CS 33

Data Representation

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Many of the slides in this lecture are either from or adapted from slides provided by the authors of the textbook "Computer Systems: A Programmer's Perspective." 2^{nd} Edition and are provided from the website of Carnegie-Mellon University, course 15-213, taught by Randy Bryant and David O'Hallaron in Fall 2010. These slides are indicated "Supplied by CMU" in the notes section of the slides.

Number Representation

- Hindu-Arabic numerals
  - developed by Hindus starting in 5th century
    - positional notation
    - symbol for 0
  - adopted and modified somewhat later by Arabs
    - known by them as “Rakam Al-Hind” (Hindu numeral system)
  - 1999 rather than MCMXCIX
    - (try doing long division with Roman numerals!)

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Which Base?

- 1999
  - base 10
    - » $9\cdot10^0+9\cdot10^1+9\cdot10^2+1\cdot10^3$
  - base 2
    - » 11111001111
      - $1\cdot2^0+1\cdot2^1+1\cdot2^2+1\cdot2^3+0\cdot2^4+0+2^5+1\cdot2^6+1\cdot2^7+1\cdot2^8+1\cdot2^9+1\cdot2^{10}$
  - base 8
    - » 3717
      - $7\cdot8^0+1\cdot8^1+7\cdot8^2+3\cdot8^3$
    - » why are we interested?
  - base 16
    - » 7CF
      - $15\cdot16^0+12\cdot16^1+7\cdot16^2$
    - » why are we interested?

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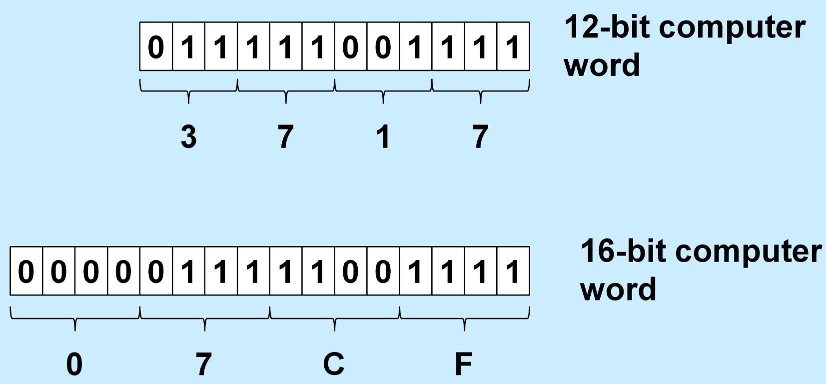
Base 2 is known as “binary” notation.

Base 8 is known as “octal” notation.

Base 10 is known as “decimal” notation.

Base 16 is known as “hexadecimal” notation. Note that “hexa” is derived from the Greek language and “decimal” is derived from the Latin language. Many people feel you shouldn’t mix languages when you invent words, but IBM, who coined the term “hexadecimal” in the 1960s, didn’t think their corporate image could withstand “sexadecimal”.

Words …



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Note that a byte consists of two hexadecimal digits, which are sometimes known as “nibbles”. A 32-bit computer word would then have eight nibbles; a 64-bit computer word would have sixteen nibbles.

Algorithm …

```
void baseX(unsigned int num, unsigned int base) {
   char digits[] = {'0', '1', '2', '3', '4', '5', '6', … };
   char buf[8*sizeof(unsigned int)+1];
   int i;

   for (i = sizeof(buf) - 2; i >= 0; i--) {
      buf[i] = digits[num%base];
      num /= base;
      if (num == 0)
         break;
   }

   buf[sizeof(buf) - 1] = '\0';
   printf("%s\n", &buf[i]);
}
```


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This routine prints the base *base* representation of *num*. (Note that the "…" is not heretofore unexplained C syntax, but is shorthand for "fill this in to the extent needed.")

Or …

```
$ bc
obase=16
1999
7CF
$
```

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“bc” (it stands for basic calculator, or perhaps better calculator) is a standard Unix command that handles arbitrary-precision arithmetic. Among its features is the ability to specify which base to use for input and output of numbers. The default base for both input and output is ten. Setting *obase* to 16 sets the base for output to 16. Similarly, one can change the base for input numbers by setting *ibase*.

Encoding Byte Values

- Byte = 8 bits
  - binary 00000000_2 to 11111111_2
  - decimal: 0_{10} to 255_{10}
  - hexadecimal 00_{16} to FF_{16}
    - » base 16 number representation
    - » use characters ‘0’ to ‘9’ and ‘A’ to ‘F’
    - » write $FA1D37B_{16}$ in C as
      - 0xFA1D37B
      - 0xfa1d37b

Hex	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

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Data Representations

C Data Type	Typical 32-bit	Intel IA32	x86-64
char	1	1	1
short	2	2	2
int	4	4	4
long	4	4	8
long long	8	8	8
float	4	4	4
double	8	8	8
long double	8	10/12	10/16
pointer	4	4	8

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The chart gives the number of bytes used to represent the indicated C data types. A long double actually required just ten bytes of storage; however, due to alignment constraints (a topic we discuss later in the course), it is given 12 bytes (aligned to a 4-byte boundary) on the 32-bit IA32 and 16 bytes (aligned to an 8-byte boundary) on the 64-bit x86-64. Note that this data type exists on the x86 architecture, but, in general, does not appear on other architectures.

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Boolean Algebra

- Developed by George Boole in 19th Century
  - algebraic representation of logic
    - » encode “true” as 1 and “false” as 0

And

- A&B = 1 when both A=1 and B=1

&	0	1
0	0	0
1	0	1

Or

- A|B = 1 when either A=1 or B=1

|	0	1
0	0	1
1	1	1

Not

- ~A = 1 when A=0

~	
0	1
1	0

Exclusive-Or (Xor)

- A^B = 1 when either A=1 or B=1, but not both

^	0	1
0	0	1
1	1	0

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General Boolean Algebras

- Operate on bit vectors
  - operations applied bitwise

```
  01101001    01101001    01101001
& 01010101  | 01010101  ^ 01010101  ~ 01010101
  01000001    01111101    00111100    10101010
```

- All of the properties of boolean algebra apply

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Example: Representing & Manipulating Sets

- Representation
  - width-w bit vector represents subsets of {0, …, w–1}
  - $a_j = 1$ iff $j \in A$

01101001	{ 0, 3, 5, 6 }
76543210	
01010101	{ 0, 2, 4, 6 }
76543210	

- Operations

&	intersection	01000001	{ 0, 6 }
|	union	01111101	{ 0, 2, 3, 4, 5, 6 }
^	symmetric difference	00111100	{ 2, 3, 4, 5 }
~	complement	10101010	{ 1, 3, 5, 7 }

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Bit-Level Operations in C

- Operations &, |, ~, ^ available in C
  - apply to any "integral" data type
    - » long, int, short, char
  - view arguments as bit vectors
  - arguments applied bit-wise
- Examples (char datatype)

~0x41 → 0xBE

~01000001_2 → 10111110_2

~0x00 → 0xFF

~00000000_2 → 11111111_2

0x69 & 0x55 → 0x41

01101001_2 & 01010101_2 → 01000001_2

0x69 | 0x55 → 0x7D

01101001_2 | 01010101_2 → 01111101_2

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Contrast: Logic Operations in C

- Contrast to Logical Operators
  - &&, ||, !
    - » view 0 as "false"
    - » anything nonzero as "true"
    - » always return 0 or 1
    - » early termination/short-circuited execution
- Examples (char datatype)

!0x41 → 0x00
!0x00 → 0x01
!!0x41 → 0x01

0x69 && 0x55 → 0x01
0x69 || 0x55 → 0x01
p && *p (avoids null pointer access)

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Contrast: Logic Operations in C

- Contrast to Logical Operators
  - &&, ||, !
    - » vie … “false”

Watch out for && vs. & (and || vs. |)… One of the more common oopsies in C programming

- E…

!0x41 → 0x00
!0x00 → 0x01
!!0x41 → 0x01

0x69 && 0x55 → 0x01
0x69 || 0x55 → 0x01
p && *p (avoids null pointer access)

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Shift Operations

- **Left Shift: x << y**
  - **shift bit-vector x left y positions**
    - throw away extra bits on left
    - fill with 0's on right
- **Right Shift: x >> y**
  - **shift bit-vector x right y positions**
    - throw away extra bits on right
  - **logical shift**
    - fill with 0's on left
  - **arithmetic shift**
    - replicate most significant bit on left
- **Undefined Behavior**
  - **shift amount < 0 or ≥ word size**

Argument x	01100010
<< 3	00010*000*
Log. >> 2	*00*011000
Arith. >> 2	*00*011000

Argument x	10100010
<< 3	00010*000*
Log. >> 2	*00*101000
Arith. >> 2	*11*101000

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The distinction between logical and arithmetic shifts should be clear by the end of this lecture.

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Signed Integers

- **Sign-magnitude**

b_{w-1}	b_{w-2}	b_{w-3}	…	b_2	b_1	b_0
sign	magnitude					

value = $(-1)^{b_{w-1}} \cdot \sum_{i=0}^{w-2} b_i \cdot 2^i$

  - **two representations of zero!**

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Signed Integers

- **Ones' complement**
  - **negate a number by forming its bitwise complement**
    - » **e.g., (-1)·01101011 = 10010100**

value = $-b_{w-1}(2^{w-1}-1)+\sum_{i=0}^{w-2} b_i \cdot 2^i$

$$\left.\begin{aligned} &= \sum_{i=0}^{w-2} b_i \cdot 2^i && \text{if } b_{w-1}=0 \\ &= \sum_{i=0}^{w-2} (b_i-1) \cdot 2^i && \text{if } b_{w-1}=1 \end{aligned}\right\} \text{two zeroes!}$$

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Note that the most-significant bit serves as the sign bit.

Signed Integers

- **Two's complement**

  $b_{w-1} = 0 \Rightarrow$ non-negative number

  $$\textbf{value} = \sum_{i=0}^{w-2} b_i \cdot 2^i$$

  $b_{w-1} = 1 \Rightarrow$ negative number

  $$\textbf{value} = (-1) \cdot 2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

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Note there's only one zero!

Two's complement is used on pretty much all of today's computers to represent signed integers.

Signed Integers

- **Negating two's complement**

$$value = -b_{w-1}2^{w-1} + \sum_{i=0}^{w-2} b_i 2^i$$

  - **how to compute –*value*?**
    (~value)+1

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Signed Integers

- **Negating two's complement (continued)**

$$\textit{value} + (\sim\!\textit{value} + 1)$$

$$= (\textit{value} + \sim\!\textit{value}) + 1$$

$$= (2^w - 1) + 1$$

$$= 2^w$$

$$= \; 1\,\overbrace{0\;0\;0\;\ldots\;0\;0\;0}^{w}$$

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If we add to the two's complement representation of a w-bit number the result of adding one to its bitwise complement, we get a w+1-bit number whose low-order w bits are zeroes and whose high-order bit is one. However, since we're constrained to only w bits, the result is a w-bit value of all zeroes, plus an overflow. If we ignore the overflow, the result is zero.

Numeric Ranges

- **Unsigned Values**
  - *UMin* = 0
    000…0
  - *UMax* **=** $2^w - 1$
    111…1

- **Two's Complement Values**
  - *TMin* = -2^{w-1}
    100…0
  - *TMax* **=** $2^{w-1} - 1$
    011…1
- **Other Values**
  - Minus 1
    111…1

Values for *W* = 16

	Decimal	Hex	Binary
UMax	**65535**	**FF FF**	11111111 11111111
TMax	**32767**	7**F FF**	01111111 11111111
TMin	**-32768**	80 00	10000000 00000000
-1	**-1**	**FF FF**	11111111 11111111
0	**0**	00 00	00000000 00000000

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Values for Different Word Sizes

	W			
	8	**16**	**32**	**64**
UMax	255	65,535	4,294,967,295	18,446,744,073,709,551,615
TMax	127	32,767	2,147,483,647	9,223,372,036,854,775,807
TMin	-128	-32,768	-2,147,483,648	-9,223,372,036,854,775,808

- **Observations**
  - $|TMin| = TMax + 1$
    - » Asymmetric range
  - $UMax = 2 * TMax + 1$

- **C Programming**
  - **#include** <limits.h>
  - declares constants, e.g.,
    - ULONG_MAX
    - LONG_MAX
    - LONG_MIN
  - values platform-specific

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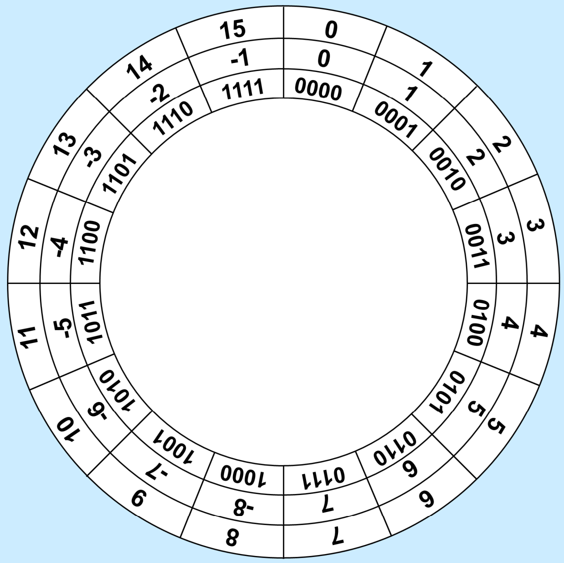
4-Bit Computer Arithmetic



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Unsigned computer arithmetic is performed modulo 2 to the power of the computer's word size. The outer ring of the figure demonstrates arithmetic modulo 2^4. To see the result, for example, of adding 3 to 2, start at 2 and go around the ring three units in the clockwise direction. If we add 5 to 14, we start at 14 and move 5 units clockwise, to 3. Similarly, to subtract 3 from 1, we start at one and move three units counterclockwise to 14.

What about two's-complement computer arithmetic? We know that the values encoded in a 4-bit computer word range from -8 to 7. How do we arrange them in the ring? As shown in the second ring, it makes sense for the non-negative numbers to be in the same positions as the corresponding unsigned values. It clearly makes sense for the integer coming just before 0 to be -1, the integer just before -1 to be -2, etc. Thus, since we have a ring, the integer following 7 is -8. Now we can see how arithmetic works for two's-complement numbers. Adding 3 to 2 works just as it does for unsigned numbers. Subtracting 3 from 1 results in -2. But adding 3 to 6 results in -7; and adding 5 to -2 results in 3.

The innermost ring shows the bit encodings for the unsigned and two's-complement values. The point of all this is that, with only one implementation of arithmetic, we can handle both unsigned and two's-complement values. Thus adding unsigned 5 and 9 is equivalent to adding two's-complement 5 and -7. The result will 1110, which, if interpreted as an unsigned value is 14, but if interpreted as a two's-complement value is -2.

Signed vs. Unsigned in C

- **Constants**
  - by default are considered to be signed integers
  - unsigned if have "U" as suffix

```
0U, 4294967259U
```

- **Casting**
  - explicit casting between signed & unsigned

```
int tx, ty;
unsigned int ux, uy; // "unsigned" means "unsigned int"
tx = (int) ux;
uy = (unsigned int) ty;
```

  - implicit casting also occurs via assignments and procedure calls

```
tx = ux;
uy = ty;
```

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Casting Surprises

- Expression evaluation
  - if there is a mix of unsigned and signed in single expression, *signed values implicitly cast to unsigned*
  - including comparison operations <, >, ==, <=, >=
  - examples for $W = 32$: TMIN = -2,147,483,648 , TMAX = 2,147,483,647

Constant$_1$	Constant$_2$	Relation	Evaluation
0	0U	==	unsigned
-1	0	<	signed
-1	0U	>	unsigned
2147483647	-2147483647-1	>	signed
2147483647U	-2147483647-1	<	unsigned
-1	-2	>	signed
(unsigned)-1	-2	>	unsigned
2147483647	2147483648U	<	unsigned
2147483647	(int) 2147483648U	>	signed

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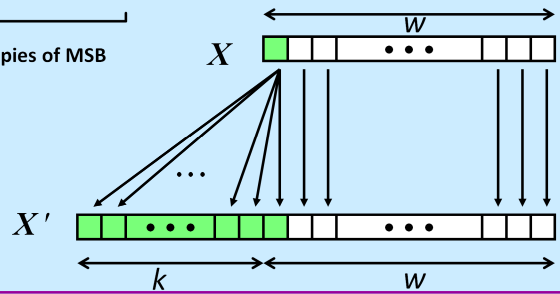
Sign Extension

- **Task:**
  - **given *w*-bit signed integer *x***
  - **convert it to *w*+*k*-bit integer with same value**
- **Rule:**
  - **make *k* copies of sign bit:**
  - $X' = \underbrace{x_{w-1}, \ldots, x_{w-1}}_{k \text{ copies of MSB}}, x_{w-1}, x_{w-2}, \ldots, x_0$



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Sign Extension Example

```
short int x =  15213;
int       ix = (int) x;
short int y = -15213;
int       iy = (int) y;
```

	Decimal	Hex	Binary
x	15213	3B 6D	00111011 01101101
ix	15213	00 00 3B 6D	00000000 00000000 00111011 01101101
y	-15213	C4 93	11000100 10010011
iy	-15213	FF FF C4 93	11111111 11111111 11000100 10010011

- **Converting from smaller to larger integer data type**
  - **C automatically performs sign extension**

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Does it Work?

$$val_w = -2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

$$val_{w+1} = -2^w + 2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

$$= -2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

$$val_{w+2} = -2^{w+1} + 2^w + 2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

$$= -2^w + 2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

$$= -2^{w-1} + \sum_{i=0}^{w-2} b_i \cdot 2^i$$

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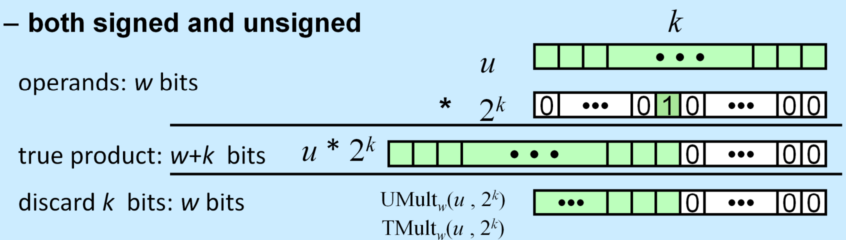
Sign extension clearly works for positive and zero values (where the sign bit is zero). But does it work for negative values? The first line of the slide shows the computation of the value of a w-bit item with a sign bit of one (i.e., it's negative). The next two lines show what happens if we extend this to a w+1-bit item, extending the sign bit. What had been the sign bit becomes one of the value bits, and its contribution to the value is now positive rather than negative. But this is compensated by the new sign bit, whose contribution is a negative value, twice as large as the original sign bit. Thus the net effect is for there to be no change in the value.

We do this again, extending to a w+2-bit item, and again, the resulting value is the same as what we started with.

Power-of-2 Multiply with Shift

- **Operation**
  - `u << k` **gives** `u * 2`^{k}
  - **both signed and unsigned**



- **Examples**

```
u << 3 ==     u * 8
u << 5 - u << 3 == u * 24
```

  - **most machines shift and add faster than multiply**
    - » **compiler generates this code automatically**

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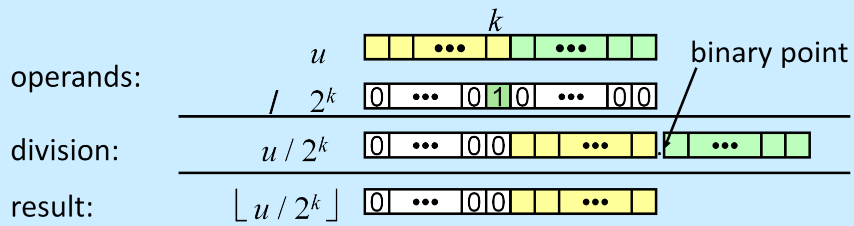
Unsigned Power-of-2 Divide with Shift

- **Quotient of unsigned by power of 2**
  - `u >> k` gives $\lfloor$ `u / ` $2^k \rfloor$
  - uses logical shift



	Division	Computed	Hex	Binary
`x`	15213	15213	`3B 6D`	`00111011 01101101`
`x >> 1`	7606.5	7606	`1D B6`	`00011101 10110110`
`x >> 4`	950.8125	950	`03 B6`	`00000011 10110110`
`x >> 8`	59.4257813	59	`00 3B`	`00000000 00111011`

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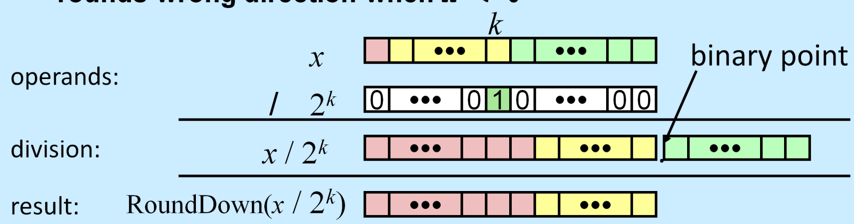
Signed Power-of-2 Divide with Shift

- **Quotient of signed by power of 2**
  - **`x >> k` gives $\lfloor$ `x / `2^k $\rfloor$**
  - **uses arithmetic shift**
  - **rounds wrong direction when `x < 0`**



	Division	Computed	Hex	Binary
`y`	-15213	-15213	`C4 93`	`11000100 10010011`
`y >> 1`	-7606.5	-7607	`E2 49`	`11100010 01001001`
`y >> 4`	-950.8125	-951	`FC 49`	`11111100 01001001`
`y >> 8`	-59.4257813	-60	`FF C4`	`11111111 11000100`

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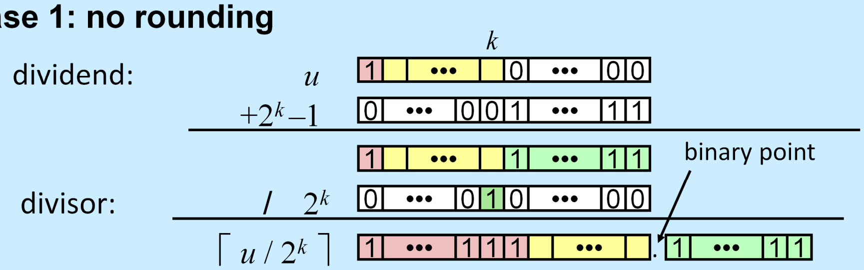
Correct Power-of-2 Divide

- **Quotient of negative number by power of 2**
  - **want $\lceil$ `x / 2`^{k} $\rceil$ (round toward 0)**
  - **compute as $\lfloor$ `(x+2`^{k}`-1)/ 2`^{k} $\rfloor$**
    - » **in C: `(x + (1<<k)-1) >> k`**
    - » **biases dividend toward 0**

Case 1: no rounding



Biasing has no effect

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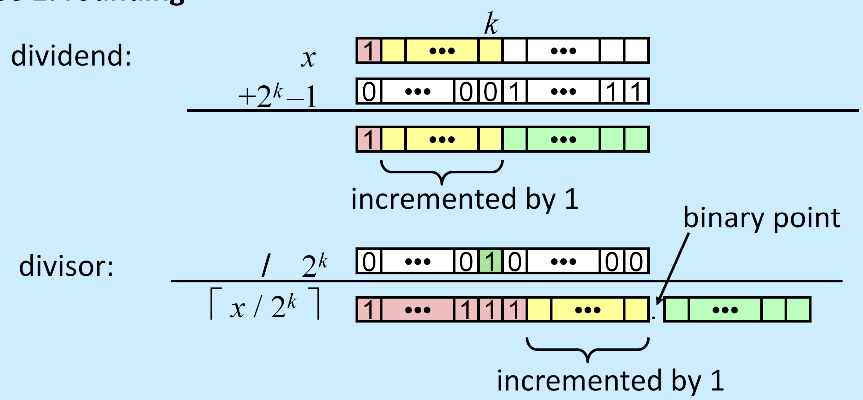
Correct Power-of-2 Divide (Cont.)

Case 2: rounding



Biasing adds 1 to final result

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Why Should I Use Unsigned?

- ***Don't*** **use just because number nonnegative**
  - **easy to make mistakes**

```
unsigned i;
for (i = cnt-2; i >= 0; i--)
  a[i] += a[i+1];
```

  - **can be very subtle**

```
#define DELTA sizeof(int)
int i;
for (i = CNT; i-DELTA >= 0; i-= DELTA)
  . . .
```

- ***Do*** **use when performing modular arithmetic**
  - **multiprecision arithmetic**
- ***Do*** **use when using bits to represent sets**
  - **logical right shift, no sign extension**

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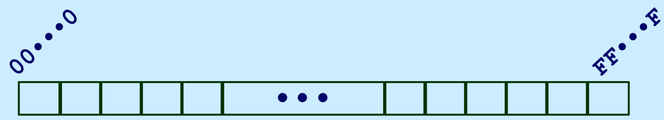
(Recall that, when mixing signed and unsigned items in an expression, the result will be unsigned.)

Byte-Oriented Memory Organization



- **Programs refer to data by address**
  - **conceptually, envision it as a very large array of bytes**
    - » in reality, it's not, but can think of it that way
  - **an address is like an index into that array**
    - » and, a pointer variable stores an address

- **Note: system provides private address spaces to each "process"**
  - **think of a process as a program being executed**
  - **so, a program can clobber its own data, but not that of others**

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Machine Words

- Any given computer has a "word size"
  - – nominal size of integer-valued data
    - » and of addresses
  - – until recently, most machines used 32 bits (4 bytes) as word size
    - » limits addresses to 4GB (2^{32} bytes)
    - » become too small for memory-intensive applications
      - • leading to emergence of computers with 64-bit word size
  - – machines still support multiple data formats
    - » fractions or multiples of word size
    - » always integral number of bytes

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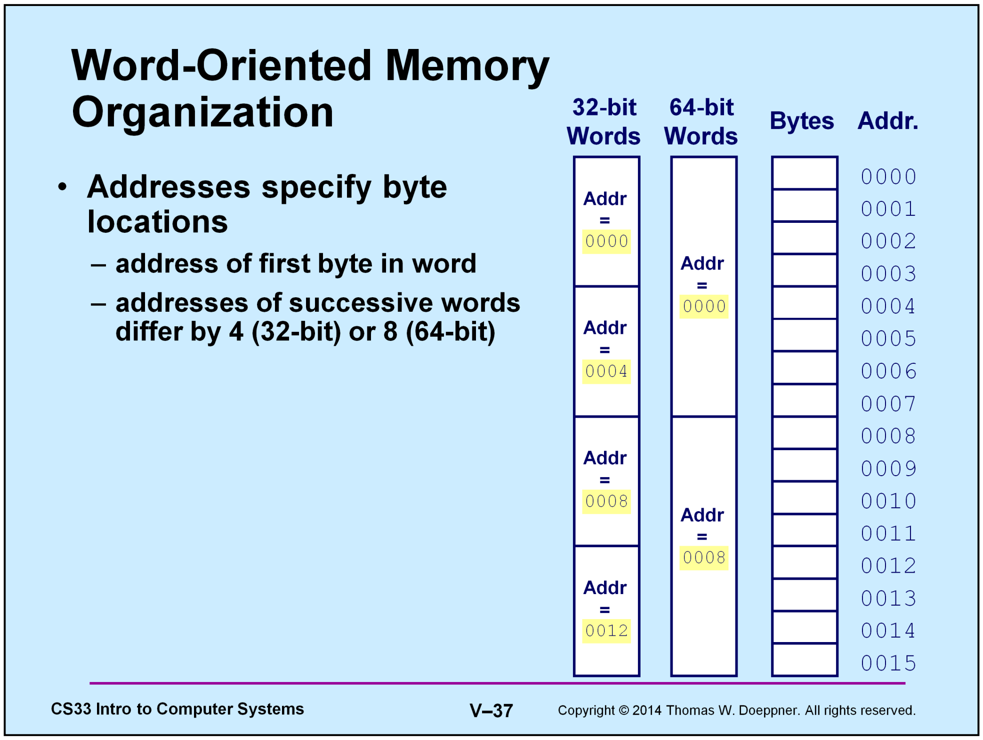

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Byte Ordering

- Four-byte integer
  - 0x01234567
- Stored at location 0x100
  - which byte is at 0x100?
  - which byte is at 0x103?

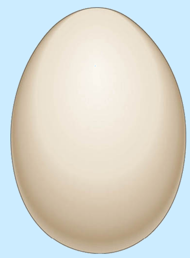
?

67	45	23	01
0x100	0x101	0x102	0x103

Little-endian

01	23	45	67
0x100	0x101	0x102	0x103

Big-endian



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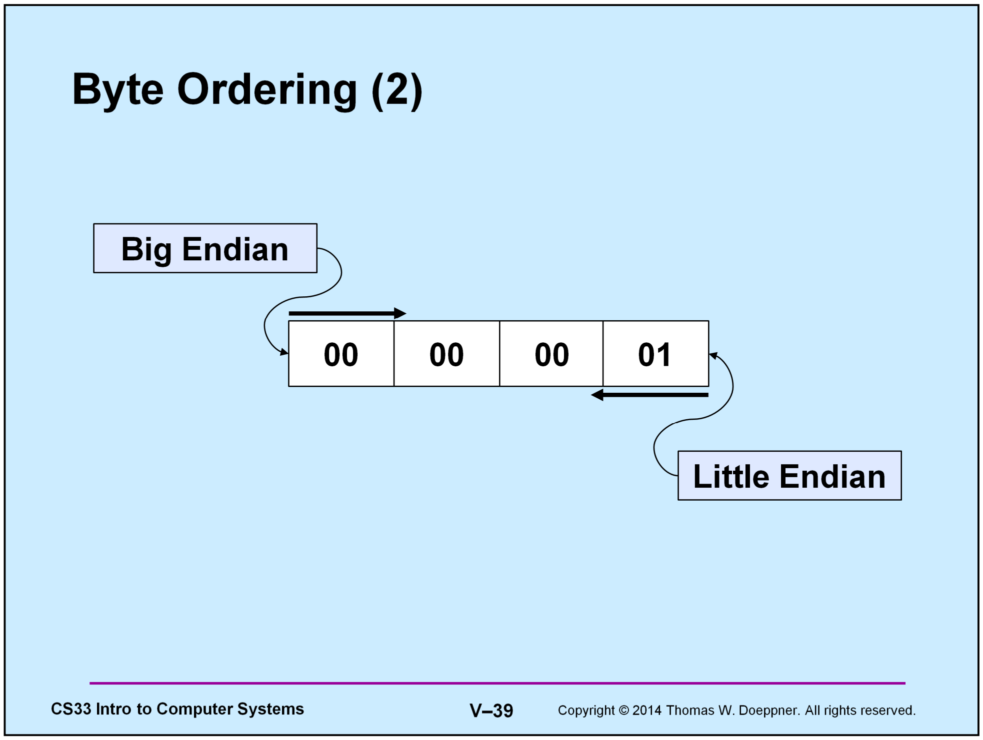
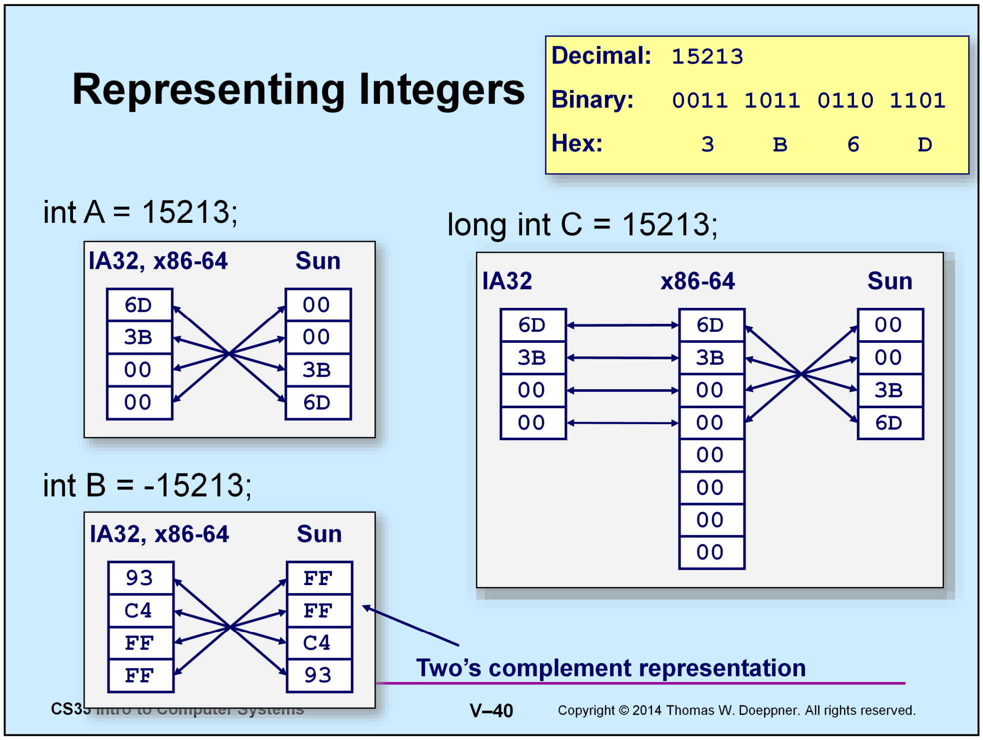


Here we have a four-byte integer one. In the big-endian representation, the address of the integer is the address of the byte containing its most-significant bits (the big end), while in the little-endian representation, the address of the integer is the address of the byte containing its least-significant bits (the little end). Suppose we pass a pointer to this integer to some procedure. However, in a type-mismatch, the procedure assumes that what is passed it is a two-byte integer. On a big-endian system, it would think it was passed a zero, but on a little-endian system, it would think it was passed a one.

This is not an argument in favor of either approach, but simply an observation that behaviors could be different.



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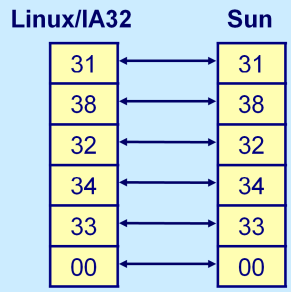
Representing Strings

```
char S[6] = "18243";
```

- **Strings in C**
  - represented by array of characters
  - each character encoded in ASCII format
    - » standard 7-bit encoding of character set
    - » character “0” has code 0x30
      - digit *i* has code 0x30+*i*
  - string should be null-terminated
    - » final character = 0
- **Compatibility**
  - byte ordering not an issue



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