2. Mathematical Methods

Can you recall?

1. What is the difference between a scalar and a vector?
2. Which of the following are scalars or vectors?
   (i) displacement (ii) distance travelled (iii) velocity
   (iv) speed (v) force (vi) work done (vii) energy

2.1 Introduction:

You will need certain mathematical tools to understand the topics covered in this book. Vector analysis and elementary calculus are two among these. You will learn calculus in details, in mathematics, in the XIIth standard. In this Chapter, you are going to learn about vector analysis and will have a preliminary introduction to calculus which should be sufficient for you to understand the physics that you will learn in this book.

2.2 Vector Analysis:

In the previous Chapter, you have studied different aspects of physical quantities like their division into fundamental and derived quantities and their units and dimensions. You also need to understand that all physical quantities may not be fully described by their magnitudes and units alone. For example if you are given the time for which a man has walked with a certain speed, you can find the distance travelled by the man, but you cannot find out where exactly the man has reached unless you know the direction in which the man has walked.

Therefore, you can say that some physical quantities, which are called scalars, can be described with magnitude alone, whereas some other physical quantities, which are called vectors, need to be described with magnitude as well as direction. In the above example the distance travelled by the man is a scalar quantity while the final position of the man relative to his initial position, i.e., his displacement can be described by magnitude and direction and is a vector quantity. In this Chapter you will study different aspects of scalar and vector quantities.

2.2.1 Scalars:

Physical quantities which can be completely described by their magnitude are called scalars, i.e. they are specified by a number and a unit. For example when we say that a given metal rod has a length 2 m, it indicates that the rod is two times longer than a certain standard unit *metre*. Thus the number 2 is the magnitude and metre is the unit; together they give us a complete idea about the length of the rod. Thus length is a scalar quantity. Similarly mass, time, temperature, density, etc., are examples of scalars. Scalars can be added or subtracted by rules of simple algebra.

2.2.2 Vectors:

Physical quantities which need magnitude as well as direction for their complete description are called vectors. Examples of vectors are displacement, velocity, force etc.

A vector can be represented by a directed line segment or by an arrow. The length of the line segment drawn to scale gives the magnitude of the vector, e.g., displacement of a body from P to Q can be represented as P $\longrightarrow$ Q, where the starting point P is called the tail and the end point Q (arrow head) is called the head of the vector. Symbolically we write it as $\overrightarrow{PQ}$. Symbolically vectors are also represented by a single capital letter with an arrow above it, e.g., $\vec{X}$, $\vec{A}$, etc. Magnitude of a vector $\vec{X}$ is written as $|\vec{X}|$.

Let us see a few examples of different types of vectors.

(a) Zero vector (Null vector): A vector having zero magnitude with a particular direction (arbitrary) is called zero vector. Symbolically it is represented as $\vec{0}$.

(1) Velocity vector of a stationary particle is a zero vector.

(2) The acceleration vector of an object

moving with uniform velocity is a zero vector.

(b) Resultant vector: The resultant of two or more vectors is that single vector, which produces the same effect, as produced by all the vectors together.

(c) Negative vector (opposite vector): A negative vector of a given vector is a vector of the same magnitude but opposite in direction to that of the given vector.

In Fig. 2.1, $\vec{B}$ is a negative vector to $\vec{A}$.

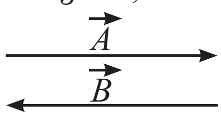


Fig. 2.1: Negative vector.

(d) Equal vector: Two vectors A and B representing same physical quantity are said to be equal if and only if they have the same magnitude and direction. Two equal vectors are shown in Fig. 2.2.

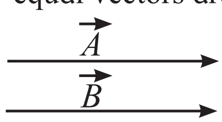


Fig. 2.2: Equal vectors.

(e) Position vector: A vector which gives the position of a particle at a point with respect to the origin of a chosen co-ordinate system is called the position vector of the particle.

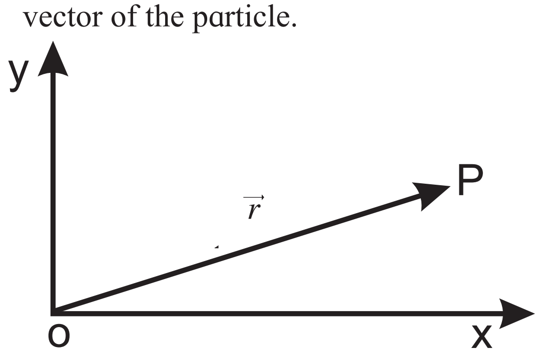


Fig 2.3: Position vector.

In Fig 2.3, $\vec{r} = \overrightarrow{OP}$ is the position vector of the particle present at P.

(f) Unit vector: A vector having unit magnitude in a given direction is called a unit vector in that direction. If $\vec{M}$ is a non-zero vector i.e. its magnitude $M = |\vec{M}|$ is not zero, the unit vector along $\vec{M}$ is written as $\hat{u}_M$ and is given by

$$\vec{M} = \hat{u}_M M \qquad \text{--- (2.1)}$$

$$or,\ \hat{u}_M = \frac{\vec{M}}{M} \qquad \text{--- (2.2)}$$

Hence $\hat{u}_M$ has magnitude unity and has the same direction as that of $\vec{M}$. We use $\hat{i}$, $\hat{j}$, and $\hat{k}$, respectively, as unit vectors along the x, y and z directions of a rectangular (three dimensional) coordinate system.

$$\hat{u}_x = \hat{i}, \hat{u}_y = \hat{j} \text{ and } \hat{u}_z = \hat{k}$$

$$\therefore \hat{i} = \frac{\vec{x}}{x}, \hat{j} = \frac{\vec{y}}{y} \text{ and } \hat{k} = \frac{\vec{z}}{z} \qquad \text{--- (2.3)}$$

Here $\vec{x}$, $\vec{y}$ and $\vec{z}$ are vectors along x, y and z axes, respectively.

2.3 Vector Operations:

2.3.1 Multiplication of a Vector by a Scalar:

Multiplying a vector $\vec{P}$ by a scalar quantity, say s, yields another vector. Let us write

$$\vec{Q} = s\vec{P} \qquad \text{--- (2.4)}$$

$\vec{Q}$ will be a vector whose direction is the same as that of $\vec{P}$ and magnitude is s times the magnitude of $\vec{P}$.

2.3.2 Addition and Subtraction of Vectors:

The addition or subtraction of two or more vectors of the same type, i.e., describing the same physical quantity, gives rise to a single vector, such that the effect of this single vector is the same as the net effect of the vectors which have been added or subtracted .

It is important to understand that only vectors of the same type (describing same physical quantity) can be added or subtracted e.g. force $\vec{F_1}$ and force $\vec{F_2}$ can be added to give the resultant force $\vec{F} = \vec{F_1} + \vec{F_2}$. But a force vector can not be added to a velocity vector.

It is easy to find addition of vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ having the same or opposite direction but different magnitudes. If individual vectors are parallel (i.e., in the same direction), the magnitude of their resultant is the addition of individual magnitudes, i.e., $|\overrightarrow{AC}| = |\overrightarrow{AB}| + |\overrightarrow{BC}|$

and direction of the resultant is the same as that of the individual vectors as shown in Fig 2.4 (a). However, if the individual vectors are anti-parallel (i.e., in the opposite direction), the magnitude of their resultant is the difference of the individual magnitudes, and the direction is that of the larger vector i.e., $\left|\overline{AC}\right| = \left\|\overline{AB}\right| - \left|\overline{BC}\right\|$ as shown in Fig. 2.4 (b).

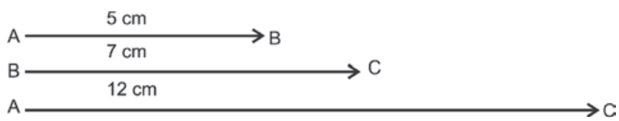


Fig. 2.4 (a): Resultant of parallel displacements.

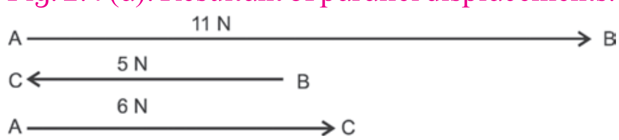


Fig 2.4 (b): Resultant of anti-parallel forces.

2.3.3 Triangle Law for Vector Addition:

When vectors of a given type do not act in the same or opposite directions, the resultant can be determined by using the triangle law of vector addition which is stated as follows:

If two vectors describing the same physical quantity are represented in magnitude and direction by the two sides of a triangle taken in order, then their resultant is represented in magnitude and direction by the third side of the triangle drawn in the opposite sense (from the starting point of first vector to the end point of the second vector).

Let $\vec{A}$ and $\vec{B}$ be two vectors in the plane of paper as shown in Fig. 2.5 (a). The sum of these two vectors can be obtained by using the triangle law described above as shown in Fig. 2.5 (b). The resultant vector is indicated by $\vec{C}$.

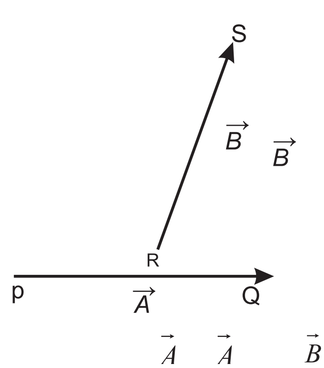


Fig. 2.5 (a): Two vectors $\vec{A}$ and $\vec{B}$ in a plane,

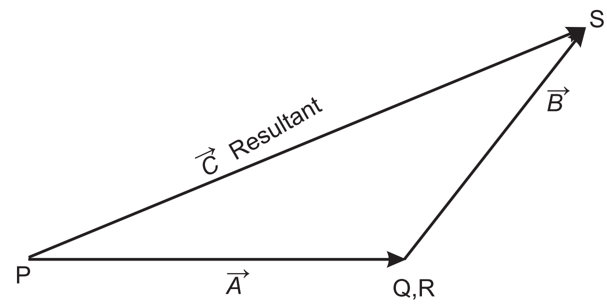


Fig. 2.5 (b): Resultant vector $\vec{C} = \vec{A} + \vec{B}$.

We can use the triangle law for showing that

(a) Vector addition is commutative.

For any two vectors $\vec{P}$ and $\vec{Q}$,

$$\vec{P} + \vec{Q} = \vec{Q} + \vec{P} \qquad \text{--- (2.5)}$$

Figure 2.6 (a) shows addition of the two vector $\vec{P}$ and $\vec{Q}$ in two different ways. Triangle OAB shows $\vec{P} + \vec{Q} = \vec{R} = \overline{OB}$, while triangle OCB shows $\vec{Q} + \vec{P} = \vec{R} = \overline{OB}$.

$\therefore \vec{P} + \vec{Q} = \vec{Q} + \vec{P}$

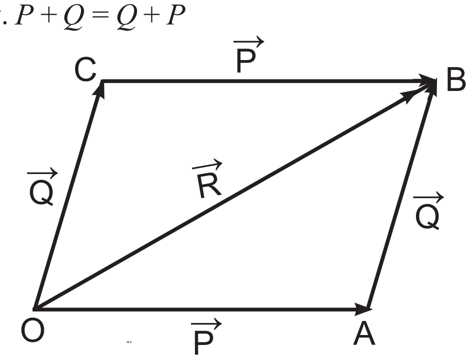


Fig. 2.6 (a): Commutative law.

(b) Vector addition is associative

If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are three vectors then

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

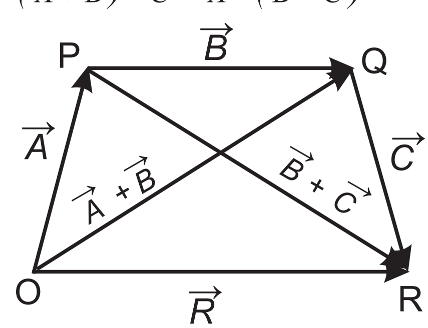


Fig. 2.6 (b): Associative law.

Figure 2.6 (b) shows addition of 3 vectors

$\vec{A}$, $\vec{B}$ and $\vec{C}$ in two different ways to give resultant $\vec{R}$.

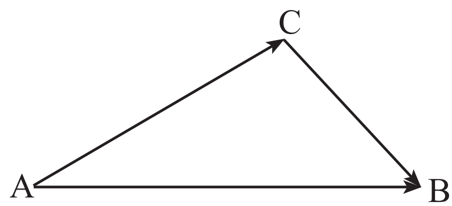
$$\vec{R} = (\vec{A}+\vec{B})+\vec{C} \quad \text{--- from triangle OQR}$$

$$\vec{R} = \vec{A}+(\vec{B}+\vec{C}) \quad \text{--- from triangle OPR}$$

i.e., $(\vec{A}+\vec{B})+\vec{C} = \vec{A}+(\vec{B}+\vec{C})$ --- (2.6)

Thus the Associative law is proved.

Example 2.1: Express vector $\overrightarrow{AC}$ in terms of vectors $\overrightarrow{AB}$ and $\overrightarrow{CB}$ shown in the following figure.



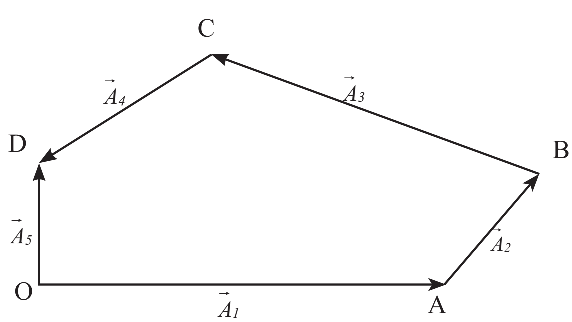
Solution: Using the triangle law of addition of vectors we can write

$$\overrightarrow{AC} + \overrightarrow{CB} = \overrightarrow{AB}$$

$$\therefore \overrightarrow{AC} = \overrightarrow{AB} - \overrightarrow{CB}$$

Example 2.2: From the following figure, determine the resultant of four forces

$\vec{A}_1, \vec{A}_2, \vec{A}_3$ and $\vec{A}_4$



Solution: Join $\overrightarrow{OB}$ to complete Δ OAB as shown in (a)

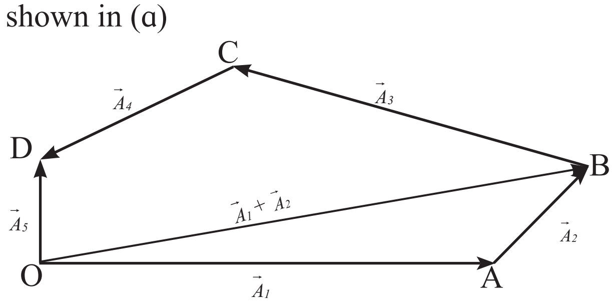


Fig. (a)

Now, $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \vec{A}_1 + \vec{A}_2$

Join $\overrightarrow{OC}$ to complete triangle OBC as shown in (b).

Now, $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \vec{A}_1 + \vec{A}_2 + \vec{A}_3$

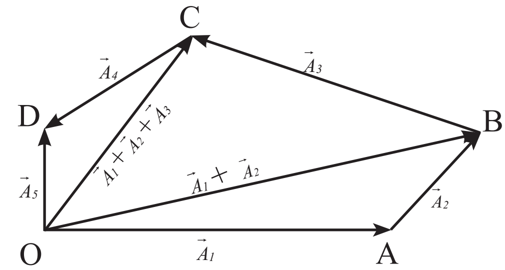


Fig. (b)

From triangle OCD,

$$\overrightarrow{OD} = \vec{A}_5 = \overrightarrow{OC} + \overrightarrow{CD} = \vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4$$

Thus $\overrightarrow{OD}$ is the resultant of the four vectors, $\vec{A}_1, \vec{A}_2, \vec{A}_3$ and $\vec{A}_4$, represented by

$\overrightarrow{OA}, \overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{CD}$, respectively.

2.3.4 Law of parallelogram of vectors:

Another geometrical method of adding two vectors is called parallelogram law of vector addition which is stated as follows:

If two vectors of the same type, originating from the same point (tails at the same point) are represented in magnitude and direction by two adjacent sides of a parallelogram, their resultant vector is given in magnitude and direction by the diagonal of the parallelogram starting from the same point as shown in Fig. 2.7.

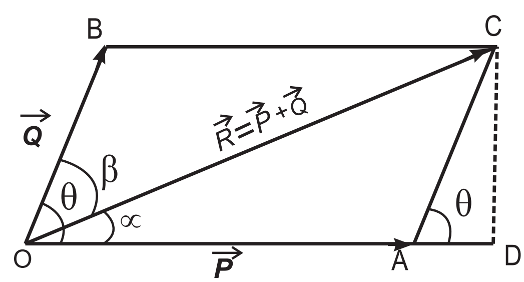


Fig 2.7: Parallelogram law of vector addition.

In Fig. 2.7, vector $\overrightarrow{OA} = \vec{P}$ and vector $\overrightarrow{OB} = \vec{Q}$, represent two vectors originating from point O, inclined to each other at an angle θ. If we complete the parallelogram, then according to this law, the diagonal $\overrightarrow{OC} = \vec{R}$ represents the resultant vector.

To find the magnitude of $\vec{R}$, drop a perpendicular from C to reach OA (extended) at D. In right angled triangle ODC, by application by Pythagoras theorem,

$$OC^2 = OD^2+DC^2$$

$$= (OA+AD)^2 + DC^2$$

$$OC^2 = OA^2+2OA\cdot AD+AD^2+DC^2$$

In the right angled triangle ADC, by application of Pythagoras theorem

$$AD^2+DC^2=AC^2$$

$$\therefore OC^2=OA^2+2OA.\ AD+ AC^2 \quad \text{--- (2.7)}$$

Also,

$$\overline{OA} = \vec{P},\ \overline{AC} = \overline{OB} = \vec{Q} \text{ and } \overline{OC} = \vec{R}$$

In Δ ADC, $\cos\theta = AD/AC$

$\therefore$ $AD=AC\cos\theta = Q\cos\theta$

Substituting in Eq. (2.7)

$$R^2 = P^2+Q^2+2\,P\,Q\cos\theta.$$

$$\therefore R = \sqrt{P^2+Q^2+2\,P\,Q\cos\theta} \quad \text{--- (2.8)}$$

Equation (2.8) gives us the magnitude of resultant vector $\vec{R}$.

To find the direction of the resultant vector $\vec{R}$, we will have to find the angle (α) made by $\vec{R}$ with $\vec{P}$.

In ΔODC, $\tan\alpha = \dfrac{DC}{OD}$

$$= \frac{DC}{OA+AD} \quad \text{--- (2.9)}$$

From the figure, $\sin\theta = \dfrac{DC}{AC}$

$$\therefore DC = AC\sin\theta = Q\sin\theta$$

Also,

$AD = AC\cos\theta = Q\cos\theta$

and $OA = \vec{P}$,

Substituing in Eq. (2.9), we get

$$\tan\alpha = \frac{Q\sin\theta}{P+Q\cos\theta}$$

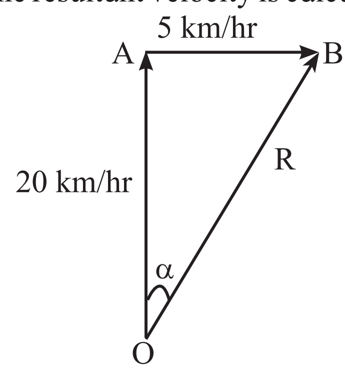
$$\therefore \alpha = \tan^{-1}\left(\frac{Q\sin\theta}{P+Q\cos\theta}\right) \quad \text{--- (2.10)}$$

Equation (2.10) gives us the direction of resultant vector $\vec{R}$.

If β is the angle between $\vec{R}$ and $\vec{Q}$, it can be similarly derived that $\beta = \tan^{-1}\left(\dfrac{P\sin\theta}{Q+P\cos\theta}\right)$

Example 2.3: Water is flowing in a stream with velocity 5 km/hr in an easterly direction relative to the shore. Speed of a boat is relative to still water is 20 km/hr. If the boat enters the stream heading North, with what velocity will the boat actually travel?

Solution: The resultant velocity $\vec{R}$ of the boat can be obtained by adding the two velocities using Δ OAB shown in the figure. Magnitude of the resultant velocity is calculated as follows:



$$R = \sqrt{20^2+5^2}$$

$$= \sqrt{425} = 20.61\ \text{km/hr}$$

The direction of the resultant velocity is

$$= \tan^{-1}\left(\frac{5}{20}\right) = \tan^{-1}(0.25)$$

$$= 14^0 04'$$

$\therefore$ The velocity of the boat is 20.61 km/hr in a direction $14^0 04'$ east of north.

2.4 Resolution of vectors:

A vector can be written as a sum of two or more vectors along certain fixed directions. Thus a vector $\vec{V}$ can be written as

$$\vec{V} = V_1\hat{\alpha} + V_2\hat{\beta} + V_3\hat{\gamma} \quad \text{--- (2.11)}$$

where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ are unit vectors along chosen directions. V_1, V_2 and V_3 are known as components of $\vec{V}$ along the three directions $\hat{\alpha}, \hat{\beta}$ and $\hat{\gamma}$.

The process of splitting a given vector into its components is called resolution of the vector. The components can be found along directions at any required angles, but if these components are found along the directions which are mutually perpendicular, they are called rectangular components.

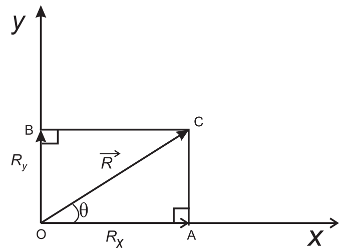


Fig. 2.8 : Resolution of a vector.

Let us see how to find rectangular components in two dimensions.

Consider a vector $\vec{R} = \overrightarrow{OC}$, originating from the origin of a rectangular co-ordinate system as shown in Fig. 2.8.

Drop perpendiculars from C that meet the x-axis at A and y-axis of at B.

$\overrightarrow{OA} = \vec{R}_x$ and $\overrightarrow{OB} = \vec{R}_y$; $\vec{R}_x$ and $\vec{R}_y$ being the components of $\overrightarrow{OC}$ along the x and y axes, respectively.

Then by the law of parallelogram of vectors,

$$\vec{R} = \vec{R}_x + \vec{R}_y$$

$$\vec{R} = R_x\hat{i} + R_y\hat{j}$$

where $\hat{i}$ and $\hat{j}$ are unit vectors along the x and y axes respectively, and R_x and R_y are the magnitudes of the two components of $\vec{R}$.

Let θ be the angle made by $\vec{R}$ with the x-axis, then

$$\cos\theta = \frac{R_x}{R}$$

$$\therefore R_x = R\cos\theta \qquad \text{--- (2.12)}$$

$$\sin\theta = \frac{R_y}{R}$$

$$\therefore R_y = R\sin\theta \qquad \text{--- (2.13)}$$

Squaring and adding Eqs. (2.12) and (2.13), we get

$$R^2\cos^2\theta + R^2\sin^2\theta = R_x^{\,2} + R_y^{\,2}$$

$$\therefore R^2 = R_x^{\,2} + R_y^{\,2}$$

$$\text{or, } R = \sqrt{R_x^{\,2} + R_y^{\,2}} \qquad \text{--- (2.14)}$$

Equation (2.14) gives the magnitude of $\vec{R}$. To find the direction of $\vec{R}$, from Fig. 2.8,

$$\tan\theta = \frac{R_y}{R_x}$$

$$\therefore \theta = \tan^{-1}\left(\frac{R_y}{R_x}\right) \qquad \text{--- (2.15)}$$

Similarly, if $\vec{R}_x$, $\vec{R}_y$ and $\vec{R}_z$ are the rectangular components of $\vec{R}$ along the x, y and z axes of the rectangular Cartesian co-ordinate system in three dimensions, then

$$\vec{R} = \vec{R}_x + \vec{R}_y + \vec{R}_z = R_x\hat{i} + R_y\hat{j} + R_z\hat{k}$$

$$\text{or, } |\vec{R}| = \sqrt{R_x^{\,2} + R_y^{\,2} + R_z^{\,2}} \qquad \text{---- (2.16)}$$

If two vectors are equal, it means that their corresponding components are also equal and vice versa.

If $\vec{A} = \vec{B}$

i.e., if $A_x\hat{i} + A_y\hat{j} + A_z\hat{k} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$, then

$$A_x = B_x,\ A_y = B_y \text{ and } A_z = B_z$$

Example 2.4: Find a unit vector in the direction of the vector $3\hat{i} + 4\hat{j}$

Solution:

Let $\vec{V} = 3\hat{i} + 4\hat{j}$

Magnitude of $\vec{V} = |\vec{V}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$

$\vec{V} = \hat{\alpha}\,|\vec{V}|$, where $\hat{\alpha}$ is a unit vector along $\vec{V}$.

$$\hat{\alpha} = \frac{\vec{V}}{|\vec{V}|} = \frac{3}{5}\hat{i} + \frac{4}{5}\hat{j}$$

Example 2.5: Given $\vec{a} = \hat{i} + 2\hat{j}$ and $\vec{b} = 2\hat{i} + \hat{j}$, what are the magnitudes of the two vectors? Are these two vectors equal?

Solution:

$$|\vec{a}| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$|\vec{b}| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

The magnitudes of $\vec{a}$ and $\vec{b}$ are equal. However, their corresponding components are not equal i.e., $a_x \neq b_x$ and $a_y \neq b_y$. Hence, the two vectors are not equal.

2.5 Multiplication of Vectors:

We saw that we can add or subtract vectors of the same type to get resultant vectors of the same type. However, when we multiply vectors of the same or different types, we get a new physical quantity which may either be a scalar (scalar product) or a vector (vector product). Also note that the multiplication of a scalar with a scalar is always a scalar and the multiplication of scalar with a vector is always a vector. Let us now study the characteristics of a scalar product and vector product of two vectors.

2.5.1 Scalar Product (Dot Product):

The scalar product or dot product of two nonzero vectors $\vec{P}$ and $\vec{Q}$ is defined as the product of magnitudes of the two vectors and the cosine of the angle θ between the two vectors. The scalar product of $\vec{P}$ and $\vec{Q}$ is written as,

$$\vec{P}\cdot\vec{Q} = PQ\cos\theta, \qquad \text{--- (2.17)}$$

where θ is the angle between $\vec{P}$ and $\vec{Q}$.

Characteristics of scalar product

(1) The scalar product of two vectors is equivalent to the product of magnitude of one vector with the magnitude of the component of the other vector in the direction of the first.

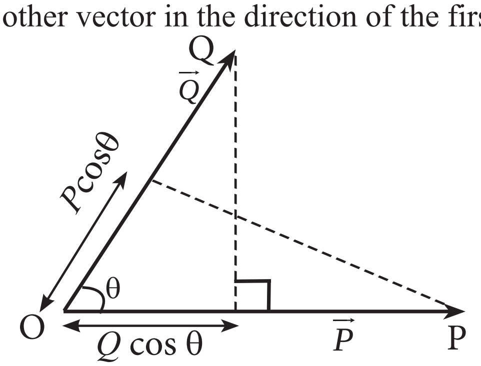


Fig. 2.9: Projection of vectors.

From Fig. 2.9,

$\vec{P}\cdot\vec{Q} = PQ\cos\theta$

$= P\,(Q\cos\theta)$

$= P$ (component of $\vec{Q}$ in the direction of $\vec{P}$)

Similarly $\vec{P}\cdot\vec{Q} = Q\,(P\cos\theta)$

$= Q$ (component of $\vec{P}$ in the direction of $\vec{Q}$)

(2) Scalar product obeys the commutative law of vector multiplication.

$$\vec{P}\cdot\vec{Q} = P\,Q\cos\theta = Q\,P\cos\theta = \vec{Q}\cdot\vec{P}$$

(3) Scalar product obeys the distributive law of multiplication

$$\vec{P}\cdot(\vec{Q}+\vec{R}) = \vec{P}\cdot\vec{Q}+\vec{P}\cdot\vec{R}$$

(4) Special cases of scalar product $\vec{P}\cdot\vec{Q} = P\,Q\cos\theta$

(i) If θ = 0, i.e., the two vectors $\vec{P}$ and $\vec{Q}$ are parallel to each other, then

$$\vec{P}\cdot\vec{Q} = P\,Q\cos\theta = P\,Q$$

Thus, $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

Do you know ?

Scalar and vector products are very useful in physics. They make mathematical formulae and their derivation very elegant.

Figure below shows a toy car pulled through a displacement $\vec{S}$. The force $\vec{F}$ responsible for this is not in the direction of $\vec{S}$ but is at an angle θ to it. Component of displacement along the direction of force $\vec{F}$ is $S\cos\theta$. According to the definition, the work done by a force is the product of the force and the displacement in the direction of force. $\therefore W = FS\cos\theta$. According to the definition of scalar product,

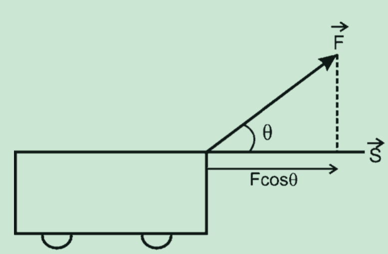
$$\vec{F}\cdot\vec{S} = F\,S\cos\theta$$

$$\therefore W = \vec{F}\cdot\vec{S}$$

Also $W = F\,(S\cos\theta) = (F\cos\theta)\,S$

Hence dot or scalar product is the product of magnitude of one of the vectors and component of the other vector in the direction of the first.

Power is the rate of doing work on a body by an external force $\vec{F}$ assumed to be constant in time. If $\vec{v}$ is the velocity of the body under the action of the force then power P is given by the scalar product of $\vec{F}$ and $\vec{v}$ i.e., $P = \vec{F}\cdot\vec{v}$.



(ii) If $\theta = 180°$, i.e., the two vectors $\vec{P}$ and $\vec{Q}$ are anti-parallel, then

$$\vec{P}\cdot\vec{Q} = P\,Q\cos 180° = -P\,Q$$

(iii) If $\theta = 90°$, i.e., the two vectors are perpendicular to each other, then

$$\vec{P}\cdot\vec{Q} = P\,Q\cos 90° = 0$$

Thus, $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

(5) If $\vec{P} = \vec{Q}$ then $\vec{P}\,.\vec{Q} = P^2 = Q^2$

(6) Scalar product of vectors expressed in terms of rectangular components :

Let $\vec{P} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}$

and $\vec{Q} = Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k}$

Then $\vec{P}\cdot\vec{Q} = P_x\,Q_x + P_y\,Q_y + P_z\,Q_z$

Proof :

$$\vec{P}\cdot\vec{Q} = (P_x\hat{i} + P_y\hat{j} + P_z\hat{k})\cdot(Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k})$$

$$= P_x\hat{i}\cdot(Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k})$$

$$+P_y\hat{j}\cdot(Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k})$$

$$+P_z\hat{k}\cdot(Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k})$$

$$= (\hat{i}\cdot\hat{i})\,P_xQ_x + (\hat{i}\cdot\hat{j})\,P_xQ_y + (\hat{i}\cdot\hat{k})P_xQ_z$$

$$+(\hat{j}\cdot\hat{i})\,P_yQ_x + (\hat{j}\cdot\hat{j})\,P_yQ_y + (\hat{j}\cdot\hat{k})P_yQ_z$$

$$+(\hat{k}\cdot\hat{i})\,P_zQ_x + (\hat{k}\cdot\hat{j})\,P_zQ_y + (\hat{k}\cdot\hat{k})\,P_zQ_z$$

Since, $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

and $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{j} = 0$

$$\therefore \vec{P}\cdot\vec{Q} = P_xQ_x + 0 + 0$$

$$+ 0 + P_yQ_y + 0$$

$$+ 0 + 0 + P_zQ_z$$

$$\therefore \vec{P}\cdot\vec{Q} = P_xQ_x + P_yQ_y + P_zQ_z$$

(7) If $\vec{a}\cdot\vec{b} = \vec{a}\cdot\vec{c}$, where $\vec{a} \neq 0$, it is not necessary that $\vec{b} = \vec{c}$. Using the distributive law, we can write $\vec{a}\cdot\left(\vec{b} - \vec{c}\right) = 0$. It implies that either $\vec{b} - \vec{c}$ = 0 or $\vec{a}$ is perpendicular to $\vec{b} - \vec{c}$. It does not necessarily imply that $\vec{b} - \vec{c} = 0$

Example 2.6: Find the scalar product of the two vectors

$$\vec{v}_1 = \hat{i} + 2\hat{j} + 3\hat{k} \text{ and } \vec{v}_2 = 3\hat{i} + 4\hat{j} - 5\hat{k}$$

Solution:

$$\vec{v}_1\cdot\vec{v}_2 = (\hat{i} + 2\hat{j} + 3\hat{k})\cdot(3\hat{i} + 4\hat{j} - 5\hat{k})$$

$$= 1\times 3 + 2\times 4 + 3\times(-5)$$

$$= -4$$

as $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1,$

and $\hat{i}\cdot\hat{j} = \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = \hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{i} = \hat{k}\cdot\hat{j} = 0$

2.5.2 Vector Product (cross product):

The vector product or cross product of two vectors ($\vec{P}$ and $\vec{Q}$) is a vector whose magnitude is equal to the product of magnitudes of the two vectors and sine of the smaller angle (θ) between the two vectors. The direction of the product vector is given by $\hat{u}_r$ which is a unit vector perpendicular to the plane containing the two vectors and is given by the right hand screw rule. This is shown in Fig. 2.10 (a) and (b)

a) $\vec{R} = \vec{P}\times\vec{Q} = PQ\sin\theta\,\hat{u}_r$ --- (2.18)

b) $\vec{S} = \vec{Q}\times\vec{P} = PQ\sin\theta\,\hat{u}_s$ --- (2.19)

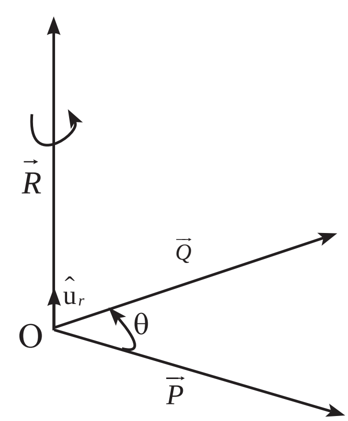


Fig. 2.10 (a): Vector product $\vec{R} = \vec{P}\times\vec{Q}$.

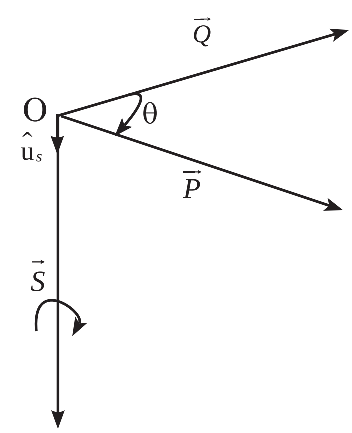


Fig. 2.10 (b): Vector product $\vec{S} = \vec{Q}\times\vec{P}$.

According to the right hand screw rule, if the screw is rotated in a direction from $\vec{P}$ to $\vec{Q}$ through the smaller angle, then the direction in which the tip of the screw advances is the direction of $\vec{R}$, perpendicular to the plane containing $\vec{P}$ and $\vec{Q}$. One example of vector or cross product is the force $\vec{F}$ experienced by a charge q moving with velocity $\vec{v}$ through a uniform magnetic field of magnetic induction $\vec{B}$. It is an empirical law (experimentally determined) given by $\vec{F} = q\vec{v} \times \vec{B}$.

Do you know ?

1. As linear displacement $\vec{x}$ is the distance travelled by a body along the line of travel, angular displacement $\vec{\theta}$ is the angle swept by a body about a given axis. The rate of change of angular displacement is the angular velocity denoted by $\vec{\omega}$. If a body is rotating about as axis, it possesses an angular velocity $\vec{\omega}$. If at a point at a distance $\vec{r}$ from the axis of rotation the body has linear velocity $\vec{v}$, then $\vec{v} = \vec{\omega} \times \vec{r}$.
2. An external force is needed to move a body from one point to other. Similarly to rotate a body about an axis passing through it, torque is required. Torque is a vector with its direction along the axis of rotation and magnitude describing the turning effect of force $\vec{F}$ acting on the body to rotate it about the given axis. Torque $\vec{\tau}$ is given as $\vec{\tau} = \vec{r} \times \vec{F}$, $\vec{r}$ being the perpendicular distance of a point on the body where the force is applied from the axis of rotation.

Characteristics of Vector Product:

(1) Vector product does not obey commutative law of multiplication.

$$\vec{P} \times \vec{Q} \neq \vec{Q} \times \vec{P} \qquad \text{--- (2.20)}$$

However, $|\vec{P} \times \vec{Q}| = |\vec{Q} \times \vec{P}|$ i.e., the magnitudes are the same but the directions are opposite to each other.

(2) The vector product obeys the distributive law of multiplication.

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \qquad \text{--- (2.21)}$$

(3) Special cases of cross product

$$|\vec{P} \times \vec{Q}| = P\,Q \sin\theta \qquad \text{--- (2.22)}$$

(i) If $\theta = 0$. i.e., if the two nonzero vectors are parallel to each other, their vector product is a zero vector $|\vec{P} \times \vec{Q}| = P\,Q \cdot 0 = 0$

(ii) If $\theta = 180°$, i.e., if the two nonzero vectors are anti-parallel, their vector product is a zero vector $|\vec{P} \times \vec{Q}| = P\,Q \sin 180° = P\,Q \sin \pi = 0$

(iii) If $\theta = 90°$, i.e., if the two nonzero vectors are perpendicular to each other, the magnitude of their vector product is equal to the product of magnitudes of the two vectors.

$$|\vec{P} \times \vec{Q}| = P\,Q \sin 90° = P\,Q$$

Thus $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$ and $\hat{k} \times \hat{i} = \hat{j}$

(4) If $\vec{P} = \vec{Q}$ then $|\vec{P} \times \vec{Q}| = |\vec{P} \times \vec{P}| = |\vec{Q} \times \vec{Q}| = 0$.

Thus $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$

(5) Let $\vec{P} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}$

and $\vec{Q} = Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k}$

$$\vec{P} \times \vec{Q} = \left(P_x\hat{i} + P_y\hat{j} + P_z\hat{k}\right) \times \left(Q_x\hat{i} + Q_y\hat{j} + Q_z\hat{k}\right)$$

$$= P_xQ_x\left(\hat{i} \times \hat{i}\right) + P_xQ_y\left(\hat{i} \times \hat{j}\right) + P_xQ_z\left(\hat{i} \times \hat{k}\right)$$

$$+ P_yQ_x\left(\hat{j} \times \hat{i}\right) + P_yQ_y\left(\hat{j} \times \hat{j}\right) + P_yQ_z\left(\hat{j} \times \hat{k}\right)$$

$$+ P_zQ_x\left(\hat{k} \times \hat{i}\right) + P_zQ_y\left(\hat{k} \times \hat{j}\right) + P_zQ_z\left(\hat{k} \times \hat{k}\right)$$

Now $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$, and

$$\hat{i} \times \hat{k} = -\hat{j},\ \hat{j} \times \hat{i} = -\hat{k},\ \hat{k} \times \hat{j} = -\hat{i}$$

$$\hat{i} \times \hat{j} = \hat{k},\ \hat{j} \times \hat{k} = \hat{i},\ \hat{k} \times \hat{i} = \hat{j}.$$

$$\therefore \vec{P} \times \vec{Q} = 0 + P_xQ_y\hat{k} - P_xQ_z\hat{j}$$

$$- P_yQ_x\hat{k} + 0 + P_yQ_z\hat{i}$$

$$+ P_zQ_x\hat{j} - P_zQ_y\hat{i} + 0$$

$$= (P_yQ_z - P_zQ_y)\,\hat{i}$$

$$+ (P_zQ_x - P_xQ_z)\,\hat{j}$$

$$+ (P_xQ_y - P_yQ_x)\,\hat{k}$$

This can be written in a determinant form as

$$\vec{P}\times\vec{Q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ P_x & P_y & P_z \\ Q_x & Q_y & Q_z \end{vmatrix} \quad \text{--- (2.23)}$$

(6) The magnitude of cross product of two vectors is numerically equal to the area of a parallelogram whose adjacent sides represent the two vectors.

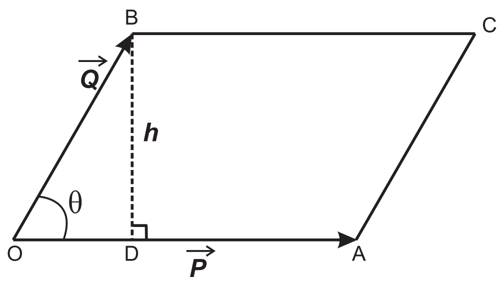


Fig 2.11: Area of parallelogram and vector product.

As shown in fig. 2.11,

$\vec{P} = \overrightarrow{OA}$, $\vec{Q} = \overrightarrow{OB}$, $\vec{P}$ and $\vec{Q}$ are inclined at an angle θ.

Perpendicular BD, of length *h* drawn on OA, gives the height of the parallelogram with OA as base.

Area of parallelogram

= base × height

$= \text{OA} \times \text{BD}, \text{ as } \sin\theta = \dfrac{\text{BD}}{\text{OB}}$

$= P\,Q\sin\theta$

$= \left|\vec{P}\times\vec{Q}\right|$

= magnitude of the vector product --- (2.24)

Example 2.7: The angular momentum $\vec{L} = \vec{r}\times\vec{p}$, where $\vec{r}$ is a position vector and $\vec{p}$ is linear momentum of a body.

If $\vec{r} = 4\hat{i}\times 6\hat{j} - 3\hat{k}$ and $\vec{p} = 2\hat{i} + 4\hat{j} - 5\hat{k}$, find $\vec{L}$

Solution:

$$\vec{L} = \vec{r}\times\vec{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 6 & -3 \\ 2 & 4 & -5 \end{vmatrix}$$

$$\therefore \vec{L} = (-30 + 12)\hat{i} + (-6 + 20)\hat{j} + (16 - 12)\vec{k}$$

$$= -18\hat{i} + 14\hat{j} + 4\hat{k}.$$

Example 2.8: If $\vec{A} = 5\hat{i} + 6\hat{j} + 4\hat{k}$ and $\vec{B} = 2\hat{i} - 2\hat{j} + 3\hat{k}$, determine the angle between $\vec{A}$ and $\vec{B}$.

Solution: $\vec{A}\cdot\vec{B} = A\,B\cos\theta = A_xB_x + A_yB_y + A_zB_z$

$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{A\,B}$$

$$\cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}\sqrt{B_x^2 + B_y^2 + B_z^2}}$$

$$\cos\theta = \frac{(5)(2) + (6)(-2) + (4)(3)}{\sqrt{25 + 36 + 16}\sqrt{4 + 4 + 9}}$$

$$= \frac{10}{\sqrt{77}.\sqrt{17}} = 0.2764$$

$\theta = \cos^{-1} 0.2765 = 73°58'$

Example 2.9: Given $\vec{P} = 4\hat{i} - \hat{j} + 8\hat{k}$ and $\vec{Q} = 2\hat{i} - m\hat{j} + 4\hat{k}$, find *m* if $\vec{P}$ and $\vec{Q}$ have the same direction.

Solution: Since $\vec{P}$ and $\vec{Q}$ have the same direction, their corresponding components must be in the same proportion, i.e.,

$$\frac{P_x}{Q_x} = \frac{P_y}{Q_y} = \frac{P_z}{Q_z}$$

$$\frac{4}{2} = \frac{-1}{-m} = \frac{8}{4}$$

$$\therefore m = \frac{1}{2}$$

2.6 Introduction to Calculus:

Calculus is the study of continuous (not discrete) changes in mathematical quantities. This branch of mathematics was first developed by G.W Leibnitz and Sir Issac Newton in the 17th century and is extensively used in several branches of science. You will study calculus in mathematics in XIIth standard. Here we will learn the basics of the two branches of calculus namely differential and integral calculus. These are necessary to understand the topics covered in this book.

2.6.1 Differential Calculus:

Let us consider a function $y = f(x)$. Here x is called an independent variable and $f(x)$ gives the value of y for different values of x and is the dependent variable. For example x could be the position of a particle moving along x-axis and $y = f(x)$ could be its velocity at that position x. We can thus draw a graph of y against x as shown in Fig. 2.12 (a). Let A and B be two points on the curve giving values of y at $x = x_0$ and $x = x_0 + \Delta x$, where Δx is a small increment in x. The slope of the straight line joining A and B is given by $\tan\theta = \dfrac{\Delta y}{\Delta x}$.

If we make Δx smaller, the point B will come closer to A and if we keep making Δx smaller and smaller, we will ultimately reach a stage when B will coincide with A. This process is called taking the limit Δx going to zero and is written as $\lim\limits_{\Delta x\to 0}$. In this limit the line AB extended on both sides to P and Q will become the tangent to the curve at A, i.e., at

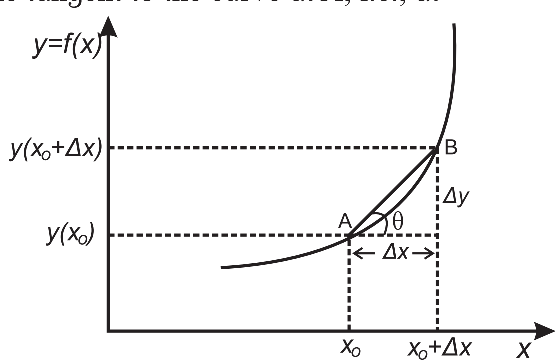


Fig. 2.12 (a): Average rate of change of y with respect to x.

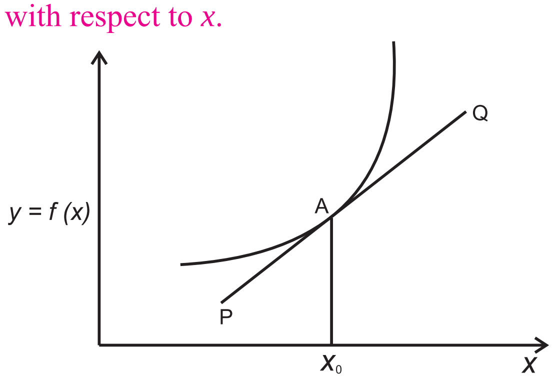


Fig. 2.12 (b): Rate of change of y with respect to x at x_0

$x = x_0$. In this limit both Δx and Δy will go to zero. However, when two quantities tend to zero, their ratio need not go to zero. In fact $\lim\limits_{\Delta x\to 0}\left(\dfrac{\Delta y}{\Delta x}\right)$ becomes the slope of the tangent shown by PQ in Fig. 2.12 (b). This is written as dy/dx at $x = x_0$.

Thus,

$$\left.\frac{dy}{dx}\right|_{x_0} = \lim_{\Delta x\to 0}\frac{(y+\Delta y)-y}{\Delta x}$$

$$\left.\frac{df(x)}{dx}\right|_{x_0} = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

We can drop the subscript zero and write a general formula which will be valid for all values of x as

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{df(x)}{dx} \quad \text{--- (2.25)}$$

In XII[th] standard you will learn about the properties of derivatives and how to find derivatives of different functions. Here we will just list the properties as we will need them in later Chapter s. dy/dx is called the derivative of y with respect to x (which is the rate of change of y with respect to change in x) and the process of finding the derivative is called differentiation. Let $f_1(x)$ and $f_2(x)$ be two different functions of x and let s be a constant. Some of the properties of differentiation are

1. $$\frac{d(sf(x))}{dx} = s\frac{df(x)}{dx} \quad \text{--- (2.26)}$$

2. $$\frac{d}{dx}(f_1(x)+f_2(x)) = \frac{df_1(x)}{dx}+\frac{df_2(x)}{dx} \quad \text{--- (2.27)}$$

3. $$\frac{d}{dx}(f_1(x)\times f_2(x)) = f_1(x)\frac{df_2(x)}{dx}+f_2(x)\frac{df_1(x)}{dx} \quad \text{--- (2.28)}$$

4. $$\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) = \frac{1}{f_2(x)}\frac{df_1(x)}{dx} - \frac{f_1(x)}{f_2^{\,2}(x)}\frac{df_2(x)}{dx} \quad \text{--- (2.29)}$$

5. If x depends on time another variable t then,

$$\frac{df(x)}{dt} = \frac{df(x)}{dx}\frac{dx}{dt} \quad \text{--- (2.30)}$$

6.

$$\frac{d}{dx}f\left(g[x]\right) = f'\left(g(x)\right)\times g'(x)$$

where $f'\left(g(x)\right) = \dfrac{df}{dg}$

or $\dfrac{dy}{dx} = \dfrac{dy}{dv}\dfrac{dv}{dx}$

The derivatives of some simple functions of x are given below.

1. $\dfrac{d}{dx}(x^n) = n\,x^{n-1}$ --- (2.31)

2. $\dfrac{d(e^x)}{dx} = e^x$ and $\dfrac{d(e^{ax})}{dx} = ae^{ax}$ --- (2.32)

3. $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$ --- (2.33)

4. $\dfrac{d}{dx}(\sin x) = \cos x$ --- (2.34)

5. $\dfrac{d}{dx}(\cos x) = -\sin x$ --- (2.35)

6. $\dfrac{d}{dx}(\tan x) = \sec^2 x$ --- (2.36)

7. $\dfrac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$ --- (2.37)

8. $\dfrac{d}{dx}(\sec x) = \tan x \sec x$ --- (2.38)

9. $\dfrac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ --- (2.39)

Example 2.10: Find the derivatives of the functions.

(a) $f(x) = x^8$ (b) $f(x) = x^3 + \sin x$

(c) $f(x) = x^3 \sin x$

Solution :

(a) Using $\dfrac{dx^n}{dx} = nx^{n-1}$,

$$\frac{d(x^8)}{dx} = 8x^7$$

(b) Using

$$\frac{d}{dx}(f_1(x) + f_2(x)) = \frac{df_1(x)}{dx} + \frac{df_2(x)}{dx} \quad \text{and}$$

$$\frac{d(\sin x)}{dx} = \cos x$$

$$\frac{d}{dx}(x^3 + \sin x) = \frac{d(x^3)}{dx} + \frac{d(\sin x)}{dx}$$

$$= 3x^2 + \cos x$$

c) Using

$$\frac{d}{dx}(f_1(x)\, f_2(x)) = f_1(x)\frac{df_2(x)}{dx} + \frac{df_1(x)}{dx} f_2(x)$$

and $\dfrac{d(\sin x)}{dx} = \cos x$

$$\frac{d}{dx}(x^3 \sin x) = x^3 \frac{d(\sin x)}{dx} + \frac{d(x^3)}{dx} \sin x$$

$$= x^3 \cos x + 3x^2 \sin x$$

2.6.2 Integral calculus

Integral calculus is the branch of mathematics dealing with properties of integrals and their applications. Physical interpretation of integral of a function $f(x)$, i.e., $\int f(x)dx$ is the area under the curve $f(x)$ versus x. It is the reverse process of differentiation as we will see below.

We know how to find the area of a rectangle, triangle etc. In Fig. 2.13(a) we have shown y which is a function of x, A and B being two points on it.

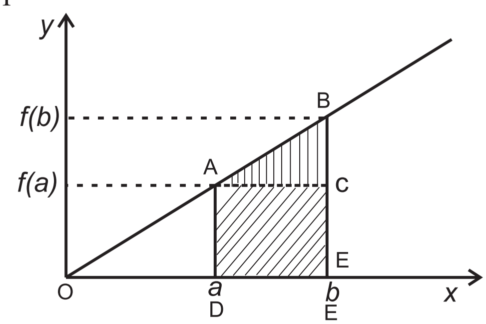


Fig. 2.13 (a): Area under a straight line.

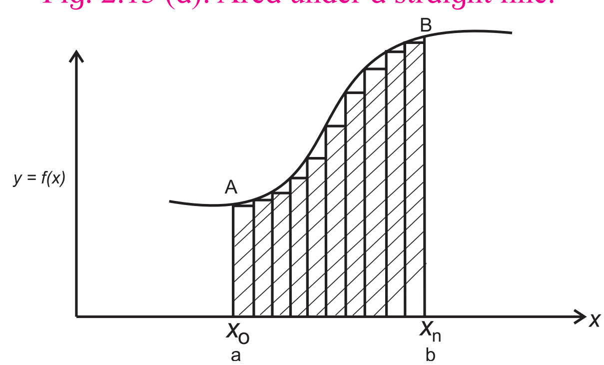


Fig. 2.13 (b): Area under a curve.

The area under the curve (straight line) from $x = a$ to $x = b$ is shown by shaded area. This can be obtained as sum of the area of the rectangle ADEC = $f(a)\,(b-a)$ and the area of the triangle ABC = 1/2 $(b-a)\,(f(b)-f(a))$

Figure 2.13(b) shows another function of x. We do not have a simple formula to calculate the area under this curve. For this calculation, we use a simple trick. We divide the area into a large number of vertical strips as shown in the figure. We assume thickness (width) of each strip to be so small that it can be assumed to be a rectangle as shown in the figure and add the areas of these rectangles. Thus the area under the curve is given by

Area under the curve

$$= \sum_{i=1}^{n} \Delta A_i = \sum_{i=1}^{n} (x_i - x_{i-1})\, f(x_i)$$

where n is the number of strips and ΔA_i is the area of the i[th] strip.

As the strips are not really rectangles, the area calculated above is not exactly equal to the area under the curve. However as we increase n, the sum of areas of rectangles gets closer to the actual area under the curve and becomes equal to it in the limit $n \to \infty$. Thus we can write,

Area under the curve

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (x_i - x_{i-1})\, f(x_i) \quad \text{--- (2.40)}$$

Integration helps us in getting exact area if the change is really continuous, i.e., n is really infinite. It is represented as $\int_{x=a}^{x=b} f(x)dx$ and is called the definite integral of $f(x)$ from $x = a$ to $x = b$.

Thus, $$\int_{x=a}^{x=b} f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} (x_i - x_{i-1})\, f(x_i) \quad \text{--- (2.41)}$$

The process of obtaining the integral is called integration. We can also write

$$F(x) = \int f(x)dx \quad \text{--- (2.42)}$$

$F(x)$ is called the indefinite (without any limits on x) integral of $f(x)$. Differentiation is the reverse process to that of integration. Therefore,

$$f(x) = \frac{d}{dx}(F(x)) \quad \text{--- (2.43)}$$

$$\therefore F(x)\Big|_a^b = F(b) - F(a) = \int_a^b f(x)dx \quad \text{--- (2.44)}$$

Properties of integration

1. $\int \left(f_1(x) + f_2(x)\right) dx = \int f_1(x)dx + \int f_2(x)dx$ --- (2.45)

2. $\int K\, f(x)dx = K \int f(x)dx$ for K = constant --- (2.46)

Indefinite integrals of some basic functions are given below. Their definite integrals can be obtained by using the Eq. (2.44)

1. $\int x^n dx = \frac{x^{n+1}}{n+1}$ --- (2.47)
2. $\int \frac{1}{x} dx = \ln x$ --- (2.48)
3. $\int \sin x\, dx = -\cos x$ --- (2.49)
4. $\int \cos x\, dx = \sin x$ --- (2.50)
5. $\int e^x\, dx = e^x$ --- (2.51)

Example 2.11: Evaluate the following integrals:

(a) $\int x^8 dx$

(b) $\int_2^5 x^2 dx$

(c) $\int (x + \sin x)\, dx$

Solution: (a) Using formula

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \quad \int x^8 dx = \frac{x^9}{9}$$

(b) Using Eq. (2.44),

$$\int_2^5 x^2 dx = \frac{x^3}{3}\Big|_2^5 = \frac{5^3}{3} - \frac{2^3}{3} = \frac{125-8}{3} = \frac{117}{3}$$

(c) Using Eq. (2.45),

$$\int \left(f_1(x) + f_2(x)\right) dx = \int f_1(x)dx + \int f_2(x)dx$$

and $\int \sin x\, dx = \cos x$, we get $\int (x + \sin x)\, dx$

$$\int x\, dx + \int \sin x\, dx = \frac{x^2}{2} - \cos x$$

Internet my friend

1. hyperphysics.phy-astr.gsu.edu/hbase/vect.html#veccon
2. hyperphysics.phy-astr.gsu.edu/hbase/hframe.html

Exercises

1. Choose the correct option.

i) The resultant of two forces 10 N and 15 N acting along $+x$ and $-x$-axes respectively, is

(A) 25 N along $+x$-axis
(B) 25 N along $-x$-axis
(C) 5 N along $+x$-axis
(D) 5 N along $-x$-axis

ii) For two vectors to be equal, they should have the

(A) same magnitude
(B) same direction
(C) same magnitude and direction
(D) same magnitude but opposite direction

iii) The magnitude of scalar product of two unit vectors perpendicular to each other is

(A) zero (B) 1
(C) -1 (D) 2

iv) The magnitude of vector product of two unit vectors making an angle of 60° with each other is

(A) 1 (B) 2
(C) 3/2 (D) $\sqrt{3}/2$

v) If $\vec{A}, \vec{B}$ and $\vec{C}$ are three vectors, then which of the following is not correct?

(A) $\vec{A}\cdot\left(\vec{B}+\vec{C}\right)=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$

(B) $\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}$

(C) $\vec{A}\times\vec{B}=\vec{B}\times\vec{A}$

(D) $\vec{A}\times\left(\vec{B}+\vec{C}\right)=\vec{A}\times\vec{B}+\vec{B}\times\vec{C}$

2. Answer the following questions.

i) Show that $\vec{a}=\dfrac{\hat{i}-\hat{j}}{\sqrt{2}}$ is a unit vector.

ii) If $\vec{v_1}=3\hat{i}+4\hat{j}+\hat{k}$ and $\vec{v_2}=\hat{i}-\hat{j}-\hat{k}$, determine the magnitude of $\vec{v_1}+\vec{v_2}$.

[Ans: 5]

iii) For $\vec{v_1}=2\hat{i}-3\hat{j}$ and $\vec{v_2}=-6\hat{i}+5\hat{j}$, determine the magnitude and direction of $\vec{v_1}+\vec{v_2}$.

$\left[\text{Ans: } 2\sqrt{5},\ \theta=\tan^{-1}\left(-\dfrac{1}{2}\right)\text{with } x\text{-axis}\right]$

iv) Find a vector which is parallel to $\vec{v}=\hat{i}-2\hat{j}$ and has a magnitude 10.

$\left[\text{Ans: }\dfrac{10}{\sqrt{5}}\hat{i}-\dfrac{20}{\sqrt{5}}\hat{j}\right]$

v) Show that vectors $\vec{a}=2\hat{i}+5\hat{j}-6\hat{k}$ and $\vec{b}=\hat{i}+\dfrac{5}{2}\hat{j}-3\hat{k}$ are parallel.

3. Solve the following problems.

i) Determine $\vec{a}\times\vec{b}$, given $\vec{a}=2\hat{i}+3\hat{j}$ and $\vec{b}=3\hat{i}+5\hat{j}$.

$\left[\text{Ans: }\hat{k}\right]$

ii) Show that vectors $\vec{a}=2\hat{i}+3\hat{j}+6\hat{k}$, $\vec{b}=3\hat{i}-6\hat{j}+2\hat{k}$ and $\vec{c}=6\hat{i}+2\hat{j}-3\hat{k}$ are mutually perpendicular.

iii) Determine the vector product of $\vec{v_1}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{v_2}=\hat{i}+2\hat{j}-3\hat{k}$,

$\left[\text{Ans: }-7\hat{i}+5\hat{j}+\hat{k}\right]$

iv) Given $\vec{v_1}=5\hat{i}+2\hat{j}$ and $\vec{v_2}=a\hat{i}-6\hat{j}$ are perpendicular to each other, determine the value of a.

$\left[\text{Ans: }\dfrac{12}{5}\right]$

v) Obtain derivatives of the following functions:

(i) $x\sin x$ (ii) $x^4+\cos x$
(iii) $x/\sin x$

$\left[\text{Ans: (i) }\sin x+x\cos x,\ \text{(ii) }4x^3-\sin x,\ \text{(iii) }\dfrac{1}{\sin x}-\dfrac{x\cos x}{\sin^2 x}\right]$

vi) Using the rule for differentiation for quotient of two functions, prove that

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)=\sec^2 x$$

vii) Evaluate the following integral:

(i) $\int_0^{\pi/2}\sin x\,dx$ (ii) $\int_1^5 x\,dx$

$\left[\text{Ans: (i) }1,\text{(ii) }12\right]$

***