Proof by induction of the Inequality of Harmonic numbers: $H_{2^n} \geq 1 + \frac{n}{2}$

My question is, for the question below, in the inductive step, where does $\frac{1}{2^{(k+1)}}$ come from?And where does 2^k come from in the third last step?

EXAMPLE 7 **An Inequality for Harmonic Numbers** The **harmonic numbers** H_j, $j = 1, 2, 3, \ldots$, are defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}.$$

For instance,

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}.$$

Use mathematical induction to show that

$$H_{2^n} \geq 1 + \frac{n}{2},$$

whenever n is a nonnegative integer.

Solution: To carry out the proof, let $P(n)$ be the proposition that $H_{2^n} \geq 1 + \frac{n}{2}$.

BASIS STEP: $P(0)$ is true, because $H_{2^0} = H_1 = 1 \geq 1 + \frac{0}{2}$.

INDUCTIVE STEP: The inductive hypothesis is the statement that $P(k)$ is true, that is, $H_{2^k} \geq 1 + \frac{k}{2}$, where k is a nonnegative integer. We must show that if $P(k)$ is true, then $P(k+1)$, which states that $H_{2^{k+1}} \geq 1 + \frac{k+1}{2}$, is also true. So, assuming the inductive hypothesis, it follows that

$$\begin{aligned}
H_{2^{k+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the definition of harmonic number} \\
&= H_{2^k} + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the definition of } 2^k\text{th harmonic number} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2^k+1} + \cdots + \frac{1}{2^{k+1}} && \text{by the inductive hypothesis} \\
&\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} && \text{because there are } 2^k \text{ terms each} \geq 1/2^{k+1} \\
&\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2} && \text{canceling a common factor of } 2^k \text{ in second term} \\
&= 1 + \frac{k+1}{2}.
\end{aligned}$$

This establishes the inductive step of the proof.

(proof-writing) (induction) (harmonic-numbers)

edited Apr 19 '15 at 6:56
Martin Sleziak
40.4k ● 5 ■ 102 ▲ 220

asked Apr 19 '15 at 5:16
null
132 ■ 1 ▲ 8

2 Answers

the $1/2^{k+1}$ comes in because that's what the 2^{k+1}-th term would be (they want to prove the inequality for $H_{2^{k+1}}$). The 2^k on the third to last line comes in by them saying that

$$\frac{1}{2^k+1} + \frac{1}{2^k+2} + \ldots + \frac{1}{2^{k+1}} \geq \frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \ldots + \frac{1}{2^{k+1}}.$$

Going from $2^k + 1$ to 2^{k+1}, there are 2^k terms. And so the right hand side of the inequality (i.e. the third line from the bottom) simplifies to $2^k \cdot \frac{1}{2^{k+1}}$.

answered Apr 19 '15 at 5:34

Brent
1,190 2 10

Because there are 2^k terms, each at least $\frac{1}{2^{k+1}}$.

These terms are $\frac{1}{2^k+1}, \frac{1}{2^k+2}, \frac{1}{2^k+3}, \ldots,$
$\frac{1}{2^k+2^k} = \frac{1}{2^{k+1}}$.

answered Apr 19 '15 at 5:35

marty cohen
59.8k 3 42 107