1 Mathematical logic

Let's Study

- **Statement**
- **Logical connectives**
- **Quantifiers and quantified statements**
- **Statement patterns and logical equivalence**
- **Algebra of statements**
- **Venn diagrams**

Introduction:

Mathematics is an exact science. Every statement must be precise. There has to be proper reasoning in every mathematical proof. Proper reasoning involves Logic. Logic related to mathematics has been developed over last 100 years or so. The axiomatic approach to logic was first propounded by the English philosopher and mathematician George Boole. Hence it is known as Boolean logic or mathematical logic or symbolic logic.

The word 'logic' is derived from the Greek word 'Logos' which means reason. Thus Logic deals with the method of reasoning. Aristotle (382-322 B.C.), the great philosopher and thinker laid down the foundations of study of logic in a systematic form. The study of logic helps in increasing one's ability of systematic and logical reasoning and develop the skill of understanding validity of statements.

1.1 Statement:

A statement is a declarative sentence which is either true or false but not both simultaneously. Statements are denoted by letters like p,q,r, ...

For example:

i) 2 is a prime number.

ii) Every rectangle is a square.

iii) The Sun rises in the West.

iv) Mumbai is the capital of Maharashtra.

Truth value of a statement:

A statement is either true or false. The truth value of a 'true' statement is denoted by T (TRUE) and that of a false statement is denoted by F (FALSE).

Example 1: Observe the following sentences.

i) The Sun rises in the East.

ii) The square of a real number is negative.

iii) Sum of two odd numbers is odd.

iv) Sum of opposite angles in a cyclic rectangle is 180°.

Here, the truth value of statements (i) and (iv) is T, and that of (ii) and (iii) is F.

Note: The sentences like exclamatory, interrogative, imperative are not considered as statements.

Example 2: Observe the following sentences.

i) May God bless you!

ii) Why are you so unhappy?

iii) Remember me when we are parted.

iv) Don't ever touch my phone.

v) I hate you!

vi) Where do you want to go today?

The above sentences cannot be assigned truth values, so none of them is a statement.

The sentences (i) and (v) are exclamatory.

The sentences (ii) and (vi) are interrogative.

The sentences (iii) and (iv) are imperative.

Open sentences:

An open sentence is a sentence whose truth can vary according to some conditions which are not stated in the sentence.

Example 3: Observe the following.

i) $x + 4 = 8$

ii) Chinese food is very tasty

Each of the above sentences is an open sentence, because truth of (i) depends on the value of x; if $x = 4$, it is true and if $x \neq 4$, it is false and that of (ii) varies as degree of tasty food varies from individual to individual.

Note:

i) An open sentence is not considered a statement in logic.

ii) Mathematical identities are true statements.

For example:

$a + 0 = 0 + a = a$, for any real number a.

Activity:

Determine whether the following sentences are statements in logic and write down the truth values of the statements.

Sr. No.	Sentence	Whether it is a statement or not (yes/No)	If 'No' then reason	Truth value of statement
1.	$\sqrt{-9}$ is a rational number	Yes	—	False 'F'.
2.	Can you speak in French?	No	Interrogative	—
3.	Tokyo is in Gujrat	Yes	—	False 'F'.
4.	Fantastic, let's go!	No	Exclamatory	—
5.	Please open the door quickly.	No	Imperative	—
6.	Square of an even number is even.			True 'T'
7.	$x + 5 < 14$			
8.	5 is a perfect square			
9.	West Bengal is capital of Kolkata.			
10.	$i^2 = -1$			

(**Note:** Complete the above table)

EXERCISE 1.1

State which of the following sentences are statements. Justify your answer if it is a statement. Write down its truth value.

i) A triangle has 'n' sides

ii) The sum of interior angles of a triangle is 180°

iii) You are amazing!

iv) Please grant me a loan.

v) $\sqrt{-4}$ is an irrational number.

vi) $x^2 - 6x + 8 = 0$ implies $x = -4$ or $x = -2$.

vii) He is an actor.

viii) Did you have lunch?

ix) Have a cup of tea.

x) $(x + y)^2 = x^2 + 2xy + y^2$ for all $x, y \in R$.

xi) Every real number is a complex number.

xii) 1 is a prime number.

xiii) With the sunset the day ends.

xiv) $1 ! = 0$

xv) $3 + 5 > 11$

xvi) The number Π is an irrational number.

xvii) $x^2 - y^2 = (x + y)(x - y)$ for all $x, y \in R$.

xviii) The number 2 is the only even prime number.

xix) Two co-planar lines are either parallel or intersecting.

xx) The number of arrangements of 7 girls in a row for a photograph is 7!.

xxi) Give me a compass box.

xxii) Bring the motor car here.

xxiii) It may rain today.

xxiv) If $a + b < 7$, where $a \geq 0$ and $b \geq 0$ then $a < 7$ and $b < 7$.

xxv) Can you speak in English?

1.2 Logical connectives:

A logical connective is also called a logical operator, sentential connective or sentential operator. It is a symbol or word used to connect two or more sentences in a grammatically valid way.

Observe the following sentences.

i) Monsoon is very good this year and the rivers are rising.

ii) Sneha is fat or unhappy.

iii) If it rains heavily, then the school will be closed.

iv) A triangle is equilateral if and only if it is equiangular.

The words or group of words such as “and”, “or”, “if ... then”, “If and only if”, can be used to join or connect two or more simple sentences. These connecting words are called logical connectives.

Note: ‘not’ is a logical operator for a single statement. It changes the truth value from T to F and F to T.

Compound Statement:

A compound statement is a statement which is formed by combining two or more simple statements with the help of logical connectives.

The above four sentences are compound sentences.

Note:

i) Each of the statements that comprise a compound statement is called a sub-statement or a component statement.

ii) Truth value of a compound statement depends on the truth values of the sub-statements i.e. constituent simple statements and connectives used. Every simple statement has its truth value either ‘T’ or ‘F’. Thus, while determining the truth value of a compound statement, we have to consider all possible combinations of truth values of the simple statements and connectives. This can be easily expressed with the help of a truth table.

Table 1.1: Logical connectives

Sr. No.	Connective	Symbol	Name of corresponding compound statement
1.	and	$\wedge$	conjunction
2.	or	$\vee$	disjunction
3.	not	$\sim$	Negation
4.	If ... then	$\rightarrow$ (or $\Rightarrow$)	conditional or implication
5.	If and only if or iff	$\Leftrightarrow$ (or $\Leftrightarrow$)	Biconditional or double implication

A) Conjuction ($\wedge$):

If p and q are any two statements connected by the word "and", then the resulting compound statement "p and q" is called the conjunction of p and q, which is written in the symbolic form as '$p \wedge q$'.

For example:

Let $p : \sqrt{2}$ is a rational number

$q : 4 + 3i$ is a complex number

The conjunction of the above two statements is $p \wedge q$ i.e. $\sqrt{2}$ is a rational number and $4 + 3i$ is a complex number.

Consider the following simple statements:

i) $5 > 3$; Nagpur is in Vidarbha.

$p : 5 > 3$

q : Nagpur is in Vidarbha

The conjunction is

$p \wedge q : 5 > 3$ and Nagpur is in Vidarbha.

ii) $p : a+bi$ is irrational number for all $a, b \in R$; $q : 0\,! = 1$

The conjuction is

$p \wedge q : a + bi$ is irrational number,

for all $a, b \in R$ and $0\,! = 1$

Truth table of conjunction ($p \wedge q$)

Table 1.2

p	q	p $\wedge$ q
T	T	T
T	F	F
F	T	F
F	F	F

From the last column, the truth values of above four combinations can be decided.

Remark:

i) Conjunction is true if both sub-statements are true. Otherwise it is false.

ii) Other English words such as "but", "yet", "though", "still", "moreover" are also used to join two simple statements instead of "and".

iii) Conjunction of two statements corresponds to the "intersection of two sets" in set theory.

SOLVED EXAMPLES

Ex. 1: Write the following statements in symbolic form.

i) An angle is a right angle and its measure is 90°.

ii) Jupiter is a planet and Mars is a star.

iii) Every square is a rectangle and $3 + 5 < 2$.

Solution:

i) Let p : An angle is right angle.

q : Its measure is 90°.

Then, $p \wedge q$ is the symbolic form.

ii) Let p : Jupiter is plannet

q : Mars is a star.

Then, $p \wedge q$ is the symbolic form.

iii) Let p : ....

q : ....

Then, ...

Ex. 2: Write the truth value of each of the following statements.

i) Patna is capital of Bihar and $5i$ is an imaginary number.

ii) Patna is capital of Bihar and $5i$ is not an imaginary number.

iii) Patna is not capital of Bihar and $5i$ is an imaginary number.

iv) Patna is not capital of Bihar and $5i$ is not an imaginary number.

Solution: Let p : Patna is capital of Bihar

$q : 5i$ is an imaginary number

p is true; q is true.

i) True (T), since both the sub-statements are true i.e. both Patna is capital of Bihar and $5i$ is an imaginary number are true.

(As $T \wedge T = T$)

ii) False (F), since first sub-statement "Patna is capital of Bihar" is true and second sub-statement $5i$ is not an imaginary number is False. (As $T \wedge F = F$)

iii) False (F), since first sub-statement "Patna is not capital of Bihar" is False and second sub-statement $5i$ is an imaginary number is True. (As $F \wedge T = F$)

iv) False (F), since both sub-statement "Patna is not capital of Bihar" and "$5i$ is not an imaginary number" are False.

(As $F \wedge F = F$)

B) Disjunction ($\vee$):

If p and q are two simple statements connected by the word 'or' then the resulting compound statement 'p or q' is called the disjunction of p and q, which is written in the symbolic form as '$p \vee q$'.

Note: The word 'or' is used in English language in two distinct senses, one is exclusive and the other is inclusive.

For example: Consider the following statements.

i) Throwing a coin will get a head or a tail.

ii) The amount should be paid by cheque or by demand draft.

In the above statements 'or' is used in the sense that only one of the two possibilities exists, but not both. Hence it is called exclusive sense of 'or'.

Also consider the statements:

i) Graduate or employee persons are eligible to apply for this post.

ii) The child should be accompanied by father or mother.

In the above statements 'or' is used in the sense that first or second or both possibilities exist. Hence it is called inclusive sense of 'or'. In mathematics 'or' is used in the inclusive sense. Thus p or q ($p \vee q$) means p or q or both p and q.

Example: Consider the followinig simple statements.

i) $3 > 2$; $2 + 3 = 5$ $p : 3 > 2$

$q : 2 + 3 = 5$

The disjunction is $p \vee q : 3 > 2$ or $2 + 3 = 5$

ii) New York is in U.S.; $6 > 8$

p : New York is in U.S.

$q : 6 > 8$

The disjunction is $p \vee q$: New York is in U.S. or $6 > 8$.

Truth table of disjunction ($p \vee q$)

Table 1.3

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Note:

i) The disjunction is false if both sub-statements are false. Otherwise it is true.

ii) Disjunction of two statements is equivalent to 'union of two sets' in set theory.

SOLVED EXAMPLES

Ex. 1: Express the following statements in the symbolic form.

i) Rohit is smart or he is healthy.

ii) Four or five students did not attend the lectures.

Solution:

i) Let p : Rohit is smart

q : Rohit is healthy

Then, $p \vee q$ is symbolic form.

ii) In this sentence 'or' is used for indicating approximate number of students and not as a connective. Therefore, it is a simple statement and it is expressed as

p : Four or five students did not attend the lectures.

Ex. 2: Write the truth values of the following statements.

i) India is a democratic country or China is a communist country.

ii) India is a democratic country or China is not a communist country.

iii) India is not a democratic country or China is a communist country.

iv) India is not a democratic country or China is not a communist country.

Solution: p : India is a democratic country.

q : China is a communist country.

p is true; q is true.

i) True (T), since both the sub-statements are true i.e. both "India is a democratic country" and "China is a communist country" are true. (As $T \vee T = T$)

ii) True (T), since first sub-statements "India is a democratic country" is true and second sub-statement "China is not a communist country" is false. (As $T \vee F = T$)

iii) True (T), since first sub-statements "India is not a democratic country" is false and second sub-statement "China is a communist country" is true. (As $F \vee T = T$)

iv) False (F), since both the sub-statements "India is not a democratic country" and "China is not a communist country" are false. (As $F \vee F = F$)

EXERCISE 1.2

Ex. 1: Express the following statements in symbolic form.

i) e is a vowel or $2 + 3 = 5$

ii) Mango is a fruit but potato is a vegetable.

iii) Milk is white or grass is green.

iv) I like playing but not singing.

v) Even though it is cloudy, it is still raining.

Ex. 2: Write the truth values of following statements.

i) Earth is a planet and Moon is a star.

ii) 16 is an even number and 8 is a perfect square.

iii) A quadratic equation has two distinct roots or 6 has three prime factors.

iv) The Himalayas are the highest mountains but they are part of India in the North East.

C) Negation (~):

The denial of an assertion contained in a statement is called its negation.

The negation of a statement is generally formed by inserting the word "not" at some proper place in the statement or by prefixing the statement with "it is not the case that" or "it is false that" or "it is not true that".

The negation of a statement p is written as $\sim p$ (read as "negation p" or "not p") in symbolic form.

For example:

Let p : 2 is an even number

$\sim p$: 2 is not an even number.

or $\sim p$: It is not the case that 2 is an even number

or $\sim p$: It is false that 2 is an even number

The truth table of negation ($\sim p$)

Table 1.4

p	$\sim$ p
T	F
F	T

Note: Negation of a statement is equivalent to the complement of a set in set theory.

SOLVED EXAMPLES

Ex. 1: Write the negation of the following statements.

i) p : He is honest.

ii) q : π is an irrational number.

Solution:

i) $\sim p$: He is not honest

or $\sim p$: It is not the case that he is honest

or $\sim p$: It is false that he is honest.

ii) $\sim q$: π is not an irrational number.

or $\sim q$: $\boxed{}$

or $\sim q$: $\boxed{}$

EXERCISE 1.3

1. Write the negation of each of the following statements.

   i) All men are animals.

   ii) – 3 is a natural number.

   iii) It is false that Nagpur is capital of Maharashtra

   iv) $2 + 3 \neq 5$

2. Write the truth value of the negation of each of the following statements.

   i) $\sqrt{5}$ is an irrational number

   ii) London is in England

   iii) For every $x \in N$, $x + 3 < 8$.

D) Conditional statement (Implication, $\rightarrow$)

If two simple statements p and q are connected by the group of words "If ... then ...", then the resulting compound statement "If p then q" is called a conditional statement (implication) and is written in symbolic form as "$p \rightarrow q$" (read as "p implies q").

For example:

i) Let p : There is rain

q : The match will be cancelled

then, $p \rightarrow q$: If there is rain then the match will be cancelled.

ii) Let p : r is a rational number.

q : r is a real number.

then, $p \rightarrow q$: If r is a rational number then r is a real number.

The truth table for conditional statement ($p \rightarrow q$)

Table 1.5

p	q	p $\rightarrow$ q
T	T	T
T	F	F
F	T	T
F	F	T

SOLVED EXAMPLES

Ex. 1: Express the following statements in the symbolic form.

i) If the train reaches on time, then I can catch the connecting flight.

ii) If price increases then demand falls.

Solution:

i) Let p : The train reaches on time

q : I can catch the connecting flight.

Therefore, $p \rightarrow q$ is symbolic form.

ii) Let p : price increases

q : demand falls

Therefore, $p \rightarrow q$ is symbolic form.

Ex. 2: Write the truth value of each of the following statements.

i) If Rome is in Italy then Paris is in France.

ii) If Rome is in Italy then Paris is not in France.

iii) If Rome is not in Italy then Paris is in France.

iv) If Rome is not in Italy then Paris not in France.

Solution:

p : Rome is in Italy

q : Paris is in France

p is true ; q is true.

i) True (T), since both the sub-statements are true. i.e. Rome is in Italy and Paris is in France are true. (As $T \rightarrow T = T$)

ii) False (F), since first sub-statement Rome is in Italy is true and second sub-statement Paris in not in France is false.

(As $T \rightarrow F = F$)

iii) True (T), since first sub-statement Rome is not in Italy is false and second sub-statement Paris is in France is true. (As $F \rightarrow T = T$)

iv) True (T), since both the sub-statements are false. i.e. Rome is not in Italy and Paris is not in France both are false. (As $F \rightarrow F = T$)

E) Biconditional (Double implication) ($\leftrightarrow$) or ($\Leftrightarrow$):

If two statements p and q are connected by the group of words "If and only if" or "iff", then the resulting compound statement "p if and only if q" is called biconditional of p and q, is written in symbolic form as $p \leftrightarrow q$ and read as "p if and only if q".

For example:

i) Let p : Milk is white

q : the sky is blue

Therefore, $p \leftrightarrow q$: Milk is white if and only if the sky is blue.

ii) Let $p : 3 < 5$

$q : 4\sqrt{2}$ is an irrational number.

Therefore, $p \leftrightarrow q : 3 < 5$ if and only if $4\sqrt{2}$ is an irrational number.

Truth table for biconditional ($p \leftrightarrow q$)

Table 1.6

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

SOLVED EXAMPLES

Ex. 1: Translate the following statements (verbal form) to symbolic form.

i) Price increases if and only if demand falls.

ii) $5 + 4 = 9$ if and only if $3 + 2 = 7$

Solution:

i) Let p : Price increases

q : demand falls

Therefore, $p \leftrightarrow q$ is the symbolic form.

ii) Let $p : 5 + 4 = 9$

$q : 3 + 2 = 7$

Therefore, $p \leftrightarrow q$ is the symbolic form.

Ex. 2: Write the truth value of each of the following statements.

i) The Sun rises in the East if and only if $4 + 3 = 7$

ii) The Sun rises in the East if and only if $4 + 3 = 10$

iii) The Sun rises in the West if and only if $4 + 3 = 7$

iv) The Sun rises in the West if and only if $4 + 3 = 10$

Solution:

p : The Sun rises in the East;

$q : 4 + 3 = 7$;

p is true, q is true.

The truth value of each statement is given by

i) True (T), since both the sub-statements (i.e. "The Sun rises in the East" and "$4 + 3 = 7$") are true. (As $T \leftrightarrow T = T$)

ii) False (F), since both the sub-statements have opposite truth values (i.e. "The Sun rises in the East" is true but "$4 + 3 = 10$" is false.). (As $T \leftrightarrow F = F$)

iii) False (F), since both the sub-statements have opposite truth values (i.e. "The Sun rises in the East" is false but "$4 + 3 = 7$" is true.). (As $F \leftrightarrow T = F$)

iv) True (T), since both the sub-statements have same truth values (i.e. they are false.) (As $F \leftrightarrow F = T$)

Therefore, $p \leftrightarrow q$ is the symbolic form.

Note:

i) The biconditional statement $p \leftrightarrow q$ is the compound statement "$p \rightarrow q$" and "$q \rightarrow p$" of two compound statements.

ii) $p \leftrightarrow q$ can also be read as –

a) q if and only if p.

b) p is necessary and sufficient for q.

c) q is necessary and sufficient for p.

d) p implies q and q implies p.

e) p implies and is implied by q.

Ex. 1: Express the following in symbolic form using logical connectives.

i) If a quadrilateral is a square then it is not a rhombus.

ii) It is false that Nagpur is capital of India iff $3 + 2 = 4$

iii) ABCD is a parallelogram but it is not a quadrilateral.

iv) It is false that $3^2 + 4^2 = 5^2$ or $\sqrt{2}$ is not a rational number but $3^2 + 4^2 = 5^2$ and $8 > 3$.

Solution:

i) Let p : quadrilateral is a square

q : quadrilateral is a rhombus.

Then, $p \rightarrow \sim q$ is symbolic form.

ii) Let p : Nagpur is capital of India

$q : 3 + 2 = 4$

Then, $\sim p \leftrightarrow q$ is the symbolic form.

iii) Let p : ABCD is a parallelogram

q : ABCD is a quadrilateral

Then, $p \wedge \sim q$ is the symbolic form.

iv) Let $p : 3^2 + 4^2 = 5^2$

$q : \sqrt{2}$ is a rational number

$r : 8 > 3$.

Therefore, $(\sim p \vee \sim q) \wedge (p \wedge r)$ is the symbolic form of the required statement.

Ex. 2: Express the following statements in symbolic form and write their truth values.

i) It is not true that $\sqrt{2}$ is a rational number.

ii) 4 is an odd number iff 3 is not a prime factor of 6.

iii) It is not true that i is a real number.

Solution:

i) Let $p : \sqrt{2}$ is a rational number.

Then $\sim p$ is the symbolic form.

Given statement, p is false F.

$\therefore \sim p \equiv T$

$\therefore$ the truth value of given statement is T.

ii) Let p : 4 is an odd number.

q : 3 is a prime factor of 6.

$\sim q$: 3 is not a prime factor of 6.

Therefore, $p \leftrightarrow (\sim q)$ is the symbolic form.

Given statement, p is false F.

q is true T.

$\therefore \sim q$ is false F.

$\therefore p \leftrightarrow (\sim q) \equiv F \leftrightarrow F \equiv T$

$\therefore$ the truth value of given statement is T.

iii) Let p : i is a real number.

$\therefore \sim p$: It is not true that i is a real number.

Therefore, $\sim p$: is the symbolic form.

p is false F.

$\therefore \sim p$: is true T.

$\therefore$ the truth value of the given statement is T.

Ex. 3: If p and q are true and r and s are false, find the truth value of each of the following.

i) $(p \leftrightarrow \sim q) \wedge (r \leftrightarrow \sim s)$

ii) $(p \rightarrow r) \vee (q \rightarrow s)$

iii) $\sim [(p \wedge \sim s) \vee (q \wedge \sim r)]$

Solution:

i) Without truth table : $(p \leftrightarrow \sim q) \wedge (r \leftrightarrow \sim s)$

$= (T \leftrightarrow \sim T) \wedge (F \leftrightarrow \sim F)$

$= (T \leftrightarrow F) \wedge (F \leftrightarrow T)$

$= F \wedge F$

$= F$

Construct truth table

Table 1.7

p	q	r	s	$\sim q$	$\sim s$	$p \leftrightarrow \sim q$	$r \leftrightarrow \sim s$	$(p \leftrightarrow \sim q) \wedge (r \leftrightarrow \sim s)$
T	T	F	F	F	T	F	F	F

ii) Without truth table :

$(p \rightarrow r) \vee (q \rightarrow s) = (T \rightarrow F) \vee (T \rightarrow F)$

$= F \vee F$

$= F$

Construct truth table

Table 1.8

p	q	r	s	$p \rightarrow r$	$q \rightarrow s$	$(p \rightarrow r) \vee (q \rightarrow s)$
T	T	F	F	F	F	F

ii) Without truth table : (Activity)

$\sim [(p \wedge \sim s) \vee (q \wedge \sim r)]$

$= \sim [(\square \wedge \sim \square) \vee (\square \wedge \square)]$

$= \sim [(\square \wedge T) \vee (T \wedge T)]$

$= \sim (\square \vee \square)$

$= \sim (\square)$

$= \square$

Construct truth table

Table 1.9

(**Note:** Construct truth table and complete your solution)

EXERCISE 1.4

Ex. 1: Write the following statements in symbolic form.

i) If triangle is equilateral then it is equiangular.

ii) It is not true that "i" is a real number.

iii) Even though it is not cloudy, it is still raining.

iv) Milk is white if and only if the sky is not blue.

v) Stock prices are high if and only if stocks are rising.

vi) If Kutub-Minar is in Delhi then Taj-Mahal is in Agra.

Ex. 2: Find truth value of each of the following statements.

i) It is not true that $3 - 7i$ is a real number.

ii) If a joint venture is a temporary partnership, then discount on purchase is credited to the supplier.

iii) Every accountant is free to apply his own accounting rules if and only if machinery is an asset.

iv) Neither 27 is a prime number nor divisible by 4.

v) 3 is a prime number and an odd number.

Ex. 3: If p and q are true and r and s are false, find the truth value of each of the following compound statements.

i) $p \wedge (q \wedge r)$

ii) $(p \to q) \vee (r \wedge s)$

iii) $\sim [(\sim p \vee s) \wedge (\sim q \wedge r)]$

iv) $(p \to q) \leftrightarrow \sim (p \vee q)$

v) $[(p \vee s) \to r] \vee \sim [\sim (p \to q) \vee s]$

vi) $\sim [p \vee (r \wedge s)] \wedge \sim [(r \wedge \sim s) \wedge q]$

Ex. 4: Assuming that the following statements are true,

p : Sunday is holiday,

q : Ram does not study on holiday,

find the truth values of the following statements.

i) Sunday is not holiday or Ram studies on holiday.

ii) If Sunday is not holiday then Ram studies on holiday.

iii) Sunday is a holiday and Ram studies on holiday.

Ex. 5: If p : He swims

q : Water is warm

Give the verbal statements for the following symbolic statements.

i) $p \leftrightarrow \sim q$

ii) $\sim (p \vee q)$

iii) $q \to p$

iv) $q \wedge \sim p$

1.2.1 Quantifiers and Quantified statements:

i) For every $x \in R$, x^2 is non negative. We shall now see how to write this statement using symbols. '$\forall x$' is used to denote "For all x".

Thus, the above statement may be written in mathematical notation $\forall\ z \in R$, $z^2 \geq 0$. The symbol '$\forall$' stands for "For all values of". This is known as universal quantifier.

ii) Also we can get $x \in N$ such that $x + 4 = 7$. To write this in symbols we use the symbol $\exists\, x$ to denote "there exists x". Thus, we have $\exists\, x \in N$ such that $x + 4 = 7$.

The symbol $\exists$ stands for "there exists". This symbol is known as existential quantifier.

Thus, there are two types of quantifiers.

a) Universal quantifier ($\forall$)

b) Existential quantifier ($\exists$)

Quantified statement:

An open sentence with a quantifier becomes a statement and is called a quantified statement.

SOLVED EXAMPLES

Ex. 1: Use quantifiers to convert each of the following open sentences defined on N, into a true statement.

i) $2x + 3 = 11$

ii) $x^3 < 64$

iii) $x + 5 < 9$

Solution:

i) $\exists\, x \in N$ such that $2x + 3 = 11$. It is a true statement, since $x = 4 \in N$ satisfies $2x + 3 = 11$.

ii) $x^3 < 64$ $\exists\, x \in N$ such that it is a true statement, since $x = 1$ or 2 or $3 \in N$ satisfies $x^3 < 64$.

iii) $\exists\, x \in N$ such that $x + 5 < 9$. It is a true statement for $x = 1$ or 2 or $3 \in N$ satisfies $x + 5 < 9$.

Ex. 2: If A = {1, 3, 5, 7} determine the truth value of each of the following statements.

i) $\exists\, x \in A$, such that $x^2 < 1$.

ii) $\exists\, x \in A$, such that $x + 5 \le 10$

iii) $\forall\, x \in A$, $x + 3 < 9$

Solution:

i) No number in set A satisfies $x^2 < 1$, since the square of every natural number is 1 or greater than 1.

$\therefore$ the given statement is false, hence its truth value is F.

ii) Clearly, $x = 1$, 3 or 5 satisfies $x + 5 \le 10$. So the given statement is true, hence truth value is T.

iii) Since $x = 7 \in A$ does not satisfy $x + 3 < 9$, the given statement is false. Hence its truth value is F.

EXERCISE 1.5

Ex. 1: Use quantifiers to convert each of the following open sentences defined on N, into a true statement.

i) $x^2 + 3x - 10 = 0$

ii) $3x - 4 < 9$

iii) $n^2 \ge 1$

iv) $2n - 1 = 5$

v) $Y + 4 > 6$

vi) $3y - 2 \le 9$

Ex. 2: If B = {2, 3, 5, 6, 7} determine the truth value of each of the following.

i) $\forall\, x \in B$ such that x is prime number.

ii) $\exists\, n \in B$, such that $n + 6 > 12$

iii) $\exists\, n \in B$, such that $2n + 2 < 4$

iv) $\forall\, y \in B$ such that y^2 is negative

v) $\forall\, y \in B$ such that $(y - 5) \in N$

1.3 Statement Patterns and Logical Equivalence:

A) Statement Patterns:

Let p, q, r, ... be simple statements. A compound statement obtained from these simple statements by using one or more of the connectives $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, is called a statement pattern.

For example:

i) $(p \vee q) \rightarrow r$

ii) $p \wedge (q \wedge r)$

iii) $\sim (p \vee q)$ are statement patterns

Note: While preparing truth tables of the given statement patterns, the following points should be noted.

i) If a statement pattern involves n component statements p, q, r, ... and each of p, q, r, ... has 2 possible truth values namely T and F, then the truth table of the statement pattern consists of 2^n rows.

ii) If a statement pattern contains "m" connectives and "n" component statements then the truth table of the statement pattern consists of $(m + n)$ columns.

iii) Parentheses must be introduced whenever necessary.

For example:

$\sim (p \rightarrow q)$ and $\sim p \rightarrow q$ are not the same.

SOLVED EXAMPLES

Ex. 1: Prepare the truth table for each of the following statement patterns.

i) $[(p \rightarrow q) \vee p] \rightarrow p$

ii) $\sim [p \vee q] \rightarrow \sim (p \wedge q)$

iii) $(\sim p \vee q) \vee \sim q$

iv) $(p \wedge r) \vee (q \wedge r)$

Solution:

i) $[(p \rightarrow q) \vee p) \rightarrow p$

Truth table 1.10

p	q	$p \rightarrow q$	$(p \rightarrow q) \vee p$	$[(p \rightarrow q) \vee p] \rightarrow p$
T	T	T	T	T
T	F	F	T	T
F	T	T	T	F
F	F	T	T	F

ii) $\sim (p \vee q) \rightarrow \sim (p \wedge q)$

Truth table 1.11

p	q	$p \vee q$	$\sim (p \vee q)$	$(p \wedge q)$	$\sim (p \wedge q)$	$\sim (p \vee q) \rightarrow \sim (p \wedge q)$
T	T	T	F	T	F	T
T	F	T	F	F	T	T
F	T	T	F	F	T	T
F	F	F	T	F	T	T

iii) $(\sim p \vee q) \vee \sim q$

Truth table 1.12

p	q	$\sim p$	$\sim q$	$\sim p \vee q$	$(\sim p \vee q) \vee \sim q$
T	T	F	F	T	T
T	F	F	T	F	T
F	T	T	F	T	T
F	F	T	T	T	T

iv) $(p \wedge r) \vee (q \wedge r)$

Complete the following truth table :

Truth table 1.13

p	q	r	$p \wedge r$	$q \wedge r$	$(p \wedge r) \vee (q \wedge r)$
T	□	T	T	□	T
T	□	□	F	F	□
□	F	T	T	□	□
□	□	□	F	F	□
F	□	T	□	□	F
□	T	F	□	□	F
□	□	T	□	□	□
F	□	F	F	□	□

B) Logical Equivalence:

Two or more statement patterns are said to be logically equivalent if and only if the truth values in their respective columns in the joint truth table are identical.

If s_1 , s_2 , s_3 , ... are logically equivalent statement patterns, we write $s_1 \equiv s_2 \equiv s_3 \equiv ...$

For example: Using a truth table, verify that

i) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

ii) $\sim (p \vee q) \equiv \sim p \wedge \sim q$

iii) $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

iv) $\sim r \rightarrow \sim (p \wedge q) \equiv \sim (q \rightarrow r) \rightarrow \sim p$

Solution:

i) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

Truth table 1.14

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$\sim p \vee \sim q$
1	2	3	4	5	6	7
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

From the truth table 1.14, we observe that all entires in 6^{th} and 7^{th} columns are identical.

$\therefore \sim (p \wedge q) \equiv \sim p \vee \sim q$

ii) $\sim (p \vee q) \equiv \sim p \wedge \sim q$

Truth table 1.15

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim (p \vee q)$	$\sim p \wedge \sim q$
1	2	3	4	5	6	7
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

From the truth table 1.15, we observe that all entires in 6^{th} and 7^{th} columns are identical.

$\therefore \sim (p \vee q) \equiv \sim p \wedge \sim q.$

iii) **Activity**

$p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

Truth table 1.16

p	q	r	$q \vee r$	$P \vee (q \vee r)$	$p \vee q$	$p \vee r$	$(p \vee q) \vee (p \vee r)$
1	2	3	4	5	6	7	8
T	T	□	T	□	T	□	□
T	T	□	T	□	□	T	□
□	F	T	□	T	□	□	T
□	F	F	□	□	T	T	T
F	□	□	T	T	□	□	□
F	□	□	T	T	T	F	T
□	□	□	T	□	F	T	T
□	□	F	□	□	□	□	F

From the truth table 1.16, we observe that all entires in 5th and 8th columns are identical.

$\therefore\ p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

iv) **Activity**

$\sim \mathrm{r} \rightarrow \sim (p \wedge q) \equiv \sim (q \rightarrow r) \rightarrow \sim p$

Prepare the truth table 1.17

p	q	r	$\sim p$	$\sim r$	$p \wedge q$	$\sim (p \wedge q)$	$\sim r \rightarrow \sim (p \wedge q)$	$q \rightarrow r$	$\sim (q \rightarrow r)$	$\sim (q \rightarrow r) \rightarrow \sim p$

C) Tautology, contradiction, contingency:

Tautology:

A statement pattern always having the truth value 'T', irrespective of the truth values of its component statements is called a tautology.

For example:

Consider $(p \wedge q) \rightarrow (p \vee q)$

Truth table 1.18

p	q	$p \wedge q$	$p \vee q$	$(p \wedge q) \to (p \vee q)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

In the above table, all the entries in the last column are T. Therefore, the given statement pattern is a tautology.

Contradiction:

A statement pattern always having the truth value 'F' irrespective of the truth values of its component statements is called a contradiction.

For example:

Consider $p \wedge \sim p$

Truth table 1.19

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

In the above truth table, all the entries in the last column are F. Therefore, the given statement pattern is a contradiction.

Contingency:

A statement pattern which is neither a tautology nor a contradiction is called a contingency.

For example:

Consider $(p \to q) \wedge (q \to p)$

Truth table 1.20

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

In the table 1.20, the entries in the last column are not all T and not all F. Therefore, the given statement pattern is a contingency.

SOLVED EXAMPLES

Ex. 1: Using the truth table, examine whether the following statement patterns are tautology, contradictions or contingency.

i) $(p \wedge q) \to p$

ii) $(\sim p \vee \sim q) \leftrightarrow \sim (p \wedge q)$

iii) $(\sim q \wedge p) \wedge q$

iv) $p \to (\sim q \vee r)$

Solution:

i) The truth table for $(p \wedge q) \to p$

Truth table 1.21

p	q	$p \wedge q$	$(p \wedge q) \to p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

In the table 1.21, all the entries in the last column are T. Therefore, the given statement pattern is a tautology.

ii) **Activity:**

Prepare the truth table for $(\sim p \vee \sim q) \leftrightarrow \sim (p \wedge q)$

Truth table 1.22

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim (p \wedge q)$	$(\sim p \vee \sim q)$	$(\sim p \vee \sim q) \leftrightarrow \sim (p \wedge q)$

iii) The truth table for $(\sim q \wedge p) \wedge q$

Truth table 1.23

p	q	$\sim q$	$\sim q \wedge p$	$(\sim q \wedge p) \wedge q$
T	T	F	F	F
T	F	T	T	F
F	T	F	F	F
F	F	T	F	F

In the table 1.23, all the entries in the last column are F.

Therefore, the given statement pattern is a contradiction.

iv) The truth table for $p \to (\sim q \vee r)$

Truth table 1.24

p	q	r	$\sim q$	$\sim q \vee r$	$p \to (\sim q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

In the table 1.24, the entries in the last column are neither all T nor all F.

Therefore, the given statement pattern is a contingency.

EXERCISE 1.6

1. Prepare truth tables for the following statement patterns.

   i) $p \to (\sim p \vee q)$

   ii) $(\sim p \vee q) \wedge (\sim p \vee \sim q)$

   iii) $(p \wedge r) \to (p \vee \sim q)$

   iv) $(p \wedge q) \vee \sim r$

2. Examine whether each of the following statement patterns is a tautology, a contradiction or a contingency

   i) $q \vee [\sim (p \wedge q)]$

   ii) $(\sim q \wedge p) \wedge (p \wedge \sim p)$

   iii) $(p \wedge \sim q) \to (\sim p \wedge \sim q)$

   iv) $\sim p \to (p \to \sim q)$

3. Prove that each of the following statement pattern is a tautology.

   i) $(p \wedge q) \to q$

   ii) $(p \to q) \leftrightarrow (\sim q \to \sim p)$

   iii) $(\sim p \wedge \sim q) \to (p \to q)$

   iv) $(\sim p \vee \sim q) \leftrightarrow \sim (p \wedge q)$

4. Prove that each of the following statement pattern is a contradiction.

   i) $(p \vee q) \wedge (\sim p \wedge \sim q)$

   ii) $(p \wedge q) \wedge \sim p$

   iii) $(p \wedge q) \wedge (\sim p \vee \sim q)$

   iv) $(p \to q) \wedge (p \wedge \sim q)$

5. Show that each of the following statement pattern is a contingency.

   i) $(p \wedge \sim q) \to (\sim p \wedge \sim q)$

   ii) $(p \to q) \leftrightarrow (\sim p \vee q)$

   iii) $p \wedge [(p \to \sim q) \to q]$

   iv) $(p \to q) \wedge (p \to r)$

6. Using the truth table, verify

   i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

   ii) $p \to (p \to q) \equiv \sim q \to (p \to q)$

   iii) $\sim (p \to \sim q) \equiv p \wedge \sim (\sim q) \equiv p \wedge q$

   iv) $\sim (p \vee q) \vee (\sim p \wedge q) \equiv \sim p$

7. Prove that the following pairs of statement patterns are equivalent.

   i) $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$

   ii) $p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$

   iii) $p \to q$ and $\sim q \to \sim p$ and $\sim p \vee q$

   iv) $\sim (p \wedge q)$ and $\sim p \vee \sim q$.

D) Duality:

Two compound statements s_1 and s_2 are said to be duals of each other, if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$, and c by t and t by c, where t denotes tautology and c denotes contradiction.

Note:

i) Dual of a statement is unique.

ii) The symbol ~ is not changed while finding the dual.

iii) Dual of the dual is the original statement itself.

iv) The connectives $\wedge$ and $\vee$, the special statements t and c are duals of each other.

v) T is changed to F and vice-versa.

For example:

i) Consider the distributive laws,

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ ... (1)

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ ... (2)

Observe that (2) can be obtained from (1) by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$ i.e. interchanging $\wedge$ and $\vee$.

Hence (1) is the dual of (2).

Similarly, (1) can be obtained from (2) by replacing $\vee$ by $\wedge$ and $\wedge$ by $\vee$. Hence, (2) is the dual of (1).

Therefore, statements (1) and (2) are called duals of each other.

ii) Consider De-Morgan's laws :

$\sim (p \wedge q) \equiv \sim p \vee \sim q$ ... (1)

$\sim (p \vee q) \equiv \sim p \wedge \sim q$ ... (2)

Statements, (1) and (2) are duals of each other.

SOLVED EXAMPLES

Ex. 1: Write the duals of the following statements:

i) $\sim (p \wedge q) \vee (\sim q \wedge \sim p)$

ii) $(p \vee q) \wedge (r \vee s)$

iii) $[(p \wedge q) \vee r] \wedge [(q \wedge r) \vee s]$

Solution: The duals are given by

i) $\sim (p \vee q) \wedge (\sim q \vee \sim p)$

ii) $(p \wedge q) \vee (r \wedge s)$

iii) $[(p \vee q) \wedge r] \vee [(q \vee r) \wedge s]$

Ex. 2: Write the duals of the following statements:

i) All natural numbers are integers or rational numbers.

ii) Some roses are red and all lillies are white.

Solution: The duals are given by

i) All natural numbers are integers and rational numbers.

ii) Some roses are red or all lillies are white.

EXERCISE 1.7

1. Write the dual of each of the following :

   i) $(p \vee q) \vee r$

   ii) $\sim (p \vee q) \wedge [p \vee \sim (q \wedge \sim r)]$

   iii) $p \vee (q \vee r) \equiv (p \vee q) \vee r$

   iv) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

2. Write the dual statement of each of the following compound statements.

   i) 13 is prime number and India is a democratic country.

   ii) Karina is very good or every body likes her.

   iii) Radha and Sushmita can not read Urdu.

   iv) A number is real number and the square of the number is non negative.

E) Negation of a compound statement:

We have studied the negation of simple statements. Negation of a simple statement is obtained by inserting "not" at the appropriate place in the statement e.g. the negation of "Ram is tall" is "Ram is not tall". But writing negations of compound statements involving conjunction., disjunction, conditional, biconditional etc. is not straight forward.

1) Negation of conjunction:

In section 1.3(B) we have seen that $\sim (p \wedge q) \equiv \sim p \vee \sim q$. It means that negation of the conjunction of two simple statements is the disjunction of their negation.

Consider the following conjunction.

"Parth plays cricket and chess."

Let p : Parth plays cricket.

q : Parth plays chess.

Given statement is $p \wedge q$.

You know that $\sim (p \wedge q) \equiv \sim p \vee \sim q$

$\therefore$ negation is Parth doesn't play cricket or he doesn't play chess.

2) Negation of disjunction:

In section 1.3(B) we have seen that $\sim (p \vee q) \equiv \sim p \wedge \sim q$. It means that negation of the disjunction of two simple statements is the conjunction of their negation.

For ex: The number 2 is an even number or the number 2 is a prime number.

Let p : The number 2 is an even number.

q : The number 2 is a prime number.

$\therefore$ given statement : $p \vee q$.

You know that $\sim (p \vee q) \equiv \sim p \wedge \sim q$

$\therefore$ negation is "The number 2 is not an even number and the number 2 is not a prime number".

3) Negation of negation:

Let p be a simple statement.

Truth table 1.25

p	$\sim p$	$\sim (\sim p)$
T	F	T
F	T	F

From the truth table 1.25, we see that

$\sim (\sim p) \equiv p$

Thus, the negation of negation of a statement is the original statement $\sim (\sim p) \equiv p$.

For example:

Let p : $\sqrt{5}$ is an irrational number.

The negation of p is given by

$\sim p$: $\sqrt{5}$ is not an irrational number.

$\sim (\sim p)$: $\sqrt{5}$ is an irrational number.

Therefore, negation of negation of p is $\sim (\sim p)$ i.e. it is not the case that $\sqrt{5}$ is not an irrational number.

OR it is false that $\sqrt{5}$ is not an irrational number.

OR $\sqrt{5}$ is an irrational number.

4) Negation of Conditional (Implication):

You know that $p \rightarrow q \equiv \sim p \vee q$

$\therefore \sim (p \rightarrow q) \equiv \sim (\sim p \vee q)$

$\equiv \sim (\sim p) \wedge \sim q$ ... by De-Morgan's law

$\therefore \sim (p \rightarrow q) \equiv p \wedge \sim q$

We can also prove this result by truth table.

Truth table 1.26

p	q	$p \rightarrow q$	$\sim (p \rightarrow q)$	$\sim q$	$p \wedge \sim q$
1	2	3	4	5	6
T	T	T	F	F	F
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	F	F	F

All the entries in the columns 4 and 6 of table 1.26 are identical.

$\therefore \sim (p \rightarrow q) \equiv p \wedge \sim q$

e.g. If every planet moves around the Sun then every Moon of the planet moves around the Sun.

Negation of the given statement is, Every planet moves around the Sun but (and) every Moon of the planet does not move around the Sun.

5) Negation of Biconditional (Double implication):

Consider the biconditional $p \leftrightarrow q$.

Method 1:

We have seen that

$(p \leftrightarrow q) \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$\therefore \sim (p \leftrightarrow q) \equiv \sim [(p \rightarrow q) \wedge (q \rightarrow p)]$

$\equiv \sim (p \rightarrow q) \vee \sim (q \rightarrow p)$

... by De-Morgans law

$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

... by negation of the conditional statement

$\therefore \sim (p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

Method 2:

We also prove this by using truth table 1.27.

Truth Table 1.27

p	q	$p \leftrightarrow q$	$\sim (p \leftrightarrow \sim q)$	$\sim p$	$\sim q$	$p \wedge \sim q$	$q \wedge \sim p$	$(p \wedge \sim q) \vee (q \wedge \sim p)$
1	2	3	4	5	6	7	8	9
T	T	T	F	F	F	F	F	F
T	F	F	T	F	T	T	F	T
F	T	F	T	T	F	F	T	T
F	F	T	F	T	T	F	F	F

Since all the entries in the columns 4 and 9 of truth table 1.27 are identical.

$\therefore \sim (p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$.

For example: $2n$ is divisible by 4 if and only if n is an even integer.

Let p : $2n$ is divisible by 4

q : n is an even integer.

Therefore, negation of the given statement is "$2n$ is divisible by 4 and n is not an even integer or n is an even integer and $2n$ is not divisible by 4".

Note:

Negation of a statement pattern involving one or more of the simple statements $p, q, r, \ldots$ and one or more of the three connectives $\wedge, \vee, \sim$ can be obtained by replacing $\wedge$ by $\vee$, $\vee$ by $\wedge$ and replaicng p, q, r ....by $\sim p, \sim q, \sim r$. ....

For example: Consider the statement pattern $(\sim p \wedge q) \vee (p \vee \sim q)$. Its negation is given by :

i.e. $\sim [(\sim p \wedge q) \vee (p \vee \sim q)]$

$\equiv (p \vee \sim q) \wedge (\sim p \wedge q)$

6) Negation of a quantified statement:

While forming negation of a quantified statement, we replace the word 'all' by 'some', "for every" by "there exists" and vice versa.

SOLVED EXAMPLES

Ex. 1: Write negation of each of the following statements :

i) All girls are sincere

ii) If India is playing in world cup and Rohit is the captain, then we are sure to win.

iii) Some bureaucrats are efficient.

Solution:

i) The negation is,

Some girls are not sincere

OR, There exists a girl, who is not sincere.

ii) Let p : India is playing world cup

q : Rohit is the captain

r : We win.

The given compound statement is

$(p \wedge q) \rightarrow r$

Therefore, the negation is,

$\sim [(p \wedge q) \rightarrow r] \equiv (p \wedge q) \wedge \sim r$

India is playing world cup and Rohit is the captain and we are not sure to win.

iii) The negation is, all bureaucrats are not efficient.

Converse, Inverse and contrapositive:

Let p and q be simple statements and let $p \rightarrow q$ be the implication of p and q.

Then, i) The converse of $p \rightarrow q$ is $q \rightarrow p$.

ii) Inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.

iii) Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

For example: Write the converse, inverse and contrapositive of the following compound statements.

i) If a man is rich then he is happy.

ii) If the train reaches on time then I can catch the connecting flight.

Solution:

i) Let p : A man is rich.

q : He is happy.

Therefore, the symbolic form of the given statement is $p \rightarrow q$.

Converse: $q \rightarrow p$
i.e. If a man is happy then he is rich.

Inverse: $\sim p \rightarrow \sim q$
i.e. If a man is not rich then he is not happy.

Contrapositive: $\sim q \rightarrow \sim p$ i.e. If a man is not happy then he is not rich.

ii) Let p : The train reaches on time.

q : I can catch the connecting flight.

Therefore, the symbolic form of the given statement is $p \rightarrow q$.

Converse, $q \rightarrow p$ i.e. $\boxed{}$

Inverse $\boxed{}$ i.e. $\boxed{}$

Contrapositive $\boxed{}$ i.e. $\boxed{}$

Ex. 3: Using the rules of negation, write the negation of the following :

i) $(\sim p \wedge r) \vee (p \vee \sim r)$

ii) $(p \vee \sim r) \wedge \sim q$

iii) The crop will be destroyed if there is a flood.

Solution:

i) The negation of $(\sim p \wedge r) \vee (p \vee \sim r)$ is $\sim [(\sim p \wedge r) \vee (p \vee \sim r)]$

$\equiv \sim (\sim p \wedge r) \wedge \sim (p \vee \sim r)$

... by De-Morgan's law

$\equiv (p \vee \sim r) \wedge (\sim p \wedge r)$

... by De-Morgan's law and $\sim (\sim p) \equiv p$ and $\sim (\sim r) = r$.

ii) The negation of $(p \vee \sim r) \wedge \sim q$ is

$\sim [(p \vee \sim r) \wedge \sim q]$

$\equiv \sim (p \vee \sim r) \vee \sim (\sim q)$

... by De Morgan's law

$\equiv (\sim p \wedge r) \vee q$

... by De Morgan's law and $\sim (\sim q) \equiv q$.

iii) Let p : The crop will be destroyed.

q : There is a flood.

Therefore, the given statement is $q \to p$ and its negation is $\sim(q \to p) \equiv q \wedge \sim p$

i.e. the crop will not be destroyed and there is a flood.

EXERCISE 1.8

1. Write negation of each of the following statements.
   i) All the stars are shining if it is night.
   ii) $\forall$ n $\in$ N, $n + 1 > 0$
   iii) $\exists$ n $\in$ N, $(n^2 + 2)$ is odd number
   iv) Some continuous functions are differentiable.

2. Using the rules of negation, write the negations of the following :
   i) $(p \to r) \wedge q$
   ii) $\sim(p \vee q) \to r$
   iii) $(\sim p \wedge q) \wedge (\sim q \vee \sim r)$

3. Write the converse, inverse and contrapositive of the following statements.
   i) If it snows, then they do not drive the car.
   ii) If he studies, then he will go to college.

4. With proper justification, state the negation of each of the following.
   i) $(p \to q) \vee (p \to r)$
   ii) $(p \leftrightarrow q) \vee (\sim q \to \sim r)$
   iii) $(p \to q) \wedge r$

1.4 Algebra of statements:

The statement patterns, under the relation of logical equivalence, satisfy various laws. We have already proved a majority of them and the rest are obvious. Now, we list these laws for ready reference.

1.	$p \vee p \equiv p$	Idempotent laws	$p \wedge p \equiv p$
2.	$p \vee (q \vee r)$ $\equiv (p \vee q) \vee r$ $\equiv p \vee q \vee r$	Associative laws	$p \wedge (q \wedge r)$ $\equiv (p \wedge q) \wedge r$ $\equiv p \wedge q \wedge r$
3.	$(p \vee q) \equiv q \vee p$	Commutative laws	$p \wedge q \equiv q \wedge p$
4.	$p \vee (q \wedge r)$ $\equiv (p \vee q) \wedge (p \vee r)$	Distributive laws	$p \wedge (q \vee r)$ $\equiv (p \wedge q) \vee (p \wedge r)$
5.	$p \vee c \equiv p$ $p \vee t \equiv t$	Identity laws	$p \wedge c \equiv c$ $p \wedge t \equiv p$
6.	$p \vee \sim p \equiv t$ $\sim t \equiv c$	Complement laws	$p \wedge \sim p \equiv c$ $\sim c \equiv t$
7.	$\sim(\sim p) \equiv p$	Involution law (law of double negation)	
8.	$\sim(p \vee q)$ $\equiv \sim p \wedge \sim q$	DeMorgan's laws	$\sim(p \wedge q)$ $\equiv \sim p \vee \sim q$
9.	$p \to q$ $\equiv \sim q \to \sim p$	Contrapositive law	

Note: In case of three simple statements p,q,r, we note the following :

i) $p \wedge q \wedge r$ is true if and only if p, q, r are all true and $p \wedge q \wedge r$ is false even if any one of p, q, r is false.

ii) $p \vee q \vee r$ is false if and only if p, q, r are all false, otherwise it is true.

SOLVED EXAMPLES

Ex. 1: Without using truth table, show that

i) $p \vee (q \wedge \sim q) \equiv p$

ii) $\sim (p \vee q) \vee (\sim p \wedge q) \equiv \sim p$

iii) $p \vee (\sim p \wedge q) \equiv p \vee q$

Solution:

i) $p \vee (q \wedge \sim q)$

$\equiv p \vee c$ ... by complement law

$\equiv p$ ... by Identity law

ii) $\sim (p \vee q) \vee (\sim p \wedge q)$

$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$

... by De Morgans law

$\equiv \sim p \wedge (\sim q \vee q)$

... by Distributive law

$\equiv \sim p \wedge t$

... by Complement law

$\equiv \sim p$ ... by Identity law

iii) $p \vee (\sim p \wedge q)$

$\equiv (p \vee \sim p) \wedge (p \vee q)$

... by Distributive law

$\equiv t \wedge (p \vee q)$ ... by Complement law

$\equiv p \vee q$ ... by Identity law

Ex. 2: Without using truth table, prove that $[(p \vee q) \wedge \sim p] \rightarrow q$ is a tautology.

Solution:

$[(p \vee q) \wedge \sim p] \rightarrow q$

$\equiv [(p \wedge \sim p) \vee (q \wedge \sim p)] \rightarrow q$

... by Distributive law

$\equiv [(c \vee (q \wedge \sim p)] \rightarrow q$

... by Complement law

$\equiv (\sim p \wedge q) \rightarrow q$ ... by Commutative law

$\equiv \sim (\sim p \wedge q) \vee q$

... by $\sim \sim (p \rightarrow q) \equiv \sim (p \wedge \sim q) \equiv \sim p \vee q$

$\equiv [(p \vee \sim q) \vee q$ ... by De Morgan's law

$\equiv p \vee (\sim q \vee q)]$ ... by Associative law

$\equiv p \vee t$

$\equiv t$

EXERCISE 1.9

1. Without using truth table, show that

i) $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

ii) $p \wedge [(\sim p \vee q) \vee \sim q] \equiv p$

iii) $\sim [(p \wedge q) \rightarrow \sim q] \equiv p \wedge q$

iv) $\sim r \rightarrow \sim (p \wedge q) \equiv [\sim (q \rightarrow r)] \rightarrow \sim p$

v) $(p \vee q) \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

2. Using the algebra of statement, prove that

i) $[p \wedge (q \vee r)] \vee [\sim r \wedge \sim q \wedge p] \equiv p$

ii) $(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q) \equiv p \vee \sim q)$

iii) $(p \vee q) \wedge (\sim p \vee \sim q) \equiv (p \vee \sim q) \wedge (\sim p \vee q)$

1.5 Venn Diagrams:

We have already studied Venn Diagrams while studying set theory. Now we try to investigate the similarity between rules of logical connectives and those of various operations on sets.

The rules of logic and rules of set theory go hand in hand.

i) Disjunction in logic is equivalent to the union of sets in set theory.

ii) Conjunction in logic is equivalent to the intersection of sets in set theory.

iii) Negation of a statement in logic is equivalent to the complement of a set in set theory.

iv) Implication of two statements in logic is equivalent to ‘subset’ in set theory.

v) Biconditional of two statements in logic is equivalent to “equality of two sets” in set theory.

Main object of this discussion is actually to give analogy between algebra of statements in logic and operations on sets in set theory.

Let A and B be two nonempty sets

i) The union of A and B is defined as

$A \cup B = \{x \,/\, x \in A \text{ or } x \in B\}$

ii) The intersection of A and B is defined as

$A \cap B = \{x \,/\, x \in A \text{ and } x \in B\}$

iii) The difference of A and B (relative complement of B in set A) is defined as

$A - B = \{x \,/\, x \in A,\ x \notin B\}$

Note: One of the possible relationships between two sets A and B holds .

i) $A \subset B$

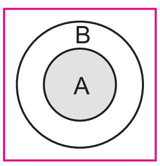
ii) $B \subset A$

iii) $A = B$

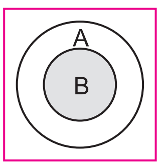
iv) $A \not\subset B$, $B \not\subset A$ and $A \cap B \neq \phi$

v) $A \not\subset B$, $B \not\subset A$ and $A \cap B = \phi$

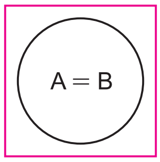
Figure:

i) 

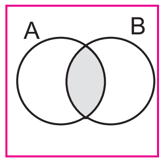
$A \subset B$

ii) 

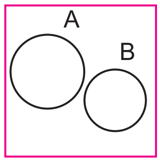
$B \subset A$

iii) 

$A = B$

iv) 

$A \cap B \neq \phi$

v) 

$A \cap B = \phi$

Fig. 1.1

Observe the following four statements

i) a) All professors are educated.

b) Equiangular triangles are precisely equilateral triangles.

ii) No policeman is a thief.

iii) Some doctors are rich.

iv) Some students are not scholars.

These statements can be generalized respectively as

a) All x’s are y’s
c) Some x’s are y’s

b) No x’s are y’s
d) Some x’s are not y’s

a) Diagram for “All x’s are y’s”

There are two possibilities

i) All x’s are y’s i.e. $x \subset y$

ii) x’s are precisely y’s i.e. $x = y$

For example:

i) Consider the statement

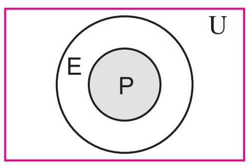
“All professors are educated”

Let p : The set of all professors.

E : The set of all educated people.

Let us choose the universal set as

u : The set of all human beings.



$P \subset E$

Fig. 1.2

The Venn diagram (fig. 1.2) represents the truth of the statement i.e. $P \subset E$.

ii) Consider the statement

India will be prosperous if and only if its citizens are hard working.

Let P : The set of all prosperous Indians.

H : The set of all hard working Indians.

U : The set of all human beings.

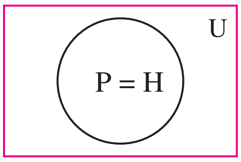


Fig. 1.3

The Venn diagram (fig. 1.3) represents the truth of the statement i.e. $P = H$.

b) Diagram for "No X's are Y's"

i) Consider the statement

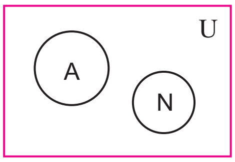
No naval person is an airforce person.

Let N : The set of all naval persons.

A : The set of all airforce persons.

Let us choose the universal set as

U : The set all human beings.



$N \cap A = \phi$

Fig. 1.4

The Venn diagram (fig. 1.4) represents the truth of the statement i.e. $N \cap A = \phi$.

ii) Consider the statement

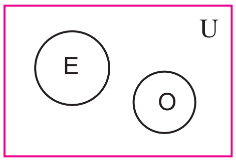
No even number is an odd numbers.

Let E : The set of all even numbers.

O : The set of all odd numbers.

Let us choose the universal set as

U : The set of all numbers.



$E \cap O = \phi$

Fig. 1.5

The Venn diagram (fig. 1.5) represents the truth of the statement i.e. $E \cap O = \phi$.

c) Diagram for "Some X's are Y's"

i) Consider the statement

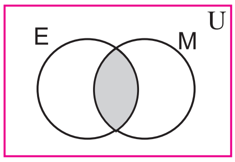
Some even numbers are multiple of seven.

Let E : The set of all even numbers.

M : The set of all numbers which are multiple of seven.

Let us choose the universal set as

U : The set of all natural numbers.



$E \cap M \neq \phi$

Fig. 1.6

The Venn diagram represents the truth of the statement i.e. $E \cap M \neq \phi$

ii) Consider the statement

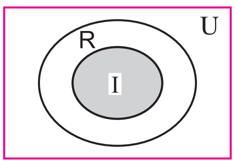
Some real numbers are integers.

Let R : The set of all real numbers.

I : The set of all integers.

Let us choose the universal set as

U : The set of all complex numbers.



$I \subset R$

Fig. 1.7

The Venn diagram (fig. 1.7) represents the truth of the statement i.e. I⊂R.

d) Diagram for "Some X's are not Y's"

i) Consider the statement

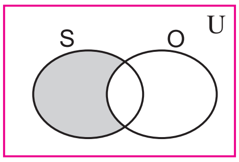
Some squares of integers are not odd numbers.

Let S : The set of all squares of integers.

O : The set of all odd numbers.

Let us choose the universal set as

U : The set of all integers.



$S - O \neq \phi$

Fig. 1.8

The Venn diagram (fig. 1.8) represents the truth of the statement i.e. $S - O \neq \phi$.

ii) Consider the statement

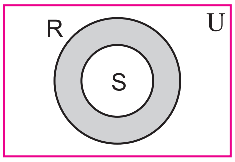
Some rectangles are not squares.

Let R : The set of all rectangles.

S : The set of all squares.

Let us choose the universal set as

U : The set of all quadrilaterals.



$R - S \neq \phi$

Fig. 1.9

The Venn diagram (fig. 1.9) represents the truth of the statement i.e. $R - S \neq \phi$.

SOLVED EXAMPLES

Ex. 1: Express the truth of each of the following statement by Venn diagram.

i) Equilateral triangles are isosceles.

ii) Some rectangles are squares.

iii) No co-operative industry is a proprietary firm.

iv) All rational numbers are real numbers.

v) Many servants are not graduates.

vi) Some rational numbers are not integers.

vii) Some quadratic equations have equal roots.

viii) All natural numbers are real numbers and x is not a natural number.

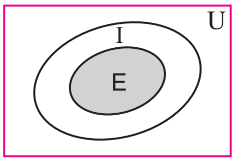
Solution:

i) Let us choose the universal set.

U : The set of all triangles.

Let I : The set of all isosceles triangles.

E : The set of all equilateral triangles.



$E \subset I$

Fig. 1.10

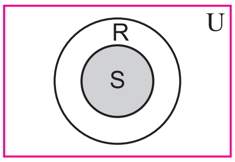
The Venn diagram (fig. 1.10) represents the truth of the given statement i.e. $E \subset I$.

ii) Let us choose the universal set.

U : The set of all quadrilaterals.

Let R : The set of all rectangles.

S : The set of all squares.



$R \cap S \neq \phi$

Fig. 1.11

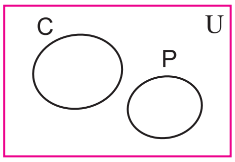
The Venn diagram (fig. 1.11) represents the truth of the given statement i.e. $R \cap S \neq \phi$.

iii) Let us choose the universal set.

U : The set of all industries.

Let C : The set of all co-operative industries.

P : The set of all proprietary firms.



$C \cap P = \phi$

Fig. 1.12

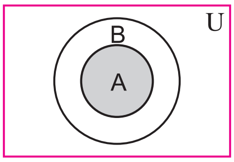
The Venn diagram (fig. 1.12) represents the truth of the given statement i.e. $C \cap P = \phi$.

iv) Let us choose the universal set.

U : The set of all complex numbers.

Let A : The set of all rational numbers.

B : The set of all real numbers.



$A \subset B$

Fig. 1.13

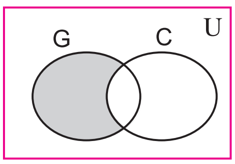
The Venn diagram (fig. 1.13) represents truth of the given statement i.e. $A \subset B$.

v) Let us choose the universal set.

U : The set of all human beings.

Let G : The set of all servants.

C : The set of all graduate people.



$G - C \neq \phi$

Fig. 1.14

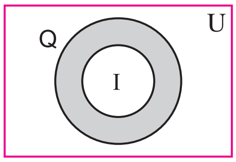
The Venn diagram (fig. 1.14) represents truth of the given statement i.e. $G - C \neq \phi$.

vi) Let us choose the universal set.

U : The set of all real numbers.

Let Q : The set of all rational numbers.

I : The set of all integers.



$Q - I \neq \phi$

Fig. 1.15

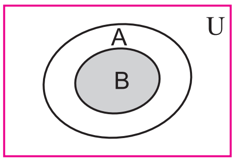
The Venn diagram (fig. 1.15) represents truth of the given statement i.e. $Q - I \neq \phi$ shaded portion.

vii) Let us choose the universal set.

U : The set of all equations.

Let A : The set of all quadratic equations.

B : The set of all quadratic equations having equal roots.



$B \subset A$

Fig. 1.16

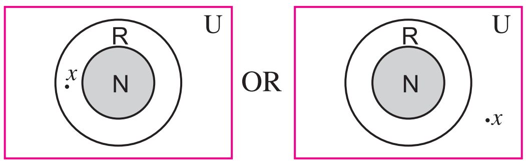
The Venn diagram (fig. 1.16) represents the truth of the given statement i.e. $B \subset A$.

viii) Let us choose the universal set.

U : The set of all complex numbers.

Let N : The set of all natural numbers.

R : The set of all real numbers.



(a) (b)

Fig. 1.17

The Venn diagram (fig. 1.17) represents the truth of the given statement.

Ex. 2: Draw the Venn diagram for the truth of the following statements.

i) There are students who are not scholars.

ii) There are scholars who are students.

iii) There are persons who are students and scholars.

Solution:

Let us choose the universal set.

U : The set of all human beings.

Let S : The set of all scholars.

T : The set of all students.

i) ii) iii)

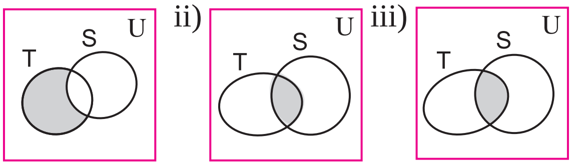


Fig. 1.18

We observe that (by Venn diagram) truth set of statements (ii) and (iii) are equal. Hence, statements (ii) and (iii) are logically equivalent.

Ex. 3: Using the Venn diagram, examine the logical equivalence of the following statements.

i) Some politicians are actors.

ii) There are politicians who are actors.

iii) There are politicians who are not actors.

Solution:

Let us choose the universal set.

U : The set of all human beings.

Let P : The set of all politicians.

A : The set of all actors.

i) ii) iii)

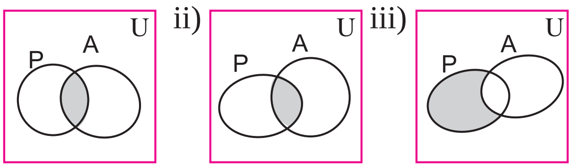


Fig. 1.19

By Venn diagrams (fig. 1.19), we observe that truth set of statements (i) and (ii) are equal.

Hence, statements (i) and (ii) are logically equivalent.

EXERCISE 1.10

1. Represent the truth of each of the following statements by Venn diagrams.
   i) Some hardworking students are obedient.
   ii) No circles are polygons.
   iii) All teachers are scholars and scholars are teachers.
   iv) If a quadrilateral is a rhombus, then it is a parallelogram.

2. Draw a Venn diagram for the truth of each of the following statements.
   i) Some sharebrokers are chartered accountants.
   ii) No wicket keeper is bowler, in a cricket team.

3. Represent the following statements by Venn diagrams.
   i) Some non resident Indians are not rich.
   ii) No circle is rectangle.
   iii) If n is a prime number and $n \neq 2$, then it is odd.

Let's Remember

1. **Statement:** Declarative sentence which is either true or false, but not both symultaneously.
   * Imperative, exclamatory, interrogative and open sentences are not statements.
   * The symbol '$\forall$' stands for "all values of". It is universal quantifier.
   * The symbol '$\exists$' stands for "there exists". It is known as existential quantifier.
   * An open sentence with a quantifier becomes a quantified statement.

2. Logical connectives:

Sr. No.	Name of the Compound statement	Connective	Symbolic form	Negation
1.	Conjunction	and	$p \wedge q$	$\sim p \vee \sim q$
2.	Disjunction	or	$p \vee q$	$\sim p \wedge \sim q$
3.	Negation	not	$\sim p$	$\sim (\sim p)$ $= p$
4.	Conditional or implication	If ... then	$p \rightarrow q$ or $p \Rightarrow q$	$p \wedge \sim q$
5.	Biconditional or double implication	If and only if ... iff ...	$p \leftrightarrow q$ (or $p \Leftrightarrow q$)	$(p \wedge \sim q) \vee$ $(\sim p \wedge q)$

3. Tautology: A statement pattern which is always true (T) is called a tautology (t).

Contradiction: A statement pattern which is always false (F) is called a contradiction (c).

Contingency \: A statement pattern which is neither a tautology nor contradiction is called a contingency.

4. Algebra of statements:

(Some standard equivalent statements)

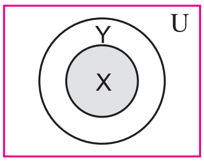
1.	$p \vee p \equiv p$	Idempotent laws	$p \wedge p \equiv p$
2.	$p \vee (q \vee r)$ $\equiv (p \vee q) \vee r$ $\equiv p \vee q \vee r$	Associative laws	$p \wedge (q \wedge r)$ $\equiv (p \wedge q) \wedge r$ $\equiv p \wedge q \wedge r$
3.	$p \vee q \equiv q \vee p$	Commutative laws	$p \wedge q \equiv q \wedge p$
4.	$p \vee (q \wedge r)$ $\equiv (p \vee q) \wedge (p \vee r)$	Distributive laws	$p \wedge (q \vee r)$ $\equiv (p \wedge q) \vee (p \wedge r)$
5.	$p \vee c \equiv p$ $p \vee t \equiv t$	Identity laws	$p \wedge c \equiv c$ $p \wedge t \equiv p$
6.	$p \vee \sim p \equiv t$ $\sim t \equiv c$	Complement laws	$p \wedge \sim p \equiv c$ $\sim c \equiv t$
7.	$\sim (\sim p) \equiv p$	Involution law (law of double negation)	
8.	$\sim (p \vee q)$ $\equiv \sim p \wedge \sim q$	DeMorgan's laws	$\sim (p \wedge q)$ $\equiv \sim p \vee \sim q$
9.	$p \rightarrow q$ $\equiv \sim q \rightarrow \sim p$	Contrapositive law	

i) $p \to q \equiv \sim q \to \sim p \equiv \sim p \vee q$

ii) $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$\equiv (\sim p \vee q) \wedge (\sim q \vee p)$

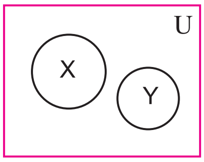
5. Venn-diagrams :

i) All x's are y's



$X \cap Y = X \neq \phi$

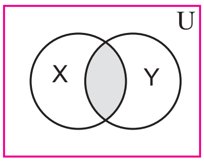
ii) "No x's are y's"



or

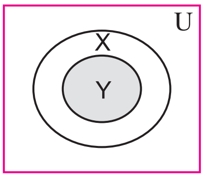
$X \cap Y = \phi$

iii) "Some x's are y's"



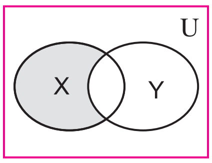
$X \cap Y \neq \phi$

or



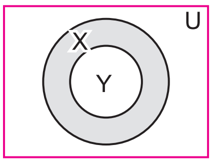
$X \cap Y = Y \neq \phi$

iv) "Some x's are not y's"



$X - Y \neq \phi$

or



$X - Y \neq \phi$

MISCELLANEOUS EXERCISE - 1

I) Choose the correct alternative.

1. Which of the following is not a statement?
   a) 2 + 2 = 5.
   b) 2 + 2 = 4.
   c) 2 is the only even prime number.
   d) Come here.

2. Which of the following is an open statement?
   a) x is a natural number.
   b) Give me a glass of water.
   c) Wish you best of luck.
   d) Good morning to all.

3. Let $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$. Then, this law is known as.
   a) commutative law
   b) associative law
   c) De-Morgan's law
   d) distributive law.

4. The false statement in the following is
   a) $p \wedge (\sim p)$ is contradiction
   b) $(p \to q) \leftrightarrow (\sim q \to \sim p)$ is a contradiction.
   c) $\sim (\sim p) \leftrightarrow p$ is a tautology
   d) $p \vee (\sim p) \leftrightarrow p$ is a tautology

5. For the following three statements
   p : 2 is an even number.
   q : 2 is a prime number.
   r : Sum of two prime numbers is always even.
   Then, the symbolic statement
   $(p \wedge q) \to \sim r$ means.
   a) 2 is an even and prime number and the sum of two prime numbers is always even.
   b) 2 is an even and prime number and the sum of two prime numbers is not always even.
   c) If 2 is an even and prime number, then the sum of two prime numbers is not always even.
   d) If 2 is an even and prime number, then the sum of two prime numbers is also even.

6. If p : He is intelligent.

   q : He is strong

   Then, symbolic form of statement "It is wrong that, he is intelligent or strong" is

   a) $\sim p \vee \sim p$ b) $\sim (p \wedge q)$

   c) $\sim (p \vee q)$ d) $p \vee \sim q$

7. The negation of the proposition "If 2 is prime, then 3 is odd", is

   a) If 2 is not prime, then 3 is not odd.

   b) 2 is prime and 3 is not odd.

   c) 2 is not prime and 3 is odd.

   d) If 2 is not prime, then 3 is odd.

8. The statement $(\sim p \wedge q) \vee \sim q$ is

   a) $p \vee q$ b) $p \wedge q$

   c) $\sim (p \vee q)$ d) $\sim (p \wedge q)$

9. Which of the following is always true?

   a) $(p \rightarrow q) \equiv \sim q \rightarrow \sim p$

   b) $\sim (p \vee q) \equiv \sim p \vee \sim q$

   c) $\sim (p \rightarrow q) \equiv p \wedge \sim q$

   d) $\sim (p \vee q) \equiv \sim p \wedge \sim q$

10. $\sim (p \vee q) \vee (\sim p \wedge q)$ is logically equivalent to

   a) $\sim p$ b) p

   c) q d) $\sim q$

11. If p and q are two statements then

   $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ is

   a) contradiction

   b) tautology

   c) Neither (i) not (ii)

   d) None of the these

12. If p is the sentence 'This statement is false' then

   a) truth value of p is T

   b) truth value of p is F

   c) p is both true and false

   d) p is neither true nor false.

13. Conditional $p \rightarrow q$ is equivalent to

   a) $p \rightarrow \sim q$

   b) $\sim p \vee q$

   c) $\sim p \rightarrow \sim q$

   d) $p \vee \sim q$

14. Negation of the statement "This is false or That is true" is

   a) That is true or This is true

   b) That is true and This is false

   c) This is true and That is false

   d) That is false and That is true

15. If p is any statement then $(p \vee \sim p)$ is a

   a) contingency

   b) contradiction

   c) tautology

   d) None of them.

II) Fill in the blanks:

i) The statement $q \rightarrow p$ is called as the ________ of the statement $p \rightarrow q$.

ii) Conjunction of two statement p and q is symbolically written as ________.

iii) If $p \vee q$ is true then truth value of $\sim p \vee \sim$ q is ________.

iv) Negation of "some men are animal" is ________.

v) Truth value of if $x = 2$, then $x^2 = -4$ is ________.

vi) Inverse of statement pattern $p \leftrightarrow q$ is given by ________.

vii) $p \leftrightarrow q$ is false when p and q have ________ truth values.

viii) Let p : the problem is easy. r : It is not challenging then verbal form of $\sim p \to r$ is ————.

ix) Truth value of 2 + 3 = 5 if and only if $-3 > -9$ is ————.

III) State whether each of the following is True or False:

i) Truth value of $2 + 3 < 6$ is F.

ii) There are 24 months in year is a statement.

iii) $p \vee q$ has truth value F is both p and q has truth value F.

iv) The negation of 10 + 20 = 30 is, it is false that $10 + 20 \neq 30$.

v) Dual of $(p \wedge \sim q) \vee t$ is $(p \vee \sim q) \vee C$.

vi) Dual of "John and Ayub went to the forest" is "John and Ayub went to the forest".

vii) "His birthday is on 29th February" is not a statement.

viii) $x^2 = 25$ is true statement.

ix) Truth value of $\sqrt{5}$ is not an irrational number is T.

x) $p \wedge t = p$.

IV) Solve the following:

1. State which of the following sentences are statements in logic.

i) Vanilla Ice cream is my favourite.

ii) $x + 3 = 8$; x is variable.

iii) Read a lot to improve your writing skill.

iv) z is a positive number.

v) $(a + b)^2 = a^2 + 2ab + b^2$ for all $a, b \in$ R.

vi) $(2 + 1)^2 = 9$.

vii) Why are you sad?

viii) How beautiful the flower is!

ix) The square of any odd number is even.

x) All integers are natural numbers.

xi) If x is real number then $x^2 \geq 0$.

xii) Do not come inside the room.

xiii) What a horrible sight it was!

2. Which of the following sentences are statements? In case of a statement, write down the truth value.

i) What is happy ending?

ii) The square of every real number is positive.

iii) Every parallelogram is a rhombus.

iv) $a^2 - b^2 = (a + b)(a - b)$ for all $a, b \in$ R.

v) Please carry out my instruction.

vi) The Himalayas is the highest mountain range.

vii) $(x - 2)(x - 3) = x^2 - 5x + 6$ for all $x \in$ R.

viii) What are the causes of rural unemployment?

ix) 0! = 1

x) The quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ always has two real roots.

3. Assuming the first statement p and second as q. Write the following statements in symbolic form.

i) The Sun has set and Moon has risen.

ii) Mona likes Mathematics and Physics.

iii) 3 is prime number iff 3 is perfect square number.

iv) Kavita is brilliant and brave.

v) If Kiran drives the car, then Sameer will walk.

vi) The necessary condition for existence of a tangent to the curve of the function is continuity.

vii) To be brave is necessary and sufficient condition to climb the Mount Everest.

viii) $x^3 + y^3 = (x + y)^3$ iff $xy = 0$.

ix) The drug is effective though it has side effects.

x) If a real number is not rational, then it must be irrational.

xi) It is not true that Ram is tall and handsome.

xii) Even though it is not cloudy, it is still raining.

xiii) It is not true that intelligent persons are neither polite nor helpful.

xiv) If the question paper is not easy then we shall not pass.

4. If p : Proof is lengthy.

   q : It is interesting.

   Express the following statements in symbolic form.

   i) Proof is lengthy and it is not interesting.

   ii) If proof is lengthy then it is interesting.

   iii) It is not true that the proof is lengthy but it is interesting.

   iv) It is interesting iff the proof is lengthy.

5. Let p : Sachin wins the match.

   q : Sachin is a member of Rajya Sabha.

   r : Sachin is happy.

   Write the verbal statement for each of the followings.

   i) $(p \wedge q) \vee r$ ii) $p \rightarrow r$

   iii) $\sim p \vee q$ iv) $p \rightarrow (p \wedge r)$

   v) $p \rightarrow q$ vi) $(p \wedge q) \wedge \sim r$

   vii) $\sim (p \vee q) \wedge r$

6. Determine the truth value of the following statements.

   i) $4 + 5 = 7$ or $9 - 2 = 5$

   ii) If $9 > 1$ then $x^2 - 2x + 1 = 0$ for $x = 1$

   iii) $x + y = 0$ is the equation of a straight line if and only if $y^2 = 4x$ is the equation of the parabola.

   iv) It is not true that $2 + 3 = 6$ or $12 + 3 = 5$

7. Assuming the following statements.

   p : Stock prices are high.

   q : Stocks are rising.

   to be true, find the truth value of the following.

   i) Stock prices are not high or stocks are rising.

   ii) Stock prices are high and stocks are rising if and only if stock prices are high.

   iii) If stock prices are high then stocks are not rising.

   iv) It is false that stocks are rising and stock prices are high.

   v) Stock prices are high or stocks are not rising iff stocks are rising.

8. Rewrite the following statements without using conditional –

   (Hint : $p \rightarrow q \equiv \sim p \vee q$)

   i) If price increases, then demand falls.

   ii) If demand falls, then price does not increase.

9. If p, q, r are statements with truth values T, T, F respectively determine the truth values of the following.

   i) $(p \wedge q) \rightarrow \sim p$

   ii) $p \leftrightarrow (q \rightarrow \sim p)$

   iii) $(p \wedge \sim q) \vee (\sim p \wedge q)$

   iv) $\sim (p \wedge q) \rightarrow \sim (q \wedge p)$

   v) $\sim [(p \rightarrow q) \leftrightarrow (p \wedge \sim q)]$

10. Write the negation of the following.

   i) If ΔABC is not equilateral, then it is not equiangular.

   ii) Ramesh is intelligent and he is hard working.

   iii) An angle is a right angle if and only if it is of measure 90°.

iv) Kanchanganga is in India and Everest is in Nepal.

v) If $x \in A \cap B$, then $x \in A$ and $x \in B$.

11. Construct the truth table for each of the following statement pattern.

i) $(p \wedge \sim q) \leftrightarrow (q \to p)$

ii) $(\sim p \vee q) \wedge (\sim p \wedge \sim q)$

iii) $(p \wedge r) \to (p \vee \sim q)$

iv) $(p \vee r) \to \sim (q \wedge r)$

v) $(p \vee \sim q) \to (r \wedge p)$

12. What is tautology? What is contradiction? Show that the negation of a tautology is a contradiction and the negation of a contradiction is a tautology.

13. Determine whether following statement pattern is a tautology, contradiction, or contingency.

i) $[(p \wedge q) \vee (\sim p)] \vee [p \wedge (\sim q)]$

ii) $[(\sim p \wedge q) \wedge (q \wedge r)] \vee (\sim q)$

iii) $[\sim (p \vee q) \to p] \leftrightarrow [(\sim p) \wedge (\sim q)]$

iv) $[\sim (p \wedge q) \to p] \leftrightarrow [(\sim p) \wedge (\sim q)]$

v) $[p \to (\sim q \vee r)] \leftrightarrow \sim [p \to (q \to r)]$

14. Using the truth table, prove the following logical equivalences.

i) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

ii) $[\sim (p \vee q) \vee (p \vee q)] \wedge r \equiv r$

iii) $p \wedge (\sim p \vee q) \equiv p \wedge q$

iv) $p \leftrightarrow q \equiv \sim (p \wedge \sim q) \wedge \sim (q \wedge \sim p)$

v) $\sim p \wedge q \equiv (p \vee q)] \wedge \sim p$

15. Write the converse, inverse, contrapositive of the following statements.

i) If $2 + 5 = 10$, then $4 + 10 = 20$.

ii) If a man is bachelor, then he is happy.

iii) If I do not work hard, then I do not prosper.

16. State the dual of each of the following statements by applying the principle of duality.

i) $(p \wedge \sim q) \vee (\sim p \wedge q) \equiv (p \vee q) \wedge \sim (p \wedge q)$

ii) $p \vee (q \vee r) \equiv \sim [(p \wedge q) \vee (r \vee s)]$

iii) 2 is even number or 9 is a perfect square.

17. Rewrite the following statements without using the connective 'If ... then'.

i) If a quadrilateral is rhombus then it is not a square.

ii) If $10 - 3 = 7$ then $10 \times 3 \neq 30$.

iii) If it rains then the principal declares a holiday.

18. Write the dual of each of the following.

i) $(\sim p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$

ii) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

iii) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (q \vee r)$

iv) $\sim (p \vee q) \equiv \sim p \wedge \sim q$

19. Consider the following statements.

i) If D is dog, then D is very good.

ii) If D is very good, then D is dog.

iii) If D is not very good, then D is not a dog.

iv) If D is not a dog, then D is not very good.

Identify the pairs of statements having the same meaning. Justify.

20. Express the truth of each of the following statements by Venn diagrams.

i) All men are mortal.

ii) Some persons are not politician.

iii) Some members of the present Indian cricket are not committed.

iv) No child is an adult.

21. If A = {2, 3, 4, 5, 6, 7, 8}, determine the truth value of each of the following statements.

i) $\exists x \in A$, such that $3x + 2 > 9$.

ii) $\forall x \in A$, $x^2 < 18$.

iii) $\exists x \in A$, such that $x + 3 < 11$.

iv) $\forall x \in A$, $x^2 + 2 \geq 5$.

22. Write the negation of the following statements.

i) 7 is prime number and Tajmahal is in Agra.

ii) $10 > 5$ and $3 < 8$.

iii) I will have tea or coffee.

iv) $\forall n \in N$, $n + 3 > 9$.

v) $\exists x \in A$, such that $x + 5 < 11$.

Activities

1 : Complete truth table for $\sim [p \vee (\sim q)] \equiv \sim p \wedge q$; Justify it.

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$\sim [p \vee (\sim q)]$	$\sim p \wedge q$
1	2	3	4	5	6	7
T	T	$\square$	F	$\square$	$\square$	F
$\square$	F	F	$\square$	$\square$	$\square$	$\square$
F	$\square$	$\square$	F	$\square$	$\square$	F
F	F	T	$\square$	T	$\square$	$\square$

Justification : --

2: If $p \leftrightarrow q$ and $p \rightarrow q$ both are true then find truth values of following with the help of activity.

i) $p \vee q$

ii) $p \wedge q$

$p \leftrightarrow q$ and $p \rightarrow q$ are true if p and q has truth values $\square$, $\square$ or $\square$, $\square$

i) $p \vee q$

a) If both p and q are true, then $p \vee q = \square \vee \square = \square$

b) If both p and q are false, then $p \wedge q = \square \wedge \square = \square$

ii) $p \wedge q$

a) - - - - - - - - - - - - - - -

b) - - - - - - - - - - - - - - -

3 : Represent following statement by Venn diagram.

i) Many students are not hard working.

ii) Some students are hard working.

iii) Sunday implies holiday.

4: You have given following statements.

$p : 9 \times 5 = 45$

q : Pune is in Maharashtra.

r : 3 is smallest prime number.

Then write truth values by activity.

i) $(p \wedge q) \wedge r = (\square \wedge \square) \wedge \square$
$= \square \wedge \square$
$= \square$

ii) $\sim [p \wedge r] = \sim (\square \wedge \square)$
$= \sim \square$
$= \square$

iii) $p \rightarrow q = \square \rightarrow \square$
$= \square$

iv) $p \rightarrow r = \square \leftrightarrow \square$
$= \square$