Revision for mid-term test

Investments (Lancaster University)

Risk Aversion and Capital Allocation (w1, ch_5.4-5.8, ch_8)

Expected Return and Standard Deviation

Expected (mean) return:
- n — states
- s — denotes a state
- $p(s)$ — probability of state s
- $R(s)$ — asset's return in state s

$$E(R) = \sum_{s=1}^{n} p(s)R(s)$$

Variance of the return:

$$\sigma_R^2 = \sum_{s=1}^{n} p(s)[R(s) - E(R)]^2$$

The standard deviation of the rate of return (σ) is a measure of risk. It is defined as the square root of the variance, which in turn is the expected value of the squared deviations from the expected return. The higher the volatility in outcomes, the higher will be the average value of these squared deviations. Therefore, variance and standard deviation provide one measure of the uncertainty of outcomes.

Portfolio Construction

Two main steps invloves:
- Selection of risky assets (e.g. stocks and bonds).
- Decision how much to invest in the risky portfolio against the risk-free asset.

- We need to know the expected return and risk of the portfolio to decide how much to allocate between the risk-free asset and the risky portfolio.
- The decision of how much to invest depends on the individual investor's personal risk and expected return preferences.

- A risk averse investor is the one who prefers less variance for the same expected return.
- We assume that all investors are risk averse.

Utility Function

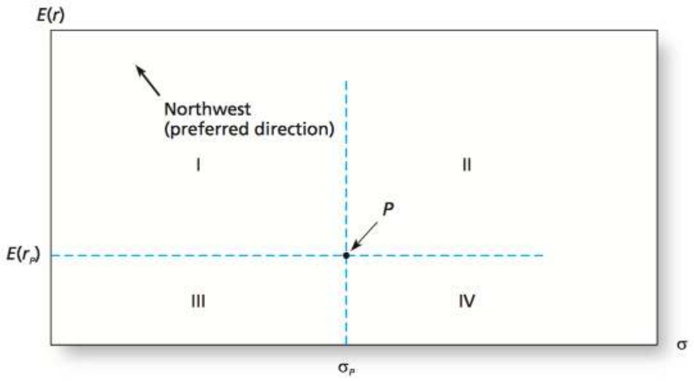
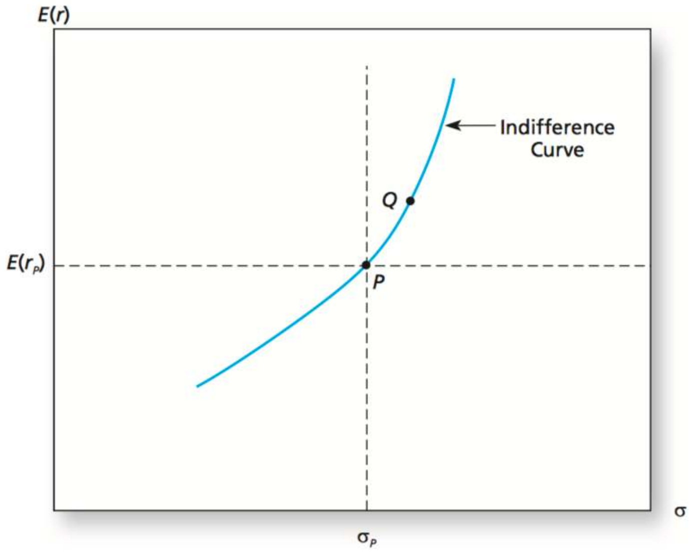
- Measures investor's preferences.
- Because we assume that an investor cares about mean and variance of portfolio returns only, the utility function naturally depends on these two terms.
- The utility function must also reflect risk aversion.

Mean-variance utility function

$$U = E(R) - \frac{1}{2}A\sigma_R^2$$

- U — utility value
- E(R) — expected return of a portfolio/asset
- A — investor's risk aversion coefficient
- σ_R^2 — variance of a portfolio's/asset's return

- ***Risk averse*: A > 0**: σ_R^2 rises, U decreases
- *Risk neutral*: A = 0: utility only depends on E(R)
- *Risk lover*: A < 0: σ_R^2 rises, U rises

Risk-Return trade-off	*Indifference Curve*
Mean-variance (M-V) criterion Portfolio A dominates B if: - $E(R_A) \geq E(R_B)$: higher expected return - $\sigma_A \leq \sigma_B$: lower standard deviation (risk volatility)	• Links all point with the same utility value on a digram. • Is steeper for more risk-averse investors. • Higher (in NW direction) for greater utility level.
	

Capital Allocation across Risky and Risk-Free Portfolios

The most fundamental decision of investing is the allocation of your assets: How much should you own in stock? How much should you own in bonds? How much should you own in cash reserves?

Combining the risk-free asset and a risky portfolio

The expected return and standard deviation of the combined portfolio C:

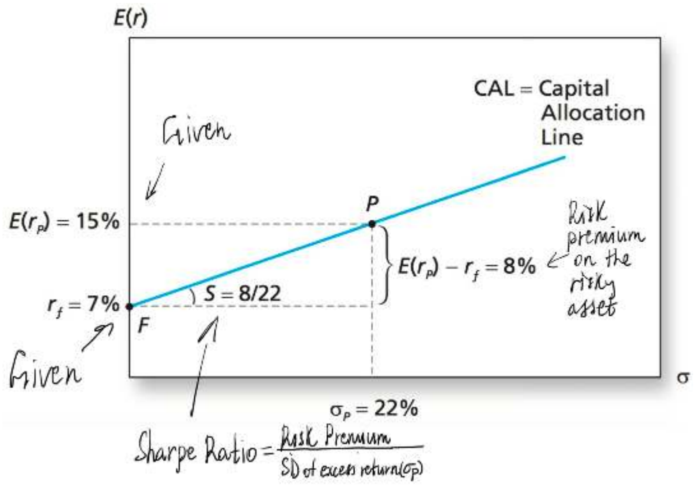
- y — the weigh on the risky portfolio

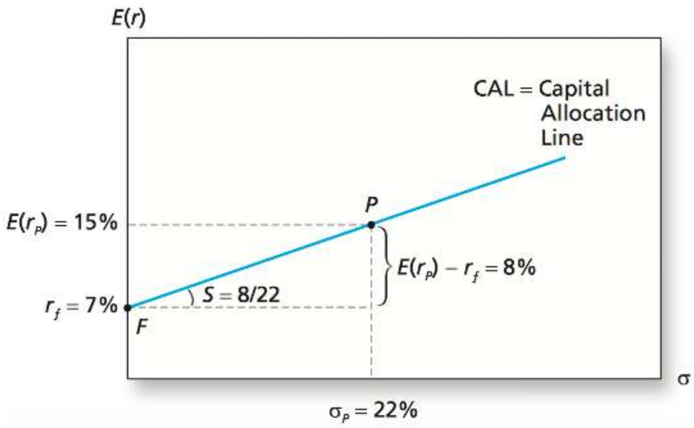
$$E(R_C) = yE(R_p) + (1-y)R_f$$

$$\sigma_C = y\sigma_p$$

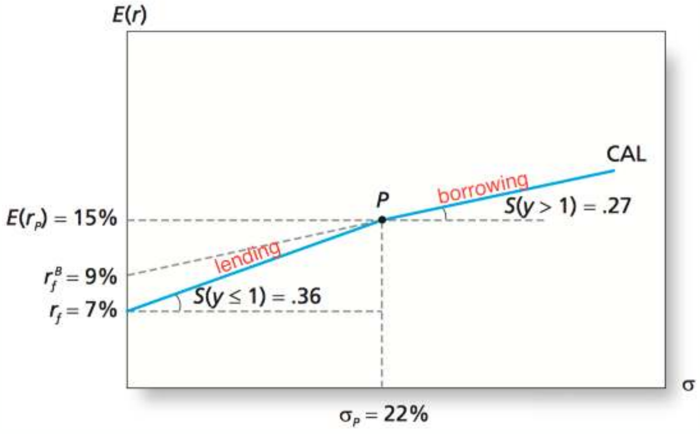
Plotting the rewritten equation will get

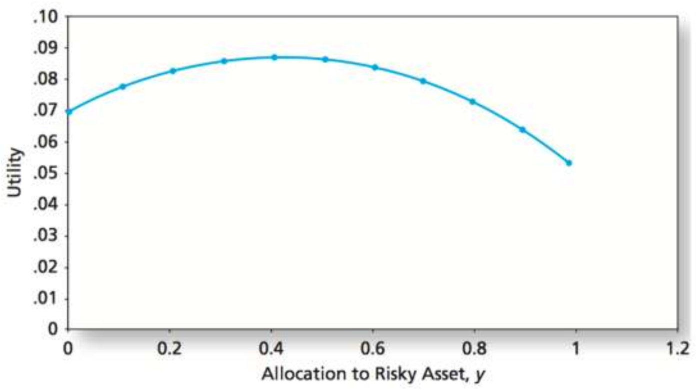
Rewritten: $E(R_C) = R_f + \dfrac{E(R_p) - R_f}{\sigma_p}\sigma_C$





This straight line is called the capital allocation line (CAL). It depicts all the risk-return combinations available to investors. The slope of the CAL, denoted S, equals the increase in the expected return of the complete portfolio per unit of additional standard deviation, in other words, incremental return per incremental risk. For this reason, the slope is called the reward-to-volatility ratio. It also is called the Sharpe ratio.

Combining the risk-free asset and a risky portfolio (cont.)	
The CAL kink when investors borrow to invest in more risky assets (leveraged portfolio).	

Risk Tolerance and Asset Allocation	
Investors choose the allocation to the risky asset, y, that maximises their utility function as given by: $U = E(r) - \frac{1}{2}A\sigma^2$ For a risk aversion A, as the weight y on the risky portfolio increases: • Utility increases up to a certain level but eventually declines - this is because volatility catches up to offset gains in expected returns	
Which combination of the risk-free asset and the risky portfolio gives maximum utility?	$y^* = \frac{E(R_p) - R_f}{A\sigma_p^2}$ For the given level of A, the indifference curve is tangent to the CAL at y^* (optimum portfolio).
1. The solution come from the following maximisation problem:	$\max_y U = E(R_C) - \frac{1}{2}A\sigma_C^2 \Rightarrow$ $\max_y U = yE(R_p) + (1-y)R_f - \frac{1}{2}Ay^2\sigma_p^2$
2. FOC (U w.r.t. y)	$\frac{dU}{dy} = E(R_p) - R_f - Ay\sigma_p^2 = 0$

Passive Strategies: The Capital Market Line (CML)
We call the capital allocation line provided by 1-month T-bills and a broad index of common stocks the capital market line (CML). A passive strategy generates an investment opportunity set that is represented by the CML.

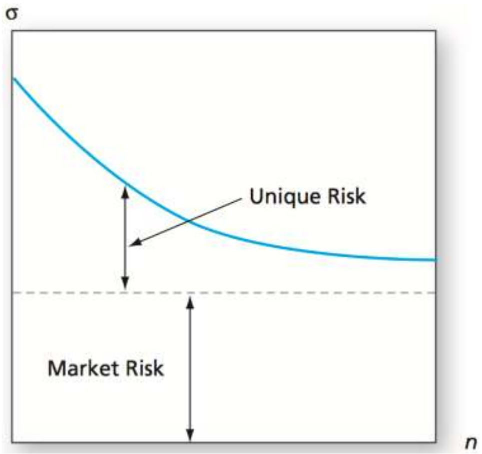
Modern Portfolio Theory (w2, ch_7)

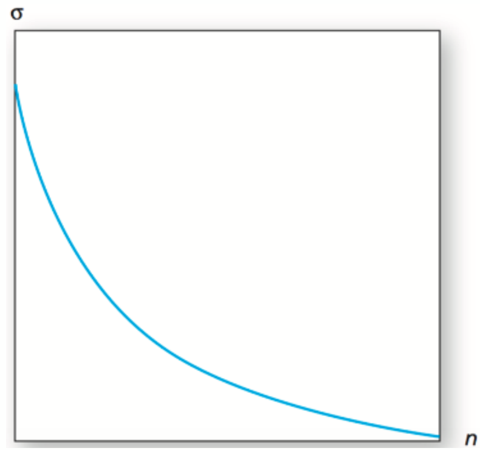
The Investment Decision

1. Capital allocation between the risky portfolio and risk-free assets.
2. The optimal capital allocation depends on risk aversion and the risk-return trade-off of the optimal risky portfolio.
   - Asset allocation across wide asset classes (stocks, international stocks, bonds). How much in cash reserves?
3. Security selection within each asset classes.

Diversification and Portfolio Risk

Market Risk	***Firm-Specific Risk***
- Marketwide risk that remains even after extensive diversification - Systematic/Non-diversifiable	- Can be eliminated by diversification - Diversifiable/Non-systematic - Fully hedged (ε - eliminated)





Portfolios of Two Risky Assets

A proportion denoted by w_D is invested in the bond fund, and the remainder, $1-w_D$, denoted w_E, is invested in the stock fund.

Rate of return: $r_p = w_D r_D + w_E r_E$

Expected return: $E(r_p) = w_D E(r_D) + w_E E(r_E)$

Portfolio variance:

$$\sigma_p^2 = w_D^2\sigma_D^2 + w_E^2\sigma_E^2 + 2w_D w_E Cov(r_D, r_E)$$

$$Cov(r_D, r_E) = \rho_{DE}\sigma_D\sigma_E$$

$$Cov(r_A, r_B) = \sum p\left\{\left[r_A - E(r_A)\right]\left[r_B - E(r_B)\right]\right\}$$

The lowest possible value of the correlation coefficient is -1, representing perfect negative correlation. In this case, the equation simplifies to $\sigma_p^2 = (w_D\sigma_D - w_E\sigma_E)^2$.

This solution uses the minimisation techniques of calculus. Write out the expression for portfolio variance, substitute $1-w_D$ for w_E, differentiate the result with respect to w_D, set the derivative equal to zero, and solve for w_D to obtain:

$$w_{Min}(D) = \frac{\sigma_E^2 - Cov(r_D, r_E)}{\sigma_D^2 + \sigma_E^2 - 2Cov(r_D, r_E)}$$

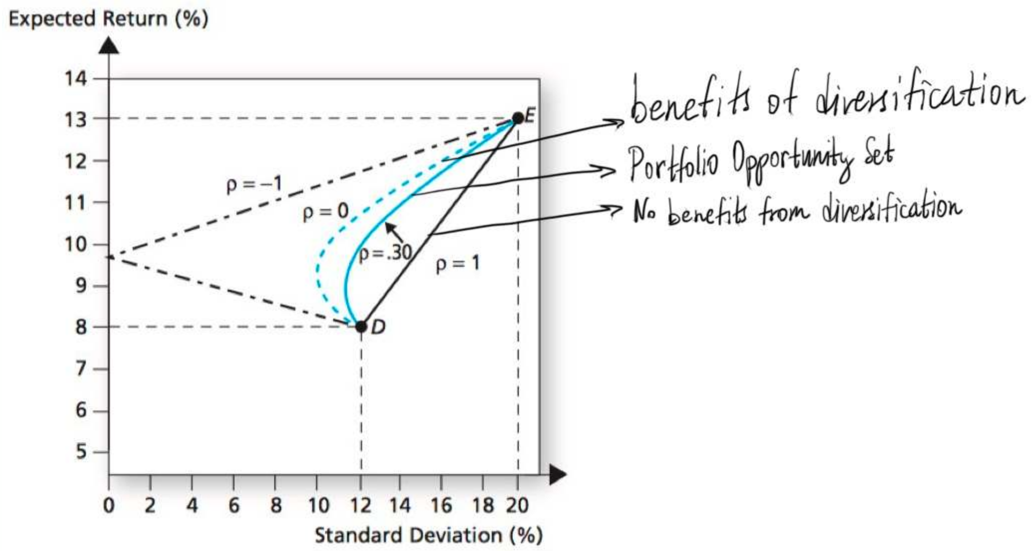
When $\rho = -1$, a perfectly hedged position can be obtained by choosing the portfolio proportions to solve $w_D\sigma_D - w_E\sigma_E = 0$.

The solution to this equation is:

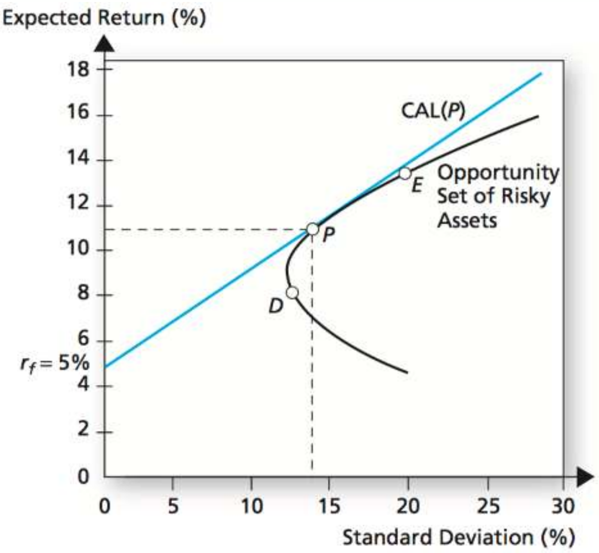
$$w_D = \frac{\sigma_E}{\sigma_D + \sigma_E}$$

$$w_E = \frac{\sigma_D}{\sigma_D + \sigma_E} = 1 - w_D$$

Portfolios of Two Risky Assets (cont.)



Asset Allocation: Risky and Risk-Free Assets



The objective is to find the weights w_D and w_E that result in the highest slope of the capital allocation line (CAL) for the ***optimal risky portfolio***.

$$Max(S_p) = \frac{E(r_p) - r_f}{\sigma_p}$$

$$E(r_p) = w_D E(r_D) + w_E E(r_E)$$

$$\sigma_p = \left[w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E Cov(r_D, r_E) \right]^{\frac{1}{2}}$$

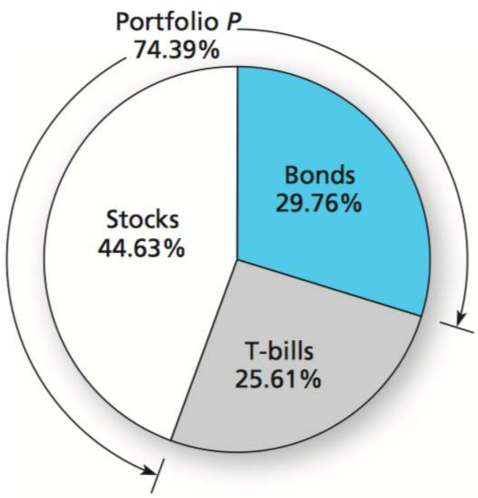
$$w_D = \frac{\left(E(r_D) - r_f\right)\sigma_E^2 - \left(E(r_E) - r_f\right)Cov(r_D, r_E)}{\left(E(r_D) - r_f\right)\sigma_E^2 + \left(E(r_E) - r_f\right)\sigma_D^2 - \left(E(r_D) - r_f + E(r_E) - r_f\right)Cov(r_D, r_E)}$$

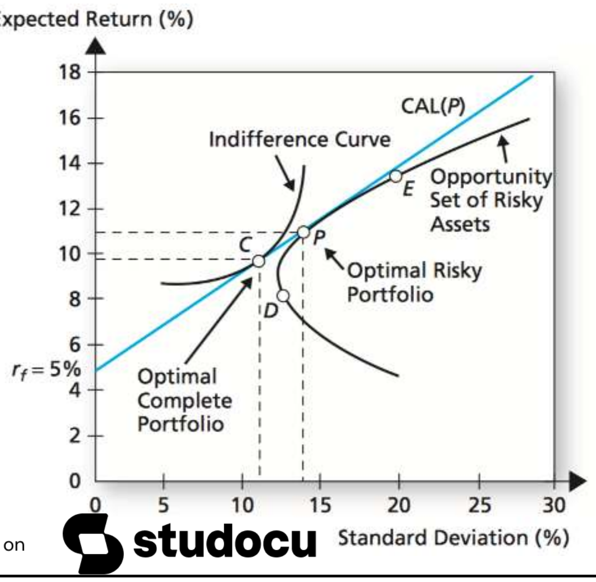
$$w_E = 1 - w_D$$

$$U = E(r) - \frac{1}{2} A\sigma^2$$

$$y = \frac{E(r_p) - r_f}{A\sigma_p^2}$$

Remember, portfolio P consists of stocks and bonds, thus need to further calculate yw_D and yw_E.





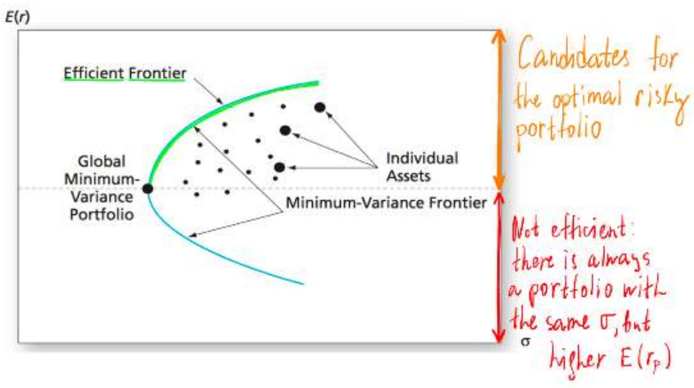
Summary: how to construct the Complete Portfolio

1. Find the return characteristics of all securities.
2. Establish the risky portfolio:
    3. Calculate the optimal risky portfolio
    4. Calculate the expected return and standard deviation of the optimal risky portfolio using the weights obtained from the previous step.
5. Allocate funds between the risky portfolio and the risk-free asset:
    6. Calculate the proportion of the complete portfolio allocated to the optimal risk portfolio and to the risk-free asset.
    7. Calculate the share of the complete portfolio invested in each risky asset and in the risk-free asset.

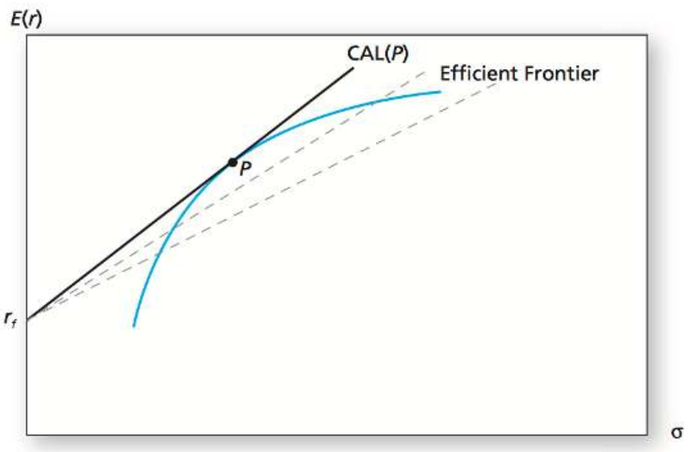
The Markowitz Portfolio Optimisation Model

The minimum-variance frontier is a graph of the lowest possible variance that can be attained for a given portfolio expected return.

The minimum-variance frontier of risky assets

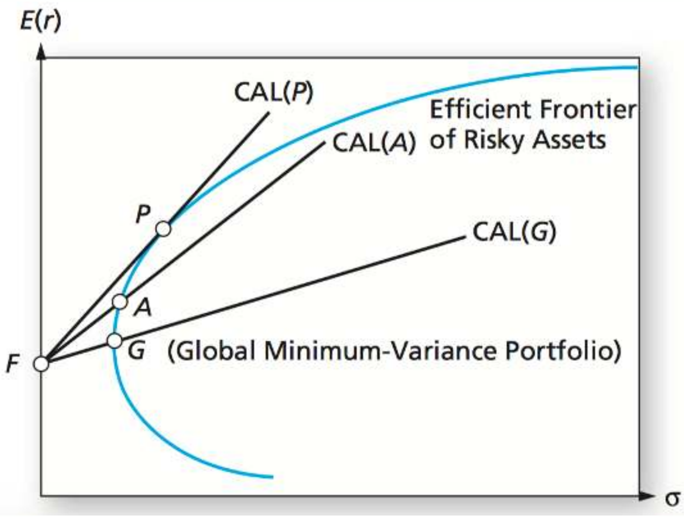


The efficient frontier of risky assets with the optimal CAL



Separation Property

- A portfolio manager will offer the same risky portfolio, P, to all clients regardless of their risk aversion.
- The degree of risk aversion of the client comes into play only in capital allocation, the selection of the desired point along the CAL.
- This result is called a separation property; it tells us that the portfolio choice problem may be separated into two independent tasks:
    1. Determination of the optimal risky portfolio (purely technical).
    2. Capital allocation, depends on personal risk. Here client is the decision maker.



Separation Property Remark

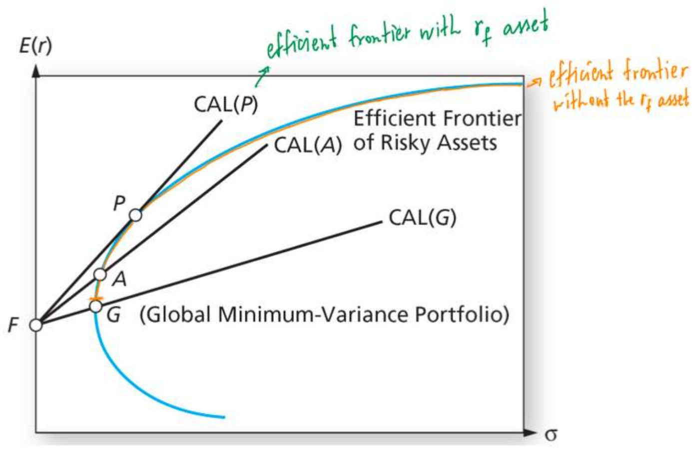
- Without any restrictions on the security selection problem, the portfolio P is the optimal risky portfolio and all investors will choose it.
- However, once we start to introduce restrictions or constraints (for example, taxes), different investors might choose portfolio A as the optimal risky portfolio.

Capital Asset Pricing Model (w3, ch_9)

The Capital Asset Pricing Model (CAPM)

Markowitz Portfolio Selection Model:

- Derives the efficient frontier of risky assets.
- Provides a framework for optimally combining risky assets
  - However, does not provide guidance with respect to the risk-return relationship for individual assets — due to a ***partial*** equilibrium analysis.
    - Considers an individual portfolio optimisation problem.



CAPM

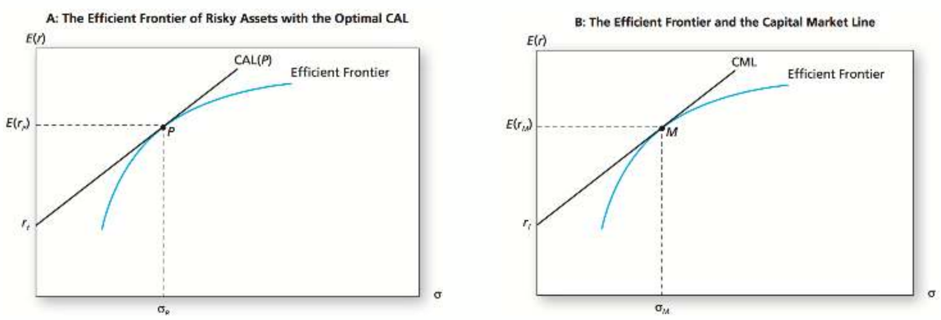
- The CAPM is a ***general*** equilibrium model of expected returns on risky assets — extends the idea of diversification with extra assumptions.
  - Considers many investors.
  - All investors are making optimal decisions.
  - Their optimal decisions must clear markets.

Assumptions

The CAPM assumes that investors:

- Are price-takers.
- Have single period investment horizon.
- Are limited to traded financial assets.
- Bear no tax on return and transaction costs.
- Are rational mean-variance optimisers following.
- Markowitz portfolio selection algorithm.
- Bear no information costs.
- Have homogenous expectations.

- A key insight of the CAPM is this:
  - because the market portfolio is the aggregation of all of these identical risky portfolios, it too will have the same weights. Therefore, if all investors choose the same risky portfolio, it must be the market portfolio, that is, the value-weighted portfolio of all assets in the investable universe. Therefore, the capital allocation line based on each investor's optimal risky portfolio will in fact also be the capital market line. This implication will allow us to say much about the risk–return trade-off.



$$w_i^M = \frac{\textit{market value of asset } i}{\textit{total market value of all assets}}$$

$$y^* = \frac{E(R_M - R_f)}{A\sigma_M^2}\ ; (y^* = 1) \Rightarrow$$

$$E(R_M) - R_f = \overbrace{\underbrace{\bar{A}}_{\textit{price of risk}} \times \underbrace{\sigma_M^2}_{\textit{amount of risk}}}^{\textit{Risk Premium}}$$

- Risk premium of the market portfolio depends on:
  - risk of the market portfolio — σ_M^2
  - risk aversion of market participants — $\bar{A}$

$$\textit{Risk Premium} = \textit{price of risk} \times \textit{amount of risk}$$

Deriving the CAPM

- The investment perspective: the market portfolio is on the efficient frontier and is the tangent point of the optimal CAL with the minimum variance frontier.
  - Since the CML is the optimal CAL, investors can skip the trouble of doing security analysis and obtain an efficient portfolio simply by holding the tangent portfolio.
  - Thus the passive strategy of investing in a market index portfolio seems to be efficient.
    - The remaining issue is the allocation between risk-free asset and the market portfolio, which depends on investors' risk aversion.

Expected Returns on Individual Securities

Risk premium of individual assets depends on its covariance with the market portfolio.

Because market return: $R_M = \sum_{i=1}^{n} w_i R_i$, $\Rightarrow$ $Cov(R_M, R_{GE}) = Cov(R_{GE}, \sum_{i=1}^{n} w_i R_i) = \sum_{i=1}^{n} w_i Cov(R_i, R_{GE})$

GE's ***reward-to-market ratio*** $= \dfrac{\textit{GE contribution to portfolio risk premium}}{\textit{GE contribution to portfolio variance}} = \dfrac{w_{GE}[E(R_{GE}) - R_f]}{w_{GE}Cov(R_{GE}, R_M)} = \dfrac{E(R_{GE}) - R_f}{\underbrace{Cov(R_{GE}, R_M)}_{\textit{undiversifiable risk}}}$

- The part of GE's risk uncorrelated with the market is diversifiable.

- In equilibrium, all assets should offer the same reward-to-risk ratio (incl. the market portfolio). Otherwise, if GE has a higher reward-to-risk ration than the market portfolio, the market portfolio cannon be efficient.
- The beta of a security is the appropriate measure of its risk because beta is proportional to the risk the security contributes to the optimal risky portfolio.
- Only beta can determine the expected return.
  - CML doesn't tell how to determine $E(R)$.
  - SML gives the $E(R)$.

$$\textit{Market reward} - \textit{to} - \textit{risk ratio} = \frac{E(R_M) - R_f}{\sigma_M^2}$$

$$\frac{E(R_{GE}) - R_f}{Cov(R_{GE}, R_M)} = \frac{E(R_M) - R_f}{\sigma_M^2}$$

$$E(R_{GE}) - R_f = \frac{Cov(R_{GE}, R_M)}{\sigma_M^2}[E(R_M) - R_f]$$

$$\boxed{\begin{gathered} E(R_{GE}) - R_f = \beta_{GE}[E(R_M) - R_f] \\ \beta_{GE} = \frac{Cov(R_{GE}, R_M)}{\sigma_M^2} \end{gathered}}$$

Violations of the CAPM

Tests of the CAPM do not support its validity:

- The relation between estimated beta and average historical return is much weaker than the CAPM suggests.
- The market capitalisation of a firm is a predictor of its average historical return even after accounting for beta. (*Size*)
- Stocks with low market-to-book ratios tend to have higher returns than stocks with high market-to-book ratios, again, after controlling for beta. (*Value/growth*).
- Stocks that have performed well over the past 6 months tend to have high expected returns over the following six months. (*Momentum*).
- Firms with high P/E ratio have lower return.
- Stocks with high dividend yield have higher returns.

Limitations of CAPM

- Unrealistic assumptions to establish the model.
- Since the market portfolio is unobservable, it is difficult to empirically test the model (*Roll's critique*).
- Markets may be close to the CAPM equilibrium but not necessarily to be fully in equilibrium.

The Security Market Line (SML)

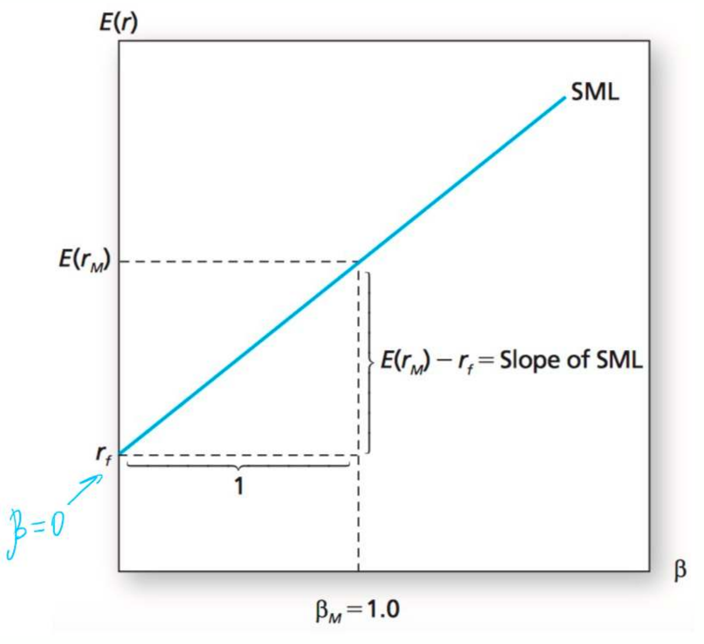
The CAPM relationship: for any asset i,

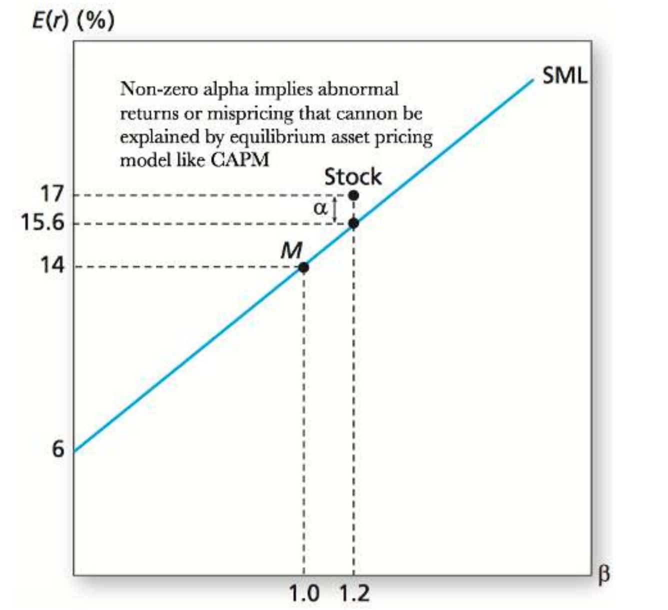
$$\underbrace{E(R_i)-R_f}_{\text{risk premium}} = \underbrace{\beta_i}_{\text{amount of risk}} \underbrace{[E(R_M)-R_f]}_{\text{price of risk}}$$

$$\beta_i = \frac{Cov(R_i, R_M)}{\sigma_M^2}$$

- The security market line depicts the expected return-beta relationship.
  - Slope — the market risk premium.
  - β_i measures/captures stock i's systematic risk and determines the expected return. *(Is the slope coefficient from a regression of the return of a stock in the return of the market).
  - σ_i cannot determine the expected return.

- In equilibrium, all securities should align to the SML.
  - The price adjustment mechanism ensures that all assets in the optimal risky portfolio have appropriate price.
  - Demand and supply cause *under*(over) valued assets to *increase*(decrease) in price and *decrease*(increase) in expected returns and converge toward the SML.





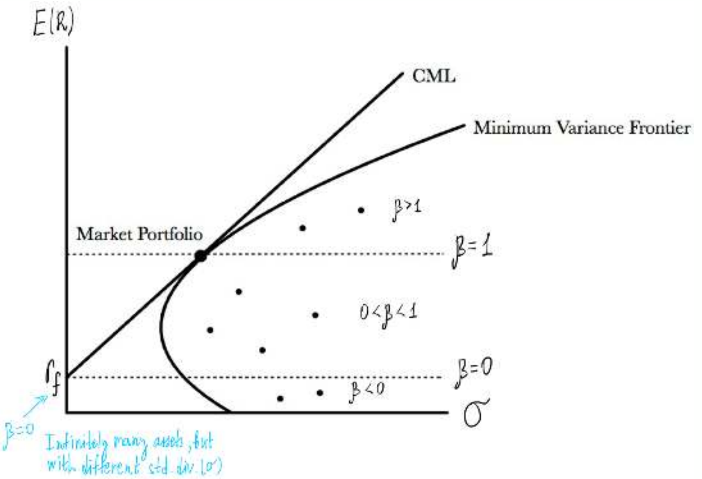
- Investors are compensated for holding systematic risk in the form of higher returns.
- The size of the compensation depends on the market risk premium $E(R_M)-R_f$.

- The market risk premium is increasing in:
  - the market volatility.
  - the degree of risk aversion of the representative investor.

- In order to implement the CAPM, we need to estimate the risk-free return, beta, and the market risk premium.
  - Risk-free return is based on US short-term treasury bills as a proxy.

Underpriced: $E(R_i) > E(R_i)_{CAPM} \Rightarrow a_i > 0$

Overpriced: $E(R_i) < E(R_i)_{CAPM} \Rightarrow a_i < 0$



Index Models (w4, ch_8)

- In this chapter we introduce index models that simplify estimation of the covariance matrix and greatly enhance the analysis of security risk premiums. By allowing us to explicitly decompose risk into systematic and firm-specific components, these models also shed considerable light on both the power and the limits of diversification. Further, they allow us to measure these components of risk for particular securities and portfolios.

Index Models

Practical Drawbacks of the Markowitz Model

- The Markowitz procedure introduced in the preceding chapter suffers from two drawbacks. First, the model requires a huge number of estimates to fill the covariance matrix. Second, the model does not provide any guideline to the forecasting of the security risk premiums that are essential to construct the efficient frontier of risky assets. Because past returns are unreliable guides to expected future returns, this drawback can be telling.

Index Models

- Without specifying the a model, we need to estimate $(N^2 - N)/2$ covariance terms.
- Index models simplify the estimation of covariances, but they cannot really improve the analysis of security risk premiums.

- Although index models are simpler to implement than the Markowitz procedure, they can still be used to construct the efficient frontier and portfolio optimisation.
- The advantages of index models are:
  - Reduces the number of direct inputs.
  - The framework is more convenient for security analysts who specialise in one type of stock.

The Single Factor Model

- With the Markowitz model, a portfolio of $n = 50$ securities requires:
  - 50 (n) estimates of expected return
  - 50 (n) estimates of variances
  - $1225 = 50 \times 49 / 2$ estimates of covariances $n(n-1)/2$

- A single-factor model of the economy classifies sources of uncertainty as systematic (macroeconomic) factors or firm-specific (microeconomic) factors. The index model assumes that the macro factor can be represented by a broad index of stock returns.
- An index model can reduce the number of parameter estimates (by reducing the number of estimates of covariances).

- We can express the rate of return on a security as the sum of its expected and unexpected parts:

$$R_i = E[R_i] + \beta_i m + e_i$$

  - R_i — asset return, random variable.
  - $E[R_i]$ — expected return, constant
  - m — common (systematic) risk factor affecting all assets' returns, random variable, $E[m] = 0$, (captures aggregate risk).
  - β_i — sensitivity coefficient to the common risk factor m, constant
  - e_i — idiosyncratic uncertainty about asset (firm) i, firm-specific risk, random variable, $E[e_i] = 0$.

- Properties of m:
  - A common risk factor affecting all securities.
  - Not correlated with firm-specific risk: $Cov(m, e_i) = 0$ or equivalently $E(me_i) = 0$.

- The variance of R_i is, then:

$$\sigma_i^2 = Var(R_i) = Var(E[R_i] + \beta_i m + e_i) = Var(\beta_i m + e_i)$$
$$= Var(\beta_i m) + Var(e_i) + 2Cov(\beta_i m, e_i) = \underbrace{\beta_i^2 \sigma_m^2}_{systematic} + \underbrace{\sigma^2(e_i)}_{firm-specific}$$

Since all securities are affected by m and firm-specific uncertainty (e_i) is uncorrelated across securities, the covariance between any two securities is:

$$Cov(R_i, R_j) = Cov(E[R_i] + \beta_i m + e_i,\ E[R_j] + \beta_j m + e_j)$$
$$= Cov(\beta_i m + e_i, \beta_j m + e_j)$$
$$= Cov(\beta_i m, \beta_j m) = \beta_i \beta_j \sigma_m^2$$

Assumption: $Cov(m, e_i) = 0$, $Cov(m, e_j) = 0$, $Cov(e_i, e_j) = 0$

Note that: $\rho = \sqrt{R^2}$. And $\text{cov}(r_A, r_M) = \rho \sigma_A \sigma_M$

The Single Index Model

- A reasonable approach to making the single-factor model operational is to assert that the rate of return on a broad index of securities such as the S&P 500 is a valid proxy for the common macroeconomic factor.
- This approach leads to an equation similar to the single factor model, which is called a ***single-index model*** because it uses the market index to proxy for the common factor.

- We can use a broad market index like S&P 500 for m.
- We need to estimate systematic risk and beta
  - Easy, because we have past data for the index.

$$\left.\begin{aligned} R_i^{ex} &= R_i - R_f \\ R_M^{ex} &= R_M - R_f \end{aligned}\right\} \textit{Excess return}$$

$$R_{i,t}^{ex} = \alpha_i + \beta_i R_{M,t}^{ex} + e_{i,t} \Big\} \textit{Put this in our index factor model}$$

- We estimate the above regression, by collecting historical data for R_i^{ex} and R_M^{ex}.
- The intercept is the mispricing of the security (when the market excess return is zero).

- Security's β is the security's sensitivity to the index.
- $e_{i,t}$ is the residual term (also called the firm-specific risk)

- ***The Single Index Model***: $R_{i,t}^{ex} = \alpha_i + \beta_i R_{M,t}^{ex} + e_{i,t}$
- Taking the expected return of the regression equation, we have: $E(R_{i,t}^{ex}) = \alpha_i + \beta_i E(R_{M,t}^{ex})$

- This expression says that part of a security's risk premium is from the risk premium of the index.
- α — the rest of the risk premium is firm-specific.
- Fund managers try to find securities with non-zero alphas.

- Merit of single-index model:
  - the number of parameters to be estimated is greatly reduced:
    - Achieved by assuming residuals across assets are uncorrelated: e_i is not correlated with e_i for $i \neq j$
    - Thus, two assets are correlated only to the extent of their correlations with the market index:
      - this correlation is captured by β_i and β_j.
- Estimation of single-index model: liner regression.
- The single-index model creates a framework that separates two different sources of return variation.

- Unlike the Markowitz model, the single index model requires the following estimates:
  - n estimates of the extra-market expected excess returns, α_i
  - n estimates of beta, β_i
  - n estimates of firm specific variances, $\sigma^2(e_i)$
  - One estimate for the market risk premium, $E(R_{M,t}^{ex})$
  - One estimate for the variance of the market excess return, σ_M^2
- We need $(3n+2)$ estimates in total with the single index model
  - For a 50 security portfolio, this means 152 estimates are needed.

The Single Index Model vs. Makrowitz Model

- The single index model places restrictions on the structure of asset return uncertainty.
- The classification of uncertainty into macro versus firm-risk oversimplifies the real-world uncertainty (i.e., industry-risk is ignored).
- Thus, the optimal portfolio derived from the single-index model might be inferior to the one obtained from the Markowitz model.

Summary: The index model is estimated by applying regression analysis to excess rates of return. The slope of the regression curve is the beta of an asset, whereas the intercept is the asset's alpha during the sample period. The regression line is also called the security characteristic line.

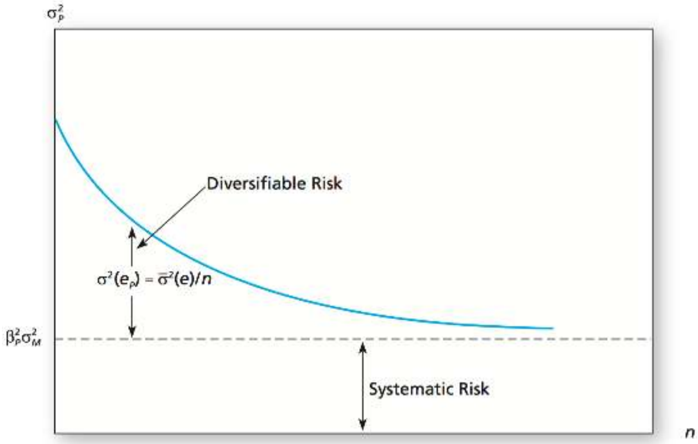
The Single Index Model and Diversification

- As in the Markowitz model, investors can also achieve diversification with the single index model.
- However, similar to the Markowitz model, firm-specific risk can be eliminated whereas market risk (systematic risk) remains.
- Consider an equally-weighted portfolio with a weight of $1/n$ on each asset:

$$R_p = \frac{1}{n}\sum_{i=1}^{n}\alpha_i + \left(\frac{1}{n}\sum_{i=1}^{n}\beta_i\right) + \frac{1}{n}\sum_{i=1}^{n}e_i$$

- $\beta_P = \frac{1}{n}\sum_{i=1}^{n}\beta_i$ — portfolio's sensitivity.
- $\alpha_P = \frac{1}{n}\sum_{i=1}^{n}\alpha_i$ — a non-market return component.
- $e_P = \frac{1}{n}\sum_{i=1}^{n}e_i$ — zero mean variable, which is the average of the firm-specific components.

- Hence, the portfolio's variance is: $\sigma_P^2 = \beta_P^2\sigma_M^2 + \sigma^2(e_P)$.
  - The systematic risk component of the portfolio variance, which we defined as the component that depends on market wide movements, is $\beta_P^2\sigma_M^2$ and depends on the sensitivity coefficients of the individual securities. This part of the risk depends on portfolio beta and σ_M^2 and will persist regardless of the extent of portfolio diversification. No matter how many stocks are held, their common exposure to the market will be reflected in portfolio systematic risk.



CAPM in Practice: The Single Index Model

- The CAPM predicts relationships among expected return.
  - We are interested in testing the theoretical asset pricing equation.
- An index model uses historically realises returns.
  - We are preparing the inputs list for portfolio analysis.
- The CAMP relies on the unobserved market portfolio, which includes all assets (real estate, human capital, etc).
  - Empirically testing the CAPM is very difficult because of *unobserved* market portfolio.
  - In reality we don't have an index which covers all stocks on the market.
  - Don't have a perfect proxy of market portfolio.
- An index model uses actual index portfolios, such as the S&P 500 index to represent systematic factors.

- The CAPM states:

$$E(R_i) - R_f = \beta_i[E(R_M) - R_f]$$

- The single-index model states:

$$E(R_i) - R_f = \alpha_i + \beta_i[R_M - R_f] + e_i$$

Example

- Estimate the regression equation for HP:

$$R_{HP}^{ex} = \alpha_{HP} + \beta_{HP}R_{SP500}^{ex} + e_{HP}$$

Regression Statistics	
Multiple *R*	.7238
R-square	.5239
Adjusted *R*-square	.5157
Standard error	.0767
Observations	60

52% of the variation in the HP is explained by the variation in the S&P500 excess return. =0.3752/(0.3752+0.3410)

Monthly standard deviation of HP's residuals.

ANOVA

	df	SS	MS
Regression	1	.3752	.3752
Residual	58	.3410	.0059
Total	59	.7162	

Portion of the variance of the HP's excess return explained by the S&P500 excess return.

	Coefficients	Standard Error	*t*-Stat	*p*-Value
Intercept (Alpha)	0.0086	.0099	0.8719	.3868
S&P 500 (Beta)	2.0348	.2547	7.9888	.0000

If t<2, a=0
If t>2, significant

Regression Refresher (Handout)

- Alpha — a model meant to fit the asset with its x's is usually designed.
  - Implying an alpha of zero.
- Beta — for the market index, most stocks should have a beta between roughly 0.5 and 2.
  - The regression p-value on beta, however, is based on the null hypothesis that beta=0.
  - Standard Errors — small variance and sample size drive the standard errors.
- R_i^2 — $= \dfrac{\text{variance in } y \text{ explained by } x\text{'s}}{\text{variance in } y} = \dfrac{\beta_i^2\sigma_M^2}{\sigma_i^2}$

Active and Passive Strategies

Optimal active portfolios constructed from the index model include analysed securities in proportion to their information ratios. The full risky portfolio is a mixture of the active portfolio and the passive market index portfolio. The index portfolio is used to enhance the diversification of the overall risky position.

- An '***active***' strategy — tries to beat the market by stock picking, or other methods.
- But CAPM implies that:
  - Security analysis is unnecessary, i.e. every investor should just buy a mix of the risk-free security and the market portfolio.
- Markets aren't perfectly efficient - temporary mispricing of securities can occur.

Stock Selection

- The CAPM predicts that $E(R_i) - R_f = \beta_i[E(R_M) - R_f]$.
- Security analysts compare their individual return expectations with the fair expected return from the CAPM model.
- A stock's alpha denotes the abnormal expected return of a security in excess of the expected return predicted by the CAPM model.
- Stocks with high alphas are undervalued and should be acquired in a portfolio.
  - A security's alpha is defined as: $\alpha_i = E(R_i) - R_f - \beta_i[E(R_M) - R_f]$
- Some fund managers try to buy positive-alpha stocks and (short) sell negative-alpha stocks.
- The CAPM predicts that all alphas are zero.

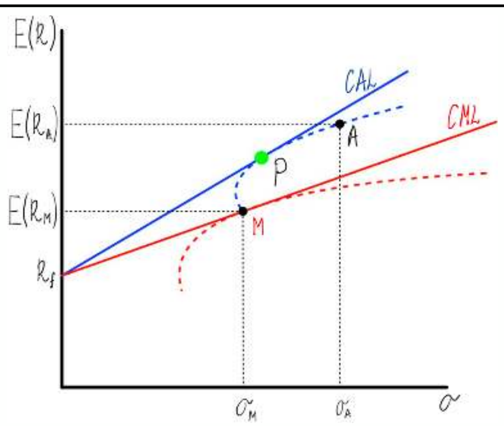
Active Investment

- A general hierarchical analysis:

1. Macroeconomic analysis — estimate the market risk premium and market variance.
2. Regressions — estimate the α's, β's and residual variances $\sigma(e_i)$ for all securities.
3. Compute expected return of securities implied by the CAPM ($\beta_i E[R_M]$) (absent any contribution from security analysis) — this market-driven expected return provides a benchmark.
4. What really makes a security attractive or unattractive to a portfolio manager is its alpha value.

- The market index might be included in a portfolio as an additional asset to avoid inadequate diversification.
- In such a portfolio the objective is to maximise the Sharpe ratio by using the portfolio weights.
- Such a portfolio is a departure from Markowitz's efficient diversification.
- However, the portfolio might be attractive to an investor if it has securities with non-zero alphas.
- Thus, the optimal risky portfolio is composed of:
  - An active portfolio, and
  - The market index portfolio (passive portfolio).

- The active portfolio of misplaced securities
  - Advantage: yields higher return
  - Disadvantage: less diversified than the passive portfolio
  - Solution: combine active (A) and passive (M) portfolios into optimal risky portfolio (P).
- Inputs to construct combines optimal risky portfolio (P):
  - Risk premium and standard deviation of market index
  - n sets of security-specific estimates of betas, residual variances, alphas.



- Active portfolio A is not diversified enough as it focus only on mispriced securities.
  - Combined portfolio P achieves optimal diversification.

Active Investment (cont.)

- Constructing a risky portfolio with $n+1$ securities:
  - The first n securities are analysed using the single index model
  - The $n+1^{th}$ security is the market index
- The portfolio can be written as:

$$\begin{aligned} R_P^{ex} &= w_1 R_1^{ex} + \ldots + w_{n+1} R_{n+!}^{ex} \\ &= w_1(\alpha_1 + \beta_1 R_M^{ex} + e_1) + \ldots + w_{n+1}(\alpha_{n+1} + \beta_{n+1} R_M^{ex} + e_{n+1}) \\ &= \sum_{i=1}^{n+1} w_i(\alpha_i + \beta_i R_M^{ex} + e_i) \end{aligned}$$

  - Where the second equality holds by plugging the single index model and $\sum_{i=1}^{n+1} w_i = 100\%$

$$R_P^{ex} = \sum_{i=1}^{n+1} w_i(\alpha_i + \beta_i R_M^{ex} + e_i) = \alpha_P + \beta_P R_M^{ex} + e_P$$

- Portfolio alpha: $\alpha_P = \sum_{i=1}^{n+1} w_i \alpha_i$
- Portfolio beta: $\beta_P = \sum_{i=1}^{n+1} w_i \beta_i$
- Portfolio residual variance: $\sigma^2(e_P) = \sum_{i=1}^{n+1} w_i^2 \sigma^2(e_i)$

  - Also note:
- Market index has zero alpha: $\alpha_{n+1} = \alpha_M = 0$
- Market index has $\beta_{n+1} = \beta_M = 1$
- Market index has zero residual variance: $\sigma^2(e_{n+1}) = \sigma^2(e_M) = 0$

Expected excess return on the portfolio: $E(R_P^{ex}) = \alpha_P + \beta_P E(R_M^{ex}) = \sum_{i=1}^{n+1} w_i \alpha_i + E(R_M^{ex}) \sum_{i=1}^{n+1} w_i \beta_i$

Variance of the portfolio: $\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sigma^2(e_P) = \sigma_M^2 \left(\sum_{i=1}^{n+1} w_i \beta_i \right)^2 + \sum_{i=1}^{n+1} w_i^2 \sigma^2(e_i)$

The objective: maximise the Sharpe ratio of this portfolio: $\max_{w_1, w_2, \ldots, w_{n+1}} \frac{E(R_P^{ex})}{\sigma_P}$

Weeks 1 and 2

Expected (mean) return:

$$E(R)=\sum_{s=1}^{n} p(s)R(s)$$

- n — states
- s — denotes a state
- $p(s)$ — probability of state s
- $R(s)$ — asset's return in state s

Variance of the return:

$$\sigma_R^2=\sum_{s=1}^{n} p(s)[R(s)-E(R)]^2$$

Mean-variance utility function:

$$U=E(R)-\frac{1}{2}A\sigma_R^2$$

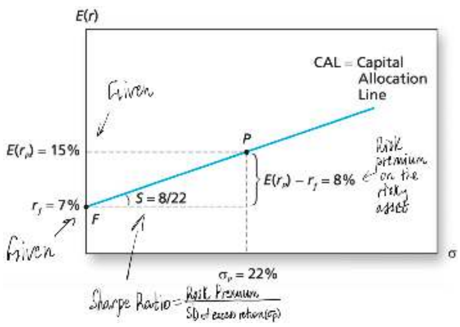
- U — utility value
- A — investor's risk aversion coefficient
- Risk averse: A > 0

Combining the risk-free asset and a risky portfolio:

$$E(R_C)=yE(R_p)+(1-y)R_f$$

$$\sigma_C=y\sigma_p$$

$$E(R_C)=R_f+\frac{E(R_p)-R_f}{\sigma_p}\sigma_C$$



Optimum portfolio:

$$y^*=\frac{E(R_p)-R_f}{A\sigma_p^2}$$

$$\max_y U=E(R_C)-\frac{1}{2}A\sigma_C^2 \Rightarrow$$

$$\max_y U=yE(R_p)+(1-y)R_f-\frac{1}{2}Ay^2\sigma_p^2$$

$$\frac{dU}{dy}=E(R_p)-R_f-Ay\sigma_p^2=0$$

Portfolio of two risky assets:

Rate of return: $r_p=w_Dr_D+w_Er_E$

Expected: $E(r_p)=w_DE(r_D)+w_EE(r_E)$

Portfolio variance:

$$\sigma_p^2=w_D^2\sigma_D^2+w_E^2\sigma_E^2+2w_Dw_ECov(r_D,r_E)$$

$$Cov(r_D,r_E)=\rho_{DE}\sigma_D\sigma_E$$

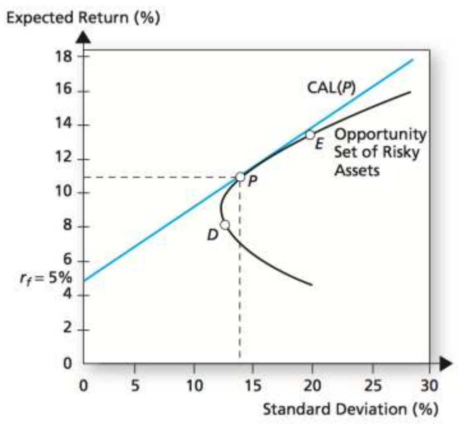
$$Cov(r_A,r_B)=\sum p\{[r_A-E(r_A)][r_B-E(r_B)]\}$$

$$w_{Min}(D)=\frac{\sigma_E^2-Cov(r_D,r_E)}{\sigma_D^2+\sigma_E^2-2Cov(r_D,r_E)}$$

$$w_D=\frac{\sigma_E}{\sigma_D+\sigma_E}$$

$$w_E=\frac{\sigma_D}{\sigma_D+\sigma_E}=1-w_D$$

Asset Allocation:
Risky and Risk-Free Assets:
(optimal risk portfolio)



$$Max(S_p)=\frac{E(r_p)-r_f}{\sigma_p}$$

$$E(r_p)=w_DE(r_D)+w_EE(r_E)$$

$$\sigma_p=\left[w_D^2\sigma_D^2+w_E^2\sigma_E^2+2w_Dw_ECov(r_D,r_E)\right]^{\frac{1}{2}}$$

$$w_D=\frac{\left(E(r_D)-r_f\right)\sigma_E^2-\left(E(r_E)-r_f\right)Cov(r_D,r_E)}{\left(E(r_D)-r_f\right)\sigma_E^2+\left(E(r_E)-r_f\right)\sigma_D^2-\left(E(r_D)-r_f+E(r_E)-r_f\right)Cov(r_D,r_E)}$$

$$w_E=1-w_D$$

$$U=E(r)-\frac{1}{2}A\sigma^2$$

$$y=\frac{E(r_p)-r_f}{A\sigma_p^2}$$

Remember, portfolio P consists of stocks and bonds, thus need to further calculate yw_D and yw_E.

Week 3

$$y^* = \frac{E(R_M - R_f)}{A\sigma_M^2}\ ;\ (y^* = 1) \Rightarrow$$

$$E(R_M) - R_f = \overbrace{\underbrace{\bar{A}}_{\text{price of risk}} \times \underbrace{\sigma_M^2}_{\text{amount of risk}}}^{\text{Risk Premium}}$$

$$w_i^M = \frac{\text{market value of asset } i}{\text{total market value of all assets}}$$

- Risk premium of the market portfolio depends on:
  - risk of the market portfolio — σ_M^2
  - risk aversion of market participants — $\bar{A}$

$$\text{Risk Premium} = \text{price of risk} \times \text{amount of risk}$$

Expected Returns on Individual Securities:

$$R_M = \sum_{i=1}^{n} w_i R_i\ ,\ \Rightarrow\ Cov(R_M, R_{GE}) = Cov(R_{GE}, \sum_{i=1}^{n} w_i R_i) = \sum_{i=1}^{n} w_i Cov(R_i, R_{GE})$$

GE's ***reward-to-market ratio***

$$= \frac{\text{GE contribution to portfolio risk premium}}{\text{GE contribution to portfolio variance}} = \frac{w_{GE}[E(R_{GE}) - R_f]}{w_{GE} Cov(R_{GE}, R_M)} = \frac{E(R_{GE}) - R_f}{\underbrace{Cov(R_{GE}, R_M)}_{\text{undiversifiable risk}}}$$

$$\text{Market reward-to-risk ratio} = \frac{E(R_M) - R_f}{\sigma_M^2}$$

$$\frac{E(R_{GE}) - R_f}{Cov(R_{GE}, R_M)} = \frac{E(R_M) - R_f}{\sigma_M^2}$$

$$E(R_{GE}) - R_f = \frac{Cov(R_{GE}, R_M)}{\sigma_M^2}[E(R_M) - R_f]$$

$$\boxed{E(R_{GE}) - R_f = \beta_{GE}[E(R_M) - R_f] \qquad \beta_{GE} = \frac{Cov(R_{GE}, R_M)}{\sigma_M^2}}$$

The Security Market Line (SML):

The CAPM relationship: for any asset i,

$$\overbrace{E(R_i) - R_f}^{\text{risk premium}} = \overbrace{\beta_i}^{\text{amount of risk}}\ \overbrace{[E(R_M) - R_f]}^{\text{price of risk}}$$

$$\beta_i = \frac{Cov(R_i, R_M)}{\sigma_M^2}$$

- The security market line depicts the expected return-beta relationship:

- Slope — the market risk premium.
- β_i measures/captures stock i's systematic risk and determines the expected return. *(Is the slope coefficient from a regression of the return of a stock in the return of the market).
- σ_i cannot determine the expected return.

Underpriced:

$$E(R_i) > E(R_i)_{CAPM} \Rightarrow a_i > 0$$

Overpriced:

$$E(R_i) < E(R_i)_{CAPM} \Rightarrow a_i < 0$$

Week 4

Single Factor Model:

$$R_i = E[R_i] + \beta_i m + e_i$$

- R_i — asset return, random variable.
- $E[R_i]$ — expected return, constant
- m — common (systematic) risk factor affecting all assets' returns, random variable, $E[m]=0$, (captures aggregate risk).
- β_i — sensitivity coefficient to the common risk factor m, constant
- e_i — idiosyncratic uncertainty about asset (firm) i, firm-specific risk, random variable, $E[e_i]=0$.

$$\sigma_i^2 = Var(R_i) = Var(E[R_i] + \beta_i m + e_i) = Var(\beta_i m + e_i)$$
$$= Var(\beta_i m) + Var(e_i) + 2Cov(\beta_i m, e_i) = \underbrace{\beta_i^2 \sigma_m^2}_{systematic} + \underbrace{\sigma^2(e_i)}_{firm-specific}$$

$$Cov(R_i, R_j) = Cov(E[R_i] + \beta_i m + e_i,\ E[R_j] + \beta_j m + e_j)$$
$$= Cov(\beta_i m + e_i, \beta_j m + e_j)$$
$$= Cov(\beta_i m, \beta_j m) = \beta_i \beta_j \sigma_m^2$$

Note that:

$$\rho = \sqrt{R^2}$$

And:

$$\text{cov}(r_A, r_M) = \rho \sigma_A \sigma_M$$

Single Index Model:

- $R_{i,t}^{ex} = \alpha_i + \beta_i R_{M,t}^{ex} + e_{i,t}$
- Taking the expected return of the regression equation, we have:
  $E(R_{i,t}^{ex}) = \alpha_i + \beta_i E(R_{M,t}^{ex})$
- The CAPM states:
  $E(R_i) - R_f = \beta_i [E(R_M) - R_f]$
- The single-index model states:
  $E(R_i) - R_f = \alpha_i + \beta_i [R_M - R_f] + e_i$

The Single Index Model and Diversification:

$$R_p = \frac{1}{n}\sum_{i=1}^{n} \alpha_i + (\frac{1}{n}\sum_{i=1}^{n} \beta_i) + \frac{1}{n}\sum_{i=1}^{n} e_i$$

-
- portfolio's variance is:
  $\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sigma^2(e_P)$
- $R_i^2 = \dfrac{\beta_i^2 \sigma_M^2}{\sigma_i^2}$

- $\beta_P = \frac{1}{n}\sum_{i=1}^{n} \beta_i$ — portfolio's sensitivity.
- $\alpha_P = \frac{1}{n}\sum_{i=1}^{n} \alpha_i$ — a non-market return component.
- $e_P = \frac{1}{n}\sum_{i=1}^{n} e_i$ — zero mean variable, which is the average of the firm-specific components.

Active investment:

- The portfolio can be written as:

$$R_P^{ex} = w_1 R_1^{ex} + \ldots + w_{n+1} R_{n+!}^{ex}$$
$$= w_1(\alpha_1 + \beta_1 R_M^{ex} + e_1) + \ldots + w_{n+1}(\alpha_{n+1} + \beta_{n+1} R_M^{ex} + e_{n+1})$$
$$= \sum_{i=1}^{n+1} w_i(\alpha_i + \beta_i R_M^{ex} + e_i)$$

- Where the second equality holds by plugging the single index model and $\sum_{i=1}^{n+1} w_i = 100\%$

$$R_P^{ex} = \sum_{i=1}^{n+1} w_i(\alpha_i + \beta_i R_M^{ex} + e_i) = \alpha_P + \beta_P R_M^{ex} + e_P$$

Expected excess return on the portfolio:

$$E(R_P^{ex}) = \alpha_P + \beta_P E(R_M^{ex}) = \sum_{i=1}^{n+1} w_i \alpha_i + E(R_M^{ex}) \sum_{i=1}^{n+1} w_i \beta_i$$

Variance of the portfolio:

$$\sigma_P^2 = \beta_P^2 \sigma_M^2 + \sigma^2(e_P) = \sigma_M^2 \left(\sum_{i=1}^{n+1} w_i \beta_i \right)^2 + \sum_{i=1}^{n+1} w_i^2 \sigma^2(e_i)$$

The objective: maximise the Sharpe ratio of this portfolio: $\max\limits_{w_1, w_2, \ldots, w_{n+1}} \dfrac{E(R_P^{ex})}{\sigma_P}$

- Portfolio alpha: $\alpha_P = \sum_{i=1}^{n+1} w_i \alpha_i$
- Portfolio beta: $\beta_P = \sum_{i=1}^{n+1} w_i \beta_i$
- Portfolio residual variance:
  $\sigma^2(e_P) = \sum_{i=1}^{n+1} w_i^2 \sigma^2(e_i)$