. We omit the proof here, but it is formalized in Lean.  $\square$

**Proposition 2.5.9.** *Let  $(M, \|\cdot\|)$  be a complete normed group, regarded as a topological group. Then for any profinite set  $S$ , one has  $H^i(S, \underline{M}) = 0$  for  $i > 0$ .*

*Proof.* This follows Proposition 2.5.7 and the part of [Sch20, Proposition 8.19] that is already formalized.  $\square$

**Lemma 2.5.10.** *Let  $0 < r < r' < 1$  be real numbers. Let  $S$  be a profinite set, and let  $V$  be a  $r$ -normed (Banach?)  $\mathbb{Z}[T^{\pm 1}]$ -module. Then  $\mathrm{Ext}_{\mathrm{Mod}_{\mathbb{Z}[T^{\pm 1}]}}^i(\overline{\mathcal{L}}_{r'}(S), V) = 0$  for all  $i \geq 0$ . In other words,*

$$\mathrm{Ext}_{\mathbb{Z}}^i(\overline{\mathcal{L}}_{r'}(S), V) \xrightarrow{[T^{-1}]_L - [T^{-1}]_V} \mathrm{Ext}_{\mathbb{Z}}^i(\overline{\mathcal{L}}_{r'}(S), V)$$

*is a bijection for all  $i$ .*

*Proof.* With Proposition 2.4.3, it suffices to prove the following assertion. Pick  $1 > r' > r > 0$ , a profinite  $S$ , and some  $r$ -Banach  $\mathbb{Z}[T^{\pm 1}]$ -module  $V$  as before. Then we want to prove that

$$\mathrm{Ext}_{\mathbb{Z}[T^{-1}]}^i(Q'(\overline{\mathcal{M}}_{r'}(S)), V) = 0$$

for all  $i \geq 0$ .

At this point, it is profitable to rewrite this again as the bijectivity of

$$(T^{-1})_V - (T^{-1})_{\mathcal{M}} : \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S)), V) \rightarrow \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S)), V).$$

Now these Ext-groups can be computed! More precisely, recall that  $Q'(\overline{\mathcal{M}}_{r'}(S))$  is a complex of the form

$$\dots \rightarrow \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)^2] \rightarrow \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)] \rightarrow 0.$$

Termwise, the Ext-groups turn into cohomology groups

$$H^i(\overline{\mathcal{M}}_{r'}(S)^{2^j}, V).$$

Unfortunately,  $\overline{\mathcal{M}}_{r'}(S)$  itself is not profinite, so we cannot directly apply Proposition 2.5.9. To get around this last cliff, we write  $Q'(\overline{\mathcal{M}}_{r'}(S))$  as a filtered colimit of complexes

$$Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq c} : \dots \rightarrow \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)_{\leq \kappa_1 c}^2] \rightarrow \mathbb{Z}[\overline{\mathcal{M}}_{r'}(S)_{\leq \kappa_0 c}] \rightarrow 0$$

where the constants  $\kappa_0 = 1, \kappa_1, \dots$  are positive and chosen so that all differentials are well-defined. (The possibility of choosing such constants has already been formalized; TODO include pointer.) It suffices to prove that

$$(T^{-1})_V - (T^{-1})_{\mathcal{M}} : \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq c}, V) \rightarrow \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq r'c}, V)$$

is a pro-isomorphism in  $c$ , as then the final result follows by passing to a derived limit over  $c$ , see Lemma 2.5.11 below. This final pro-isomorphism assertion can finally be written out, and it unravels to the statement of Theorem 1.7.1.

In passing to the derived limit over  $c$ , we use the following lemma.  $\square$

**Lemma 2.5.11.** *Assume that in each degree  $i$ , the map*

$$(T^{-1})_V - (T^{-1})_{\mathcal{M}} : \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq c}, V) \rightarrow \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq r'c}, V)$$

*is a pro-isomorphism in  $c$  (i.e., pro-systems of kernels, and of cokernels, are pro-zero). Then*

$$(T^{-1})_V - (T^{-1})_{\mathcal{M}} : \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S)), V) \rightarrow \mathrm{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S)), V).$$

*is an isomorphism.*



*Proof.* We have

$$Q'(\overline{\mathcal{M}}_{r'}(S)) = \bigcup_n Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n},$$

inducing a resolution

$$0 \rightarrow \bigoplus_n Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n} \rightarrow \bigoplus_n Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n} \rightarrow Q'(\overline{\mathcal{M}}_{r'}(S)) \rightarrow 0.$$

Passing to a corresponding long exact sequence reduces one to checking that the squares

$$\begin{array}{ccc} \prod_n \text{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n}, V) & \longrightarrow & \prod_n \text{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n}, V) \\ \downarrow & & \downarrow \\ \prod_n \text{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n}, V) & \longrightarrow & \prod_n \text{Ext}^i(Q'(\overline{\mathcal{M}}_{r'}(S))_{\leq n}, V) \end{array}$$

are bicartesian (here, horizontal maps are shift minus identity, and vertical maps are  $(T^{-1})_V - (T^{-1})_{\mathcal{M}}$ ). Equivalently, the horizontal maps become isomorphisms on vertical kernels, and vertical cokernels. But the vertical kernels and vertical cokernels induce pro-zero systems of abelian groups, and then the horizontal kernels and cokernels compute  $\varprojlim_n$  and  $\varprojlim_n^1$  of their systems, which vanish.  $\square$

**Proposition 2.5.12.** *Let  $0 < r < 1$  be a real number, and let  $S$  be a profinite set. Decomposing  $\mathbb{Z}((T))_r$  into positive and nonpositive coefficients yields a direct sum decomposition*

$$\mathbb{Z}((T))_r = T\mathbb{Z}[[T]]_r \oplus \mathbb{Z}[T^{-1}].$$

*This extends to a decomposition of spaces of measures*

$$\mathcal{L}_r(S) = \mathcal{L}(S, T\mathbb{Z}((T))_r) = \mathcal{L}(S, T\mathbb{Z}[[T]]_r) \oplus \mathcal{L}(S, \mathbb{Z}[T^{-1}])$$

*where  $\mathcal{M}(S, \mathbb{Z}[T^{-1}]) = \mathbb{Z}[T^{-1}][S]$  is the free condensed  $\mathbb{Z}[T^{-1}]$ -module on  $S$ . Letting  $\overline{\mathcal{L}}_r(S) = \mathcal{L}(S, T\mathbb{Z}[[T]]_r)$ , we get a short exact sequence of condensed  $\mathbb{Z}[T^{-1}]$ -modules*

$$0 \rightarrow \mathbb{Z}[T^{-1}][S] \rightarrow \mathcal{L}_r(S) \rightarrow \overline{\mathcal{L}}_r(S) \rightarrow 0.$$

*Proof.* On  $\mathbb{Z}((T))_{r, \leq c}$ , only finitely many nonpositive coefficients can possibly be nonzero, and each of them is bounded. This shows that the nonpositive summand of  $\mathbb{Z}((T))_r$  is given by  $\mathbb{Z}[T^{-1}]$ . To pass to profinite  $S$ , use Proposition 2.5.6.  $\square$

**Lemma 2.5.13.** *Let  $0 < r < r' < 1$  be real numbers. Let  $S$  be a profinite set, and let  $V$  be an  $r$ -normed  $\mathbb{Z}[T^{\pm 1}]$ -module. Then  $\text{Ext}_{\text{Mod}_{\mathbb{Z}[T^{-1}]}}^i(\mathcal{L}_{r'}(S), V) = 0$  for all  $i > 0$ . In other words,*

$$\text{Ext}_{\mathbb{Z}}^i(\mathcal{L}_{r'}(S), V) \xrightarrow{[T^{-1}]_L - [T^{-1}]_V} \text{Ext}_{\mathbb{Z}}^i(\mathcal{L}_{r'}(S), V)$$

*is a bijection for all  $i > 0$  and a surjection for  $i = 0$ .*

*Proof.* Consider the long exact sequence of Ext-groups arising from the short exact sequence (Lemma 2.5.12)

$$0 \rightarrow \mathbb{Z}[T^{-1}][S] \rightarrow \mathcal{L}_{r'}(S) \rightarrow \overline{\mathcal{L}}_{r'}(S) \rightarrow 0$$

by applying  $\text{Ext}^*(\_, V)$ .

By Lemma 2.5.7 all groups  $\mathrm{Ext}_{\mathrm{Cond}(\mathrm{Ab})}^i(\mathbf{Z}[S], V)$  vanish for  $i > 0$ . And by Lemma 2.5.10 all groups  $\mathrm{Ext}_{\mathrm{Mod}_{\mathbb{Z}[T^{-1}]}^{\mathrm{cond}}}^i(\overline{\mathcal{L}}_{r'}(S), V)$  vanish for  $i \geq 0$ . The result follows.

The "In other words" version can be proved without mentioning  $\mathbb{Z}[T^{-1}]$ -linear Ext groups, by using the same ingredients and the five lemma.  $\square$

**Lemma 2.5.14.** *Let  $0 < p' < 1$  be a real number, let  $S$  be a profinite set, and let  $r'$  denote  $(\frac{1}{2})^{p'}$ . There is a short exact sequence of condensed  $\mathbb{Z}[T^{-1}]$ -modules*

$$0 \rightarrow \mathcal{L}_{r'}(S) \rightarrow \mathcal{L}_{r'}(S) \rightarrow \mathcal{M}_{p'}(S) \rightarrow 0$$

where the first map is multiplication by  $2T - 1$ , and the second is evaluation at  $T = \frac{1}{2}$ .

*Proof.* TODO  $\square$

**Theorem 2.5.15** (Clausen–Scholze). *Let  $0 < p' < p \leq 1$  be real numbers, let  $S$  be a profinite set, and let  $V$  be a  $p$ -Banach space. Let  $\mathcal{M}_{p'}(S)$  be the space of rel  $p'$ -measures on  $S$ . Then*

$$\mathrm{Ext}_{\mathrm{Cond}(\mathrm{Ab})}^i(\mathcal{M}_{p'}(S), V) = 0$$

for  $i \geq 1$ .

*Proof.* Recall from Lemma 2.5.14 the short exact sequence

$$0 \rightarrow \mathcal{L}_{r'}(S) \rightarrow \mathcal{L}_{r'}(S) \rightarrow \mathcal{M}_{p'}(S) \rightarrow 0.$$

Apply to this  $\mathrm{Ext}^*(\_, V)$  to obtain a long exact sequence. Note that  $T$  acts on  $V$  via multiplication by  $\frac{1}{2}$  (by Lemma 2.1.2. Hence we can use LemmaExt-L to obtain isomorphisms between the Ext-groups involving  $\mathcal{L}_{r'}(S)$ , for  $i > 0$ , and a surjection for  $i = 0$ . The result follows.  $\square$

## REFERENCES

- [Sch19] P. Scholze. Lectures on Condensed Mathematics. 2019.
- [Sch20] P. Scholze. Lectures on Analytic Geometry. 2020.