5. Gravitation

Can you recall?

1. When released from certain height why do objects tend to fall vertically downwards?
2. What is the shape of the orbits of planets? 3. What are Kepler's laws?

5.1 Introduction:

All material objects have a natural tendency to get attracted towards the Earth. In many natural phenomena like coconut falling from trees, raindrops falling from the clouds, etc., the same tendency is observed. All bodies are attracted towards the Earth with constant acceleration. This fact was recognized by Italian physicist Galileo. He is said to have demonstrated it by releasing two balls of different masses from top of the leaning tower of Pisa which reached the ground at the same time.

Indian astronomer and mathematician Aryabhatta (476-550 A.D.) studied the motion of the moon, Earth and other planets in the 5^{th} century A.D. In his book 'Aryabhatiya', he concluded that the Earth revolves about its own axis and it moves in a circular orbit around the Sun. Also the moon revolves in a circular orbit around the Earth. Almost a thousand years after Aryabhatta, Tycho Brahe (1546-1601) and Johannes Kepler (1571-1630) studied planetary motion through careful observations. Kepler analysed the huge data meticulously recorded by Tycho Brahe and established three laws of planetary motion. He showed that the motion of planets follow these laws. The reason why planets obey these laws was provided by Newton. He explained that gravitation is the phenomenon responsible for keeping planets in their orbits around the Sun. The moon also revolves around the Earth due to gravitation. Gravitation compels dispersed matter to coalesce, hence the existence of the Earth, the Sun and all material macroscopic objects in the universe.

Every massive object in the universe experiences gravitational force. It is the force of mutual attraction between any two objects by virtue of their masses. It is always an attractive force with infinite range. It does not depend upon intervening medium. It is much weaker than other fundamental forces. Gravitational force is 10^{-39} times weaker than strong nuclear force.

5.2 Kepler's Laws:

Kepler's laws of planetary motion describe the orbits of the planets around the Sun. He published first two laws in 1609 and the third law in 1619. These laws are the result of the analysis of the data collected by Tycho Brahe through years of observations of the planetary motion.

Do you know ?

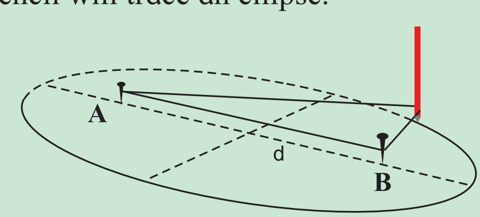
Drawing an ellipse

An ellipse is the locus of the points in a plane such that the sum of their distances from two fixed points, called the foci, is constant.

You can draw an ellipse by the following procedure.

1) Insert two tacks or drawing pins, A and B, as shown in the figure into a sheet of drawing paper at a distance 'd' apart.
2) Tie the two ends of a piece of thread whose length is greater than '2d' and place the loop around AB as shown in the figure.
3) Place a pencil inside the loop of thread, pull the thread taut and move the pencil sidewise, keeping the thread taut.

The pencil will trace an ellipse.



1 Law of orbit

All planets move in elliptical orbits around the Sun with the Sun at one of the foci of the ellipse.

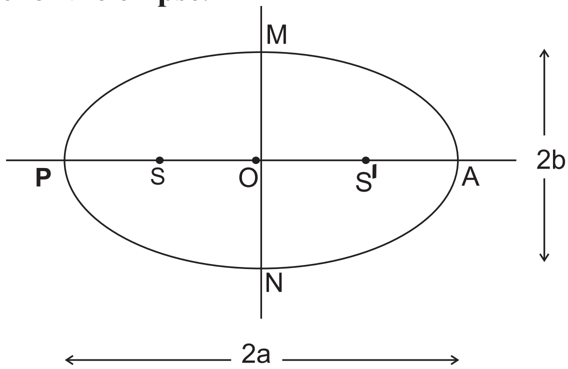


Fig. 5.1: An ellipse traced by a planet with the Sun at the focus.

The orbit of a planet around the Sun is shown in Fig. 5.1.

Here, S and S′ are the foci of the ellipse the Sun being at S.

P is the closest point along the orbit from S and is, called 'Perihelion'.

A is the farthest point from S and is, called 'Aphelion'.

PA is the major axis = $2a$.

PO and AO are the semimajor axes = a.

MN is the minor axis =$2b$.

MO and ON are the semiminor axes = b

2. Law of areas

The line that joins a planet and the Sun sweeps equal areas in equal intervals of time.

Kepler observed that planets do not move around the Sun with uniform speed. They move faster when they are nearer to the Sun while they move slower when they are farther from the Sun. This is explained by this law.

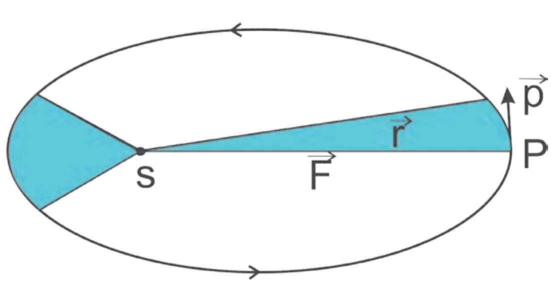


Fig. 5.2: The orbit of a planet P moving around the Sun.

Fig 5.2 shows the orbit of a planet. The shaded areas are the areas swept by SP, the line joining the planet and the Sun, in fixed intervals of time. These are equal according to the second law.

The law of areas can be understood as an outcome of conservation of angular momentum. It is valid for any central force. A central force on an object is a force which is always directed along the line joining the position of object and a fixed point usually taken to be the origin of the coordinate system. The force of gravity due to the Sun on a planet is always along the line joining the Sun and the planet (Fig. 5.2). It is thus a central force. Suppose the Sun is at the origin. The position of planet is denoted by $\vec{r}$ and the perpendicular component of its momentum is denoted by $\vec{p}$ (component $\perp$ $\vec{r}$). The area swept by the planet of mass m in given interval Δt is $\overrightarrow{\Delta A}$ which is given by

$$\overrightarrow{\Delta A} = \frac{1}{2}(\vec{r} \times \vec{v}\Delta t) \quad \text{--- (5.1)}$$

As for small Δt, $\vec{v}$ is perpendicular to $\vec{r}$ and this is the area of the triangle.

$$\therefore \frac{\overrightarrow{\Delta A}}{\Delta t} = \frac{1}{2}(\vec{r} \times \vec{v}) \quad \text{--- (5.2)}$$

Linear momentum ($\vec{p}$) is the product of mass and velocity.

$$\vec{p} = m\vec{v} \quad \text{--- (5.3)}$$

$\therefore$ putting $\vec{v} = \vec{p}/m$ in the above equation, we get

$$\frac{\overrightarrow{\Delta A}}{\Delta t} = \frac{1}{2}\left(\vec{r} \times \frac{\vec{p}}{m}\right) \quad \text{--- (5.4)}$$

Angular momentum $\vec{L}$ is the rotational equivalent of linear momentum and is defined as

$$\therefore \vec{L} = \vec{r} \times \vec{p} \quad \text{--- (5.5)}$$

For central force the angular momentum is conserved.

$$\therefore \frac{\overrightarrow{\Delta A}}{\Delta t} = \frac{\vec{L}}{2m} = \text{constant} \quad \text{--- (5.6)}$$

$\therefore$ This proves the law of areas. This is a consequence of the gravitational force being a central force.

3. Law of periods

The square of the time period of revolution of a planet around the Sun is proportional to the cube of the semimajor axis of the ellipse traced by the planet.

If r is length of semimajor axis then, this law states that

$$T^2 \propto r^3$$

$$\text{or} \quad \frac{T^2}{r^3} = \text{constant} \qquad \text{--- (5.7)}$$

Kepler's laws were based on regular observations of the motion of planets. Kepler did not know why the planets obey these laws,. i.e. he had not derived these laws.

Table 5.1 gives data from measurements of planetary motions which confirm Kepler's law of periods.

Table 5.1: Kepler's third law

Planet	Semi-major axis in units of 10^{10} m	Period in years	T^2/r^3 in units of 10^{-34} y^2m^{-3}
Mercury	5.79	0.24	2.95
Venus	10.8	0.615	3.00
Earth	15.0	1	2.96
Mars	22.8	1.88	2.98
Jupiter	77.8	11.9	3.01
Saturn	143	29.5	2.98
Uranus	287	84.	2.98
Neptune	450	165	2.99
Pluto	590	248	2.99

Example 5.1: What would be the average duration of year if the distance between the Sun and the Earth becomes

(A) thrice the present distance.

(B) twice the present distance.

Solution:

(A) Let r_1 = Present distance between the Earth and Sun

T = 365 days.

If $r_2 = 3r_1$, T_2 = ?

According to Kepler's law of period

$T_1^2 \propto r_1^3$ and $T_2^2 \propto r_2^3$

$$\therefore \frac{T_2^2}{T_1^2} = \frac{r_2^3}{r_1^3}$$

$$\therefore \frac{T_2}{T_1} = \left(\frac{r_2}{r_1}\right)^{3/2}$$

$$= \left(\frac{3r_1}{r_1}\right)^{3/2}$$

$$\therefore \frac{T_2}{T_1} = \sqrt{27}$$

$$T_2 = T_1 \times \sqrt{27}$$

$$= 365 \times \sqrt{27}$$

$$= 1897 \text{ days}$$

(B) If $r_2 = 2r_1$, T_2 = ?

$$\frac{T_2^2}{T_1^2} = \frac{r_2^3}{r_1^3}$$

$$\frac{T_2^2}{T_1^2} = \left(\frac{2r_1}{r_1}\right)^3$$

$$\therefore \frac{T_2}{T_1} = \sqrt{8}$$

$$T_2 = T_1\sqrt{8}$$

$$= 365\sqrt{8}$$

$$= 1032 \text{ days.}$$

5.3 Universal Law of Gravitation:

When objects are released near the surface of the Earth, they always fall down to the ground, i.e., the Earth attracts objects towards itself. Galileo (1564-1642) pointed out that heavy and light objects, when released from the same height, fall towards the Earth at the same speed, i.e., they have the same acceleration. Newton went beyond (the Earth and objects falling on it) and proposed that the force of attraction between masses is universal. Newton stated the universal law of gravitation which led to an explanation of terrestrial gravitation. It also explains the Kepler's laws and provides the reason behind the observed motion of planets around the Sun.

In 1665, Newton studied the motion of moon around the Earth. It was known that the moon completes one revolution about the Earth in 27.3 days. The distance from the Earth to the moon is 3.85×10^5 km. The motion of the moon is in almost a circular orbit around the Earth with constant angular speed ω. As it is a circular motion, the moon must be constantly acted upon by a force directed towards the Earth which is at the centre of the circle. This force is the centripetal force, and is given by

$$F = mr\omega^2 \qquad \text{--- (5.8)}$$

where m is the mass of the moon and r is the distance between the centres of the moon and the Earth.

Also we have $F = ma$ from Newton's laws of motion.

$$\therefore ma = mr\omega^2$$

$$\therefore a = r\omega^2 \qquad \text{--- (5.9)}$$

As angular velocity in terms of time period is given as

$$\omega = \frac{2\pi}{T}$$

we get

$$a = r\left(\frac{2\pi}{T}\right)^2 \qquad \text{--- (5.10)}$$

Substituting values of r and T, we get

$$a = \frac{3.85\times10^5\times10^3 4\pi^2\,\mathrm{m}}{(27.3\times24\times60\times60)^2\,\mathrm{s}^2}$$

$$\therefore a = 0.0027\,\mathrm{m/s^2}$$

This is the acceleration of the moon which is towards the centre of the Earth, i.e., centre of orbit in which the moon revolves. What could be the force which produces this acceleration?

Do you know ?

The value of acceleration due to gravity can be assumed to be constant when we are dealing with objects close to the surface of the Earth. This is because the difference in their distances from the centre of the Earth is negligible.

Newton assumed that the laws of nature are the same for Earthly objects and for celestial bodies. As this acceleration is much smaller than the acceleration felt by bodies near the surface of the Earth (while falling on Earth), he concluded that the acceleration felt by an object due to the gravitational force of the Earth must be decreasing with distance of the body from the Earth. (Remember that the value of acceleration due to Earth's gravity at the surface is 9.8 m/s^2)

We have,

$$\frac{a_{object}}{a_{moon}} = \frac{9.8\,\mathrm{m/s^2}}{0.0027\,\mathrm{m/s^2}} \approx 3600$$

Also,

$$\frac{\text{distance of moon from the Earth's centre}}{\text{distance of object from the Earth's centre}}$$

$$= \frac{3.85\times10^5\,\mathrm{km}}{6378\,\mathrm{km}} \approx 60$$

Thus from the above two equations we get

$$\therefore \frac{a_{object}}{a_{moon}} = \left[\frac{\text{distance of moon}}{\text{distance of object}}\right]^2 \qquad \text{--- (5.11)}$$

Newton therefore concluded that the acceleration of an object towards the Earth is inversely proportional to the square of distance of object from the centre of the Earth.

$$\therefore a \propto \frac{1}{r^2}$$

As, $F = ma$

Therefore, the force exerted by the Earth on an object of mass m at a distance r from it is

$$F \propto \frac{m}{r^2}$$

Similarly an object also exerts a force on the Earth which is

$$F_E \propto \frac{M}{r^2}$$

where M is the mass of the Earth.

According to Newton's third law of motion, the force on a body due to the Earth has to be equal to the force on the Earth due to the object. Hence the force F is also proportional to the mass of the Earth. Hence Newton concluded that the gravitational force between the Earth and an object of mass m is

$$\mathrm{F} \propto \frac{\mathrm{Mm}}{\mathrm{r}^2}$$

He then generalized it to gravitational force between any two objects and stated his

Universal law of gravitation as follows.

Every particle of matter attracts every other particle of matter with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

This law is applicable to all material objects in the universe. Hence it is known as the universal law of gravitation.

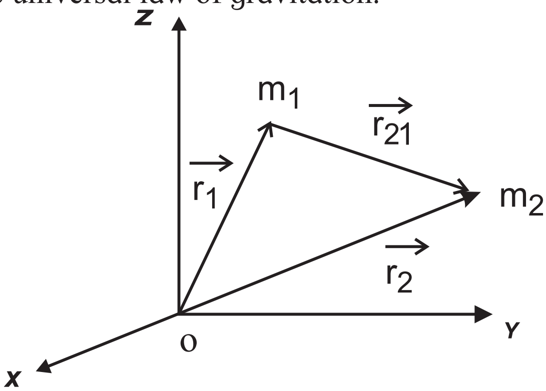


Fig. 5.3: Gravitational force between masses m_1 and m_2.

If two bodies of masses m_1 and m_2 are separated by a distance r, then the gravitational force of attraction between them can be written as

$$F \propto \frac{m_1 m_2}{r^2}$$

or, $$F = G\frac{m_1 m_2}{r^2} \quad \text{--- (5.12)}$$

where G is a constant known as the universal gravitational constant. Its value in SI units is given by

$$G = 6.67\times10^{-11}\text{N m}^2/\text{kg}^2$$

and its dimensions are

$$[G] = [L^3M^{-1}T^{-2}].$$

The gravitational force is an attractive force and it acts along the line joining the two bodies. The forces exerted by two bodies on each other have same magnitude but have opposite directions, they form an action-reaction pair.

Example 5.2: The gravitational force between two bodies is 1 N. If distance between them is doubled, what will be the gravitational force between them?

Solution: Let the masses of the two bodies be m_1 and m_2 and the distance between them be r.

The force between them, $F_1 = \frac{Gm_1m_2}{r^2}$

When the distance between them is doubled the force becomes, $F_2 = \frac{Gm_1m_2}{(2r)^2} = \frac{Gm_1m_2}{4r^2}$

$$\therefore \frac{F_1}{F_2} = \frac{Gm_1m_2}{r^2}\times\frac{4r^2}{Gm_1m_2}$$

$$\therefore \frac{F_1}{F_2} = 4$$

$$\therefore F_2 = \frac{1}{4}\text{N} \quad (\because F_1 = 1\text{N})$$

$$F_2 = 0.25\text{ N}$$

$\therefore$ Force become one forth (0.25 N)

Figure 5.3 shows two point masses m_1 and m_2 with position vectors $\vec{r}_1$ and $\vec{r}_2$ respectively from origin O. The position vector of m_2 with respect to m_1 is then given by $\vec{r}_{21} = \vec{r}_2 - \vec{r}_1$. Similarly $\vec{r}_{12} = \vec{r}_1 - \vec{r}_2 = -\vec{r}_{21}$ and if $r = |\vec{r}_{12}| = |\vec{r}_{21}|$, the formula for force on m_2 due to m_1 can be expressed in vector form as,

$$\vec{F}_{21} = G\frac{m_1m_2}{r^2}(-\hat{r}_{21}) \quad \text{--- (5.13)}$$

where $\hat{r}_{21}$ is the unit vector from m_1 to m_2. The force $\vec{F}_{21}$ is directed from m_2 to m_1. Similarly, force experienced by m_1 due to m_2 is $\vec{F}_{12}$

$$\vec{F}_{12} = G\frac{m_1m_2}{r^2}(-\hat{r}_{12}) \quad \text{--- (5.14)}$$

$$\therefore \vec{F}_{12} = -\vec{F}_{21} \quad \text{--- (5.15)}$$

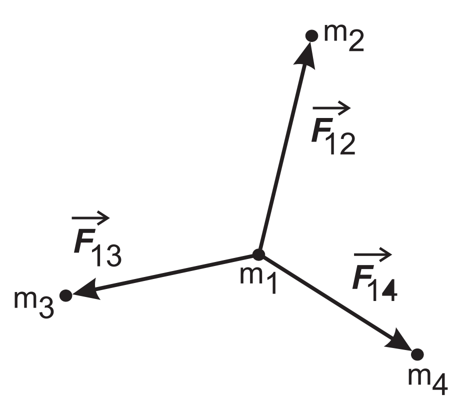


Fig. 5.4: gravitational force due to a collection of masses.

This law refers to two point masses. For a collection of point masses, the force on any one of them is the vector sum of the gravitational forces exerted by all the other point masses. As shown in Fig. 5.4, the resultant force on point mass m_1 is the vector sum of forces $\vec{F}_{12}$, $\vec{F}_{13}$ and $\vec{F}_{14}$ due to point masses m_2, m_3 and m_4 respectively. Masses m_2, m_3 and m_4 are also attracted towards mass m_1 and there is also mutual attraction between masses m_2, m_3 and m_4 but these forces are not shown in the figure.

For n particles, force on ith mass $\vec{F}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \vec{F}_{ij}$

where $\vec{F}_{ij}$ is the force on ith particle due to jth particle.

The gravitational force between an extended object like the Earth and a point mass A can be obtained by obtaining the vector sum of forces on the point mass A due to each of the point mass which make up the extended object. We can consider the following two special cases, for which we can get a simple result. We will state the result here and show how it can be understood qualitatively.

(1) The gravitational force of attraction due to a hollow, thin spherical shell of uniform density, on a point mass situated inside it is zero.

This can be qualitatively understood as follows. First let us consider the case when the point mass A, is at the centre of the hollow thin shell. In this case as every point on the shell is equidistant from A, all points exert force of equal magnitude on A but the directions of these forces are different. Now consider the forces on A due to two diametrically opposite points on the shell. The forces on A due to them will be of equal magnitude but will be in opposite directions and will cancel each other. Thus forces due to all pairs of points diametrically opposite to each other will cancel and there will be no net force on A due to the shell. When the point object is situated elsewhere inside the shell, the situation is not so symmetric. Gravitational force varies directly with mass and inversely with square of the distance. Some part of the shell may be closer to point A, but its mass is less. Remaining part will then have larger mass but its centre of mass is away from A. However mathematically it can be shown that the net gravitational force on A is still zero, so long as it is inside the shell. In fact, the gravitational force at any point inside any hollow closed object of any shape is zero.

(2) The gravitational force of attraction between a hollow spherical shell or solid sphere of uniform density and a point mass situated outside is just as if the entire mass of the shell or sphere is concentrated at the centre of the shell or sphere.

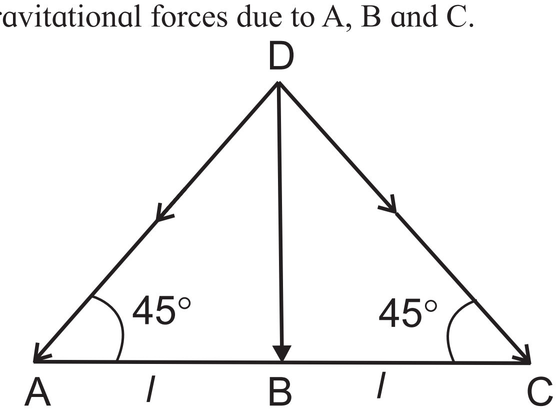
Gravitational force caused by different regions of shell can be resolved into components along the line joining the point mass to the centre and along a direction perpendicular to this line. The components perpendicular to this line cancel each other and the resultant force remains along the line joining the point to the centre. By mathematical calculations it can be shown to be equal to the force that would have been exerted if the entire mass of the shell was present at the centre of the shell.

It is obvious that case (2) is applicable for any uniform sphere (solid or hollow), so long as the point is outside the sphere.

Example 5.3: Three particles A, B, and C each having mass m are kept along a straight line with AB = BC = l. A fourth particle D is kept on the perpendicular bisecter of AC at a distance l from B. Determine the gravitational force on D.

Solution : CD = AD = $\sqrt{AB^2 + BD^2} = \sqrt{2}\ l$

Gravitational force on D = Vector sum of gravitational forces due to A, B and C.



Force due to A $= \frac{Gmm}{(AD)^2} = \frac{Gm^2}{2l^2}$. This will be along $\overrightarrow{DA}$

Force due to C $= \frac{Gmm}{(CD)^2} = \frac{Gm^2}{2l^2}$. This is along $\overrightarrow{DC}$

Force due to B $= \frac{Gmm}{(BD)^2} = \frac{Gm^2}{l^2}$. This is along $\overrightarrow{DB}$

We can resolve the forces along horizontal and vertical directions.

Let the unit vector along horizontal direction $\overrightarrow{AC}$ be $\hat{i}$ and along the vertical direction $\overrightarrow{BD}$ be $\hat{j}$

Net horizontal force on D

$$= \frac{Gm^2}{2l^2}\cos 45°(-\hat{i}) + \frac{Gm^2}{l^2}\cos 90°(\hat{i}) + \frac{Gm^2}{2l^2}\cos 45°(\hat{i})$$

$$= \frac{-Gm^2}{2\sqrt{2}l^2} + \frac{Gm^2}{2\sqrt{2}l^2} = 0$$

Net vertical force on D

$$= \frac{Gm^2}{2l^2}\cos 45°(-\hat{j}) + \frac{Gm^2}{l^2}(-\hat{j}) + \frac{Gm^2}{2l^2}\cos 45°(-\hat{j})$$

$$= \frac{-Gm^2}{l^2}\left(\frac{1}{\sqrt{2}}+1\right)(\hat{j})$$

$$= \frac{Gm^2}{l^2}\left(\frac{1}{\sqrt{2}}+1\right)(-\hat{j})$$

$(-\hat{j})$ shows that the net force is directed along DB

5.4 Measurement of the Gravitational Constant (*G*):

The magnitude of the gravitational constant *G* can be found by measuring the force of gravitational attraction between two bodies of masses m_1 and m_2 separated by certain distance '*L*'.This can be measured by using the Cavendish balance.

The Cavendish balance consists of a light rigid rod. It is supported at the centre by a fine vertical metallic fibre about 100 cm long. Two small spheres s_1 and s_2 of lead having equal mass *m* and diameter about 5 cm are mounted at the ends of the rod and a small mirror M is fastened to the metallic fibre as shown in Fig. 5.5. The mirror can be used to reflect a beam of light onto a scale and thereby measure the angel through which the wire will be twisted.

Two large lead spheres L_1 and L_2 of equal mass *M* and diameter of about 20 cm are brought close to the small spheres on opposite side as shown in Fig. 5.5. The big spheres attract the nearby small spheres by equal and opposite force. Let $\vec{F}$ be the force of attraction between a big sphere and small sphere near to it. Hence a torque will be generated without exerting any net force on the bar. Due to this torque the bar turns and the suspension wire gets twisted till the restoring torque due to the elastic property of the wire becomes equal to the gravitational torque.

The gravitational force between the spherical balls is the same as if their masses are concentrated at their centres. If *r* is the initial distance of separation between the centres of the big and the neighbouring small sphere, then the magnitude of the force between them is

$$F = G\frac{mM}{r^2}$$

If length of the rod is L, then the magnitude of the torque arising out of these forces is

$$\tau = FL = G\frac{mM}{r^2}L \qquad \text{--- (5.16)}$$

At equilibrium, it is equal and opposite to the restoring torque.

$$\therefore G\frac{mM}{r^2}L = K\theta \qquad \text{--- (5.17)}$$

where *K* is the restoring torque per unit angle and θ is the angle of twist.

By applying a known torque τ_1 and measuring the corresponding angle of twist α, the restoring torque per unit twist can be determined as $K = \tau_1/\alpha$.

Thus, in actual experiment measuring θ and knowing values of τ, *m*, *M* and *r*, the value of *G* can be calculated from Eq. (5.17). The gravitational constant measured in this way is found to be

$$G = 6.67\times10^{-11} \text{N m}^2/\text{kg}^2$$

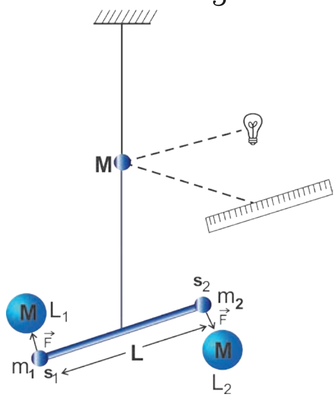


Fig 5.5 : The Cavendish balance.

5.5 Acceleration due to Gravity:

We have seen in section 5.3 that the magnitude of the gravitational force on a point object of mass *m* due to another point object of mass *M* at a distance *r* from it is given by the equation.

$$F = G\frac{mM}{r^2}$$

This formula can be used to calculate the gravitational force on an object due to the Earth. We know that the Earth is an extended object. In many practical applications Earth can be assumed to be a uniform sphere. As seen in section 5.3 its entire mass can be assumed to be concentrated at is centre. Thus if the mass of the Earth is *M* and that of the point object is *m* and the distance of the point object from the centre of the Earth is *r* then the force of attraction between them is given by

$$F = G\frac{Mm}{r^2}$$

If the point object is not acted upon by any other force, it will be accelerated towards the centre of the Earth under the action of this force. Its acceleration can be calculated by using Newton's second law *F = ma.*

Acceleration due to the gravity of the Earth =

$$G\frac{Mm}{r^2}\times\frac{1}{m}$$

$$= \frac{GM}{r^2} \qquad \text{--- (5.18)}$$

This is known as the acceleration due to gravity of the Earth and denoted by *g*.

If the object is close to the surface of the Earth, $r \cong R$, the radius of the Earth then

$$g_{\text{Earth's surface}} = \frac{GM}{R^2} \qquad \text{--- (5.19)}$$

Example 5.4: Calculate mass of the Earth from given data,

Acceleration due to gravity $g = 9.81 \text{m/s}^2$
Radius of the Earth $R_E = 6.37\times10^6$ m
$G = 6.67\times10^{-11}$ N m^2/kg^2

Solution:

$$g = \frac{GM_E}{R_E^2}$$

$$\therefore M_E = \frac{gR_E^2}{G}$$

$$\therefore M_E = \frac{9.81\times(6.37\times10^6)^2}{6.67\times10^{-11}}$$

$$\therefore M_E = 5.97\times10^{24}\,\text{kg}$$

The value of *g* depends only on the properties of the Earth and does not depend on the mass of the object. This is exactly what Galileo had found from his experiments of dropping objects with different masses from the same height.

Do you know ?

An object of mass *m* (much smaller than the mass of the Earth) is attracted towards the Earth and falls on it. The Earth is also attracted by the same force (magnitude) toward the mass *m*. However, its acceleration towards *m* will be

$$a_{earth} = \frac{\left(G\frac{Mm}{r^2}\right)}{M} = \frac{Gm}{r^2}$$

$$\therefore \frac{a_{earth}}{g} = \frac{m}{M} \quad \left(\text{as } g = \frac{GM}{r^2}\right)$$

As $m << M$, $a_{Earth} << g$ and is nearly zero. Thus, practically only the mass *m* moves towards the Earth.

Example 5.5: Calculate the acceleration due to gravity on the surface of moon if mass of the moon is 1/80 times that of the Earth and diameter of the moon is 1/4 times that of the Earth (g =9.8 m/s^2)

Solution:

M_m = Mass of the moon = $M/80$,

where M is mass of the Earth.

R_m = Radius of the moon = $R/4$,

where R is Radius of the Earth.

Acceleration due to gravity on the surface of the Earth, $g = GM/R^2$ --- (1)

Acceleration due to gravity on the surface of the moon, $g_m = GM_m/R_m^2$ --- (2)

$\therefore$ From equation (1) and (2)

$$\frac{g_m}{g} = \frac{M_m}{M} \times \left[\frac{R}{R_m}\right]^2$$

$$\frac{g_m}{g} = \frac{1}{80} \times \left[\frac{4}{1}\right]^2$$

$$\therefore \frac{g_m}{g} = \frac{1}{5}$$

$$\therefore g_m = \frac{g}{5} = \frac{9.8}{5}$$

$$\therefore g_m = 1.96 \text{ m/s}^2$$

Example 5.6: Find the acceleration due to gravity on a planet that is 10 times as massive as the Earth and with radius 20 times of the radius of the Earth ($g = 9.8$ m/s^2).

Solution : Let mass of the planet be M_p, radius of the Earth and that of the planet be R_E and R_p respectively. Let mass of the Earth be M_E and g_p be acceleration due to gravity on the planet.

$$M_p = 10\, M_E$$

$$R_p = 20\, R_E, \quad g = 9.8 \text{ m/s}^2$$

$$g_p = ?$$

$$g = \frac{GM_E}{R_E^2}, \quad g_P = \frac{GM_P}{R_P^2}$$

$$\therefore g_P = \frac{G(10M_E)}{(20R_E)^2}$$

$$= \frac{10GM_E}{400R_E^2}$$

$$= \frac{1}{40} g$$

$$= 0.245 \text{ m/s}^2$$

5.6 Variation in the Acceleration due to Gravity with Altitude, Depth, Latitude and Shape:

(A) Variation in g with Altitude:

Consider a body of mass m on the surface of the Earth. The acceleration due to gravity on the Earth's surface is given by,

$$g = \frac{GM}{R^2}$$

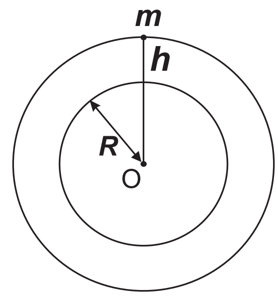


Fig. 5.6 Acceleration due to gravity at height h above the Earth's surface.

When the body is at height h above the surface of the Earth as shown in Fig. 5.6, acceleration due to gravity changes to

$$g_h = \frac{GM}{(R+h)^2}$$

$$\therefore \frac{g_h}{g} = \frac{\frac{GM}{(R+h)^2}}{\frac{GM}{R^2}}$$

$$\therefore \frac{g_h}{g} = \frac{R^2}{(R+h)^2}$$

$$\therefore g_h = \frac{g\,R^2}{(R+h)^2} \quad \text{--- (5.20)}$$

This equation shows that, the acceleration due to gravity goes on decreasing with increase in altitude of body from the surface of the Earth. We can rewrite Eq. (5.20) as

$$g_h = \frac{g\,R^2}{R^2\left(1+\frac{h}{R}\right)^2}$$

$$\therefore g_h = g\left(1+\frac{h}{R}\right)^{-2}$$

For small altitude h, i.e., for $\frac{h}{R} << 1$,

$$\therefore g_h \simeq g\left(1-\frac{2h}{R}\right) \quad \text{--- (5.21)}$$

(by neglecting higher power terms of $\frac{h}{R}$ as $\frac{h}{R} << 1$)

This expression can be used to calculate the value of g at height h above the surface of the Earth as long as $h << R$.

Example 5.7 : At what distance above the surface of Earth the acceleration due to gravity decreases by 10% of its value at the surface? (Radius of Earth = 6400 km). Assume the distance above the surface to be small compared to the radius of the Earth.

Solution : g_h = 90% of g (g decreases by 10% hence it becomes 90%)

or, $\frac{g_h}{g} = \frac{90}{100} = 0.9$

From Eq. (5.21)

$$g_h = g\left[1 - \frac{2h}{R}\right]$$

$$\therefore \frac{g_h}{g} = 1 - \frac{2h}{R}$$

$$\therefore 0.9 = 1 - \frac{2h}{R}$$

$$h = \frac{R}{20}$$

$$h = 320 \text{ km}$$

(B) Variation in g with Depth:

The Earth can be imagined to be a sphere made of large number of concentric uniform spherical shells. The total mass of the Earth is the combined mass of all the shells. When an object is on the surface of the Earth it experiences the gravitational force as if the entire mass of the Earth is concentrated at its centre.

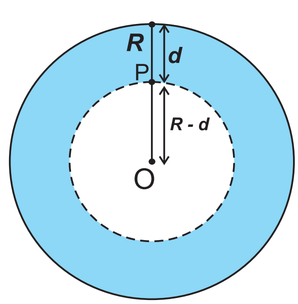


Fig. 5.7 Acceleration due to gravity at depth *d* below the surface of the Earth.

The acceleration due to gravity according to eq. (5.19) is

$$g = \frac{GM}{R^2}$$

Assuming that the density of the Earth is uniform, it is given by

$$\rho = \frac{\text{Mass } (M)}{\text{Volume}(V)}$$

$$\therefore M = \frac{4}{3}\pi R^3 \rho$$

$$\therefore g = \frac{G \times \frac{4}{3}\pi R^3 \rho}{R^2}$$

$$\therefore g = \frac{4}{3}\pi R \rho G \quad \text{--- (5.22)}$$

Consider a body at a point P at the depth *d* below the surface of the Earth as shown in Fig. 5.7. Here the force on a body at P due to the material outside the inner sphere shown by shaded region, can be shown to cancel out due to symmetry. The net force on P is only due to the material inside the inner sphere of radius OP = $R - d$. Acceleration due to gravity because of this sphere is

$$g_d = \frac{GM'}{(R-d)^2}$$

where M' = volume of the inner sphere×density

$$\therefore M' = \frac{4}{3}\pi (R-d)^3 \times \rho$$

$$\therefore g_d = \frac{G \times \frac{4}{3}\pi (R-d)^3 \rho}{(R-d)^2}$$

$$\therefore g_d = G \times \frac{4}{3}\pi (R-d)\rho \quad \text{--- (5.23)}$$

Dividing Eq. (5.23) by Eq. (5.22) we get,

$$\therefore \frac{g_d}{g} = \frac{R-d}{R}$$

$$\therefore \frac{g_d}{g} = 1 - \frac{d}{R}$$

$$\therefore g_d = g\left[1 - \frac{d}{R}\right] \quad \text{--- (5.24)}$$

This equation gives acceleration due to gravity at depth d below the Earth's surface.

It shows that the acceleration due to gravity decreases with depth.

Special case :

At the centre of the Earth, where $d = R$, Eq. (5.24) gives $g_d = 0$

Hence, a body of mass m if taken to the centre of Earth, will not experience the force of gravity due to the Earth. This can also be understood to be due to symmetry. The case is similar to the force of gravity on an object placed at the centre of a spherical shell as seen in section 5.3.

Thus, the value of acceleration due to gravity is maximum at the surface of the Earth. The value goes on decreasing with

1) increase in depth below the Earth's surface. (varies linearly with $(R\text{-}d) = r$)

2) increase in height above the Earth's surface. (varies inversely with $(R+h)^2 = r^2$)

Graphically the variation of acceleration due to gravity according to depth and height can be expressed as follows. We have plotted the value of g as a function of r, the distance from the centre of the Earth, in Fig. 5.8. For $r < R$ i. e. below the surface of the Earth, we use Eq. (5.24), according to which $g_d = g\left(1 - \frac{d}{R}\right)$

Writing $R - d = r$, the distance from the centre of the Earth, we get the value of g as a function of r, $g(r) = g\frac{r}{R}$ which is the equation of a straight line with slope g/R and passing through the origin.

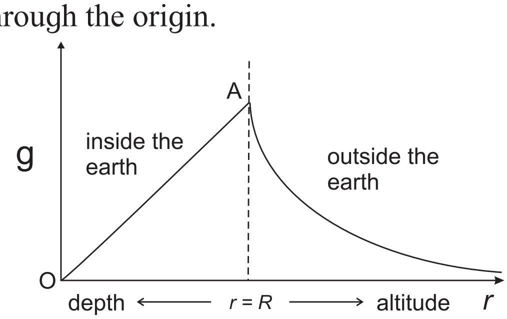


Fig 5.8 - Variation of g due to depth and altitude from the Earth's surface.

For $r > R$ we have to use Eq. (5.20). Writing $R + h = r$ we have

$g(r) = \frac{g\,R^2}{r^2}$ which is plotted in Fig. 5.8

(C) Variation in *g* with Latitude and Rotation of the Earth:

Latitude is an angle made by radius vector of any point from centre of the Earth with the equatorial plane. Obviously it ranges from 0^0 at the equator to 90^0 at the poles.

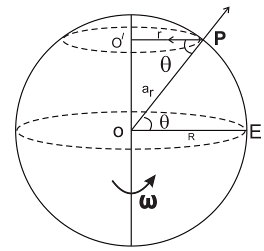


Fig. 5.9 Variation of g with latitude.

The Earth rotates about its polar axis from west to east with uniform angular velocity ω. Hence every point on the surface of the Earth (except the poles) moves in a circle parallel to the equator. The motion of a mass m at point P on the Earth is shown by the dotted circle with centre at O′ in Fig. 5.9. Let the latitude of P be θ and radius of the circle be r.

$PO' = r$

$\angle EOP = \theta$, E being a point on the equator

$\therefore \angle OPO' = \theta$

In Δ OPO′, $\cos\theta = \frac{PO'}{PO} = \frac{r}{R}$

$\therefore r = R\cos\theta$

The centripetal acceleration for the mass m, directed along PO′ is

$a = r\omega^2$

$a = R\omega^2\cos\theta$

The component of this centripetal acceleration along PO, i.e., towards the centre of the Earth is

$a_r = a\cos\theta$

$\therefore a_r = R\omega^2\cos\theta.\cos\theta$

$a_r = R\omega^2\cos^2\theta$

Part of the gravitational force of attraction on P acting towards PO is utilized in providing this components of centripetal acceleration.

Thus the effective force of gravitational attraction on m at P can be written as

$mg' = mg - mR\omega^2\cos^2\theta$

g' being the effective acceleration due to gravity at P i.e., at latitude θ. This is thus given by $g' = g - R\omega^2\cos^2\theta$ --- (5.25)

As the value of θ increases, cosθ decreases. Therefore g' will increase as we move away from equator towards any pole due to the rotation of the Earth.

special case I At equator $\theta = 0$

$$\cos\theta = 1$$

$$g' = g - R\omega^2$$

The effective acceleration due to gravity is minimum at equator, as here it is reduced by maximum amount. The reduction here is $g - g' = R\omega^2$

$R = 6.4 \times 10^6$ m ---Radius of the Earth and

ω = Angular velocity of rotation of the Earth

$$\omega = \frac{2\pi}{T}$$

$$\therefore \omega = \frac{2\pi}{24 \times 60 \times 60}$$

$$\therefore \omega = 7.275 \times 10^{-5}\,\mathrm{s}^{-1}$$

$$\therefore g - g' = R\omega^2$$

$$\therefore g - g' = 0.03386\,\mathrm{m/s^2}$$

Case II At poles $\theta = 90^0$

$$\cos\theta = 0$$

$$\therefore g' = g - R\omega^2\cos\theta$$

$$= g - 0$$

$$= g$$

There is no reduction in acceleration due to gravity at poles, due to the rotation of the Earth as the poles are lying on the axis of rotation and do not revolve.

Variation of g with latitudes at sea level is given in the following table.

Table 5.2: Variation of g with latitude

Latitude (°)	g (m/s²)
0	9.7804
10	9.7819
20	9.7864
30	9.7933
40	9.8017
50	9.8107
60	9.8192
70	9.8261
80	9.8306
90	9.8322

Effect of the shape of the Earth: Quite often we assume the Earth to be a sphere. However, it is actually on ellipsoid; bulged at equator. Hence equatorial radius of Earth (6378 km) is greater than the polar radius (6356 km). Thus, on the equator, there is combined effect of greater radius and rotation in reducing the force of gravity. As a result, the acceleration due to gravity on the equator is $g_E = 9.7804\,\mathrm{m/s^2}$ and on the poles it is $g = 9.8322\,\mathrm{m/s^2}$.

Weight of an object is the force with which the Earth attracts that object. Thus, weight $w = mg$ where m is the mass of the object. As the value of g changes with altitude, depth and latitude, the weight also changes. Weight of an object is minimum at the equator. Similarly, the weight of an object reduces with increasing height above the Earth's surface and with increasing depth below the its surface.

5.7 Gravitational Potential and Potential Energy:

In earlier standards, you have studied potential energy as the energy possessed by an object on account of its position or configuration. The word *configuration* corresponds to the distribution of the particles in the object. More specifically, potential energy is the work done *against conservative force* (or forces) *in achieving* a certain position or configuration *of a given system*. It always depends upon the relative positions of the particles in that system. There is a universal principle that states, **Every system always configures itself in order to have minimum potential energy or every system tries to minimize its potential energy.** Obviously, in order to change the configuration, you will have to do work.

Examples:

(I) A spring in its natural state, possesses minimum potential energy. Whenever we stretch it or compress it, we perform work against the conservative force (in this case, the elastic restoring force). Due to this work, the relative distances between the particles of the system change (configuration changes) and its potential energy increases. The spring finally regains its original configuration of minimum potential energy on removal of the applied force.

(II) When an object is lying on the Earth, the system of that object and the Earth has minimum potential energy. This is the gravitational potential energy of the system as these two are bound by the gravitational force. While lifting the object to some height (new position), we do work against the conservative gravitational force in order to achieve the new position.

In its new position, the object is at rest due to balanced forces. If you are holding the object, the force of static friction between the object and your fingers balances the gravitational force. If kept on a surface, the normal reaction force given by the surface balances the gravitational force. However, now, the object has a ***capacity*** to acquire kinetic energy, when given an opportunity (when allowed to fall). We call this increase in the capacity as the potential energy *gained* by the system. As we raise it more and more, this capacity, and hence potential energy of the system, increases. It falls on the Earth to achieve the configuration of minimum potential energy on dropping it from the new position.

Thus, in general, we can write **work done against a conservative force acting on an object = Increase in the potential energy of the system.**

$$\therefore \vec{F}.d\vec{x} = dU$$

Here dU is the change in potential energy while displacing the object through $d\vec{x}$, $\vec{F}$ being the force acting on the object

It should be remembered that potential energy is *always* of the system as a whole. For an object on the Earth, it is of the system of the object and the Earth and not only of that object. There is no meaning to potential energy of an isolated object in the intergalactic (gravity free) space, in the absence of any conservative force acting upon it.

5.7.1 Expression for Gravitational Potential Energy:

Work done against gravitational force $\vec{F}_g$, in displacing an object through a small displacement $d\vec{r}$, appears as increase in the potential energy of the system.

$$\therefore dU = -\vec{F}_g.d\vec{r}$$

Negative sign appears because dU is the work done by us (external agent) *against* the gravitational force $\vec{F}_g$

For displacement of the object from an initial position $\vec{r}_i$ to the final position $\vec{r}_f$, the change in potential energy ΔU, can be obtained by integrating dU.

$$\therefore \Delta U = \int_{r_i}^{r_f} (dU) = \int_{r_i}^{r_f} \left(-\vec{F}_g.d\vec{r}\right)$$

Gravitational force of the Earth, $\vec{F}_g = -\frac{GMm}{r^2}\hat{r}$ where $\hat{r}$ is the unit vector in the direction of $\vec{r}$. Negative sign appears here because $\vec{r}$ is from centre of the Earth to the object and $\vec{F}_g$ is directed towards centre of the Earth.

$\therefore$ For 'Earth and mass' system,

$$\Delta U = \int_{r_i}^{r_f} dU = \int_{r_i}^{r_f} -\left(-\frac{GMm}{r^2}\hat{r}\right).d\vec{r}$$

$$= GMm\int_{r_i}^{r_f}\left(\frac{dr}{r^2}\right) \text{ as } d\vec{r} \text{ is along } \hat{r}$$

$$= GMm\left(-\frac{1}{r}\right)_{r_i}^{r_f}$$

$$= GMm\left(\frac{1}{r_i} - \frac{1}{r_f}\right) \qquad \text{---(5.26)}$$

Change in potential energy corresponds to the work done against conservative forces. The absolute value of potential energy is not defined. It is logical as well as convenient to choose the point of zero potential energy to be the point of zero force. For gravitational force, such point is taken at $r = \infty$. This point should be chosen as the initial point so that initially the potential energy is zero. $\therefore U(r_i) = 0$ at $r_i = \infty$

Final point r_f is obviously the point where we need to determine the potential energy of the system. $\therefore r_f = r$

$$\therefore U(\mathrm{r})_{grav.} = GMm\left(\frac{1}{r_i} - \frac{1}{r_f}\right)$$

$$= GMm\left(\frac{1}{\infty} - \frac{1}{r}\right)$$

$$= -\frac{GMm}{r} \qquad \text{--- (5.27)}$$

This is gravitational potential energy of the system of object of mass m and Earth of mass M having separation r (between their centres of mass).

Example 5.8: What will be the change in potential energy of a body of mass m when it is raised from height R_E above the Earth's surface to 5/2 R_E above the Earth's surface? R_E and M_E are the radius and mass of the Earth respectively.

Solution:

$$\Delta U = GmM_E\left(\frac{1}{r_i}-\frac{1}{r_f}\right)$$

$$= GmM_E\left(\frac{1}{2R_E}-\frac{1}{3.5R_E}\right)$$

$$= \frac{GmM_E}{R_E}\times\frac{1.5}{2\times3.5} = \frac{2.14\,GmM_E}{R_E}$$

5.7.2 Connection of potential energy formula with *mgh*:

If the object is on the surface of Earth, $r = R$

$$\therefore U_1 = -\frac{GMm}{R}$$

If the object is lifted to height h above the surface of Earth, the potential energy becomes

$$U_2 = -\frac{GMm}{R+h}$$

Increase in the potential energy is given by

$$\Delta U = U_2 - U_1$$

$$= GMm\left(-\frac{1}{R+h}-\left[-\frac{1}{R}\right]\right)$$

$$= GMm\left(\frac{h}{R(R+h)}\right)$$

$$= \frac{GMmh}{R(R+h)}$$

If g is acceleration due to the Earth on the surface of Earth, $GM = gR^2$

$$\therefore \Delta U = mgh\left(\frac{R}{R+h}\right) \quad \text{--- (5.28)}$$

Eq. (5.28) gives the work to be done (or energy to be supplied) to raise an object of mass m to a height h, above the surface of the Earth.

If $h \ll R$, we can use $R+h \cong R$. Only in this case $\Delta U = mgh$ --- (5.29)

Thus, mgh is increase in the gravitational potential energy of the Earth -mass system if an object of mass m is lifted to a height h, provided h is negligible compared to radius of the Earth (up to a few kilometers).

5.7.3 Concept of Potential:

From eq. (5.27), the gravitational potential energy of the system of Earth and ***any*** mass m at a distance r from the centre of the Earth is given by

$$U = -\frac{GMm}{r}$$

$$= \left(-\frac{GM}{r}\right)m$$

$$= \left[\left(V_E\right)_r\right]m \quad \text{--- (5.30)}$$

The factor $-\frac{GM}{r} = \left(V_E\right)_r$ depends only upon mass of Earth and the location. Thus, it is the same for any mass m bound to the Earth. Conveniently, this is defined as the gravitational potential of Earth at distance r from its centre. In terms of potential, we can write the potential energy of the Earth-mass system as

Gravitational potential energy, U=Gravitational potential $V_r \times$ mass m **or** Gravitational potential is Gravitational potential energy per unit mass, i.e., $V_r = \frac{U}{m}$. The concept of potential can be defined on similar lines for any conservative force field.

Gravitational potential difference between any two points in gravitational field can be written as

$$V_2 - V_1 = \left(\frac{U_2 - U_1}{m}\right) = \frac{dW}{m} \quad \text{--- (5.31)}$$

= Work done (or change in potential energy) per unit mass

In general, for a system of any two masses m_1 and m_2, separated by r, we can write

Gravitational potential energy,

$$U = -\frac{Gm_1m_2}{r} = \left(V_1\right)m_2 = \left(V_2\right)m_1 \quad \text{---(5.32)}$$

Here V_1 and V_2 are gravitational potentials at r due to m_1 and m_2 respectively.

5.7.4 Escape Velocity:

When any object is thrown vertically up, it falls back to the Earth after reaching a certain height. Higher the speed with which the object is thrown up, greater will be the height. If we keep on increasing the velocity, a stage will come when the object will reach heights so large that it will escape the gravitational field of the Earth and will not fall back on the Earth. This initial velocity is called the escape velocity.

Thus, the minimum velocity with which a body should be thrown vertically upwards from the surface of the Earth so that it escapes the Earth's gravitational field, is called the escape velocity (v_e) of the body. Obviously, as the gravitational force due to Earth becomes zero only at infinite distance, the object has to reach infinite distance in order to escape.

Let us consider the kinetic and potential energies of an object thrown vertically upwards with escape velocity v_e, when it is at the surface of the Earth and when it reaches infinite distance. On the surface of the Earth,

$$K.E. = \frac{1}{2}mv_e^2$$

$$P.E. = -\frac{GMm}{R}$$

Total energy = $P.E. + K.E.$

$$= \frac{1}{2}mv_e^2 - \frac{GMm}{R} \qquad \text{--- (5.33)}$$

The kinetic energy of the object will go on decreasing with time as it is pulled back by Earth's gravitational force. It will become zero when it reaches infinity. Thus at infinite distance from the Earth

$$K.E. = 0$$

Also, $P.E. = -\frac{GMm}{\infty} = 0$

$\therefore$ Total energy = $P.E. + K.E. = 0$

As energy is conserved

$$\frac{1}{2}mv_e^2 - \frac{GMm}{R} = 0$$

or, $$v_e = \sqrt{\frac{2GM}{R}} \qquad \text{--- (5.34)}$$

Using the numerical values of G, M and R. the escape velocity is 11.2 km/s.

5.8 Earth Satellites:

The objects which revolve around the Earth are called Earth satellites. moon is the only natural satellite of the Earth. It revolves in almost a circular orbit around the Earth with period of revolution of nearly 27.3 days. Artificial satellites have been launched by several countries including India. These satellites have different periods of revolution according to their practical use like navigation, surveillance, communication, looking into space and monitoring the weather.

Communication Satellites: These are geostationary satellites. They revolve around the Earth in equatorial plane. They have same sense of rotation as that of the Earth and the same period of rotation as that of the Earth, i. e., one day or 24 hours. Due to this, they appear stationary from the Earth's surface. Hence they are called geostationary satellites or geosynchronous satellites. These are used for communication, television transmission, telephones and radiowave signal transmission, e.g., INSAT group of satellites launched by India.

Polar Satellites: These satellites are placed in lower polar orbits. They are at low altitude 500 km to 800 km. Polar satellites are used for weather forecasting and meteorological purpose. They are also used for astronomical observations and study of Solar radiations.

Period of revolution of polar satellite is nearly 85 minutes, so it can orbit the Earth16 time per day. They go around the poles of the Earth in a north-south direction while the Earth rotates in an east-west direction about its own axis. The polar satellites have cameras fixed on them. The camera can view small stripes of the Earth in one orbit. In entire day the whole Earth can be viewed strip by strip. Polar and equatorial regions at close distances can be viewed by these satellites.

5.8.1 Projection of Satellite:

For the projection of an artificial satellite, it is necessary for the satellite to have a certain velocity and a minimum two stage rocket. A single stage rocket can not achieve this. When the fuel in first stage of rocket is ignited on the surface of the Earth, it raises the satellite vertically. The velocity of projection of satellite normal to the surface of the Earth is the vertical velocity. If this vertical velocity is less that the escape velocity (v_e), the satellite returns to the Earth's surface. While, if the vertical velocity is greater than or equal to the escape velocity, the satellite will escape from Earth's gravitational influence and go to infinity. Hence launching of a satellite in an orbit round the Earth can not take place by use of single stage rocket. It requires minimum two stage rocket.

With the help of first stage of rocket, satellite can be taken to a desired height above the surface of the Earth. Then the launcher is rotated in horizontal direction i.e. through 90^0 using remote control and the first stage of the rocket is detached. Then with the help of second stage of rocket, a specific horizontal velocity (v_h) is given to satellite so that it can revolve in a circular path round the Earth. The exact horizontal velocity of projection that must be given to a satellite at a certain height so that it can revolve in a circular orbit round the Earth is called the **critical velocity** or **orbital velocity** (v_c)

A satellite follows different paths depending upon the horizontal velocity provided to it. Four different possible cases are shown in Fig. 5.10.

Case (I) $v_h<v_c$:

If tangential velocity of projection v_h is less than the critical velocity, the orbit of satellite is an ellipse with point of projection as apogee (farthest from the Earth) and Earth at one of the foci.

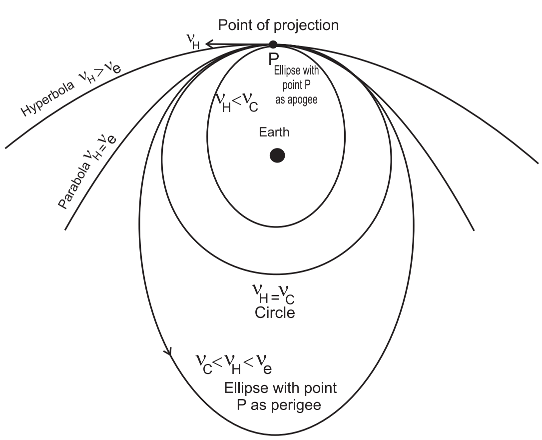


Fig. 5.10: Various possible orbits depending on the value of v_h.

During this elliptical path, if the satellite passes through the Earth's atmosphere, it experiences a nonconservative force of air resistance. As a result it loses energy and spirals down to the Earth.

Case (II) $v_h=v_c$

If the horizontal velocity is exactly equal to the critical velocity, the satellite moves in a stable circular orbit round the Earth.

Case (III) $v_c<v_h<v_e$

If horizontal velocity is greater than the critical velocity and less than the escape velocity at that height, the satellite again moves in an elliptical orbit round the Earth with the point of projection as perigee (point closest to the Earth).

Case (IV) $v_h = v_e$

If horizontal speed of projection is equal to the escape speed at that height, the satellite travels along parabolic path and never returns to the point of projection. Its speed will be zero at infinity.

Case (V) $v_h > v_e$

If horizontal velocity is greater than the escape velocity, the satellite escapes from gravitational influence of Earth transversing a hyperbolic path.

Expression for critical speed

Consider a satellite of mass *m* revolving round the Earth at height *h* above its surface.

Let *M* be the mass of the Earth and *R* be its radius. If the satellite is moving in a circular orbit of radius ($R+h$) = r, its speed must be the magnitude of critical velocity v_c.

The centripetal force necessary for circular motion of satellite is provided by gravitational force exerted by the Earth on the satellite.

$\therefore$ Centripetal force = Gravitational force

$$\frac{m v_c^2}{r} = \frac{GMm}{r^2}$$

$$\therefore v_c^2 = \frac{GM}{r}$$

$$\therefore v_c = \sqrt{\frac{GM}{r}}$$

$$\therefore v_c = \sqrt{\frac{GM}{(R+h)}} = \sqrt{g_h(R+h)} \quad \text{--- (5.35)}$$

This is the expression for critical speed in the orbit of radius $(R + h)$

It is clear that the critical speed of a satellite is independent of the mass of the satellite. It depends upon the mass of the Earth and the height at which the satellite is revolving or gravitational acceleration at that altitude. The critical speed of a satellite decreases with increase in height of satellite.

Special case

When the satellite is revolving close to the surface of the Earth, the height is very small as compared to the radius of the Earth. Hence the height can be neglected and radius of the orbit is nearly equal to R (i.e $R >> h$, $R+h \approx R$)

$$\therefore \text{ Critical speed } v_c = \sqrt{\frac{GM}{R}}$$

As G is related to acceleration due to gravity by the relation,

$$g = \frac{GM}{R^2}$$

$$\therefore GM = gR^2$$

$\therefore$ Critical speed in terms of acceleration due to gravity can be obtained as

$$v_c = \sqrt{\frac{gR^2}{R}} = \sqrt{gR}$$

$$= 7.92 \text{ km/s}$$

Obviously, this is the maximum possible critical speed. This is at least 25 times the speed of the fastest passenger aeroplanes.

Example 5.9: Show that the critical velocity of a body revolving in a circular orbit very close to the surface of a planet of radius R and mean density ρ is $2R\sqrt{\frac{G\pi\rho}{3}}$.

Solution : Since the body is revolving very close to the planet, $h = 0$

$$\text{density } \rho = \frac{M}{V} = \frac{M}{\frac{4}{3}\pi R^3}$$

$$\therefore M = \frac{4}{3}\pi R^3 \rho$$

Critical Velocity

$$v_c = \sqrt{\frac{GM}{R}} = \sqrt{\frac{G\frac{4}{3}\pi R^3 \rho}{R}}$$

$$\therefore v_c = 2R\sqrt{\frac{G\pi\rho}{3}}$$

When a satellite revolves very close to the surface of the Earth, motion of satellite gets affected by the friction produced due to resistance of air. In deriving the above expression the resistance of air is not considered.

5.8.2 Weightlessness in a Satellite:

According to Newton's second law of motion, $F = ma$, where F is the net force acting on an object having acceleration a.

Let us consider the example of a lift or elevator from an inertial frame of reference.

Whether the lift is at rest or in motion, a passenger in it experiences only two forces: (i) Gravitational force mg directed vertically downwards (towards centre of the earth) and (ii) normal reaction force N directed vertically upwards, exerted by the floor of the lift. As these forces are oppositely directed, the net force in the downward direction will be $F = ma - N$.

Though the weight of a body (passenger, in this case) is the gravitational force acting upon it, we experience or *feel* our weight only due to the normal reaction force N exerted by the floor. This, in turn, is equal and opposite to the relative force between the body and the lift. If you are standing on a weighing machine in a lift, the force recorded by the weighing machine is nothing but the normal reaction N.

Case I: Lift having zero acceleration

This happens when the lift is at rest or is moving upwards or downwards with constant velocity:

The net force $F = 0 = mg - N \therefore mg = N$

Hence in this case we feel our normal weight mg.

Case II: Lift having net upward acceleration a_u

This happens when the lift just starts moving upwards or is about to stop at a lower floor during its downward motion (remember, while stopping during downward motion, the acceleration must be upwards).

As the net acceleration is upwards, the upward force must be greater.

$\therefore F = ma_u = N - mg \therefore N = mg + ma_u$, i.e., $N > mg$, hence, we *feel* heavier.

It should also be remembered that this is not an apparent feeling. The weighing machine really records a reading greater than mg.

Case III (a): Lift having net downward acceleration a_d

This happens when the lift just starts moving downwards or is about to stop at a higher floor during its upward motion (remember, while stopping during upward motion, the acceleration must be downwards).

As the net acceleration is downwards, the downward force must be greater.

$\therefore F = ma_d = mg - N \therefore N = mg - ma_d$, i.e., $N < mg$, hence, we *feel* lighter.

It should be remembered that this is not an apparent feeling. The weighing machine really records a reading less than mg.

Case III (b): State of free fall: This will be possible if the cables of the lift are cut. In this case, the downward acceleration $a_d = g$.

If the downward acceleration becomes equal to the gravitational acceleration g, we get, $N = mg - ma_d = 0$.

Thus, there will not be any feeling of weight. This is the state of *total* weightlessness and the weighing machine will record *zero*.

In the case of a revolving satellite, the satellite is performing a circular motion. The acceleration for this motion is centripetal, which is provided by the gravitational acceleration g at the location of the satellite. In this case, $a_d = g$, or the satellite (along with the astronaut) is in the state of free fall. Obviously, the apparent weight will be zero, giving the feeling of *total* weightlessness. Perhaps you might have seen in some videos that the astronauts are floating inside the satellite. It is really difficult for them to change their position.

In spite of free fall, why is the satellite not falling on the earth? The reason is that the revolving satellite is having a tangential velocity which manages to keep it moving in a circular orbit at that height.

5.8.3 Time Period of a Satellite:

The time taken by a satellite to complete one revolution round the Earth is its time period.

Consider a satellite of mass m projected to height h and provided horizontal velocity equal to the critical velocity. The satellite revolves in a circular orbit of radius $(R+h) = r$.

The distance traced by satellite in one revolution is equal to the circumference of the circular orbit within periodic time T.

$$\therefore \text{Critical speed} = \frac{\text{Circumference of the orbit}}{\text{Time period}}$$

$$v_c = \frac{2\pi r}{T}$$

but we have, $v_c = \sqrt{\frac{Gm}{r}}$

$$\therefore \sqrt{\frac{Gm}{r}} = \frac{2\pi r}{T}$$

or, $$\frac{GM}{r} = \frac{4\pi^2 r^2}{T^2}$$

$$\therefore T^2 = \frac{4\pi^2 r^3}{GM}$$

As π^2, G and M are constant, $T^2 \propto r^3$, i.e., the square of period of revolution of satellite is directly proportional to the cube of the radius of orbit.

$$T = 2\pi\sqrt{\frac{r^3}{GM}}$$

$$\therefore T = 2\pi\sqrt{\frac{(R+h)^3}{GM}} \quad \text{--- (5.36)}$$

This is an expression for period of satellite revolving in a *circular orbit* round the Earth. Period of a satellite does not depend on its mass. It depends on mass of the Earth, radius of the Earth and the height of the satellite. If the height of projection is increased, period of the satellite increases. Period of the satellite can also be obtained in terms of acceleration due to gravity.

As $GM = g_h(R+h)^2$

$$\therefore T = 2\pi\sqrt{\frac{(R+h)^3}{g_h(R+h)^2}}$$

$$\therefore T = 2\pi\sqrt{\frac{R+h}{g_h}}$$

$$\therefore T = 2\pi\sqrt{\frac{r}{g_h}} \quad \text{--- (5.37)}$$

Special case :

When satellite revolves close to the surface of the Earth, $R + h \approx R$ and $g_h \approx g$. Hence the minimum period of revolution is

$$(T)_{min} = 2\pi\sqrt{\frac{R}{g}} \quad \text{--- (5.38)}$$

Example 5.9: Calculate the period of revolution of a polar satellite orbiting close to the surface of the Earth. Given $R = 6400$ km, $g = 9.8$ m/s^2.

Solution : h is negligible as satellite is close to the Earth surface.

$$\therefore R + h \approx R$$

$$g_h \approx g$$

$$R = 6400 \text{ km} = 6.4\times10^6 \text{ m.}$$

$$T = 2\pi\sqrt{\frac{R}{g}}$$

$$= 2\times3.14\sqrt{\frac{6.4\times10^6}{9.8}}$$

$$= 5.075\times10^3 \text{ second}$$

$$= 85 \text{ minute (approximately)}$$

Example 5.10: An artificial satellite revolves around a planet in circular orbit close to its surface. Obtain the formula for period of the satellite in terms of density ρ and radius R of planet.

Solution : Period of satellite is given by,

$$T = 2\pi\sqrt{\frac{(R+h)^3}{GM}} \quad \text{--- (1)}$$

Here, the satellite revolves close to the surface of planet, hence h is negligible, hence $R + h \simeq R$

$$\text{density}(\rho) = \frac{\text{mass}(M)}{\text{volume}(V)}$$

$$\therefore M = \rho V \quad \text{--- (2)}$$

As planet is spherical in shape, volume of planet is given as

$$V = \frac{4}{3}\pi R^3$$

$$\therefore M = \frac{4}{3}\pi R^3\rho \quad \text{--- (3)}$$

Substituting the values form eq. (2) and (3) in Eq. (1), we get

$$T = 2\pi\sqrt{\frac{R^3}{G\times\frac{4}{3}\pi R^3\rho}}$$

$$\therefore T = \sqrt{\frac{3\pi}{G\rho}}$$

5.8.4 Binding Energy of an orbiting satellite:

The minimum energy required by a satellite to escape from Earth's gravitational influence is the binding energy of the satellite.

Expression for Binding Energy of satellite revolving in circular orbit round the Earth

Consider a satellite of mass m revolving at height h above the surface of the Earth in a circular orbit. It possesses potential energy as well as kinetic energy. Let M be the mass of the Earth, R be the Radius of the Earth, v_c be critical velocity of satellite, $r = (R+h)$ be the radius of the orbit.

∴Kinetic energy of satellite

$$= \frac{1}{2}mv_c^{\,2}$$

$$= \frac{1}{2}\frac{GMm}{r} \quad \text{--- (5.39)}$$

The gravitational potential at a distance r from the centre of the Earth is $-\frac{GM}{r}$

∴ Potential energy of satellite = Gravitational potential × mass of satellite

$$= -\frac{GMm}{r} \quad \text{--- (5.40)}$$

The total energy of satellite is given as

$T.E. = K.E. + P.E.$

$$= \frac{1}{2}\frac{GMm}{r} - \frac{GMm}{r}$$

$$= -\frac{1}{2}\frac{GMm}{r} \qquad \text{--- (5.41)}$$

Total energy of a circularly orbiting satellite is negative. Negative sign indicates that the satellite is bound to the Earth, due to gravitational force of attraction. For the satellite to be free from the Earth's gravitational influence its total energy should become non-negative (zero or positive). Hence the minimum energy to be supplied to unbind the satellite is $+\frac{1}{2}\frac{GMm}{r}$ This is the binding energy of a satellite.

Internet my friend

hyperphysics.phy-astr.gsu.edu/hbase/grav.html#grav

Exercises

1. Choose the correct option.

i) The value of acceleration due to gravity is maximum at ______

(A) the equator of the Earth .

(B) the centre of the Earth.

(C) the pole of the Earth.

(D) slightly above the surface of the Earth.

ii) The weight of a particle at the centre of the Earth is ______

(A) infinite.

(B) zero.

(C) same as that at other places.

(D) greater than at the poles.

iii) The gravitational potential due to the Earth is minimum at ______

(A) the centre of the Earth.

(B) the surface of the Earth.

(C) a points inside the Earth but not at its centre.

(D) infinite distance.

iv) The binding energy of a satellite revolving around planet in a circular orbit is 3×10^{9} J. Its kinetic energy is ______

(A) 6×10^{9}J

(B) -3 $\times10^{9}$J

(C) -6 $\times10^{+9}$J

(D) 3 $\times10^{+9}$J

2. Answer the following questions.

i) State Kepler's law equal of area.

ii) State Kepler's law of period.

iii) What are the dimensions of the universal gravitational constant?

iv) Define binding energy of a satellite.

v) What do you mean by geostationary satellite?

vi) State Newton's law of gravitation.

vii) Define escape velocity of a satellite.

viii) What is the variation in acceleration due to gravity with altitude?

ix) On which factors does the escape speed of a body from the surface of Earth depend?

x) As we go from one planet to another planet, how will the mass and weight of a body change?

xi) What is periodic time of a geostationary satellite?

xii) State Newton's law of gravitation and express it in vector form.

xiii) What do you mean by gravitational constant? State its SI units.

xiv) Why is a minimum two stage rocket necessary for launching of a satellite?

xv) State the conditions for various possible orbits of a satellite depending upon the tangential speed of projection.

2. Answer the following questions in detail.

i) Derive an expression for critical velocity of a satellite.

ii) State any four applications of a communication satellite.

iii) Show that acceleration due to gravity at height h above the Earth's surface is $g_h = g\left(\frac{R}{R+h}\right)^2$

iv) Draw a labelled diagram to show different trajectories of a satellite depending upon the tangential projection speed.

v) Derive an expression for binding energy of a body at rest on the Earth's surface.

vi) Why do astronauts in an orbiting satellite have a feeling of weightlessness?

vii) Draw a graph showing the variation of gravitational acceleration due to the depth and altitude from the Earth's surface.

viii) At which place on the Earth's surface is the gravitational acceleration maximum? Why?

ix) At which place on the Earth surface the gravitational acceleration minimum? Why?

x) Define the binding energy of a satellite. Obtain an expression for binding energy of a satellite revolving around the Earth at certain attitude.

xi) Obtain the formula for acceleration due to gravity at the depth 'd' below the Earth's surface.

xii) State Kepler's three laws of planetary motion.

xiii) State the formula for acceleration due to gravity at depth 'd' and altitude 'h' Hence show that their ratio is equal to $\left(\frac{R-d}{R-2h}\right)$ by assuming that the altitude is very small as compared to the radius of the Earth.

xiv) What is critical velocity? Obtain an expression for critical velocity of an orbiting satellite. On what factors does it depend?

xv) Define escape speed. Derive an expression for the escape speed of an object from the surface of the each.

xvi) Describe how an artificial satellite using two stage rocket is launched in an orbit around the Earth.

4. Solve the following problems.

i) At what distance below the surface of the Earth, the acceleration due to gravity decreases by 10% of its value at the surface, given radius of Earth is 6400 km.

[Ans: 640 km].

ii) If the Earth were made of wood, the mass of wooden Earth would have been 10% as much as it is now (without change in its diameter). Calculate escape speed from the surface of this Earth.

[Ans: 3.54 km/s]

iii) Calculate the kinetic energy, potential energy, total energy and binding energy of an artificial satellite of mass 2000 kg orbiting at a height of 3600 km above the surface of the Earth.

Given:- $G = 6.67\times10^{-11}$ Nm^2/kg^2

$R = 6400$ km

$M = 6\times10^{24}$ kg

[Ans: $KE = 40.02\times10^9$J,

$PE = -80.09\times10^9$J,

$TE = -40.07\times10^9$J,

$BE = 40.02\times10^9$J]

iv) Two satellites A and B are revolving around a planet. Their periods of revolution are 1 hour and 8 hours respectively. The radius of orbit of satellite B is 4×10^4 km. find radius of orbit of satellite A .

[Ans: 1×10^4 km]

v) Find the gravitational force between the Sun and the Earth.

Given Mass of the Sun = 1.99×10^{30} kg

Mass of the Earth = 5.98×10^{24} kg

The average distance between the Earth and the Sun = 1.5×10^{11} m.

[Ans: 3.5×10^{22} N]

vi) Calculate the acceleration due to gravity at a height of 300 km from the surface of the Earth. ($M = 5.98 \times10^{24}$ kg, $R = 6400$ km).

[Ans :- 8.889 m/s^2]

vii) Calculate the speed of a satellite in an orbit at a height of 1000 km from the Earth's surface. $M_E=5.98\times10^{24}$ kg, $R = 6.4\times10^6$ m.

[Ans : 7.34×10^3 m/s]

viii) Calculate the value of acceleration due to gravity on the surface of Mars if the radius of Mars = 3.4×10^3 km and its mass is 6.4×10^{23} kg.

[Ans : 3.69 m/s^2]

ix) A planet has mass 6.4×10^{24} kg and radius 3.4×10^6 m. Calculate energy required to remove on object of mass 800 kg from the surface of the planet to infinity.

[Ans : 5.02×10^{10}J]

x) Calculate the value of the universal gravitational constant from the given data. Mass of the Earth = 6×10^{24} kg, Radius of the Earth = 6400 km and the acceleration due to gravity on the surface = 9.8 m/s^2

[Ans : 6.69×10^{-11} N m^2/kg^2]

xi) A body weighs 5.6 kg wt on the surface of the Earth. How much will be its weight on a planet whose mass is 7 times the mass of the Earth and radius twice that of the Earth's radius.

[Ans: 9.8 kg-wt]

xii). What is the gravitational potential due to the Earth at a point which is at a height of $2R_E$ above the surface of the Earth, Mass of the Earth is 6×10^{24} kg, radius of the Earth = 6400 km and G = 6.67×10^{-11} Nm^2 kg^{-2}.

[Ans: 2.08×10^7 J]

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