9 DIFFERENTIATION

Let's :Learn

- The meaning of rate of change.
- Definition of derivative and the formula associated with it.
- Derivatives of some standard functions.
- Relationship between Continuity and Differentiability.

Let's Recall

- Real valued functions on R.
- Limits of functions.
- Continuity of a function at a point and over an interval.

9.1.1 INTRODUCTION :

Suppose we are travelling in a car from Mumbai to Pune. We are displacing ourselves from the origin (Mumbai) from time to time. We know that the average speed of the car

$$= \frac{\text{Total distance travelled}}{\text{Time taken to travel that distance}}$$

But at different times the speed of the car can be different. It is the ratio of a very small distance travelled, with the small time interval required to travel that distance. The limit of this ratio as the time interval tends to zero is the speed of the car at that time. This process of obtaining the speed is given by the differentiation of the distance function with respect to time. This is an example of derivative or differentiation. This measures how quickly the car moves with time. Speed is the rate of change of distance with time.

When we speak of velocity, it is the speed with the direction of movements. In problems with no change in direction, words speed and velocity may be interchanged.

The rate of change in a function at a point with respect to the variable is called the derivative of the function at that point. The process of finding a derivative is called differentiation

9.1.2 DEFINITION OF DERIVATIVE AND DIFFERENTIABILITY

Let $f(x)$ be a function defined on an open interval containing the point 'a'. If

$\lim\limits_{\delta x \to 0}\left(\frac{f(a+\delta x)-f(a)}{\delta x}\right)$ exists, then f is said to be differentiable at $x = a$ and this limit is said to be the derivative of f at a and is denoted by $f'(a)$.

We can calculate derivative of 'f' at any point x in the domain of f.

Let $y = f(x)$ be a function. Let there be a small increment in the value of 'x', say δx, then correspondingly there will be a small increment in the value of y say δy.

$$\therefore\ y+\delta y = f(x+\delta x)$$

$$\therefore\ \delta y = f(x+\delta x) - y$$

$$\delta y = f(x+\delta x) - f(x) \qquad \text{....}\ [\because\ y = f(x)]$$

As δx is a small increment and $\delta x \neq 0$, so dividing throughout by δx, we get $\frac{\delta y}{\delta x} = \frac{f(x+\delta x)-f(x)}{\delta x}$

Now, taking the limit as $\delta x \to 0$ we get

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{f(x+\delta x)-f(x)}{\delta x}\right)$$

If the above limit exists, then that limiting value is called the derivative of the function and it is symbolically represented as, $\frac{dy}{dx}$

so $\frac{dy}{dx} = f'(x)$

We can consider the graph of $f(x)$ i.e. $\{(x, y) / y = f(x)\}$ and write the differentiation in terms of y and x

NOTE : (1) If $y = f(x)$ is a differentiable function of x then $\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{dy}{dx}$ and

$$\lim_{\delta x \to 0}\left(\frac{f(x+\delta x)-f(x)}{\delta x}\right) = f'(x)$$

(2) Let $\delta x = h$, Suppose that

$\lim_{h \to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$ exists. It implies that

$$\lim_{h \to 0^-}\left(\frac{f(a+h)-f(a)}{h}\right) = \lim_{h \to 0^+}\left(\frac{f(a+h)-f(a)}{h}\right)$$

$\lim_{h \to 0^-}\left(\frac{f(a+h)-f(a)}{h}\right)$ is called the Left Hand Derivative (LHD) at $x = a$

$\lim_{h \to 0^+}\left(\frac{f(a+h)-f(a)}{h}\right)$ is called the Right Hand Derivative (RHD) at $x = a$

Generally LHD at $x = a$ is represented as $f'(a^-)$ or L $f'(a)$, and RHD at $x = a$ is represented as $f'(a^+)$ or R $f'(a)$

9.1.3 DERIVATIVE BY METHOD OF FIRST PRINCIPLE.

The process of finding the derivative of a function from the definition of derivative is known as derivatives from first principle. Just for the sake of convenience δx can be replaced by h.

If $f(x)$ is a given function on an open interval, then the derivative of $f(x)$ with respect to x by method of first principle is given by

$$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \frac{dy}{dx}$$

The derivative of $y = f(x)$ with respect to x at $x = a$ by method of first principle is given by

$$f'(a) = \lim_{h \to 0}\left(\frac{f(a+h)-f(a)}{h}\right) = \left(\frac{dy}{dx}\right)_{x=a}$$

9.1.4 DERIVATIVES OF SOME STANDARD FUNCTIONS

(1) Find the derivative of x^n w. r. t. x. for $n \in N$

Solution :

Let $f(x) = x^n$

$f(x+h) = (x+h)^n$

By method of first principle,

$$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x) = \lim_{h \to 0}\left(\frac{(x+h)^n - x^n}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{x^n + {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \dots\dots + h^n - x^n}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{{}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + {}^nC_3 x^{n-3}h^3 + \dots\dots + h^n}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{h({}^nC_1 x^{n-1} + {}^nC_2 x^{n-2}h + {}^nC_3 x^{n-3}h^2 + \dots\dots + h^{n-1})}{h}\right)$$

$$= \lim_{h \to 0}\left(nx^{n-1} + {}^nC_2 x^{n-2}h + {}^nC_3 x^{n-3}h^2 + \dots\dots + h^{n-1}\right)$$

$= nx^{n-1} + 0 + 0 + \dots + 0$ (as $h \to 0$, $h \neq 0$)

$\therefore$ if $f(x) = x^n$, $f'(x) = nx^{n-1}$

(2) Find derivative of $\sin x$ w. r. t. x.

Solution :

Let $f(x) = \sin x$

$f(x+h) = \sin(x+h)$

By method of first principle,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x) = \lim_{h\to 0}\left(\frac{\sin(x+h)-\sin x}{h}\right)$$

$$\lim_{h\to 0}\left(\frac{2\cos\left(\frac{2x+h}{2}\right).\sin\left(\frac{h}{2}\right)}{h}\right)$$

$$= \quad 2\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\lim_{h\to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)\left(\frac{1}{2}\right)$$

$$= \quad 2\lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\lim_{h\to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)\left(\frac{1}{2}\right)$$

$$= \quad 2\cos x.(1)\left(\frac{1}{2}\right)\text{.......}\left[\because \lim_{\theta\to 0}\left(\frac{\sin p\theta}{p\theta}\right)=1\right]$$

$\therefore$ if $f(x) = \sin x, f'(x) = \cos x$

(3) Find the derivative of $\tan x$ w. r. t. x.

Solution:

Let $f(x) = \tan x$

$f(x+h) = \tan(x+h)$

From the definition,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x) = \lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\frac{\sin(x+h).\cos x-\cos(x+h)\sin x}{\cos(x+h).\cos x}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin h}{h.\cos(x+h).\cos x}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin h}{h}\right).\lim_{h\to 0}\left(\frac{1}{\cos(x+h).\cos x}\right)$$

$$= (1).\left(\frac{1}{\cos^2 x}\right)\text{........}\left[\because \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1\right]$$

$\therefore$ if $f(x) = \tan x, f'(x) = \sec^2 x$

(4) Find the derivative of $\sec x$ w. r. t. x.

Solution:

Let $f(x) = \sec x$

$f(x+h) = \sec(x+h)$

From the definition,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x) = \lim_{h\to 0}\left(\frac{\sec(x+h)-\sec x}{h}\right)$$

$$\lim_{h\to 0}\left(\frac{\frac{1}{\cos(x+h)}-\frac{1}{\cos x}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\frac{\cos x-\cos(x+h)}{\cos(x+h).\cos x}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{-2\sin\left(\frac{2x+h}{2}\right).\sin\left(\frac{-h}{2}\right)}{h.\cos(x+h).\cos x}\right)$$

$$= 2\lim_{h\to 0}\left(\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h).\cos x}\right)\lim_{h\to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)\left(\frac{1}{2}\right)$$

$$= \frac{2\sin x}{\cos^2 x}.(1).\left(\frac{1}{2}\right).......\left[\because \lim_{\theta\to 0}\left(\frac{\sin p\theta}{p\theta}\right)=1\right]$$

$\therefore$ if $f(x) = \sec x, f'(x) = \sec x \tan x$

(5) Find the derivative of $\log x$ w. r. t. $x.(x > 0)$

Solution:

Let $f(x) = \log x$

$f(x+h) = \log(x+h)$

From the definition,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x) = \lim_{h\to 0}\left(\frac{\log(x+h)-\log(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log\left(\frac{x+h}{x}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log\left(1+\frac{h}{x}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\right)\times\left(\frac{1}{x}\right)$$

$$= 1.\left(\frac{1}{x}\right) \qquad \left[\because \lim_{x\to 0}\left(\frac{\log(1+x)}{x}\right)=1\right]$$

$\therefore$ if $f(x) = \log x, f'(x) = \dfrac{1}{x}$

(6) Find then derivative of a^x w. r. t. x. $(a > 0)$

Solution:

Let $f(x) = a^x$

$f(x+h) = a^{x+h}$

From the definition,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x) = \lim_{h\to 0}\left(\frac{a^{(x+h)}-a^x}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{a^x(a^h-1)}{h}\right)$$

$$= a^x\lim_{h\to 0}\left(\frac{a^h-1}{h}\right)$$

$$f'(x) = a^x \log a.........\left[\because \lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\log a\right]$$

$\therefore$ if $f(x) = a^x,\ f'(x) = a^x.\log a$

Try the following

(1) If $f(x)=\frac{1}{x^n}$, for $x \neq 0$, n$\in$N, then prove that

$$f'(x)=-\frac{n}{x^{n+1}}$$

(2) If f(x) = cos x, then prove that

$$f'(x)=-\sin x$$

(3) If f(x) = cot x, then prove that

$$f'(x)=-\cos ec^2 x$$

(4) If f(x) = cosec x, then prove that

$$f'(x)=-\cos ecx.\cot x$$

(5) If $f(x)=e^x$, then prove that $f'(x)=e^x$

SOLVED EXAMPLES

Ex. 1. Find the derivatives of the following from the definition,

(i) $\sqrt{x}$ (ii) cos (2x+3) (iii) 4^x (iv) log(3x–2)

Solution :

(i) Let $f(x)=\sqrt{x}$

$$f(x+h)=\sqrt{x+h}$$

From the definition,

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x)=\lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$=\lim_{h\to 0}\left(\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\right)$$

$$=\lim_{h\to 0}\left(\frac{h}{h(\sqrt{x+h}+\sqrt{x})}\right)$$

$$=\lim_{h\to 0}\left(\frac{1}{\sqrt{x+h}+\sqrt{x}}\right) \text{...[As } h\to 0,\ h\neq 0]$$

$$=\frac{1}{2\sqrt{x}}$$

(ii) Let $f(x)$ = cos (2x + 3)

$f(x+h)$ = cos (2(x+h)+3) =cos ((2x+3)+2h)

From the definition,

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x)=\lim_{h\to 0}\left(\frac{\cos[(2x+3)+2h]-\cos(2x+3)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{-2\sin\left(\frac{2(2x+3)+2h}{2}\right)\sin(h)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{-2\sin(2x+3+h)\sin(h)}{h}\right)$$

$$=\lim_{h\to 0}[-2\sin(2x+3+h)]\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

$$=[-2\sin(2x+3)](1)\text{.......}\left[\because \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1\right]$$

$$=-2\sin(2x+3)$$

(iii) Let $f(x)=4^x$

$$f(x+h)=4^{x+h}$$

From the definition

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x)=\lim_{h\to 0}\left(\frac{4^{x+h}-4^{x}}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{4^{x}(4^{h}-1)}{h}\right)$$

$$=4^{x}\lim_{h\to 0}\left(\frac{4^{h}-1}{h}\right)$$

$$=4^{x}\log 4\text{......}\left[\because \lim_{x\to 0}\left(\frac{a^{x}-1}{x}\right)=\log a\right]$$

(iv) Let $f(x)=\log(3x-2)$

$$f(x+h)=\log[3(x+h)-2]=\log[(3x-2)+3h]$$

From the definition,

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x)=\lim_{h\to 0}\left(\frac{\log[(3x-2)+3h]-\log(3x-2)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\log\left(\frac{(3x-2)+3h}{3x-2}\right)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\log\left(1+\frac{3h}{3x-2}\right)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\log\left(1+\frac{3h}{3x-2}\right)}{\frac{3h}{3x-2}}\right)\times\left(\frac{3}{3x-2}\right)$$

$$=(1)\times\left(\frac{3}{3x-2}\right)\quad\left[\because \lim_{x\to 0}\left(\frac{\log(1+px)}{px}\right)=1\right]$$

$$=\frac{3}{3x-2}$$

Ex. 2. Find the derivative of $f(x) = \sin x$, at $x=\pi$

Solution:

$$f(x)=\sin x$$

$$f(\pi)=\sin\pi=0$$

$$f(\pi+h)=\sin(\pi+h)=-\sin h$$

From the definition,

$$f'(a)=\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$$

$$f'(\pi)=\lim_{h\to 0}\left(\frac{f(\pi+h)-f(\pi)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{-\sin h-0}{h}\right)=\lim_{h\to 0}\left(-\frac{\sin h}{h}\right)$$

$$=-\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

$$=-1\quad\left[\because \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1\right]$$

Ex. 3. Find the derivative of $x^{2}+x+2$, at $x=-3$

Solution :

Let $f(x)=x^{2}+x+2$

For $x = -3$, $f(-3) = (-3)^2 - 3+2 = 9 - 3 + 2 = 8$

$$f(-3+h)=(-3+h)^{2}+(-3+h)+2$$
$$=h^{2}-6h+9-3+h+2=h^{2}-5h+8$$

From the definition,

$$f'(a)=\lim_{h\to0}\left(\frac{f(a+h)-f(a)}{h}\right)$$

$$f'(-3)=\lim_{h\to0}\left(\frac{f(-3+h)-f(-3)}{h}\right)$$

$$=\lim_{h\to0}\left(\frac{h^2-5h+8-8}{h}\right)$$

$$=\lim_{h\to0}\left(\frac{h^2-5h}{h}\right)$$

$$=\lim_{h\to0}\left(\frac{h(h-5)}{h}\right)$$

$$=\lim_{h\to0}(h-5)......[h\to0,h\neq0]$$

$$=-5$$

9.1.5 RELATIONSHIP BETWEEN DIFFERENTIABILITY AND CONTINUITY

Theorem : Every differentiable function is continuous.

Proof : Let $f(x)$ be differentiable at $x = a$.

Then, $f'(a)=\lim_{h\to0}\left(\frac{f(a+h)-f(a)}{h}\right)$ .... (1)

we have to prove that $f(x)$ is continuous at $x = a$.

i.e. we have to prove that $\lim_{x\to a}f(x)=f(a)$

Let $x = a + h$, $x\to a$, $h\to0$

We need to show that

$$\lim_{h\to0}f(a+h)=f(a)$$

The equation (1) can also be written as

$$\lim_{h\to0}\left(\frac{f(a+h)-f(a)}{h}\right)=\lim_{h\to0}f'(a)$$

As $h\to0$, $h\neq0$

Multiplying both the sides of above equation by h we get

$$\lim_{h\to0}\left[h\left(\frac{f(a+h)-f(a)}{h}\right)\right]=\lim_{h\to0}\left[hf'(a)\right]$$

$$\lim_{h\to0}[f(a+h)-f(a)]=0[f'(a)]=0$$

$$\therefore\ \lim_{h\to0}f(a+h)=f(a)$$

This proves that $f(x)$ is continuous at $x = a$.

Hence every differentiable function is continuous.

Note that a continuous function need not be differentiable.

This can be proved by an example.

Ex.: Let $f(x)=|x|$ be defined on R.

$$f(x) = -x \quad \text{for } x<0$$
$$= x \quad \text{for } x\geq0$$

Consider, $\lim_{x\to0^-}f(x)=\lim_{x\to0^-}(-x)=0$

$$\lim_{x\to0^+}f(x)=\lim_{x\to0^+}(x)=0$$

For, $x = 0$, $f(0) = 0$

$$\lim_{x\to0^-}f(x)=\lim_{x\to0^+}(x)=f(0)=0$$

Hence $f(x)$ is continuous at $x = 0$.

Now, we have to prove that $f(x)$ is not differentiable at $x = 0$ i.e. $f'(0)$ doesn't exist.

i.e. we have to prove that,

$$\lim_{h\to0^-}\left(\frac{f(0+h)-f(0)}{h}\right)\neq\lim_{h\to0^+}\left(\frac{f(0+h)-f(0)}{h}\right)$$

We have, L. H. D. at $x = 0$, is $f'(0^-)$

$$=\lim_{h\to0^-}\left(\frac{f(0+h)-f(0)}{h}\right)$$

$$=\lim_{h\to0^-}\left(\frac{-h}{h}\right)=\lim_{h\to0^-}(-1)$$

$f'(0^-) = -1$ .... (I)

Now, R. H. D. at $x = 0$, is $f'(0^+)$

$$= \lim_{h \to 0^+} \left(\frac{f(0+h) - f(0)}{h} \right)$$

$$= \lim_{h \to 0^-} \left(\frac{h}{h} \right) = \lim_{h \to 0^+} (1)$$

$f'(0^+) = 1$.......... (II)

Therefore from (I) and (II), we get

$f'(0^-) \neq f'(0^+)$ that is

$$\lim_{h \to 0^-} \left(\frac{f(0+h) - f(0)}{h} \right) \neq \lim_{h \to 0^+} \left(\frac{f(0+h) - f(0)}{h} \right)$$

Though $f(x)$ is continuous at $x = 0$, it is not differentiable at $x = 0$.

SOLVED EXAMPLES

Ex. 1. Test whether the function $f(x) = (3x-2)^{\frac{2}{5}}$ is differentiable at $x = \frac{2}{3}$

Solution :

Given that, $f(x) = (3x-2)^{\frac{2}{5}}$

$$f'(a) = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right)$$

Note that $f\left(\frac{2}{3}\right) = 0$

For, $x = \frac{2}{3}$, $f'\left(\frac{2}{3}\right) = \lim_{h \to 0} \left(\frac{f\left(\frac{2}{3}+h\right) - f\left(\frac{2}{3}\right)}{h} \right)$

$$= \lim_{h \to 0} \left(\frac{\left[3\left(\frac{2}{3}+h\right) - 2\right]^{\frac{2}{5}}}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{[2+3h-2]^{\frac{2}{5}}}{h} \right)$$

$$= \lim_{h \to 0} \left(\frac{(3h)^{\frac{2}{5}}}{h} \right) = 3^{\frac{2}{5}} \lim_{h \to 0} \left(\frac{h^{\frac{2}{5}}}{h} \right)$$

$$= 3^{\frac{2}{5}} \lim_{h \to 0} \left(\frac{1}{h^{\frac{3}{5}}} \right)$$

This limit does not exist.

$\therefore$ $f(x)$ is not differentiable at $x = \frac{2}{3}$

Ex. 2. Examine the differentiability of

$f(x) = (x-2)|x-2|$ at $x = 2$

Solution : Given that $f(x) = (x-2)|x-2|$

That is $f(x) = -(x-2)^2$ for $x < 2$

$= (x-2)^2$ for $x \geq 2$

$$Lf'(2) = \lim_{h \to 0^-} \left(\frac{f(2+h) - f(2)}{h} \right)$$

$$= \lim_{h \to 0^-} \left(\frac{-(2+h-2)^2 - (2-2)^2}{h} \right)$$

$$= \lim_{h \to 0^-} \left(\frac{-h^2}{h} \right) = \lim_{h \to 0^-} (-h)$$

$Lf'(2) = 0$

$$Rf'(2) = \lim_{h \to 0^+} \left(\frac{f(2+h) - f(2)}{h} \right)$$

$$= \lim_{h \to 0^+} \left(\frac{(2+h-2)^2 - (2-2)^2}{h} \right)$$

$$= \lim_{h \to 0^+} \left(\frac{h^2}{h} \right) = \lim_{h \to 0^+} (h)$$

$Rf'(2) = 0$

So, $Lf'(2) = Rf'(2) = 0$

Hence the function $f(x)$ is differentiable at $x = 2$.

Ex. 3. Show the function $f(x)$ is continuous at $x = 3$, but not differentiable at $x = 3$. if

$$f(x) = 2x + 1 \quad \text{for } x \le 3$$
$$= 16 - x^2 \quad \text{for } x > 3.$$

Solution : $f(x) = 2x + 1$ for $x \le 3$

$$= 16 - x^2 \quad \text{for } x > 3.$$

For $x = 3$, $f(3) = 2(3) + 1 = 7$

$$\lim_{x\to3^-} f(x) = \lim_{x\to3}(2x+1) = 2(3)+1 = 7$$

$$\lim_{x\to3^+} f(x) = \lim_{x\to3}(16-x^2) = 16-(3)^2 = 7$$

$$\lim_{x\to3^-} f(x) = \lim_{x\to3^+} f(x) = f(3) = 7$$

$\therefore$ $f(x)$ is continuous at $x = 3$.

$$Lf'(3) = \lim_{h\to0^-}\left(\frac{f(3+h)-f(3)}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{2(3+h)+1-7}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{6+2h+1-7}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{2h}{h}\right) = \lim_{h\to0}(2)$$

$Lf'(3) = 2$ .... (1)

$$Rf'(3) = \lim_{h\to0^+}\left(\frac{f(3+h)-f(3)}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{16-(3+h)^2-7}{h}\right) = \lim_{h\to0}\left(\frac{16-9-6h-h^2-7}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{-h(6+h)}{h}\right) = \lim_{h\to0}(-(6+h))$$

$Rf'(3) = -6$ .... (2)

from (1) and (2), $Lf'(3) \ne Rf'(3)$

$\therefore$ $f(x)$ is not differentiable at $x = 3$.

Hence $f(x)$ is continuous at $x = 3$, but not differentiable at $x = 3$.

Ex. 4. Show that the function $f(x)$ is differentiable at $x = -3$ where, $f(x) = x^2 + 2$.

Solution :

For $x = -3$, $$f'(-3) = \lim_{h\to0}\left(\frac{f(-3+h)-f(-3)}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{(-3+h)^2+2-11}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{9-6h+h^2+2-11}{h}\right)$$

$$= \lim_{h\to0}\left(\frac{h^2-6h}{h}\right)$$

$$= \lim_{h\to0}(h-6) \quad (\because h\to0,\ h\ne0)$$

$$= -6$$

$f'(-3)$ exists so, $f(x)$ is differentiable at $x = -3$.

EXERCISE 9.1

(1) Find the derivatives of the following w. r. t. x by using method of first principle.

(a) x^2+3x-1 (b) $\sin(3x)$

(c) e^{2x+1} (d) 3^x (e) $\log(2x+5)$

(f) $\tan(2x+3)$ (g) $\sec(5x-2)$

(h) $x\sqrt{x}$

(2) Find the derivatives of the following w. r. t. x. at the points indicated against them by using method of first principle

(a) $\sqrt{2x+5}$ at $x = 2$ (b) $\tan x$ at $x = \pi/4$

(c) 2^{3x+1} at $x = 2$ (d) $\log(2x+1)$ at $x = 2$

(e) e^{3x-4} at $x = 2$ (f) $\cos x$ at $x = \frac{5\pi}{4}$

(3) Show that the function f is not differentiable at $x = -3$,

where $f(x) = x^2 + 2$ for $x < -3$

$= 2 - 3x$ for $x \geq -3$

(4) Show that $f(x) = x^2$ is continuous and differentiable at $x = 0$.

(5) Discuss the continuity and differentiability of

(i) $f(x) = x|x|$ at $x = 0$

(ii) $f(x) = (2x+3)|2x+3|$ at $x = -3/2$

(6) Discuss the continuity and differentiability of $f(x)$ at $x = 2$

$f(x) = [x]$ if $x \in [0,4)$. [where [*] is a greatest integer (floor) function]

(7) Test the continuity and differentiability of $f(x) = 3x+2$ if $x > 2$

$= 12 - x^2$ if $x \leq 2$ at $x = 2$.

(8) If $f(x) = \sin x - \cos x$ if $x \leq \pi/2$
$= 2x - \pi + 1$ if $x > \pi/2$. Test the continuity and differentiability of f at $x = \pi/2$

(9) Examine the function

$$f(x) = x^2 \cos\left(\frac{1}{x}\right), \text{ for } x \neq 0$$

$$= 0, \quad \text{for } x = 0$$

for continuity and differentiability at $x = 0$.

9.2 RULES OF DIFFERENTIATION

9.2.1. Theorem 1. Derivative of Sum of functions.

If u and v are differentiable functions of x such that $y = u + v$, then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Proof: Given that, $y = u + v$ where u and v are differentiable functions of x

Let there be a small increment in the value of x, say δx, then u changes to $(u + \delta u)$ and v changes to $(v + \delta v)$ respectively, correspondingly y changes to $(y + \delta y)$

$$\therefore (y + \delta y) = (u + \delta u) + (v + \delta v)$$

$$\therefore \delta y = (u + \delta u) + (v + \delta v) - y$$

$$\therefore \delta y = (u + \delta u) + (v + \delta v) - (u + v)$$

$[\because y = u + v]$

$$\therefore \delta y = \delta u + \delta v$$

As δx is small increment in x and $\delta x \neq 0$, dividing throughout by x we get,

$$\frac{\delta y}{\delta x} = \frac{\delta u + \delta v}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}$$

Taking the limit as $\delta x \to 0$, we get,

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}\right)$$

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right) + \lim_{\delta x \to 0}\left(\frac{\delta v}{\delta x}\right) \quad \text{...... (I)}$$

Since u and v are differentiable function of x

$$\therefore \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right) = \frac{du}{dx} \text{ and } \lim_{\delta x \to 0}\left(\frac{\delta v}{\delta x}\right) = \frac{dv}{dx} \quad \text{..... (II)}$$

$$\therefore \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{du}{dx} + \frac{dv}{dx} \quad \text{[From (I) and (II)]}$$

i.e. $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

9.2.2 Theorem 2. Derivative of Difference of functions.

If u and v are differentiable functions of x such that $y = u - v$, then $\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

[Left for students to prove]

Corollary : If $u, v, w.....$ are finite number of differentiable functions of x such that $y = k_1u \pm k_2v \pm k_3w \pm .....$

then $\frac{dy}{dx} = k_1\frac{du}{dx} \pm k_2\frac{dv}{dx} \pm k_3\frac{dw}{dx}$ ......

9.2.3 Theorem 3. Derivative of Product of functions.

If u and v are differentiable functions of x such that $y = u.v$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Proof : Given that y = uv

Let there be a small increment in the value of x, say δx, then u changes to $(u+\delta u)$ and v changes to $(v+\delta v)$ respectively, correspondingly y changes to $(y+\delta y)$

$\therefore$ $y+\delta y = (u+\delta u)(v+\delta v)$

$\therefore$ $\delta y = (u+\delta u)(v+\delta v) - y$

$\therefore$ $\delta y = uv + u\delta v + v\delta u + \delta u\delta v - uv$

$\therefore$ $\delta y = u\delta v + v\delta u + \delta u.\delta v$

As δx is small increment in x and $\delta x \neq 0$, dividing throughout by δx We get,

$$\frac{\delta y}{\delta x} = \frac{u\delta v + v\delta u + \delta u\delta v}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta v}{\delta x} + \frac{\delta u\delta v}{\delta x}$$

Taking the limit as $\delta x \to 0$, we get,

$$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x\to 0}\left(u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u\delta v}{\delta x}\right)$$

$$u\lim_{\delta x\to 0}\left(\frac{\delta v}{\delta x}\right) + v\lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x}\right) + \lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x}\right)\lim_{\delta x\to 0}(\delta v)$$

As $\delta x \to 0$, we get $\delta v \to 0$

$$\lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = u\lim_{\delta x\to 0}\left(\frac{\delta v}{\delta x}\right) + v\lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x}\right)$$

$$+\lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x}\right)\lim_{\delta v\to 0}(\delta v) \quad .... (1)$$

Given that, u and v are differentiable functions of x

$$\therefore \lim_{\delta x\to 0}\left(\frac{\delta u}{\delta x}\right) = \frac{du}{dx} \text{ and } \lim_{\delta x\to 0}\left(\frac{\delta v}{\delta x}\right) = \frac{dv}{dx} \quad ..... (2)$$

$$\therefore \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = u\frac{dv}{dx} + v\frac{du}{dx} + \frac{du}{dx}(0)$$

[From (1) & (2)]

$$\therefore \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right) = u\frac{dv}{dx} + v\frac{du}{dx}$$

i.e. $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Corollary : If u, v and w are differentiable functions of x such that $y = uvw$ then

$$\frac{dy}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

9.2.4 Theorem 4. Derivative of Quotient of functions.

If u and v are differentiable functions of x such that

$$y = \frac{u}{v} \text{ where } v \neq 0 \text{ then } \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Proof : Given that, $y = \frac{u}{v}$, where u and v are differentiable functions of x

Let there be a small increment in the value of x say δx then u changes to $(u+\delta u)$ and v changes to $(v+\delta v)$ respectively, correspondingly y changes to $(y+\delta y)$

$$\therefore y + \delta y = \frac{u+\delta u}{v+\delta v}$$

$$\therefore \delta y = \frac{u+\delta u}{v+\delta v} - y$$

$$\therefore \delta y = \frac{u+\delta u}{v+\delta v} - \frac{u}{v} \ldots\ldots \left[\because \; y = \frac{u}{v} \right]$$

$$\therefore \delta y = \frac{v(u+\delta u) - u(v+\delta v)}{(v+\delta v).v} = \frac{v.u + v.\delta u - u.v - u.\delta v}{(v+\delta v).v}$$

$$\delta y = \frac{v.\delta u - u.\delta v}{(v+\delta v).v}$$

As δx is small increment in x and $\delta x \neq 0$, dividing throughout by δx We get,

$$\frac{\delta y}{\delta x} = \frac{v.\delta u - u.\delta v}{\delta x.(v+\delta v).v} = \frac{v.\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v^2 + v.\delta v}$$

Taking the limit as $\delta x \to 0$, we get,

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{v.\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v^2 + v.\delta v}\right)$$

As $\delta x \to 0$, we get $\delta v \to 0$

$$= \frac{v.\lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right) - u.\lim_{\delta x \to 0}\left(\frac{\delta v}{\delta x}\right)}{v^2 + v.\lim_{\delta v \to 0}(\delta v)} \quad \ldots\ldots\ldots (1)$$

Since, u and v are differentiable functions of$x$

$$\lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right) = \frac{du}{dx} \text{ and } \lim_{\delta x \to 0}\left(\frac{\delta v}{\delta x}\right) = \frac{dv}{dx} \quad \ldots.. (2)$$

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2 + v.(0)} \quad \ldots \text{[From (1) and (2)]}$$

$$\therefore \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

i.e. $\dfrac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

SOLVED EXAMPLES

Find the derivatives of the following functions

Ex. 1. 1) $y = x^{\frac{3}{2}} + \log x - \cos x$

2) $f(x) = x^5 \operatorname{cosec} x + \sqrt{x}\, \tan x$ 3) $y = \dfrac{e^x - 5}{e^x + 5}$

4) $y = \dfrac{x \sin x}{x + \sin x}$

Solution :

1) Given, $y = x^{\frac{3}{2}} + \log x - \cos x$

Differentiate w.r.t.x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{3}{2}} + \log x - \cos x\right)$$

$$= \frac{d}{dx}\left(x^{\frac{3}{2}}\right) + \frac{d}{dx}(\log x) - \frac{d}{dx}(\cos x)$$

$$= \frac{3}{2}\, x^{\frac{1}{2}} + \frac{1}{x} - (-\sin x)$$

$$\frac{dy}{dx} = \frac{3}{2}\sqrt{x} + \frac{1}{x} + \sin x$$

2) Given $f(x) = x^5 \operatorname{cosec} x + \sqrt{x}\, \tan x$

Differentiate w.r.t.x.

$$f'(x) = \frac{d}{dx}\left(x^5 \operatorname{cosec} x + \sqrt{x}\, \tan x\right)$$

$$= \frac{d}{dx}\left(x^5 \operatorname{cosec} x\right) + \frac{d}{dx}\left(\sqrt{x}\, \tan x\right)$$

$$= x^5 \times \frac{d}{dx}(\operatorname{cosec} x) + \operatorname{cosec} x \times \frac{d}{dx}(x^5) +$$

$$\sqrt{x} \times \frac{d}{dx}(\tan x) + \tan x \times \frac{d}{dx}\sqrt{x}$$

$$= x^5 \times (-\operatorname{cosec} x \times \cot x) + \operatorname{cosec} x \times (5x^4) +$$

$$\sqrt{x} \times (\sec^2 x) + \tan x \times \left(\frac{1}{2\sqrt{x}}\right)$$

$= -x^5 \ cosecx.cotx + 5x^4 \ cosecx + \sqrt{x} \times (sec^2x) + \dfrac{1}{2\sqrt{x}} \tan x$

3) Given that $y = \left(\dfrac{e^x - 5}{e^x + 5}\right)$

Differentiate w.r.t.x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x - 5}{e^x + 5}\right)$$

$$= \frac{(e^x+5)\times\frac{d}{dx}(e^x-5) - (e^x-5)\times\frac{d}{dx}(e^x+5)}{(e^x+5)^2}$$

$$= \frac{(e^x+5).(e^x) - (e^x-5).(e^x)}{(e^x+5)^2}$$

$$= \frac{e^{2x} + 5(e^x) - e^{2x} + 5(e^x)}{(e^x+5)^2}$$

$$= \frac{10e^x}{(e^x+5)^2}$$

4) $y = \dfrac{x \sin x}{x + \sin x}$

Differentiate w.r.t.x.,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x\sin x}{x+\sin x}\right)$$

$$= \frac{(x+\sin x)\times\frac{d}{dx}(x\sin x) - (x\sin x)\times\frac{d}{dx}(x+\sin x)}{(x+\sin x)^2}$$

$$= \frac{(x+\sin x)\times\left(x\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(x)\right) - (x\sin x).(1+\cos x)}{(x+\sin x)^2}$$

$$= \frac{(x+\sin x).(x\cos x + \sin x) - (x\sin x).(1+\cos x)}{(x+\sin x)^2}$$

$$= \frac{x^2\cos x + x\sin x + x\sin x\cos x + \sin^2 x - x\sin x - x\sin x\cos x}{(x+\sin x)^2}$$

$$= \frac{x^2\cos x + \sin^2 x}{(x+\sin x)^2}$$

Ex. 2) If $f(x) = p \tan x + q \sin x + r$, $f(0) = -4\sqrt{3}$, $f\left(\dfrac{\pi}{3}\right) = -7\sqrt{3}$, $f'\left(\dfrac{\pi}{3}\right) = 3$ then find p, q and r.

Solution :

Given that $f(x) = p \tan x + q \sin x + r$ ...(1)

$f'(x) = p \sec^2 x + q \cos x$ ...(2)

$f(0) = -4\sqrt{3}$

put $x = 0$ in (1)

$f(0) = p \tan 0 + q \sin 0 + r = r \ \therefore r = -4\sqrt{3}$

$f\left(\dfrac{\pi}{3}\right) = -7\sqrt{3}$,

$\therefore$ from (1) $p \tan\left(\dfrac{\pi}{3}\right) + q \sin\left(\dfrac{\pi}{3}\right) + r = -7\sqrt{3}$

$p\sqrt{3} + q\dfrac{\sqrt{3}}{2} - 4\sqrt{3} = -7\sqrt{3} \ \therefore \ 2p + q = -6$ ...(3)

$f'\left(\dfrac{\pi}{3}\right) = 3$

$\therefore$ from (2), $p\sec^2\left(\dfrac{\pi}{3}\right) + q\cos\left(\dfrac{\pi}{3}\right) = 3$

$4p + \dfrac{q}{2} = 3 \ \therefore \ 8p + q = 6$ ...(4)

(4) – (3) gives $6p = 12 \ \therefore \ p = 2$, put $p = 2$ in (3), we get $q = -10 \ \therefore \ p = 2$, $q = -10$ and $r = -4\sqrt{3}$

9.2.5 Derivatives of Algebraic Functions

Sr.No.	$f(x)$	$f'(x)$
01	**c**	**0**
02	x^n	nx^{n-1}
03	$\dfrac{1}{x^n}$	$-\dfrac{n}{x^{n+1}}$
04	$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$

9.2.6 Derivatives of Trigonometric functions

Sr.No.	$y = f(x)$	dy/dx = $f'(x)$
01	$\sin x$	$\cos x$
02	$\cos x$	$-\sin x$
03	$\tan x$	$\sec^2 x$
04	$\cot x$	$-\operatorname{cosec}^2 x$
05	$\sec x$	$\sec x \tan x$
06	$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

9.2.7 Derivatives of Logarithmic and Exponential functions

Sr.No.	$y = f(x)$	dy/dx = $f'(x)$
01	$\log x$	$1/x$
02	e^x	e^x
03	a^x	$a^x \log a$

EXERCISE 9.2

(I) Differentiate the following w.r.t.x

1) $y = x^{\frac{4}{3}} + e^x - \sin x$

2) $y = \sqrt{x} + \tan x - x^3$

3) $y = \log x - \operatorname{cosec} x + 5^x - \dfrac{3}{x^{\frac{3}{2}}}$

4) $y = x^{\frac{7}{3}} + 5x^{\frac{4}{5}} - \dfrac{5}{x^{\frac{2}{5}}}$

5) $y = 7^x + x^7 - \dfrac{2}{3}x\sqrt{x} - \log x + 7^7$

6) $y = 3\cot x - 5e^x + 3\log x - \dfrac{4}{x^{\frac{3}{4}}}$

(II) Differentiate the following w.r.t.x.

1) $y = x^5 \tan x$

2) $y = x^3 \log x$

3) $y = (x^2 + 2)^2 \sin x$

4) $y = e^x \log x$

5) $y = x^{\frac{3}{2}} e^x \log x$

6) $y = \log e^{x^3} \log x^3$

(III) Differentiate the following w.r.t.x.

1) $y = x^2\sqrt{x} + x^4 \log x$

2) $y = e^x \sec x - x^{\frac{5}{3}} \log x$

3) $y = x^4 + x\sqrt{x}\cos x - x^2 e^x$

4) $y = (x^3 - 2)\tan x - x\cos x + 7^x . x^7$

5) $y = \sin x \log x + e^x \cos x - e^x\sqrt{x}$

6) $y = e^x \tan x + \cos x \log x - \sqrt{x}\, 5^x$

(IV) Differentiate the following w.r.t.x.

1) $y = \dfrac{x^2+3}{x^2-5}$

2) $y = \dfrac{\sqrt{x}+5}{\sqrt{x}-5}$

3) $y = \dfrac{xe^x}{x+e^x}$

4) $y = \dfrac{x\log x}{x+\log x}$

5) $y = \dfrac{x^2 \sin x}{x+\cos x}$

6) $y = \dfrac{5e^x-4}{3e^x-2}$

(V) (1) If $f(x)$ is a quadratic polynomial such that $f(0) = 3$, $f'(2) = 2$ and $f'(3) = 12$ then find $f(x)$

(2) If $f(x) = a\sin x - b\cos x$, $f'\left(\dfrac{\pi}{4}\right) = \sqrt{2}$ and $f'\left(\dfrac{\pi}{6}\right) = 2$, then find $f(x)$.

(VI) Fill in the blanks. (Activity Problems)

(1) $y = e^x . \tan x$

diff. w.r.t.x.

$$\frac{dy}{dx}=\frac{d}{dx}(e^x \tan x)$$

$$=\square\frac{d}{dx}\tan x+\tan x\frac{d}{dx}\square$$

$$=\square\,\square+\tan x\cdot\square$$

$$=e^x[\square+\square]$$

(2) $y=\dfrac{\sin x}{x^2+2}$

diff. w.r.t.x.

$$\frac{dy}{dx}=\frac{\square\frac{d}{dx}(\sin x)-\sin x\frac{d}{dx}\square}{(x^2+2)^2}$$

$$=\frac{\square\square-\sin x\square}{(x^2+2)^2}$$

$$=\frac{\square-\square}{(x^2+2)^2}$$

3) $y = (3x^2 + 5)\cos x$

Diff. w.r.t.x

$$\frac{dy}{dx}=\frac{d}{dx}\left[(3x^2+5)\cos x\right]$$

$$=\left(3x^2+5\right)\frac{d}{dx}[\ \square\ ]+\cos x\frac{d}{dx}[\ \square\ ]$$

$$=\left(3x^2+5\right)[\ \square\ ]+\cos x[\ \square\ ]$$

$$\therefore\ \frac{dx}{dy}=\left(3x^2+5\right)[\ \square\ ]+[\ \square\ ]\cos x$$

4) Differentiate $\tan x$ and $\sec x$ w.r.t.x. using the formulae for differentiation of $\frac{\text{u}}{\text{v}}$ and $\frac{1}{\text{v}}$ respectively.

Brief idea of L' Hospital Rule

Consider the functions $f(x)$ and $g(x)$, If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$ and if $\lim\limits_{x\to a} f'(x) = p$ and $\lim\limits_{x\to a} g'(x) = q$ where $q \neq 0$ then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}=\dfrac{p}{q}$

If $\lim\limits_{x\to a} g'(x) = 0$, then provided $\lim\limits_{x\to a} f'(x) = 0$, we can study $\lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ using the same rule.

Ex. 1 : $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x^2}\right)$

Here $f(x) = \sin x$, $\lim\limits_{x\to 0} f(x) = 0$ and

$g(x) = x^2$, $\lim\limits_{x\to 0} g(x) = 0$

$f'(x) = \cos x$, $\lim\limits_{x\to 0} f'(x) = \cos 0 = 1 \neq 0$

$g'(x) = 2x$, $\lim\limits_{x\to 0} g'(x) = 2(0) = 0$

Since $\lim\limits_{x\to 0} g'(x) = 0$. L' Hospital Rule cannot be applied.

Ex. 2 : $\lim\limits_{x\to 2}\left[\dfrac{x^2-7x+10}{x^2+2x-8}\right]$

Here $f(x) = x^2 - 7x + 10$, $\lim\limits_{x\to 2} f(x) = 0$ and $g(x) = x^2 + 2x - 8$ $\lim\limits_{x\to 2} g(x) = 0$

$f'(x) = 2x - 7$, $\lim\limits_{x\to 2} f'(x) = 2(2) - 7 = -3 \neq 0$

$g'(x) = 2x + 2$, $\lim\limits_{x\to 2} g'(x) = 2(2) + 2 = 6 \neq 0$,

So L 'Hospital' s rule is applicable.

$$\therefore\ \lim_{x\to 2}\left[\frac{x^2-7x+10}{x^2+2x-8}\right]$$

$$=\lim_{x\to 2}\left[\frac{2x-7}{2x+2}\right]=\frac{-3}{6}=-\frac{1}{2}$$

Let's Remember

- $f(x)$ is differentiable at $x = a$ if $Lf'(a) = Rf'(a)$

- Derivative by First Principle :

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

- Derivatives of standard functions :

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
c(constant)	0
x^n	nx^{n-1}
$\frac{1}{x^n}$	$-\frac{n}{x^{n+1}}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\log_e x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x\log a}$
e^x	e^x
a^x	$a^x \log a$
$u \pm v$	$u' \pm v'$
uv	$uv' + u'v$
$\frac{u}{v}$	$\frac{vu'-uv'}{v^2}$

MISCELLANEOUS EXERCISE-9

I) Select the appropriate option from the given alternative.

1) If $y = \frac{x-4}{\sqrt{x+2}}$, then $\frac{dy}{dx}$

(A) $\frac{1}{x+4}$ (B) $\frac{\sqrt{x}}{(\sqrt{x+2})^2}$

(C) $\frac{1}{2\sqrt{x}}$ (D) $\frac{x}{(\sqrt{x}+2)^2}$

2) If $y = \frac{ax+b}{cx+d}$, then $\frac{dy}{dx} =$

(A) $\frac{ab-cd}{(cx+d)^2}$ (B) $\frac{ax-c}{(cx+d)^2}$

(C) $\frac{ac-bd}{(cx+d)^2}$ (D) $\frac{ad-bc}{(cx+d)^2}$

3) If $y = \frac{3x+5}{4x+5}$, then $\frac{dy}{dx} =$

(A) $-\frac{15}{(3x+5)^2}$ (B) $-\frac{15}{(4x+5)^2}$

(C) $-\frac{5}{(4x+5)^2}$ (D) $-\frac{13}{(4x+5)^2}$

4) If $y = \frac{5\sin x-2}{4\sin x+3}$, then $\frac{dy}{dx} =$

(A) $\frac{7\cos x}{(4\sin x+3)^2}$ (B) $\frac{23\cos x}{(4\sin x+3)^2}$

(C) $-\frac{7\cos x}{(4\sin x+3)^2}$ (D) $-\frac{15\cos x}{(4\sin x+3)^2}$

5) Suppose $f(x)$ is the derivative of $g(x)$ and $g(x)$ is the derivative of $h(x)$.

If $h(x) = a\sin x + b\cos x + c$ then $f(x) + h(x) =$

(A) 0 (B) c (C) –c (D) $-2(a\sin + b\cos x)$

6) If $f(x) = 2x + 6$ for $0 \le x \le 2$

$= ax^2 + bx$ for $2 < x \le 4$

is differentiable at $x = 2$ then the values of a and b are.

(A) $a = -\frac{3}{2}, b = 3$ (B) $a = \frac{3}{2}, b = 8$

(C) $a = \frac{1}{2}, b = 8$ (D) $a = -\frac{3}{2}, b = 8$

7) If $f(x) = x^2 + \sin x + 1$ for $x \leq 0$

$= x^2 - 2x + 1$ for $x \leq 0$ then

(A) f is continuous at $x = 0$, but not differentiable at $x = 0$

(B) f is neither continuous nor differentiable at $x = 0$

(C) f is not continuous at $x = 0$, but differentiable at $x = 0$

(D) f is both continuous and differentiable at $x = 0$

8) If, $f(x) = \frac{x^{50}}{50} + \frac{x^{49}}{49} + \frac{x^{48}}{48} + ... + \frac{x^2}{2} + x + 1$, then $f'(1) =$

(A) 48 (B) 49 (C) 50 (D) 51

II)

(1) Determine whether the following function is differentiable at $x = 3$ where,

$f(x) = x^2 + 2$, for $x \geq 3$

$= 6x - 7$, for $x < 3$.

(2) Find the values of p and q that make function $f(x)$ differentiable everywhere on R

$f(x) = 3 - x$, for $x < 1$

$= px^2 + qx$, for $x \geq 1$.

(3) Determine the values of p and q that make the function $f(x)$ differentiable on R where

$f(x) = px^3$, for $x < 2$

$= x^2 + q$, for $x \geq 2$.

(4) Determine all real values of p and q that ensure the function

$f(x) = px + q$ for, $x \leq 1$

$= \tan\left(\frac{\pi x}{4}\right)$, for $1 < x < 2$ is differentiable at $x = 1$.

(5) Discuss whether the function

$f(x) = |x+1| + |x-1|$ is differentiable $\forall x \in$ R

(6) Test whether the function

$f(x) = 2x - 3$, for $x \geq 2$

$= x - 1$, for $x < 2$

is differentiable at $x = 2$.

(7) Test whether the function

$f(x) = x^2 + 1$, for $x \geq 2$

$= 2x + 1$, for $x < 2$

is differentiable at $x = 2$.

(8) Test whether the function

$f(x) = 5x - 3x^2$ for $x \geq 1$

$= 3 - x$, for $x < 1$

is differentiable at $x = 1$.

(9) If $f(2) = 4$, $f'(2) = 1$ then find

$$\lim_{x \to 2} \left[\frac{xf(2) - 2f(x)}{x - 2}\right]$$

10) If $y = \frac{e^x}{\sqrt{x}}$ find $\frac{dy}{dx}$ when $x = 1$.