Growth of functions: asymptotic notation

To characterize the time cost of algorithms, we focus on functions that map input size to (typically, worst-case) running time. (Similarly for space costs.)

We are interested in precise notation for characterizing **running-time differences that are likely to be significant across different platforms and different implementations of the algorithms**.

This naturally leads to an interest in the **"asymptotic growth"** of functions. We focus on how the function behaves as its input grows large.

Asymptotic notation is a standard means for describing families of functions that share similar asymptotic behavior.

Asymptotic notation allows us to ignore small input sizes, constant factors, lower-order terms in polynomials, and so forth.

Big O notation (asymptotic upper bound)

Definition $f(n) \in O(g(n))$ if there are positive c, n_0 s.t.

$$|f(n)| \leq c|g(n)| \quad \text{for all } n \geq n_0 \, .$$

We assume that these functions map reals to reals.
They may be partial functions, but, as we'll explain shortly, we will assume that they are "asymptotically-defined on $\mathcal{N}$".

It is common to write $f(n) = O(g(n))$ instead of $f(n) \in O(g(n))$.

Beware: this is *not* the same as writing $O(g(n)) = f(n)$.

What about the awkwardness of writing $f(n)$ instead of f, and $g(n)$ instead of g? (That is, writing an expression for the *value* of function f at n, when we wish to name the function f itself.)

It corresponds to the standard (and convenient) practice of writing, say, $O(n^2)$ to stand for the set of functions $f : \mathcal{R} \rightarrow \mathcal{R}$ s.t. for some positive c, n_0,

$$|f(n)| \leq c|n^2| \quad \text{for all } n \geq n_0 \, .$$

Is $g(n) = O(g(n))$?

Asymptotically defined on $\mathcal{N}$

We say we're interested in functions from $\mathcal{R}$ to $\mathcal{R}$, but ...

... what about a function like $\log_2 n$?

Problem: not defined for $n \leq 0$.

An approach that couldn't talk about log-space and log-time would not be very useful.

And what about a function like $n!$? (Commonly defined with domain $\mathcal{N}$.)

It is convenient to consider any (partial) function from $\mathcal{R}$ to $\mathcal{R}$ that is "asymptotically defined on $\mathcal{N}$" — that is, defined for all natural numbers $n \geq n_0$, for some constant n_0.

So let's assume that **in asymptotic notation we consider (partial) functions from $\mathcal{R}$ to $\mathcal{R}$ that are asymptotically defined on $\mathcal{N}$.**

Ignoring constants

Asymptotic notation is supposed to allow us to ignore constants. Let's look at how this works...

$$\text{Is } n + k = O(n)?$$

$f(n) \in O(g(n))$ if there are positive c, n_0 s.t.

$$|f(n)| \leq c|g(n)| \quad \text{for all } n \geq n_0 \, .$$

If $k = 0$, the result is trivial. (Take $c = 1, n_0 = 1$.)

If k is positive:
We can take $c = 2$ and $n_0 = k$. Notice that

$$|n + k| \leq 2|n| \quad \text{for all } n \geq k \, .$$

What if k is negative?

Then we can take $c = 1$ and $n_0 = -k$, and notice that

$$|n + k| \leq |n| \quad \text{for all } n \geq -k \, .$$

So, for all $k \in \mathcal{R}$,

$$n + k = O(n) \, .$$

If $f(n) = O(g(n))$, is $kf(n) = O(g(n))$?

$f(n) \in O(g(n))$ if there are positive c, n_0 s.t.

$$|f(n)| \leq c|g(n)| \quad \text{for all } n \geq n_0 .$$

If $k = 0$, the result is immediate, so assume otherwise.

Assume $f(n) = O(g(n))$.

So there are positive constants c', n_0' s.t.

$$|f(n)| \leq c'|g(n)| \quad \text{for all } n \geq n_0' .$$

Take $c = c'|k|$ and $n_0 = n_0'$.

Observe that, for all $n \geq n_0$,

$$\begin{array}{rcl} |kf(n)| & = & |k| \cdot |f(n)| \\ & \leq & |k| \cdot c'|g(n)| \\ & = & c|g(n)| \end{array}$$

So

$$kf(n) = O(g(n))$$

whenever $f(n) = O(g(n))$.

If $f(n) = O(g(n))$, is $f(n) + k = O(g(n))$?

$f(n) \in O(g(n))$ if there are positive c, n_0 s.t.

$$|f(n)| \le c|g(n)| \quad \text{for all } n \ge n_0 .$$

Assume that $f(n) = O(g(n))$.

We need positive c and n_0 s.t.

$$|f(n) + k| \le c|g(n)| \quad \text{for all } n \ge n_0 .$$

In general, such c and n_0 may fail to exist. But here's a simple condition that will guarantee their existence...

Assume that $|g(n)|$ has an asymptotic nonzero lower bound. That is, for some positive b and m_0,

$$|g(n)| \ge b \quad \text{for all } n \ge m_0 .$$

Since $f(n) = O(g(n))$, there are c', n_0' s.t.

$$f(n) \le c' g(n) \quad \text{for all } n \ge n_0' .$$

It suffices to take $c = |k/b| + c'$ and $n_0 = m_0 + n_0'$. Let's check that...

Given: For some positive b and m_0,

$$|g(n)| \geq b \quad \text{for all } n \geq m_0 \,,$$

and, for some positive c', n_0',

$$|f(n)| \leq c'|g(n)| \quad \text{for all } n \geq n_0' \,.$$

NTS: There are positive c and n_0 s.t.

$$|f(n) + k| \leq c|g(n)| \quad \text{for all } n \geq n_0 \,.$$

Take $c = |k/b| + c'$ and $n_0 = m_0 + n_0'$.

Notice that, for all $n \geq n_0$,

$$\begin{array}{rcl} |f(n) + k| & \leq & |f(n)| + |k| \\ & \leq & c'|g(n)| + |k| \\ & \leq & c'|g(n)| + |k| \cdot \frac{|g(n)|}{b} \\ & = & c'|g(n)| + |k/b| \cdot |g(n)| \\ & = & (c' + |k/b|) \cdot |g(n)| \\ & = & c \cdot |g(n)| \end{array}$$

So, if $|g(n)|$ has a nonzero asymptotic lower bound, then

$$f(n) + k = O(g(n))$$

whenever $f(n) = O(g(n))$.

Polynomials and big O

Our results thus far can explain, for instance, the well-known fact that

$$an^k = O(n^k)$$

and, similarly, that

$$n^k + b = O(n^k) \quad (k \geq 0)$$

But what about drawing the conclusion that

$$an^k + b = O(n^k) \quad (k \geq 0)$$

It looks like this should be true, but we'd need another argument.

There is a much more general fact about polynomials and Big O that you should know:

Any polynomial expression whose highest order term is

$$an^k$$

belongs to

$$O(n^k)\,.$$

Let's prove it. . .

Polynomials of degree at most k belong to $O(n^k)$

Consider an arbitrary expression of the form

$$a_k n^k + a_{k-1} n^{k-1} + \cdots + a_2 n^2 + a_1 n^1 + a_0 n^0$$

with $k \in \mathcal{N}$ and $a_k, \ldots, a_0 \in \mathcal{R}$. (Notice that any of $a_k, \ldots, a_0$ may equal zero. If all do, the result is trivial, so we'll assume that at least one does not.)

For convenience, we'll write this expression in the form

$$\sum_{i=0}^{k} a_i n^i \,.$$

Notice that, for all $n \geq 1$,

$$\begin{array}{rcl} \left| \sum_{i=0}^{k} a_i n^i \right| & \leq & \sum_{i=0}^{k} |a_i n^i| \\ & = & \sum_{i=0}^{k} |a_i| n^i \\ & \leq & \left(\sum_{i=0}^{k} |a_i| \right) \cdot n^k \\ & = & \left(\sum_{i=0}^{k} |a_i| \right) \cdot |n^k| \end{array}$$

which shows that

$$\sum_{i=0}^{k} a_i n^i = O(n^k) \,.$$

(Take $n_0 = 1$ and $c = \sum_{i=0}^{k} |a_i|$.)

If $f(n) = O(g(n))$ and $g(n) = O(h(n))$, then $f(n) = O(h(n))$

Assume $f(n) = O(g(n))$ and $g(n) = O(h(n))$.

It follows that there are positive c', n_0', c'', n_0'' s.t.

$$|f(n)| \leq c'|g(n)| \quad \text{for all } n \geq n_0'$$

and

$$|g(n)| \leq c''|h(n)| \quad \text{for all } n \geq n_0''\,.$$

Take $c = c'c''$ and $n_0 = n_0' + n_0''$. Notice that

$$|f(n)| \leq c'|g(n)| \leq c'c''|h(n)| = c|h(n)| \quad \text{for all } n \geq n_0\,.$$

If $f(n) = O(h(n))$ and $g(n) = O(h(n))$, then $f(n) + g(n) = O(h(n))$

Assume $f(n) = O(h(n))$ and $g(n) = O(h(n))$.

It follows that there are positive c', n_0', c'', n_0'' s.t.

$$|f(n)| \leq c'|h(n)| \quad \text{for all } n \geq n_0'$$

and

$$|g(n)| \leq c''|h(n)| \quad \text{for all } n \geq n_0''\,.$$

Take $c = c' + c''$ and $n_0 = n_0' + n_0''$. Then

$$|f(n) + g(n)| \leq |f(n)| + |g(n)| \leq c|h(n)| \quad \text{for all } n \geq n_0\,.$$

That is, $f(n) + g(n) = O(h(n))$.

It follows that if $f_i(n) = O(g(n))$ for all $i \in \mathcal{N}$, then

$$\sum_{i=0}^{k} f_i(n) = O(g(n))$$

for any $k \in \mathcal{N}$.

If $f(n) + g(n) = O(h(n))$ and $f(n)$ and $g(n)$ are asymptotically nonnegative, then $f(n) = O(h(n))$

Assume that $f(n) + g(n) = O(h(n))$ and that both $f(n)$ and $g(n)$ are asymptotically nonnegative.

It follows that there are positive c', n_0', m_0, m_0' s.t.

$$|f(n) + g(n)| \le c'|h(n)| \quad \text{for all } n \ge n_0'$$

and

$$0 \le f(n) \quad \text{for all } n \ge m_0$$

and

$$0 \le g(n) \quad \text{for all } n \ge m_0'$$

Take $c = c'$ and $n_0 = n_0' + m_0 + m_0'$. Then

$$|f(n)| \le c|h(n)| \quad \text{for all } n \ge n_0 .$$

What if one of $f(n)$,$g(n)$ is not asymptotically nonnegative?

(Consider, for example, $f(n) = n$, $g(n) = -n$, $h(n) = 0$.)

If $f(n) \cdot g(n) = O(h(n))$ and $g(n)$ has a positive asymptotic lower bound, then $f(n) = O(h(n))$

Assume that $f(n) \cdot g(n) = O(h(n))$ and that $g(n)$ has a positive asymptotic lower bound. So there are positive c', n_0', b, m_0 s.t.

$$|f(n) \cdot g(n)| \leq c'|h(n)| \quad \text{for all } n \geq n_0'$$

and

$$b \leq g(n) \quad \text{for all } n \geq m_0 \,.$$

Take $c = c'/b$ and $n_0 = n_0' + m_0$. Notice that

$$b|f(n)| = |bf(n)| \leq |f(n) \cdot g(n)| \leq c'|h(n)| \quad \text{for all } n \geq n_0$$

Therefore,

$$|f(n)| \leq \frac{c'}{b}|h(n)| = c|h(n)| \quad \text{for all } n \geq n_0$$

Is it true that if $f(n) \cdot g(n) = O(h(n))$, then at least one of $f(n), g(n)$ belongs to $O(h(n))$? Suppose

$$f(n) = \begin{cases} n & \text{, if } n \text{ is even} \\ 0 & \text{, otherwise} \end{cases} \qquad g(n) = \begin{cases} n & \text{, if } n \text{ is odd} \\ 0 & \text{, otherwise} \end{cases} \qquad h(n) = 0$$

If $f(n) = O(g(n))$ and $f'(n) = O(g'(n))$, then $f(n) \cdot f'(n) = O(g(n) \cdot g'(n))$

Assume $f(n) = O(g(n))$ and $f'(n) = O(g'(n))$.

So there are positive c', n_0', c'', n_0'' s.t.

$$|f(n)| \leq c'|g(n)| \quad \text{for all } n \geq n_0'$$

and

$$|f'(n)| \leq c''|g'(n)| \quad \text{for all } n \geq n_0''.$$

Take $c = c'c''$ and $n_0 = n_0' + n_0''$. Then, for all $n \geq n_0$,

$$\begin{array}{rcl} |f(n) \cdot f'(n)| & = & |f(n)| \cdot |f'(n)| \\ & \leq & c'|g(n)| \cdot c''|g'(n)| \\ & = & (c' \cdot c'') \cdot |g(n)| \cdot |g'(n)| \\ & = & c(|g(n)| \cdot |g'(n)|) \\ & = & c|g(n) \cdot g'(n)| \end{array}$$

It follows that if $f_i(n) = O(g_i(n))$ for all $i \in \mathcal{N}$, then

$$\prod_{i=0}^{k} f_i(n) = O\left(\prod_{i=0}^{k} g_i(n)\right)$$

for any $k \in \mathcal{N}$.

$n^j \neq O(n^k)$ if $j > k$

Assume $j > k$. Suppose there are positive c, n_0 s.t.

$$|n^j| \leq c|n^k| \quad \text{for all } n \geq n_0\,.$$

Then

$$n^{j-k} \leq c \quad \text{for all } n \geq n_0\,.$$

But this is not possible, since $j - k$ is positive. Contradiction. We conclude that the supposition was false. Hence $n^j \neq O(n^k)$ when $j > k$.

Asymptotic lower bound, asymptotically tight bound

Definition (asymptotic lower bound)
$f(n) \in \Omega(g(n))$ if there are positive c, n_0 s.t.

$$c|g(n)| \leq |f(n)| \quad \text{for all } n \geq n_0 \,.$$

Claim: $f(n) = \Omega(g(n))$ iff $g(n) = O(f(n))$. Easy to show. (Try it.)

Definition (asymptotically tight bound)
$f(n) \in \Theta(g(n))$ if there are positive c, d, n_0 s.t.

$$c|g(n)| \leq |f(n)| \leq d|g(n)| \quad \text{for all } n \geq n_0 \,.$$

Theorem $\quad f(n) = \Theta(g(n))$ iff $f(n) = \Omega(g(n))$ and $f(n) = O(g(n))$.

$f(n) \stackrel{?}{=} \Omega(f(n))$

$f(n) \stackrel{?}{=} \Theta(f(n))$

You might want to consider whether other properties of big O hold for Ω, Θ.

Other asymptotic notations

Definition (asymptotically strict upper bound)
$f(n) \in o(g(n))$ if for every positive c there is a positive n_0 s.t.

$$|f(n)| < c|g(n)| \quad \text{for all } n \geq n_0 \,.$$

Notice: If $f(n) = o(g(n))$ then $f(n) = O(g(n))$.

Claim: If $f(n) = o(g(n))$ then $f(n) \neq \Omega(g(n))$.

Definition (asymptotically strict lower bound)
$f(n) \in \omega(g(n))$ if for every positive c there is a positive n_0 s.t.

$$c|g(n)| < |f(n)| \quad \text{for all } n \geq n_0 \,.$$

Notice: If $f(n) = \omega(g(n))$ then $f(n) = \Omega(g(n))$.

Claim: If $f(n) = \omega(g(n))$ then $f(n) \neq O(g(n))$.

Claim: $f(n) = o(g(n))$ iff $g(n) = \omega(f(n))$.

A few more laws

$$\text{If } f(n) = \Theta(g(n)) \text{ and } g(n) = \left\{ \begin{array}{c} o(h(n)) \\ O(h(n)) \\ \Theta(h(n)) \\ \Omega(h(n)) \\ \omega(h(n)) \end{array} \right\} \quad \text{then } f(n) = \left\{ \begin{array}{c} o(h(n)) \\ O(h(n)) \\ \Theta(h(n)) \\ \Omega(h(n)) \\ \omega(h(n)) \end{array} \right\}.$$

$$\text{If } f(n) = \left\{ \begin{array}{c} o(g(n)) \\ \omega(g(n)) \end{array} \right\} \text{ and } g(n) = \left\{ \begin{array}{c} O(h(n)) \\ \Omega(h(n)) \end{array} \right\}$$

$$\text{then } f(n) = \left\{ \begin{array}{c} o(h(n)) \\ \omega(h(n)) \end{array} \right\}.$$

$$\text{If } f(n) = \left\{ \begin{array}{c} O(g(n)) \\ \Omega(g(n)) \end{array} \right\} \text{ and } g(n) = \left\{ \begin{array}{c} o(h(n)) \\ \omega(h(n)) \end{array} \right\}$$

$$\text{then } f(n) = \left\{ \begin{array}{c} o(h(n)) \\ \omega(h(n)) \end{array} \right\}.$$

Some facts about hierarchy of asymptotic classes

n^n beats factorial:

$$n! = o(n^n)$$

Factorial beats exponential:

$$k^n = o(n!)$$

Exponential beats polynomial:

$$n^k = o(b^n) \quad \text{if } b > 1$$

Higher-degree polynomial beats lower-degree polynomial:

$$n^j = o(n^k) \quad \text{if } 0 \leq j < k$$

Polynomial beats (poly)logarithmic:

$$\log_b^a n = o(n^k) \quad \text{if } b > 1 \text{ and } k > 0$$

Note: $\log_b^a n$ is an abbreviation for $(\log_b n)^a$.

The base of a (poly)logarithm doesn't matter:

$$\log_a^k n = \Theta(\log_b^k n) \quad \text{if } a, b > 1$$

The base of an exponential matters:

$$a^n = o(b^n) \quad \text{if } 1 \leq a < b$$

Adding a constant to the exponent (of an exponential) doesn't matter:

$$a^n = \Theta(a^{n+k})$$

A constant factor in the exponent matters:

$$a^n = o(a^{bn}) \quad \text{if } a, b > 1$$

A hierarchy of asymptotically strict bounds

$$2^{2^n}$$
$$n^n$$
$$n!$$
$$3^n$$
$$n \cdot 2^n$$
$$2^n$$
$$n^3$$
$$n^2 \log n$$
$$n^2$$
$$n \log n$$
$$n$$
$$\sqrt{n}$$
$$\log^3 n$$
$$\log^2 n$$
$$\log n$$
$$\sqrt{\log n}$$
$$\log \log n$$
$$\log \log \log n$$
$$1$$
$$1/n$$
$$0$$