CBSE-XII-2016 EXAMINATION

SET-1

Series ONS

MATHEMATICS

Paper & Solution

Code: 65/1/S

Time: 3 Hrs.

Max. Marks: 100

General Instructions:

*(i) **All** questions are **compulsory**.*

*(ii) Please check that this question paper contains **26** questions.*

*(iii) Questions **1 - 6** in **Section A** are very short-answer type questions carrying **1** mark each.*

*(w) Questions **7 - 19** in **Section B** are long-answer **I** type questions carrying **4** marks each.*

*(v) Questions **20 - 26** in Section C are long-answer **II** type questions carrying **6** marks each.*

(vi) Please write down the serial number of the question before attempting it.

SECTION – A

Question numbers 1 to 6 carry 1 mark each.

1. If vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = \frac{1}{2}, |\vec{b}| = \frac{4}{\sqrt{3}}$ and $|\vec{a}\times\vec{b}| = \frac{1}{\sqrt{3}}$, then find $|\vec{a}\cdot\vec{b}|$.

Solution:

Getting $\sin\theta = \dfrac{\frac{1}{\sqrt{3}}}{\frac{1}{2}\cdot\frac{4}{\sqrt{3}}} = \frac{1}{2}$

Hence $|\vec{a}.\vec{b}| = \frac{1}{2}\cdot\frac{4}{\sqrt{3}}\cdot\frac{\sqrt{3}}{2} = 1$

2. If $\vec{a}$ and $\vec{b}$ are unit vectors, then what is the angle between $\vec{a}$ and $\vec{b}$ for $\vec{a}$ and $\sqrt{2}\vec{b}$ to be a unit vector?

Solution:

$|\vec{a} - \sqrt{2}\vec{b}|^2 = 1 \Rightarrow \vec{a}.\vec{b} = \frac{1}{\sqrt{2}}.$

$\therefore$ Angle between a and $b = \frac{\pi}{4}.$

3. Find the distance between the planes $\vec{r}\cdot\left(2\hat{i} - 3\hat{j} + 6\hat{k}\right) - 4 = 0$ and $\vec{r}\cdot\left(6\hat{i} - 9\hat{j} + 18\hat{k}\right) + 30 = 0$

Solution:

Writing or using, that given planes are parallel

$d = \dfrac{|4+10|}{\sqrt{4+9+36}} = 2$ units

4. If A is a square matrix such that $|A| = 5$, write the value of $|AA^T|$.

Solution:

$|AA^T| = |A||A^T| = |A|^2$

$= 25$

5. If $A=\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix}$ and $B=\begin{pmatrix}1 & -4\\ 3 & -2\end{pmatrix}$, find $|AB|$.

Solution:

Getting AB $=\begin{pmatrix}7 & -8\\ 0 & -10\end{pmatrix}$

$|AB| = -70$

6. If A= $A=\begin{pmatrix}0 & 3\\ 2 & -5\end{pmatrix}$ and $KA=\begin{pmatrix}0 & 4a\\ -8 & 5b\end{pmatrix}$ find the values of k and a.

Solution:

k (2) = –8 $\Rightarrow$ k = –4

–4(3) = 4a $\Rightarrow$ a = –3

SECTION – B

Question numbers 7 to 19 carry 4 marks each.

7. Differentiate $(\sin 2x)^x + \sin^{-1}\sqrt{3x}$ with respect to x.

OR

Differentiate $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$ with respect to $\cos^{-1}x^2$.

Solution:

$$y=(\sin 2x)^x+\sin^{-1}(\sqrt{3}x)=u+v$$

$$\therefore \frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

$$u=(\sin 2x)^x \Rightarrow \log u = x\log\sin 2x$$

$$\frac{1}{u}\frac{du}{dx}=2x.\cot 2x+\log\sin 2x$$

$$\therefore \frac{du}{dx}=(\sin 2x)^x[2x\cot 2x+\log\sin 2x]$$

$$\frac{dv}{dx}=\frac{1}{\sqrt{1-3x}}\frac{\sqrt{3}}{2\sqrt{x}}$$

$$\therefore \frac{dy}{dx}=(\sin 2x)^x[2x\cot 2x+\log\sin 2x]+\frac{\sqrt{3}}{2\sqrt{x}\sqrt{1-3x}}$$

OR

Let $y=\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right)$ and $z=\cos^{-1}x^2$

$$z=\cos^{-1}x^2 \Rightarrow \cos z \Rightarrow y=\tan^{-1}\left(\frac{\sqrt{1+\cos z}-\sqrt{1-\cos z}}{\sqrt{1+\cos z}+\sqrt{1-\cos z}}\right)$$

$$\therefore\ y=\tan^{-1}\left(\frac{\cos\frac{z}{2}-\sin\frac{z}{2}}{\cos\frac{z}{2}+\sin\frac{z}{2}}\right)=\tan^{-1}\left(\frac{1-\tan\frac{z}{2}}{1+\tan\frac{z}{2}}\right)$$

$$\therefore\ y=\tan-1\left[\tan\left(\frac{\pi}{4}-\frac{z}{2}\right)\right]=\frac{\pi}{4}-\frac{z}{2}$$

$$\Rightarrow \frac{dy}{dz}=-\frac{1}{2}$$

8. Find k, if $f(x)=\begin{cases} k\sin\dfrac{\pi}{2}(x+1), & x\le 0 \\ \dfrac{\tan x-\sin x}{x^3}, & x>0 \end{cases}$ is continuous at x = 0.

Solution:

LHL = $\lim\limits_{x\to 0^-} k.\sin\frac{\pi}{2}(x+1)=k$

RHL = $\lim\limits_{x\to 0^+} \frac{\tan x(1-\cos x)}{x^3}$

$\lim\limits_{x\to 0^+} \frac{\tan x}{x}.2\left(\frac{\sin x/2}{2x/2}\right)^2=\frac{1}{2}$

$\Rightarrow k=\frac{1}{2}$

9. Find equation of normal to the curve $ay^2 = x^3$ at the point whose x coordinate is am^2.

Solution:

When $x = am^2$, we get $y=\pm am^3$

$ay^2=x^3 \Rightarrow 2ay\frac{dy}{dx}=3x^2 \Rightarrow \frac{dy}{dx}=\frac{3x^2}{2ay}$

slope of normal $=\mp\frac{2a}{3}\frac{am^3}{a^2m^4}=\mp\frac{2}{3m}$

$\therefore$ Equation of normal is $y\mp am^3=\mp\frac{2}{3m}(x-am^2)$

[Full marks may be given, if only one value for point, slope and equation is derived]

10. Find: $\int\frac{1-\sin x}{\sin x\,(1+\sin x)}dx$

Solution:

Writing $\int\frac{1-\sin x}{\sin x(1+\sin x)}dx=\int\frac{(1+\sin x)-2\sin x}{\sin x(1+\sin x)}dx$

$$= \int \frac{1}{\sin x} dx - 2\int \frac{1}{(1+\sin x)} dx$$

$$= \int \text{cosec}\, dx - 2\int \frac{(1-\sin x)}{\cos 2x}\, dx$$

$$= \log|\text{cosec}\, x - \cot x| - 2\int (\sec^2 x - \sec x \tan x) dx$$

$$= \log|\text{cosec}\, x - \cot x| - 2(\tan x - \sec x) + C$$

11. Find: $\int [\log(\log x) + \frac{1}{(\log x)^2}\ ] dx$

Solution:

$$I = \int \left[\log(\log x) + \frac{1}{(\log x)^2} \right] dx$$

$$= \int \log(\log x).1\, dx + \int \frac{1}{(\log x)^2} dx$$

$$= \log(\log x).x - \int \frac{1}{\log x}.\frac{1}{x}.x\, dx + \int \frac{1}{(\log x)^2} dx$$

$$= x.\log(\log x) - \left[\frac{1}{\log x}.x - \int \frac{-1}{(\log x)^2}.\frac{1}{x}.x\, dx \right] + \int \frac{1}{(\log x)^2} dx$$

$$= x\log(\log x) - \frac{x}{\log x} + C$$

12. Evaluate: $\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\, dx$

OR

Evaluate: $\int_0^1 \cot^{-1}(1 - x + x^2) dx$

Solution:

$$I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x} dx \qquad ....(i)$$

$$I = \int_0^{\pi/2} \frac{\sin^2(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)} dx = \int_0^{\pi/2} \frac{\cos^2 x}{\cos x + \sin x} dx = \qquad ....(ii)$$

$$2I = \int_0^{\pi/2} \frac{1}{\sin x + \cos x} dx$$

$$\Rightarrow I = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \frac{1}{\cos x \frac{1}{\sqrt{2}} + \sin x \frac{1}{\sqrt{2}}} dx = \frac{1}{2\sqrt{2}} \int_0^{\pi/2} \sec\left(x - \frac{\pi}{4} \right) dx$$

$$=\frac{1}{2\sqrt{2}}\left[\log\left|\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right|\right]_0^{\pi/2}$$

$$=\frac{1}{2\sqrt{2}}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| \text{ or } \frac{1}{\sqrt{2}}\log|\sqrt{2}+1|$$

OR

$$I=\int_0^1 \cot^{-1}(1-x+x^2)dx=\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx$$

$$=\int_0^1 \tan^{-1}\frac{x+(1-x)}{1-x(1-x)}dx=\int_0^1 \tan^{-1}x\,dx+\int_0^1 \tan^{-1}(1-x)\,dx$$

$$=2\int_0^1 \tan^{-1}x\,dx$$

$$=2\left[(\tan^{-1}x.x)_0^1-\int_0^1\frac{x}{1+x^2}dx\right]$$

$$=2\left[x\tan^{-1}x-\frac{1}{2}\log|1+x^2|\right]_0^1$$

$$=2\left[\frac{\pi}{4}-\frac{1}{2}\log 2\right] \text{ or } \frac{\pi}{2}-\log 2$$

13. Solve the differential equation: $(x+1)\dfrac{dy}{dx}-y=e^{3x}(x+1)^3$

Solution:

The given differential equation can be written as

$$\frac{dy}{dx}-\frac{1}{x+1}y=(x+1)^2.e^{3x}$$

Here, integrating factor = $e^{\int -\frac{1}{x+1}dx}=\dfrac{1}{x+1}$

$\therefore$ Solution is $y\dfrac{1}{x+1}=\int(x+1)e^{3x}dx$

$$\therefore\ \frac{y}{y+1}=(x+1)\frac{e^{3x}}{3}-\frac{e^{3x}}{9}+C$$

or $y=\left[\dfrac{1}{3}(x+1)^2-\dfrac{x+1}{9}\right]e^{3x}+C(x+1)$

14. Solve the differential equation: $2y\,e^{x/y}\,dx+\left(y-2x\,e^{x/y}\right)dy=0$

Solution:

From the given differential equation, we can write

$$\frac{dx}{dy}=\frac{2xe^{x/y}-y}{2ye^{x/y}}=\frac{2x/ye^{x/y}-1}{2e^{x/y}}$$

Putting $\frac{x}{y}=v \Rightarrow \frac{dx}{dy}=v+y\frac{dv}{dy}$

$\therefore \quad v+y\frac{dv}{dy}=\frac{2ve^{v}-1}{2e^{v}} \Rightarrow y\frac{dv}{dy}=-\frac{1}{2e^{v}}$

$\Rightarrow 2\int e^{v}dv=-\int\frac{dv}{y}$

$\therefore \quad 2e^{v}+\log|y|=C \Rightarrow 2e^{x/y}+\log|y|=C$

15. Ishan wants to donate a rectangular plot of land for a school in his village. When he was asked to give dimensions of the plot, he told that if its length is decreased by 50 m and breadth is increased by 50 m, then its area will remain same, but if length is decreased by 10 m and breadth is decreased by 20 m, then its area will decrease by 5300 m^2. Using matrices, find the dimensions of the plot. Also give reason why he wants to donate the plot for a school

Solution:

Let length be x m and breadth be y m

$\therefore \ (x-50)(y+50)=xy \Rightarrow 50x-50y=2500$ or $x-y=50$

and $(x-10)(y-20)=xy-5300 \Rightarrow 2x+y=550$

$$\begin{pmatrix}1 & -1\\ 2 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}50\\ 550\end{pmatrix} \Rightarrow \begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1 & 1\\ -2 & 1\end{pmatrix}\begin{pmatrix}50\\ 500\end{pmatrix}$$

$\Rightarrow x=\frac{1}{3}(600)=200\ m,\ \ y=\frac{1}{3}(450)=150\,m$

“Helping the children of his village to learn” (or any other relavent value)

16. Prove that $2\sin^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{17}{31}\right)=\frac{\pi}{4}$

OR

Solve the equation for x: $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$

Solution:

LHS $=2\sin^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{17}{31}\right)$

$=2\tan^{-1}\left(\frac{3}{4}\right)-\tan^{-1}\left(\frac{17}{31}\right)$

$$=\tan^{-1}\left(\frac{2.\frac{3}{4}}{1-\frac{9}{16}}\right)-\tan^{-1}\left(\frac{17}{31}\right)=\tan^{-1}\left(\frac{24}{7}\right)-\tan^{-1}\left(\frac{17}{31}\right)$$

$$\tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}.\frac{17}{31}}\right)=\tan^{-1}\left(\frac{17}{31}\right)=\tan^{-1}\left(\frac{24}{7}\right)-\tan^{-1}\left(\frac{17}{31}\right)$$

$$\tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}.\frac{17}{31}}\right)=\tan^{-1}\left(\frac{625}{625}\right)=\tan^{-1}(1)=\frac{\pi}{4}=\text{RHS}$$

OR

$$\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$$

$$\Rightarrow\frac{1}{\sqrt{1+x^2}}=\frac{4}{5}\text{ or }\sqrt{1+x^2}=\frac{5}{4}$$

$$\Rightarrow x^2=\frac{9}{16}\Rightarrow x=\pm\frac{3}{4}$$

17. There are two bags A and B. Bag A contains 3 white and 4 red balls whereas bag B contains 4 white and red balls. Three balls are drawn at random (without replacement) from one of the bags and are found to be two white and one red. Find the probability that these were drawn from bag B.

Solution:

Let E_1: selecting bag A, E_2: selecting bag B

A: getting 2 white and 1 red out of 3 drawn (without replacement)

$$\therefore\ P(E_1)=P(E_2)=\frac{1}{2}$$

$$P\left(\frac{A}{E_1}\right)=\frac{{}^3C_2.{}^4C_1}{{}^7C_3}=\frac{12}{35}$$

$$P\left(\frac{A}{E_2}\right)=\frac{{}^4C_2.{}^3C_1}{{}^7C_3}=\frac{18}{35}$$

$$P\left(\frac{E_2}{A}\right)=\frac{P(E_2)P\left(\frac{A}{E_2}\right)}{P(E_1)P\left(\frac{A}{E_1}\right)+P(E_2)P\left(\frac{A}{E_2}\right)}$$

$$=\frac{\frac{1}{2}.\frac{18}{35}}{\frac{1}{2}.\frac{12}{35}+\frac{1}{2}.\frac{18}{35}}=\frac{3}{5}$$

18. Given that vectors $\vec{a},\vec{b},\vec{c}$ form a triangle such that $\vec{a}=\vec{b}+\vec{c}$. Find p, q, r, s such that area of triangle is $5\sqrt{6}$ where $\vec{a}=p\hat{i}+q\hat{j}+r\hat{k}$, $\vec{b}=s\hat{i}+3\hat{j}+4\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}-2\hat{k}$.

Solution:

$\vec{a}=\vec{b}+\vec{c}\Rightarrow p\hat{i}+q\hat{j}+r\hat{k}=(s+3)\hat{i}+4\hat{j}+2\hat{k}$

$p = s + 3,\ q = 4,\ r = 2$

area $=\frac{1}{2}|\vec{b}\times\vec{c}|=5\sqrt{6}$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ s & 3 & 4\\ 3 & 1 & -2\end{vmatrix}=-10\hat{i}+(2s+12)\hat{j}+(x-9)\hat{k}$$

$\therefore 100+(2s+12)^2+(s-9)^2=(10\sqrt{6})^2=600$

$\Rightarrow s^2+6s+55=0\Rightarrow s=-11, p=-8, \text{ or } s=5, p=8$

19. Find the equation of plane passing through the points $A(3,2,1)$, $B(4,2,-2)$ and $C(6,5,-1)$ and hence find the value of λ for which $A(3,2,1)$, $B(4,2,-2)$, $C(6,5,-1)$ and $D(\lambda,5,5)$ are coplanar.

OR

Find the co-ordinates of the point where the line $\vec{r}=(-\hat{i}-2\hat{j}-3\hat{k})+\lambda(3\hat{i}+4\hat{j}+3\hat{k})$ meets the plane which is perpendicular to the vector $\vec{n}=\hat{i}+\hat{j}+3\hat{k}$ and at a distance of $\frac{4}{\sqrt{11}}$ from origin

Solution:

Equation of plane passing through A, B and C is

$$\begin{vmatrix}x-3 & y-2 & z-1\\ 1 & 0 & -3\\ 2 & 3 & 1\end{vmatrix}=0$$

$\Rightarrow(x-3)9-(y-2)7+(z-1)3=0\Rightarrow 9x-7y+3z=16 \qquad ...(i)$

If A, B, C and D are coplanar, D must lie on (i)

$\Rightarrow 9\lambda-35+15-16=0\Rightarrow\lambda=4.$

OR

Equation of plane, perpendicular to $\vec{n}=\hat{i}+\hat{j}+3\hat{k}$ and at a distance $\frac{4}{\sqrt{11}}$ from origin is

$\vec{r}=\frac{(\hat{i}+\hat{j}+3\hat{k})}{\sqrt{11}}=\frac{4}{\sqrt{11}}$ or $\vec{r}.(\hat{i}+\hat{j}+3\hat{k})=4 \qquad ....(i)$

Any point on the line $\vec{r}=(-\hat{i}-2\hat{j}-3\hat{k})+\lambda(3\hat{i}+4\hat{j}+3\hat{k})$ is

$(-1+3\lambda)\hat{i}+(-2+4\lambda)\hat{j}+(-3+3\lambda)\hat{k} \qquad ...(ii)$

If this point is the point of intersection of the plane and the line then,

$(-1+3\lambda)1+(-2+4\lambda)1+(-3+3\lambda)3=4$

$\Rightarrow\lambda=1.$

Hence the point of intersection is (2, 2, 0)

SECTION – C

Question numbers 20 to 26 carry 6 marks each.

20. Let f: $N \to N$ be a function defined as $f(x) = 4x^2 + 12x + 15$. Show that $f: N \to S$ is invertible (where S is range of *f*). Find the inverse of *f* and hence find $f^{-1}(31)$ and $f^{-1}(87)$.

Solution:

Let $x_1, x_2 \in N$ and $f(x_1) = f(x_2)$

$\Rightarrow 4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15$

$\Rightarrow 4(x_1^2 - x_2^2) + 12(x_2 - x_2) = 0 \Rightarrow (x_1 - x_2)(4x_1 + 4x_2 + 12) = 0$

$\Rightarrow x_1 - x_2 = 0$ or $x_1 = x_2$ as $4x_1 + 4x_2 + 12 \neq 0, x_1, x_2 \in \mathrm{N}$

$\therefore f$ is a $1-1$ function

$f: \mathrm{N} \to \mathrm{S}$ is onto as co-domain = range

Hence *f* is invertible.

$$y = 4x^2 + 12x + 15 = (2x+3)^2 + 6 \Rightarrow x = \frac{\sqrt{y-6} - 3}{2}$$

$$\therefore\ f^{-1}(y) = \frac{\sqrt{y-6} - 3}{2},\ y \in S.$$

$$f^{-1}(31) = \frac{\sqrt{31-6} - 3}{2} = 1$$

$$f^{-1}(87) = \frac{\sqrt{87-6} - 3}{2} = 3$$

21. Using properties of determinants prove that:

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)\,(b-c)\,(c-a)(a+b+c)\,(a^2+b^2+c^2)$$

OR

Using elementary row operation, find the inverse of the following matrix:

$$A = \begin{pmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{pmatrix}$$

Solution:

Let $\Delta = \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$

$$C_1 \to C_1 + C_2 - 2C_3 \Rightarrow \Delta = (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$$

$$R_1 \to R_1 - R_2, \text{ and } R_2 \to R_2 - R_3 \Rightarrow \Delta = (a^2+b^2+c^2)\begin{vmatrix} 0 & a^2-b^2 & c(b-a) \\ 0 & b^2-c^2 & a(c-b) \\ 1 & c^2 & ab \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} 0 & a+b & -c \\ 0 & b+c & -a \\ 1 & c^2 & ab \end{vmatrix}$$

$$R_2 \to R_2 - R_1 \Rightarrow \Delta = (a^2+b^2+c^2)(a-b)(b-c)\begin{vmatrix} 0 & a+b & -c \\ 0 & c-a & c-a \\ 1 & c^2 & ab \end{vmatrix}$$

$$\therefore\ \Delta = (a^2+b^2+c^2)(a-b)(b-c)(c-a)\begin{vmatrix} 0 & a+b & -c \\ 0 & 1 & 1 \\ 1 & c^2 & ab \end{vmatrix}$$

Expanding by C_1 to get $\Delta = (a^2+b^2+c^2)(a-b)(b-c)(c-a)(a+b+c)$

OR

Let A = IA $\therefore \begin{pmatrix} 2 & -1 & 3 \\ -5 & 3 & 1 \\ -3 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$

$$R_2 \to R_2 - 2R_3 \Rightarrow \begin{pmatrix} 2 & -1 & 3 \\ 1 & -1 & -5 \\ -3 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix} A$$

$$\begin{matrix} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 + 3R_2 \end{matrix} \Rightarrow \begin{pmatrix} 0 & 1 & 13 \\ 1 & -1 & -5 \\ 0 & -1 & -12 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 3 & -5 \end{pmatrix} A$$

$$R_1 \leftrightarrow R_2 \Rightarrow \begin{pmatrix} 1 & -1 & -5 \\ 0 & 1 & 13 \\ 0 & -1 & -12 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -2 \\ 1 & -2 & 4 \\ 0 & 3 & -5 \end{pmatrix} A$$

$$\begin{cases} R_1 \to R_1 + R_2 \\ R_3 \to R_3 + R_2 \end{cases} \Rightarrow \begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 13 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 1 & -2 & 4 \\ 1 & 1 & -1 \end{pmatrix} A$$

$$\begin{cases} R_1 \to R_1 - 8R_3 \\ R_2 \to R_2 - 13R_3 \end{cases} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{pmatrix} A$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -7 & -9 & 10 \\ -12 & -15 & 17 \\ 1 & 1 & -1 \end{pmatrix}$$

22. Determine the intervals in which the function $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$ is strictly increasing or strictly decreasing.

OR

Find the maximum and minimum values of $f(x) = \sec x + \log \cos^2 x,\ o < x < 2\pi.$

Solution:

$f'(x) = 4x^3 - 24x^2 + 44x - 24$

$= 4(x^3 - 6x^2 + 11x - 6) = 4(x - 1)(x - 2)(x - 3)$

$f'(x) = 0 \Rightarrow x = 1, x = 2, x = 3$

The intervals are $(-\infty, 1), (1, 2), (2, 3), (3, \infty)$

since $f'(x) > 0$ in $(1, 2)$ and $(3, \infty)$

$\therefore$ is strictly increasing in $(1, 2) \cup (3, \infty)$

and strictly decreasing in $(-\infty, 1) \cup (2, 3)$

OR

$f(x) = \sec x + 2 \log |\cos x|$

$f'(x) = \sec x \tan x - 2 \tan x = \tan x (\sec x - 2)$

$f'(x) = 0 \Rightarrow \tan x = 0$ or $\sec x = 2$

$\Rightarrow x = \pi,\ x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

$f''(x) = \sec x \tan^2 x + (\sec x - 2)\sec^2 x$

$$\left.\begin{array}{l} f''(\pi/3) = 6(+ve) \Rightarrow f(x) \text{ is minimum at } x = \pi/3 \\ f''(\pi) = -3(-ve) \Rightarrow f(x) \text{ is minimum at } x = \pi \\ f''(5\pi/3) = 6(+ve) \Rightarrow f(x) \text{ is minimum at } x = 5\pi/3 \end{array}\right\}$$

Maximum value $= f(\pi) = -1.$

Minimum value $= f(\pi/3) = f(5\pi/3) =$ $2 - 2\log 2$ or $2 + \log(1/4)$

23. Using integration find the area of the region $\{(x, y) : y^2 \le 6ax \text{ and } x^2 + y^2 \le 16a^2\}$

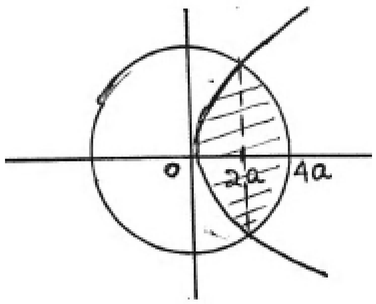
Solution:

Solving $y^2 = 6ax$ and $x^2 + y^2 = 16a^2$

we get $x^2 + 6ax - 16a^2 = 0$

$(x + 8a)(x - 2a) = 0$

$x = -8a, x = 2a$



Required area = $2\left[\int_0^{2a}\sqrt{6a}\sqrt{x}\,dx+\int_{2a}^{4a}\sqrt{16a^2-x^2}\,dx\right]$

$=2\left[\left(\sqrt{6}\sqrt{a}\frac{2}{3}x^{3/2}\right)_0^{2a}+\left(\frac{x}{2}\sqrt{16a^2-x^2}+8a^2\sin^{-1}\frac{x}{4a}\right)_{2a}^{4a}\right]$

$=2\left[\frac{8\sqrt{3}a^2}{3}+8a^2\frac{\pi}{2}-2a^2\sqrt{3}-8a^2\frac{\pi}{6}\right]$

$=2\left[\frac{2\sqrt{3}a^2}{3}+8a^2\frac{\pi}{3}\right]$ sq. units

24. Find the equation of the plane containing two parallel lines $\frac{x-1}{2}=\frac{y+1}{-1}=\frac{z}{3}$ and $\frac{x}{4}=\frac{y-2}{-2}=\frac{z+1}{6}$. also, find if the plane thus obtained contains the line $\frac{x-2}{3}=\frac{y-1}{1}=\frac{z-2}{5}$ or not.

Solution:

Points on the lines are a_1 = (1, –1, 0), a_2 = (0, 2, –1)

and the direction of lines is $2\hat{i}-\hat{j}+3\hat{k}$

let the equation of plane through a_1 be

$a(x-1)+b(y+1)+c(z)=0$ …(i)

(0, 2, –1) lies on it, $\therefore -a+3b-c=0$ …(ii)

and a, b, c are DR's of a line $\perp$ to the line with DR's 2, –1, 3

$\therefore 2a-b+3c=0$ …(iii)

Solving (ii) & (iii) we get $\frac{a}{8}=\frac{b}{1}=\frac{c}{-5}$

$\therefore$ Equation of plane is $8(x-1)+1(y+1)-5z=0$

$\Rightarrow 8x+y-5z=7$ …(iv)

For the line $\frac{x-2}{3}=\frac{y-1}{1}=\frac{z-2}{5}$, since the point (2, 1, 2) lies on plane (iv) as 8(2) + 1 – 5(2) = 7

and 3(8) + 1(1) + 5(–5) = 25 – 25 = 0

$\therefore$ The plane (iv) contains the given line

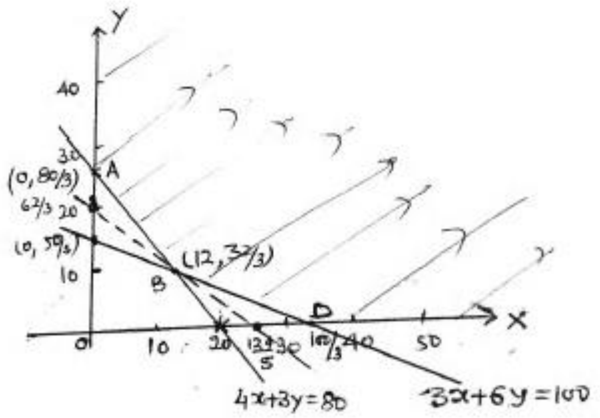
25. A diet is to contain at least 80 units of Vitamin A and 100 units of minerals. Two foods F_1 and F_2 are available costing ₹ 5 per unit and ₹ 6 per unit respectively. One unit of food F_1 contains 4 units of vitamin A and 3 units of minerals whereas one unit of food F_2 contains 3 units of vitamin A and 6 units of minerals. Formulate this as a linear programming problem. Find the minimum cost of diet that consists of mixture of these two foods and also meets minimum nutritional requirement.

Solution:

Let x units of F_1 and y units of F_2 be mixed

$\therefore$ We have Minimize cost (C) = $5x+6y$

$$\text{subject to} \left.\begin{array}{l} 4x+3y\geq 80 \\ 3x+6y\geq 100 \\ x\geq 0,\ \ y\geq 0 \end{array}\right\}$$



Correct Figure

C (A) = 160

C (B) = 60 + 64 = 124

$C(D)=\dfrac{500}{3}$

$5x+6y\leq 124$ Passes through B only

$\therefore$ Minimum cost = ₹ 124

F_1 = 12 units

$F_2=\dfrac{32}{3}$ units

26. Three numbers are selected at random (without replacement) from first six positive integers. If X denotes the smallest of the three numbers obtained, find the probability distribution of X. Also find the mean and variance of the distribution.

Solution:

Total number of ways = ${}^6C_3$ = 20

X :	1	2	3	4
P(X) :	$\frac{10}{20}$	$\frac{6}{20}$	$\frac{3}{20}$	$\frac{1}{20}$
XP(X) :	$\frac{10}{20}$	$\frac{12}{20}$	$\frac{9}{20}$	$\frac{4}{20}$
X^2 P(X) :	$\frac{10}{20}$	$\frac{24}{20}$	$\frac{27}{20}$	$\frac{16}{20}$

Mean $\sum XP(X)=\dfrac{35}{20}=\dfrac{7}{4}$

Variance = $\sum X^2P(X)-\left[\sum XP(X)\right]^2=\dfrac{77}{20}-\dfrac{49}{16}=\dfrac{63}{80}$