6 Line and Plane

Let's Study

Let's recall.

A line in space is completely determined by a point on it and its direction. Two points on a line determine the direction of the line. Let us derive equations of lines in different forms and discuss parallel lines.

6.1 Vector and Cartesian equations of a line:

Line in space is a locus. Points on line have position vectors. Position vector of a point determines the position of the point in space. In this topic position vector of a variable point on line will be denoted by $\bar{r}$.

6.1.1 Equation of a line passing through a given point and parallel to given vector:

Theorem 6.1 :

The vector equation of the line passing through A($\bar{a}$) and parallel to vector $\bar{b}$ is $\bar{r} = \bar{a} + \lambda \bar{b}$.

Proof:

Let L be the line which passes through A($\bar{a}$) and parallel to vector $\bar{b}$.

Let $P(\bar{r})$ be a variable point on the line L.

$\therefore$ $\overline{AP}$ is parallel to $\bar{b}$.

$\therefore$ $\overline{AP} = \lambda \bar{b}$, where λ is a scalar .

$\therefore$ $\overline{OP} - \overline{OA} = \lambda \bar{b}$

$\therefore$ $\bar{r} - \bar{a} = \lambda \bar{b}$

$\therefore$ $\bar{r} = \bar{a} + \lambda \bar{b}$

is the required vector equation of the line.

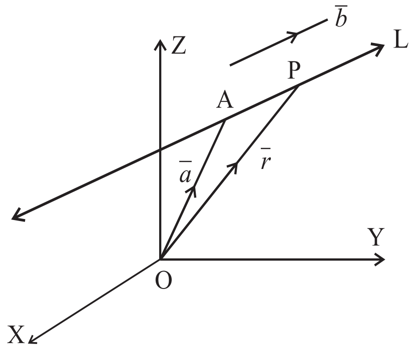

Fig. 6.1

Remark: Each real value of λ corresponds to a point on line L and conversely each point on L determines unique value of λ. There is one to one correspondence between points on L and values of λ. Here λ is called a parameter and equation $\bar{r} = \bar{a} + \lambda \bar{b}$ is called the **parametric form of vector equation** of line.

Activity: Write position vectors of any three points on the line $\bar{r} = \bar{a} + \lambda \bar{b}$.

Remark: The equation of line passing through A($\bar{a}$) and parallel to vector $\bar{b}$ can also be expressed as $(\bar{r} - \bar{a}) \times \bar{b} = \bar{0}$. This equation is called the non-parametric form of vector equation of line.

Theorem 6.2 :

The Cartesian equations of the line passing through A(x_1, y_1, z_1) and having direction ratios a, b, c are $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$.

Proof: Let L be the line which passes through A(x_1, y_1, z_1) and has direction ratios a, b, c.

Let P(x, y, z) be a variable point on the line L other than A.

$\therefore$ Direction ratios of L are $x - x_1$, $y - y_1$, $z - z_1$.

But direction ratios of line L are a, b, c.

$\therefore$ $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$ are the required Cartesian equations of the line.

In Cartesian form line cannot be represented by a single equation.

Remark :

- If $\bar{b} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ then a_1, b_1, c_1 are direction ratios of the line and conversely if a_1, b_1, c_1 are direction ratios of a line then $\bar{b} = a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$ is parallel to the line.
- The equations $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c} = \lambda$ are called the symmetric form of Cartesian equations of line.
- The equations $x = x_1 + \lambda a$, $y = y_1 + \lambda b$, $z = z_1 + \lambda c$ are called parametric form of the Cartesian equations of line.

- The co-ordinates of variable point P on the line are $(x_1 + \lambda a, y_1 + \lambda b, z_1 + \lambda c)$.
- Corresponding to each real value of λ there is one point on the line and conversely corresponding to each point on the line there is unique real value of λ.
- Whenever we write the equations of line in symmetric form, it is assumed that, none of a, b, c is zero. If atleast one of them is zero then we write the equations of line in parametric form and not in symmetric form.
- In place of direction ratios a, b, c if we take direction cosines l, m, n then the co-ordinates of the variable point P will be $(x_1 + \lambda l, y_1 + \lambda m, z_1 + \lambda n)$, A (x_1, y_1, z_1)

  $\therefore$ $AP^2 = (x_1 + \lambda l - x_1)^2 + (y_1 + \lambda m - y_1)^2 + (z_1 + \lambda n - z_1)^2$

  $= (\lambda l)^2 + (\lambda m)^2 + (\lambda n)^2 = \lambda^2\{(l)^2 + (m)^2 + (n)^2\} = \lambda^2$

  $AP^2 = \lambda^2$ $\therefore$ $AP = |\lambda|$ and $\lambda = \pm AP$

  Thus parameter of point on a line gives its distance from the base point of the 1ine.

6.1.2 Equation of a line passing through given two points.

Theorem 6.3 : The equation of the line passing through A $(\bar{a})$ and B$(\bar{b})$ is $\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$.

Proof: Let L be the line which passes through A$(\bar{a})$ and B $(\bar{b})$.

Let P$(\bar{r})$ be a variable point on the line L other than A.

$\therefore$ $\overline{AP}$ and $\lambda\overline{AB}$ are collinear.

$\therefore$ $\overline{AP} = \lambda\overline{AB}$, where λ is a scalar .

$\therefore$ $\bar{r} - \bar{a} = \lambda(\bar{b} - \bar{a})$ $\quad \bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$

$\therefore$ $\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$ is the required equation of the line.

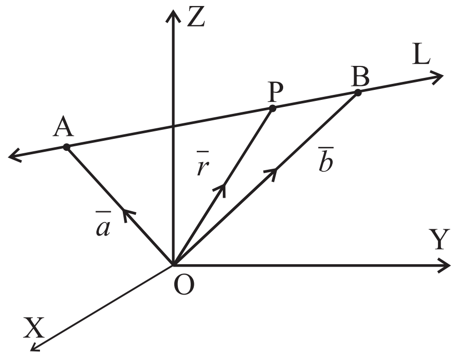

Fig. 6.2

Remark: The equation of the line passing through A$(\bar{a})$ and B$(\bar{b})$ can also be expressed as $(\bar{r} - \bar{a}) \times (\bar{b} - \bar{a}) = \bar{0}$.

Theorem 6.4 : The Cartesian equations of the line passing through $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is
$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

Proof: Let L be the line which passes through $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$

Let $P(x, y, z)$ be a variable point on the line L other than A.

$\therefore$ Direction ratios of L are $x - x_1, y - y_1, z - z_1$

But direction ratios of line L are $x_2 - x_1, y_2 - y_1, z_2 - z_1$

$\therefore$ $\dfrac{x - x_1}{x_2 - x_1} = \dfrac{y - y_1}{y_2 - y_1} = \dfrac{z - z_1}{z_2 - z_1}$ are the required Cartesian equations of the line.

Solved Examples

Ex.(l) Verify that point having position vector $4\hat{i} - 11\hat{j} + 2\hat{k}$ lies on the line

$\bar{r} = (6\hat{i} - 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 7\hat{j} + 3\hat{k})$.

Solution: Replacing $\bar{r}$ by $4\hat{i} - 11\hat{j} + 2\hat{k}$ we get,

$4\hat{i} - 11\hat{j} + 2\hat{k} = (6\hat{i} - 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 7\hat{j} + 3\hat{k})$

$\therefore$ $6 + 2\lambda = 4, -4 + 7\lambda = -11, 5 + 3\lambda = 2$

From each of these equations we get the same value of λ.

$\therefore$ Given point lies on the given line.

Alternative Method: Equation $\bar{r} = \bar{a} + \lambda\bar{b}$ can be written as $\bar{r} - \bar{a} = \lambda\bar{b}$

Thus point P($\bar{r}$) lies on the line if and only if $\bar{r} - \bar{a}$ is a scalar multiple of $\bar{b}$.

Equation of line is $\bar{r} = (6\hat{i} - 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 7\hat{j} + 3\hat{k})$.

$\therefore$ $\bar{a} = 6\hat{i} - 4\hat{j} + 5\hat{k}$ and $\bar{b} = 2\hat{i} + 7\hat{j} + 3\hat{k}$

The position vector of given point is $\bar{r} = 4\hat{i} - 11\hat{j} + 2\hat{k}$

$\bar{r} - \bar{a} = (4\hat{i} - 11\hat{j} + 2\hat{k}) - (6\hat{i} - 4\hat{j} + 5\hat{k}) = -2\hat{i} - 7\hat{j} - 3\hat{k}$

$= -(2\hat{i} + 7\hat{j} + 3\hat{k})$

$= -1\bar{b}$, a scalar multiple of $\bar{b}$.

$\therefore$ Given point lies on the given line.

Ex.(2) Find the vector equation of the line passing through the point having position vector $4\hat{i} - \hat{j} + 2\hat{k}$ and parallel to vector $-2\hat{i} - \hat{j} + \hat{k}$

Solution: The equation of the line passing through A($\bar{a}$) and parallel to vector $\bar{b}$ is $\bar{r} = \bar{a} + \lambda\bar{b}$.
The equation of the line passing through $4\hat{i} - \hat{j} + 2\hat{k}$ and parallel to vector $-2\hat{i} - \hat{j} + \hat{k}$ is
$\bar{r} = (4\hat{i} - \hat{j} + 2\hat{k}) + \lambda(-2\hat{i} - \hat{j} + \hat{k})$.

Ex.(3) Find the vector equation of the line passing through the point having position vector $2\hat{i} + \hat{j} - 3\hat{k}$ and perpendicular to vectors $\hat{i} + \hat{j} + \hat{k}$. and $\hat{i} + 2\hat{j} - \hat{k}$

Solution:

Let $a = 2\hat{i} + \hat{j} - 3\hat{k}$, $\bar{b} = \hat{i} + \hat{j} + \hat{k}$ and $c = \hat{i} + 2\hat{j} - \hat{k}$

We know that $\bar{b} \times \bar{c}$ is perpendicular to both $\bar{b}$ and $\bar{c}$.

$\therefore$ $\bar{b} \times \bar{c}$ is parallel to the required line.

$$\bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -3\hat{i} + 2\hat{j} + \hat{k}$$

Thus required line passes through $a = 2\hat{i} + \hat{j} - 3\hat{k}$ and parallel to $-3\hat{i} + 2\hat{j} + \hat{k}$.

$\therefore$ Its equation is $\bar{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \lambda(-3\hat{i} + 2\hat{j} + \hat{k})$

Ex.(4) Find the vector equation of the line passing through $2\hat{i} + \hat{j} - \hat{k}$ and parallel to the line joining points $-\hat{i} + \hat{j} + 4\hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$.

Solution :

Let A, B, C be points with position vectors $\bar{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\bar{b} = -\hat{i} + \hat{j} + 4\hat{k}$ and

$\bar{c} = \hat{i} + 2\hat{j} + 2\hat{k}$ respectively.

$\overline{BC} = \bar{c} - \bar{b} = (\hat{i} + 2\hat{j} + 2\hat{k}) - (-\hat{i} + \hat{j} + 4\hat{k}) = 2\hat{i} + \hat{j} - 2\hat{k}$

The required line passes through $2\hat{i} + \hat{j} - \hat{k}$ and is parallel to $2\hat{i} + \hat{j} - 2\hat{k}$

Its equation is $\bar{r} = (2\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} + \hat{j} - 2\hat{k})$

Ex.(5) Find the vector equation of the line passing through A(1, 2, 3) and B(2, 3, 4).

Solution: Let position vectors of points A and B be $\bar{a}$ and $\bar{b}$.

$\therefore$ $\bar{a} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\bar{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore$ $\bar{b} - \bar{a} = (2\hat{i} + 3\hat{j} + 4\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = (\hat{i} + \hat{j} + \hat{k})$

The equation of the line passing through A $(\bar{a})$ and B $(\bar{b})$ is $\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$.

The equation of the required line is $\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} + \hat{j} + \hat{k})$

Activity: Verify that position vector of B satisfies the above equation.

Ex.(6) Find the Cartesian equations of the line passing through A(1, 2, 3) and having direction ratios 2, 3, 7.

Solution:

The Cartesian equations of the line passing through A(x_1, y_1, z_1) and having direction ratios a, b, c are $\dfrac{x-x_1}{a} = \dfrac{y-y_1}{b} = \dfrac{z-z_1}{c}$.

Here $(x_1, y_1, z_1) = (1, 2, 3)$ and direction ratios are 2, 3, 7.

Required equation $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{7}$.

Ex.(7) Find the Cartesian equations of the line passing through A(l, 2, 3) and B(2, 3. 4).

Solution: The Cartesian equations of the line passing through A(x_1, y_1, z_1) and B(x_2, y_2, z_2) are $\dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} = \dfrac{z-z_1}{z_2-z_1}$

Here $(x_1, y_1, z_1) = (1, 2, 3)$ and $(x_2, y_2, z_2) = (2, 3, 4)$.

$\therefore$ Required Cartesian equations are $\dfrac{x-1}{2-1} = \dfrac{y-2}{3-2} = \dfrac{z-3}{4-3}$.

$\therefore$ $\dfrac{x-1}{1} = \dfrac{y-2}{1} = \dfrac{z-3}{1}$.

$\therefore$ $x - 1 = y - 2 = z - 3$.

Activity: Verify that co-ordinates of B satisfy the above equation.

Ex.(8) Find the Cartesian equations of the line passing through the point A(2, 1, –3) and perpendicular to vectors $\bar{b} = \hat{i} + \hat{j} + \hat{k}$ and $\bar{c} = \hat{i} + 2\hat{j} - \hat{k}$

Solution: We know that $\bar{b} \times \bar{c}$ is perpendicular to both b and c.

$$\therefore \bar{b} \times \bar{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} = -3\hat{i} + 2\hat{j} + \hat{k} \text{ is parallel to the required line.}$$

The direction ratios of the required line are –3, 2, 1 and it passes through A(2, 1, –3).

$\therefore$ Its Cartesian equations are $\dfrac{x-2}{-3} = \dfrac{y-1}{2} = \dfrac{z+3}{1}$.

Ex.(9) Find the angle between lines $\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$

and $\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} + \hat{k})$

Solution: Let $\bar{b}$ and $\bar{c}$ be vectors along given lines .

$\therefore \quad \bar{b} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\bar{c} = \hat{i} + 2\hat{j} + 2\hat{k}$

Angle between lines is same as the angle between $\bar{b}$ and $\bar{c}$.

The angle between $\bar{b}$ and $\bar{c}$ is given by,

$$\cos\theta = \frac{\bar{b}\cdot\bar{c}}{|\bar{b}|\cdot|\bar{c}|} = \frac{(2\hat{i}-2\hat{j}+\hat{k})\cdot(\hat{i}+2\hat{j}+2\hat{k})}{3\times3} = \frac{0}{9} = 0$$

$\therefore \cos\theta = 0 \qquad \therefore \theta = 90^\circ$

Lines are perpendicular to each other.

Ex.(10) Show that lines $\bar{r} = (-\hat{i} - 3\hat{j} + 4\hat{k}) + \lambda\ (-10\hat{i} - \hat{j} + \hat{k})$ and

$\bar{r} = (-10\hat{i} - \hat{j} + \hat{k}) + \mu(-\hat{i} - 3\hat{j} + 4\hat{k})$ intersect each other.

Find the position vector of their point of intersection.

Solution: The position vector of a variable point on the line $\bar{r} = (-\hat{i} - 3\hat{j} + 4\hat{k}) + \lambda(-10\hat{i} - \hat{j} + \hat{k})$ is

$(-1 - 10\lambda)\hat{i} + (-3 - \lambda)\hat{j} + (4 + \lambda)\hat{k}$

The position vector of a variable point on the line

$\bar{r} = (-10\hat{i} - \hat{j} + \hat{k}) + \mu(-\hat{i} - 3\hat{j} + 4\hat{k})$ is $(-10 - 1\mu)\hat{i} + (-1 - 3\mu)\hat{j} + (1 + 4\mu)\hat{k}$

Given lines intersect each other if there exist some values of λ and μ for which

$(-1 - 10\lambda)\hat{i} + (-3 - \lambda)\hat{j} + (4 + \lambda)\hat{k} = (-10 - 1\mu)\hat{i} + (-1 - 3\mu)\hat{j} + (1 + 4\mu)\hat{k}$

$\therefore \quad -1 - 10\lambda = -10 - 1\mu,\ -3 - \lambda = -1 - 3\mu$ and $4 + \lambda = 1 + 4\mu$

$\therefore \quad 10\lambda - \mu = 9\ ,\ \lambda - 3\mu = -2$ and $\lambda - 4\mu = -3 \qquad \ldots\ \ldots\ \ldots (1)$

Given lines intersect each other if this system is consistent

$$\text{As} \begin{vmatrix} 10 & -1 & 9 \\ 1 & -3 & -2 \\ 1 & -4 & -3 \end{vmatrix} = 10(9 - 8) + 1\,(-3 + 2) + 9\,(-4 + 3) = 10 - 1 - 9 = 0$$

$\therefore$ The system (1) is consistent and lines intersect each other.

Solving any two equations in system (1), we get. $\lambda = 1$, $\mu = 1$

Substituting this value of λ in $(-1-10\lambda)\hat{i} + (-3-\lambda)\hat{j} + (4+\lambda)\hat{k}$ we get, $-11\hat{i} - 4\hat{j} + 5\hat{k}$

$\therefore$ The position vector of their point of intersection is $-11\hat{i} - 4\hat{j} + 5\hat{k}$.

Ex.(11) Find the co-ordinates of points on the line $\dfrac{x+1}{2} = \dfrac{y-2}{3} = \dfrac{z+3}{6}$, which are at 3 unit distance from the base point A(–1, 2, –3).

Solution: Let Q $(2\lambda - 1, 3\lambda + 2, 6\lambda - 3)$ be a point on the line which is at 3 unit distance from the point A(–1,2,–3) $\therefore$ AQ = 3

$$\therefore \sqrt{(2\lambda)^2 + (3\lambda)^2 + (6\lambda)^2} = 3 \quad \therefore \quad (2\lambda)^2 + (3\lambda)^2 + \left((6\lambda)^2\right) = 9 \quad \therefore \; 49\lambda^2 = 9$$

$$\therefore \quad \lambda = -\frac{3}{7} \, or \, \frac{3}{7}$$

$\therefore$ There are two points on the line which are at a distance of 3 units from P.

Their co-ordinates are Q $(2\lambda - 1, 3\lambda + 2, 6\lambda - 3)$

Hence, the required points are

$$\left(-\frac{1}{7}, 3\frac{2}{7}, -\frac{3}{7}\right) \; and \; \left(-1\frac{6}{7}, \frac{5}{7}, -5\frac{4}{7}\right).$$

Exercise 6.1

(1) Find the vector equation of the line passing through the point having position vector $-2\hat{i} + \hat{j} + \hat{k}$ and parallel to vector $4\hat{i} - \hat{j} + 2\hat{k}$.

(2) Find the vector equation of the line passing through points having position vectors $3\hat{i} + 4\hat{j} - 7\hat{k}$ and $6\hat{i} - \hat{j} + \hat{k}$.

(3) Find the vector equation of line passing through the point having position vector $5\hat{i} + 4\hat{j} + 3\hat{k}$ and having direction ratios –3, 4, 2.

(4) Find the vector equation of the line passing through the point having position vector $\hat{i} + 2\hat{j} + 3\hat{k}$ and perpendicular to vectors $\hat{i} + \hat{j} + \hat{k}$ and $2\hat{i} - \hat{j} + \hat{k}$.

(5) Find the vector equation of the line passing through the point having position vector $-\hat{i} - \hat{j} + 2\hat{k}$ and parallel to the line $\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} + \hat{k})$.

(6) Find the Cartesian equations of the line passing through A(–1, 2, 1) and having direction ratios 2, 3, 1.

(7) Find the Cartesian equations of the line passing through A(2, 2,1) and B(l, 3, 0).

(8) A(–2, 3, 4), B(l, 1, 2) and C(4, –1, 0) are three points. Find the Cartesian equations of the line AB and show that points A, B, C are collinear.

(9) Show that lines $\dfrac{x+1}{-10} = \dfrac{y+3}{-1} = \dfrac{z-4}{1}$ and $\dfrac{x+10}{-1} = \dfrac{y+1}{-3} = \dfrac{z-1}{4}$ intersect each other. Find the co-ordinates of their point of intersection.

(10) A line passes through (3, –1, 2) and is perpendicular to lines $\bar{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and $\bar{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \mu(\hat{i} - 2\hat{j} + 2\hat{k})$. Find its equation.

(11) Show that the line $\frac{x-2}{1}=\frac{y-4}{2}=\frac{z+4}{-2}$ passes through the origin.

6.2 Distance of a point from a line:

Theorem 6.5:

The distance of point P($\bar{\alpha}$) from the line $\bar{r} = \bar{a}+\lambda\bar{b}$ is

$$\sqrt{|\bar{\alpha}-\bar{a}|^2-\left[\frac{(\bar{\alpha}-\bar{a})\cdot\bar{b}}{|\bar{b}|}\right]^2}$$

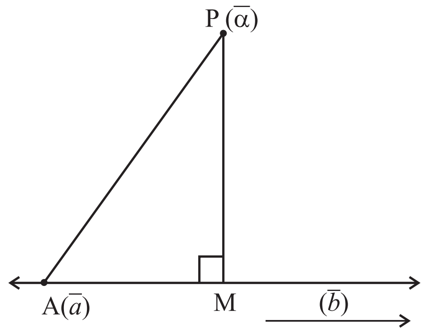

Fig. 6.3

Proof: The line $\bar{r} = \bar{a}+\lambda\bar{b}$ passes through A($\bar{a}$).

Let M be the foot of the perpendicular drawn from P to the line $\bar{r} = \bar{a}+\lambda\bar{b}$

$\therefore$ $AM = |\overline{AM}|$ = the projection of $\overline{AP}$ on the line.

= the projection of $\overline{AP}$ on $\bar{b}$. (As line is parallel to $\bar{b}$)

$$=\frac{\overline{AP}\cdot\bar{b}}{|\bar{b}|}$$

Now ΔAM P is a right angled triangle. $\therefore$ $PM^2 = AP^2 - AM^2$

$$PM^2=|\overline{AP}|^2-|\overline{AM}|^2 = |\bar{\alpha}-\bar{a}|^2-\left[\frac{(\bar{\alpha}-\bar{a})\cdot\bar{b}}{|\bar{b}|}\right]^2$$

$$\therefore \quad PM=\sqrt{|\bar{\alpha}-\bar{a}|^2-\left[\frac{(\bar{\alpha}-\bar{a})\cdot\bar{b}}{|\bar{b}|}\right]^2}$$

The distance of point P($\bar{a}$) from the line $\bar{r} = (\bar{a}+\lambda\bar{b})$ is $PM=\sqrt{|\bar{\alpha}-\bar{a}|^2-\left[\frac{(\bar{\alpha}-\bar{a})\cdot\bar{b}}{|\bar{b}|}\right]^2}$

Ex.(12) : Find the length of the perpendicular drawn from the point P(3, 2,1) to the line $\bar{r}=(7\hat{i}+7\hat{j}+6\hat{k})+\lambda(-2\hat{i}+2\hat{j}+3\hat{k})$

Solution:

The length of the perpendicular is same as the distance of P from the given line.

The distance of point P($\bar{\alpha}$) from the line $\bar{r}=\bar{a}+\lambda\bar{b}$ is $\sqrt{|\bar{\alpha}-\bar{a}|^2-\left[\frac{(\bar{\alpha}-\bar{a})\cdot\bar{b}}{|\bar{b}|}\right]^2}$

Here $\bar{\alpha} = 3\hat{i} + 2\hat{j} + \hat{k}$, $\bar{a} = 7\hat{i} + 7\hat{j} + 6\hat{k}$, $\bar{b} = -2\hat{i} + 2\hat{j} + 3\hat{k}$

$\therefore\ \bar{\alpha} - \bar{a} = (3\hat{i} + 2\hat{j} + \hat{k}) - (7\hat{i} + 7\hat{j} + 6\hat{k}) = -4\hat{i} - 5\hat{j} - 5\hat{k}$

$$|\bar{\alpha} - \bar{a}| = \sqrt{(-4)^2 + (-5)^2 + (-5)^2} = \sqrt{16 + 25 + 25} = \sqrt{66}$$

$$(\bar{\alpha} - \bar{a}).\bar{b} = (-4\hat{i} - 5\hat{j} - 5\hat{k}).(-2\hat{i} + 2\hat{j} + 3\hat{k}) = 8 - 10 - 15 = -17$$

$$|\bar{b}| = \sqrt{(-2)^2 + (2)^2 + (3)^2} = \sqrt{17}$$

The require length =

$$\sqrt{|\bar{a} - \bar{a}|^2 - \left[\frac{(\bar{a} - \bar{a}).\bar{b}}{|\bar{b}|}\right]^2} = \sqrt{66 - \left|\frac{-17}{\sqrt{17}}\right|^2} = \sqrt{66 - 17} = \sqrt{49} = 7 \text{ unit}$$

Ex.(13) : Find the distance of the point P(0, 2, 3)from the line $\frac{x+3}{5} = \frac{y-1}{2} = \frac{z+4}{3}$.

Solution:

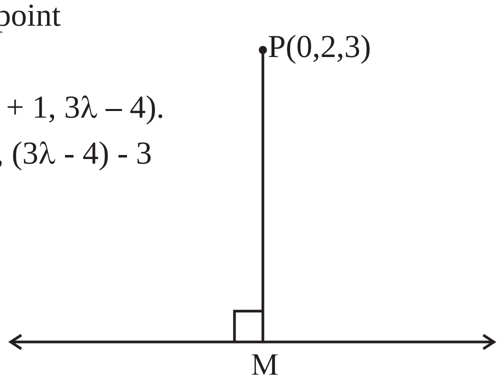

Fig. 6.4

Let M be the foot of the perpendicular drawn from the point P(0, 2, 3) to the given line.

M lies on the line. Let co-ordinates of M be $(5\lambda - 3, 2\lambda + 1, 3\lambda - 4)$.

The direction ratios of PM are $(5\lambda - 3) - 0$, $(2\lambda + 1) - 2$, $(3\lambda - 4) - 3$

i.e. $5\lambda - 3$, $2\lambda - 1$, $3\lambda - 7$

The direction ratios of given line are 5, 2, 3 and

PM is perpendicular to the given line .

$\therefore\ 5(5\lambda - 3) + 2(2\lambda - 1) + 3(3\lambda - 7) = 0$

$\therefore\ \lambda = 1$

$\therefore$ The co-ordinates of M are (2,3, –1).

The distance of P from the line is PM = $\sqrt{(2-0)^2 + (3-2)^2 + (-1-3)^2} = \sqrt{21}$ unit

6.3 Skew lines:

If two lines in space intersect at a point then the shortest distance between them is zero. If two lines in space are parallel to each other then the shortest distance between them is the perpendicular distance between them.

A pair of lines in space which neither intersect each other nor are parallel to each other are called skew lines. Skew lines are non-coplanar. Lines in the same plane either intersect or are parallel to each other.

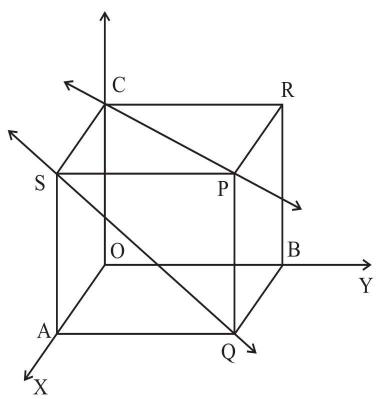

Fig. 6.5

In the figure 6.5, line CP that goes diagonally across the plane CSPR and line SQ passes across the plane SAQP are skew lines The shortest distance between skew lines is the length of the segment which is perpendicular to both the lines.

6.3.1 Distance between skew lines :

Theorem 6.6: The distance between lines $\bar{r} = \bar{a}_1 + \lambda_1\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2\bar{b}_2$ *is* $\left|\dfrac{(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|}\right|$

Proof: Let L_1 and L_2 be the lines whose equations are $\bar{r} = \bar{a}_1 + \lambda_1 \bar{b}_1$, $\bar{r} = \bar{a}_2 + \lambda_2 \bar{b}_2$ respectively. Let PQ be the segment which is perpendicular to both L_1 and L_2.

To find the length of segment PQ.

Lines L_1 and L_2 pass through points A($\bar{a}_1$) and B($\bar{a}_2$) respectively.

Lines L_1 and L_2 are parallel to $\bar{b}_1$ and $\bar{b}_2$ respectively.

As PQ is perpendicular to both L_1 and L_2, it is parallel to $\bar{b}_1 \times \bar{b}_2$

The unit vector along $\overline{PQ}$ = unit vector along $\bar{b}_1 \times \bar{b}_2 = \hat{n}$ (say)

PQ = The projection of $\overline{AB}$ on $\overline{PQ}$ = $\overline{AB} \cdot \hat{n}$

$$PQ = \left|\frac{(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|}\right|$$

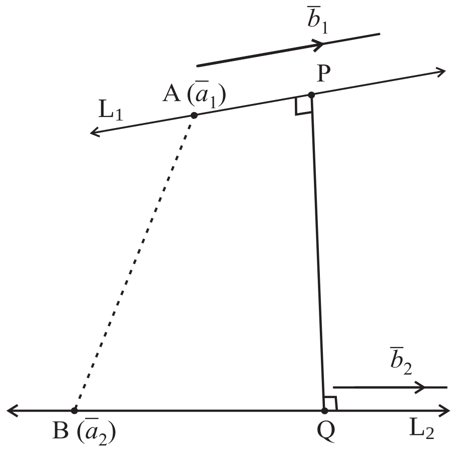

Fig. 6.6

The shortest distance between lines $\bar{r} = \bar{a}_1 + \lambda_1 \bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2 \bar{b}_2$ is $\left|\dfrac{(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|}\right|$

Remark

- Two lines intersect each other if and only if the shortest distance between them is zero.
- Lines $\bar{r} = \bar{a}_1 + \lambda_1 \bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2 \bar{b}_2$ intersect each other if and only if $(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times \bar{b}_2) = 0$

  Lines $\dfrac{x-x_1}{a_1} = \dfrac{y-y_1}{b_1} = \dfrac{z-z_1}{c_1}$ = and $\dfrac{x-x_2}{a_2} = \dfrac{y-y_2}{b_2} = \dfrac{z-z_2}{c_2}$ intersect each other if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Ex.(14) Find the shortest distance between lines $\bar{r} = (2\hat{i} - \hat{j}) + \lambda(2\hat{i} + \hat{j} - 3\hat{k})$ and $\bar{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + \hat{j} - 5\hat{k})$

Solution: The shortest distance between lines $\bar{r} = \bar{a}_1 + \lambda_1 \bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2 \bar{b}_2$ is $\left|\dfrac{(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|}\right|$

Here $\bar{a}_1 = 2\hat{i} - \hat{j}$, $\bar{a}_2 = \hat{i} - \hat{j} + 2\hat{k}$, $\bar{b}_1 = 2\hat{i} + \hat{j} + 3\hat{k}$, $\bar{b}_2 = 2\hat{i} + \hat{j} - 5\hat{k}$

$\bar{a}_2 - \bar{a}_1 = (\hat{i} - \hat{j} + 2\hat{k}) - (2\hat{i} - \hat{j}) = -\hat{i} + 2\hat{k}$

And $\bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -3 \\ 2 & 1 & -5 \end{vmatrix} = -2\hat{i} + 4\hat{j}$

$\therefore$ $|\bar{b}_1 \times \bar{b}_2| = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$

$(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2) = (-\hat{i} + 2\hat{k})\cdot(-2\hat{i} + 4\hat{j}) = 2$

The required shortest distance $\left|\dfrac{(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|}\right| = \left|\dfrac{2}{2\sqrt{5}}\right| = \dfrac{1}{\sqrt{5}}$ unit.

Ex.(15) Find the shortest distance between lines $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ and $\dfrac{x-2}{3} = \dfrac{y-4}{4} = \dfrac{z-5}{5}$

Solution:

The vector equations of given lines are $\bar{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$ and $\bar{r} = (2\hat{i} + 4\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 4\hat{j} + 5\hat{k})$

The shortest distance between lines $\bar{r} = \bar{a}_1 + \lambda\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda\bar{b}_2$ is $\left|\dfrac{(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|}\right|$ and

Here $\bar{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$, $\bar{a}_2 = 2\hat{i} + 4\hat{j} + 5\hat{k}$, $\bar{b}_1 = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\bar{b}_2 = 3\hat{i} + 4\hat{j} + 5\hat{k}$

$\therefore$ $\bar{a}_2 - \bar{a}_1 = (2\hat{i} + 4\hat{j} + 5k) - (\hat{i} + 2\hat{j} + 3\hat{k}) = \hat{i} + 2\hat{j} + 2\hat{k}$

And $\bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = -\hat{i} + 2\hat{j} - \hat{k}$

$|\bar{b}_1 \times \bar{b}_2| = \sqrt{1+4+1} = \sqrt{6}$

$(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2) = (\hat{i} + 2\hat{j} + 2\hat{k}).(-\hat{i} + 2\hat{j} - \hat{k}) = 1$

$\therefore$ The required shortest distance $\left|\dfrac{(a_2 - a_1).(\bar{b}_1 \times \bar{b}_2)}{|\bar{b}_1 \times \bar{b}_2|}\right| = \dfrac{1}{\sqrt{6}}$ unit.

Ex.(16) Show that lines :

$\bar{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and $\bar{r} = (4\hat{i} - 3\hat{j} + 2\hat{k}) + \mu(\hat{i} - 2\hat{j} + 2\hat{k})$ intersect each other.

Solution: Lines $\bar{r} = \bar{a}_1 + \lambda_1 \bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2 \bar{b}_2$ intersect each other if and only if

$$(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2) = 0$$

Here $\bar{a}_1 = \hat{i} + \hat{j} - \hat{k}$, $\bar{a}_2 = 4\hat{i} - 3\hat{j} + 2\hat{k}$, $\bar{b}_1 = 2\hat{i} - 2\hat{j} + \hat{k}$, $\bar{b}_2 = \hat{i} - 2\hat{j} + 2\hat{k}$,

$\bar{a}_2 - \bar{a}_1 = 3\hat{i} - 4\hat{j} + 3\hat{k}$

$$(\bar{a}_2 - \bar{a}_1).(\bar{b}_1 \times \bar{b}_2) = \begin{vmatrix} 3 & -4 & 3 \\ 2 & -2 & 1 \\ 1 & -2 & 2 \end{vmatrix} = 3(-2) + 4(3) + 3(-2) = -6 + 12 - 6 = 0$$

Given lines intersect each other.

6.3.2 Distance between parallel lines:

Theorem 6.7: The distance between parallel lines $\bar{r} = \bar{a}_1 + \lambda\bar{b}$ and $\bar{r} = \bar{a}_2 + \lambda\bar{b}$ is $|(\bar{a}_2 - \bar{a}_1) \times \hat{b}|$

Proof: Let lines represented by $\bar{r} = \bar{a}_1 + \lambda\bar{b}$ and $\bar{r} = \bar{a}_2 + \lambda\bar{b}$ be L_1 and L_2

Lines L_1 and L_2 pass through $A(\bar{a}_1)$ and $B(\bar{a}_2)$ respectively.

Let BM be perpendicular to L_1. To find BM

ΔAMB is a right angle triangle. Let m $\angle BAM = \theta$

$$\sin\theta = \frac{BM}{AB}$$

$\therefore$ BM = AB sin θ = AB.1 sin θ = AB.$|\hat{b}|$. sin θ

$$|\overline{AB} \times \hat{b}| = |(\bar{a}_2 - \bar{a}_1) \times \hat{b}|$$

$\therefore$ The distance between parallel lines $\bar{r} = \bar{a}_1 + \lambda\bar{b}$ and $\bar{r} = \bar{a}_2 + \lambda_1\bar{b}$ is given by

$$d = BM = |(\bar{a}_2 - \bar{a}_1) \times \hat{b}|$$

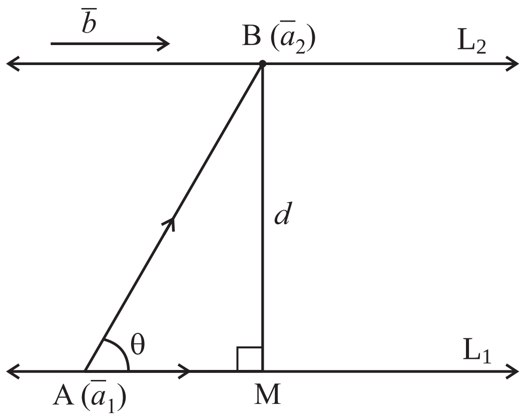

Fig. 6.7

Ex.(17) Find the distance between parallel lines $\bar{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} + \hat{j} - 2\hat{k})$ and $\bar{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + \hat{j} - 2\hat{k})$

Solution: The distance between parallel lines $\bar{r} = \bar{a}_1 + \lambda\bar{b}$ and $\bar{r} = \bar{a}_2 + \lambda\bar{b}$ given by

$$d = |(\bar{a}_2 - \bar{a}_1) \times \hat{b}|$$

Here $\bar{a}_1 = 2\hat{i} - \hat{j} + \hat{k}$, $\bar{a}_2 = \hat{i} - \hat{j} + 2\hat{k}$, $\bar{b} = 2\hat{i} + \hat{j} - 2\hat{k}$

$$\therefore \bar{a}_2 - \bar{a}_1 = \left(\hat{i} - \hat{j} + 2\hat{k}\right) - \left(2\hat{i} - \hat{j} + \hat{k}\right) = -\hat{i} + \hat{k} \text{ and } \hat{b} = \frac{2\hat{i} + \hat{j} - 2\hat{k}}{3}$$

$$\therefore (\bar{a}_2 - \bar{a}_1) \times \hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 1 \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{vmatrix} = \frac{1}{3}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & 1 \\ 2 & 1 & -2 \end{vmatrix} = \frac{1}{3}\left\{-\hat{i} - \hat{k}\right\}$$

$$d = \left|(\bar{a}_2 - \bar{a}_1) \times \hat{b}\right| = \frac{\sqrt{2}}{3} \text{ unit}$$

Alternative Method:

The distance between parallel lines $\bar{r} = \bar{a}_1 + \lambda \bar{b}$ and $\bar{r} = \bar{a}_2 + \lambda \bar{b}$ is same as the distance of point $A(\bar{a}_1)$ from the line $\bar{r} = \bar{a}_2 + \lambda \bar{b}$. This distance is given by

$$d = \sqrt{\left|\bar{a}_2 - \bar{a}_1\right|^2 - \left[\frac{(\bar{a}_2 - \bar{a}_1) \cdot \bar{b}}{|\bar{b}|}\right]^2}$$

Now $\bar{a}_2 - \bar{a}_1 = \left(\hat{i} - \hat{j} + 2\hat{k}\right) - \left(2\hat{i} - \hat{j} + \hat{k}\right) = -\hat{i} + \hat{k}$

$\therefore \left|\bar{a}_2 - \bar{a}_1\right| = \sqrt{2}$

As $(\bar{a}_2 - \bar{a}_1) \cdot \bar{b} = \left(-\hat{i} + \hat{k}\right) \cdot \left(2\hat{i} + \hat{j} - 2\hat{k}\right) = -2 + 0 - 2 = -4$

$\bar{b} = 2\hat{i} + \hat{j} - 2\hat{k}, \qquad |\bar{b}| = 3$

$$d = \sqrt{\left|\bar{a}_1 - \bar{a}_2\right|^2 - \left[\frac{(\bar{a}_1 - \bar{a}_2) \cdot \bar{b}}{|\bar{b}|}\right]^2} = \sqrt{2 - \left(-\frac{4}{3}\right)^2} = \sqrt{2 - \frac{16}{9}} = \frac{\sqrt{2}}{3} \text{ unit}$$

Ex.(18) Find the distance between parallel lines $\frac{x}{2} = \frac{y}{-1} = \frac{z}{2}$ and $\frac{x-1}{2} = \frac{y-1}{-1} = \frac{z-1}{2}$

Solution:

The vector equations of given lines are $\bar{r} = \lambda\left(2\hat{i} - \hat{j} + 2\hat{k}\right)$

and $\bar{r} = \left(\hat{i} + \hat{j} + \hat{k}\right) + \mu\left(2\hat{i} - \hat{j} + 2\hat{k}\right)$

The distance between parallel lines $\bar{r} = \bar{a}_1 + \lambda \bar{b}$ and $\bar{r} = \bar{a}_2 + \lambda \bar{b}$ is given by $d = \left|(\bar{a}_2 - \bar{a}_1) \times \hat{b}\right|$

Here $\bar{a}_1 = \bar{0}$, $\bar{a}_2 = \hat{i} + \hat{j} + \hat{k}$, $\bar{b} = 2\hat{i} - \hat{j} + 2\hat{k}$

$\therefore \bar{a}_2 - \bar{a}_1 = \hat{i} + \hat{j} + \hat{k}$

And $\hat{b} = \dfrac{2\hat{i} - \hat{j} + 2\hat{k}}{3}$

$$\therefore (\bar{a}_2 - \bar{a}_1) \times \hat{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ \frac{2}{3} & \frac{-1}{3} & \frac{2}{3} \end{vmatrix} = \frac{1}{3}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & -1 & 2 \end{vmatrix} = \frac{1}{3}\left\{3\hat{i} - 3\hat{k}\right\} = \hat{i} - \hat{k}$$

$$d = \left|(\bar{a}_2 - \bar{a}_1) \times \hat{b}\right| = \sqrt{2} \text{ unit}$$

Exercise 6.2

(1) Find the length of the perpendicular from (2, –3, 1) to the line $\frac{x+1}{2} = \frac{y-3}{3} = \frac{z+1}{-1}$

(2) Find the co-ordinates of the foot of the perpendicular drawn from the point $2\hat{i} - \hat{j} + 5\hat{k}$ to the line $\bar{r} = \left(11\hat{i} - 2\hat{j} - 8\hat{k}\right) + \lambda\left(10\hat{i} - 4\hat{j} - 11\hat{k}\right)$. Also find the length of the perpendicular.

(3) Find the shortest distance between the lines $\bar{r} = \left(4\hat{i} - \hat{j}\right) + \lambda\left(\hat{i} + 2\hat{j} - 3\hat{k}\right)$ and

$\bar{r} = \left(\hat{i} - \hat{j} + 2\hat{k}\right) + \mu\left(\hat{i} + 4\hat{j} - 5\hat{k}\right)$

(4) Find the shortest distance between the lines $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

(5) Find the perpendicular distance of the point (1, 0, 0) from the line $\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$

Also find the co-ordinates of the foot of the perpendicular.

(6) A(1, 0, 4), B(0, –11, 13), C(2, –3, 1) are three points and D is the foot of the perpendicular from A to BC. Find the co-ordinates of D.

(7) By computing the shortest distance, determine whether following lines intersect each other.

(i) $\bar{r} = \left(\hat{i} - \hat{j}\right) + \lambda\left(2\hat{i} + \hat{k}\right)$ and $\bar{r} = \left(2\hat{i} - \hat{j}\right) + \mu\left(\hat{i} + \hat{j} - \hat{k}\right)$

(ii) $\frac{x-5}{4} = \frac{y-7}{-5} = \frac{z+3}{-5}$ and $\frac{x-8}{7} = \frac{y-7}{1} = \frac{z-5}{3}$

(8) If lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect each other then find k.

Miscellaneous Exercise 6 A

(1) Find the vector equation of the line passing through the point having position vector $3\hat{i} + 4\hat{j} - 7\hat{k}$ and parallel to $6\hat{i} - \hat{j} + \hat{k}$.

(2) Find the vector equation of the line which passes through the point (3, 2, 1) and is parallel to the vector $2\hat{i} + 2\hat{j} - 3\hat{k}$.

(3) Find the Cartesian equations of the line which passes through the point (–2, 4, –5) and parallel to the line $\frac{x+2}{3}=\frac{y-3}{5}=\frac{z+5}{6}$.

(4) Obtain the vector equation of the line $\frac{x+5}{3}=\frac{y+4}{5}=\frac{z+5}{6}$.

(5) Find the vector equation of the line which passes through the origin and the point (5, –2, 3).

(6) Find the Cartesian equations of the line which passes through points (3, –2, –5) and (3, –2, 6).

(7) Find the Cartesian equations of the line passing through A(3, 2, l) and B(1, 3, 1).

(8) Find the Cartesian equations of the line passing through the point A(1, 1, 2) and perpendicular to vectors $\bar{b}=\hat{i}+2\hat{j}+\hat{k}$ and $\bar{c}=3\hat{i}+2\hat{j}-\hat{k}$.

(9) Find the Cartesian equations of the line which passes through the point (2, 1, 3) and perpendicular to lines $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-3}{3}$ and $\frac{x}{-3}=\frac{y}{2}=\frac{z}{5}$.

(10) Find the vector equation of the line which passes through the origin and intersect the line $x-1=y-2=z-3$ at right angle.

(11) Find the value of λ so that lines $\frac{1-x}{3}=\frac{7y-14}{2\lambda}=\frac{z-3}{2}$ and $\frac{7-7x}{3\lambda}=\frac{y-5}{1}=\frac{6-z}{5}$ are at right angle.

(12) Find the acute angle between lines $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-3}{2}$ and $\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{1}$.

(13) Find the acute angle between lines $x=y, z=0$ and $x=0, z=0$.

(14) Find the acute angle between lines $x=-y, z=0$ and $x=0, z=0$.

(15) Find the co-ordinates of the foot of the perpendicular drawn from the point (0, 2, 3) to the line $\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}$.

(16) By computing the shortest distance determine whether following lines intersect each other.

(i) $\bar{r}=\left(\hat{i}+\hat{j}-\hat{k}\right)+\lambda\left(2\hat{i}-\hat{j}+\hat{k}\right)$ and $\bar{r}=\left(2\hat{i}+2\hat{j}-3\hat{k}\right)+\mu\left(\hat{i}+\hat{j}-2\hat{k}\right)$

(ii) $\frac{x-5}{4}=\frac{y-7}{5}=\frac{z+3}{5}$ and $x-6=y-8=z+2$.

(17) If lines $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$ and $\frac{x-2}{1}=\frac{y+m}{2}=\frac{z-2}{1}$ intersect each other then find m.

(18) Find the vector and Cartesian equations of the line passing through the point (–1, –1, 2) and parallel to the line $2x-2=3y+1=6z-2$.

(19) Find the direction cosines of the line $\bar{r}=\left(-2\hat{i}+\frac{5}{2}\hat{j}-\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}\right)$.

(20) Find the Cartesian equation of the line passing through the origin which is perpendicular to $x-1=y-2=z-1$ and intersects the $\frac{x-1}{2}=\frac{y+1}{3}=\frac{z-1}{4}$.

(21) Write the vector equation of the line whose Cartesian equations are $y=2$ and $4x-3z+5=0$.

(22) Find the co-ordinates of points on the line $\frac{x-1}{1} = \frac{y-2}{-2} = \frac{z-3}{2}$ which are at the distance 3 unit from the base point A(l, 2, 3).

6.4 Equations of Plane :

Introduction : A plane is a surface such that the line joining any two points on it lies entirely on it.

Plane can be determined by

(i) two intersecting lines

(ii) two parallel lines

(iii) a line and a point outside it

(iv) three non collinear points.

Definition: A line perpendicular to a plane is called a normal to the plane. A plane has several normals. They all have the proportional direction ratios. We require only direction ratios of normal therefore we refer normal as **the normal** to a plane.

Direction ratios of the normal to the XY plane are 0, 0, 1.

6.4.1 Equation of plane passing through a point and perpendicular to a vector.

Theorem 6.8: The equation of the plane passing through the point $A(\bar{a})$ and perpendicular to vector $\bar{n}$ is $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$.

Proof: Let $P(\bar{r})$ be any point on the plane.

$\therefore \overline{AP}$ is perpendicular to $\bar{n}$.

$\therefore \overline{AP} \cdot \bar{n} = 0$

$\therefore (\bar{r} - \bar{a}) \cdot \bar{n} = 0$

$\therefore \bar{r} \cdot \bar{n} - \bar{a} \cdot \bar{n} = 0$

$\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$

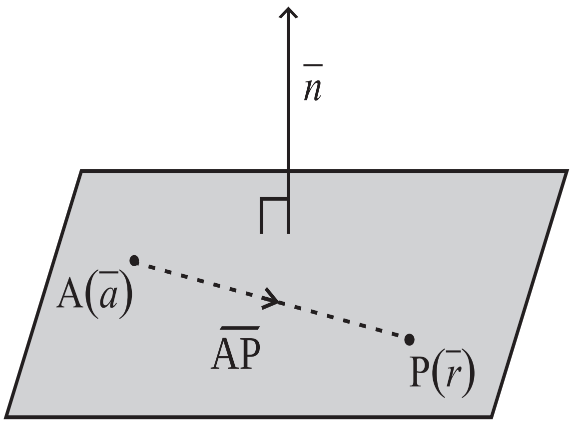

Fig. 6.8

This is the equation of the plane passing through the point $A(\bar{a})$ and perpendicular to the vector $\bar{n}$.

Remark :

- Equation $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$ is called the **vector equation of plane in scalar product form .**
- If $\bar{a} \cdot \bar{n} = d$ then equation $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$ takes the form $\bar{r} \cdot \bar{n} = d$.

Cartesian form :

Theorem 6.9 : The equation of the plane passing through the point A(x_1, y_1, z_1) and direction ratios of whose normal are a, b, c is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

Proof: Let P(x, y, z) be any point on the plane.

The direction ratios of AP are $x - x_1$, $y - y_1$, $z - z_1$.

The direction ratios of the normal are a, b, c. And AP is perpendicular to the normal.

$\therefore a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.. This is the required equation of plane.

Remark: Above equation may be written as $ax + by + cz + d = 0$

Ex.(1) Find the vector equation of the plane passing through the point having position vector $2\hat{i} + 3\hat{j} + 4\hat{k}$ and perpendicular to the vector $2\hat{i} + \hat{j} - 2\hat{k}$.

Solution: We know that the vector equation of the plane passing through $A(\bar{a})$ and normal to vector $\bar{n}$ is given by $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$.

Here $\bar{a} = 2\hat{i} + 3\hat{j} + 4\hat{k}$, $\bar{n} = 2\hat{i} + \hat{j} - 2\hat{k}$

$$\bar{a} \cdot \bar{n} = \left(2\hat{i} + 3\hat{j} + 4\hat{k}\right) \cdot \left(2\hat{i} + \hat{j} - 2\hat{k}\right) = 4 + 3 - 8 = -1$$

The vector equation of the plane is $\bar{r} \cdot \left(2\hat{i} + \hat{j} - 2\hat{k}\right) = -1$.

Ex.(2) Find the Cartesian equation of the plane passing through A(1, 2, 3) and the direction ratios of whose normal are 3, 2, 5.

Solution: The equation of the plane passing through A(x_1, y_1, z_1) and normal to the line having direction ratios a, b, c is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.

Here $(x_1, y_1, z_1) \equiv (1, 2, 3)$ and direction ratios of the normal are 3, 2. 5.

The Cartesian equation of the plane is $3(x - 1) + 2(y - 2) + 5(z - 3) = 0$.

$\therefore 3x + 2y + 5z - 22 = 0$.

Ex.(3) The foot of the perpendicular drawn from the origin to a plane is M(2, 1, -2). Find vector equation of the plane.

Solution: OM is normal to the plane.

$\therefore$ The direction ratios of the normal are 2, 1, -2.

The plane passes through the point having position vector $2\hat{i} + \hat{j} - 2\hat{k}$ and vector $\overline{\text{OM}} = 2\hat{i} + \hat{j} - 2\hat{k}$ is normal to it.

Its vector equation is $\bar{r} \cdot \bar{n} = \bar{a} \cdot \bar{n}$.

$\bar{r}\cdot\left(2\hat{i}+\hat{j}-2\hat{k}\right)=\left(2\hat{i}+\hat{j}-2\hat{k}\right)\cdot\left(2\hat{i}+\hat{j}-2\hat{k}\right)$

$\bar{r}\cdot\left(2\hat{i}+\hat{j}-2\hat{k}\right)=9$.

6.4.2 The vector equation of the plane passing through point $A(\bar{a})$ and parallel to $\bar{b}$ and $\bar{c}$:

Theorem 6.10 : The vector equation of the plane passing through the point $A(\bar{a})$ and parallel to non-zero and non-parallel vectors $\bar{b}$ and $\bar{c}$ is $(\bar{r}-\bar{a})\cdot\left(\bar{b}\times\bar{c}\right)=0$.

Proof: As vectors $\bar{b}$ and $\bar{c}$ are parallel to the plane, vector $\bar{b}\times\bar{c}$ is normal to the plane. Plane passes through $A(\bar{a})$.

Let $P(\bar{r})$ be any point on the plane.

$\therefore \overline{AP}$ is perpendicular to $\bar{b}\times\bar{c}$

$\therefore \overline{AP}\cdot\left(\bar{b}\times\bar{c}\right)=0$

$\therefore (\bar{r}-\bar{a})\cdot\left(\bar{b}\times\bar{c}\right)=0$

$\therefore \bar{r}\cdot\left(\bar{b}\times\bar{c}\right)=\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)$ is the required equation.

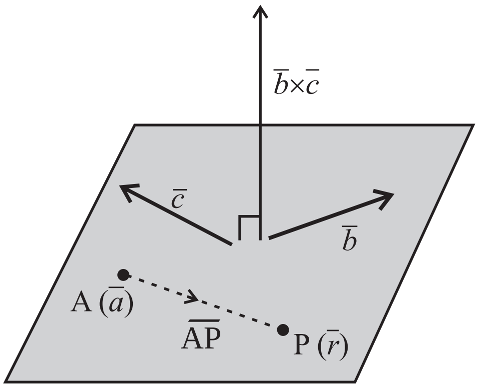

Fig. 6.9

Remark: As $\overline{AP}$, $\bar{b}$ and $\bar{c}$ are parallel to the same plane, they are coplanar vectors. Therefore $\overline{AP}$ can be expressed as the linear combination of $\bar{b}$ and $\bar{c}$. Hence $\overline{AP}=\lambda\bar{b}+\mu\bar{c}$ for some scalars λ and μ.

$\therefore \bar{r}-\bar{a}=\lambda\bar{b}+\mu\bar{c}$

$\therefore \bar{r}=\bar{a}+\lambda\bar{b}+\mu\bar{c}$ This equation is called the **vector equation of plane in parametric form.**

Ex(4) Find the vector equation of the plane passing through the point A(–1, 2, –5) an parallel to vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $\hat{i}+\hat{j}-\hat{k}$.

Solution: The vector equation of the plane passing through point $A(\bar{a})$ and parallel to $\bar{b}$ and $\bar{c}$ is $\bar{r}\cdot\left(\bar{b}\times\bar{c}\right)=\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)$.

Here $\bar{a}=-\hat{i}+2\hat{j}-5\hat{k}$, $\bar{b}=4\hat{i}-\hat{j}+3\hat{k}$, $\bar{c}=\hat{i}+\hat{j}-\hat{k}$

$$\bar{b}\times\bar{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & -1 & 3\\ 1 & 1 & -1\end{vmatrix}=-2\hat{i}+7\hat{j}+5\hat{k}$$

$$\bar{a}\cdot\left(\bar{b}\times\bar{c}\right)=\left(-\hat{i}+2\hat{j}-5\hat{k}\right)\cdot\left(-2\hat{i}+7\hat{j}+5\hat{k}\right)=-9$$

The required equation is $\bar{r}\cdot\left(-2\hat{i}+7\hat{j}+5\hat{k}\right)=-9$

Ex.(5) Find the Cartesian equation of the plane $\bar{r}=\left(\hat{i}-\hat{j}\right)+\lambda\left(\hat{i}+\hat{j}+\hat{k}\right)+\mu\left(\hat{i}-2\hat{j}+3\hat{k}\right)$

Solution: Given plane is perpendicular to vector $\bar{n}$, where

$$\bar{n}=\bar{b}\times\bar{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ 1 & -2 & 3\end{vmatrix}=5\hat{i}-2\hat{j}-3\hat{k}$$

$\therefore$ The direction ratios of the normal are 5, –2, –3.

And plane passes through A(1, -1, 0).

$\therefore$ Its Cartesian equation is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\therefore 5(x - 1) - 2(y + 1) - 3(z - 0) = 0$

$\therefore 5x - 2y - 3z - 7 = 0$

6.4.3 The vector equation of plane passing through three non-collinear points :

Theorem 6.11 : The vector equation of the plane passing through non-collinear points $A\left(\bar{a}\right)$, $B\left(\bar{b}\right)$ and $C\left(\bar{c}\right)$ is $\left(\bar{r}-\bar{a}\right)\cdot\left(\bar{b}-\bar{a}\right)\times\left(\bar{c}-\bar{a}\right)=0$

Proof: Let $P\left(\bar{r}\right)$ be any point on the plane passing through non-collinear points $A\left(\bar{a}\right)$, $B\left(\bar{b}\right)$ and $C\left(\bar{c}\right)$.

$\therefore\ \overline{AP},\ \overline{AB}$ and $\overline{AC}$ are coplanar .

$\therefore\ \overline{AP}\cdot\overline{AB}\times\overline{AC}=0$

$\therefore\left(\bar{r}-\bar{a}\right)\cdot\overline{AB}\times\overline{AC}=0$

$\therefore\left(\bar{r}-\bar{a}\right)\cdot\left(\bar{b}-\bar{a}\right)\times\left(\bar{c}-\bar{a}\right)=0$

$\therefore\left[\bar{r}-\bar{a}\quad \bar{b}-\bar{a}\quad \bar{c}-\bar{a}\right]=0$

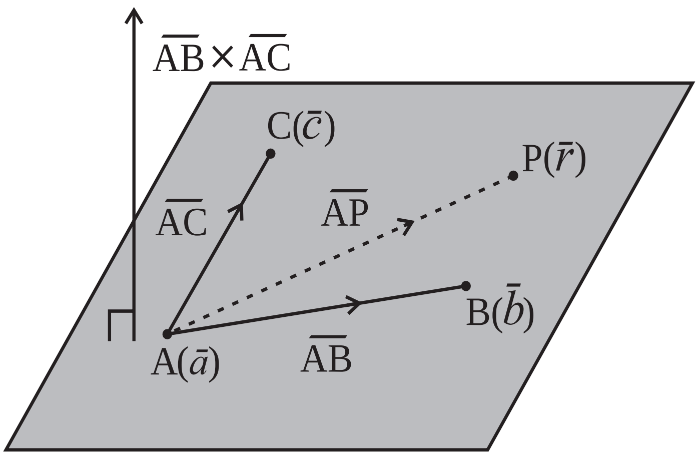

Fig. 6.10

This is the required equation of plane.

Cartesian form of the above equation is

$$\begin{vmatrix}x-x_1 & y-y_1 & z-z_1\\ x_2-x_1 & y_2-y_1 & z_2-z_1\\ x_3-x_1 & y_3-y_1 & z_3-z_1\end{vmatrix}=0$$

Ex.(6) Find the vector equation of the plane passing through points A (l, 1, 2), B (0, 2, 3) and C (4, 5, 6).

Solution; Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be position vectors of points A, B and C respectively .

$$\therefore \bar{a} = \hat{i} + \hat{j} + 2\hat{k},\ \bar{b} = 2\hat{j} + 3\hat{k},\ \bar{c} = 4\hat{i} + 5\hat{j} + 6\hat{k}$$

$$\therefore \bar{b} - \bar{a} = -\hat{i} + \hat{j} + \hat{k} \text{ and } \bar{c} - \bar{a} = 3\hat{i} + 4\hat{j} + 4\hat{k}$$

$$\therefore \left(\bar{b} - \bar{a}\right) \times \left(\bar{c} - \bar{a}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 1 \\ 3 & 4 & 4 \end{vmatrix} = 7\hat{j} - 7\hat{k}$$

The plane passes through $\bar{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $7\hat{j} - 7\hat{k}$ is normal to the plane.

$\therefore$ Its equation is $\left(\bar{r} - \bar{a}\right) \cdot \left(7\hat{j} - 7\hat{k}\right) = 0$

$$\therefore \left(\bar{r} - \left(\hat{i} + \hat{j} + 2\hat{k}\right)\right) \cdot \left(7\hat{j} - 7\hat{k}\right) = 0$$

$$\therefore \bar{r} \cdot \left(7\hat{j} - 7\hat{k}\right) = \left(\hat{i} + \hat{j} + 2\hat{k}\right) \cdot \left(7\hat{j} - 7\hat{k}\right)$$

$\therefore \bar{r} \cdot \left(7\hat{j} - 7\hat{k}\right) = -7$ is the required equation.

6.4.4 The normal form of equation of plane.

Theorem 6.12: The equation of the plane at distance p unit from the origin and to which unit vector $\hat{n}$ is normal is $\bar{r} \cdot \hat{n} = p$

Proof: Let ON be the perpendicular to the plane $\therefore$ ON = p

As $\hat{n}$ is the unit vector along ON, $\overline{ON} = p\hat{n}$

Let $P(\bar{r})$ be any point on the plane.

$\therefore \overline{NP} \perp \hat{n}$ $\qquad$ $\therefore \overline{NP} \cdot \hat{n} = 0$

$\therefore \left(\bar{r} - p\hat{n}\right) \cdot \hat{n} = 0$ $\qquad$ $\therefore \bar{r} \cdot \hat{n} - p\hat{n} \cdot \hat{n} = 0$

$\therefore \bar{r} \cdot \hat{n} - p = 0$ $\qquad$ $\therefore \bar{r} \cdot \hat{n} = p$

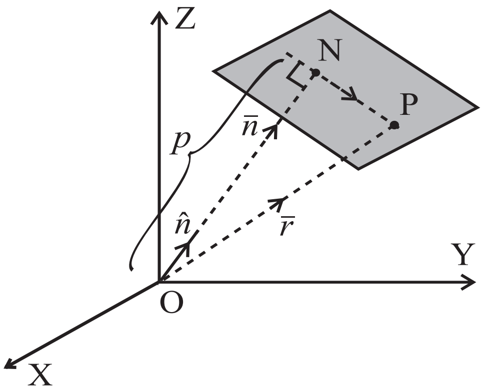

Fig. 6.11

This is called the **normal form of vector equation** of plane.

Remark:

- If l, m, n are direction cosines of the normal to a plane then $\hat{n} = l\hat{i} + m\hat{j} + n\hat{k}$.
- If N is the foot of the perpendicular drawn from the origin to the plane and ON = p then the co-ordinates of N are (pl, pm, pn).
- The equation of the plane is $lx + my + nz = p$. This is the **normal form of the Cartesian equation** of the plane.
- There are two planes at distance p units from origin and having $\hat{n}$ as unit vector along normal, namely $\bar{r} \cdot \hat{n} = \pm p$

Ex.(7) Find the vector equation of the plane which is at a distance of 6 unit from the origin and to which the vector $2\hat{i}-\hat{j}+2\hat{k}$ is normal.

Solution: Here p = 6 and $\bar{n}=2\hat{i}-\hat{j}+2\hat{k}$ $\quad\quad \therefore |\bar{n}|=3$

$$\therefore \hat{n}=\frac{\bar{n}}{|\bar{n}|}=\frac{2\hat{i}-\hat{j}+2\hat{k}}{3}$$

The required equation is $\therefore \bar{r}\cdot\hat{n}=p$

$$\therefore \bar{r}\cdot\left(\frac{2\hat{i}-\hat{j}+2\hat{k}}{3}\right)=6$$

$$\therefore \bar{r}\cdot\left(2\hat{i}-\hat{j}+2\hat{k}\right)=18$$

Ex.(8) Find the perpendicular distance of the origin from the plane $x-3y+4z-6=0$

Solution : First we write the given Cartesian equation in normal form.

i.e. in the form $lx + my + nz = p$

Direction ratios of the normal are 1, –3, 4.

$\therefore$ Direction cosines are $\frac{1}{\sqrt{26}},\frac{-3}{\sqrt{26}},\frac{4}{\sqrt{26}}$

Given equation can be written as $\frac{1}{\sqrt{26}}x-\frac{-3}{\sqrt{26}}y+\frac{4}{\sqrt{26}}z=\frac{6}{\sqrt{26}}$

$\therefore$ The distance of the origin from the plane is $\frac{6}{\sqrt{26}}$

Ex.(9) Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $2x+y-2z=18$

Solution:

The Direction ratios of the normal to the plane $2x+y-2z=18$ are 2, 1, –2.

$\therefore$ Direction cosines are $\frac{2}{3},\frac{1}{3},-\frac{2}{3}$

The normal form of the given Cartesian equation is $\frac{2}{3}x+\frac{1}{3}y-\frac{2}{3}z=6$

$\therefore$ $p=6$

The coordinates of the foot of the perpendicular are $(lp, mp, np)=\left(6\left(\frac{2}{3}\right),6\left(\frac{1}{3}\right),6\left(-\frac{2}{3}\right)\right)\equiv(4,2,-4)$

Ex.(10) Reduce the equation $\bar{r}\cdot\left(3\hat{i}-4\hat{j}+12\hat{k}\right)=8$ to the normal form and hence find

(i) the length of the perpendicular from the origin to the plane

(ii) direction cosines of the normal.

Solution: Here $\bar{n}=3\hat{i}-4\hat{j}+12\hat{k}$ $\quad \therefore |\bar{n}|=13$

The required normal form is $\bar{r}\cdot\dfrac{\left(3\hat{i}-4\hat{j}+12\hat{k}\right)}{13}=\dfrac{8}{13}$

(i) the length of the perpendicular from the origin to the plane is $\dfrac{8}{13}$

(ii) direction cosines of the normal are $\dfrac{3}{13},\dfrac{-4}{13},\dfrac{12}{13}$.

6.4.5 Equation of plane passing through the intersection of two planes :

If planes $\left(\bar{r}\cdot\bar{n}_1-d_1\right)=0$ and $\bar{r}\cdot\bar{n}_2-d_2=0$ intersect each other, then for every real value of λ equation $\bar{r}\cdot\left(\bar{n}_1+\lambda\bar{n}_2\right)=\left(d_1+\lambda d_2\right)$ represents a plane passing through the line of their intersection

If planes $a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$ intersect each other, then for every real value of λ, equation $\left(a_1x+b_1y+c_1z+d_1\right)+\lambda\left(a_2x+b_2y+c_2z+d_2\right)=0$ represents a plane passing through the line of their intersection.

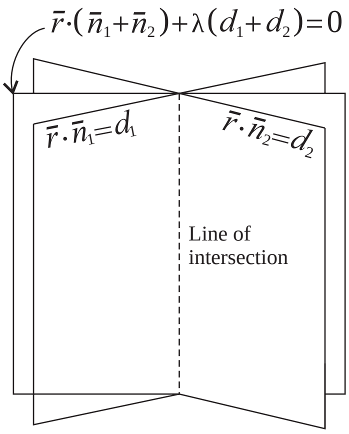

Fig. 6.12

Ex.(11) Find the vector equation of the plane passing through the point (1, 0. 2) and the line of intersection of planes $\bar{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=8$ and $\bar{r}\cdot\left(2\hat{i}+3\hat{j}+4\hat{k}\right)=3$

Solution: The equation of the required plane is of the form equation $\therefore \bar{r}\cdot\left(\bar{n}_1+\lambda\bar{n}_2\right)-\left(d_1+\lambda d_2\right)=0$

$$\therefore \bar{r}\cdot\left[\left(\hat{i}+\hat{j}+\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}+4\hat{k}\right)\right]=8+3\lambda \quad \text{... (1)}$$

$$\therefore \bar{r}\cdot\left(\left(1+2\lambda\right)\hat{i}+\left(1+3\lambda\right)\hat{j}+\left(1+4\lambda\right)\hat{k}\right)=8+3\lambda$$

The plane passes through the point (1, 0, 2).

$$\therefore \left(\hat{i}+2\hat{k}\right)\cdot\left(\left(1+2\lambda\right)\hat{i}+\left(1+3\lambda\right)\hat{j}+\left(1+4\lambda\right)\hat{k}\right)=8+3\lambda$$

$$\therefore \left(1+2\lambda\right)+2\left(1+4\lambda\right)=8+3\lambda$$

$$\therefore 1+2\lambda+2+8\lambda=8+3\lambda$$

$\therefore 7\lambda = 5$

$\therefore \lambda = \frac{5}{7}$ ... (2)

From (1) and (2) we get

$$\therefore \bar{r}\cdot\left(\left(\hat{i}+\hat{j}+\hat{k}\right)+\frac{5}{7}\left(2\hat{i}+3\hat{j}+4\hat{k}\right)\right)=8+3\left(\frac{5}{7}\right)$$

$$\therefore \bar{r}\cdot\left(17\hat{i}+22\hat{j}+27\hat{k}\right)=71$$

Exercise 6.3

(1) Find the vector equation of a plane which is at 42 unit distance from the origin and which is normal to the vector $2\hat{i}+\hat{j}-2\hat{k}$.

(2) Find the perpendicular distance of the origin from the plane $6x-2y+3z-7=0$.

(3) Find the coordinates of the foot of the perpendicular drawn from the origin to the plane $2x+6y-3z=63$.

(4) Reduce the equation $\bar{r}\cdot\left(3\hat{i}+4\hat{j}+12\hat{k}\right)=78$ to normal form and hence find

(i) the length of the perpendicular from the origin to the plane (ii) direction cosines of the normal.

(5) Find the vector equation of the plane passing through the point having position vector $\hat{i}+\hat{j}+\hat{k}$ and perpendicular to the vector $4\hat{i}+5\hat{j}+6\hat{k}$.

(6) Find the Cartesian equation of the plane passing through A(–1, 2, 3), the direction ratios of whose normal are 0, 2, 5.

(7) Find the Cartesian equation of the plane passing through A(7, 8, 6) and parallel to the XY plane.

(8) The foot of the perpendicular drawn from the origin to a plane is M(1,0,0). Find the vector equation of the plane.

(9) Find the vector equation of the plane passing through the point A(–2, 7, 5) and parallel to vectors $4\hat{i}-\hat{j}+3\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$.

(10) Find the Cartesian equation of the plane $\bar{r}=\left(5\hat{i}-2\hat{j}-3\hat{k}\right)+\lambda\left(\hat{i}+\hat{j}+\hat{k}\right)+\mu\left(\hat{i}-2\hat{j}+3\hat{k}\right)$.

(11) Find the vector equation of the plane which makes intercepts 1, 1, 1 on the co-ordinates axes.

6.5 Angle between planes: In this article we will discuss angles between two planes, angle between a line and a plane.

6.5.1 Angle between planes: Angle between planes can be determined from the angle between their normals. Planes $\bar{r}\cdot\bar{n}_1 = d_1$ and $\bar{r}\cdot\bar{n}_2 = d_2$ are perpendicular to each other if and only if $\bar{n}_1\cdot\bar{n}_2 = 0$ Planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are perpendicular to each other if and only if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Definition : If two planes are not perpendicular to each other then **the angle between them** is defined as the **acute angle** between their normals.

The angle between the planes $\bar{r}\cdot\bar{n}_1 = d_1$ and $\bar{r}\cdot\bar{n}_2 = d_2$ is given by $\cos\theta = \left|\dfrac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1|\cdot|\bar{n}_2|}\right|$

Ex.(12) Find the angle between planes $\bar{r}\cdot\left(\hat{i}+\hat{j}-2\hat{k}\right)=8$ and $\bar{r}\cdot\left(-2\hat{i}+\hat{j}+\hat{k}\right)=3$

Solution: Normal to the given planes are $\bar{n}_1 = \hat{i}+\hat{j}-2\hat{k}$ and $\bar{n}_2 = -2\hat{i}+\hat{j}+\hat{k}$

The acute angle θ between normal is given by

$$\cos\theta = \left|\frac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1|\cdot|\bar{n}_2|}\right|$$

$$\therefore \cos\theta = \left|\frac{\left(\hat{i}+\hat{j}-2\hat{k}\right)\cdot\left(-2\hat{i}+\hat{j}+\hat{k}\right)}{\sqrt{6}\cdot\sqrt{6}}\right| = \left|\frac{-3}{6}\right| = \frac{1}{2}$$

$$\therefore \cos\theta = \frac{1}{2} \qquad\qquad \therefore \theta = 60^\circ = \frac{\pi}{3}$$

The acute angle between normals $\bar{n}_1$ and $\bar{n}_2$ is 60°.

∴ The angle between given planes is 60°.

6.5.2 Angle between a line and a plane: Line $\bar{r} = \bar{a} + \lambda\bar{b}$ is perpendicular to the plane $\bar{r}\cdot\bar{n} = d$ if and only if $\bar{b}$ and $\bar{n}$ are collinear. *i.e.* if $\bar{b} = t\bar{n}$ for some $t \in$ R Line $\bar{r} = \bar{a} + \lambda\bar{b}$ is parallel to the plane of $\bar{r}\cdot\bar{n} = d$ and only if $\bar{b}$ and $\bar{n}$ are perpendicular to each other. *i.e.* if $\bar{b}\cdot\bar{n} = 0$.

Definition : The angle between a line and a plane is defined as the complementary angle of the acute angle between the normal to the plane and the line.

Because of the definition, the angle between a line and a plane can't be obtuse.

If θ is the angle between the line $\bar{r} = \bar{a} + \lambda\bar{b}$ and the plane $\bar{r}\cdot\bar{n} = d$ then the acute angle between the line and the normal to the plane is $\dfrac{\pi}{2} - \theta$.

$$\therefore \cos\left(\frac{\pi}{2} - \theta\right) = \left|\frac{\bar{b}\cdot\bar{n}}{|\bar{b}|\cdot|\bar{n}|}\right|$$

$$\sin\theta = \left|\frac{\bar{b}\cdot\bar{n}}{|\bar{b}|\cdot|\bar{n}|}\right|$$

Ex.(13) Find the angle between the line $\bar{r} = \left(\hat{i} + 2\hat{j} + \hat{k}\right) + \lambda\left(\hat{i} + \hat{j} + \hat{k}\right)$ and the plane $\bar{r} \cdot \left(2\hat{i} - \hat{j} + \hat{k}\right) = 8$

Solution: The angle between the line $\bar{r} = \bar{a} + \lambda\bar{b}$ and the plane $\bar{r} \cdot \bar{n} = d$ is given by $\sin\theta = \left|\dfrac{\bar{b} \cdot \bar{n}}{|\bar{b}| \cdot |\bar{n}|}\right|$

Here $\bar{b} = \hat{i} + \hat{j} + \hat{k}$ and $\bar{n} = 2\hat{i} - \hat{j} + \hat{k}$

$\therefore \bar{b} \cdot \bar{n} = \left(\hat{i} + \hat{j} + \hat{k}\right) \cdot \left(2\hat{i} - \hat{j} + \hat{k}\right) = 2 - 1 + 1 = 2$

$|\bar{b}| = \sqrt{1+1+1} = \sqrt{3}$ and $|\bar{n}| = \sqrt{4+1+1} = \sqrt{6}$

$$\therefore \sin\theta = \left|\frac{\bar{b} \cdot \bar{n}}{|\bar{b}| \cdot |\bar{n}|}\right| = \frac{2}{\sqrt{3} \cdot \sqrt{6}} = \frac{\sqrt{2}}{3}$$

$$\therefore \theta = \sin^{-1}\left(\frac{\sqrt{2}}{3}\right)$$

6.6 Coplanarity of two lines:

We know that two parallel lines are always coplanar. If two non-parallel lines are coplanar then the shortest distance between them is zero. Conversely if the distance between two non-parallel lines is zero then they are coplanar.

Thus lines $\bar{r} = \bar{a}_1 + \lambda_1\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2\bar{b}_2$ are coplanar if and only if $\left(\bar{a}_2 - \bar{a}_1\right) \cdot \left(\bar{b}_1 \times \bar{b}_2\right) = 0$

The plane determined by them passes through $A\left(\bar{a}_1\right)$ and $\bar{b}_1 \times \bar{b}_2$. is normal to the plane .

$\therefore$ Its equation is $\left(\bar{r} - \bar{a}_1\right) \cdot \left(\bar{b}_1 \times \bar{b}_2\right) = 0$.

Lines $\dfrac{x - x_1}{a_1} = \dfrac{y - y_1}{b_1} = \dfrac{z - z_1}{c_1}$ and $\dfrac{x - x_2}{a_2} = \dfrac{y - y_2}{b_2} = \dfrac{z - z_2}{c_2}$ are coplanar if and only if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

The equation of the plane determined by them is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$.

Ex.(14) Show that lines $\bar{r} = \left(\hat{i}+\hat{j}-\hat{k}\right)+\lambda\left(2\hat{i}-2\hat{j}+\hat{k}\right)$ and $\bar{r} = \left(4\hat{i}-3\hat{j}+2\hat{k}\right)+\mu\left(\hat{i}-2\hat{j}+2\hat{k}\right)$ are coplanar. Find the equation of the plane determined by them.

Solution: Lines $\bar{r} = \bar{a}_1 + \lambda_1\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2\bar{b}_2$ are coplanar if and only if $\left(\bar{a}_2 - \bar{a}_1\right)\cdot\left(\bar{b}_1 \times \bar{b}_2\right) = 0$

Here $\bar{a}_1 = \hat{i}+\hat{j}-\hat{k},\ \bar{a}_2 = 4\hat{i}-3\hat{j}+2\hat{k}$

$\bar{b}_1 = 2\hat{i}-2\hat{j}+\hat{k},\ \bar{b}_2 = \hat{i}-2\hat{j}+2\hat{k}$

$\therefore \bar{a}_2 - \bar{a}_1 = 3\hat{i}-4\hat{j}+3\hat{k}$

$$\left(\bar{a}_2 - \bar{a}_1\right)\cdot\left(\bar{b}_1 \times \bar{b}_2\right) = \begin{vmatrix} 3 & -4 & 3 \\ 2 & -2 & 1 \\ 1 & -2 & 2 \end{vmatrix} = 3(-2)+4(3)+3(-2) = -6+12-6 = 0$$

$\therefore$ Given lines are coplanar.

$$\text{Now } \bar{b}_1 \times \bar{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & -2 & 2 \end{vmatrix} = -2\hat{i}-3\hat{j}-2\hat{k}$$

The equation of the plane determined by them is $\left(\bar{r}-\bar{a}_1\right)\cdot\left(\bar{b}_1 \times \bar{b}_2\right) = 0$

$\therefore \bar{r}\cdot\left(\bar{b}_1 \times \bar{b}_2\right) = \bar{a}_1\cdot\left(\bar{b}_1 \times \bar{b}_2\right)$

$\therefore \bar{r}\cdot\left(-2\hat{i}-3\hat{j}-2\hat{k}\right) = \left(\hat{i}+\hat{j}-\hat{k}\right)\cdot\left(-2\hat{i}-3\hat{j}-2\hat{k}\right)$

$\therefore \bar{r}\cdot\left(-2\hat{i}-3\hat{j}-2\hat{k}\right) = -3$

$\therefore \bar{r}\cdot\left(2\hat{i}+3\hat{j}+2\hat{k}\right) = 3$

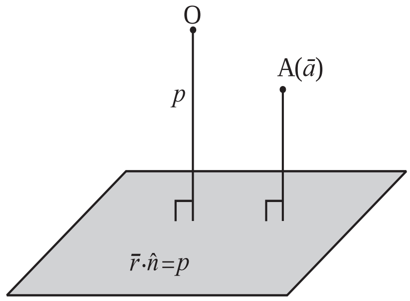

Fig. 6.13

6.7 Distance of a point from a plane.

To find the distance of the point $A\left(\bar{a}\right)$ from the plane $\bar{r}\cdot\hat{n} = p$

The distance of the origin from the plane $\bar{r}\cdot\hat{n} = p$ is p.

The equation of the plane passing through $A\left(\bar{a}\right)$ and parallel to the plane

$\bar{r}\cdot\hat{n} = p$ is $\bar{r}\cdot\hat{n} = \bar{a}\cdot\hat{n}$.

The distance of the origin from the plane $\bar{r}\cdot\hat{n} = \bar{a}\cdot\hat{n}$ is $\bar{a}\cdot\hat{n}$

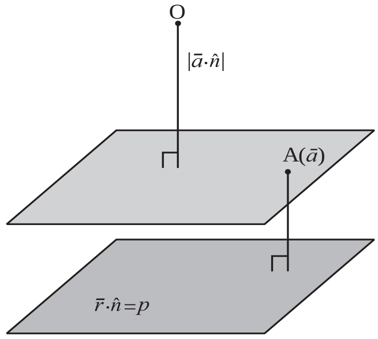

Fig. 6.14

The distance of the point $A(\bar{a})$ from the plane $\bar{r}\cdot\hat{n}=p$ is given by $\left|p-\bar{a}\cdot\hat{n}\right|$.

Remark : For finding distance of a point from a plane, the equation of the plane must be in the normal form.

Ex.(15) Find the distance of the point $4\hat{i}-3\hat{j}+2\hat{k}$ from the plane $\bar{r}\,.\left(-2\hat{i}+\hat{j}-2\hat{k}\right)=6$

Solution: Here $\bar{a}=4\hat{i}-3\hat{j}+2\hat{k}, \bar{n}=-2\hat{i}+\hat{j}-2\hat{k}$

$$\therefore |\bar{n}|=\sqrt{(-2)^2+(1)^2+(-2)^2}=3$$

$$\therefore \hat{n}=\frac{\left(-2\hat{i}+\hat{j}-2\hat{k}\right)}{3}$$

The normal form of the equation of the given plane is

$$\bar{r}\cdot\frac{\left(-2\hat{i}+\hat{j}-2\hat{k}\right)}{3}=2 \quad \therefore p=2$$

Now, $\bar{a}\cdot\hat{n}=\left(4\hat{i}-3\hat{j}+2\hat{k}\right)\cdot\dfrac{\left(-2\hat{i}+\hat{j}-2\hat{k}\right)}{3}$

$$=\frac{\left(4\hat{i}-3\hat{j}+2\hat{k}\right)\cdot\left(-2\hat{i}+\hat{j}-2\hat{k}\right)}{3}=\frac{15}{3}=-5$$

$$\therefore \bar{a}\cdot\hat{n}=-5$$

The required distance is given by $\left|p-\bar{a}\cdot\hat{n}\right|=|2-(-5)|=|7|=7$

Therefore the distance of the point $4\hat{i}-3\hat{j}+2\hat{k}$ from the plane $\left(-2\hat{i}+\hat{j}-2\hat{k}\right)=6$ is 7 unit.

Exercise 6.4

(1) Find the angle between planes $\bar{r}\cdot\left(\hat{i}+\hat{j}+2\hat{k}\right)=13$ and $\bar{r}\cdot\left(2\hat{i}-\hat{j}+\hat{k}\right)=31$.

(2) Find the acute angle between the line $\bar{r}\cdot\left(\hat{i}+2\hat{j}+2\hat{k}\right)+\lambda\left(2\hat{i}+3\hat{j}-6\hat{k}\right)$ and the plane $\bar{r}\cdot\left(2\hat{i}-\hat{j}+\hat{k}\right)=0$.

(3) Show that lines $\bar{r}=\left(2\hat{j}-3\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}+3\hat{k}\right)$ and $\bar{r}=\left(2\hat{i}+6\hat{j}+3\hat{k}\right)+\mu\left(2\hat{i}+3\hat{j}+4\hat{k}\right)$ are coplanar. Find the equation of the plane determined by them.

(4) Find the distance of the point $4\hat{i}-3\hat{j}+\hat{k}$ from the plane $\bar{r}\cdot\left(2\hat{i}+3\hat{j}-6\hat{k}\right)=21$.

(5) Find the distance of the point (1, 1, –l) from the plane $3x+4y-12z+20=0$.

Remember This: Line

- The vector equation of the line passing through $A(\bar{a})$ and parallel to vector $\bar{b}$ is $\bar{r} = \bar{a} + \lambda\bar{b}$
- The vector equation of the line passing through $A(\bar{a})$ and $B(\bar{b})$ is $\bar{r} = \bar{a} + \lambda(\bar{b} - \bar{a})$.
- The Cartesian equations of the line passing through A(x_1, y_1, z_1) and having direction ratios *a*, *b*, *c* are $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$
- The distance of point $P(\bar{\alpha})$ from the line $\bar{r} = \bar{a} + \lambda\bar{b}$ is given by $\sqrt{\left|\bar{\alpha} - \bar{a}\right|^2 - \left[\frac{(\bar{\alpha} - \bar{a})\cdot\bar{b}}{\left|\bar{b}\right|}\right]^2}$
- The shortest distance between lines $\bar{r} = \bar{a}_1 + \lambda_1\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2\bar{b}_2$ is given by $d = \left|\frac{(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2)}{\left|\bar{b}_1 \times \bar{b}_2\right|}\right|$
- Lines $\bar{r} = \bar{a}_1 + \lambda_1\bar{b}_1$ and $\bar{r} = \bar{a}_2 + \lambda_2\bar{b}_2$ intersect each other if and only if $(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1 \times \bar{b}_2) = 0$
- Lines $\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$ and $\frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$ intersect each other if and only if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$
- The distance between parallel lines $\bar{r} = \bar{a}_1 + \lambda\bar{b}$ and $\bar{r} = \bar{a}_2 + \lambda\bar{b}$ is given by $d = \left|(\bar{a}_2 - \bar{a}_1)\times\hat{b}\right|$

Plane

- The vector equation of the plane passing through $A(\bar{a})$ and normal to vector $\bar{n}$ *is* $\bar{r}\cdot\bar{n} = \bar{a}\cdot\bar{n}$
- Equation $\bar{r}\cdot\bar{n} = \bar{a}\cdot\bar{n}$ is called the **veetor equation of plane in scalar product form.**
- If $\bar{a}\cdot\bar{n} = d$ then equation $\bar{r}\cdot\bar{n} = \bar{a}\cdot\bar{n}$ takes the form $\bar{r}\cdot\bar{n} = d$
- The equation of the plane passing through $A(x_1, y_1, z_1)$ and normal to the line having direction ratios *a*,*b*,*c* is $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$.

- The vector equation of the plane passing through point $A(\bar{a})$ and parallel to $\bar{b}$ and $\bar{c}$ is $\bar{r}\cdot(\bar{b}\times\bar{c}) = a\cdot(\bar{b}\times\bar{c})$
- Equation $\bar{r} = \bar{a} + \lambda\bar{b} + \mu\bar{c}$ is called the vector equation of plane in parametric form.
- The vector equation of the plane passing through non-collinear points $A(\bar{a})$, $B(\bar{b})$ and $C(\bar{c})$ is $(\bar{r}-\bar{a})\cdot(\bar{b}-\bar{a})\times(\bar{c}-\bar{a}) = 0$
- Cartesian from of the above equation is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$
- The equation of the plane at distance *p* unit from the origin and to which unit vector $\hat{n}$ is normal is $\bar{r}\cdot\hat{n} = p$
- If *l, m, n* are direction cosines of the normal to a plane which is at distance *p* unit from the origin then its equation is $lx + my + nz = p$.
- If *N* is the foot of the perpendicular drawn from the origin to a plane and $ON = p$ then the co-ordinates of *N* are *(pl, pm, pn)*.
- If planes $(\bar{r}\cdot\bar{n}_1 - d_1) = 0$ and $\bar{r}\cdot\bar{n}_2 - d_2 = 0$ intersect each other, then for every real value of λ,equation $\bar{r}\cdot(\bar{n}_1 + \lambda\bar{n}_2) - (d_1 + \lambda d_2) = 0$ represents a plane passing through the line of their intersection.
- If planes and $a_1x+b_1y+c_1z+d=0$ and $a_2x+b_2y + c_2z + d_2 = 0$ intersect each other, then for every real value of λ, equation $(a_1x+b_1y + c_1z + d_1) + \lambda\,(a_2x+b_2y\ c_2z + d_2) = 0$ represents a plane passing through the line of their intersection.
- The angle between the planes $\bar{r}\cdot\bar{n}_1 = d_1$ and $\bar{r}\cdot\bar{n}_2 = d_2$ is given by $\cos\theta = \left|\dfrac{\bar{n}_1\cdot\bar{n}_2}{|\bar{n}_1||\bar{n}_2|}\right|$
- The acute angle between the line $\bar{r} = \bar{a} + \lambda\bar{b}$ and the plane $\bar{r}\cdot\bar{n} = d$ is given by $\sin\theta = \left|\dfrac{\bar{b}\cdot\bar{n}}{|\bar{b}_1||\bar{n}|}\right|$
- Lines $\bar{r} = a_1 + \lambda_1 b_1$ and $\bar{r} = \bar{a}_2 + \lambda_2\bar{b}_2$ are coplanar if and only if $(\bar{a}_2 - \bar{a}_1)\cdot(\bar{b}_1\times\bar{b}_2) = 0$ and the quation of the plane determined by them is $(\bar{r} - \bar{a}_1)\cdot(\bar{b}_1\times\bar{b}_2) = 0$
- Lines $\dfrac{x-x_1}{a_1} = \dfrac{y-y_1}{b_1} = \dfrac{z-z_1}{c_1}$ and $\dfrac{x-x_2}{a_2} = \dfrac{y-y_2}{b_2} = \dfrac{z-z_2}{c_2}$.

are coplanar if and only if $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$, and the equation of the plane determined by them is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

- The distance of the point $A(\bar{a})$ from the plane $\bar{r}\cdot\hat{n} = p$ is given by $\left|p - \bar{a}\cdot\hat{n}\right|$

Miscellenous Exercise 6 (B)

I **Choose correct alternatives.**

1) If the line $\frac{x}{3} = \frac{y}{4} = z$ is perpendicular to the line $\frac{x-1}{k} = \frac{y+2}{3} = \frac{z-3}{k-1}$ then the value of k is:

A) $\frac{11}{4}$ B) $-\frac{11}{4}$ C) $\frac{11}{2}$ D) $\frac{4}{11}$

2) The vector equation of line $2x - 1 = 3y + 2 = z - 2$ is

A) $\bar{r} = \left(\frac{1}{2}\hat{i} - \frac{2}{3}\hat{j} + 2\hat{k}\right) + \lambda\left(3\hat{i} + 2\hat{j} + 6\hat{k}\right)$

B) $\bar{r} = \hat{i} - \hat{j} + \left(2\hat{i} + \hat{j} + \hat{k}\right)$

C) $\bar{r} = \left(\frac{1}{2}\hat{i} - \hat{j}\right) + \lambda\left(\hat{i} - 2\hat{j} + 6\hat{k}\right)$

D) $\bar{r} = \left(\hat{i} + \hat{j}\right) + \lambda\left(\hat{i} - 2\hat{j} + 6\hat{k}\right)$

3) The direction ratios of the line which is perpendicular to the two lines

$\frac{x-7}{2} = \frac{y+17}{-3} = \frac{z-6}{1}$ and $\frac{x+5}{1} = \frac{y+3}{2} = \frac{z-6}{-2}$ are

A) 4,5,7 B) 4, -5, 7 C) 4, -5,-7 D) -4, 5, 8

4) The length of the perpendicular from (1, 6,3) to the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$

A) 3 B) $\sqrt{11}$ C) $\sqrt{13}$ D) 5

5) The shortest distance between the lines $\bar{r}=\left(\hat{i}+2\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}-\hat{k}\right)$ and $\bar{r}=\left(2\hat{i}-\hat{j}-\hat{k}\right)+\mu\left(2\hat{i}+\hat{j}+2\hat{k}\right)$ is

A) $\frac{1}{\sqrt{3}}$ B) $\frac{1}{\sqrt{2}}$ C) $\frac{3}{\sqrt{2}}$ D) $\frac{\sqrt{3}}{2}$

6) The lines $\frac{x-2}{1}=\frac{y-3}{1}=\frac{z-4}{-k}$ and $\frac{x-1}{k}=\frac{y-4}{2}=\frac{z-5}{1}$ are coplanar if

A) k = 1 or -1 B) k = 0 or -3 C) k = $\pm$ 3 D) k = 0 or -1

7) The lines $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ *and* $\frac{x-1}{-2}=\frac{y-2}{-4}=\frac{z-3}{6}$ and are

A) perpendicular B) inrersecting C) skew D) coincident

8) Equation of X-axis is

A) x = y = z B) y = z

C) y = 0, z = 0 D) x = 0, y = 0

9) The angle between the lines 2x = 3y = –z and 6x = –y = –4z is

A) 45° B) 30° C) 0° D) 90°

10) The direction ratios of the line 3x + 1 = 6y–2 = 1–z are

A) 2,1,6 B) 2,1,–6 C) 2,–1,6 D) –2,1,6

11) The perpendicular distance of the plane 2x + 3y - z = *k* from the origin is $\sqrt{14}$ units, the value of *k* is

A) 14 B) 196 C) $2\sqrt{14}$ D) $\frac{\sqrt{14}}{2}$

12) The angle between the planes and $\bar{r}.\left(\bar{i}-2\bar{j}+3\bar{k}\right)+4=0$ and $\bar{r}\cdot\left(2\bar{i}+\bar{j}-3\bar{k}\right)+7=0$ is

A) $\frac{\pi}{2}$ B) $\frac{\pi}{3}$ C) $\cos^{-1}\left(\frac{3}{4}\right)$ D) $\cos^{1}\left(\frac{9}{14}\right)$

13) If the planes $\bar{r}\cdot\left(2\bar{i}-\lambda\bar{j}+\bar{k}\right)=3$ and $\bar{r}\cdot\left(4\bar{i}-\bar{j}+\mu\bar{k}\right)=5$ are parallel, then the values of λ and μ are respectively.

A) $\frac{1}{2},-2$ B) $-\frac{1}{2},2$ C) $-\frac{1}{2},-2$ D) $\frac{1}{2},2$

14) The equation of the plane passing through (2, -1, 3) and making equal intercepts on the coordinate axes is

A) $x + y + z = 1$ B) $x + y + z = 2$ C) $x + y + z = 3$ D) $x + y + z = 4$

15) Measure of angle between the planes $5x-2y+3z-7=0$ and $15x-6y+9z+5=0$ is

A) 0º B) 30º

C) 45º D) 90º

16) The direction cosines of the normal to the plane 2x-y+2z=3 are

A) $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$ B) $\frac{-2}{3}, \frac{1}{3}, \frac{-2}{3}$

C) $\frac{2}{3}, \frac{1}{3}, \frac{2}{3}$ D) $\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}$

17) The equation of the plane passing through the points (1, -1, 1), (3, 2, 4) and parallel to Y-axis is :

A) 3x+2z –1= 0 B) 3x –2z=1 C) 3x+2z+1=0 D) 3x+2z=2

18) The equation of the plane in which the line $\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}$ and $\frac{x-8}{7}=\frac{y-4}{1}=\frac{z+5}{3}$ lie, is

A) 17x - 47y - 24z + 172 = 0

B) 17x + 47y - 24z + 172 = 0

C) 17x + 47y + 24z +172 = 0

D) 17x - 47y + 24z + 172 = 0

19) If the line $\frac{x+1}{2}=\frac{y-m}{3}=\frac{z-4}{6}$ lies in the plane 3x-14y+6z+49=0, then the value of *m* is:

A) 5 B) 3 C) 2 D) -5

20) The foot of perpendicular drawn from the point (0,0,0) to the plane is (4, -2, -5) then the equation of the plane is

A) 4x + y + 5z = 14 B) 4x - 2y - 5z = 45

C) x - 2y - 5z = 10 D) 4x + y + 6z = 1l

II. Solve the following :

(1) Find the vector equation of the plane which is at a distance of 5 unit from the origin and which is normal to the vector $2\hat{i}+\hat{j}+2\hat{k}$

(2) Find the perpendicular distance of the origin from the plane 6x+2y+3z-7=0

(3) Find the coordinates of the foot of the perpendicular drawn from the origin to the plane 2x+3y+6z=49.

(4) Reduce the equation $\bar{r}.\left(6\hat{i}+8\hat{j}+24\hat{k}\right)=13$ to normal form and hence find

(i) the length of the perpendicular from the origin to the plane

(ii) direction cosines of the normal.

(5) Find the vector equation of the plane passing through the points A(l, -2,l), B(2, -1, -3) and C(0,l,5).

(6) Find the Cartesian equation of the plane passing through A(1, - 2, 3) and the direction ratios of whose normal are 0, 2, 0.

(7) Find the Cartesian equation of the plane passing through A(7, 8, 6) and parallel to the plane $\bar{r}.\left(6\hat{i}+8\hat{j}+7\hat{k}\right)=0$

(8) The foot of the perpendicular drawn from the origin to a plane is M(l, 2,0). Find the vector equation of the plane.

(9) A plane makes non zero intercepts a, b, c on the co-ordinates axes. Show that the vector equation of the plane is $\bar{r}.\left(bc\hat{i}+ca\hat{j}+ab\hat{k}\right)=abc$

(10) Find the vector equation of the plane passing through the point A(-2, 3,5) and parallel to vectors $4\hat{i}+3\hat{k}$ and $\hat{i}+\hat{j}$

(11) Find the Cartesian equation of the plane $\bar{r}=\lambda\left(\hat{i}+\hat{j}-\hat{k}\right)+\mu\left(\hat{i}+2\hat{j}+3\hat{k}\right)$

(12) Find the vector equations of planes which pass through A(l, 2,3), B(3, 2, 1) and make equal intercepts on the co-ordinates axes.

(13) Find the vector equation of the plane which makes equal non-zero intercepts on the co-ordinates axes and passes through (1,1,1).

(14) Find the angle between planes $\bar{r}\cdot\left(-2\hat{i}+\hat{j}+2\hat{k}\right)=17$ and $\bar{r}\cdot\left(2\hat{i}+2\hat{j}+\hat{k}\right)=71$.

(15) Find the acute angle between the line $\bar{r}=\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$ and the plane $\bar{r}\cdot\left(2\hat{i}-\hat{j}+\hat{k}\right)=23$

(16) Show that lines $\bar{r}=\left(\hat{i}+4\hat{j}\right)+\lambda\left(\hat{i}+2\hat{j}+3\hat{k}\right)$ and $.\bar{r}=\left(3\hat{j}-\hat{k}\right)+\mu\left(2\hat{i}+3\hat{j}+4\hat{k}\right)$

are coplanar. Find the equation of the plane determined by them.

(17) Find the distance of the point $3\hat{i}+3\hat{j}+\hat{k}$ from the plane $\bar{r}\cdot\left(2\hat{i}+3\hat{j}+6\hat{k}\right)=21$

(18) Find the distance of the point (13, 13,-13) from the plane 3x + 4y - 12z = 0.

(19) Find the vector equation of the plane passing through the origln and containing the line $\bar{r}=\left(\hat{i}+4\hat{j}+\hat{k}\right)+\lambda\left(\hat{i}+2\hat{j}+\hat{k}\right)$

(20) Find the vector equation of the plane which bisects the segment joining A(2,3, 6) and B(4,3, -2) at right angle.

(21) Show that lines $x=y$, $z=0$ and $x+y=0$, $z=0$ intersect each other. Find the vector equation of the plane determined by them.

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