CBSE-XII-2014 EXAMINATION

MATHEMATICS

Paper & Solution

Code: 65/1

Time: 3 Hrs. Max. Marks: 100

General Instructions:

(i) **All** question are compulsory.

(ii) The question paper consists of **29** questions divided into three sections A, B and C. Section A comprises of **10** questions of **one mark** each, Section B comprises of **12** questions of **four marks** each and Section C comprises of **7** questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is **not** permitted. You may ask for logarithmic tables, if required.

SECTION A

Question numbers 1 to 10 carry 1 mark each.

1. If $R=\{(x,y):x+2y=8\}$ is a relation on N, write the range of R.

Solution:

$R=\{(x,y):x+2y=8\}$ is a relation on N

Then we can say 2y = 8 – x

$$y=4-\frac{x}{2}$$

so we can put the value of x, x = 2, 4, 6 only

we get y = 3 at x = 2
we get y = 2 at x = 4
we get y = 1 at x = 6
so range = {1, 2, 3} Ans.

2. *If* $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}, xy<1,$ then write the value of x + y + xy.

Solution:

$$\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\frac{x+y}{1-xy}=\frac{\pi}{4}$$

$$\Rightarrow \frac{x+y}{1-xy}=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{x+y}{1-xy}=1\,or,\, x+y=1-xy$$

$$or,\, x+y+xy=1\, Ans.$$

3. If A is a square matrix such that $A^2 = A$, then write the value of $7A - (I + A)^3$, where I is an identity matrix.

Solution:

$A^2 = A$

$7A - (I + A)^3$

$7A - [(I + A)^2(I + A)] = 7A - [(I I + AA + 2AI) (I + A)]$

$= 7A - [I + A^2 + 2AI] [I + A]$

$= 7A - [I + A + 2A] [I + A]$

$= 7A - [I + 3A] [I + A]$

$= 7A - [I I + IA + 3AI + 3A^2]$

$= 7A - [I + A + 3A + 3A]$

$= 7A - [I + 7A]$

$= - I$ Ans.

4. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, find the value of x + y.

Solution:

If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$ then x + y = ?

we can compare the element of 2 matrices. so

$x - y = -1 \ldots (1)$

$2x - y = 0 \ldots (2)$

On solving both eq^n we get $\rightarrow x = 1, y = 2$

so $x + y = 3$ Ans.

5. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, find the value of x.

Solution:

$\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$

on expanding both determinants we get

$12x + 14 = 32 - 42$

$12x + 14 = -10$

$12x = -24$

$x = -2$ Ans.

6. If $f(x) = \int_0^x t \sin t \, dt$, then write the value of $f'(x)$.

Solution:

$f(x) = \int_0^x t \sin t \, dt$

$\Rightarrow f'(x) = 1 \cdot x \sin x - 0$

= x sin x Ans.

7. Evaluate :

$$\int_2^4 \frac{x}{x^2+1}dx$$

Solution:

$$I = \int_2^4 \frac{x}{x^2+1}dx$$

Put $x^2+1=t$

$$\Rightarrow 2xdx = dt \qquad xdx = \frac{1}{2}dt$$

$$\left| \begin{array}{l} at\ x=2 \\ t=5 \\ at\ x=4 \\ t=17 \end{array} \right.$$

$$\therefore I = \int_4^{17} \frac{1/2}{t}dt$$

$$= \frac{1}{2}\left[\log|t|\right]_4^{17}$$

$$= \frac{1}{2}\left[\log 17 - \log 4\right]$$

$$= \frac{1}{2}\log(17/4)\ Ans.$$

8. Find the value of 'p' for which the vectors $3\hat{i}+2\hat{j}+9\hat{k}$ *and* $\hat{i}-2p\hat{j}+3\hat{k}$ are parallel.

Solution:

Let $\vec{a} = 3\hat{i}+2\hat{j}+9\hat{k}, \vec{b} = \hat{i}-2p\hat{j}+3\hat{k}$

If $\vec{a}, \vec{b}$ are parallel vector then their exist a, λ such that

$\vec{a} = \lambda\vec{b}$

So $\left(3\hat{i}+2\hat{j}+9\hat{k}\right) = \lambda\left(\hat{i}-2p\hat{j}+3\hat{k}\right)$

compare $3 = \lambda$ | $2 = -2p\lambda$ | $9 = 3\lambda$, $\lambda = 3$

put λ = 3 in 2= –2pλ

2 = –2p.3

$$p = -\frac{1}{3}\ Ans.$$

9. Find $\vec{a}.(\vec{b}\times\vec{c})$, *if* $\vec{a} = 2\hat{i}+\hat{j}+3\hat{k}, \vec{b} = -\hat{i}+2\hat{j}+\hat{k}$ *and* $\vec{c} = 3\hat{i}+\hat{j}+2\hat{k}$.

Solution:

If $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}, \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}, \vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$

Then $\vec{a}.\left(\vec{b} \times \vec{c}\right) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$

expand along $R_1 = 2[4 - 1] - 1[-2 - 3] + 3[-1 - 6]$

$= 6 + 5 - 21 = -10$

10. If the Cartesian equations of a line are $\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$, write the vector equation for the line.

Solution:

Cartesian eq^n of line is $\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$,

we can write it as $\frac{x-3}{-5} = \frac{y+4}{7} = \frac{z-3}{2}$

so vector eq^n is $\vec{r} = \left(3i - 4j + 3k\right) + \lambda\,(-5\hat{i} + 7\hat{j} + 2\hat{k})$

where λ is a constant

SECTION B

Question numbers 11 to 22 carry 4 marks each.

11. If the function f : R → R be given by $f(x) = x^2 + 2$ and g : R → R be given by $g(x) = \frac{x}{x-1}$, $x \neq 1$, find fog and gof and hence find fog (2) and gof (–3).

Solution:

$f : R \to R;\ f(x) = x^2 + 2$

$g : R \to R;\ g(x) = \frac{x}{x-1}, x \neq 1$

$fog = f(g(x))$

$= f\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^2 + 2$

$= \frac{x^2}{(x-1)^2} + 2$

$= \frac{x^2 + 2(x-1)^2}{(x-1)^2}$

$= \frac{x^2 + 2x^2 - 4x + 2}{(x-1)^2}$

$= \frac{3x^2 - 4x + 2}{(x-1)^2}$

$$gof = g(f(x))$$
$$= g(x^2+2)$$
$$\frac{\left(x^2+2\right)}{\left(x^2+2\right)-1} = \frac{x^2+2}{x^2+1} = 1+\frac{1}{x^2}$$

$$\therefore\ fog(2) = \frac{3(2)^2-4(2)+2}{(2-1)^2} = 6$$

$$gof(-3) = 1+\frac{1}{(-3)^2+1} = \frac{11}{10} = 1\frac{1}{10}$$

12. Prove that $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ \frac{-1}{\sqrt{2}} \le x \le 1$

OR

If $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$, find the value of x.

Solution:

$$\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x,\ \frac{1}{\sqrt{2}} \le x \le 1$$

In LHS

put x = cos 2θ

$$\tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right]$$

$$= \tan^{-1}\left[\frac{\sqrt{1+2\cos^2\theta-1}-\sqrt{1-1+2\sin^2\theta}}{\sqrt{1+2\cos^2\theta-1}+\sqrt{1-1+2\sin^2\theta}}\right]$$

$$= \tan^{-1}\left[\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right]$$

$$= \tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right]$$

$$= \tan^{-1}\left[\frac{\tan(\pi/4)-\tan\theta}{1+\tan(\pi/4).\tan\theta}\right]$$

$$= \tan^{-1}\left[\tan(\pi/4)-\theta\right]$$

$$= \frac{\pi}{4}-\theta \quad as\left\{\begin{matrix} x=\cos 2\theta \\ so,\ \theta\frac{\cos^{-1}x}{2}\end{matrix}\right\}$$

$$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x = RHS \qquad proved$$

OR

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4} \quad (1)$$

Use formula, $\tan^{-1}\left[\dfrac{\dfrac{x-2}{x-4}+\dfrac{x+2}{x+4}}{1-\left(\dfrac{x-2}{x-4}\right).\left(\dfrac{x+2}{x+4}\right)}\right]=\dfrac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left[\frac{(x-2)(x+4)+(x+2).(x-4)}{(x-4).(x+4)-(x-2).(x+2)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \frac{(x-2)(x+4)+(x+2).(x-4)}{(x-4).(x+4)-(x-2).(x+2)}=1$$

$$\Rightarrow \frac{x^2-8+2x+x^2-8-2x}{x^2-16-x^2+4}=1$$

$$\Rightarrow \frac{2x^2-16}{-12}=1$$

$$\Rightarrow 2x^2=-12+16=4$$

$$\Rightarrow x^2=2 \qquad \Rightarrow x=\pm\sqrt{2}$$

13. Using properties of determinants, prove that

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}=x^3$$

Solution:

To prove, $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}=x^3$

LHS $=\begin{vmatrix} x & x & x \\ 5x & 4x & 2x \\ 10x & 8x & 3x \end{vmatrix}+\begin{vmatrix} y & x & x \\ 4y & 4x & 2x \\ 8y & 8x & 3x \end{vmatrix}$

$$=x^3\begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 2 \\ 10 & 8 & 3 \end{vmatrix}+yx^2\begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 2 \\ 8 & 8 & 3 \end{vmatrix}$$

Applying $C_1 \to C_1 - C_2$, $C_2 \to C_2 - C_3$ in the first determinant

$$= x^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 5 & 3 \end{vmatrix} + yx^2 \times 0$$

As the first two columns of the 2[nd] determinant are same.
Expanding the first determinant through R_1

$$= x^3.1.\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = x^3(5-4)$$

$= x^3 = RHS$ thus proved

14. Find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$, if $x = ae^{\theta}(\sin\theta - \cos\theta)$ and $y = ae^{\theta}(\sin\theta + \cos\theta)$.

Solution:

$y = ae^{\theta}(\sin\theta + \cos\theta)$
$x = ae^{\theta}(\sin\theta - \cos\theta)$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ (Applying parametric differentiation) … (1)

Now, $\frac{dy}{d\theta} = ae^{\theta}(\cos\theta - \sin\theta) + ae^{\theta}(\sin\theta + \cos\theta)$

$= 2ae^{\theta}(\cos\theta)$ (Applying product Rule)

$\frac{dx}{d\theta} = ae^{\theta}(\cos\theta + \sin\theta) + ae^{\theta}(\sin\theta - \cos\theta)$
$= 2ae^{\theta}(\sin\theta)$

Substituting the values of $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ in (1)

$$\frac{dy}{dx} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta} = \cot\theta$$

Now $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$

$$[\cot\theta]_{\theta=\pi/4} = \cot\frac{\pi}{4} = 1.$$

15. If y = $Pe^{ax} + Qe^{bx}$, show that

$$\frac{d^2y}{dx^2}-(a+b)\frac{dy}{dx}+aby=0.$$

Solution:

If y = $Pe^{ax}+Qe^{bx}$ ...(1)

$$\frac{dy}{dx}=aPe^{ax}+bQe^{bx} \quad ...(2)$$

$$\frac{d2y}{dx^2}=a^2Pe^{ax}+b^2Qe^{bx} \quad ...(3)$$

multiplying ... (1) by ab
we get, aby = $abPe^{ax}+abQe^{bx}$ ... (4)
multiplying (2) by (a + b)

we get,, $(a+b)\frac{dy}{dx}=(a+b)(aPe^{ax}+bQe^{bx})=(a^2Pe^{ax}+b^2Pe^{bx})+(abPe^{ax}+abQe^{bx})$

or, $(a^2bPe^{ax}+b^2Qe^{bx})-(a+b)\frac{dy}{dx}+(abPe^{ax}+abQe^{bx})$

or, $\frac{d^2y}{dx^2}-(a+b)\frac{dy}{dx}+aby=0$

16. Find the value(s) of x for which y = $[x(x-2)]^2$ is an increasing function.

OR

Find the equations of the tangent and normal to the curve $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ at the point $\left(\sqrt{2}a,b\right)$.

Solution:

y = $[x(x-2)]^2$

we know, for increasing function we have $f'(x)\geq 0$

$$\therefore\ f'(x)=2[x(x-2)]\left[\frac{d}{dx}x(x-2)\right]$$

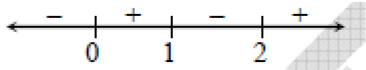
Or, $f'(x)=2[x(x-2)]\frac{d}{dx}(x^2-2x)$

= 2x(x – 2) (2x – 2)
= 4x(x – 2) (x – 1)

For $f'(x)\geq 0$

i.e., 4x(x – 1) (x – 2) ≥ 0

the values of x are :



$x \in [0,1] \cup [2, \infty]$

OR

The slope of the tangent at $\left(\sqrt{2}\,a, b\right)$ to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0$$

$$\Rightarrow y' = \left.\frac{b^2 x}{a^2 y}\right]_{(\sqrt{2}a, b)} = \frac{b^2 \sqrt{2}a}{a^2 b} = \frac{b\sqrt{2}}{a}$$

The equation of the tangent :

$$y - b = \frac{b\sqrt{2}}{a}(x - \sqrt{2}a) \text{ \{using point-slope form : } y - y_1 = m(x - x_1)\}$$

$$ay - ab = b\sqrt{2}x - 2ab$$

or $b\sqrt{2}x - ay - ab = 0$

Normal :

The slope of the normal = $\frac{-1}{dy/dx}$

$$= \frac{-1}{\frac{b\sqrt{2}}{a}} = -\frac{a}{b\sqrt{2}}$$

Equation of Normal :

$$y - b = \frac{-a}{b\sqrt{2}}(x - \sqrt{2}a)$$

$$yb\sqrt{2} - b^2\sqrt{2} = -ax + \sqrt{2}a^2$$

or $ax + b\sqrt{2}y - \sqrt{2}(a^2 + b^2) = 0$

17. Evaluate :

$$\int_0^{\pi} \frac{4x \sin x}{1 + \cos^2 x} dx$$

OR

Evaluate :

$$\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$$

Solution:

$$I=\int_0^{\pi}\frac{4x\sin x}{1+\cos^2 x}dx$$

$$I=\int_0^{\pi}\frac{4(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx \quad \left\{Applying \int f(a-x)=\int f(x)\right.$$

$$I=\int_0^{\pi}\frac{4\pi\sin x}{1+\cos^2 x}dx-\int_0^{\pi}\frac{4x\sin x}{1+\cos^2 x}dx$$

Or,

$$I=\int_0^{\pi}\frac{4\pi\sin x}{1+\cos^2 x}dx-I$$

$$2I=4\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx$$

$$2I=4\pi.2\times\int_0^{\pi/2}\frac{\sin x}{1+\cos^2 x}dx \quad \left\{\text{Applying} \int_0^{2a} f(x)dx=2\int_0^{2} f(x)dx \quad \text{if f (2a – x)= f (x)}\right.$$

$$I=4\pi\int_0^{\pi/2}\frac{\sin x}{1+\cos^2 x}dx$$

put cos x = t $\Rightarrow -\sin x dx = dt$

as well for x = 0, $x=\pi/2$

t = 1 t = 0

$$\therefore I=4\pi\int_1^{0}\frac{-dt}{1+t^2}$$

$$I=4\pi\int_0^{1}\frac{dt}{1+t^2} \quad \left\{\int_a^b f(x)dx=-\int_b^a f(x)dx\right.$$

$$I=4\pi\left[\tan^{-1}1-\tan^{-1}0\right]$$

$$=4\pi\times\frac{\pi}{4}=\pi^2.$$

OR

$$\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$$

put , x + 2 $=\lambda\left(\frac{d}{dx}(x^2+5x+6)\right)+\mu$

$$x+2=2\lambda x+5\lambda+\mu$$

comparing coefficients of x both sides

$$1=2\lambda \Rightarrow \lambda=1/2$$

comparing constant terms both sides,

$$2=5\lambda+\mu$$

or, $2=5\left(\frac{1}{5}\right)+\mu$

or, $\mu=2-\frac{5}{2}=\frac{-1}{2}$

$$\therefore \int \frac{x+2}{\sqrt{x^2+5x+6}}dx=\int \frac{\frac{1}{2}(2x+5)-\frac{1}{2}}{\sqrt{x^2+5x+6}}dx \quad \{as\ x+2=\lambda(2x+5)+\mu\}$$

$$\therefore I=\underset{(I_1)}{\int \frac{\frac{1}{2}(2x+5)}{\sqrt{x^2+5x+6}}dx}-\underset{(I_2)}{\frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+6}}}$$

$$\therefore I=I_1-I_2$$

$I_1=\frac{1}{2}\int \frac{(2x+5)}{\sqrt{x^2+5x+6}}dx,$ put $x^2+5x+6=t$

$$\therefore\ (2x+5)dx=dt$$

$$=\frac{1}{2}\int \frac{dt}{\sqrt{t}}=\frac{1}{2}\left(\frac{t^{-1/2+1}}{-\frac{1}{2}+1}\right)+C=t^{1/2}+C=\sqrt{x^2+5x+6}+C$$

$$I_2=\frac{1}{2}\frac{dx}{\sqrt{x^2+5x+6}}$$

$$=\frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+\frac{25}{4}-\frac{25}{4}+6}}=\frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$\frac{1}{2}.\log\left[\left(x+\frac{5}{2}\right)+\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right]+C$$

$$\frac{1}{2}.\log\left[\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right]+C$$

Substituting the values of I_1 and I_2 in (1)
we get,

$$I=\sqrt{x^2+5x+6}+\frac{1}{2}\log\left[\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right]+c$$

18. Find the particular solution of the differential equation $\frac{dy}{dx}=1+x+y+xy$, given that y = 0 when x = 1.

Solution:

$$\frac{dy}{dx}=(1+x)+y(1+x)$$

Or, $\frac{dy}{dx}=(1+y)(1+x)$

Or, $\frac{dy}{1+y}=(1+x)dx$

$$\int\frac{dy}{1+y}=\int(1+x)dx$$

$$\log|1+y|=x+\frac{x^2}{2}+C$$

given y = 0 when x = 1

i.e., $\log|1+0|=1+\frac{1}{2}+C$

$$\Rightarrow C=-\frac{3}{2}$$

$\therefore$ The particular solution is

$$\log|1+y|=\frac{x^2}{2}+x-\frac{3}{2}.$$

or the answer can expressed as

$$\log|1+y| = \frac{x^2+2x-3}{2}$$

or $1+y = e^{(x^2+2x-3)/2}$

or, $y = e^{(x^2+2x-3)} - 1.$

19. Solve the differential equation $(1+x^2)\frac{dy}{dx} + y = e^{\tan -1x}$.

Solution:

$$(1+x^2)\frac{dy}{dx} + y = e^{\tan -1x}$$

$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e\tan^{-1}}{1+x^2}$$

It is a linear differential equation of 1st order.
comparing with standard LDE

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$P(x) = \frac{1}{1+x^2}; Q(x) = \frac{e\tan^{-1x}}{1+x^2}$$

Integrating factor $IF = e^{\int Pdx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan -1x}$

Solution of LDE

$$y.IF = \int IF\ Q(x)dx + C$$

$$\therefore y.e^{\tan -1x} = \int e^{\tan -1x}.\frac{e^{\tan -1x}}{1+x^2}dx + C$$

$$y.e^{\tan^{-1}x} = \int \frac{(e^{\tan^{-1}x})^2}{1+x^2}dx + C \quad ....(1)\ \text{y}$$

To solving $\int \frac{(e^{\tan^{-1}x})^2}{1+x^2}dx$

Put $e^{\tan^{-1}x} = t$

or $e^{\tan^{-1}x}.\frac{1}{1+x^2} = dt$

$$\therefore \int \frac{e^{\tan^{-1}x} . e^{\tan^{-1}x}}{1+x^2} dx = \int t dt$$

$$= \frac{t^2}{2} + C = \frac{(e^{\tan^{-1}x})^2}{2} + C$$

Substituting in (1)

$$y . e^{\tan^{-1}x} = \frac{(e^{\tan^{-1}x})^2}{2} + C$$

20. Show that the four points A, B, C and D with position vectors $4\hat{i}+5\hat{j}+\hat{k}, -\hat{j}-\hat{k}, 3\hat{i}+9\hat{j}+4\hat{k}$ and $4(-\hat{i}+\hat{j}+\hat{k})$ respectively are coplanar.

OR

The scalar product of the vector $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $\vec{b} = 2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c} = \lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b}+\vec{c}$.

Solution:

If P.V of $\vec{A} = 4\hat{i}+5\hat{j}+\hat{k}$

$\vec{B} = -\hat{j}-\hat{k}$

$\vec{C} = 3\hat{i}+9\hat{j}+4\hat{k}$

$\vec{D} = 4(-\hat{i}+\hat{j}+\hat{k})$

Points $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ all Coplanar if $\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right] = 0 \quad \Rightarrow (1)$

So, $\overrightarrow{AB} = P.V.\text{ of } \vec{B} - P.V.\text{ of } \vec{A} = \ -4\hat{i}-6\hat{j}-2\hat{k}$

$\overrightarrow{AC} = P.V.\text{ of } \vec{C} - P.V.\text{ of } \vec{A} = \ -\hat{i}-4\hat{j}+3\hat{k}$

$\overrightarrow{AD} = P.V.\text{ of } \vec{D} - P.V.\text{ of } \vec{A} = \ -8\hat{i}-\hat{j}+3\hat{k}$

So, so for $\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]$

$$= \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

expand along $R_1 \rightarrow$

– 4[12 + 3] + 6[–3 + 24] – 2[1 +32]
= – 60 + 126 – 66

= 0
So, we can say that point A, B, C, D are Coplanar proved

OR

Given $\rightarrow \quad \vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$

$\vec{c} = \lambda\hat{i} + 2\hat{j} - 3\hat{k}$

So, $\vec{b} + \vec{c} = (2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$

Unit vector along $(\vec{b} + \vec{c}) = \dfrac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 36 + 4}}$

$= \dfrac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 40}}$

given that dot product of $\vec{a}$ with the unit vector of $\vec{b} + \vec{c}$ is equal to 1
So, apply given condition

$$\frac{(2+\lambda) + 6 - 2}{\sqrt{(2+\lambda)^2 + 40}} = 1$$

$\Rightarrow 2 + \lambda + 4 = \sqrt{(2+\lambda)^2 + 40}$

Squaring $36 + \lambda^2 + 12\lambda = 4 + \lambda^2 + 4\lambda + 40$

$\Rightarrow 8\lambda = 8$

$\Rightarrow \lambda = 1.$

21. A line passes through (2, –1, 3) and is perpendicular to the lines

$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and

$\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$. Obtain its equation in vector and Cartesian form.

Solution:

Line L is passing through point $= (2\hat{i} - \hat{j} + 3\hat{k})$

If $L_1 \Rightarrow \vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$

$L_2 \Rightarrow \vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

Let dr of line L = a_1, a_2, a_3
The eq^n of L in vector form $\Rightarrow$

$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \mathrm{k}(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

k is any constant.
so by condition that L1is perpendicular to L $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$2a_1 - 2a_2 + a_3 = 0$ … (1)

and also

$L \perp L_2$

so, $a_1 + 2a_2 + 2a_3 = 0$ …(2)
Solve (1), (2)
$3a_1 + 3a_3 = 0$

$\Rightarrow\ a_3 = -a_1$

put it in (2)
$a_1 + 2a_2 - 2a_1 = 0$

$a_2 = \frac{a_1}{2}$ let

so dr of L = $\left(a_1, \frac{a_1}{2}, -a_1\right)$

so we can say dr of L $= \left(1, \frac{1}{2}, -1\right)$

so eq^n of L in vector form

$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + k\left(\hat{i} + \frac{\hat{j}}{2} - \hat{k}\right)$

3-D form $\rightarrow \frac{x-2}{1} = \frac{y+1}{1/2} = \frac{z-3}{-1}$

22. An experiment succeeds thrice as often as it fails. Find the probability that in the next five trials, there will be at least 3 successes.
Solution:

In Binomial distribution

$(p+q)^n = {}^nC_0.p^n + {}^nC_1.p^{n-1}.q^1 + {}^nC_2.p^{n-2}.q^2 + ... + {}^nC_n.q^n$

if p = probability of success
q = prob. of fail
given that p = 3q …(1)
we know that p + q = 1

so, 3q + q = 1

$q = \frac{1}{4}$

So, $p = \frac{3}{4}$

Now given $\Rightarrow$ n = 5 we required minimum 3 success
$(p + q)^5 = {}^5C_0.p^5 + {}^5C_1.p^4.q^1 + {}^5C_2.p^3.q^2$

$$= {}^5C_0.\left(\frac{3}{4}\right)^5 + {}^5C_1.\left(\frac{3}{4}\right)^4.\left(\frac{1}{4}\right) + {}^5C_2.\left(\frac{3}{4}\right)^3.\left(\frac{1}{4}\right)^2$$

$$= \frac{3^5}{4^5} + \frac{5.3^4}{4^5} + \frac{10.3^3}{4^5}$$

$$= \frac{3^5 + 5.3^4 + 10.3^3}{4^5} = \frac{3^3[9+15+10]}{4^5} = \frac{34 \times 27}{16 \times 64} = \frac{459}{512}.$$

SECTION C

Question numbers 23 to 29 carry 6 marks each.

23. Two schools A and B want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school A wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to 3, 2 and 1 students respectively with a total award money of ₹ 1,600. School B wants to spend ₹ 2,300 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is ₹ 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award.

Solution:

Let Matrix D represents number of students receiving prize for the three categories :

D =

Number of students of school	SINCERITY	TRUTHFULNESS	HELPFULNESS
A	3	2	1
B	4	1	3
One student for each value	1	1	1

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where x, y and z are rupees mentioned as it is the question, for sincerity, truthfulness and helpfulness respectively.

$E = \begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$ is a matrix representing total award money for school A, B and for one prize for each value.

We can represent the given question in matrix multiplication as :

DX = E

or $\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$

Solution of the matrix equation exist if $|D| \neq 0$

i.e., $\begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 3[1-3]-2[4-3]+1[4-1]$

$= -6 - 2 + 3$

$= -5$

therefore, the solution of the matrix equation is

$X = D^{-1} E$

To find D^{-1}; $D^{-1} = \frac{1}{|D|} adj(D)$

Cofactor Matrix of D

$= \begin{bmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{bmatrix}$

Adjoint of D = adj (D)

$= \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$

{transpose of Cofactor Matrix}

$\therefore D^{-1} = \frac{1}{-5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$

Now, $X = D^{-1}E$

$= \frac{1}{-5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$

$\therefore$ x = 200, y = 300, z = 400. Ans.

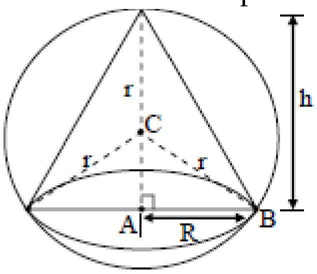
Award can also be given for Punctuality.

24. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also show that the maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.

Solution:

Let R and h be the radius and height of the cone.

r be the radius of sphere.



To show $h = \frac{4r}{3}$

and Maximum Volume of Sphere

$= \frac{8}{27}$ Volume of Sphere

In ΔABX, $AC = h - r$

$\therefore (h - r)^2 + R^2 = r^2$ {Pythagorus Theorem}

$\Rightarrow R^2 = r^2 - (h-r)^2$

Volume of cone : $V = \frac{1}{3}\pi R^2 h$

or, $V = \frac{1}{3}\pi(r^2 - (h-r)^2)h$

$V = \frac{1}{3}\pi[r^2 - h^2 - r^2 + 2hr]h$

$V = \frac{1}{3}\pi[2h^2 r - h^3]$

For maxima or minima, $\frac{dV}{dh} = 0$

Now, $\frac{dV}{dh} = \frac{1}{3}\pi[4hr - 3h^2]$

Putting, $\frac{dV}{dh} = 0$

We get $4hr = 3h^2$

$\Rightarrow h = \frac{4r}{3}$

$\frac{d^2V}{dh^2} = \frac{1}{3}\pi[4r - 6h]$

Putting $h = \frac{4r}{3}$

$\frac{d^2V}{dh^2} = \frac{1}{3}\pi\left(4r - \frac{6.4r}{3}\right)$

$= -\frac{1}{3}\pi[4r]$

Which is less than zero, therefore

$h = \frac{4r}{3}$ is a Maxima

and the Volume of the cone at $h = \frac{4r}{3}$

will be maximum,

$$V = \frac{1}{3}\pi R^2 h$$

$$= \frac{1}{3}\pi[r^2 - (h-r)^2]h$$

$$= \frac{1}{3}\pi\left[r^2 - \left(\frac{4r}{3} - r\right)^2\right]\left[\frac{4r}{3}\right]$$

$$= \frac{1}{3}\pi\left[\frac{8r^2}{9}\right]\left[\frac{4r}{3}\right]$$

$$= \frac{8}{27}\left(\frac{4\pi r^3}{3}\right)$$

$= \frac{8}{27}$ (Volume of the sphere)

25. Evaluate :

$$\int \frac{1}{\cos^4 x + \sin^4 x} dx$$

Solution:

$$\int \frac{1}{\cos^4 x + \sin^4 x} dx$$

$$= \int \frac{\frac{1}{\cos^4 x} dx}{1 + \tan^4 x}$$

$$= \int \frac{\sec^2 x \sec^2 x dx}{1 + \tan^4 x}$$

$$= \int \frac{(1 + \tan^2 x)\sec^2 x dx}{1 + \tan^4 x}$$

put tan x = t ⇒ $\sec^2$x dx = dt

$$= \int \frac{(1 + t^2)dt}{1 + t^4}$$

$$= \int \frac{(1/t^2 + 1)dt}{\frac{1}{t^2} + t^2} \{\text{dividing each by } t^2\}$$

$$= \int \frac{\left(1 + 1/t^2\right)dt}{\left(t - 1/t\right)^2 + 2}$$

Put $t - \frac{1}{t} = z \quad \Rightarrow \left(1 + \frac{1}{t^2}\right) dt = dz$

$= \int \frac{dz}{z^2 + 2} = \frac{1}{\sqrt{2}} \tan^{-1} z + C$

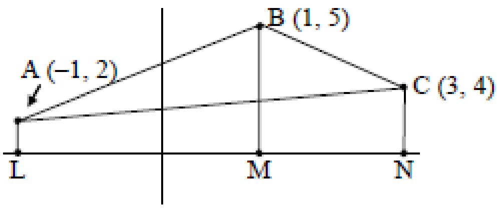
$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\tan x - \frac{1}{\tan x}\right) + C$

$= \frac{1}{\sqrt{2}} \tan^{-1}(\tan x - \cot x) + C$

26. Using integration, find the area of the region bounded by the triangle whose vertices are (– 1, 2), (1, 5) and (3, 4).

Solution:

Let A = (–1, 2)
B = (1, 5)
C = (3, 4)



We have to find the area of Δ ABC

Find eq^n of Line AB $\rightarrow y - 5 = \left(\frac{2-5}{-1-1}\right) \cdot (x - 1)$

$y - 5 = \frac{3}{2}(x - 1)$

2y – 10 = 3x – 3
3x – 2y + 7 = 0 …(1)

$y = \frac{3x + 7}{2}$

Eq^n of BC $\rightarrow \ y - 4 = \left(\frac{5-4}{1-3}\right) \cdot (x - 3)$

$y - 4 = \frac{1}{-2}(x - 3)$

$2y - 8 = -x + 3$

$x + 2y - 11 = 0$ ....(2)

$y = \frac{11 - x}{2}$

Eq^n of AC $\rightarrow \ y - 4 = \left(\frac{2-4}{-1-3}\right) \cdot (x - 3)$

$y - 4 = \frac{1}{2}(x - 3) \Rightarrow 2y - 8 = x - 3$

x – 2y + 5 = 0 …(3)

$\Rightarrow y = \frac{x+5}{2}$

So, required area = $\int_{-1}^{1}\left(\frac{3x+7}{2}\right)dx + \int_{1}^{3}\left(\frac{11-x}{2}\right)dx - \int_{-1}^{3}\left(\frac{x+5}{2}\right)dx$

$= \frac{1}{2}\left[\frac{3x^2}{2}+7x\right]_{-1}^{1} + \frac{1}{2}\left[11x - \frac{x^2}{2}\right]_{1}^{3} - \frac{1}{2}\left[\frac{x^2}{2}+5x\right]_{-1}^{3}$

$= \frac{1}{2}\left[\left(\frac{3}{2}+7\right)-\left(\frac{3}{2}-7\right)\right] + \frac{1}{2}\left[\left(33-\frac{9}{2}\right)-\left(11-\frac{1}{2}\right)\right] - \frac{1}{2}\left[\left(\frac{9}{2}+15\right)-\left(\frac{1}{2}-5\right)\right]$

$= \frac{1}{2}[14+22-4-24] = \frac{1}{2}[36-28] = 4$ square unit

27. Find the equation of the plane through the line of intersection of the planes x + y + z = 1 and 2x + 3y + 4z = 5 which is perpendicular to the plane x – y + z = 0. Also find the distance of the plane obtained above, from the
origin.

OR

Find the distance of the point (2, 12, 5) from the point of intersection of the line
$\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = 0$.

Solution:

Eq^n of given planes are

$P_1 \Rightarrow$ x + y + z – 1 = 0

$P_2 \Rightarrow$ 2x + 3y + 4z – 5 = 0

Eq^n of plane through the line of intersection of planes P_1, P_2 is

$P_1 + \lambda P_2 = 0$

(x + y + z – 1) + λ(2x + 3y + 4z – 5) = 0

(1 + 2λ) x + (1 + 3λ) y + (1 + 4λ) z + (– 1 – 5λ) = 0 … (1)

given that plane represented by eq^n (1) is perpendicular to plane

x – y + z = 0

so we use formula $a_1a_2 + b_1b_2 + c_1c_2 = 0$

so (1 + 2λ).1 + (1 + 3λ). (– 1) + (1 + 4λ).1 = 0

1 + 2λ – 1 – 3λ + 1 + 4λ = 0

3λ + 1 = 0

$\lambda = \frac{-1}{3}$

Put $\lambda = -\frac{1}{3}$ in eq^n (1) so we get

$\left(1-\frac{2}{3}\right)x + (1-1)y + \left(1-\frac{4}{3}\right)z + \frac{2}{3} = 0$

$\frac{x}{3} - \frac{z}{3} + \frac{2}{3} = 0$

x – z + 2 = 0 Ans.

OR

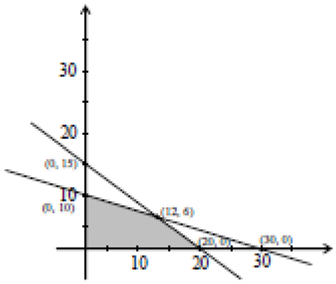
General points on the line:

x = 2 + 3λ, y = – 4 + 4λ, z = 2 + 2λ
The equation of the plane :
$\vec{r}.(\hat{i}-2\hat{j}+\hat{k})=0$
The point of intersection of the line and the plane :
Substituting general point of the line in the equation of plane and finding the particular value of λ.
$[(2+3\lambda)\hat{i}+(-4+4\lambda)\hat{j}+(2+2\lambda)\hat{k}].(\hat{i}-2\hat{j}+\hat{k})=0$
$(2+3\lambda).1+(-4+4\lambda)(-2)+(2+2\lambda).1=0$
$12-3\lambda=0$ or, $\lambda=4$
∴ the point of intersection is :
(2 + 3 (4), –4 +4(4), 2 + 2(4)) = (14, 12, 10)
Distance of this point from (2, 12, 5) is
$=\sqrt{(14-2)^2+(12-12)^2+(10-5)^2}$ {Applying distance formula}
$=\sqrt{12^2+5^2}$
= 13 Ans.

28. A manufacturing company makes two types of teaching aids A and B of Mathematics for class XII. Each type of A requires 9 labour hours of fabricating and 1 labour hour for finishing. Each type of B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of ₹ 80 on each piece of type A and ₹120 on each piece of type B. How many pieces of type A and type B should be manufactured per week to get a maximum profit? Make it as an LPP and solve graphically. What is the maximum profit per week?
Solution:
Let pieces of type A manufactured per week = x
Let pieces of type B manufactured per week = y
Companies profit function which is to be maximized : Z = 80x + 120y

	Fabricating hours	Finishing hours
A	9	1
B	12	3

Constraints : Maximum number of fabricating hours = 180
$\therefore 9x+12y\le 180 \Rightarrow 3x+4y\le 60$ K
Where 9x is the fabricating hours spent by type A teaching aids, and 12y hours spent on type B.
and Maximum number of finishing hours = 30
$\therefore x+3y\le 30$
where x is the number of hours spent on finishing aid A while 3y on aid B.
So, the LPP becomes :
Z (MAXIMISE) = 80x + 120 y
Subject to $3x+4y\le 60$
$x+3y\le 30$
$x\ge 0$
$y\ge 0$
Solving it Graphically :



Z = 80x + 120y at (0, 15)
= 1800
Z = 1200 at (0, 10)
Z = 1600 at (20, 0)
Z = 960 + 720 at (12, 6)
= 1680
Maximum profit is at (0, 15)
∴ Teaching aid A = 0
Teaching aid B = 15
Should be made

29. There are three coins. One is a two-headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the times and third is also a biased coin that comes up tails 40% of the times. One of the three coins is chosen at random and tossed, and it shows heads. What is the probability that it was the two-headed coin?

OR

Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find the probability distribution of the random variable X, and hence find the mean of the distribution.

Solution:
If there are 3 coins.
Let these are A, B, C respectively
For coin A $\rightarrow$ Prob. of getting Head P(H) = 1
For coin B $\rightarrow$ Prob. of getting Head P(H) = $\frac{3}{4}$
For coin C $\rightarrow$ Prob. of getting Head P(H) = 0.6
we have to find $P\left({}^{A}\!/_{H}\right)$ = Prob. of getting H by coin A
So, we can use formula

$$P\left({}^{A}\!/_{H}\right)=\frac{P\left({}^{H}\!/_{A}\right).P(A)}{P\left({}^{H}\!/_{A}\right).P(A)+P\left({}^{H}\!/_{B}\right).P(B)+P\left({}^{H}\!/_{C}\right).P(C)}$$

Here P(A) = P(B) = P(C) = $\frac{1}{3}$ (Prob. of choosing any one coin)

$$P\left({}^{H}\!/_{A}\right)=1,\ P\left({}^{H}\!/_{B}\right)=\frac{3}{4},\ P\left({}^{H}\!/_{C}\right)=0.6$$

Put value in formula so

$$P\left(A/H\right) = \frac{1.\frac{1}{3}}{1.\frac{1}{3}+\frac{3}{4}.\frac{1}{3}+\frac{1}{3}(0.6)} = \frac{1}{1+0.75+0.6}$$

$$= \frac{100}{235}$$

$$= \frac{20}{47} \textit{Ans.}$$

OR

First six numbers are 1, 2, 3, 4, 5, 6.
X is bigger number among 2 number so

Variable (X)	2	3	4	5	6
Probability P(X)					

if X = 2
for P(X) = Prob. of event that bigger of the 2 chosen number is 2
So, Cases = (1, 2)

So, $P(X) = \frac{1}{{}^6C_2} = \frac{1}{15}$....(1)

if X = 3
So, favourable cases are = (1, 3), (2, 3)

$$P(x) = \frac{2}{{}^6C_2} = \frac{2}{15}....(2)$$

if X = 4 $\Rightarrow$ favourable casec = (1, 4), (2, 4), (3, 4)

$$P(X) = \frac{3}{15}....(3)$$

if X = 5 $\Rightarrow$ favourable casec = (1, 5), (2, 5), (3, 5), (4, 5)

$$P(X) = \frac{4}{15}....(4)$$

if X = 6 $\Rightarrow$ favourable casec = (1, 6), (2, 6), (3, 6), (4, 6), (5, 6)

$$P(X) = \frac{5}{15}....(5)$$

We can put all value of P(X) in chart, So

Variable (X)	2	3	4	5	6
Probability P(X)	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

and required mean $= 2.\left(\frac{1}{15}\right)+3.\left(\frac{2}{15}\right)+4\left(\frac{3}{15}\right)+5.\left(\frac{4}{15}\right)+6.\left(\frac{5}{15}\right)$

$$= \frac{70}{15} = \frac{14}{3} \text{Ans.}$$