

BLUEPRINT FOR THE LIQUID TENSOR EXPERIMENT

PETER SCHOLZE (ALL RESULTS JOINT WITH DUSTIN CLAUSEN), EDITED BY JOHAN COMMELIN

Remark 0.1. This text is based on the lecture notes on Analytic Geometry [Sch20], by Peter Scholze. The final section is copy-pasted from those lecture notes almost verbatim. This text is meant as a blueprint for the Liquid Tensor Experiment.

1. BREEN–DELIBNE DATA

The goal of this section is to give a precise statement of the Breen–Deligne resolution. We first give the statement, and provide details later.

Theorem 1.1 (Breen–Deligne). *For an abelian group A , there is a resolution, functorial in A , of the form*

$$\dots \rightarrow \bigoplus_{i=1}^{n_i} \mathbb{Z}[A^{r_{ij}}] \rightarrow \dots \rightarrow \mathbb{Z}[A^3] \oplus \mathbb{Z}[A^2] \rightarrow \mathbb{Z}[A^2] \rightarrow \mathbb{Z}[A] \rightarrow A \rightarrow 0.$$

I (Johan Commelin) have not figured out the details. But it seems to be possible to avoid the $\bigoplus_{i=1}^{n_i}$, so we will aim for something like the following statement.

Theorem 1.2. *For an abelian group A , there is a resolution, functorial in A , of the form*

$$\dots \rightarrow \mathbb{Z}[A^{n_i}] \rightarrow \dots \rightarrow \mathbb{Z}[A^2] \rightarrow \mathbb{Z}[A] \rightarrow A \rightarrow 0.$$

What does a homomorphism $f: \mathbb{Z}[A^m] \rightarrow \mathbb{Z}[A^n]$ that is functorial in A look like?

Well, such a “universal map” is specified by what it does to the generators $(a_1, a_2, a_3, \dots, a_m)$. It can send such an element to an arbitrary element of $\mathbb{Z}[A^n]$, but it has to be “universal”.

In the end, this means that f will be a \mathbb{Z} -linear combination of “basic universal maps”, where a “basic universal map” is one that sends (a_1, a_2, \dots, a_m) to (b_1, \dots, b_n) , where b_i is a \mathbb{Z} -linear combination $c_{i,1} \cdot a_1 + \dots + c_{i,m} \cdot a_m$. So a “basic universal map” is specified by the $n \times m$ -matrix c .

Definition 1.3. A *basic universal map* from exponent m to n , is an $n \times m$ -matrix with coefficients in \mathbb{Z} .

Definition 1.4. A *universal map* from exponent m to n , is a formal \mathbb{Z} -linear combination of basic universal maps from exponent m to n .

We point out that basic universal maps can be composed by matrix multiplication, and this formally induces a composition of universal maps.

Definition 1.5. The addition on A^n induces a universal map $\sigma_\alpha: \mathbb{Z}[(A^n)^2] \rightarrow \mathbb{Z}[A^n]$, namely the formal generator $(I_n I_n)$, where I_n denotes the $n \times n$ identity matrix. (Here α stands for “addition”.)

Definition 1.6. The formal sum of the two projections $(A^n)^2 \rightarrow A^n$ induces a universal map $\sigma_\pi: \mathbb{Z}[(A^n)^2] \rightarrow \mathbb{Z}[A^n]$, namely the formal sum $(I_n 0) + (0 I_n)$, where I_n denotes the $n \times n$ identity matrix, and 0 the $n \times n$ zero matrix. (Here π stands for “projections”.)

Definition 1.7. Let f be a universal map from exponent m to n . Then $f \oplus f$ denotes the universal map from exponent $2m$ to $2n$, that applies f componentwise. If f is a generator (i.e. a basic universal map) then $f \oplus f$ is

$$\begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}.$$

Definition 1.8. A tuple (n, f) of Breen–Deligne data consists of a sequence of exponents $n_0, n_1, n_2, \dots \in \mathbb{N}$, and universal maps f_i from exponent n_{i+1} to n_i .

Such a tuple is a *complex* if for all i we have $f_i \circ f_{i+1} = 0$.

Definition 1.9. If (n, f) is a tuple of Breen–Deligne data, then $(n, f) \oplus (n, f)$ is the tuple consisting of exponents $2n_i$ and universal maps $f_i \oplus f_i$.

The two universal maps σ_α and σ_π explained in the examples above, can be viewed as universal maps of complexes: $(n, f) \oplus (n, f) \rightarrow (n, f)$.

Definition 1.10. A *homotopy* for a tuple (n, f) of Breen–Deligne data is a homotopy between the maps of complexes

$$\sigma_\alpha, \sigma_\pi: (n, f) \oplus (n, f) \rightarrow (n, f)$$

In other words, it consists of universal maps h_i from exponent $2n_i$ to n_{i+1} , such that

$$f_{i+1} \circ h_{i+1} + h_i \circ (f_i \oplus f_i) = \sigma_\alpha - \sigma_\pi.$$

Definition 1.11. A *Breen–Deligne package* is a triple (n, f, h) , such that (n, f) is Breen–Deligne data that is a complex, and h is a homotopy for (n, f) .

Definition 1.12. We will now construct an example of a Breen–Deligne package. In some sense, it is the “easiest” solution to the conditions posed above. The exponents will be $n_i = 2^i$, and the homotopies h_i will be the identity. Under these constraints, we inductively construct the universal maps f_i :

$$f_0 = \sigma_\alpha - \sigma_\pi, \quad f_{i+1} = (\sigma_\alpha - \sigma_\pi) - (f_i \oplus f_i).$$

We leave it as exercise for the reader, to verify that with these definitions (n, f, h) forms a Breen–Deligne package.

2. SPACES OF CONVERGENT POWER SERIES

Definition 2.1. A *pseudo-normed group* is an abelian group A , together with an increasing filtration $A_c \subseteq A$ of subsets A_c indexed by $\mathbb{R}_{\geq 0}$, such that each A_c contains 0, is closed under negation, and $A_{c_1} + A_{c_2} \subseteq A_{c_1+c_2}$.

Definition 2.2. Let $r > 0$ be a real number, and let S be a finite set. Denote by $\overline{\mathcal{M}}_r(S)$ the set

$$\left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} \mid \sum_{n \geq 1, s \in S} |a_{n,s}| r^n < \infty \right\}$$

Definition 2.3. Let $r > 0$ and $c \geq 0$ be real numbers, and let S be a finite set. Denote by $\overline{\mathcal{M}}_r(S)_{\leq c}$ the set

$$\left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} \left| \sum_{n \geq 1, s \in S} |a_{n,s}| r^n \leq c \right. \right\}$$

Lemma 2.4. The space $\overline{\mathcal{M}}_r(S)$ is a pseudo-normed group, with filtration $\overline{\mathcal{M}}_r(S)_{\leq c}$.

Proof. Obvious. □

Lemma 2.5. Let $r > 0$ and $c \geq 0$ be real numbers, and let S be a finite set. The space $\overline{\mathcal{M}}_r(S)_{\leq c}$ is the profinite limit of the finite sets

$$\overline{\mathcal{M}}_r(S)_{\leq c, \leq N} = \left\{ \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} \left| \sum_{1 \leq n \leq N, s \in S} |a_{n,s}| r^n \leq c \right. \right\}$$

This endows $\overline{\mathcal{M}}_r(S)_{\leq c}$ with the profinite topology.

Proof. Formalised, but omitted from this text. □

For the remainder of this section, let $r > 0, c \geq 0$ be real numbers, and let S be a finite set.

Definition 2.6. There is a natural action of T^{-1} on $\overline{\mathcal{M}}_r(S)$, via

$$T^{-1} \cdot \left(\sum_{n \geq 1} a_{n,s} T^n \right)_{s \in S} = \left(\sum_{n \geq 1} a_{n+1,s} T^n \right)_{s \in S}.$$

Definition 2.7. The natural action of T^{-1} on $\overline{\mathcal{M}}_r(S)$ restricts to maps

$$T^{-1} \cdot _ : \overline{\mathcal{M}}_r(S)_{\leq c} \rightarrow \overline{\mathcal{M}}_r(S)_{\leq c'}$$

for all $c' \geq c/r$.

Lemma 2.8. For all $c' \geq c/r$, the natural maps

$$T^{-1} \cdot _ : \overline{\mathcal{M}}_r(S)_{\leq c} \rightarrow \overline{\mathcal{M}}_r(S)_{\leq c'}$$

are continuous.

Proof. Formalised, but omitted from this text. □

Lemma 2.9. Let f be a basic universal map from exponent m to n . We get an induced homomorphism of pseudo-normed groups

$$\overline{\mathcal{M}}_r(S)^m \rightarrow \overline{\mathcal{M}}_r(S)^n$$

bounded by the maximum (over all i) of $\sum_j |f_{ij}|$, where the f_{ij} are the coefficients of the $n \times m$ -matrix representing f .

Proof. Omitted. □

Definition 2.10. Let f be a basic universal map from exponent m to n . Let $c_1, c_2 \in \mathbb{R}_{\geq 0}$. We say that (c_1, c_2) is f -suitable, if for all i

$$\sum_j c_1 |f_{ij}| \leq c_2.$$

Lemma 2.11. *Let f be a basic universal map from exponent m to n , and let (c_1, c_2) be f -suitable. We get an induced map*

$$\overline{\mathcal{M}}_r(S)_{\leq c_1}^m \rightarrow \overline{\mathcal{M}}_r(S)_{\leq c_2}^n$$

that is continuous.

Proof. Formalised, but omitted from this text. \square

3. COMPLETIONS OF LOCALLY CONSTANT FUNCTIONS

Definition 3.1. Let V be a normed abelian group, and X a compact topological space. We denote by $\widehat{V}(X)$ the completion of the normed abelian group of locally constant functions $X \rightarrow V$ with respect to the sup norm.

This construction is functorial in bounded group homomorphisms $V \rightarrow V'$ and contravariantly functorial in continuous maps $X \rightarrow X'$.

We continue to use the notation of before: let $r > 0, c \geq 0$ be real numbers, and let S be a finite set.

Definition 3.2. Let f be a basic universal map from exponent m to n , and let (c_1, c_2) be f -suitable. We get an induced map

$$\widehat{V}(f): \widehat{V}(\overline{\mathcal{M}}_r(S)_{\leq c_1}^m) \rightarrow \widehat{V}(\overline{\mathcal{M}}_r(S)_{\leq c_2}^n).$$

Definition 3.3. Let f be a universal map from exponent m to n . Let $c_1, c_2 \in \mathbb{R}_{\geq 0}$. We say that (c_1, c_2) is f -suitable, if for all basic universal maps g that occur in the formal sum f , the pair of nonnegative reals (c_1, c_2) is g -suitable.

Definition 3.4. Let f be a universal map from exponent m to n , and let (c_1, c_2) be f -suitable. We get an induced map

$$\widehat{V}(f): \widehat{V}(\overline{\mathcal{M}}_r(S)_{\leq c_1}^m) \rightarrow \widehat{V}(\overline{\mathcal{M}}_r(S)_{\leq c_2}^n)$$

that is the sum $\sum n_g V(g)$, if f is the formal sum $\sum n_g g$ of basic universal maps.

Definition 3.5. Let $r > 0$ be a real number. An r -normed $\mathbb{Z}[T^{\pm 1}]$ -module is a normed abelian group V endowed with an automorphism $T: V \rightarrow V$ such that for all $v \in V$ we have $\|T(v)\| = r\|v\|$.

Lemma 3.6. *Let $r \in \mathbb{R}_{\geq 0}$, and let V be an r -normed $\mathbb{Z}[T^{\pm 1}]$ -module. Let X be a compact space. Then $\widehat{V}(X)$ is naturally an r -normed $\mathbb{Z}[T^{\pm 1}]$ -module, with the action of T given by post-composition.*

Proof. Formalised, but omitted from this text. \square

Definition 3.7. Let $r > 0, r' > 0, c \geq 0$ be real numbers. and let V be an r -normed $\mathbb{Z}[T^{\pm 1}]$ -module. Assume $r' \leq 1$. Let S be a finite set.

There are two natural actions of T^{-1} on $\widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c})$. The first comes from Definition 2.7, which gives a continuous map

$$\overline{\mathcal{M}}_{r'}(S)_{\leq cr'} \rightarrow \overline{\mathcal{M}}_{r'}(S)_{\leq c}$$

and thus a map

$$(T^{-1})^*: \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c}) \rightarrow \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq cr'}).$$

The other comes from Lemma 3.6, using the r -normed $\mathbb{Z}[T^{\pm 1}]$ -module V . We get a map

$$[T^{-1}]: \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c}) \rightarrow \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c}),$$

that we can compose with the map

$$\widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c}) \rightarrow \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq cr'}),$$

obtained from the natural inclusion

$$\overline{\mathcal{M}}_{r'}(S)_{\leq cr'} \rightarrow \overline{\mathcal{M}}_{r'}(S)_{\leq c}.$$

We thus end up with two maps

$$(T^{-1})^*, [T^{-1}]: \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c}) \rightarrow \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq cr'}).$$

and we define $\widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c})^{T^{-1}}$ to be the equalizer of $(T^{-1})^*$ and $[T^{-1}]$. In other words, the kernel of $(T^{-1})^* - [T^{-1}]$.

Definition 3.8. Let $r, r' \in \mathbb{R}_{>0}$, and let V be an r -normed $\mathbb{Z}[T^{\pm 1}]$ -module. Assume $r' \leq 1$. Let S be a finite set. Let f be a universal map from exponent m to n , and let (c_1, c_2) be f -suitable.

The natural map from Definition 3.4 restricts to a map

$$\widehat{V}(f)^{T^{-1}}: \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c_2}^n)^{T^{-1}} \rightarrow \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c_1}^m)^{T^{-1}}$$

4. SOME NORMED HOMOLOGICAL ALGEBRA

Definition 4.1. A *system of complexes* of normed abelian groups is for each sufficiently large c (i.e. all $c \geq c_0$ for some $c_0 > 0$), a complex

$$C_c^\bullet: C_c^0 \rightarrow C_c^1 \rightarrow \dots$$

of normed abelian groups together with maps of complexes $\text{res}_{c',c}^i: C_{c'}^\bullet \rightarrow C_c^\bullet$, for $c' \geq c$, satisfying the obvious associativity condition.

By convention, for every system of complexes C , we will set $C_c^{-1} = 0$ for all c . This will come up each time we write C_c^{i-1} and i could be 0.

In this section, given $x \in C_{c'}^\bullet$ and $c \leq c'$ we will use the notation $x|_c := \text{res}_{c',c}(x)$.

Definition 4.2. A system of complexes is *admissible* if all differentials and maps $\text{res}_{c',c}^i$ are norm-nonincreasing.

Definition 4.3. Let $(C_c^\bullet)_c$ be a system of complexes. For integers $m \geq 0$ and constants $k \geq 1$, $c'_0 > 0$, the datum $(C_c^\bullet)_c$ is *$\leq k$ -exact in degrees $\leq m$ and for $c \geq c'_0$ with bound K* if the following condition is satisfied. For all $c \geq c'_0$ and all $x \in C_{kc}^i$ with $i \leq m$ there is some $y \in C_c^{i-1}$ such that

$$\|x|_c - dy\| \leq K\|dx\|.$$

We will also need a version where the inequality is relaxed by some arbitrary small additive constant.

Definition 4.4. Let $(C_c^\bullet)_c$ be a system of complexes. For integers $m \geq 0$ and constants $k \geq 1$, $c'_0 > 0$, the datum $(C_c^\bullet)_c$ is *weakly $\leq k$ -exact in degrees $\leq m$ and for $c \geq c'_0$ with bound K* if the following condition is satisfied. For all $c \geq c'_0$, all $x \in C_{kc}^i$ with $i \leq m$ and any $\varepsilon > 0$ there is some $y \in C_c^{i-1}$ such that

$$\|x|_c - dy\| \leq K\|dx\| + \varepsilon.$$

We first note that the difference between those two definitions is only about cocycles if we are ready to loose a tiny something on the norm bound K .

Lemma 4.5. *Let $(C_c^\bullet)_c$ be a system of complexes. If $(C_c^\bullet)_c$ is weakly $\leq k$ -exact in degrees $\leq m$ and for $c \geq c'_0$ with bound K and if, for all $c \geq c'_0$ and all $x \in C_{kc}^i$ with $i \leq m$ such that $dx = 0$ there is some $y \in C_c^{i-1}$ such that $x|_c = dy$ then, for every positive δ , $(C_c^\bullet)_c$ is $\leq k$ -exact in degrees $\leq m$ and for $c \geq c'_0$ with bound $K + \delta$.*

Proof. Let δ be some positive real number. Let x be an element of C_{kc}^i for some $c \geq c'_0$ and $i \leq m$. If $dx = 0$ then the assumption we made about exact elements is exactly what we want.

Assume now that $dx \neq 0$. The weak exactness assumption applied to $\varepsilon = \delta\|dx\|$ gives some $y \in C_c^{i-1}$ such that

$$\begin{aligned} \|x|_c - dy\| &\leq K\|dx\| + \delta\|dx\| \\ &= (K + \delta)\|dx\| \end{aligned}$$

□

A more important observation is that, in both definitions, we can also ask some control on the norm of y if we are ready to square the restriction depth factor k .

Lemma 4.6. *Let $(C_c^\bullet)_c$ be a system of complexes which is weakly $\leq k$ -exact in degrees $\leq m$ and for $c \geq c'_0$ with bound K . For all $c \geq c'_0$, all $x \in C_{k^2c}^i$ with $i \leq m$, all $\varepsilon > 0$ and all $\delta > 0$ there is some $y \in C_c^{i-1}$ such that*

$$\|x|_c - dy\| \leq K\|dx\| + \varepsilon \quad \text{and} \quad \|y\| \leq K(K + 1)\|x\| + \delta.$$

Proof. Fix x , ε and δ . The weak exactness assumption applied to x and some η to be chosen later gives us $w \in C_{kc}^{i-1}$ such that

$$\|x|_{kc} - dw\| \leq K\|dx\| + \eta.$$

Then the weak exactness assumption applied to w and some τ to be chosen later gives us $z \in C_c^{i-2}$ such that

$$\|w|_c - dz\| \leq K\|dw\| + \tau.$$

We set $y = w|_c - dz$. Since $dy = dw|_c$, we get the first required estimate as long as $\eta \leq \varepsilon$. And we have:

$$\begin{aligned} \|y\| &\leq K\|dw\| + \tau \\ &\leq K(\|x|_{kc}\| + K\|dx\| + \eta) + \tau \\ &\leq K(K + 1)\|x\| + K\eta + \tau \end{aligned}$$

which is fine as long as $K\eta + \tau \leq \delta$. So we set $\eta = \min(\varepsilon, \delta/(2K))$ and $\tau = \delta/2$. □

Lemma 4.7. *Let M_c^\bullet be an admissible collection of complexes of normed abelian groups, where $c \geq c_0$.*

Assume that M_c^\bullet is weakly $\leq k$ -exact in degrees $\leq m$ for $c \geq c_0$ with bound K . Then the completion $\overline{M_c^\bullet}$ is weakly $\leq k^2$ -exact in degrees $\leq m$ for $c \geq c_0$ with bound K .

Proof. Let $x \in \overline{M_{k^2c}^i}$, where $c \geq c_0$ and $i \leq m$ and let $\varepsilon > 0$. We can write $x = \sum_j x^j$ where

- $x^j \in M_{k^2c}^i$ for all $j \geq 0$,
- $\|x - x^0\| \leq \varepsilon_0$ for some positive ε_0 to be chosen later. This implies that $\|dx - dx^0\| \leq \varepsilon_0$ and in particular $\|dx^0\| \leq \|dx\| + \varepsilon_0$,

- $\|x^j\| \leq \varepsilon_j$ if $j > 0$, for some positive ε_j to be chosen later. This implies $\|dx^j\| \leq \varepsilon_j$ for all $j > 0$.

Using Lemma 4.6, we get a sequence y^j in M_c^{i-1} such that

$$\|x_{|c}^j - dy^j\| \leq K\|dx^j\| + \delta_j \quad \text{and} \quad \|y^j\| \leq K(K+1)\|x^j\| + \tau_j.$$

for positive sequences δ and τ to be chosen later.

Since M_c^{i-1} is complete, the series $\sum y^j$ converges as soon as we can guarantee that $\sum \|y^j\|$ converges. Our estimates ensure this convergence as soon as the sum of the $K(K+1)\varepsilon_j + \tau_j$ converges so here we only need ε and τ to be summable.

We then set $y = \sum y^j$ and compute:

$$\begin{aligned} \|x_{|c} - dy\| &= \left\| \sum_{j \geq 0} x_{|c}^j - dy^j \right\| \\ &\leq \sum_{j \geq 0} \|x_{|c}^j - dy^j\| \\ &\leq \sum_{j \geq 0} K\|dx^j\| + \delta_j \\ &\leq K\|dx\| + K\varepsilon_0 + \delta_0 + \sum_{j > 0} (K\varepsilon_j + \delta_j) \end{aligned}$$

So everything is fine as long as $\sum_{j \geq 0} (K\varepsilon_j + \delta_j) \leq \varepsilon$, say $\varepsilon_j = \varepsilon 2^{-j-2}/K$ and $\delta_j = \varepsilon 2^{-j-2}$. \square

Lemma 4.8. *Let M_c^\bullet be an admissible collection of complexes of complete normed abelian groups, where $c \geq c_0$.*

Assume that M_c^\bullet is weakly $\leq k$ -exact in degrees $\leq m$ for $c \geq c_0$ with bound K . Then M_c^\bullet , for every $\delta > 0$, it is $\leq k^2$ -exact in degrees $\leq m$ for $c \geq c_0$ with bound $K + \delta$.

Proof. Lemma 4.5 ensures we only need to care about cocycles of M . More precisely, let x be a cocycle in $M_{k^2c}^i$ for some $i \leq m$ and $c \geq c_0$. We need to find $y \in M_c^{i-1}$ such that $dy = x_{|c}$.

By weak $\leq k$ -exactness applied to x and a sequence ε_j to be chosen later, we can find a sequence $w^j \in M_{kc}^{i-1}$ such that

$$\|x_{kc} - dw^j\| \leq \varepsilon_j.$$

Then, by weak $\leq k$ -exactness applied to each $w^{j+1} - w^j$ and a sequence δ_j to be chosen later, we can find a sequence $z^j \in M_c^{i-2}$ such that

$$\|(w^{j+1} - w^j)_{|c} - dz^j\| \leq K\|dw^{j+1} - dw^j\| + \delta_j.$$

We set $y^j := w_{|c}^j - \sum_{l=0}^{j-1} dz^l \in M_c^{i-1}$.

We have

$$\begin{aligned} \|y^{j+1} - y^j\| &= \|(w^{j+1} - w^j)_{|c} - dz^j\| \\ &\leq K\|dw^{j+1} - dw^j\| + \delta_j \\ &\leq 2K\varepsilon_j + \delta_j. \end{aligned}$$

So y_j is a Cauchy sequence as long as we make sure $2K\varepsilon_j + \delta_j \leq 2^{-j}$ for instance. Since M_c^{i-1} is complete, this sequence converges to some y . Because $dy^j = dw_{|c}^j$, we get that $\|x_{|c} - dy^j\| \leq \varepsilon_j$ and in the limit $x_{|c} = dy$. \square

Proposition 4.9. *Let M_c^\bullet and $M_c'^\bullet$ be two admissible collections of complexes of complete normed abelian groups, where $c \geq c_0$. Let $f_c^\bullet : M_c^\bullet \rightarrow M_c'^\bullet$ be a collection of maps between these collections of complexes that is strictly compatible with the norm and commutes with restriction maps, and assume that it satisfies*

$$\|x_{|c}\| \leq K'' \|f(x)\|$$

for all $i \leq m+1$ and all $x \in M_{kk''c}^i$. Let $N_c^\bullet = M_c'^\bullet / M_c^\bullet$ be the collection of quotient complexes, with the quotient norm; this is again an admissible collection of complexes.

Assume that M_c^\bullet (resp. $M_c'^\bullet$) is weakly $\leq k$ -exact (resp. $\leq k'$ -exact) in degrees $\leq m$ for $c \geq c_0$ with bound K (resp. K'). Then N_c^\bullet is weakly $\leq kk'k''$ -exact in degrees $\leq m-1$ for $c \geq c_0$ with bound $K'(KK'' + 1)$.

Proof. Let $n \in N_{kk'k''c}^{i-1}$ for $i \leq m-1$. We fix $\varepsilon > 0$. We need to find an element $y \in N_c^{i-1}$ such that

$$\|n_{|c} - dy\| \leq K'(KK'' + 1)\|dn\| + \varepsilon.$$

Pick any preimage $m' \in M_{kk'k''c}^{i-1}$ of n . In particular dm' is a preimage of dn . By definition of the quotient norm, we can find $m_1 \in M_{kk'k''c}^{i+1}$ and $m_1'' \in (M')_{kk'k''c}^{i+1}$ such that

$$dm' = f(m_1) + m_1''$$

with $\|m_1''\| \leq \|dn\| + \varepsilon_1$, for some positive ε_1 to be chosen later.

Applying the differential to the last displayed equation, and using that this kills the image of d , and that f is a map of complexes, we see that

$$f(dm_1) = -dm_1''.$$

Using the norm bound on f , we get

$$\begin{aligned} \|dm_{1|kk'c}\| &\leq K'' \|f(dm_1)\| = K'' \|dm_1''\| \\ &\leq K'' \|m_1''\| \leq K'' \|dn\| + K'' \varepsilon_1. \end{aligned}$$

On the other hand, weak exactness of M applied to $m_{1|kk'c}$ gives $m_0 \in M_{kk'c}^i$ such that

$$\|m_{1|kk'c|k'c} - dm_0\| \leq K \|dm_{1|kk'c}\| + \varepsilon_1$$

which combines with the previous estimate to give:

$$\|m_{1|k'c} - dm_0\| \leq KK'' \|dn\| + (KK'' + 1)\varepsilon_1.$$

Now let $m'_{\text{new}} = m'_{|k'c} - f(m_0) \in M_{k'c}^{i-1}$; this is a lift of $n_{|k'c}$. Then

$$dm'_{\text{new}} = dm'_{|k'c} - f(m_{1|k'c}) + f(m_{1|k'c} - dm_0) = m''_{1|k'c} + f(m_{1|k'c} - dm_0).$$

In particular,

$$\|dm'_{\text{new}}\| \leq (KK'' + 1)\|dn\| + (KK'' + 2)\varepsilon_1.$$

Now weak exactness of M' gives $x \in M_c^{i-1}$ such that

$$\|m'_{\text{new}|c} - dx\| \leq K' \|dm'_{\text{new}}\| + \varepsilon_1 \leq K'((KK'' + 1)\|dn\| + (KK'' + 2)\varepsilon_1) + \varepsilon_1.$$

In particular, letting $y \in N_c^{i-1}$ be the image of x , we get

$$\|n|_c - dy\| \leq K'(KK'' + 1)\|dn\| + (K'(KK'' + 2) + 1)\varepsilon_1,$$

which exactly what we wanted if we choose $\varepsilon_1 = \varepsilon/(K'(KK'' + 2) + 1)$. \square

Proposition 4.10. *Let M_c^\bullet and $M'_c{}^\bullet$ be two admissible collections of complexes of complete normed abelian groups, where $c \geq c_0$. Let $f_c^\bullet : M_c^\bullet \rightarrow M'_c{}^\bullet$ be a collection of maps between these collections of complexes that is strictly compatible with the norm and commutes with restriction maps, and assume that it satisfies*

$$\|x|_c\| \leq K''\|f(x)\|$$

for all $i \leq m+1$ and all $x \in M_{k''c}^i$. Let $N_c^\bullet = M_c^\bullet / \overline{M_c^\bullet}$ be the collection of quotient complexes, with the quotient norm; this is again an admissible collection of complexes.

Assume that M_c^\bullet (resp. $M'_c{}^\bullet$) is $\leq k$ -exact (resp. $\leq k'$ -exact) in degrees $\leq m$ for $c \geq c_0$ with bound K (resp. K'). Then, for every $\delta > 0$, N_c^\bullet is $\leq (kk'k'')^2$ -exact in degrees $\leq m-1$ for $c \geq c_0$ with bound $K'(KK'' + 1) + \delta$.

Proof. The exactness assumptions on M and M' give the corresponding weak exactness condition. Hence Proposition 4.9 ensures that N_c^\bullet is weakly $\leq kk'k''$ -exact in degrees $\leq m-1$ for $c \geq c_0$ with bound $K'(KK'' + 1)$. Since N_c^\bullet is a complex of complete groups, Lemma 4.8 gives the required exactness. \square

Proposition 4.11. *Fix an integer $m \geq 0$ and a constant k . Then there exists an $\epsilon > 0$ and a constant k_0 , depending (only) on k and m , with the following property.*

Consider an admissible system of double complexes $M_c^{p,q}$, $p, q \geq 0$, $c \geq c_0$, of complete normed abelian groups as well as some $k' \geq k_0$ and some $H > 0$, such that

$$\begin{array}{ccccccc} M_c^{0,0} & \longrightarrow & M_c^{0,1} & \longrightarrow & M_c^{0,2} & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ M_c^{1,0} & \longrightarrow & M_c^{1,1} & \longrightarrow & M_c^{1,2} & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ M_c^{2,0} & \longrightarrow & M_c^{2,1} & \longrightarrow & \ddots & & \\ \downarrow & & \downarrow & & & & \\ \vdots & & \vdots & & & & \end{array}$$

- (1) for $j = 0, \dots, m$, the columns $M_c^{p,j}$ are $\leq k$ -exact in degrees $\leq m$ for $c \geq c_0$;
- (2) for $i = 0, \dots, m+1$, the rows $M_c^{i,q}$ are $\leq k$ -exact in degrees $\leq m-1$ for $c \geq c_0$;
- (3) for $q = 0, \dots, m$ and $c \geq c_0$, there is a map $h_{k'c}^q : M_{k'c}^{0,q+1} \rightarrow M_c^{1,q}$ with

$$\|h_{k'c}^q(x)\|_{M_c^{1,q}} \leq H\|x\|_{M_{k'c}^{0,q+1}}$$

for all $x \in M_{k'c}^{0,q+1}$, and such that for all $c \geq c_0$ and $x \in M_{k'c}^{0,q}$, one has

$$(4.1) \quad \|\text{res}_{k'^2c, k'c}^{1,q}(d^{0,q}(x)) \pm h_{k'^2c}^q(d_{k'^2c}^{0,q}(x)) \pm d_{k'c}^{1,q-1}(h_{k'^2c}^{q-1}(x))\|_{M_{k'c}^{1,q}} \leq \epsilon \|\text{res}_{k'^2c, c}^{0,q}(x)\|_{M_c^{0,q}}.$$

Then the first row is $\leq \max(k'^2, 2k_0H)$ -exact in degrees $\leq m$ for $c \geq c_0$.

We note that the bound on the homotopy is of a peculiar nature, in that the bound only depends on a deep restriction of x .

Proof. First, we treat the case $m = 0$. If $m = 0$, we claim that one can take $\epsilon = \frac{1}{2k}$ and $k_0 = k$. We have to prove exactness at the first step. Let $x_{k'^2c} \in M_{k'^2c}^{0,0}$ and denote $x_{k'c} = \text{res}_{k'^2c, k'c}^{0,0}(x)$ and $x_c = \text{res}_{k'^2c, c}^{0,0}(x)$. Then by assumption (1) (and $k' \geq k$), we have

$$\|x_c\|_{M_c^{0,0}} \leq k \|d_{k'c}^{0,0}(x_{k'c})\|_{M_{k'c}^{1,0}}.$$

On the other hand, by (3),

$$\|\text{res}_{k'^2c, k'c}^{1,0}(d_{k'^2c}^{0,0}(x)) \pm h_{k'^2c}^0(d_{k'^2c}'^{0,0}(x))\|_{M_{k'c}^{1,0}} \leq \epsilon \|x_c\|_{M_c^{0,0}}.$$

In particular, noting that $\text{res}_{k'^2c, k'c}^{1,0}(d_{k'^2c}^{0,0}(x)) = d_{k'c}'^{0,0}(x_{k'c})$, we get

$$\|x_c\|_{M_c^{0,0}} \leq k \|d_{k'c}'^{0,0}(x_{k'c})\|_{M_{k'c}^{1,0}} \leq k\epsilon \|x_c\|_{M_c^{0,0}} + kH \|d_{k'^2c}'^{0,0}(x)\|_{M_{k'^2c}^{0,1}}.$$

Thus, taking $\epsilon = \frac{1}{2k}$ as promised, this implies

$$\|x_c\|_{M_c^{0,0}} \leq 2kH \|d_{k'^2c}'^{0,0}(x)\|_{M_{k'^2c}^{0,1}}.$$

This gives the desired $\leq \max(k'^2, 2k_0H)$ -exactness in degrees $\leq m$ for $c \geq c_0$.

Now we argue by induction on m . Consider the complex $N^{p,q}$ given by $M^{p,q+1}$ for $q \geq 1$ and $N^{p,0} = M^{p,1}/\overline{M^{p,0}}$ (the quotient by the closure of the image, which is also the completion of $M^{p,1}/M^{p,0}$), equipped with the quotient norm. Using the normed version of the snake lemma, Proposition 4.10 in the appendix to this lecture, one checks that this satisfies the assumptions for $m-1$, with k replaced by $\max(k^4, k^3 + k + 1)$. \square

5. MORE BREEN-DELIGNE

We need the following results about the Breen-Deligne resolution for normed abelian groups. Let us consider here abelian groups M (in any topos) equipped with an increasing filtration $M_{\leq c} \subset M$ by subobjects indexed by the positive real numbers, such that $0 \in M_{\leq c}$, $-M_{\leq c} = M_{\leq c}$ and $M_{\leq c} + M_{\leq c'} \subset M_{\leq c+c'}$; we need no further conditions. Let us call these pseudo-normed abelian groups.

Fix a choice of a functorial Breen-Deligne resolution

$$C(M) : \dots \rightarrow \mathbb{Z}[M^{a_i}] \rightarrow \dots \rightarrow \mathbb{Z}[M^{a_1}] \rightarrow \mathbb{Z}[M^{a_0}] \rightarrow M \rightarrow 0$$

of an abelian group M ; purely for notational convenience, we can and do assume that each term is of the form $\mathbb{Z}[M^{a_i}]$ (as opposed to a finite direct sum of such). The possibility of doing this follows from the proof of [Sch19, Theorem 4.10], noting that a functor of the form $A \mapsto \mathbb{Z}[A^n] \oplus \mathbb{Z}[A^m]$ admits a surjection from the functor $A \mapsto \mathbb{Z}[A^{n+m}] \oplus \mathbb{Z}$; this gives a resolution where all terms are of the form $\mathbb{Z}[A^{a_i}] \oplus \mathbb{Z}^m$. Now pass to the quotient of these complexes corresponding to the map $0 \rightarrow A$; this gives a complex all of whose terms are of the form $\mathbb{Z}[A^{a_i}]/\mathbb{Z}$. Noting that $\mathbb{Z}[A^{a_i}]$ is functorially isomorphic to $\mathbb{Z}[A^{a_i}]/\mathbb{Z} \oplus \mathbb{Z}$ (via splitting $0 \rightarrow A^{a_i} \rightarrow 0$), we can then add an acyclic complex of \mathbb{Z} 's in each degree to get a resolution all of whose terms are of the form $\mathbb{Z}[A^{a_i}]$.

We also need some homotopies. More precisely, we start with the following homotopy.

Lemma 5.1. *For an abelian group M , the maps σ_1, σ_2 from*

$$C(M^2) : \dots \rightarrow \mathbb{Z}[M^{2a_i}] \rightarrow \dots \rightarrow \mathbb{Z}[M^{2a_1}] \rightarrow \mathbb{Z}[M^{2a_0}]$$

to

$$C(M) : \dots \rightarrow \mathbb{Z}[M^{a_i}] \rightarrow \dots \rightarrow \mathbb{Z}[M^{a_1}] \rightarrow \mathbb{Z}[M^{a_0}],$$

induced by addition $M^2 \rightarrow M$, respectively the sum of the two maps induced by two projections $M^2 \rightarrow M$, are homotopic, via some functorial homotopy

$$h_i : \mathbb{Z}[M^{2a_i}] \rightarrow \mathbb{Z}[M^{a_{i+1}}].$$

If M is a pseudo-normed abelian group object in any topos, then σ_1 and σ_2 are well-defined as maps of complexes from

$$C(M^2)_{\leq c/2} : \dots \rightarrow \mathbb{Z}[M_{\leq c_i c/2}^{2a_i}] \rightarrow \dots \rightarrow \mathbb{Z}[M_{\leq c_1 c/2}^{2a_1}] \rightarrow \mathbb{Z}[M_{\leq c/2}^{2a_0}]$$

to

$$C(M)_{\leq c} : \dots \rightarrow \mathbb{Z}[M_{\leq c_i c}^{a_i}] \rightarrow \dots \rightarrow \mathbb{Z}[M_{\leq c_1 c}^{a_1}] \rightarrow \mathbb{Z}[M_{\leq c}^{a_0}]$$

for all $c > 0$. In that case, for all $i \geq 0$ there are universal constants c'_i such that h_i defines well-defined maps

$$\mathbb{Z}[M_{\leq c_i c/2}^{2a_i}] \rightarrow \mathbb{Z}[M_{\leq c'_i c_{i+1} c}^{a_{i+1}}]$$

for all $c > 0$.

Proof. This is a consequence of the proof of the existence of the Breen-Deligne resolution, proved in the same way as [Sch19, Proposition 4.17]. The existence of the constants c'_i is again formal, as in the last lemma. \square

Now we need the following generalization to adding N elements.

Lemma 5.2. *Let N be a power of 2. The maps of complexes σ_1, σ_2 from*

$$C(M^N) : \dots \rightarrow \mathbb{Z}[M^{Na_i}] \rightarrow \dots \rightarrow \mathbb{Z}[M^{Na_1}] \rightarrow \mathbb{Z}[M^{Na_0}]$$

to

$$C(M) : \dots \rightarrow \mathbb{Z}[M^{a_i}] \rightarrow \dots \rightarrow \mathbb{Z}[M^{a_1}] \rightarrow \mathbb{Z}[M^{a_0}],$$

induced by addition $M^N \rightarrow M$, respectively the sum of the N maps induced by the N projections $M^N \rightarrow M$, are homotopic, via some functorial homotopy

$$h_i^N : \mathbb{Z}[M^{Na_i}] \rightarrow \mathbb{Z}[M^{a_{i+1}}]$$

which moreover satisfies the following bound, with the same constants c'_0, c'_1, \dots as in the previous lemma:

If M is a pseudo-normed abelian group object in any topos, then σ_1 and σ_2 are well-defined as maps of complexes from

$$C(M^N)_{\leq c/N} : \dots \rightarrow \mathbb{Z}[M_{\leq c_i c/N}^{Na_i}] \rightarrow \dots \rightarrow \mathbb{Z}[M_{\leq c_1 c/N}^{Na_1}] \rightarrow \mathbb{Z}[M_{\leq c/N}^{Na_0}]$$

to

$$C(M)_{\leq c} : \dots \rightarrow \mathbb{Z}[M_{\leq c_i c}^{a_i}] \rightarrow \dots \rightarrow \mathbb{Z}[M_{\leq c_1 c}^{a_1}] \rightarrow \mathbb{Z}[M_{\leq c}^{a_0}]$$

for all $c > 0$. In that case, h_i^N defines well-defined maps

$$\mathbb{Z}[M_{\leq c_i c/N}^{Na_i}] \rightarrow \mathbb{Z}[M_{\leq c'_i c_{i+1} c}^{a_{i+1}}]$$

for all $c > 0$.

Proof. Let $N = 2^m$. For each $j = 0, \dots, m-1$, the two maps from $C(M^{2^{j+1}})$ to $C(M^{2^j})$ from the previous lemma are homotopic, and we use the homotopy from that lemma. Composing homotopies (which amounts concretely to a certain sum) we get the desired homotopy from $C(M^{2^m})$ to $C(M)$. It follows directly from this construction that the constants c'_i are unchanged. \square

6. END OF PROOF

Definition 6.1. Let (n, f, h) be a Breen–Deligne package, and let $c = (c_0, c_1, \dots)$ be a sequence of nonnegative real numbers. We say that c is (n, f, h) -suitable, if for all i , the pair (c_{i+1}, c_i) is f_i -suitable.

(Note! The order (c_{i+1}, c_i) is contravariant compared to Definition 3.3. This is because of the contravariance of $\widehat{V}(_)$; see Definition 3.8.)

Definition 6.2. Let $r, r' \in \mathbb{R}_{>0}$, and let V be an r -normed $\mathbb{Z}[T^{\pm 1}]$ -module. Assume $r' \leq 1$. Let S be a finite set. Let (n, f, h) be a Breen–Deligne package, and let $c' = (c'_0, c'_1, c'_2, \dots)$ be a sequence of constants in $\mathbb{R}_{\geq 0}$ that is (n, f, h) -suitable.

For every $c \in \mathbb{R}_{\geq 0}$, the maps from Definition 3.8 induced by the universal maps f_i from the Breen–Deligne package (n, f, h) assemble into a complex of normed abelian groups

$$\dots \rightarrow \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c_i}^{n_i})^{T^{-1}} \rightarrow \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c_{i+1}}^{n_{i+1}})^{T^{-1}} \rightarrow \dots \rightarrow 0.$$

Together, these complexes fit into a system of complexes with the natural restriction maps.

Now we state the following result, which is our main goal.

N.b.: It differs from Theorem 9.4 of [Sch20] only in one aspect: we assume that the sets S are finite, rather than profinite.

Theorem 6.3. Let (n, f, h) be a Breen–Deligne package, and let $c' = (c'_0, c'_1, c'_2, \dots)$ be a sequence of constants in $\mathbb{R}_{\geq 0}$. Fix radii $1 > r' > r > 0$. For any m there is some k and c_0 such that for all finite sets S and all r -normed $\mathbb{Z}[T^{\pm 1}]$ -modules V , the system of complexes

$$C_c^\bullet: \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c})^{T^{-1}} \rightarrow \widehat{V}(\overline{\mathcal{M}}_{r'}(S)_{\leq c'_1}^2)^{T^{-1}} \rightarrow \dots$$

is $\leq k$ -exact in degrees $\leq m$ for $c \geq c_0$.

Remark 6.4. Note: the text below is copied almost verbatim from [Sch20]. Small parts have been formalized. We expect that the text will be rewritten and expanded as the formalization project progresses.

We will prove Theorem 6.3 by induction on m . Unfortunately, the induction requires us to prove a stronger statement. Before stating it, we need another definition.

Definition 6.5. A *polyhedral lattice* is a finite free abelian group Λ equipped with a norm $\|\cdot\|_\Lambda: \Lambda \otimes \mathbb{R} \rightarrow \mathbb{R}$ (so $\Lambda \otimes \mathbb{R}$ is a Banach space) that is given by the supremum of finitely many linear functions on Λ with rational coefficients; equivalently, the “unit ball” $\{\lambda \in \Lambda \otimes \mathbb{R} \mid \|\lambda\|_\Lambda \leq 1\}$ is a rational polyhedron.

Endow $\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))$ with the subspaces

$$\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c} = \{f: \Lambda \rightarrow \overline{\mathcal{M}}_{r'}(S) \mid \forall x \in \Lambda, f(x) \in \overline{\mathcal{M}}_{r'}(S)_{\leq c\|x\|}\}.$$

As Λ is polyhedral, it is enough to check the given condition for finitely many x .

We can then define double complexes like (??). Lemma ?? stays true with the same constants. Now we claim the following generalization of Theorem 6.3.

Theorem 6.6. *Fix radii $1 > r' > r > 0$. For any m there is some k such that for all polyhedral lattices Λ there is a constant $c_0(\Lambda) > 0$ such that for all finite sets S and all r -normed $\mathbb{Z}[T^{\pm 1}]$ -modules V , the system of complexes*

$$C_{\Lambda, c}^\bullet: \widehat{V}(\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c})^{T^{-1}} \rightarrow \widehat{V}(\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c_1 c}^2)^{T^{-1}} \rightarrow \dots$$

is $\leq k$ -exact in degrees $\leq m$ for $c \geq c_0(\Lambda)$.

Proof. Use $\Lambda = \mathbb{Z}$, and the isomorphism $\text{Hom}(\mathbb{Z}, A) \cong A$. \square

We note that the constants c_1, c_2, \dots implicit in the choice of the complex are chosen once and for all (after fixing r and r'), and it can be ensured that the transition maps in the complex are norm-nonincreasing. Indeed, if the c_i are suitable as in Definition 6.1, the maps

$$\widehat{V}(\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c_i c}^{a_i}) \rightarrow \widehat{V}(\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c_{i+1} c}^{a_{i+1}})$$

will have bounded norm, independently of V (as they are a certain universal finite sum of maps induced by maps between the profinite sets in paranthesis, each of which induces a map of norm bounded by 1), so on the subspace of T^{-1} -invariants, one can shrink the norm down to 1 by shrinking c_{i+1} . We make and fix this choice of the c_i for the statement of Theorem 6.6, and the rest of the proof.

Finally, we can prove the key combinatorial lemma, ensuring that any element of $\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))$ can be decomposed into N elements whose norm is roughly $\frac{1}{N}$ of the original element. As preparation, we have the following simple result.

Lemma 6.7. *Let Λ be a finite free abelian group, let N be a positive integer, and let $\lambda_1, \dots, \lambda_m \in \Lambda$ be elements. Then there is a finite subset $A \subset \Lambda^\vee$ such that for all $x \in \Lambda^\vee = \text{Hom}(\Lambda, \mathbb{Z})$ there is some $x' \in A$ such that $x - x' \in N\Lambda^\vee$ and for all $i = 1, \dots, m$, the numbers $x'(\lambda_i)$ and $(x - x')(\lambda_i)$ have the same sign, i.e. are both nonnegative or both nonpositive.*

Proof. It suffices to prove the statement for all x such that $\lambda_i(x) \geq 0$ for all i ; indeed, applying this variant to all $\pm \lambda_i$, one gets the full statement.

Thus, consider the submonoid $\Lambda_+^\vee \subset \Lambda^\vee$ of all x that pair nonnegatively with all λ_i . This is a finitely generated monoid by standard results; let y_1, \dots, y_M be a set of generators. Then we can take for A all sums $n_1 y_1 + \dots + n_M y_M$ where all $n_j \in \{0, \dots, N-1\}$. \square

Now we have the key lemma:

Lemma 6.8. *Let Λ be a polyhedral lattice. Then for all positive integers N there is a constant d such that for all $c > 0$ one can write any $x \in \text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c}$ as*

$$x = x_1 + \dots + x_N$$

where all $x_i \in \text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c/N+d}$.

Proof. Pick $\lambda_1, \dots, \lambda_m \in \Lambda$ generating the norm. We fix a finite subset $A \subset \Lambda^\vee$ satisfying the conclusion of the previous lemma. Write

$$x = \sum_{n \geq 1, s \in S} x_{n,s} T^n[s]$$

with $x_{n,s} \in \Lambda^\vee$. Then we can decompose

$$x_{n,s} = Nx_{n,s}^0 + x_{n,s}^1$$

where $x_{n,s}^1 \in A$ and we have the same-sign property of the last lemma. Letting $x^0 = \sum_{n \geq 1, s \in S} x_{n,s}^0 T^n[s]$, we get a decomposition

$$x = Nx^0 + \sum_{a \in A} ax_a$$

with $x_a \in \overline{\mathcal{M}}_{r'}(S)$ (with the property that in the basis given by the $T^n[s]$, all coefficients are 0 or 1). Crucially, we know that for all $i = 1, \dots, m$, we have

$$\|x(\lambda_i)\| = N\|x^0(\lambda_i)\| + \sum_{a \in A} |a(\lambda_i)| \|x_a(\lambda_i)\|$$

by using the same sign property of the decomposition.

Using this decomposition of x , we decompose each term into N summands. This is trivial for the first term Nx^0 , and each summand of the second term reduces to the similar problem for $\Lambda = \mathbb{Z}$. In that case, one can take $d = 1$, as follows by decomposing any sum with terms of size at most 1 into N such partial sums whose sums differ by at most 1. (It follows that in general one can take for d the supremum over all i of $\sum_{a \in A} |a(\lambda_i)| \cdot$) \square

Proof of Theorem 6.6. We argue by induction on m , so assume the result for $m - 1$ (this is no assumption for $m = 0$, so we do not need an induction start). This gives us some $k > 1$ for which the statement of Theorem 6.6 holds true for $m - 1$; if $m = 0$, simply take any $k > 1$. In the proof below, we will increase k further in a way that depends only on m and r . After this modified choice of k , we fix ϵ and k_0 as provided by Proposition 4.11. Moreover, we let k' be the supremum of k_0 and the c'_i from Lemma 5.1 (and 5.2) for $i = 0, \dots, m$. Finally, choose a positive integer b so that $2k'(\frac{r}{r'})^b \leq \epsilon$, and let N be the minimal power of 2 that satisfies

$$k'/N \leq (r')^b.$$

Then in particular $r^b N \leq \frac{2}{k'}(\frac{r}{r'})^b \leq \epsilon$.

We consider the diagonal embedding

$$\Lambda \hookrightarrow \Lambda' = \Lambda^N,$$

where we endow Λ' with the norm

$$\|(\lambda_1, \dots, \lambda_N)\|_{\Lambda'} = \frac{1}{N}(\|\lambda_1\|_{\Lambda} + \dots + \|\lambda_N\|_{\Lambda}).$$

For any $m \geq 1$, let $\Lambda'^{(m)}$ be given by $\Lambda'^m / \Lambda \otimes (\mathbb{Z}^m)_{\sum=0}$; then $\Lambda'^{(\bullet)}$ is cosimplicial polyhedral lattice, the Čech conerve of $\Lambda \rightarrow \Lambda'$. For $m = 0$, we set $\Lambda'^{(0)} = \Lambda$. It is clear that all of these are polyhedral lattices.

In particular, for any $c > 0$, we have

$$\mathrm{Hom}(\Lambda'^{(m)}, \overline{\mathcal{M}}_{r'}(S))_{\leq c} = \mathrm{Hom}(\Lambda', \overline{\mathcal{M}}_{r'}(S))_{\leq c}^{m / \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c}},$$

the m -fold fibre product of $\mathrm{Hom}(\Lambda', \overline{\mathcal{M}}_{r'}(S))_{\leq c}$ over $\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c}$; and

$$\mathrm{Hom}(\Lambda', \overline{\mathcal{M}}_{r'}(S))_{\leq c} = \mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c/N}^N,$$

with the map to $\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c}$ given by the sum map.

Consider the collection of double complexes $C_{\Lambda'(\bullet),c}^\bullet$ associated to this cosimplicial polyhedral lattice by Dold-Kan. Up to rescaling the norms in the complex for $\Lambda'^{(m)}$ by a universal constant (something like $(m+2)!$), the differentials are strictly compatible with norms (as they are an alternating sum of $m+1$ face maps, all of which are of norm ≤ 1), so this collection of normed double complexes is admissible. By induction, the first condition of Proposition 4.11 is satisfied for all $c \geq c_0$ with c_0 large enough (depending on Λ but not V or S). By Lemma 6.8, and noting that $\text{Hom}(\Lambda'^{(\bullet)}, \overline{\mathcal{M}}_{r'}(S))_{\leq c}$ is the Čech nerve of

$$\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c/N}^N \xrightarrow{\Sigma} \text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c},$$

also the second condition is satisfied, with k the maximum of the previous k and some constant depending only on m and r , provided we take c_0 large enough so that $(k-1)r'c_i c_0/N$ is at least the d of Lemma 6.8 for all $i = 0, \dots, m$ (so this choice of c_0 again depends on Λ). Indeed, then one can splice a surjection of profinite sets between the maps

$$\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c_i c/N}^{Na} \rightarrow \text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c_i c}^a$$

and

$$\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq k c_i c/N}^{Na} \rightarrow \text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq k c_i c}^a,$$

and so the transition map between the columns of that double complex factors over a similar complex arising from a simplicial hypercover of profinite sets, so the constants are bounded by Proposition ??, Lemma ??, and Proposition 4.10. At this point, we have finalized our choice of k (and, as promised, this choice depended only on m and r), and so we also finalized the constants ϵ , k' and N from the first paragraph of the proof.

Finally, to check the third condition, we use Lemma 5.2 to find, in degrees $\leq m$, a homotopy between the two maps from the first row

$$\widehat{V}(\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c})^{T^{-1}} \rightarrow \widehat{V}(\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c_1 c}^2)^{T^{-1}} \rightarrow \dots$$

to the second row

$$\widehat{V}(\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c/N}^N)^{T^{-1}} \rightarrow \widehat{V}(\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c_1 c/N}^{2N})^{T^{-1}} \rightarrow \dots$$

respectively induced by the addition $\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c/N}^N \rightarrow \text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c}$ (which is the map that forms part of the double complex), and the map that is the sum of the N maps induced by the N projection maps

$$\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c/N}^N \rightarrow \text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c/N} \subset \text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c}.$$

By Lemma 5.2, we can find this homotopy between the complex for $k'c$ and the complex for c , by our choice of $k' \geq c'_i$ for $i = 0, \dots, m$. As N is fixed, the homotopy is the universal homotopy from Lemma 5.2, and in particular its norm is bounded by some universal constant H .

Finally, it remains to establish the estimate (4.1) on the homotopic map. We note that this takes $x \in \widehat{V}(\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq k'^2 c_i c}^{a_i})^{T^{-1}}$ (with $i = q$ in the notation of (4.1)) to the element

$$y \in \widehat{V}(\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq k' c_i c/N}^{Na_i})^{T^{-1}}$$

that is the sum of the N pullbacks along the N projection maps $\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq k' c_i c/N}^{Na_i} \rightarrow \text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq k' c_i c}^{a_i}$. We note that these actually take image in $\text{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c_i c}^{a_i}$ as $N \geq k'$,

so this actually gives a well-defined map

$$\widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c_i c}^{a_i})^{T^{-1}} \rightarrow \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq k' c_i c/N}^{N a_i})^{T^{-1}}.$$

We need to see that this map is of norm $\leq \epsilon$. Now note that by our choice of N , we actually have $k' c_i c/N \leq (r')^b c_i c$, so this can be written as the composite of the restriction map

$$\widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq c_i c}^{a_i})^{T^{-1}} \rightarrow \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq (r')^b c_i c}^{a_i})^{T^{-1}}$$

and

$$\widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq (r')^b c_i c}^{a_i})^{T^{-1}} \rightarrow \widehat{V}(\mathrm{Hom}(\Lambda, \overline{\mathcal{M}}_{r'}(S))_{\leq k' c_i c/N}^{N a_i})^{T^{-1}}.$$

The first map has norm exactly r^b , by T^{-1} -invariance, and as multiplication by T scales the norm with a factor of r on \widehat{V} .¹ The second map has norm at most N (as it is a sum of N maps of norm ≤ 1). Thus, the total map has norm $\leq r^b N$. But by our choice of N , we have $r^b N \leq \epsilon$, giving the result.

Thus, we can apply Proposition 4.11, and get the desired $\leq \max(k'^2, 2k_0 H)$ -exactness in degrees $\leq m$ for $c \geq c_0$, where k' , k_0 and H were defined only in terms of k , m , r' and r , while c_0 depends on Λ (but not on V or S). This proves the inductive step. \square

Question 6.9. Can one make the constants explicit, and how large are they?² Modulo the Breen-Deligne resolution, all the arguments give in principle explicit constants; and actually the proof of the existence of the Breen-Deligne resolution should be explicit enough to ensure the existence of bounds on the c_i and c'_i .

This completes the proof of all results announced so far.

REFERENCES

- [Sch19] P. Scholze. Lectures on Condensed Mathematics. 2019.
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¹Here is where we use $r' > r$, ensuring different scaling behaviour of the norm on source and target.

²A back of the envelope calculation seems to suggest that k is roughly doubly exponential in m , and that N has to be taken of roughly the same magnitude.