

CHAPTER 5

CONTINUITY AND DIFFERENTIATION

VERY SHORT ANSWER TYPE QUESTIONS (1 MARK)

1. For what value of x , $f(x) = |2x - 7|$ is not derivable.
2. Write the set of points of continuity of $g(x) = |x - 1| + |x + 1|$.
3. What is derivative of $|x - 3|$ at $x = -1$.
4. What are the points of discontinuity of $f(x) = \frac{(x - 1) + (x + 1)}{(x - 7)(x - 6)}$.
5. Write the number of points of discontinuity of $f(x) = [x]$ in $[3, 7]$.
6. The function, $f(x) = \begin{cases} \lambda x - 3 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ 2x & \text{if } x > 2 \end{cases}$ is a continuous function for all $x \in R$, find λ .
7. For what value of K , $f(x) = \begin{cases} \frac{\tan 3x}{\sin 2x}, & x \neq 0 \\ 2K, & x = 0 \end{cases}$ is continuous $\forall x \in R$.
8. Write derivative of $\sin x$ w.r.t. $\cos x$.
9. If $f(x) = x^2 g(x)$ and $g(1) = 6$, $g'(1) = 3$ find value of $f'(1)$.
10. Write the derivative of the following functions :
 - (i) $\log_3 (3x + 5)$
 - (ii) $e^{\log_2 x}$
 - (iii) $e^{6 \log_8 (x-1)}$, $x > 1$

(iv) $\sec^{-1}\sqrt{x} + \operatorname{cosec}^{-1}\sqrt{x}, x \geq 1.$

(v) $\sin^{-1}(x^{7/2})$

(vi) $\log_x 5, x > 0.$

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

11. Discuss the continuity of following functions at the indicated points.

(i) $f(x) = \begin{cases} \frac{x - |x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$ at $x = 0.$

(ii) $g(x) = \begin{cases} \frac{\sin 2x}{3x}, & x \neq 0 \\ \frac{3}{2}, & x = 0 \end{cases}$ at $x = 0.$

(iii) $f(x) = \begin{cases} x^2 \cos(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ at $x = 0.$

(iv) $f(x) = |x| + |x - 1|$ at $x = 1.$

(v) $f(x) = \begin{cases} x - [x], & x \neq 1 \\ 0 & x = 1 \end{cases}$ at $x = 1.$

12. For what value of k , $f(x) = \begin{cases} 3x^2 - kx + 5, & 0 \leq x < 2 \\ 1 - 3x & 2 \leq x \leq 3 \end{cases}$ is continuous

$\forall x \in [0, 3].$

13. For what values of a and b

$f(x) = \begin{cases} \frac{x+2}{|x+2|} + a & \text{if } x < -2 \\ a + b & \text{if } x = -2 \\ \frac{x+2}{|x+2|} + 2b & \text{if } x > -2 \end{cases}$ is continuous at $x = 2.$

14. Prove that $f(x) = |x + 1|$ is continuous at $x = -1$, but not derivable at $x = -1$.
15. For what value of p ,

$$f(x) = \begin{cases} x^p \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases} \text{ is derivable at } x = 0.$$

16. If $y = \frac{1}{2} \left[\tan^{-1} \left(\frac{2x}{1-x^2} \right) + 2 \tan^{-1} \left(\frac{1}{x} \right) \right]$, $0 < x < 1$, find $\frac{dy}{dx}$.

17. If $y = \sin \left[2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} \right]$ then $\frac{dy}{dx} = ?$

18. If $5^x + 5^y = 5^{x+y}$ then prove that $\frac{dy}{dx} + 5^{y-x} = 0$.

19. If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$ then show that $\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$.

20. If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ then show that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

21. If $(x+y)^{m+n} = x^m \cdot y^n$ then prove that $\frac{dy}{dx} = \frac{y}{x}$.

22. Find the derivative of $\tan^{-1} \left(\frac{2x}{1-x^2} \right)$ w.r.t. $\sin^{-1} \left(\frac{2x}{1+x^2} \right)$.

23. Find the derivative of $\log_e(\sin x)$ w.r.t. $\log_a(\cos x)$.

24. If $x^y + y^x + x^x = m^n$, then find the value of $\frac{dy}{dx}$.

25. If $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ then find $\frac{d^2y}{dx^2}$.

26. If $x = ae^t (\sin t - \cos t)$

$y = ae^t (\sin t + \cos t)$ then show that $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$ is 1.

27. If $y = \sin^{-1} [x\sqrt{1-x} - \sqrt{x}\sqrt{1-x^2}]$ then find $-\frac{dy}{dx}$.

28. If $y = x^{\log_e x} + (\log_e x)^x$ then find $\frac{dy}{dx}$.

29. Differentiate x^{x^x} w.r.t. x .

30. Find $\frac{dy}{dx}$, if $(\cos x)^y = (\cos y)^x$

31. If $y = \tan^{-1} \left(\frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}} \right)$ where $\frac{\pi}{2} < x < \pi$ find $\frac{dy}{dx}$.

32. If $x = \sin \left(\frac{1}{a} \log_e y \right)$ then show that $(1-x^2) y'' - xy' - a^2 y = 0$.

33. Differentiate $(\log x)^{\log x}$, $x > 1$ w.r.t. x

34. If $\sin y = x \sin (a + y)$ then show that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

35. If $y = \sin^{-1} x$, find $\frac{d^2 y}{dx^2}$ in terms of y .

36. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then show that $\frac{d^2 y}{dx^2} = \frac{-b^4}{a^2 y^3}$.

37. If $y = e^{a \cos^{-1} x}$, $-1 \leq x \leq 1$, show that $(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$

38. If $y^3 = 3ax^2 - x^3$ then prove that $\frac{d^2 y}{dx^2} = \frac{-2a^2 x^2}{y^5}$.

39. Verify Rolle's theorem for the function, $y = x^2 + 2$ in the interval $[a, b]$ where $a = -2$, $b = 2$.

40. Verify Mean Value Theorem for the function, $f(x) = x^2$ in $[2, 4]$

ANSWERS

1. $x = -7/2$
2. R
3. -1
4. $x = 6, 7$
5. Points of discontinuity of $f(x)$ are 4, 5, 6, 7 i.e. four points.
Note : At $x = 3$, $f(x) = [x]$ is continuous. because $\lim_{x \rightarrow 3^+} f(x) = 3 = f(3)$.
6. $\lambda = \frac{7}{2}$.
7. $k = \frac{3}{4}$.
8. $-\cot x$
9. 15
10. (i) $\frac{3}{3x+5} \log_3 e$
- (ii) $e^{\log_2 x} \frac{1}{x} \cdot \log_2 e$.
- (iii) $6(x-1)^5$
- (iv) 0
- (v) $\frac{7}{2} \frac{x^2 \sqrt{x}}{\sqrt{1-x^7}}$.
- (vi) $\frac{-\log_e 5}{x(\log_e x)^2}$.
11. (i) Discontinuous
- (ii) Discontinuous
- (iii) Continuous
- (iv) continuous
- (v) Discontinuous
12. $k = 11$
13. $a = 0, b = -1$.
15. $p > 1$.
16. 0
17. $\frac{-x}{\sqrt{1-x^2}}$.
22. 1
23. $-\cot^2 x \log_e a$
24. $\frac{dy}{dx} = \frac{-x^x(1+\log x) - yx^{y-1} - y^x \log y}{x^y \log x + xy^{x-1}}$.

$$25. \quad \frac{d^2 y}{dx^2} = \frac{1}{3a} \operatorname{cosec} \theta \sec^4 \theta.$$

$$27. \quad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

$$28. \quad x^{\log x} \frac{2 \log x}{x} + (\log x)^x \left[\frac{1}{\log x} + \log(\log x) \right].$$

$$29. \quad \frac{dy}{dx} = x^{x^x} \cdot x^x \log x \left(1 + \log x + \frac{1}{x \log x} \right).$$

$$30. \quad \frac{dy}{dx} = \frac{y \tan x + \log \cos y}{x \tan y + \log \cos x}$$

$$31. \quad \frac{dy}{dx} = -\frac{1}{2}.$$

Hint. : $\sin \frac{x}{2} > \cos \frac{x}{2}$ for $x \in \left(\frac{\pi}{2}, \pi \right)$.

$$33. \quad (\log x)^{\log x} \left[\frac{1}{x} + \frac{\log(\log x)}{x} \right], \quad x > 1$$

$$35. \quad \sec^2 y \tan y.$$