5.1.1 Maximum Height of an AVL Tree

What is the maximum height of an AVL tree having exactly n nodes? To answer this question, we will pose the following question:

What is the minimum number of nodes (sparsest possible AVL tree) an AVL tree of height h can have ?

Let F_h be an AVL tree of height h, having the minimum number of nodes. F_h can be visualized as in Figure 5.2.
Let F_l and F_r be AVL trees which are the left subtree and right subtree, respectively, of F_h. Then F_l or F_r must have height h-2.
Suppose F_l has height h-1 so that F_r has height h-2. Note that F_r has to be an AVL tree having the minimum number of nodes among all AVL trees with height of h-1. Similarly, F_r will have the minimum number of nodes among all AVL trees of height h--2. Thus we have

$$|F_h| = |F_{h-1}| + |F_{h-2}| + 1$$

where $|F_r|$ denotes the number of nodes in F_r. Such trees are called **Fibonacci trees**. See Figure 5.3. Some Fibonacci trees are shown in Figure 4.20. Note that $|F_0| = 1$ and $|F_1| = 2$.
Adding 1 to both sides, we get

$$|F_h| + 1 = (|F_{h-1}| + 1) + (|F_{h-2}| + 1)$$

Thus the numbers $|F_h| + 1$ are Fibonacci numbers. Using the approximate formula for Fibonacci numbers, we get

$$|F_h| + 1 \approx \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{h+3}$$

$\Rightarrow$

$h \approx 1.44 \log |F_n|$

$\Rightarrow$

The sparsest possible AVL tree with n nodes has height

$$h \approx 1.44 \log n$$

$\Rightarrow$

The worst case height of an AVL tree with n nodes is

$$1.44 \log n$$

Figure 5.3: Fibonacci trees

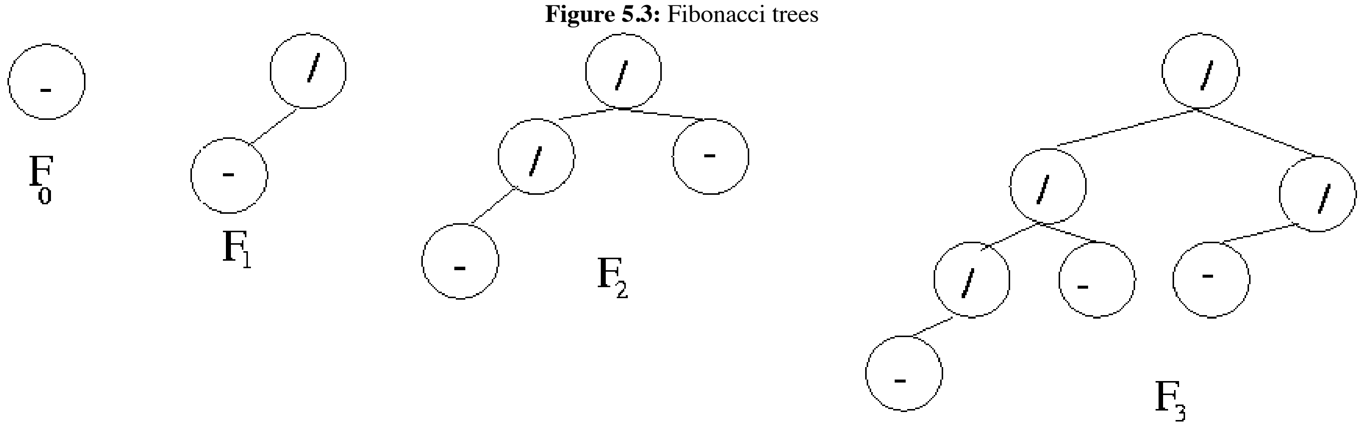

Figure 5.4: Rotations in a binary search tree