7. PROBABILITY

Let's Study

- Types of events
- Algebra of events
- Probability
- Addition theorem
- Conditional Probability

Introduction :

The dictionary meaning of Probability is "Chance". The words such as chance, probably, likely, possibly are used in our life.

For example

1) It is likely to rain today.
2) Our school Hockey team has good chance to win intercollegiate tournament.
3) The chance that a new born child is a girl is 50%

In each of the above statements there is a different degree of certainty. The probability measures the degree of certainty of an event.

There are two types of experiments, namely deterministic experiment and probabilistic experiment.

A deterministic experiment is like finding the area of a given right angled triangle.

A probabilistic experiment is like throwing a die and trying to find a certain number on the top. We will study the probabilistic experiments.

In previous standards we have defined probability as "The ratio of number of favourable outcomes to the total number of all possible outcomes".

For example, suppose the experiment is of forming a two-digit number using the digits 3,6,8,9 without repetition. What is the probability (chance) that an even number is formed?

The following set S (Sample space) shows different ways of forming a two-digit number.

S = {36, 38, 39, 63, 68, 69, 83, 86, 89, 93, 96, 98}

$\therefore$ n(S) = 12

The favourable outcomes are even numbers.

The set of favourable outcomes is

A = {36, 38, 68, 86, 96, 98}

$\therefore$ n(A) = 6

$\therefore$ Required probability is,

$$P(A) = \frac{n(A)}{n(S)} = \frac{6}{12} = \frac{1}{2}$$

Let's Recall

1) Random experiment: A random experiment is an action in which all the possible results are known in advance and are equally likely.

The exact result is unpredictable i.e. not known in advance.

Example

a) When a card is drawn from a well-shuffled pack of 52 cards, it is a random experiment because we know in advance that there are 52 equally possible results

b) Selecting any two students from a group of 5 students to play a game of chess is a random experiment because (i) we know in advance that any two students can be selected from 5 students in ${}^5C_2$ = 10 different ways i.e. we have 10 different results, (ii) we don't know in advance which two students are going to be selected.

2) Outcome: The result of a random experiment is called "outcome of experiment".

Examples

a) Suppose a random experiment is "die is thrown". Then we may get number 1,2,3,4,5, or 6 on the upper face of die that is this experiment gives 6 possible results or outcomes.

b) Suppose a random experiment is "3 students are selected from a group of 5 students S_1, S_2, S_3, S_4 and S_5". This can be done in ${}^5C_3$ = 10 different ways .

∴ This experiment has 10 different outcomes or results.

c) Suppose a random experiment is "3 books B_1,B_2, B_3 are to be arranged in a line on a shelf".

This can be done in ${}^3P_3 = 3! = 6$ different ways.

∴ This experiment has 6 different outcomes or results.

3) Equally likely outcomes: The outcomes of random experiment are said to be equally likely, if none of them is expected to occur in preference to any other.

Examples

a) If an unbiased coin is tossed, we have no reason to expect that Head will appear more often than Tail or vice versa.

b) If a baby is born, it is equally likely that it is a boy or it is a girl.

4) Sample space: The set of all possible outcomes of a random experiment is called the sample space. It is denoted by S or Ω.

Each element of the sample space is called a sample point. The number of elements in the sample space is denoted by n(S) or n(Ω).

Example

a) Suppose a random experiment is of tossing a coin 3 times. Then the sample space is

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

∴ n(S) = 8

b) Suppose a random experiment is of throwing a die. Then the sample space S = {1, 2, 3, 4, 5, 6}

∴ n(S) = 6

5) Event: A subset of the sample space is called an event.

Example

A random experiment consists of tossing a pair of dice.

Here, Sample space

S={(1,1),(1,2),(1,3),(1,4),(1,5),(1,6),
(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),
(3,1),(3,2),(3,3),(3,4),(3,5),(3,6),
(4,1), (4,2),(4,3),(4,4),(4,5),(4,6),
(5,1), (5,2),(5,3),(5,4),(5,5),(5,6),
(6,1), (6,2),(6,3),(6,4),(6,5), (6,6)}

∴ n(S) = 36

Define a subset A of S as follows

A={(x,y) | x+y is a perfect square}

∴ A = {(1,3), (2,2), (3,1), (3,6), (4,5), (5,4), (6,3)}

∴ n(A) = 7

A contains 7 favourable outcomes out of 36 possible outcomes.

Subset A is an event defined on the sample space S.

Remark: A is a subset of S ∴ $0 \leq n(A) \leq n(S)$

Let's Recall

7.1 TYPES OF EVENTS:

A) Simple or elementary event: A subset of sample space, consisting of a single element is called a simple or elementary event. Every sample point of the sample space S forms an elementary event.

Example

a) If 2 coins are tossed, then the sample space is S = {HH, HT, TH, TT}

Let A be the event that both the coins show tails.

∴ A = {TT}. In this case event A contains only one sample point of the sample space S.

Therefore, event A is a simple or elementary event.

b) If a die is thrown, then the sample space is S = {1, 2, 3, 4, 5, 6}

Let B be the event that die shows an even prime number

∴ B = {2}. In this case event B contains only one sample point of the sample space S. Therefore, event B is a simple or elementary event.

B) Occurrence and non-occurrence of event: After conducting an experiment, if the outcome belongs to subset A then we say that event A has occurred. If the outcome does not belong to subset A then we say that event A has not occurred.

Example

If a die is thrown, then the sample space S = {1, 2, 3, 4, 5, 6}

Let event A be that the die shows a number which is a multiple of 3. ∴ A = {3, 6}

After conducting the experiment, if the die shows number 3 or 6, then we say that event A has occurred. If the die shows a number say 5, then we say that event A has not occurred.

C) Sure Event: An event which contains all the sample points of the sample space S, is call as a sure event.

Example

A card is drawn at random from a pack of 52 cards. Let event A be that a card drawn is either red or black.∴ Event A is a sure event.

D) Impossible Event: An event which does not contain any sample point of the sample space S is called an impossible event. Null set ϕ represents an impossible event.

Example

A die is thrown.

∴ Sample Space S = {1, 2, 3, 4, 5, 6}

Let event B: die shows number 7.

∴ B = { } = ϕ

∴ Event B is an impossible event.

E) Complementary Event: Let A be an event defined on a sample space S. Therefore $A \subset S$. The complement of event A is the set containing all the elements in sample space S but not in A, which is denoted by A' or $\overline{A}$ or A^c.

∴ A' or $\overline{A}$ or $A^c = \{a \mid a \in S, a \notin A\}$

Ex. Two coins are tossed simultaneously.

∴ Sample Space S = {HH, HT, TH, TT}

∴ n(S) = 4

Let event A: both the coins show tails.

∴ A = {TT} ∴ n(A) = 1

∴ A’ = {HH, HT, TH} ∴n(A’) = 3

That is coins show at least one head.

n(A) + n(A’) = n(S)

7.2 Algebra of Events

Since events represent subsets, the operations in set theory are very useful to describe algebra of events and to work out problems in probability.

A) Union of two events: Let A and B be any two events defined on the sample space S. Then union of two events is denoted by $A \cup B$ and is defined as occurrence of at least one of the events. Either event A occurs or event B occurs or both occur.

∴ $A \cup B = \{x \mid x \in A \text{ or } x \in B \text{ or both}\}$

Ex. Sample Space S = All positive integers up to 50

Let event A: Set of numbers divisible by 6.

∴ A = {6, 12, 18, 24, 30, 36, 42, 48}

Let event B: Set of numbers divisible by 9.

∴ B = {9, 18, 27, 36, 45}

∴ A∪B = {6, 9, 12, 18, 24, 27, 30, 36, 42, 45, 48}.

B) Exhaustive Events: Two events A and B defined on the sample space S are said to be Exhaustive Events if A∪B = S

For example :

Let event A : die shows a number less than or equal to 4.

∴ A = {1, 2, 3, 4}

Let event B: die shows a number greater than or equal to 2.

∴ B = {2, 3, 4, 5, 6}

∴ A∪B = {1, 2, 3, 4, 5, 6} = S

∴ A and B are exhaustive events.

C) Intersection of two events: Let A and B be any two events defined on the sample space S. Then intersection of two events is denoted by A∩B and is defined as simultaneous occurrence of both the events.

∴ A∩B = {x | x ∈ A and x ∈ B}

Ex. A ticket is drawn from a box containing tickets numbered 1 to 50.

∴ Sample Space S = {1, 2, 3, 4, ... 50}

Let event A: number on ticket is divisible by 4.

∴ A = {4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48}

Let event B: number of ticket is divisible by 5.

∴ B = {5, 10, 15, 20, 25, 30, 35, 40, 45, 50}

∴ A∩B = {20, 40}

D) Mutually Exclusive Events: Two events A and B defined on the sample space are said to be mutually exclusive if they cannot occur simultaneously in a single trial that is occurrence of any one event prevents the occurrence of other event or A∩B = ϕ . It can also be said that A and B are disjoint sets.

Example : Consider the experiment, where a ticket drawn from a box containing tickets numbered 1 to 50. Sample space S = {1, 2, 3, 4, ... 50}

Let event A: number on ticket is divisible by 4.

∴ A = {4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48}

Let event B: number on ticket is divisible by 13.

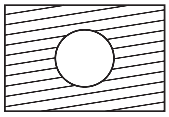
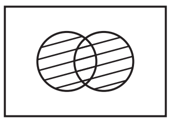
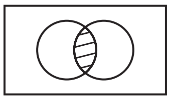
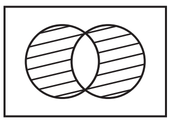
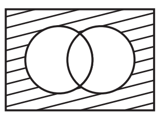
∴ B = {13, 26, 39}

∴ A∩B = { } = ϕ i.e. number of tickets which is divisible by 4 and 13, which is not possible in a single trial. ∴ A and B are mutually exclusive events.

Note: If two events A and B defined on the sample space S are mutually exclusive and exhaustive, then they are complementary events.

i.e. if A∪B = S and A∩B = ϕ then A and B are complementary events.

Table 7.1

Set Notation	Interpretation	Venn Diagram
A′ or $\overline{A}$ or A^c = S – A where ‘S’ is sample space and A ⊂ S	Complement of event A i.e. not A	
A∪B	Occurrence of at least one event	
A∩B	Occurrence of both the event	
(A’∩B)∪(A∩B’)	Occurrence of exactly one event	
A’∩B’ = (A∪B)’	Neither A nor B occur	

SOLVED EXAMPLES

Example 1: Write the sample space of the experiment 'coin and a die are thrown simultaneously'

Solution:

Sample space S = {(H,1), (H,2), (H,3), (H,4), (H,5), (H,6), (T,1), (T,2), (T,3), (T,4), (T,5), (T,6)}

Example 2: Sunita and Samrudhi who live in Mumbai wish to go on holiday to Delhi together. They can travel to Delhi from Mumbai either by car or by train or plane and on reaching Delhi they can go for city-tour either by bus or Taxi. Describe the sample space, showing all the combined outcomes of different ways they could complete city-tour from Mumbai.

Solution: Sample space

S = {(car, bus), (car, taxi), (train, bus), (train, taxi), (plane, bus), (plane, taxi)}

Example 3: Three coins are tossed. Events E_1, E_2, E_3 and E_4 are defined as follows.

E_1: Occurrence of at least two heads.

E_2: Occurrence of at least two tails.

E_3: Occurrence of at most one head.

E_4: Occurrence of two heads.

Write the sample space and events E_1, E_2, E_3 and E_4.

Find $E_1 \cup E_4$, E_3'. Also check whether

i) E_1 and E_2 are mutually exclusive

ii) E_2 and E_3 are equal

Solution: Sample space

S= {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

E_1: {HHH, HHT, HTH, THH}

E_2: {HTT, THT, TTH, TTT}

E_3: {HTT, THT, TTH, TTT}

E_4: {HHT, HTH, THH,}

$E_1 \cup E_4$ = {HTT, HTH, THH, HHH}

E'_3 = {HHH, HHT, HTH, THH}

i) $E_1 \cap E_2 = \{\} = \phi$

$\therefore$ E_1 and E_2 are mutually exclusive.

ii) E_2 and E_3 are equal.

EXERCISE 7.1

1) State the sample Space and n(S) for the following random experiments.

   a) A coin is tossed twice. If a second throw results in tail, a die is thrown.

   b) A coin is tossed twice. If a second throw results in head, a die is thrown, otherwise a coin is tossed.

2) In a bag there are three balls; one black, one red and one green. Two balls are drawn one after another with replacement. State sample space and n(S).

3) A coin and die are tossed. State sample space of following events.

   a) A: Getting a head and an even number.

   b) B: Getting a prime number.

   c) C: Getting a tail and perfect square.

4) Find total number of distinct possible outcomes n(S) for each of the following random experiments.

   a) From a box containing 25 lottery tickets any 3 tickets are drawn at random.

   b) From a group of 4 boys and 3 girls, any two students are selected at random.

   c) 5 balls are randomly placed into five cells, such that each cell will be occupied.

   d) 6 students are arranged in a row for photograph.

5) Two dice are thrown. Write favourable outcomes for the following events.

   a) P: Sum of the numbers on two dice is divisible by 3 or 4.

   b) Q: Sum of the numbers on 2 dice is 7.

c) R: Sum of the numbers on two dice is a prime number.

Also check whether i) events P and Q are mutually exclusive and exhaustive.

ii) Events Q and R are mutually exclusive and exhaustive.

6) A card is drawn at random from an ordinary pack of 52 playing cards. State the number of elements in the sample space if consideration of suits

a) is not taken into account.

b) is taken into account.

7) Box-I contains 3 red (R_{11}, R_{12}, R_{13}) and 2 blue (B_{11}, B_{12}) marbles while Box-II contains 2 red (R_{21}, R_{22}) and 4 blue (B_{21}, B_{22}, B_{23}, B_{24}) marbles. A fair coin is tossed. If the coin turns up heads, a marble is chosen from Box-I; if it turns up tails, a marble is chosen from Box-II. Describe the sample space.

8) Consider an experiment of drawing two cards at random from a bag containing 4 cards marked 5, 6, 7 and 8. Find the sample space if cards are drawn

a) with replacement.

b) without replacement.

Let's Learn

7.3 CONCEPT OF PROBABILITY :

In any random experiment there is always uncertainty as to whether a particular event will or will not occur. It is convenient to assign a number between 0 and 1, as a measure of chance, with which we can expect the event to occur. If, for example, the chance (or probability) of a particular event is 1/4, we would say that there is 25% chance it will occur and 75% chance that is will not occur. Assignment of real numbers (between 0 and 1) to the events defined in a sample space S is known as the probability assignment.

Some elementary properties of probability:

If A and B are any events of the sample space S then

1. $0 \leq P(A) \leq 1$

2. $P(\phi) = 0$ and $P(S) = 1$

3. $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \phi$

   $\therefore P(A \cup A') = P(S) = 1$

   and $P(A \cup A') = P(A) + P(A')$

   $\therefore P(A') = 1 - P(A)$

4. If $A \subset B$, then $P(A) \leq P(B)$

5. $P(A) = P(A \cap B) + P(A \cap B')$

   $\therefore P(A \cap B') = P(A) - P(A \cap B)$

SOLVED EXAMPLES

Example 1: If $A \cup B \cup C = S$ (the sample space) and A, B and C are mutually exclusive events, can the following represent probability assignment?

i) $P(A) = 0.2$, $P(B) = 0.7$, $P(C) = 0.1$

ii) $P(A) = 0.4$, $P(B) = 0.6$, $P(C) = 0.2$

Solution:

i) Since
$$P(S) = P(A \cup B \cup C) = P(A) + P(B) + P(C) = 0.2 + 0.7 + 0.1 = 1$$

and $0 \leq P(A), P(B), P(C) \leq 1$

$\therefore$ Using axiom 1, 2 and 3 of axiomatic definition of probability P(A), P(B) and P(C) represent probabilities of the events A, B and C respectively.

ii) Since

$$P(S) = P(A \cup B \cup C) = P(A) + P(B) + P(C) = 0.4 + 0.6 + 0.2 = 1.2 \neq 1$$

$\therefore$ P(A), P(B) and P(C) cannot represent probability assignment.

Example 2: A fair die is thrown. What is probability of getting a number multiple of 3?

Solution: Random experiment is , a fair die is thrown.

$\therefore$ Sample space S = {1, 2, 3, 4, 5, 6}

$\therefore$ n(S) = 6

Let A be the event that number on die is multiple of 3.

$\therefore$ A = {3, 6} $\therefore$n(A) = 2

$\therefore$ P(A) = 2/6 = 1/3

Example 3: One card is drawn at random from a pack of 52 cards. What is the probability that it is a King or Queen?

Solution: Random Experiment = One card is drawn at random from a pack of 52 cards

$\therefore n(S) = {}^{52}C_1 = 52.$

Let event A: Card drawn is King

and event B: Card drawn is Queen.

Since pack of 52 cards contains, 4 king cards from which any one king card can be drawn in ${}^{4}C_1 = 4$ ways.

$\therefore$ n(A) = 4 $\therefore$P(A) = n(A)/n(S) = 4/52 = $\frac{1}{13}$

Similarly, a pack of 52 cards contains, 4 queen cards from which any one queen card can be drawn in ${}^{4}C_1 = 4$ ways. $\therefore$ n(B) = 4

$\therefore$ P(B) = n(B)/n(S) = 4/52 = $\frac{1}{13}$

Since A and B are mutually exclusive events

$\therefore$ required probability P(king or queen)

= P(A∪B) = P(A) + P(B) = 4/52 + 4/52 = 2/13

Example 4: The sample space S of a random experiment is given by S = $\{a_1, a_2, a_3, a_4\}$ with probabilities $P(a_1) = 0.3$, $P(a_2) = 0.1$ $P(a_3) = 0.4$ and $P(a_4) = 0.2$. Consider events A and B as: A= $\{a_1, a_2\}$ and B = $\{a_2, a_3, a_4\}$. Determine the following probabilities. i) P(A) ii) P(B) iii) P(A’) iv) P(A∩B) v) P(A∪B) vi) P(A’∩B’). Also check whether A and B are mutually exclusive and exhaustive?

Solution:

i) $P(A) = P(a_1) + P(a_2) = 0.3 + 0.1 = 0.4$ (using property)

ii) $P(B) = P(a_2) + P(a_3) + P(a_4) = 0.1 + 0.4 + 0.2 = 0.6$ (using property)

iii) P(A’) = 1 – P(A) = 1-0.4 = 0.6 (using property)

iv) $A \cap B = \{a_2\}$

$\therefore$ $P(A\cap B) = P(a_2) = 0.1$

v) $A \cup B = \{a_1, a_2, a_3, a_4\}$

$\therefore$ $P(A\cup B) = P(a_1) + P(a_2) + P(a_3) + P(a_4) = 1$

Or

using property, we have

P(A∪B) = P(A) + P(B) – P(A∩B) =0.4 + 0.7 – 0.1 = 1

vi) P(A’∩B’) = P(A∪B)’ (using De Morgan’s law)

= 1 - P(A∪B) = 1 – 1 = 0

Since $A\cap B = \{a_2\} \neq \phi$ therefore A and B are not mutually exclusive events but $A\cup B = \{a_1, a_2, a_3, a_4\}$ = S, therefore A and B are exhaustive events.

Example 5: Five men in a company out of 20 are graduates. If 3 men are picked out of 20 at random, what is the probability that they are all graduates? What is the probability of at least one graduate?

Solution: Out of 20 men, any 3 men can be selected in ${}^{20}C_3$

Let event A: All 3 selected men are graduates.

Out of 5 graduate men any three men can be selected in ${}^{5}C_3$ ways.

$\therefore$ required probability

$P(A) = {}^{5}C_3 / {}^{20}C_3 = 10/1140 = 1/114$

Let event B: At least one graduate man is selected.

$\therefore$ B' is the event that no graduates man is selected.

Since out of 20 men, 5 men are graduates, therefore from the remaining 15 non-graduate men any three non-graduates can be selected in $^{15}C_3$ ways.

$\therefore$ P(B') = $^{15}C_3$/ $^{20}C_3$ = 455/1140 = 91/228

$\therefore$ required probability

P(B) = 1 - P(B') = 1 – 91/228 = 137/228

Example 6: Find the probability if letters of the word STORY are arranged, so that

a) T and Y are always together.

b) Begin with T and end with Y.

Solution: The word STORY consists of 5 different letters, which can be arranged among themselves in 5! ways.

n(s) = 5! = 120.

Let us consider T and Y as a single letter say X. Therefore, now we have four different letters X, S, O and R which can be arranged themselves in 4! = 24 different ways. After this is done, two letters T and Y can be arranged themselves in 2! = 2 ways. Therefore, by fundamental principle, total number of arrangements in which T and Y are always together is 24 × 2 = 48.

$\therefore$ required probability P(A) = n(A)/n(s) = 48/120 = 2/5

Let event B: An arrangement begins with T and ends with Y.

Remaining 3 letters in the middle can be arranged in 3! = 6 different ways.

$\therefore$required probability P(B) = 6/120 = 1/20

EXERCISE 7.2

1) A fair die is thrown two times. Find the chance that
   a) Product of the numbers on the upper face is 12.
   b) Sum of the numbers on the upper face is 10.
   c) Sum of the numbers on the upper face is at least 10.
   d) Sum of the numbers on the upper face is 4.
   e) The first throw gives an odd number and second throw gives multiple of 3.
   f) Both the time die shows same number (doublet)

2) Two cards are drawn from a pack of 52 cards. Find the probability that
   a) Both are black.
   b) Both are diamond.
   c) Both are ace cards.
   d) Both are face cards.
   e) One is spade and other is non-spade.
   f) Both are from same suit.
   g) Both are from same denomination.

3) Four cards are drawn from a pack of 52 cards. Find the probability that
   a) 3 are king and 1 is jack.
   b) All the cards are from different suit.
   c) At least one heart.
   d) All cards are club and one of them a jack.

4) A bag contains 15 balls of three different colours: Green, Black and Yellow. A ball is drawn at random from the bag. The probability of green ball is 1/3. The probability of yellow is 1/5.
   a) What is the probability of black ball?
   b) How many balls are green, black and yellow?

5) A box contains 75 tickets numbered 1 to 75. A ticket is drawn at random from the box. What is the probability that,
   a) Number on ticket is divisible by 6?
   b) Number on ticket is perfect square?
   c) Number on ticket is prime?
   d) Number on ticket is divisible by 3 and 5?

6) From a group of 8 boys and 5 girls a committee of five is to be formed. Find the probability that committee contains

a) 3 boys and 2 girls

b) At least 3 boys

7) A room has three sockets for lamps. From a collection of 10 light bulbs of which 6 are defective, a person selects 3 bulbs at random and puts them in socket. What is the probability that the room is lit?

8) The letters of the word LOGARITHM are arranged at random. Find the probability that

a) Vowels are always together.

b) Vowels are never together.

c) Exactly 4 letters between G and H.

d) Begin with O and end with T.

e) Starts with vowel and ends with consonant.

9) The letters of the word SAVITA are arranged at random. Find the probability that vowels are always together.

Let's Learn

7.4 ADDITION THEOREM

Statement: Let S be a sample space associated with the given random experiment. Let A and B be any two events defined on the sample space S. Then the probability of occurrence of at least one event is denoted by $P(A\cup B)$ and given by $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

SOLVED EXAMPLES

Example 1: Two dice are thrown together. What is the probability that,

i) Sum of the numbers is divisible by 3 or 4?

ii) Sum of the numbers is neither divisible by 3 nor 5?

Solution: Since two dice are thrown,

therefore $n(S) = 36$

i) Let event A: Sum of the numbers is divisible by 3

$\therefore$ possible sums are 3, 6, 9, 12.

$\therefore$ A = {(1, 2), (1, 5), (2, 1), (2, 4), (3, 3), (3, 6), (4, 2), (4, 5), (5, 1), (5, 4), (6, 3), (6, 6)}

$\therefore$ $n(A) = 12$ $\therefore$ $P(A) = n(A) / n(S) = 12/36$

Let event B: Sum of the numbers is divisible by 4.

$\therefore$ possible sums are 4, 8, 12

$\therefore$ B ={(1, 3), (2, 2), (2, 6), (3, 1), (3, 5), (4, 4), (5, 3), (6, 2), (6, 6)}

$\therefore$ $n(B) = 9$ $\therefore$ $P(B) = n(B) / n(S) = 9/36$

$\therefore$ Event $A\cap B$: Sum of the numbers is divisible by 3 and 4 i.e. divisible by 12.

$\therefore$ possible sum is 12

$\therefore$ $A\cap B$ = {(6, 6)}

$\therefore$ $n(A\cap B) = 1$

$\therefore$ $P(A\cap B) = n(A\cap B)/n(S) = 1/36$

$\therefore$ required probability

= P (Sum of the numbers is divisible by 3 or 4)

= $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

= $12/36 + 9/36 – 1/36 = 20/36 = 5/9$

ii) Let event X: Sum of the numbers is divisible by 3

$\therefore$ possible sums are 3, 6, 9, 12

$\therefore$ X ={(1, 2), (1, 5), (2, 1), (2, 4), (3, 3), (3, 6), (4, 2), (4, 5), (5, 1), (5, 4), (6, 3), (6, 6)}

$\therefore$ $n(X) = 12$ $\therefore$ $P(X) = n(X) / n(S) = 12/36$

Let event Y: Sum of the numbers is divisible by 5.

$\therefore$ possible sums are 5, 10

$\therefore$ Y ={(1, 4), (2, 3), (3, 2), (4, 1), (4, 6), (5, 5), (6, 4)}

$\therefore$ $n(Y) = 7$ $\therefore P(Y) = n(Y) / n(S) = 7/36$

$\therefore$ Event $X\cap Y$: sum is divisible by 3 and 5

$\therefore$ $X\cap Y = \{\} = \phi$

[X and Y are mutually exclusive events]

$\therefore$ $P(X\cap Y) = n(X\cap Y) / n(S) = 0$

$\therefore$ required probability = P(Sum of the numbers is neither divisible by 3 nor 5)

= $P(X'\cap Y') = P((X\cup Y)' = 1 - P(X\cup Y)$

$= \quad 1 - [P(X) + P(Y)]$

$= \quad 1 - 19/36 = \dfrac{17}{36}$

Example 2: The probability that a student will solve problem A is 2/3, and the probability that he will not solve problem B is 5/9. If the probability that student solves at least one problem is 4/5, what is the probability that he will solve both the problems?

Solution: Let event A: student solves problem A

$\therefore \quad P(A) = \dfrac{2}{3}$

event B: student solves problem B.

$\therefore$ event B': student will not solve problem B.

$\therefore \quad P(B') = \dfrac{5}{9}$

$\therefore \quad P(B) = 1 - P(B') = 1 - 5/9 = \dfrac{4}{9}$

Probability that student solves at least one problem $= P(A \cup B) = \dfrac{4}{5}$

$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore$ required probability = P(he will solve both the problems)

$$= \quad P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= 2/3 + 4/9 - 4/5 = \frac{14}{45}$$

EXERCISE 7.3

1) Two dice are thrown together. What is the probability that sum of the numbers on two dice is 5 or number on second die is greater than or equal to the number on the first die?

2) A card is drawn from a pack of 52 cards. What is the probability that,
   a) Card is either red or black?
   b) Card is either red or face card?

3) Two cards are drawn from a pack of 52 cards. What is the probability that,
   a) both the cards are of same colour?
   b) both the cards are either black or queens?

4) A bag contains 50 tickets, numbered from 1 to 50. One ticket is drawn at random. What is the probability that,
   a) number on the ticket is a perfect square or divisible by 4?
   b) number on the ticket is a prime number or greater than 30?

5) Hundred students appeared for two examinations. 60 passed the first, 50 passed the second and 30 passed in both. Find the probability that student selected at random
   a) passed in at least one examination
   b) passed in exactly one examination
   c) failed in both examinations.

6) If $P(A) = 1/4$, $P(B) = 2/5$ and $P(A \cup B) = 1/2$. Find the values of the following probabilities.
   a) $P(A \cap B)$ b) $P(A \cap B')$ c) $P(A' \cap B)$ d) $P(A' \cup B')$ e) $P(A' \cap B')$

7) A computer software company is bidding for computer program A and B. The probability that the company will get software A is 3/5, the probability that the company will get software B is 1/3 and the probability that company will get both A and B is 1/8. What is the probability that the company will get at least one software?

8) A card is drawn from well shuffled pack of 52 cards. Find the probability of it being heart or a queen.

9) In a group of students, there are 3 boys and 4 girls. Four students are to be selected at random from the group. Find the probability that either 3 boys and 1 girl or 3 girls and 1 boy are selected.

Let's Learn

7.5 CONDITIONAL PROBABILITY:

Let S be a sample space associated with the given random experiment. Let A and B be any two events defined on the sample space S. Then the probability of occurrence of event A under the condition that event B has already occurred and $P(B) \neq 0$ is called conditional probability of event A given B and is denoted by P(A/B).

Consider the following examples :

Example 1: A card is drawn from a pack of 52 cards, given that it is a red card, what is the probability that it is a face card.

Solution: Let event A: Red card is drawn and event B: face card is drawn

A card is drawn from a pack of 52 cards, therefore n(S) = 52. But we are given that red card is drawn, therefore our sample space reduces to event A only, which contains n(A) = 26 sample points. Event A is called reduced or truncated sample space. Out of 26 red cards, 6 cards are favourable for face cards.

∴ P[card drawn is face card given that it is a red card] = P[B/A] = 6 /26 = 3/13

Example 2: A pair of dice is thrown. If sum of the numbers is an odd number, what is the probability that sum is divisible by 3?

Solution: Let Event A: sum is an odd number.

Event B: Sum is divisible by 3.

A pair of dice is thrown, therefore n(S) = 36. But we are given that sum is odd, therefore our sample space reduces to event A only as follows:

A = {(1,2), (1,4), (1,6), (2,1), (2,3), (2,5), (3,2), (3,4), (3,6), (4,1), (4,3), (4,5), (5,2), (5,4), (5,6), (6,1), (6,3), (6,5)}

∴ n(A) = 18

Out of 18 sample points following 6 sample points are favourable for occurrence of event B

B = {(1, 2), (2, 1), (3, 6), (4, 5), (5, 4), (6, 3)}

∴ P[sum is divisible by 3 given that sum is an odd number] = P(B/A) = 6/18 = 1/3

7.5.1 Let S be a finite sample space, associated with the given random experiment, containing equally likely outcomes. Then we have the following result.

Statement: Conditional probability of event A given that event B has already occurred is given by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \; P(B) \neq 0$$

SOLVED EXAMPLES

Example 1: Find the probability that a single toss of a die will result in a number less than 4 if it is given that the toss resulted in an odd number.

Solution: Let event A: toss resulted in an odd number and

Event B: number is less than 4

∴ A = {1, 3, 5} ∴ P(A) = 3/6 = ½

B = {1, 2, 3} ∴ A∩B = {1, 3}

∴ P(A∩B) = 2/6 = 1/3

∴ P(number is less than 4 given that it is odd)

= P(B/A) = P(A∩B)/ P(A) = (1/3)/ (1/2) = 2/3

Example 2: If P(A') = 0.7, P(B) = 0.7, P(B/A) = 0.5, find P(A/B) and P(A∪B).

Solution: Since P(A') = 0.7

P(A) = 1 – P(A') = 1 - 0.7 = 0.3

Now P(B/A) = P(A∩B)/P(A)

∴ 0.5 = P(A∩B)/ 0.3

∴ P(A∩B) = 0.15

Again P(A/B) = P(A∩B)/P(B)

= 0.15/0.7

∴ P(A/B) = 3/14

Further, by addition theorem

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.3 + 0.7 - 0.15 = 0.85$

7.5.2 Multiplication theorem:

Statement: Let S be sample space associated with the given random experiment. Let A and B be any two events defined on the sample space S. Then the probability of occurrence of both the events is denoted by $P(A \cap B)$ and is given by

$$P(A \cap B) = P(A).P(B/A) = P(B).P(A/B)$$

SOLVED EXAMPLES

Example 1: Two cards are drawn from a pack of 52 cards one after other without replacement. What is the probability that both cards are ace cards?

Solution: Let event A: first card drawn is an Ace card.

Let event B: second card drawn is an Ace card.

$\therefore$ required probability = P(both are Ace cards)

$= P(A \cap B) = P(A)P(A/B)$

Now $P(A) = 4/52 = 1/13$

Since first ace card is not replaced in the pack, therefore now we have 51 cards containing 3 ace cards

$\therefore$ Probability of getting second ace card under the condition that first ace card is not replaced in the pack $= P(B/A) = 3/51 = 1/17$

$\therefore$ P(both are ace cards) $= P(A \cap B)$

$= P(A)P(B/A) = (1/13) \times (1/17) = 1/221$

Example 2: An urn contains 4 black and 6 white balls. Two balls are drawn one after the other without replacement, what is the probability that

(i) both balls are black?

(ii) the 2nd ball is black ?

Solution: Let event A: first ball drawn is black.

Event B: second ball drawn is black.

$\therefore$ required probability = P(both are black balls)

$= P(A \cap B) = P(A)P(B/A)$

Now $P(A) = 4/10$

Since first black ball is not replaced in the urn, therefore now we have 9 balls containing 3 black balls.

$\therefore$ Probability of getting second black ball under the condition that first black ball is not replaced in the pack $= P(B/A) = 3/9$

$\therefore$ P(both are black balls) $= P(A \cap B)$

$= P(A)P(B/A) = (4/10) \times (3/9) = 2/15$

The first ball can be black or non-black (white)

case 1 : The first ball is white : The probability of second ball is black is

$$\frac{6}{10} \times \frac{4}{9} = \frac{4}{15}$$

case 2 : The first ball is black : The probability of second ball is black is

$$\frac{4}{10} \times \frac{3}{9} = \frac{2}{15}$$

$\therefore$ The required probability

$$= \frac{4}{15} + \frac{2}{15} = \frac{6}{15} = \frac{2}{5}$$

7.5.3 Independent Events:

Let S be sample space associated with the given random experiment. Let A and B be any two events defined on the sample space S. If the occurrence of one event does not affect the occurrence of other event, then two events A and B are said to be independent.

i.e. if $P(A/B) = P(A/B') = P(A)$ or

$P(B/A) = P(B/A') = P(B)$

Then A and B are independent events.

Corollary : If A and B are independent events then $P(A \cap B) = P(A).P(B)$

In general, if $A_1, A_2, A_3, \ldots A_n$ are n independent events, then

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1).P(A_2)\ldots P(A_n)$$

Remark:

If A and B are independent events then

a) A and B' are also independent events

b) A' and B' are also independent events

c) A' and B are also independent events

SOLVED EXAMPLES

Example 1: Two cards are drawn at random one after the other. Given that first card drawn in non-face red card and second card is face card, check whether events are independent if the cards are drawn

i) without replacement ?
ii) with replacement ?

Solution: Let event A: first card drawn is a non-face red card and event B: second card drawn is face card.

$\therefore$ $P(A) = 20/52 = 5/13$ and $P(B) = 12/52 = 3/13$

$\therefore$ required probability = P(second card drawn is face card given that it is a red card)

i) Without replacement: Since first non-face red card is not replaced, therefore now we have 51 cards containing 12 face cards.

$\therefore$ $P(B/A) = 12/51 \neq P(B)$. In this case A and B are not independent.

ii) With replacement: Since first non-face red is replaced, therefore now again we have 52 cards containing 12 face cards.

$\therefore$ $P(B/A) = 12/52 = 3/13 = P(B)$.

In this case A and B are independent.

Example 2: If A and B are two independent events and $P(A) = 3/5$, $P(B) = 2/3$, find

i) $P(A\cap B)$ ii) $P(A\cap B')$ iii) $P(A'\cap B)$ iv) $P(A'\cap B')$ v) $P(A\cup B)$

Solution: $P(A) = 3/5 \therefore P(A') = 1 - P(A) = 2/5$

$P(B) = 2/3 \therefore P(B') = 1 - P(B) = 1/3$

i) $P(A\cap B) = P(A)P(B) = 2/5$

ii) $P(A\cap B') = P(A)P(B') = 1/5$

iii) $P(A'\cap B) = P(A')P(B) = 4/15$

iv) $P(A'\cap B') = P(A')P(B') = 2/15$

v) $P(A\cup B) = P(A) + P(B) - P(A\cap B) = 13/15$

EXERCISE 7.4

1) Two dice are thrown simultaneously, if at least one of the dice show a number 5, what is the probability that sum of the numbers on two dice is 9?

2) A pair of dice is thrown. If sum of the numbers is an even number, what is the probability that it is perfect square?

3) A box contains 11 tickets numbered from 1 to 11. Two tickets are drawn at random with replacement. If the sum is even, find the probability that both the numbers are odd.

4) A card is drawn from a well shuffled pack of 52 cards. Consider two events A and B as A: a club card is drawn. B: an ace card is drawn. Determine whether events A and B are independent or not.

5) A problem in statistics is given to three students A, B and C. Their chances of solving the problem are 1/3, 1/4 and 1/5 respectively. If all of them try independently, What is the probability that,

   a) problem is solved?

   b) problem is not solved?

   c) exactly two students solve the problem?

6) The probability that a 50-year old man will be alive till age 60 is 0.83 and the probability that a 45-year-old woman will be alive till age 55 is 0.97. What is the probability that a man whose age is 50 and his wife whose age is 45 will both be alive for next 10 years?

7) In an examination, 30% of the students have failed in subject I, 20% of the students have

failed in subject II and 10% have failed in both subject I and subject II. A student is selected at random, what is the probability that the student

a) has failed in subject I, if it is known that he is failed in subject II?

b) has failed in at least one subject?

c) has failed in exactly one subject?

8) One shot is fired from each of the three guns. Let A, B and C denote the events that the target is hit by first, second and third gun respectively. assuming that A, B and C are independent events and that P(A) = 0.5, P(B) = 0.6 and P(C) = 0.8,then find the probability that at least one hit is registered.

9) A bag contains 10 white balls and 15 black balls. Two balls are drawn in succession without replacement. What is the probability that,

a) first is white and second is black?

b) one is white and other is black?

10) An urn contains 4 black, 5 white and 6 red balls. Two balls are drawn one after the other without replacement, what is the probability that at least one ball is black?

11) Two balls are drawn from an urn containing 5 green, 3 blue and 7 yellow balls one by one without replacement. What is the probability that at least one ball is blue?

12) A bag contains 4 blue and 5 green balls. Another bag contains 3 blue and 7 green balls. If one ball is drawn from each bag, what is the probability that two balls are of the same colour?

13) Two cards are drawn one after other from a pack of 52 cards with replacement. What is the probability that both the cards drawn are face cards?

Let's Remember

- **Random experiment :** A random experiment is an action in which all the possible results are known in advance and are equally likely.
- **Outcome :** The result of a random experiment is called "Outcome of experiment".
- **Equally likely outcomes :** The outcomes of random experiment are said to be equally likely, if all of them have equal chance to occur.
- **Sample space :** The set of all possible outcomes of a random experiment is called the sample space. It is denoted by S or Ω. Each element of the sample space is called a sample point. The number of elements in the sample space is denoted by n(S) or n(Ω).
- **Event :** A subset of the sample space is called an event.
- **Simple or elementary event :** A subset of sample space, consisting of a single element is called a simple or elementary event. Every sample point of the sample space S forms an elementary event.
- **Sure Event :** An event which contains all the sample points of the sample space S, is called as a sure event.
- **Impossible Event :** An event which does not contain any sample point of the sample space S is called an impossible event. Null set ϕ represent an impossible event.
- **Complementary Event :** Let A be an event defined on a sample space S. Therefore $A \subset S$. The complement of event A is the set containing all the elements in sample space S but not in A, which is denoted by A' or $\overline{A}$ or A^c.

- **Union of two events :** Let A and B be any two events defined on the sample space S. Then union of two events is denoted by $A\cup B$ and is defined as occurrence of at least one of the events. Either event A occurs or event B occurs or both occur.

- **Exhaustive Events :** Two events A and B defined on the sample space S are said to be Exhaustive Event if $A\cup B = S$

- **Intersection of two events :** Let A and B be any two events defined on the sample space S. Then intersection of two events is denoted by $A\cap B$ and is defined as simultaneous occurrence of both the events.

- **Mutually Exclusive Events :** Two events A and B defined on the sample space are said to be mutually exclusive if they cannot occur simultaneously in a single trial that is occurrence of any one event prevents the occurrence of other event or $A\cap B = \phi$. it can also be said that A and B are disjoint sets.

- If A' is the compliment of A, then $P(A') = 1 - P(A)$

- For any event A in S, we have $0 \leq P(A) \leq 1$

- If ϕ is null set (i.e. impossible event), then $P(\phi) = 0$

- If $A\subset B$, then $P(A) \leq P(B)$

- If A and B are any two events, then $P(A\cap B') = P(A) - P(A\cap B)$

- If A and B are any two events, then $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

- **Addition theorem :** Let S be a sample space associated with the given random experiment. Let A and B be any two events defined on the sample space S. Then the probability of occurrence of at least one event is denoted by $P(A\cup B)$ and given by $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

- **Conditional Probability :**

  If A and B are any two events defined on the same sample space S, then conditional probability of event A given that event B has already occurred is denoted by P(A/B)

  $\therefore \; P(A/B) = P(A\cap B) / P(B), \; P(B) \neq 0$

  Similarly,

  $P(B/A) = P(A\cap B) / P(A),$

  $P(A) \neq 0$

- **Multiplication theorem :** Let S be sample space associated with the given random experiment. Let A and B be any two events defined on the sample space S. Then the probability of occurrence of both the events is denoted by $P(A\cap B)$ and is given by

  $$P(A\cap B) = P(A)\cdot P(B/A)$$
  $$= P(B)\cdot P(A/B)$$

- **Independent Events :** Let S be sample space associated with the given random experiment. Let A and B be any two events defined on the sample space S. If the occurrence of one event does not affect the occurrence of other event, then two events A and B are said to be independent.

  i.e. if $P(A/B) = P(A/B') = P(A)$ or

  $P(B/A) = P(B/A') = P(B)$

- If A and B are independent events then $P(A\cap B) = P(A).P(B)$

- $\dfrac{P(A)}{P(A')}$ is called odds ratio in favour of A.

- If A and B are independent events then

  a) A and B' are also independent events

  b) A' and B'are also independent events.

MISCELLANEOUS EXERCISE - 7

1) From a group of 2 men (M_1, M_2) and three women (W_1, W_2, W_3), two persons are selected. Describe the sample space of the experiment. If E is the event in which one

man and one woman are selected, then which are the cases favourable to E?

2) Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys and 1 girl and three boys. One child is selected at random from each group. What is the chance that three selected consists of 1 girl and 2 boys.

3) A room has 3 sockets for lamps. From a collection of 10 light bulbs 6 are defective. A person selects 3 at random and puts them in every socket. What is the probability that the room will be lit?

4) There are 2 red and 3 black balls in a bag. 3 balls are taken out at random from the bag. Find the probability of getting 2 red and 1 black ball or 1 red and 2 black balls.

5) A box contains 25 tickets numbered 1 to 25. Two tickets are drawn at random. What is the probability that the product of the number on tickets is even?

6) A, B and C are mutually exclusive and exhaustive events associated with the random experiment. Find P(A), given that P(B) = (3/2) P(A) and P(C) = (1/2) P(B).

7) An urn contains four tickets marked with numbers 112, 121, 211, 222 and one ticket is drawn at random. Let A_i(i = 1,2,3) be the event that i^{th} digit of the number of the ticket drawn is 1. Discuss the independence of the events A_1, A_2 and A_3.

8) The odds against a certain event are 5:2 and odds in favour of another independent event are 6:5. Find the chance that at least one of the events will happen.

9) The odds against the husband who is 55 years old living till he is 75 is 8:5 and it is 4:3 against his wife who is now 48, living till she is 68. Find the probability that

   a) the couple will be alive 20 years hence

   b) at least one of them will be alive 20 years hence.

10) Two throws are made, the first with 3 dice and second with 2 dice. The faces of each die are marked with the number 1 to 6. What is the probability that the total in first throw is not less than 15 and at the same time the total in the second throw is not less than 8?

11) Two-third of the students in a class are boys and rest are girls. It is known that the probability of girl getting first class is 0.25 and that of boy getting is 0.28. Find the probability that a student chosen at random will get first class.

12) A number of two digits is formed using the digits 1, 2, 3, ..., 9. What is the probability that the number so chosen is even and less than 60?

13) A bag contains 8 red balls and 5 white balls. Two successive draws of 3 balls are made without replacement. Find the probability that the first drawing will give 3 white balls and second drawing will give 3 red balls.

14) The odds against student X solving a business statistics problem are 8:6. and odds in favour of student Y solving the same problem are 14:16

   a) What is the chance that the problem will be solved, if they try independently?

   b) What is the probability that neither solves the problem?

Activity 7.1

A bag contains 4 blue and 5 green balls. Another bag contains 3 blue and 7 green balls. A ball is drawn from each bag. Represent this data in a tree diagram. Hence find the probabilities to find

(i) The two balls are of the same colour

(ii) The two balls are of different colour

Activity 7.2

The population is categorized according to sex and employment status as follows :

	Employment	Unemployment	Total
Male	460	40	500
Female	140	260	400
Total	600	300	900

Represent this data in a tree digram. Hence find

(i) The probability that the individual is man, knowing that the individual chosen is employed.

(ii) The probability that the individual is unemployed, knowing that the individual is a female.

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