2 Matrices

Let's Study

- Types of Matrices
- Algebra of Matrices
- Properties of Matrices
- Elementary Transformation
- Inverse of Matrix
- Application of Matrices

Let's Recall

1) Determinants

2) Properties of determinants

2.1 Introduction:

The theory of matrices was developed by the mathematician Arthur Cayley. Matrices are useful in expressing numerical information in a compact form. They are effectively used in expressing different operations. Hence they are essential in economics, finance, business and statistics.

Definition: A rectangular arrangement of *mn* numbers in m rows and n columns, enclosed in [ ] or () is called a matrix of order *m* by *n*. A matrix by itself does not have a value or any special meaning.

The order of a matrix is denoted by $m \times n$, read as *m* by *n*.

Each member of a matrix is called an element of the matrix.

Matrices are generally denoted by capital letters like A, B, C, …. and their elements are denoted by small letters like a_{ij}, b_{ij}, c_{ij}, ….. etc. where a_{ij} is the element in i^{th} row and j^{th} column of the matrix A.

For example: i) $A = \begin{bmatrix} 2 & -3 & 9 \\ 1 & 0 & -7 \\ 4 & -2 & 1 \end{bmatrix}$ here $a_{32} = -2$

A is a matrix having 3 rows and 3 columns. The order of A is 3×3. There are 9 elements in the matrix A.

ii) $B = \begin{bmatrix} -1 & -5 \\ 2 & 6 \\ 0 & 9 \end{bmatrix}$

B is a matrix having 3 rows and 2 columns. The order of B is 3×2. There are 6 elements in the matrix B.

In general, a matrix of order m × n is represented by

$$A = \left[a_{ij}\right]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3j} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Here a_{ij} = The element in i^{th} row and j^{th} column.

Ex. In matrix $A = \begin{bmatrix} 2 & -3 & 9 \\ 1 & 0 & -7 \\ 4 & -2 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$a_{11} = 2$, $a_{12} = -3$, $a_{13} = 9$, $a_{21} = 1$, $a_{22} = 0$, $a_{23} = -7$, $a_{31} = 4$, $a_{32} = -2$, $a_{33} = 1$

2.2 Types of Matrices:

1) Row Matrix : A matrix that has only one row is called a row matrix. It is of order $1 \times n$, where $n \geq 1$.

For example: i) $\begin{bmatrix} -1 & 2 \end{bmatrix}_{1\times 2}$ ii) $\begin{bmatrix} 9 & 0 & -3 \end{bmatrix}_{1\times 3}$

2) Column Matrix: A matrix that has only one column is called a column matrix. It is of order $m \times 1$, where $m \geq 1$.

For example: i) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{2\times 1}$ ii) $\begin{bmatrix} 5 \\ -9 \\ -3 \end{bmatrix}_{3\times 1}$

Note: A real number can be treated as a matrix of order 1×1. It is called a singleton matrix.

3) Zero or Null Matrix : A matrix in which every element is zero is called a zero or null matrix. It is denoted by O.

For example: $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{3\times 2}$

4) Square Matrix : A matrix with the number of rows equal to the number of columns is called a square matrix. If a square matrix is of order $n\times n$ then n is called the order of the square matrix.

For example: $A = \begin{bmatrix} 5 & -3 & i \\ 1 & 0 & -7 \\ 2i & -8 & 9 \end{bmatrix}_{3\times 3}$

Let's Note: Let $A = [a_{ij}]_{n\times n}$ be a square matrix of order n, Then

(i) the elements $a_{11}, a_{22}, a_{33}, \ldots, a_{ii}, \ldots, a_{nn}$ are called the diagonal elements of the matrix A. Note that the diagonal elements are defined only for a square matrix;

(ii) elements a_{ij}, where $i \neq j$ are called non diagonal elements of the matrix A;

(iii) elements a_{ij}, where $i < j$, are called elements above the diagonal;

(iv) elements a_{ij}, where $i > j$, are called elements below the diagonal.

Statements iii) and iv) are to be verified by looking at matrices of different orders.

5) Diagonal Matrix: A square matrix in which every non-diagonal element is zero, is called a diagonal matrix.

For example: i) $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times3}$ ii) $B = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}_{2\times2}$

Note : If a_{11}, a_{22}, a_{33} are diagonal elements of a diagonal matrix A of order 3, then we write the matrix A as $A = \text{Diag}[a_{11}, a_{22}, a_{33}]$.

6) Scalar Matrix: A diagonal matrix in which all the diagonal elements are same is called a scalar matrix.

For example: $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3\times3}$

7) Unit or Identity Matrix : A scalar matrix in which all the diagonal elements are 1(unity), is called a Unit Matrix or an Identity Matrix. Identity Matrix of order n is denoted by I_n.

For example: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note :

1. Every Identity Matrix is a scalar matrix but every scalar matrix need not be Identity Matrix. However a scalar matrix is a scalar multiple of the identity matrix.

2. Every scalar matrix is diagonal matrix but every diagonal matrix need not be scalar matrix.

8) Upper Triangular Matrix: A square matrix in which every element below the diagonal is zero is called an upper triangular matrix.

Matrix $A = [a_{ij}]_{n\times n}$ is upper triangular if $a_{ij} = 0$ for all $i > j$.

For example: i) $A = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 9 \end{bmatrix}_{3\times3}$

9) Lower Triangular Matrix: A square matrix in which every element above the diagonal is zero, is called a lower triangular matrix.

Matrix $A = [a_{ij}]_{n\times n}$ is lower triangular if $a_{ij} = 0$ for all $i < j$.

For example: $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 0 & 0 \\ -5 & 1 & 9 \end{bmatrix}_{3\times3}$

10) Triangular Martix: A square matrix is called a triangular matrix if it is an upper triangular or a lower triangular matrix.

Note: The diagonal, scalar, unit and square null matrices are also triangular matrices.

(11) Determinant of a Matrix: Determinant of a matrix is defined only for a square matrix.

If A is a square matrix, then the same arrangement of the elements of A also gives us a determinant. It is denoted by |A| or det(A).

If $A = [a_{ij}]_{n\times n}$ then |A| is of order n.

For example: i) If $A = \begin{bmatrix} 1 & 3 \\ -5 & 4 \end{bmatrix}_{2\times 2}$ then $|A| = \begin{vmatrix} 1 & 3 \\ -5 & 4 \end{vmatrix}$

ii) If $B = \begin{bmatrix} 2 & -1 & 3 \\ -4 & 1 & 5 \\ 7 & -5 & 0 \end{bmatrix}_{3\times 3}$ then $|B| = \begin{vmatrix} 2 & -1 & 3 \\ -4 & 1 & 5 \\ 7 & -5 & 0 \end{vmatrix}$

12) Singular Matrix : A square matrix A is said to be a singular matrix if

|A| = det(A) = 0. Otherwise, it is said to be a non-singular matrix.

For example: i) If $B = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}_{3\times 3}$ then $|B| = \begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{vmatrix}$

$|B| = 2(24 - 25) - 3(18 - 20) + 4(15 - 16)$

$= -2 + 6 - 4$

$= 0$

$|B| = 0$

Therefore B is a singular matrix.

ii) $A = \begin{bmatrix} 2 & -1 & 3 \\ -7 & 4 & 5 \\ -2 & 1 & 6 \end{bmatrix}_{3\times 3}$ Then $|A| = \begin{vmatrix} 2 & -1 & 3 \\ -7 & 4 & 5 \\ -2 & 1 & 6 \end{vmatrix}$

$|A| = 2(24 - 5) - (-1)(-42 + 10) + 3(-7 + 8)$

$= 38 - 32 + 3$

$= 9$

$|A| = 9$

As $|A| \neq 0$, A is a non-singular matrix.

SOLVED EXAMPLES

Ex. 1) Show that the matrix $\begin{bmatrix} x+y & y+z & z+x \\ 1 & 1 & 1 \\ z & x & y \end{bmatrix}$ is a singular matrix.

Solution : Let A = $\begin{bmatrix} x+y & y+z & z+x \\ 1 & 1 & 1 \\ z & x & y \end{bmatrix}$

$\therefore$ |A| = $\begin{vmatrix} x+y & y+z & z+x \\ 1 & 1 & 1 \\ z & x & y \end{vmatrix}$

Now $|A| = (x + y)(y - x) - (y + z)(y - z) + (z + x)(x - z)$

$= y^2 - x^2 - y^2 + z^2 + x^2 - z^2$

$= 0$

$\therefore$ A is a singular matrix.

EXERCISE 2.1

(1) Construct a matrix $A = [a_{ij}]_{3\times 2}$ whose element a_{ij} is given by

(i) $a_{ij} = \dfrac{(i-j)^2}{5-i}$ (ii) $a_{ij} = i - 3j$ (iii) $a_{ij} = \dfrac{(i+j)^3}{5}$

(2) Classify each of the following matrices as a row, a column, a square, a diagonal, a scalar, a unit, an upper traingular, a lower triangular matrix.

(i) $\begin{bmatrix} 3 & -2 & 4 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 5 \\ 4 \\ -3 \end{bmatrix}$ (iii) $\begin{bmatrix} 9 & \sqrt{2} & -3 \end{bmatrix}$ (iv) $\begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$

(v) $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -1 & 0 \\ -7 & 3 & 1 \end{bmatrix}$ (vi) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ (vii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(3) Which of the following matrices are singular or non singular?

(i) $\begin{bmatrix} a & b & c \\ p & q & r \\ 2a-p & 2b-q & 2c-r \end{bmatrix}$ (ii) $\begin{bmatrix} 5 & 0 & 5 \\ 1 & 99 & 100 \\ 6 & 99 & 105 \end{bmatrix}$

(iii) $\begin{bmatrix} 3 & 5 & 7 \\ -2 & 1 & 4 \\ 3 & 2 & 5 \end{bmatrix}$ (iv) $\begin{bmatrix} 7 & 5 \\ -4 & 7 \end{bmatrix}$

(4) Find K if the following matrices are singular.

(i) $\begin{bmatrix} 7 & 3 \\ -2 & K \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 3 & 1 \\ 7 & K & 1 \\ 10 & 9 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} K-1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & -2 & 4 \end{bmatrix}$

2.3 Algebra of Matrices:

(1) Transpose of a Matrix (2) Determinant of a Matrix (3) Equality of Matrices (4) Addition of Matrices (5) Scalar Multiplication of a Matrix and (6) Multiplication of two martices.

(1) Transpose of a Matrix: A is a matrix of order m × n. The matrix obtained by interchanging rows and columns of matrix A is called the transpose of the matrix A. It is denoted by A' or A^T. The order of A^T is n × m.

For example: i) If A = $\begin{bmatrix} -1 & 5 \\ 3 & -2 \\ 4 & 7 \end{bmatrix}_{3\times2}$ then A^T = $\begin{bmatrix} -1 & 3 & 4 \\ 5 & -2 & 7 \end{bmatrix}_{2\times3}$

ii) If B = $\begin{bmatrix} 1 & 0 & -2 \\ 8 & -1 & 2 \\ 4 & 3 & 5 \end{bmatrix}_{3\times3}$ then B^T = $\begin{bmatrix} 1 & 8 & 4 \\ 0 & -1 & 3 \\ -2 & 2 & 5 \end{bmatrix}_{3\times3}$

i) Symmetric Matrix: A square matrix A = $[a_{ij}]_{n\times n}$ in which $a_{ij} = a_{ji}$, for all i and j, is called a symmetric matrix.

For example: A = $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}_{3\times3}$

Let's Note: The diagonal matrices are symmetric. Null square matrix is symmetric.

ii) Skew-Symmetric Matrix: A square matrix A = $[a_{ij}]_{n\times n}$ in which $a_{ij} = -a_{ji}$, for all i and j, is called a skew symmetric matrix.

Here for i = j, $a_{ij} = -a_{ji}$, $\therefore$ $2a_{ii} = 0$ $\therefore$ $a_{ii} = 0$ for all i = 1, 2, 3, ... n.

In a skew symmetric matrix, each diagonal element is zero.

For example: B = $\begin{bmatrix} 0 & 4 & -7 \\ -4 & 0 & 5 \\ 7 & -5 & 0 \end{bmatrix}_{3\times3}$

Let's Note:

1) $(A^T)^T = A$

2) If A is a symmetric matrix then $A = A^T$

3) If B is a skew symmetric matrix then $B = -B^T$

4) A null square matrix is also skew symmetric.

5) $|A| = |A^T|$

(3) Equality of Two matrices: Two matrices A and B are said to be equal if (i) order of A = order of B and (ii) corresponding elements of A and B are same. That is $a_{ij} = b_{ij}$ for all i, j. Symbolically, this is written as A=B.

For example: i) If $A = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 0 & 0 \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} 2 & -1 \\ 4 & 0 \\ 1 & 0 \end{bmatrix}_{3\times2}$

Here $B^T = \begin{bmatrix} 2 & 4 & 1 \\ -1 & 0 & 0 \end{bmatrix}_{2\times3}$ In matrices A and B, $A \neq B$, but $A = B^T$.

For example: ii) If $\begin{bmatrix} 2a-b & 4 \\ -7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -7 & a+3b \end{bmatrix}$, then find a and b.

Using definition of equality of matrices, we have

$2a - b = 1$ ...... (1) and

$a + 3b = 2$ ...... (2)

Solving equation (1) and (2), we get $a = \dfrac{5}{7}$ and $b = \dfrac{3}{7}$

Let's note: If A = B, then B = A

(4) Addition of Two Matrices: A and B are two matrices of same order. Their addition, denoted by A + B, is a matrix obtained by adding the corresponding elements of A and B. Note that orders of A, B and A + B are same.

Thus if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ then $A + B = [a_{ij} + b_{ij}]_{m\times n}$

For example: i) If $A = \begin{bmatrix} 2 & 3 & 1 \\ -1 & -2 & 0 \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} -4 & 3 & 1 \\ 5 & 7 & -8 \end{bmatrix}_{2\times3}$ find A + B.

Solution: Since A and B have same order, A + B is defined and

$$A + B = \begin{bmatrix} 2+(-4) & 3+3 & 1+1 \\ -1+5 & -2+7 & 0+(-8) \end{bmatrix}_{2\times3} = \begin{bmatrix} -2 & 6 & 2 \\ 4 & 5 & -8 \end{bmatrix}_{2\times3}$$

Let's Note: If A and B are two matrices of same order then subtraction of two matrices is defined as, $A - B = A + (-B)$, where $-B$ is the negative of matrix B.

For example: i) If $A = \begin{bmatrix} -1 & 4 \\ 3 & -2 \\ 0 & 5 \end{bmatrix}_{3\times2}$ and $B = \begin{bmatrix} -1 & 5 \\ 2 & -6 \\ 4 & 9 \end{bmatrix}_{3\times2}$, Find A − B.

Solution: Since A and B have same order, A − B is defined and

$$A - B = A + (-B) = \begin{bmatrix} -1 & 4 \\ 3 & -2 \\ 0 & 5 \end{bmatrix}_{3\times 2} - \begin{bmatrix} -1 & 5 \\ 2 & -6 \\ 4 & 9 \end{bmatrix}_{3\times 2} = \begin{bmatrix} -1+1 & 4+(-5) \\ 3+(-2) & -2+6 \\ 0+(-4) & 5+(-9) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 4 \\ -4 & -4 \end{bmatrix}$$

(5) Scalar Multiplication of a Matrix: If A is any matrix and k is a scalar, then the matrix obtained by multiplying each element of A by the scalar k is called the scalar multiple of the matrix A and is denoted by kA.

Thus if $A = [a_{ij}]_{m\times n}$ and k is any scalar then $kA = [ka_{ij}]_{m\times n}$.

Here the orders of matrices A and kA are same.

For example: i) If $A = \begin{bmatrix} -1 & 5 \\ 3 & -2 \\ 4 & 7 \end{bmatrix}_{3\times 2}$ and $k = \dfrac{3}{2}$, then kA.

$$\frac{3}{2}A = \frac{3}{2}\begin{bmatrix} -1 & 5 \\ 3 & -2 \\ 4 & 7 \end{bmatrix}_{3\times 2} = \begin{bmatrix} -\frac{3}{2} & \frac{15}{2} \\ \frac{9}{2} & -3 \\ 6 & \frac{21}{2} \end{bmatrix}_{3\times 2}$$

Properties of addition and scalar multiplication: If A, B, C are three matrices conformable for addition and α, β are scalars, then

(i) A + B = B + A, that is, the matrix addition is commutative.

(ii) (A +B) + C = A + (B + C), that is, the matrix addition is associative.

(iii) For martix A, we have A + O = O + A = A, that is, zero matrix is conformable for addition and it is the identity for matrix addition.

(iv) For a matrix A, we have A + (−A) = (−A) + A = O, where O is zero matrix conformable with the matrix A for addition.

(v) $\alpha(A \pm B) = \alpha A \pm \alpha B$

(vi) $(\alpha \pm \beta)A = \alpha A \pm \beta B$

(vii) $\alpha(\beta \cdot A) = (\alpha \cdot \beta) \cdot A$

(viii) OA = O

SOLVED EXAMPLES

Ex. 1) If $A = \begin{bmatrix} 5 & -3 \\ 1 & 0 \\ -4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 7 \\ -3 & 1 \\ 2 & -2 \end{bmatrix}$, find 2A − 3B.

Solution: Let $2A - 3B = 2\begin{bmatrix} 5 & -3 \\ 1 & 0 \\ -4 & -2 \end{bmatrix} - 3\begin{bmatrix} 2 & 7 \\ -3 & 1 \\ 2 & -2 \end{bmatrix}$

$$= \begin{bmatrix} 10 & -6 \\ 2 & 0 \\ -8 & -4 \end{bmatrix} + \begin{bmatrix} -6 & -21 \\ 9 & -3 \\ -6 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 10-6 & -6-21 \\ 2+9 & 0-3 \\ -8-6 & -4+6 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -27 \\ 11 & -3 \\ -14 & 2 \end{bmatrix}$$

Ex. 2) If A = diag(2, −5, 9), B = diag(−3, 7, −14) and C = diag(1, 0, 3), find B−A−C.

Solution: B − A − C = B − (A + C)

Now, A + C = diag(2, −5, 9) + diag(1, 0, 3) = diag(3, −5, 12)

B − A − C = B − (A + C) = diag(−3, 7, −14) − diag(3, −5, 12)

= diag(−6, 12, −26)

$$= \begin{bmatrix} -6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & -26 \end{bmatrix}$$

Ex. 3) If $A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 7 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 & 6 \\ 0 & 2 & -5 \end{bmatrix}$, find the matrix X such that 3A − 2B + 4X = 5C.

Solution: Since 3A − 2B + 4X = 5C

$\therefore$ 4X = 5C − 3A + 2B

$$\therefore \quad 4X = 5\begin{bmatrix} 1 & -1 & 6 \\ 0 & 2 & -5 \end{bmatrix} - 3\begin{bmatrix} 2 & 3 & -1 \\ 4 & 7 & 5 \end{bmatrix} + 2\begin{bmatrix} 1 & 3 & 2 \\ 4 & 6 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -5 & 30 \\ 0 & 10 & -25 \end{bmatrix} + \begin{bmatrix} -6 & -9 & 3 \\ -12 & -21 & -15 \end{bmatrix} + \begin{bmatrix} 2 & 6 & 4 \\ 8 & 12 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 5-6+2 & -5-9+6 & 30+3+4 \\ 0-12+8 & 10-21+12 & -25-15-2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -8 & 37 \\ -4 & 1 & -42 \end{bmatrix}$$

$$\therefore \quad X = \frac{1}{4}\begin{bmatrix} 1 & -8 & 37 \\ -4 & 1 & -42 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{1}{4} & -2 & \frac{37}{4} \\ -1 & \frac{1}{4} & -\frac{21}{2} \end{bmatrix}$$

Ex. 4) If $\begin{bmatrix} 2x+1 & -1 \\ 3 & 4y \end{bmatrix} + \begin{bmatrix} -1 & 6 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 6 & 12 \end{bmatrix}$, find x and y.

Solution: Given $\begin{bmatrix} 2x+1 & -1 \\ 3 & 4y \end{bmatrix} + \begin{bmatrix} -1 & 6 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 6 & 12 \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} 2x & 5 \\ 6 & 4y \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 6 & 12 \end{bmatrix}$$

$\therefore$ Using definition of equality of matrices, we have $2x = 4$, $4y = 12$

$\therefore$ $x = 2$, $y = 3$

Ex. 5) Find a, b, c if the matrix A = $\begin{bmatrix} 2 & a & 3 \\ -7 & 4 & 5 \\ c & b & 6 \end{bmatrix}$ is a symmetric matrix.

Solution: Given that A = $\begin{bmatrix} 2 & a & 3 \\ -7 & 4 & 5 \\ c & b & 6 \end{bmatrix}$ is a symmetric matrix.

$\therefore$ $a_{ij} = a_{ji}$ for all i and j

$\therefore$ $a = -7$, $b = 5$, $c = 3$

Ex. 6) If $A = \begin{bmatrix} -1 & -5 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}_{3\times2}$ Find $(A^T)^T$.

Solution: Let $A = \begin{bmatrix} -1 & -5 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}_{3\times2}$

$\therefore \quad A^T = \begin{bmatrix} -1 & 2 & 3 \\ -5 & 0 & -4 \end{bmatrix}_{2\times3}$

Now $(A^T)^T = \begin{bmatrix} -1 & -5 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}_{3\times2}$

$= A$

Ex. 7) If $X + Y = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -3 & -2 \end{bmatrix}$ and $X - 2Y = \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 4 & -2 \end{bmatrix}$ then find X, Y.

Solution: Let $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -3 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 4 & -2 \end{bmatrix}$

$X + Y = A$ .... (1), $X - 2Y = B$ .... (2), Solving (1) and (2) for X and Y

Consider (1) − (2), $3Y = A - B$,

$\therefore \quad Y = \frac{1}{3}(A - B)$

$$\therefore \quad Y = \frac{1}{3}\left\{\begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -3 & -2 \end{bmatrix} - \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 4 & -2 \end{bmatrix}\right\}$$

$$= \frac{1}{3}\begin{bmatrix} 4 & -2 \\ -2 & 4 \\ -7 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} \\ -\frac{7}{3} & 0 \end{bmatrix}$$

From (1) X + Y = A,

$\therefore \quad X = A - Y,$

$$\therefore \quad X = \begin{bmatrix} 2 & -1 \\ 1 & 3 \\ -3 & -2 \end{bmatrix} - \begin{bmatrix} \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} \\ -\frac{7}{3} & 0 \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{5}{3} & \frac{5}{3} \\ -\frac{2}{3} & -2 \end{bmatrix}$$

EXERCISE 2.2

(1) If $A = \begin{bmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 2 & 2 \\ 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 3 \\ -1 & 4 \\ -2 & 1 \end{bmatrix}$

Show that (i) A + B = B + A

(ii) (A + B) + C = A + (B + C)

(2) If $A = \begin{bmatrix} 1 & -2 \\ 5 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -3 \\ 4 & -7 \end{bmatrix}$, then find the matrix A − 2B + 6I, where I is the unit matrix of order 2.

(3) If $A = \begin{bmatrix} 1 & 2 & -3 \\ -3 & 7 & -8 \\ 0 & -6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 9 & -1 & 2 \\ -4 & 2 & 5 \\ 4 & 0 & -3 \end{bmatrix}$ then find the matrix C such that A + B + C is a zero matrix.

(4) If $A = \begin{bmatrix} 1 & -2 \\ 3 & -5 \\ -6 & 0 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 \\ 4 & 2 \\ 1 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ -1 & -4 \\ -3 & 6 \end{bmatrix}$, find the matrix X such that

3A − 4B + 5X = C.

(5) If $A = \begin{bmatrix} 5 & 1 & -4 \\ 3 & 2 & 0 \end{bmatrix}$, find $(A^T)^T$.

(6) If $A = \begin{bmatrix} 7 & 3 & 1 \\ -2 & -4 & 1 \\ 5 & 9 & 1 \end{bmatrix}$, find $(A^T)^T$.

(7) Find a, b, c if $\begin{bmatrix} 1 & \frac{3}{5} & a \\ b & -5 & -7 \\ -4 & c & 0 \end{bmatrix}$ is a symmetric matrix.

(8) Find x, y, z if $\begin{bmatrix} 0 & -5i & x \\ y & 0 & z \\ \frac{3}{2} & -\sqrt{2} & 0 \end{bmatrix}$ is a skew symmetric matrix.

(9) For each of the following matrices, find its transpose and state whether it is symmetric, skew-symmetric or neither.

(i) $\begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 5 & 1 \\ -5 & 4 & 6 \\ -1 & -6 & 3 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 1+2i & i-2 \\ -1-2i & 0 & -7 \\ 2-i & 7 & 0 \end{bmatrix}$

(10) Construct the matrix $A = [a_{ij}]_{3\times3}$ where $a_{ij} = i - j$. State whether A is symmetric or skew symmetric.

(11) Solve the following equations for X and Y, if $3X - Y = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $X - 3Y = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix}$

(12) Find matrices A and B, if $2A - B = \begin{bmatrix} 6 & -6 & 0 \\ -4 & 2 & 1 \end{bmatrix}$ and $A - 2B = \begin{bmatrix} 3 & 2 & 8 \\ -2 & 1 & -7 \end{bmatrix}$

(13) Find x and y, if $\begin{bmatrix} 2x+y & -1 & 1 \\ 3 & 4y & 4 \end{bmatrix} + \begin{bmatrix} -1 & 6 & 4 \\ 3 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 5 & 5 \\ 6 & 18 & 7 \end{bmatrix}$

(14) If $\begin{bmatrix} 2a+b & 3a-b \\ c+2d & 2c-d \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, find a, b, c and d.

(15) There are two book shops own by Suresh and Ganesh. Their sales (in Rupees) for books in three subject - Physics, Chemistry and Mathematics for two months, July and August 2017 are given by two matrices A and B.

July sales (in Rupees), Physics Chemistry Mathematics

$$A = \begin{bmatrix} 5600 & 6750 & 8500 \\ 6650 & 7055 & 8905 \end{bmatrix}$$ First Row Suresh / Second Row Ganesh

August sales (in Rupees), Physics Chemistry Mathematics

$$B = \begin{bmatrix} 6650 & 7055 & 8905 \\ 7000 & 7500 & 10200 \end{bmatrix}$$ First Row Suresh / Second Row Ganesh

then, (i) Find the increase in sales in Rupees from July to August 2017.

(ii) If both book shops get 10% profit in the month of August 2017, find the profit for each book seller in each subject in that month.

(6) Multiplication of Two Matrices:

Two Matrices A and B are said to be conformable for multiplication if the number of columns in A is equal to the number of rows in B. For example, A is of order $m \times n$ and B is of order $n \times p$.

In this case the elements of the product AB form a matrix defined as follows:

$$A_{m\times n} \times B_{n\times p} = C_{m\times p}, \qquad \text{where } C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$\text{If } A = [a_{jk}]_{m\times n} = \begin{array}{cc} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2k} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3k} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} & \cdots & a_{mn} \end{bmatrix} & \rightarrow \ i^{th} \text{ row} \end{array}$$

$$B = [b_{kj}]_{n\times p} = \begin{array}{c} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2p} \\ b_{31} & b_{32} & \cdots & b_{3j} & \cdots & b_{3p} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{nj} & \cdots & b_{np} \end{bmatrix} \\ \downarrow \\ j^{th} \text{ column} \end{array} \qquad \text{then}$$

then $\quad c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots\dots\dots + a_{in}b_{nj}$

SOLVED EXAMPLES

Ex.1: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \end{bmatrix}_{1\times 3}$ and $B = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix}_{3\times 1}$ Find AB.

Solution: Since number of columns of A = number of rows of B = 3

Therefore product AB is defined and its order is 1.

$AB = [a_{11} \times b_{11} + a_{12} \times b_{21} + a_{13} \times b_{31}]$

Ex.2: Let $A = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}_{1\times 3}$ and $B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}_{3\times 1}$, find AB. Does BA exist? If yes, find it.

Solution: Product AB is defined and order of AB is 1.

$$\therefore \quad AB = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\times 3 + 3\times 2 + 2\times 1 \end{bmatrix} = \begin{bmatrix} 11 \end{bmatrix}_{1\times 1}$$

Again, number of column of B = number of rows of A = 1.

$\therefore$ product BA is also defined and the order of BA is 3.

$$BA = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}_{3\times 1} \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}_{1\times 3}$$

$$= \begin{bmatrix} 3\times 1 & 3\times 3 & 3\times 2 \\ 2\times 1 & 2\times 3 & 2\times 2 \\ 1\times 1 & 1\times 3 & 1\times 2 \end{bmatrix}_{3\times 3} = \begin{bmatrix} 3 & 9 & 6 \\ 2 & 6 & 4 \\ 1 & 3 & 2 \end{bmatrix}_{3\times 3}$$

Remark: Here AB and BA both are defined but they are different matrices.

Ex.3: $A = \begin{bmatrix} -1 & -2 \\ -3 & 2 \\ 1 & 0 \end{bmatrix}_{3\times 2}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}_{2\times 2}$ Find AB and BA if they exist.

Solution: Here A is order of 3 × 2 and B is order of 2 × 2. By conformability of product, AB is defined but BA is not defined.

$$\therefore \quad AB = \begin{bmatrix} -1 & -2 \\ -3 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -1+2 & -2+4 \\ -3-2 & -6-4 \\ 1+0 & 2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ -5 & -10 \\ 1 & 2 \end{bmatrix}$$

Ex.4: Let $A = \begin{bmatrix} 3 & 2 & -1 \\ -2 & 5 & 4 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} 3 & -3 \\ -4 & 2 \end{bmatrix}_{2\times 2}$ Find AB and BA whichever exists.

Solution: Since number of columns of A $\neq$ number of rows of B

$\therefore$ Product of AB is not defined. But number of columns of B = number of rows of A=2, the product BA is exists,

$$\therefore \quad BA = \begin{bmatrix} 3 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ -2 & 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 9+6 & 6-15 & -3-12 \\ -12-4 & -8+10 & 4+8 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -9 & -15 \\ -16 & 2 & 12 \end{bmatrix}$$

Ex.5: Let $A = \begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix}$, Find AB and BA which ever exist.

Solution: Since A and B are two matrices of same order 2×2.

$\therefore$ Both the products AB and BA exist and both the products are of same order 2×2.

$$AB = \begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -4-12 & 12+6 \\ -5+8 & 15-4 \end{bmatrix}$$

$$= \begin{bmatrix} -16 & 18 \\ 3 & 11 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 4 & -3 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -4+15 & 3+6 \\ 16-10 & -12-4 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 9 \\ 6 & -16 \end{bmatrix}$$

Here again $AB \neq BA$

Remark: 1) If AB exists, BA may or may not exist.

2) If BA exists, AB may or may not exist.

3) If AB and BA both exist they may or may not be equal.

2.4 Properties of Matrix Multiplication:

1) For matrices A and B, matrix multiplication is not commutative, that is, in general $AB \neq BA$.

2) For three matrices A, B, C, matrix multiplication is associative. That is $(AB)C = A(BC)$ if orders of matrices are suitable for multiplication.

For example: Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}$

Then $AB = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & -1-2 & 2+6 \\ 4 & -4-3 & 8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 8 \\ 4 & -7 & 17 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} 1 & -3 & 8 \\ 4 & -7 & 17 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2-9 & 1+3+16 \\ -8-21 & 4+7+34 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 20 \\ -29 & 45 \end{bmatrix} \quad \text{..... (1)}$$

$$\therefore \quad BC = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2-3 & 1+1+4 \\ -3 & 1+6 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 6 \\ -3 & 7 \end{bmatrix}$$

$$\text{Now, } A(BC) = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} -5 & 6 \\ -3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -5-6 & 6+14 \\ -20-9 & 24+21 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 20 \\ -29 & 45 \end{bmatrix} \quad \text{..... (2)}$$

From (1) and (2), (AB)C = A(BC)

3) For three matrices A, B, C, matrix multiplication is distributive over addition.

i) A(B + C) = AB + AC (left distributive law)

ii) (B + C)A = BA + CA (right distributive law)

These laws can be verified by examples.

4) For a given square matrix A, there exists a unit matrix I of the same order as that of A, such that AI = IA = A. I is called Identity matrix for matrix multiplication.

For example: Let $A = \begin{bmatrix} 3 & -2 & -1 \\ 2 & 0 & 4 \\ 1 & 3 & 2 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then AI $= \begin{bmatrix} 3 & -2 & -1 \\ 2 & 0 & 4 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 3+0+0 & 0-2+0 & 0+0-1 \\ 2+0+0 & 0+0+0 & 0+0+4 \\ 1+0+0 & 0+3+0 & 0+0+2 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -2 & -1 \\ 2 & 0 & 4 \\ 1 & 3 & 2 \end{bmatrix}$$

$= IA$

5) For any matrix A, there exists a null matrix O such that a) AO = O and b) OA = O.

6) The products of two non zero matrices can be a zero matrix. That is AB = O but $A \neq O$, $B \neq O$.

For example: Let $A = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix}$,

Here $A \neq O$, $B \neq O$ but $AB = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, that is AB = O

7) Positive integer powers of a square matrix A are obtained by repeated multiplication of A by itself. That is $A^2 = AA$, $A^3 = AAA$, ..., $A^n = AA....n$ times

SOLVED EXAMPLES

Ex.1: If $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}$, show that the matrix AB is non singular,

Solution: let $AB = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & -1 \\ 0 & 2 \end{bmatrix}$

$$= \begin{bmatrix} -2-3+0 & 1+1+4 \\ 0-3+0 & 0+1+6 \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 6 \\ -3 & 7 \end{bmatrix},$$

$\therefore |AB| = \begin{vmatrix} -5 & 6 \\ -3 & 7 \end{vmatrix} = -35 + 18 = -17 \neq 0$

$\therefore$ By definition, matrix AB is non singular.

Ex. 2: If $A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$ find $A^2 - 5A$. What is your conclusion?

Solution : Let $A^2 = A.A$

$$= \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+9+9 & 3+3+9 & 3+9+3 \\ 3+3+9 & 9+1+9 & 9+3+3 \\ 3+9+3 & 9+3+3 & 9+9+1 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 15 & 15 \\ 15 & 19 & 15 \\ 15 & 15 & 19 \end{bmatrix}$$

$$\therefore\ A^2 - 5A = \begin{bmatrix} 19 & 15 & 15 \\ 15 & 19 & 15 \\ 15 & 15 & 19 \end{bmatrix} - 5 \begin{bmatrix} 1 & 3 & 3 \\ 3 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 15 & 15 \\ 15 & 19 & 15 \\ 15 & 15 & 19 \end{bmatrix} - \begin{bmatrix} 5 & 15 & 15 \\ 15 & 5 & 15 \\ 15 & 15 & 5 \end{bmatrix}$$

$$\therefore\ A^2 - 5A = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = 14 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 14\,I$$

$\therefore$ By definition of scalar matrix, $A^2 - 5A$ is a scalar matrix.

Ex. 3: If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$, find k, so that $A^2 - kA + 2I = O$, where I is a 2×2 the identity matrix and O is null matrix of order 2.

Solution: Given $A^2 - kA + 2I = O$

$\therefore$ Here, $A^2 = AA$

$$= \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8 & -6+4 \\ 12-8 & -8+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix}$$

$\therefore$ $A^2 - kA + 2I = O$

$$\therefore \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} - k \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = O$$

$$\therefore \begin{bmatrix} 1 & -2 \\ 4 & -4 \end{bmatrix} - \begin{bmatrix} 3k & -2k \\ 4k & -2k \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1-3k+2 & -2+2k \\ 4-4k & -4+2k+2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Using definition of equality of matrices, we have

$$\begin{array}{ll} 1-3k+2=0 & \therefore 3k=3 \\ -2+2k=0 & \therefore 2k=2 \\ 4-4k=0 & \therefore 4k=4 \\ -4+2k+2=0 & \therefore 2k=2 \end{array} \Bigg\} \quad k = 1$$

Ex. 4: Find x and y, if $\begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \left\{ 3 \begin{bmatrix} 6 & 3 \\ -1 & 2 \\ 5 & 4 \end{bmatrix} + 2 \begin{bmatrix} -4 & -1 \\ 1 & 0 \\ -3 & -4 \end{bmatrix} \right\} = \begin{bmatrix} x & y \end{bmatrix}$

Solution: Given $\begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \left\{ 3 \begin{bmatrix} 6 & 3 \\ -1 & 2 \\ 5 & 4 \end{bmatrix} + 2 \begin{bmatrix} -4 & -1 \\ 1 & 0 \\ -3 & -4 \end{bmatrix} \right\} = \begin{bmatrix} x & y \end{bmatrix}$

$$\therefore \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \left\{ \begin{bmatrix} 18 & 9 \\ -3 & 6 \\ 15 & 12 \end{bmatrix} + \begin{bmatrix} -8 & -2 \\ 2 & 0 \\ -6 & -8 \end{bmatrix} \right\} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 10 & 7 \\ -1 & 6 \\ 9 & 4 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\therefore \begin{bmatrix} 20+27 & 14+12 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$$\therefore \begin{bmatrix} 47 & 26 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

$\therefore$ $x = 47$, $y = 26$ by definition of equality of matrices.

$(A + B)(A - B) = A^2 - AB + BA - B^2$

Let's Note :

Using the distributive laws discussed earlier, we can derive the following results. If A and B are square matrices of the same order, then

i) $(A + B)^2 = A^2 + AB + BA + B^2$

ii) $(A - B)^2 = A^2 - AB - BA + B^2$

Ex 5: A School purchased 8 dozen Mathematics books, 7 dozen physics books and 10 dozen chemistry books of standard XI. The price of one book of Mathematics, Physics and Chemistry are Rs.50, Rs.40 and Rs.60 respectively. Use matrix multiplication to find the total amount that the school pays the book seller.

Solution: Let A be the column matrix of books of different subjects and let B be the row matrix of prices of one book of each subject.

$$A = \begin{bmatrix} 8\times 12 \\ 7\times 12 \\ 10\times 12 \end{bmatrix} = \begin{bmatrix} 96 \\ 84 \\ 120 \end{bmatrix} \qquad B = \begin{bmatrix} 50 & 40 & 60 \end{bmatrix}$$

$\therefore$ The total amount received by the bookseller is obtained by the matrix BA.

$$\therefore\ BA = \begin{bmatrix} 50 & 40 & 60 \end{bmatrix} \begin{bmatrix} 96 \\ 84 \\ 120 \end{bmatrix}$$

$$= [50 \times 96 + 40 \times 84 + 60 \times 120]$$

$$= [4800 + 3360 + 7200]$$

$$= [15360]$$

Thus the amount received by the bookseller from the school is Rs. 15360.

EXERCISE 2.3

1) Evaluate i) $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 3 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$

2) If $A = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & -3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. State whether AB = BA? Justify your answer.

3) Show that AB = BA where, $A = \begin{bmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 9 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{bmatrix}$

4) Verify A(BC) = (AB)C, if $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \\ 0 & 4 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -2 \\ -1 & 1 \\ 0 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 0 & -2 \end{bmatrix}$

5) Verify that A(B+C) = AB + AC, if $A = \begin{bmatrix} 4 & -2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 3 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}$

6) If $A = \begin{bmatrix} 4 & 3 & 2 \\ -1 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & -2 \end{bmatrix}$ show that matrix AB is non singular.

7) If $A + I = \begin{bmatrix} 1 & 2 & 0 \\ 5 & 4 & 2 \\ 0 & 7 & -3 \end{bmatrix}$, find the product $(A + I)(A - I)$.

8) If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, show that $A^2 - 4A$ is a scalar matrix.

9) If $A = \begin{bmatrix} 1 & 0 \\ -1 & 7 \end{bmatrix}$, find k so that $A^2 - 8A - kI = O$, where I is a 2×2 unit and O is null matrix of order 2.

10) If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, prove that $A^2 - 5A + 7I = 0$, where I is 2×2 unit matrix.

11) If $A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & a \\ -1 & b \end{bmatrix}$ and if $(A + B)^2 = A^2 + B^2$, find value of a and b.

12) Find k, If $A = \begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix}$ and $A^2 = kA - 2I$.

13) Find x and y, If $\left\{ 4\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 4 \\ 2 & 1 & 1 \end{bmatrix} \right\} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

14) Find x, y, z if $\left\{ 3\begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 1 \\ -1 & 2 \\ 3 & 1 \end{bmatrix} \right\} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} x-3 \\ y-1 \\ 2z \end{bmatrix}$

15) Jay and Ram are two friends. Jay wants to buy 4 pens and 8 notebooks, Ram wants to buy 5 pens and 12 notebooks. The price of One pen and one notebook was Rs. 6 and Rs.10 respectively. Using matrix multiplication, find the amount each one of them requires for buying the pens and notebooks.

- **Properties of the transpose of a matrix:**

(i) If A and B are two matrices of same order, then $(A + B)^T = A^T + B^T$

(ii) If A is a matrix and k is a constant, then $(kA)^T = kA^T$

(iii) If A and B are conformable for the product AB, then $(AB)^T = B^T A^T$

For example: Let A = $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$, B = $\begin{bmatrix} 2 & 3 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}$, $\therefore$ AB is defined and

$$AB = \begin{bmatrix} 2+2+1 & 3+4+2 \\ 6+1+3 & 9+2+6 \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ 10 & 17 \end{bmatrix} \qquad \therefore (AB)^T = \begin{bmatrix} 5 & 10 \\ 9 & 17 \end{bmatrix} \text{ .... (1)}$$

$$\text{Now } A^T = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}, B^T = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}, \qquad \therefore B^TA^T = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\therefore B^TA^T = \begin{bmatrix} 2+2+1 & 6+1+3 \\ 3+4+2 & 9+2+6 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 9 & 17 \end{bmatrix} \text{ .... (2)}$$

$\therefore$ From (1) and (2) we have proved that, $(AB)^T = B^TA^T$

In general $(A_1A_2A_3........An)^T = An^T............ A_3^TA_2^TA_1^T$

(iv) If A is a symmetric matrix, then $A^T = A$.

For example: Let A = $\begin{bmatrix} 2 & -3 & 4 \\ -3 & 5 & -2 \\ 4 & -2 & 1 \end{bmatrix}$ be a symmetric matrix.

$$A^T = \begin{bmatrix} 2 & -3 & 4 \\ -3 & 5 & -2 \\ 4 & -2 & 1 \end{bmatrix} = A$$

(v) If A is a skew symmetric matrix, then $A^T = -A$.

For example: Let A = $\begin{bmatrix} 0 & 5 & 4 \\ -5 & 0 & -2 \\ -4 & 2 & 0 \end{bmatrix}$ be a skew symmetric matrix.

$$\therefore A^T = \begin{bmatrix} 0 & -5 & -4 \\ 5 & 0 & 2 \\ 4 & -2 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 5 & 4 \\ -5 & 0 & -2 \\ -4 & 2 & 0 \end{bmatrix} = -A, \qquad \therefore A^T = -A.$$

(vi) If A is a square matrix, then (a) $A + A^T$ is symmetric.

(b) $A - A^T$ is skew symmetric.

For example: (a) Let A = $\begin{bmatrix} 3 & 5 & 7 \\ 2 & 4 & -6 \\ 3 & 8 & -5 \end{bmatrix}$, $\therefore A^T = \begin{bmatrix} 3 & 2 & 3 \\ 5 & 4 & 8 \\ 7 & -6 & -5 \end{bmatrix}$

Now $A + A^T = \begin{bmatrix} 3 & 5 & 7 \\ 2 & 4 & -6 \\ 3 & 8 & -5 \end{bmatrix} + \begin{bmatrix} 3 & 2 & 3 \\ 5 & 4 & 8 \\ 7 & -6 & -5 \end{bmatrix} = \begin{bmatrix} 6 & 7 & 10 \\ 7 & 8 & 2 \\ 10 & 2 & -10 \end{bmatrix}$

$\therefore$ $A + A^T$ is a symmetric matrix, by definition.

(b) Let $A - A^T = \begin{bmatrix} 3 & 5 & 7 \\ 2 & 4 & -6 \\ 3 & 8 & -5 \end{bmatrix} - \begin{bmatrix} 3 & 2 & 3 \\ 5 & 4 & 8 \\ 7 & -6 & -5 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 4 \\ -3 & 0 & -14 \\ -4 & 14 & 0 \end{bmatrix}$

$\therefore$ $A - A^T$ is a skew symmetric matrix, by definition.

Let's Note: A square matrix A can be expressed as the sum of a symmetric and a skew symmetric matrix as follows.

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

For example: Let $A = \begin{bmatrix} 4 & -5 & 3 \\ -6 & 2 & 1 \\ 7 & 8 & -9 \end{bmatrix}$, $\therefore$ $A^T = \begin{bmatrix} 4 & -6 & 7 \\ -5 & 2 & 8 \\ 3 & 1 & -9 \end{bmatrix}$

$$A + A^T = \begin{bmatrix} 4 & -5 & 3 \\ -6 & 2 & 1 \\ 7 & 8 & -9 \end{bmatrix} + \begin{bmatrix} 4 & -6 & 7 \\ -5 & 2 & 8 \\ 3 & 1 & -9 \end{bmatrix} = \begin{bmatrix} 8 & -11 & 10 \\ -11 & 4 & 9 \\ 10 & 9 & -18 \end{bmatrix}$$

$$\text{Let } P = \frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 8 & -11 & 10 \\ -11 & 4 & 9 \\ 10 & 9 & -18 \end{bmatrix} = \begin{bmatrix} 4 & -\frac{11}{2} & 5 \\ -\frac{11}{2} & 2 & \frac{9}{2} \\ 5 & \frac{9}{2} & -18 \end{bmatrix}$$

The matrix P is a symmetric matrix.

$$\text{Also } A - A^T = \begin{bmatrix} 4 & -5 & 3 \\ -6 & 2 & 1 \\ 7 & 8 & -9 \end{bmatrix} - \begin{bmatrix} 4 & -6 & 7 \\ -5 & 2 & 8 \\ 3 & 1 & -9 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -4 \\ -1 & 0 & -7 \\ 4 & 7 & 0 \end{bmatrix}$$

$$\text{Let } Q = \frac{1}{2}(A - A^T) = \frac{1}{2}\begin{bmatrix} 0 & 1 & -4 \\ -1 & 0 & -7 \\ 4 & 7 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & -2 \\ -\frac{1}{2} & 0 & -\frac{7}{2} \\ 2 & \frac{7}{2} & 0 \end{bmatrix}$$

The matrix Q is a skew symmetric matrix.

Since P + Q = symmetric matrix + skew symmetric matrix.

Thus A = P + Q.

EXERCISE 2.4

(1) Find A^T, if (i) $A = \begin{bmatrix} 1 & 3 \\ -4 & 5 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & -6 & 1 \\ -4 & 0 & 5 \end{bmatrix}$

(2) If $A = [a_{ij}]_{3\times 3}$ where $a_{ij} = 2(i - j)$. Find A and A^T. State whether A and A^T both are symmetric or skew symmetric matrices ?

(3) If $A = \begin{bmatrix} 5 & -3 \\ 4 & -3 \\ -2 & 1 \end{bmatrix}$, Prove that $(A^T)^T = A$.

(4) If $A = \begin{bmatrix} 1 & 2 & -5 \\ 2 & -3 & 4 \\ -5 & 4 & 9 \end{bmatrix}$, Prove that $A^T = A$.

(5) If $A = \begin{bmatrix} 2 & -3 \\ 5 & -4 \\ -6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 4 & -1 \\ -3 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ -1 & 4 \\ -2 & 3 \end{bmatrix}$ then show that

(i) $(A + B)^T = A^T + B^T$ (ii) $(A - C)^T = A^T - C^T$

(6) If $A = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 3 \\ 4 & -1 \end{bmatrix}$, then find C^T, such that $3A - 2B + C = I$, where I is the unit matrix of order 2.

(7) If $A = \begin{bmatrix} 7 & 3 & 0 \\ 0 & 4 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -2 & 3 \\ 2 & 1 & -4 \end{bmatrix}$ then find

(i) $A^T + 4B^T$ (ii) $5A^T - 5B^T$

(8) If $A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 & -4 \\ 3 & 5 & -2 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 2 & 3 \\ -1 & -1 & 0 \end{bmatrix}$, verify that

$(A + 2B + 3C)^T = A^T + 2B^T + 3C^T$

(9) If $A = \begin{bmatrix} -1 & 2 & 1 \\ -3 & 2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \\ -1 & 3 \end{bmatrix}$, prove that $(A + B^T)^T = A^T + B$.

(10) Prove that $A + A^T$ is a symmetric and $A - A^T$ is a skew symmetric matrix, where

(i) $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 2 & 1 \\ -2 & -3 & 2 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 5 & 2 & -4 \\ 3 & -7 & 2 \\ 4 & -5 & -3 \end{bmatrix}$

(11) Express each of the following matrix as the sum of a symmetric and a skew symmetric matrix.

(i) $\begin{bmatrix} 4 & -2 \\ 3 & -5 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

(12) If $A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \\ 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 & -4 \\ 2 & -1 & 1 \end{bmatrix}$, verify that

(i) $(AB)^T = B^T A^T$ (ii) $(BA)^T = A^T B^T$

2.5 Elementary Transformations:

Let us understand the meaning and application of elementary transformations. There are three elementary transformations of a matrix. They can be used on rows and columns.

a) **Interchange of any two rows or any two columns:** If we interchange the i^{th} row and the j^{th} row of a matrix then, after this interchange, the original matrix is transformed to a new matrix.

This transformation is symbolically denoted as $R_i \leftrightarrow R_j$ or R_{ij}

The same transformation can be applied to two columns, say $C_i \leftrightarrow C_j$ or C_{ij}

For Example:

If $A = \begin{bmatrix} 5 & 7 \\ 3 & 2 \end{bmatrix}$ then $R_1 \leftrightarrow R_2$ gives new matrix $\begin{bmatrix} 3 & 2 \\ 5 & 7 \end{bmatrix}$

and $C_1 \leftrightarrow C_2$ gives new matrix $\begin{bmatrix} 7 & 5 \\ 2 & 3 \end{bmatrix}$

b) **Multiplication of the elements of any row or column by a non zero scalar:** If k is a non zero scalar and the row R_i is to be multiplied by a Scalar k, then we multiply every element of R_i by the Scalar k. Symbolically the transformation is denoted by kR_i or $R_i(k)$

For example: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 0 \\ 4 & 1 & 3 \end{bmatrix}$ then $4R_2$ gives $\begin{bmatrix} 1 & 2 & 3 \\ 8 & 20 & 0 \\ 4 & 1 & 3 \end{bmatrix}$

Similarly, if any column of the matrix is multiplied by a constant then we multiply every element of the column by the constant. It is denoted by kC_i or $C_i(k)$

If $B = \begin{bmatrix} 2 & 3 \\ -5 & 1 \end{bmatrix}$ then ${}^3C_2$ gives $\begin{bmatrix} 2 & 9 \\ -5 & 3 \end{bmatrix}$

c) **Adding the scalar multiples of all the elements of any row (column) to corresponding elements of any other row (column):** If k is a non-zero scalar and the k-multiples of the elements of $R_j(C_j)$ are to be added to the elements of $R_i(C_i)$ then the transformation is symbolically denoted as R_i+kR_j or C_i+kC_j

For example:

1) If $A = \begin{bmatrix} 2 & 5 \\ 7 & 8 \end{bmatrix}$ then $R_2 + 2R_1$ gives $= \begin{bmatrix} 2 & 5 \\ 7+2(2) & 8+2(5) \end{bmatrix}$

$$= \begin{bmatrix} 2 & 5 \\ 11 & 18 \end{bmatrix}$$

2) If $B = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix}$ then $C_1 - 2C_2$ $= \begin{bmatrix} 3-2(-2) & -2 \\ 1-2(4) & 4 \end{bmatrix}$

$$= \begin{bmatrix} 7 & -2 \\ -7 & 4 \end{bmatrix}$$

Let's Note:

1) After transformation $R_i + kR_j$, R_j remains same as in the original matrix. Similarly with the transformation $C_i + kC_j$, C_j remains same as in the original matrix.

2) The elements of a row or multiples of the element of a row can not be added to the elements of a column or conversely.

3) When any elementary row transformations are applied on both the sides of AB = C, the prefactor A changes and B remains unchanged. The same row transformations are applied on C.

For Example:

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix}$ then $AB = \begin{bmatrix} 1 & 10 \\ 1 & 20 \end{bmatrix}$ = C say

Now if we require C to be transformed to a new matrix by $R_1 \leftrightarrow R_2$

$$C \to \begin{bmatrix} 1 & 20 \\ 1 & 10 \end{bmatrix}$$

If the same transformation used for A then $A \to \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and B remains unchanged then product

$$AB = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -3+4 & 0+20 \\ -1+2 & 0+10 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 20 \\ 1 & 10 \end{bmatrix}$$

$$= C$$

SOLVED EXAMPLES

1) If $A = \begin{bmatrix} 2 & 4 \\ 1 & 7 \end{bmatrix}$ then apply the transformation $R_1 \leftrightarrow R_2$ on A.

Solution: As $A = \begin{bmatrix} 2 & 4 \\ 1 & 7 \end{bmatrix}$ $R_1 \leftrightarrow R_2$ gives $R_{12} = \begin{bmatrix} 1 & 7 \\ 2 & 4 \end{bmatrix}$

2) If $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ then apply the transformation $C_2 - 2C_1$.

Solution: As $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 1 \\ 3 & 2 & 1 \end{bmatrix}$ $C_2 - 2C_1 = \begin{bmatrix} 1 & 2-2(1) & 3 \\ 4 & 7-2(4) & 1 \\ 3 & 2-2(3) & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2-2 & 3 \\ 4 & 7-8 & 1 \\ 3 & 2-6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 1 \\ 3 & -4 & 1 \end{bmatrix}$$

3) If $A = \begin{bmatrix} 1 & -2 & -7 \\ -2 & 1 & 5 \\ 3 & 2 & 1 \end{bmatrix}$ then apply the transformation $R_2 + 2R_1$.

Solution: As $A = \begin{bmatrix} 1 & -2 & -7 \\ -2 & 1 & 5 \\ 3 & 2 & 1 \end{bmatrix}$ $R_2 + 2R_1$,

`

$$= \begin{bmatrix} 1 & -2 & -7 \\ -2+2(1) & 1+2(-2) & 5+2(-7) \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -7 \\ -2+2 & 1-4 & 5-14 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & -7 \\ 0 & -3 & -9 \\ 3 & 2 & 1 \end{bmatrix}$$

4) Convert $\begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix}$ into identity matrix by suitable row transformations.

Solution: Given $A = \begin{bmatrix} 2 & 3 \\ 1 & 6 \end{bmatrix}$ $R_1 \leftrightarrow R_2$

$= \begin{bmatrix} 1 & 6 \\ 2 & 3 \end{bmatrix}$ By $R_2 - 2R_1$ $\begin{bmatrix} 1 & 6 \\ 0 & -9 \end{bmatrix}$

$\frac{-1}{9} R_2$, $A = \begin{bmatrix} 1 & 6 \\ 0 & 1 \end{bmatrix}$

By $R_1 - 6R_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

2.6 Inverse of a matrix :

If A is a square matrix of order m and if there exists another square matrix B of the same order such that AB = BA = I, where I is the unit matrix of order m then B is called the inverse of A and is denoted by A^{-1} (read as A inverse)

Using the notation A^{-1} for B we write the above equations as $AA^{-1} = A^{-1}A = I$

Let's Note: For the existence of inverse of matrix A, it is necessary that $|A| \neq 0$, that is A is a non singular matrix.

- **Uniqueness of the inverse of a matrix:**

It can be proved that if A is a square matrix where $|A| \neq 0$, then its inverse, say A^{-1}, is unique.

Theorem: Prove that, if A is a square matrix and its inverse exists then the inverse is unique.

Proof: Let A be a square matrix of order m and let its inverse exist.

Let, if possible, B and C be two inverses of A

Then, by definition of the inverse matrix,

AB = BA = I and AC = CA = I

$$\begin{aligned}\text{Now consider } B &= BI\\ &= B(AC)\\ &= (BA)C\\ &= IC\\ B &= C\end{aligned}$$

Hence B = C, that is, the inverse of a matrix is unique.

Inverse of a matrix (if it exists) can be obtained by any of the two methods:

(1) Elementary Transformations (2) Adjoint Method.

1) Inverse of a non singular matrix by elementary transformations:

By the definition of inverse of a matrix A, if A^{-1} exists, then $AA^{-1} = A^{-1}A = I$.

To find A^{-1}, we first convert A into I. This can be done by using elementary transformations.

Hence the equation $AA^{-1} = I$ can be transformed into an equation of the type $A^{-1} = B$, by applying the same series of row transformations on both sides of the above equation. Similarly, if we start with the equation $A^{-1}A = I$ then the transformations should be applied to the columns of A. Apply column transformations to post factor and the other side, whereas prefactor remains unchanged.

$AA^{-1} = I$ (Row transformation) $A^{-1}A = I$ (column transformation)

Now if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a non singular matrix then reduce A into I

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The suitable row transformation are as follows

1) Reduce a_{11} to 1
2) Then reduce a_{21} and a_{31} to 0
3) Reduce a_{22} to 1
4) Then reduce a_{12} and a_{32} to 0
5) Reduce a_{33} to 1
6) Then reduce a_{13} and a_{23} to 0

Remember that the similar working rule (but not the same) can be used if you are using column transformations

$$\begin{array}{lll} a_{11} \rightarrow 1 & a_{12} \text{ and } a_{13} & \rightarrow 0 \\ a_{22} \rightarrow 1 & a_{21} \text{ and } a_{23} & \rightarrow 0 \\ a_{33} \rightarrow 1 & a_{31} \text{ and } a_{32} & \rightarrow 0 \end{array}$$

SOLVED EXAMPLES

1) Find which of the following matrix is invertible

(i) $\begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

Solution: i) $A = \begin{bmatrix} 1 & 2 \\ 5 & 7 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 \\ 5 & 7 \end{vmatrix}$$

$$= 7 - 10$$

$$= -3 \neq 0$$

$\therefore$ A^{-1} exists. $\therefore$ A is invertible matrix.

ii) $B = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$

$$|B| = \begin{vmatrix} 2 & 4 \\ 3 & 6 \end{vmatrix}$$

$$= 12 - 12 = 0$$

$\therefore$ B is singular matrix and hence B^{-1} does not exist.

$\therefore$ B is not invertible matrix.

iii) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

$|A| = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

$|A| = 2\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 2 & 3 \end{vmatrix}$ (By property of determinant)

$|A| = 2(0)$ (Row R_1 and R_3 are identical)

$\therefore$ A is singular matrix and hence A^{-1} does not exist.

$\therefore$ A is not invertible matrix.

2) Find the inverse of $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ by elementary transformation.

Solution: $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix}$

$= 6 - 5 = 1 \neq 0$

$\therefore$ A^{-1} is exist.

(I) $AA^{-1} = I$ By Row transformation

$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Using $R_1 \leftrightarrow R_2$

$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Using $R_2 \rightarrow R_2 - 2R_1$

$\begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$

Using $R_1 \rightarrow R_1 + 3R_2$

$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -5 \\ 1 & -2 \end{bmatrix}$

$\therefore \quad R_2 \rightarrow (-1)R_2$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$IA^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \qquad \text{............... (I)}$$

(II) $A^{-1}A = I$ By column transformations we get,

$$A^{-1}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C_1 \rightarrow \left(\frac{1}{2}\right) C_1$$

$$A^{-1}\begin{bmatrix} 1 & 5 \\ \frac{1}{2} & 3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$$

Using $C_2 \rightarrow C_2 - 5C_1$

$$A^{-1}\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-5}{2} \\ 0 & 1 \end{bmatrix}$$

Using $C_2 \rightarrow 2C_2$

$$A^{-1}\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -5 \\ 0 & 2 \end{bmatrix}$$

Using $C_1 \rightarrow C_1 - \left(\frac{1}{2}\right) C_2$

$$A^{-1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} I = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix} \qquad \text{........... (II)}$$

From I and II

A^{-1} is unique.

3) Find the inverse of $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ by using elementary row transformation.

Solution: Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= 2(3-0) - 0(15-0) - 1(5-0)$$

$$= 6 - 0 - 5$$

$$= 1 \neq 0$$

$\therefore$ A^{-1} is exist.

Consider $AA^{-1} = I$

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_1 \to 3R_1$ $\begin{bmatrix} 6 & 0 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_1 \to R_1 - R_2$ $\begin{bmatrix} 1 & -1 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_2 \to R_2 - 5R_1$ $\begin{bmatrix} 1 & -1 & -3 \\ 0 & 6 & 15 \\ 0 & 1 & 3 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ -15 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_2 \leftrightarrow R_3$ $\begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 3 \\ 0 & 6 & 16 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 0 & 1 \\ -15 & 6 & 0 \end{bmatrix}$

By $R_1 \to R_1 + R_2$ and $R_3 \to R_3 - 6R_2$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 1 \\ -15 & 6 & -6 \end{bmatrix}$

By $R_3 \to \frac{-1}{3} R_3$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 0 & 1 \\ 5 & -2 & 2 \end{bmatrix}$

By $R_2 \to R_2 - 3R_3$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

$$IA^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

2) **Inverse of a non singular matrix by Adjoint Method:** This method can be directly used for finding the inverse. However, for understanding this method we should know the definitions of minor and co-factor.

Definition: Minor of an element a_{ij} of matrix is the determinant obtained by ignoring ith row and jth column in which the element a_{ij} lies. Minor of an element a_{ij} is denoted by M_{ij}.

Definition: Cofactor of an element a_{ij} of matrix is given by $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is minor of the element a_{ij}. Cofactor of an element a_{ij} is denoted A_{ij}.

Adjoint of a Matrix:

The adjoint of a square matrix is defined as the transpose of the cofactor matrix of A.

The adjoint of a matrix A is denoted by adjA.

For Example: If A is a square matrix of order 3 then the matrix of its cofactors is

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

and the required adjoint of A is the transpose of the above matrix. Hence

$$\text{adj}A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

If $A = [a_{ij}]_{m\times m}$ is non singular square matrix then the inverse of matrix exists and given by

$$A^{-1} = \frac{1}{|A|} \text{adj}(A)$$

SOLVED EXAMPLES

1) Find the cofactor matrix of $A = \begin{bmatrix} 2 & 4 \\ -1 & 7 \end{bmatrix}$

Solution: Here $a_{11} = 2, a_{12} = 4, a_{21} = -1, a_{22} = 7$

Minor of a_{11} i.e., M_{11} $\therefore$ $A_{11} = (-1)^{1+1}M_{11} = (-1)^2 7 = 7$

Similarly we can find $M_{12} = -1$, and $A_{12} = (-1)^{1+2}M_{12} = -1(-1) = 1$

$M_{21} = 4$, and $A_{21} = (-1)^{2+1}M_{21} = -1(4) = -4$

$M_{22} = 2$, and $A_{22} = (-1)^{2+2}M_{22} = 1(2) = 2$

$\therefore$ Required cofactors are 7, 1, −4, 2

$\therefore$ Cofactor Matrix $= [A_{ij}]_{2\times 2} = \begin{bmatrix} 7 & 1 \\ -4 & 2 \end{bmatrix}$

2) Find the adjoint of $A = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$

Solution: Minor of $a_{11} = M_{11} = -6$

$\therefore$ $A_{11} = (-1)^{1+1}M_{11} = 1(-6) = -6$

Minor of $a_{12} = M_{12} = 4$

$\therefore$ $A_{12} = (-1)^{1+2}M_{12} = -1(4) = -4$

Minor of $a_{21} = M_{21} = -3$

$\therefore$ $A_{21} = (-1)^{2+1}M_{21} = -1(-3) = 3$

Minor of $a_{22} = M_{22} = 2$

$\therefore$ $A_{22} = (-1)^{2+2}M_{22} = 1(2) = 2$

$\therefore$ Cofactor of matrix $[A_{ij}]_{2\times 2} = \begin{bmatrix} -6 & -4 \\ 3 & 2 \end{bmatrix}$

$$\text{adj}(A) = [A_{ij}]^T = \begin{bmatrix} -6 & 3 \\ -4 & 2 \end{bmatrix}$$

3) If $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$ then find A^{-1} by the adjoint method.

Solution: Given $A = \begin{bmatrix} 2 & -2 \\ 4 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -2 \\ 4 & 3 \end{vmatrix} = 6 + 8 = 14 \neq 0$$

$\therefore$ A^{-1} is exist.

$M_{11} = 3$ $\therefore$ $A_{11} = (-1)^{1+1}M_{11} = 1(3) = 3$

$M_{12} = 4, \quad \therefore \quad A_{12} = (-1)^{1+2}M_{12} = -1(4) = -4$

$M_{21} = -2, \quad \therefore \quad A_{21} = (-1)^{2+1}M_{21} = -1(-2) = 2$

$M_{22} = 2, \quad \therefore \quad A_{22} = (-1)^{2+2}M_{22} = 1(2) = 2$

$$\therefore \quad \text{Cofactor matrix } [A_{ij}]_{2\times 2} = \begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$$

$$\text{adj }(A) = [A_{ij}]^T = \begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ adj }(A)$$

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 3 & 2 \\ -4 & 2 \end{bmatrix}$$

4) If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ then find A^{-1} by the adjoint method.

Solution: Given $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{vmatrix} = 2(4-1) + 1(-2+1) + 1(1-2)$$

$$= 6 - 1 - 1 = 4 \neq 0$$

$\therefore \quad A^{-1}$ exists.

For the given matrix A

$$\therefore \quad A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 1(4-1) = 3$$

$$\therefore \quad A_{12} = (-1)^{1+2}\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} = -1(-2+1) = 1$$

$$\therefore \quad A_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} = 1(1-2) = -1$$

$$\therefore \quad A_{21} = (-1)^{2+1}\begin{vmatrix} -1 & 1 \\ -1 & 2 \end{vmatrix} = -1(-2+1) = 1$$

$$\therefore \quad A_{22} = (-1)^{2+2}\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \quad = \quad 1(4-1) = 3$$

$$\therefore \quad A_{23} = (-1)^{2+3}\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} \quad = \quad -1(-2+1) = 1$$

$$\therefore \quad A_{31} = (-1)^{3+1}\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} \quad = \quad 1(1-2) = -1$$

$$\therefore \quad A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 1 \\ -1 & -1 \end{vmatrix} \quad = \quad -1(-2+1) = 1$$

$$\therefore \quad A_{33} = (-1)^{3+3}\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} \quad = \quad 1(4-1) = 3$$

$$\therefore \quad \text{Cofactor matrix } [A_{ij}]_{3x3} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\text{adj }(A) = [A_{ij}]^T = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{ adj } A$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

EXERCISE 2.5

1) Apply the given elementary transformation on each of the following matrices.

i) $\begin{bmatrix} 3 & -4 \\ 2 & 2 \end{bmatrix}$, $\quad R_1 \leftrightarrow R_2$

ii) $\begin{bmatrix} 2 & 4 \\ 1 & -5 \end{bmatrix}$, $\quad C_1 \leftrightarrow C_2$

iii) $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ $3R_2$ and $C_2 \rightarrow C_2 - 4C_1$

2) Transform $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{bmatrix}$ into an upper triangular matrix by suitable row transformations.

3) Find the cofactor of the following matrices

i) $\begin{bmatrix} 1 & 2 \\ 5 & -8 \end{bmatrix}$ ii) $\begin{bmatrix} 5 & 8 & 7 \\ -1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

4) Find the adjoint of the following matrices

i) $\begin{bmatrix} 2 & -3 \\ 3 & 5 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & 5 \\ -2 & 0 & -1 \end{bmatrix}$

5) Find the inverse of the following matrices by the adjoint method

i) $\begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & -2 \\ 4 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$

6) Find the inverse of the following matrices by the transformation method.

i) $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

7) Find the inverse $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ by elementary column transformation.

8) Find the inverse $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$ of by the elementary row transformation.

9) If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ 1 & 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$ then find matrix X such that XA = B

10) Find matrix X, If AX = B where $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

2.7 Applications of Matrices:

To find a solution of simultaneous linear equations.

Consider the following pair of simultaneous linear equations in two variables.

$$\left.\begin{aligned} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{aligned}\right\} \quad \text{.......... (i)}$$

Now consider the 2×2 matrix formed by coefficient of x and y

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, \qquad X = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Now consider the following matrix equation AX = B

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} a_1x + b_1y \\ a_2x + b_2y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \qquad \text{.................. (ii)}$$

$\therefore \quad a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$

Hence the matrix equation (ii) is equivalent to pair of simultaneous linear equations given by (i)

$\therefore$ Matrix form of $\quad a_1x + b_1y = c_1$

$a_2x + b_2y = c_2$ is

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Similarly suppose we have three simultaneous equations in three variables

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

This can be summarized by the matrix equation

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

For example: Write the following linear equations in the form of a matrix equation.

1) $3x + 5y = 2$

$-2x + y = 5$

Solution : $A = \begin{bmatrix} 3 & 5 \\ -2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$

AX=B

$$\begin{bmatrix} 3 & 5 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

2) $3x + 2y - z = 4$

$7x - 2y - 2z = 3$

$2x - 3y + 5z = 4$

Solution: $A = \begin{bmatrix} 3 & 2 & -1 \\ 7 & -2 & -2 \\ 2 & -3 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$

$$AX = B$$

$$\begin{bmatrix} 3 & 2 & -1 \\ 7 & -2 & -2 \\ 2 & -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

There are two methods for solving linear equations

(I) Method of Inversion (II) Method of Reduction

(I) Method of Inversion: Consider a system of linear equations. Suppose we express it in the matrix form AX=B, where A is non singular ($|A| \neq 0$). Then A has a unique inverse A^{-1}.

Pre multiplying AX=B by A^{-1}, we get

$A^{-1}(AX) = A^{-1}B$

$(A^{-1}A)X = A^{-1}B$

$IX = A^{-1}B$

$\therefore$ $X = A^{-1}B$. Thus, there is a unique solution to the given system of linear equations.

SOLVED EXAMPLES

1) Solve the following equations by inversion method

$2x + 3y = 5$

$6x - 2y = 4$

Solution: $A = \begin{bmatrix} 2 & 3 \\ 6 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$,

Given equations can be written in matrix form as

$AX = B$

Pre-multiplying AX = B by A^{-1} we get

$A^{-1}(AX) = A^{-1}B$

$(A^{-1}A)X = A^{-1}B$

$IX = A^{-1}B$

$X = A^{-1}B$

First we find the inverse by A by row transformation

We write $AA^{-1} = I$

$$\begin{bmatrix} 2 & 3 \\ 6 & -2 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Using $R_1 \to \left(\frac{1}{2}\right) R_1$ $\begin{bmatrix} 1 & \frac{3}{2} \\ 6 & -2 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$

Using $R_2 \to R_2 - 6R_1$ $\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & -11 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -3 & 1 \end{bmatrix}$

Using $R_2 \to \left(\frac{-1}{11}\right) R_2$ $\begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{3}{11} & \frac{-1}{11} \end{bmatrix}$

Using $R_1 \to R_1 - \frac{3}{2} R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} \frac{2}{22} & \frac{3}{22} \\ \frac{3}{11} & \frac{-1}{11} \end{bmatrix}$

$$A^{-1} = \frac{1}{11} \begin{bmatrix} 1 & \frac{3}{2} \\ 3 & -1 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{11} \begin{bmatrix} 1 & \frac{3}{2} \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

$$X = \frac{1}{11} \begin{bmatrix} 5+6 \\ 15-4 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 11 \\ 11 \end{bmatrix} = \begin{bmatrix} \frac{11}{11} \\ \frac{11}{11} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad x = 1, \; y = 1$$

Hence the solution of given linear equations are $x = 1$, $y = 1$.

2) Solve the following equations by the inversion method

$x - y + z = 4$

$2x + y - 3z = 0$

$x + y + z = 2$

Solution: The matrix equation is

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

AX=B

Pre-multiplying $AX = B$ by A^{-1} we get

$A^{-1}(AX) = A^{-1}B$

$(A^{-1}A)X = A^{-1}B$

$IX = A^{-1}B$

$X = A^{-1}B$

First we find the inverse of A by adjoint method

$$|A| = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix}$$

$|A| = 1(1 + 3) + 1(2 + 3) + 1(2 - 1)$

$= 4 + 5 + 1$

$= 10 \neq 0$

A^{-1} is exist

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$$

$M_{11} = \begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} = 1 + 3 = 4 \quad \therefore A_{11} = (-1)^2 M_{11} = 1(4) = 4$

$M_{12} = \begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} = 2+3 = 5 \quad \therefore A_{12} = (-1)^3 M_{12} = (-1)(5) = -5$

$M_{13} = \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = 2-1 = 1 \quad \therefore A_{13} = (-1)^4 M_{13} = 1(1) = 1$

$M_{21} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -1-1 = -2 \quad \therefore A_{21} = (-1)^3 M_{21} = (-1)(-2) = 2$

$M_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1-1 = 0 \quad \therefore A_{22} = (-1)^4 M_{22} = 1(0) = 0$

$M_{23} = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 1+1 = 2 \quad \therefore A_{23} = (-1)^5 M_{23} = (-1)(2) = -2$

$M_{31} = \begin{vmatrix} -1 & 1 \\ 1 & -3 \end{vmatrix} = 3-1 = 2 \quad \therefore A_{31} = (-1)^4 M_{31} = 1(2) = 2$

$M_{32} = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3-2 = -5 \quad \therefore A_{32} = (-1)^5 M_{32} = (-1)(-5) = 5$

$$M_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} = 1 + 2 = 3 \qquad \therefore A_{33} = (-1)^6 M_{33} = 1(3) = 3$$

$$\therefore \quad [A_{ij}] = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$$

$$\text{adj}(A) = [A_{ij}]^T$$

$$= \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} \text{Adj}(A)$$

$$= \frac{1}{10} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \frac{1}{10} \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4+0+6 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore \; x = 2, y = -1, z = 1$

(II) Method of Reduction:

Here we start by writing the given linear equations as the matrix equation AX = B. Then we perform suitable row transformations on the matrix A. Using the row transformations, we reduce matrix A into an upper triangular matrix or lower triangular matrix. The same row transformations are performed simultaneously on matrix B.

After this, we rewrite the equations in the form of a system of linear equations. Now they are in such a form that they can be easily solved by elimination method. The required solution is obtained in this way.

Solved Examples

1) Solve the equations $2x - y = -2$ and $3x + 4y = 3$ by the method of reduction.

Solution: The given equations can be write as

$$2x - y = -2$$

$$3x + 4y = 3$$

Hence the matrix equation is AX = B

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

By $R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

By $R_1 \leftrightarrow R_1 - R_2$

$$\begin{bmatrix} 1 & 5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

By $R_2 \leftrightarrow R_2 - 2R_1$

$$\begin{bmatrix} 1 & 5 \\ 0 & -11 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \end{bmatrix}$$

We write equations as

$x + 5y = 5$ --------------------------- (1)

$-11y = -12$ --------------------------- (2)

from (2), $y = \frac{12}{11}$ --------------------------- (3)

Put $y = \frac{12}{11}$ in equation (1) to get

$$x + 5 \times \left(\frac{12}{11}\right) = 5$$

$$x = 5 - \frac{60}{11} = \frac{55-60}{11} = \frac{-5}{11}$$

$$\therefore \quad x = \frac{-5}{11}, \; y = \frac{12}{11}$$

2) Express the following equations in matrix form and solve them by the method of reduction

$x - y + z = 1,\ 2x - y = 1,\ 3x + 3y - 4z = 2.$

Solution: The given equations can be write as

$x - y + z = 1$

$2x - y = 1$

$3x + 3y - 4z = 2$

Hence the matrix equation is AX = B

$$\therefore \quad \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 3 & 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

By $R_2 \to R_2 - 2R_1$
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 3 & 3 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

By $R_3 \to R_3 - 3R_1$
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 6 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

By $R_3 \to R_3 - 6R_2$
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}$$

We write equations as

$x - y + z = 1$ ---------------------------------- (1)

$y - 2z = -1$ ------------------------------------ (2)

$5z = 5$ --- (3)

From (3), $z = 1$

Put $z = 1$ in equation (2) $y - 2(1) = -1$ $\quad \therefore\ y = 2 - 1 = 1$

Put $y = 1, z = 1$ in equation (1) $x - 1 + 1 = 1,$ $\quad \therefore\ x = 1$

$\therefore \quad x = 1, y = 1, z = 1$

EXERCISE 2.6

1) Solve the following equations by method of inversion.

i) $x + 2y = 2,\ 2x + 3y = 3$

ii) $2x + y = 5,\ 3x + 5y = -3$

iii) $2x - y + z = 1,\ x + 2y + 3z = 8$ and $3x + y - 4z = 1$

iv) $x + y + z = 1,\ x - y + z = 2$ and $x + y - z = 3$

2) Express the following equations in matrix form and solve them by method of reduction.

i) $x + 3y = 2,\ 3x + 5y = 4$

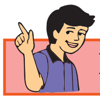
ii) $3x - y = 1\ ,\ 4x + y = 6$

iii) $x + 2y + z = 8,\ 2x + 3y - z = 11$ and $3x - y - 2z = 5$

iv) $x + y + z = 1,\ 2x + 3y + 2z = 2$ and $x + y + 2z = 4$

3) The total cost of 3 T.V. and 2 V.C.R. is Rs. 35000. The shopkeeper wants profit of Rs. 1000 per T.V. and Rs. 500 per V.C.R. He sell 2 T.V. and 1 V.C.R. and he gets total revenue as Rs. 21500. Find the cost and selling price of T.V and V.C.R.

4) The sum of the cost of one Economic book, one Co-operation book and one account book is Rs. 420. The total cost of an Economic book, 2 Co-operation books and an Account book is Rs. 480. Also the total cost of an Economic book, 3 Co-operation book and 2 Account books is Rs. 600. Find the cost of each book.



Let's Remember

- **Scalar Multiplication of a matrix:**

  If $A = [a_{ij}]_{m\times n}$ is a matrix and k is a scalar, then $kA = [ka_{ij}]$.

- **Addition of two matrices:**

  Matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to conformable for addition if orders of A and B are same.

  Then $A + B = [a_{ij} + b_{ij}]$. The order of A + B is the same as the order of A and B.

- **Multiplication of two matrices:**

  A and B are said to be conformable for multiplication if the number of columns of A is equal to the number of rows of B.

  That is, if $A = [a_{ik}]_{m\times p}$ and $B = [b_{kj}]_{p\times n}$, then AB is defined and $AB = [c_{ij}]_{m\times n}$,

  where $c_{ij} = \sum_{k=1}^{p} a_{ik}\, b_{kj}$ $\quad i = 1, 2, \ldots\ldots, m$

  $j = 1, 2, \ldots\ldots, n.$

- If $A = [a_{ij}]_{m\times n}$ is any matrix, then the transpose of A is denoted by $A^T = B = [b_{ij}]_{n\times m}$ and $b_{ij} = a_{ji}$

- If A is a square matrix, then

  i) $A + A^T$ is a symmetric matrix ii) $A - A^T$ is a skew-symmetric matrix.

- Every square matrix A can be expressed as the sum of a symmetric and a skew-symmetric matrix as

  $$A = \frac{1}{2}[A + A^T] + \frac{1}{2}[A - A^T]$$

- **Elementary Transformations:**

  a) Interchange of any two rows or any two columns

  b) Multiplication of the elements of any row or column by a non zero scalar

  c) Adding the scalar multiples of all the elements of any row (column) to the corresponding elements of any other row (column).

- If A and B are two square matrices of the same order such that AB = BA = I, then A and B are inverses of each other. A is denoted by B^{-1} and B is denoted by A^{-1}.
- For finding the inverse of A, if row transformation are to be used then we consider $AA^{-1} = I$ and if column transformation are to be used then we consider $A^{-1}A = I$.
- For finding the inverse of any nonsingular square matrix, two methods can be used

  i) Elementary transformation method. ii) Adjoint Method.
- A system of linear equations can be solved using matrices. The two methods are

  i) Method of inversion. ii) Method of reduction (Row transformations).

MISCELLANEOUS EXERCISE - 2

I. Choose the correct alternative.

1) If AX = B, where $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then X = ...

a) $\begin{bmatrix} \frac{3}{5} \\ \frac{3}{7} \end{bmatrix}$ b) $\begin{bmatrix} \frac{7}{3} \\ \frac{5}{3} \end{bmatrix}$ c) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ d) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

2) The matrix $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$ is ....

a) Identity Matrix b) scalar matrix c) null matrix d) diagonal matrix

3) The matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is ....

a) Identity matrix b) diagonal matrix c) scalar matrix d) null matrix

4) If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then $|\text{adj}.A|$ = ...

a) a^{12} b) a^9 c) a^3 d) a^{-3}

5) Adjoint of $\begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}$ is ....

a) $\begin{bmatrix} -6 & 3 \\ -4 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 6 & 3 \\ -4 & 2 \end{bmatrix}$ c) $\begin{bmatrix} -6 & -3 \\ 4 & 2 \end{bmatrix}$ d) $\begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$

6) If A = diag. $[d_1,d_2,d_3,\ldots\ldots,d_n]$, where $d_i \neq 0$, for i = 1,2,3,……,n then A^{-1} = ……….

a) diag.$[1/d_1,1/d_2,1/d_3,\ldots\ldots,1/d_n]$ b) D c) I d) O

7) If $A^2 + mA + nI = O$ & $n \neq 0$, $|A| \neq 0$, then A^{-1} = ……….

a) $\frac{-1}{m}(A+nI)$ b) $\frac{-1}{n}(A+mI)$ c) $\frac{-1}{n}(I+mA)$ d) $(A + mnI)$

8) If a 3×3 matirx B has its inverse equal to B, then B^2 = ……………

a) $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9) If A = $\begin{bmatrix} \alpha & 4 \\ 4 & \alpha \end{bmatrix}$ & $|A^3|$ = 729 then α = ……….

a) ± 3 b) ± 4 c) ± 5 d) ± 6

10) If A and B square matrices of order n×n such that $A^2 - B^2 = (A - B)(A + B)$, then which of the following will be always true ?

a) AB = BA b) either of A or B is a zero matrix

c) either of A and B is an identity matrix d) A = B

11) If A = $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ then A^{-1} = ………………

a) $\begin{bmatrix} 3 & -5 \\ 1 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$ d) $\begin{bmatrix} 3 & -5 \\ 1 & -2 \end{bmatrix}$

12) If A is a 2×2 matrix such that A(adj.A) = $\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$, then $|A|$ = ………………

a) 0 b) 5 c) 10 d) 25

13) If A is a non singular matrix, then det (A^{-1}) = ………..

a) 1 b) 0 c) det(A) d) 1/det(A)

14) If A = $\begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix}$ B = $\begin{bmatrix} -1 & 0 \\ 1 & 5 \end{bmatrix}$ then AB =

a) $\begin{bmatrix} 1 & -10 \\ 1 & 20 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 10 \\ -1 & 20 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 10 \\ 1 & 20 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 10 \\ -1 & -20 \end{bmatrix}$

15) If $x + y + z = 3$, $x + 2y + 3z = 4$, $x + 4y + 9z = 6$, then (y, z) = ……….

a) (−1,0) b) (1,0) c) (1,−1) d) (−1,1)

II. Fill in the blanks.

1) $A = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is ……………… matrix.

2) Order of matrix $\begin{bmatrix} 2 & 1 & 1 \\ 5 & 1 & 8 \end{bmatrix}$ is …………………

3) If $A = \begin{bmatrix} 4 & x \\ 6 & 3 \end{bmatrix}$ is a singular matrix then x is ………………

4) Matrix $B = \begin{bmatrix} 0 & 3 & 1 \\ -3 & 0 & -4 \\ p & 4 & 0 \end{bmatrix}$ is skew symmetric then value of p is ………………

5) If $A = [a_{ij}]_{2\times3}$ and $B = [b_{ij}]_{m\times1}$ and AB is defined then m = ……………

6) If $A = \begin{bmatrix} 3 & -5 \\ 2 & 5 \end{bmatrix}$ then cofactor of a_{12} is ……………..

7) If $A = [a_{ij}]_{m\times m}$ is non-singular matrix then $A^{-1} = \dfrac{1}{....}$ adj(A)

8) $(A^T)^T$ = …………..

9) If $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1 & -1 \\ x & 2 \end{bmatrix}$ then x = ……………..

10) If $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, then matrix form is $\begin{bmatrix} \cdots & \cdots \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cdots \\ \cdots \end{bmatrix}$

III. State whether each of the following is True or False.

1) Single element matrix is row as well as column matrix.

2) Every scalar matrix is unit matrix.

3) $A = \begin{bmatrix} 4 & 5 \\ 6 & 1 \end{bmatrix}$ is non singular matrix.

4) If A is symmetric then $A = -A^T$

5) If AB and BA both are exist then AB = BA

6) If A and B are square matrices of same order then $(A + B)^2 = A^2 + 2AB + B^2$

7) If A and B are conformable for the product AB then $(AB)^T = A^TB^T$

8) Singleton matrix is only row matrix.

9) $A = \begin{bmatrix} 2 & 1 \\ 10 & 5 \end{bmatrix}$ is invertible matrix.

10) $A(\text{adj}.A) = |A|I$, where I is the unit matrix.

IV. Solve the following.

1) Find k, if $\begin{bmatrix} 7 & 3 \\ 5 & k \end{bmatrix}$ is singular matrix.

2) Find x,y,z if $\begin{bmatrix} 2 & x & 5 \\ 3 & 1 & z \\ y & 5 & 8 \end{bmatrix}$ is symmetric matrix.

3) If $A = \begin{bmatrix} 1 & 5 \\ 7 & 8 \\ 9 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 \\ 1 & 5 \\ -8 & 6 \end{bmatrix}$ $C = \begin{bmatrix} -2 & 3 \\ 1 & -5 \\ 7 & 8 \end{bmatrix}$ then show that (A+B) + C = A + (B+C)

4) If $A = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 7 \\ -3 & 0 \end{bmatrix}$ Find matrix A – 4B + 7I where I is the unit matrix of order 2.

5) If $A = \begin{bmatrix} 2 & -3 \\ 3 & -2 \\ -1 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -3 & 4 & 1 \\ 2 & -1 & -3 \end{bmatrix}$ Verify

i) $(A + 2B^T)^T = A^T + 2B$ ii) $(3A - 5B^T)^T = 3A^T - 5B$

6) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 1 \\ -3 & 2 & -1 \\ -2 & 1 & 0 \end{bmatrix}$ then show that AB and BA are both singular matrices.

7) If $A = \begin{bmatrix} 3 & 1 \\ 1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}$, verify $|AB| = |A|\,|B|$

8) If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ then show that $A^2 - 4A + 3I = 0$

9) If $A = \begin{bmatrix} -3 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & a \\ b & 0 \end{bmatrix}$ and $(A + B)(A - B) = A^2 - B^2$, find a and b.

10) If $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, then find A^3

11) Find x, y, z if $\left\{5\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & -2 \\ 1 & 3 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x-1 \\ y+1 \\ 2z \end{bmatrix}$

12) If $A = \begin{bmatrix} 2 & -4 \\ 3 & -2 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 1 & 0 \end{bmatrix}$ then show that $(AB)^T = B^T A^T$

13) If A = $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$ then reduce it to unit matrix by row transformation.

14) Two farmers Shantaram and Kantaram cultivate three crops rice, wheat and groundnut. The sale (in Rupees.) of these crops by both the farmers for the month of April and May 2016 is given below,

April 2016 (In Rs.)

	Rice	Wheat	Groundnut
Shantaram	15000	13000	12000
Kantaram	18000	15000	8000

May 2016 (In Rs.)

	Rice	Wheat	Groundnut
Shantaram	18000	15000	12000
Kantaram	21000	16500	16000

Find i) The total sale in rupees for two months of each farmer for each crop.

ii)the increase in sale from April to May for every crop of each farmer.

15) Check whether following matrices are invertible or not.

i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & 4 & 3 \\ 1 & 1 & 0 \\ 1 & 4 & 5 \end{bmatrix}$ iv) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

16) Find inverse of the following matrices (if they exist) by elementary transformation.

i) $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ ii) $\begin{bmatrix} 2 & 1 \\ 7 & 4 \end{bmatrix}$ iii) $\begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$ iv) $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

17) Find the inverse of $\begin{bmatrix} 3 & 1 & 5 \\ 2 & 7 & 8 \\ 1 & 2 & 5 \end{bmatrix}$ by adjoint method.

18) Solve the following equations by method of inversion.

i) $4x - 3y - 2 = 0$, $3x - 4y + 6 = 0$

ii) $x + y - z = 2$, $x - 2y + z = 3$ and $2x - y - 3z = -1$

iii) $x - y + z = 4$, $2x + y - 3z = 0$ and $x + y + z = 2$

19) Solve the following equation by method of reduction.

i) $2x + y = 5$, $3x + 5y = -3$

ii) $x + 2y - z = 3$, $3x - y + 2z = 1$ and $2x - 3y + 3z = 2$

iii) $x - 3y + z = 2$, $3x + y + z = 1$ and $5x + y + 3z = 3$

20) The sum of three numbers is 6. If we multiply third number by 3 and add it the second number we get 11. By the adding first and third number we get a number which is double the second number. Use this information and find a system of linear equations. Find the three number using matrices.

Activities

1) Construct a matrix of order 2×2 where the (ij)th element given by $a_{ij} = \frac{(i+j)^2}{2+i}$

Solution: Let A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$ be the required matrix.

Given that $a_{ij} = \frac{(i+j)^2}{2+i}$, $a_{11} = \frac{(.......)^2}{....+1} = \frac{4}{3}$, $a_{12} = \frac{(.......)^2}{.......} = \frac{9}{3} = ...$

$a_{21} = \frac{(2+1)^2}{2+2} = \frac{.......}{4}$, $a_{22} = \frac{(.......)^2}{2+2} = \frac{.......}{.......} = 4$

$\therefore \quad A = \begin{bmatrix} \frac{4}{3} & ... \\ \frac{...}{...} & 4 \end{bmatrix}$

2) If A = $\begin{bmatrix} 1 & -5 \\ 6 & 7 \end{bmatrix}$, B = $\begin{bmatrix} 2 & -3 \\ 4 & 8 \end{bmatrix}$, Find AB – 2I, where I is unit matrix of order 2.

Solution: Given A = $\begin{bmatrix} 1 & -5 \\ 6 & 7 \end{bmatrix}$, B = $\begin{bmatrix} 2 & -3 \\ 4 & 8 \end{bmatrix}$

Consider AB – 2I = $\begin{bmatrix} 1 & -5 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ 4 & 8 \end{bmatrix} - 2 \begin{bmatrix} ... & ... \\ ... & ... \end{bmatrix}$

$\therefore \quad AB - 2I = \begin{bmatrix} .... & -3-40 \\ 12+28 & .... \end{bmatrix} - \begin{bmatrix} ... & 0 \\ 0 & ... \end{bmatrix} = \begin{bmatrix} ... & -43 \\ 40 & ... \end{bmatrix} - \begin{bmatrix} ... & 0 \\ 0 & ... \end{bmatrix}$

$\therefore \quad AB - 2I = \begin{bmatrix} ... & -43 \\ 40 & ... \end{bmatrix}$

3) If A = $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$ then find A^{-1} by the adjoint method.

Solution: Given A = $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

$|A| = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = ... = ... \neq 0$

$\therefore \quad A^{-1}$ is exist

$M_{11} = \ldots\ldots,\qquad \therefore A_{11} = (-1)^{1+1}M_{11} = \ldots.. = \ldots\ldots$

$M_{12} = \ldots\ldots,\qquad \therefore A_{12} = (-1)^{1+2}M_{12} = \ldots.. = \ldots\ldots$

$M_{21} = \ldots\ldots,\qquad \therefore A_{21} = (-1)^{2+1}M_{21} = \ldots\ldots = \ldots\ldots$

$M_{22} = \ldots\ldots,\qquad \therefore A_{22} = (-1)^{2+2}M_{22} = \ldots\ldots = \ldots\ldots$

$$\therefore \quad [A_{ij}]_{2\times 2} = \begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix}$$

$$\text{Adj }(A) = [A_{ij}]^T = \begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{Adj}(A)$$

$$A^{-1} = \frac{1}{\ldots.}\begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix}$$

4) Solve the following equations by inversion method.

$$x + 2y = 1$$

$$2x - 3y = 4$$

Solution: $A = \begin{bmatrix} \ldots & \ldots \\ \ldots & \ldots \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}, B = \begin{bmatrix} \ldots \\ 4 \end{bmatrix},$

Given equations can be written as AX = B

Pre-multiplying by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}B$$

$$\ldots\ldots. = A^{-1}B$$

$$IX = A^{-1}B$$

$$X = A^{-1}B$$

First we find the inverse of A by row transformation

We write $AA^{-1} = I$

Using $R_2 \to R_2 - 2R_1$ $\quad \begin{bmatrix} 1 & 2 \\ 0 & \ldots \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ \ldots & 1 \end{bmatrix}$

Using $\left(\frac{-1}{7}\right) R_2$ $\quad \begin{bmatrix} 1 & 2 \\ 0 & \ldots \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{\ldots}{\ldots} & \frac{-1}{\ldots} \end{bmatrix}$

Using $R_1 \to R_1 - 2R_2$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} \frac{\ldots}{\ldots} & \frac{\ldots}{\ldots} \\ \frac{\ldots}{\ldots} & \frac{-1}{\ldots} \end{bmatrix}$

$$A^{-1} = \frac{1}{....}\begin{bmatrix} 1 & \frac{3}{2} \\ 3 & -1 \end{bmatrix}$$

$$X = A^{-1}B = \frac{1}{....}\begin{bmatrix} ... & ... \\ ... & ... \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$= \frac{1}{....}\begin{bmatrix} ... & ... \\ ... & ... \end{bmatrix} = \frac{1}{....}\begin{bmatrix} ... & ... \\ ... & ... \end{bmatrix} = \begin{bmatrix} ... & ... \\ ... & ... \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ... \\ ... \end{bmatrix}$$

$$\therefore \quad x = \frac{11}{7}, \; y = \frac{-2}{7}$$

Hence the solution of given linear equation is $x = \frac{11}{7}$, $y = \frac{-2}{7}$

5) Express the following equations in matrix form and solve them by the method of reduction

$x + 3y + 3z = 12$, $x + 4y + 4z = 15$, $x + 3y + 4z = 13$.

Solution: The given equations can be write as

$x + 3y + 3z = 12$, $x + 4y + 4z = 15$, $x + 3y + 4z = 13$.

Hence the matrix equation is AX = B

$$\begin{bmatrix} ... & ... & ... \\ ... & ... & ... \\ ... & ... & ... \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ... \\ ... \\ ... \end{bmatrix} \qquad \text{(i.e. A}\mathbf{X} = \text{B)}$$

By $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ $\begin{bmatrix} 1 & 3 & 3 \\ ... & ... & ... \\ ... & ... & ... \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ ... \\ ... \end{bmatrix}$

We write equation as

$x + 3y + 3z = 12$ ------------------------ (1)

$y + z =$ …….. -------------------------- (2)

$z =$ ……… ---------------------------- (3)

from (3), $z = 1$

Put $z = 1$ in equation (2) $y +$ ……. =…….. $\qquad y =$ ……….

Put $y =$ ……… , $z = 1$ in equation (1) $x +$ …… + …….. = ………. , $x =$ …….

$\therefore$ $x =$ ………. , $y =$ ……….. , $z = 1$