Math 214 – Multivariable Calculus
Power Series

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Lecture 8

Definition — Power series

A power series centered at $c \in \mathbb{R}$ is an expression of the form

$$p(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \ ,$$

where $a_0, a_1, \cdots \in \mathbb{R}$ are fixed.

Goal

Use a power series to define a differentiable function. - Obstacle: How about convergence to a number?

Example

Find numbers $x \in \mathbb{R}$ for which the series below converges to a number.

$$p(x) = \sum_{n=0}^{\infty} (x-3)^n$$

Problem

For which $x \in \mathbb{R}$ does

$$p(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

converge to a number? – Find answer using ratio/root tests

Theorem — Ratio/root computation of interval of absolute convergence

The power series $p(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ converges absolutely for a given $x \in \mathbb{R}$ if and only if

1. $$|x-c| < \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| =: R$$
2. $$|x-c| < \lim_{n\to\infty} \frac{1}{\sqrt[n]{|a_n|}} =: R$$

Terminology

If the limit R above exists, it is called the radius of convergence of the p, and $I :=]c-R, c+R[$ is called the interval of absolute convergence of p.

Example Find radius and interval of absolute convergence

$$\sum_{n=1}^{\infty} n(x-2)^n$$

Example Find radius and interval of absolute convergence

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example Find radius and interval of absolute convergence

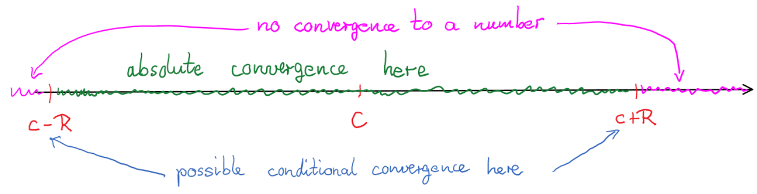
$$\sum_{n=0}^{\infty} n!x^n$$

Example Find radius and interval of absolute convergence

$$\sum_{n=1}^{\infty} \frac{x^n}{n2^n}$$

Info

Suppose $\sum_{n=0}^{\infty} a_n(x-c)^n$ has radius of convergence R, then the following hold:



1. On the open interval $]c-R, c+R[$ the series converges absolutely to a number.
2. At the boundary points $c-R$ and $c+R$ of the interval of absolute convergence, the series may converge conditionally (to a number), it may converge to $+\infty$ or to $-\infty$, or it may diverge. But it does not converge absolutely.
3. Outside of $[c-R, c+R]$ the series does not converge to a number.

Example

We already know that the radius of convergence of the power series below is $R = 1$. Investigate the series for conditional convergence at the boundary points of the interval of convergence.

$$\sum_{n=1}^{\infty} n(x-2)^n$$

Next slide

Go on to ‘Taylor polynomials / Taylor series’