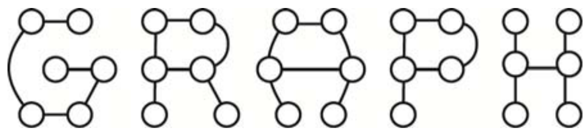
Basic Definitions and Concepts

Complement to the Prologue and to Chapter 1, "Respecting the Rules"

1. CHAINS, CYCLES AND CONNECTIVITY

Take out a sheet of paper, choose a few spots, and mark those spots with small circles or the like. Add a few links between some pairs of spots. The spots you have chosen, the little circles on your sheet, are called "vertices" and the lines you've drawn to link some of them are called "edges." These lines can be straight or curved; the important thing is whether or not a link exists between two vertices. So, for example, the drawing below is a graph that contains 29 vertices and 27 edges.

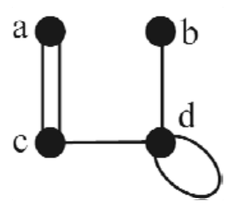


This is how Manori describes a graph. More formally, a graph is defined using an ordered pair (V, E): V is a set of elements we call "vertices" and E is a set of links, called "edges," linking certain pairs of vertices. E is therefore a subset of the Cartesian product $V \times V$. In general, we use the notation $\{x, y\}$ for an edge linking the vertices x and y.

An edge is a non-ordered pair of vertices. An edge linking the vertices x and y may be noted either as $\{x, y\}$ or as $\{y, x\}$.

Note that an edge may link a vertex to itself. In this case it is referred to as a "loop." Also, several parallel edges may link the same pair of vertices. In this case we are dealing with a "multigraph."

For example, the multigraph shown below contains four vertices, represented by black circles and tagged with the letters a, b, c and d. There is a loop around vertex d and two parallel edges linking vertices a and c.



Definition

A "chain" in a graph is a succession of edges that make it possible to link two vertices in the graph.

To illustrate this, if we place the edges {b, d} and {c, d} end-to-end in the graph above, we get a chain linking vertices b and c. There are other chains linking vertices b and c. For example, we can consider the following sequences of edges:

- {b, d}, {d, d}, {c, d}, {a, c}, {a, c}
- {b, d}, {d, d}, {d, d}, {d, d}, {d, d}, {c, d}.

Definition

A chain that touches each vertex one time at most is called "elementary."

The only elementary chain linking vertices b and c in the example above is the chain made up of edges {b, d} and {c, d}.

Definition

A chain that uses each edge one time at most is called "simple."

There are only four simple chains linking b and c in the graph above. They are made up of the following sequences of edges:

- {b, d}, {c, d}
- {b, d}, {d, d}, {c, d}
- {b, d}, {c, d}, {a, c}, {a, c}
- {b, d}, {d, d}, {c, d}, {a, c}, {a, c}.

Notes

- The last two chains described above are simple because there are two distinct edges linking vertices a and c. If there were only one edge between these two vertices, these last two chains would not be simple, because they would use the edge linking vertex a to vertex c two times.
- All elementary chains are simple, because several passages over the same edge necessarily entail several passages over the vertices at the ends of that edge.

Definition

A graph is called "connected" if there is a chain between every pair of vertices in the graph.

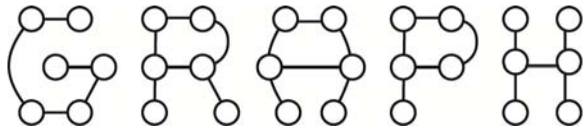
Manori defines the term "connected" for Courtel this way:

> *"If you can get to any vertex on the graph through any other vertex by following the edges, we say the graph is 'connected.' This word comes from the fact that you can connect any point to any other one using the graph's edges."*

Definition

We say that two vertices are part of a same "connected component" in a graph G if, and only if, there is a chain in G that links them together.

For example, there are five connected components in the graph below.



We can see that a graph is connected if and only if it contains only a single connected component.

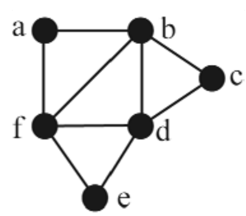
Definition

A chain whose first extremity is equal to its last extremity is called a "cycle." Once again, we talk about an "elementary" cycle when the graph's vertices are touched once at most, and we talk about a "simple" cycle if none of the edges are used twice.

Note

All elementary cycles are necessarily simple, since if we use an edge more than once, that implies that we have also touched the extremities of the edge more than once.

In the example below, there is only one elementary cycle passing by each vertex in the graph. It's the cycle made up of edges {a, b}, {b, c}, {c, d}, {d, e}, {e, f} and {f, a}. This cycle is also simple.



The same graph contains other simple cycles, for example the one made up of edges {a, b}, {b, c}, {c, d}, {d, e}, {e, f}, {f, b}, {b, d}, {d, f} and {f, a}, as illustrated below.

Elementary (and therefore simple) cycle

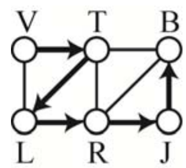
Simple, but not elementary, cycle

2. Directed graphs

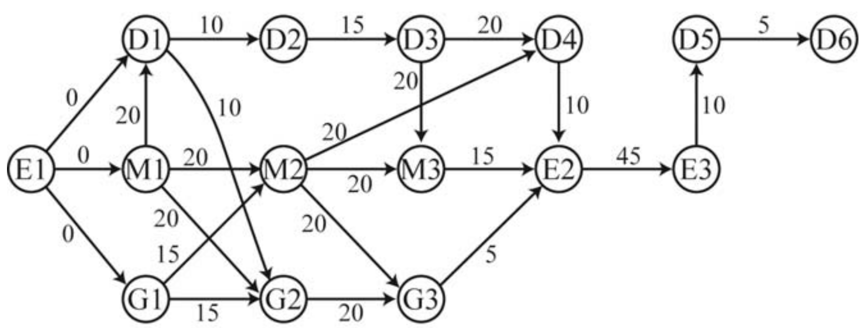
In some cases, it may be useful to associate a direction to the edges of a graph. As such, instead of having an {x, y} edge linking the x vertex and the y vertex, we might have an edge directed from x to y or from y to x. We call this an “arc” and we use the notation (x,y) to represent an arc leading from x to y.

Sometimes we may be dealing with “mixed” graphs, when some edges, but not all of them, have a direction.

In Chapter 4, Bonvin draws a mixed graph when he tries to reproduce the configuration of the treasure hunt location. The arcs represent the path that would make it possible to gather the clues left by Mr. Grumbacker.



For his part, Manori uses directed graphs in Chapter 8 to try and push back the Courtel family’s wake-up time on Thursday morning.

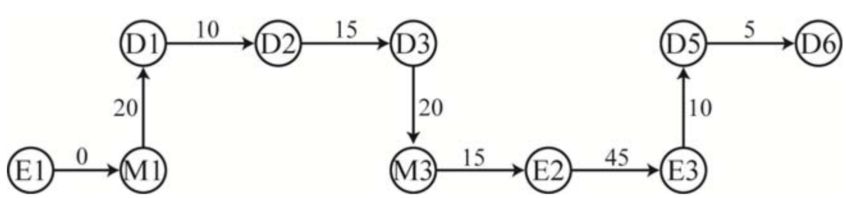


All the concepts defined above for non-directed graphs can also be extended to directed graphs. For instance, we get the following definitions.

Definition

A “path” in a directed graph is a succession of arcs that allows you to move from one vertex to another.

Example: in Chapter 8, Manori draws a path linking E1 to D6.



Definition

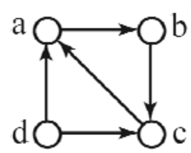
A directed graph is “strongly connected” if there is a path from x toward y and from y toward x for every x, y pair of vertices in the graph.

Example: the directed graph drawn by Manori to push back the Courtels’ wake-up time is not strongly connected because, for instance, there is no path leading from G1 to E1. To make it strongly connected, all we would have to do is add an arc from D6 toward E1.

Definition

A “circuit” in a directed graph is a path whose first extremity coincides with its final extremity.

Example: the graph below has a circuit containing vertices a, b and c.



Paths and circuits can be elementary or simple, just like chains and cycles. They are elementary if they don’t touch the same vertex twice, and they are simple if they don’t go over the same arc twice.

3. Degrees of vertices

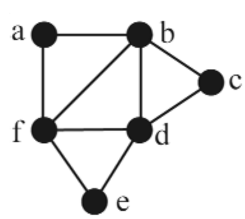
In the book’s first chapter, Manori explains to Courtel that

> *“When you’re trying to draw a graph, you can always count the number of edges touching each vertex.”*

Definition

The number of edges that touch a vertex determine its “degree.”

So for example, in the graph below, vertices a, c and e each have a degree of 2, while vertices b, d and f each have a degree of 4.



The property that permits Manori to demonstrate that the conference organizers can’t create groups within the constraints they have established is the following one.

Property

The sum of the degrees of each vertex in a graph is always an even number.

Proof

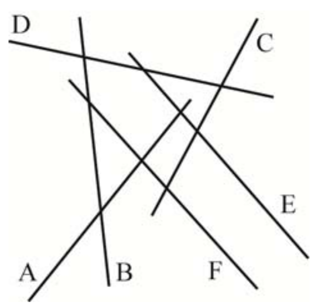
When you add together the degrees of the vertices, each edge is counted twice. Each edge {x, y} will be counted first when we consider the degree of vertex x, and a second time when we add the degree of y. This sum is therefore double the number of edges in the graph, and for that reason, it must be an even number. □

In the example above, we have three 2-degree vertices and three 4-degree vertices, which gives a total of $3\times2 + 3\times4 = 18$, or the double of 9, which is the number of edges in the graph.

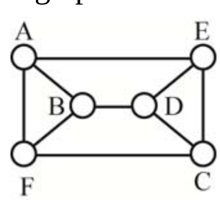
This is how Manori succeeds in convincing Sébastien that the groupings the organizers want to create cannot exist. The graph he uses includes one vertex per COPS participant, and one edge between two participants when those participants have worked together in the past. Requiring that each person be in a group of 15 including seven people they have already worked with is equivalent to trying to build a graph with 15 vertices, each with a degree of 7. Given that the total of the degrees in this hoped-for graph would be $7\times15 = 105$, which is an odd number, the graph the organizers want to build cannot exist.

Other applications:

- It is just as impossible to connect 15 computers with cables such that each computer has a direct link with exactly seven other computers. In this case the vertices are computers and the edges represent the cable links between the computers.
- It is also impossible to draw 15 line segments on the plane such that each segment has a non-empty intersection with exactly seven other segments. The vertices here correspond with segments and the edges with intersections between segments. However, if we want six segments that each touch exactly three other segments, that is possible, for example using the six segments below:



which can be represented with this graph:



Other very simple properties dealing with vertex degrees can be demonstrated. Here is another example.

Property

In every graph, there are at least two vertices of the same degree.

Proof

Consider a given graph with x number of vertices. The degree of each vertex is therefore a number chosen from the set {0, 1, …, x-1} since each vertex can only be linked at most to x-1 other vertices.

If all the degrees are different, given that {0, 1, …, x-1} contains x values and that the graph contains x vertices, there must necessarily exist a vertex with a degree of y for each y in the set {0, 1, …, x-1}.

Hence, there must be a 0-degree vertex that is not linked with any other vertex, and there must also exist a (x-1)-degree vertex which is linked to all the other vertices. But this is in fact a contradiction, because the (x-1)-degree vertex cannot be linked with the 0-degree vertex (which is not linked to any other vertex). □

Application:

Let's imagine that the COPS organizers decide to change the rules for creating their work groups. They could, for instance, decide that when two people are in a same group, they know a different number of people in the group. Manori would then have been easily able to prove that such a grouping isn't possible, since the graph associated with each group would have to include vertices that are each of different degree, which is impossible, as determined by the property set out above.