Existence of a Riemannian Metric (atlas + partition of unity)

Definition 0.1 (Partition of unity subordinate to a cover)**.** Let M be a smooth manifold and let $\{U_i\}_{i\in\mathcal{I}}$ be an open cover of M. A *smooth partition of unity on M subordinate to the cover $\{U_i\}_{i\in\mathcal{I}}$* is a locally finite family $\{\rho_i\}_{i\in\mathcal{I}}$ of smooth functions $\rho_i : M \to [0,1]$ such that

1. $\mathrm{supp}(\rho_i) \subset U_i$ for every $i \in \mathcal{I}$,
2. for every $p \in M$ the sum $\sum_{i\in\mathcal{I}} \rho_i(p) = 1$ (the sum is finite at each point by local finiteness).

Theorem 0.2. *Let M be a smooth n-dimensional manifold. Assume there exists a smooth partition of unity subordinate to any given open cover of M. Let $\{(U_i, \varphi_i)\}_{i\in\mathcal{I}}$ be an atlas of coordinate charts covering M, and let $\{\rho_i\}_{i\in\mathcal{I}}$ be a smooth partition of unity subordinate to the cover $\{U_i\}_{i\in\mathcal{I}}$. Then M admits a smooth Riemannian metric g; i.e. a smooth section*

$$g \in \Gamma(M,\, T^*M \otimes T^*M)$$

whose fibrewise bilinear forms g_p are symmetric and positive-definite.

Proof. For each index i define a local $(0,2)$-tensor $g^{(i)}$ on U_i by pulling back the Euclidean inner product on $\mathbb{R}^n$ along the chart map φ_i: for $p \in U_i$ and $v, w \in T_pM$,

$$g^{(i)}_p(v,w) = \big\langle d(\varphi_i)_p(v),\, d(\varphi_i)_p(w)\big\rangle_{\mathrm{Euc}}.$$

Each $g^{(i)}$ is smooth on U_i, symmetric, and positive-definite on each fibre T_pM.

Now use the given partition of unity $\{\rho_i\}_{i\in\mathcal{I}}$ subordinate to the cover $\{U_i\}$. Define a global $(0,2)$-tensor field g on M by the locally finite sum

$$g \;:=\; \sum_{i\in\mathcal{I}} \rho_i\, g^{(i)}.$$

Concretely, for $p \in M$ and $v, w \in T_pM$,

$$g_p(v,w) = \sum_{i\in\mathcal{I}} \rho_i(p)\, g^{(i)}_p(v,w).$$

Local finiteness of the partition ensures the sum is finite at each point, so g_p is well-defined. Each summand $\rho_i g^{(i)}$ is a smooth section with support in U_i, and near any point only finitely many summands are nonzero; hence the coordinate components of g are finite sums of smooth functions and therefore smooth. Thus $g \in \Gamma(M, T^*M \otimes T^*M)$.

Symmetry is immediate from symmetry of each $g^{(i)}$ and the scalar factors ρ_i.

To check positive-definiteness: fix $p \in M$ and a nonzero $v \in T_pM$. Since $\sum_i \rho_i(p) = 1$ and each $\rho_i(p) \geq 0$, there is at least one index i with $\rho_i(p) > 0$. For such an i, because $g^{(i)}_p$ is positive-definite, $g^{(i)}_p(v,v) > 0$. Every term $\rho_j(p) g^{(j)}_p(v,v)$ is ≥ 0, and at least one is > 0; therefore

$$g_p(v,v) = \sum_j \rho_j(p) g^{(j)}_p(v,v) > 0.$$

Thus g_p is positive-definite for every $p \in M$, hence non-degenerate. Therefore g is a Riemannian metric on M. $\square$