Math 214 – Multivariable Calculus
Taylor Polynomials

George Peschke

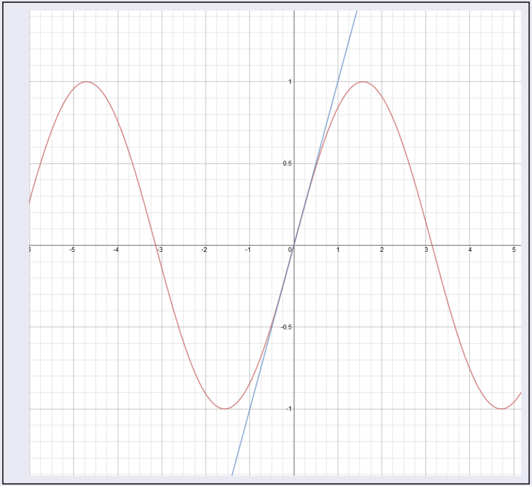
Department of Mathematical and Statistical Sciences
University of Alberta

Lecture 9

Recall about the meaning of the derivative

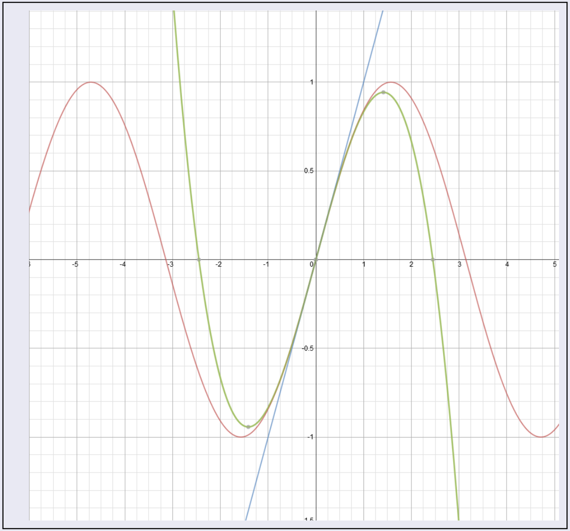
If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at c, its linear approximation is given by

$$L(x) := f(c) + f'(c)(x - c) \qquad \text{uses first derivative}$$

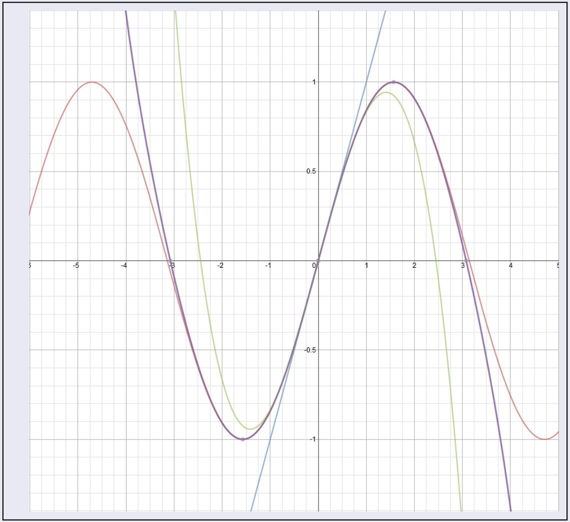


Improvement of approximation via cubic polynomial: uses 1st, 2nd, 3rd derivative

$$Q(x) := f(c) + f'(c)(x-c) + f''(c)\frac{(x-c)^2}{1\cdot 2} + f'''(c)\frac{(x-c)^3}{1\cdot 2\cdot 3}$$



Further improvement of approximation via higher order polynomials: order 7



Goal: improved approximations via polynomials of higher degree

Question: If

$$T_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdot + a_n(x-c)^n$$

is to be the best possible n-th degree polynomial approximating f near c, how do we determine its coefficients $a_0, \ldots, a_n$?

Idea

Reverse engineer: if

$$f(x) = a_0 + a_1(x-c) + \cdots + a_n(x-c)^n$$

is a polynomial of degree n, then its best possible approximation by a polynomial of degree n should be f itself. Now, observe that we can find the coefficients $a_0, a_1, ..., a_n$ by differentiation.

Result

$$a_k = \frac{1}{k!} \cdot f^{(k)}(c)$$

Definition — Taylor polynomial

If a function f is n times differentiable at $c \in \mathbb{R}$, then its n-th order Taylor polynomial at/about c is

$$T_n f(x) := \sum_{i=0}^{n} \frac{f^{(i)}(c)}{i!} \cdot (x-c)^i$$

Example

Find the 3rd order Taylor polynomial of

$$f(x) = x^3 - 2x^2 + 3x + 1$$

about $c = 2$; note $T_3(x) = f(x)$ because f is a polynomial of degree 3.

Example

Find the Taylor polynomials of order 1, 2, and 3 of

$$f(x) = \sin x \qquad \text{about} \quad c = 0.$$

Theorem — using the Taylor polynomial

If f is $(n+1)$-times differentiable on $[c, x]$, then

$$f(x) = T_n f(x) + \frac{f^{(n+1)}(t)}{(n+1)!} \cdot (x-c)^{n+1} \qquad \text{for some} \quad t \in \,]c, x[\,.$$

Recall — Mean Value Theorem

If f is differentiable on $[c, x]$, then

$$f(x) = f(c) + f'(t) \cdot (x-c) \qquad \text{for some} \quad t \in \,]c, x[\,.$$

Taylor's theorem, with $n = 0$, is the Mean Value Theorem.

Example — Computing the exponential function

Find a polynomial $p(x)$ satisfying

$$p(x) \approx e^x \qquad \text{on} \quad [-1, 1]$$

to an accuracy of 5 places.

Next slide

Go on to ‘Taylor series’