CBSE-XII-2016 EXAMINATION

SET-1

Series ONS

MATHEMATICS

Paper & Solution

Code: 65/1/E

Time: 3 Hrs.

Max. Marks: 100

General Instructions:

*(i) **All** questions are **compulsory**.*

*(ii) Please check that this question paper contains **26** questions.*

*(iii) Questions **1 - 6** in **Section A** are very short-answer type questions carrying **1** mark each.*

*(w) Questions **7 - 19** in **Section B** are long-answer **I** type questions carrying **4** marks each.*

*(v) Questions **20 - 26** in Section C are long-answer **II** type questions carrying **6** marks each.*

(vi) Please write down the serial number of the question before attempting it.

SECTION – A

Question numbers 1 to 6 carry 1 mark each.

1. Write the value of $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$.

Solution:

$R_1 \to R_1 + R_2 + R_3$ or $C1 \to C_1 + C_2 + C_3$

Ans. 0

2. If A = $\begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$ and BA = (b_{ij}), find $b_{21} + b_{32}$.

Solution:

$b_{21} = -16$, $b_{23} = -2$ [For any one correct value]

$b_{21} + b_{23} = -16 + (-2) = -18$

3. Write the number of all possible matrices of order 2 x 3 with each entry I or 2.

Solution:

2^6 or 64

4. Write the coordinates of the point which is the reflection of the point (α, β, γ) in the XZ-plane.

Solution:

$(\alpha, -\beta, \gamma)$

5. Find the position vector of the point which divides the join of points with position vectors $\vec{a}+3\vec{b}$ and $\vec{a}-\vec{b}$ internally in the ratio 1 : 3.

Solution:

$\dfrac{1(\vec{a}-\vec{b})+3(\vec{a}+3\vec{b})}{4}$ (i.e., using correct formula)

$=\vec{a}+2\vec{b}$

6. If $|\vec{a}|=4, |\vec{b}|=3$ and $\vec{a}\,.\,\vec{b}=6\sqrt{3},$ then find the value of $|\vec{a}\times\vec{b}|$.

Solution:

Finding $\cos\theta=\dfrac{\sqrt{3}}{2}$

$|\vec{a}\times\vec{b}|=6$

SECTION – B

Question numbers 7 to 19 carry 4 marks each.

7. Solve for x: $\tan^{-1}\left(\dfrac{2-x}{2+x}\right)=\dfrac{1}{2}\tan^{-1}\dfrac{x}{2}, x>0.$

OR

Prove that $2\sin^{-1}\left(\dfrac{3}{5}\right)-\tan^{-1}\left(\dfrac{17}{31}\right)=\dfrac{\pi}{4}.$

Solution:

$$\tan^{-1}\left(\frac{2-x}{2+x}\right)=\frac{1}{2}\tan^{-1}\frac{x}{2}$$

$$\Rightarrow 2\tan^{-1}\left(\frac{2-x}{2+x}\right)=\tan^{-1}\frac{x}{2}$$

$$\Rightarrow \tan^{-1}\frac{2\left(\frac{2-x}{2+x}\right)}{1-\left(\frac{2-x}{2+x}\right)^2}=\tan^{-1}\frac{x}{2}$$

$$\Rightarrow \tan^{-1}\frac{4-x^2}{4x}=\tan^{-1}\frac{x}{2}$$

$$\Rightarrow \frac{4-x^2}{4x}=\frac{x}{2}$$

$$\Rightarrow x=\frac{2}{\sqrt{3}}\ (\because x>0)$$

OR

$$2\sin^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{17}{31}\right)$$

$$= 2\tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} - \tan^{-1}\left(\frac{17}{31}\right)$$

$$= \tan^{-1}\frac{24}{7} - \tan^{-1}\frac{17}{31}$$

$$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\times\frac{17}{31}}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

8. On her birthday Seems decided to donate some money to children of an orphanage home. If there were 8 children less, everyone would have got ₹ 10 more. However, if there were 16 children more, everyone would have got ₹ 10 less. Using matrix method, find the number of children and the amount distributed by Seema. What values are reflected by Seema's decision?

Solution:

Let the number of children be x and the amount distributed by Seema for one student be ₹ y.

So, (x – 8)(y + 10) = xy

$$\Rightarrow 5x - 4y = 40 \qquad \ldots(i)$$

and (x + 16)(y – 10) = xy

$$\Rightarrow 5x - 8y = -80 \qquad \ldots(ii)$$

Here $A = \begin{matrix} 5 & -4 \\ 5 & -8 \end{matrix},\ X = \begin{matrix} x \\ y \end{matrix},\ B = \begin{matrix} 40 \\ -80 \end{matrix}$

$$AX = B \Rightarrow X = A^{-1}B$$

$$A^{-1} = -\frac{1}{20}\begin{matrix} -8 & 4 \\ -5 & 5 \end{matrix}$$

$$\Rightarrow \begin{matrix} x \\ y \end{matrix} = \begin{matrix} 32 \\ 30 \end{matrix}$$

$$\Rightarrow x = 32,\ y = 30$$

No. of students = 32

Amount given to each student = ₹ 30.

Value reflected: To help needy people.

9. If $x = e^{\cos 2t}$ and $y = e^{\sin 2t}$, prove that $\frac{dy}{dx} = -\frac{y\log x}{x\log y}$.

OR

Verify Mean Value theorem for the function $f(x) = 2\sin x + \sin 2x$ on $[0, \pi]$.

Solution:

$$\frac{dx}{dt} = e^{\cos 2t}(-2\sin 2t) \text{ or } -2x\sin 2t$$

$$\frac{dx}{dt} = e^{\cos 2t} 2\cos 2t) \text{ or } -2y\cos 2t$$

$$\frac{dx}{dt} = \frac{-e^{\sin 2t} 2\cos 2t}{e^{\cos 2t} 2\sin 2t} \text{ or } -\frac{y}{x}\frac{\cos 2t}{\sin 2t}$$

$$= \frac{-y\log x}{x\log y}$$

OR

f(x) = 2 sin x + sin 2x on $[0, \pi]$

$$\left.\begin{array}{l} f(x) \text{ is continuous in } [0, \pi] \\ f(x) \text{ is differentiable in } (0, \pi) \end{array}\right\}$$

$\therefore$ Mean value theorem is applicable

f(0) = 0, $f(\pi) = 0$

f ′(x) = 2 cos x + 2 cos 2x

f′(c) = 2 cos c + 2 cos 2c

$$f'(c) = \frac{f(\pi) - f(0)}{\pi - 0} = 0$$

$\therefore$ 2 cos c + 2 cos 2c = 0

$\Rightarrow$ cos c + 2 cos2 c – 1 = 0

$\Rightarrow$ (2 cos c – 1)(cos c + 1) = 0

$$\Rightarrow \cos c \frac{1}{2}$$

$$\Rightarrow c = \frac{\pi}{3} \in (0, \pi)$$

Hence mean value theorem is verified.

10. Show that the function *f* given by :

$$f(x) = \begin{cases} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}, & \text{if } x \neq 0 \\ -1, & \text{if } x = 0 \end{cases}$$

is discontinuous at x = 0.

Solution:

$$f(x) = \begin{array}{ll} \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} & x \neq 0 \\ -1 & x = 0 \end{array}$$

LHL: $\lim_{x\to 0^-} \dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$

$= \lim_{h\to 0} \dfrac{e^{-\frac{1}{h}}-1}{e^{-\frac{1}{h}}+1} = \dfrac{0-1}{0+1} = -1$

RHL: $\lim_{h\to 0} \dfrac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1} = \lim_{h\to 0} \dfrac{1-e^{-\frac{1}{h}}}{1+e^{-\frac{1}{h}}} = 1$

LHL $\neq$ RHL

$\therefore$ f(x) is discontinuous at x = 0

11. Find the equation of the tangent line to the curve $y=\sqrt{5x-3}-5$, which is parallel to the line 4x - 2y + 5=0.

Solution:

$y=\sqrt{5x-3}-5$

$\dfrac{dy}{dx} = \dfrac{5}{2\sqrt{5x-3}}$

Slope of line 4x – 2y + 5 = 0 is $\dfrac{-4}{-2} = 2.$

$\therefore \dfrac{5}{2\sqrt{5x-3}} = 2 \quad x \quad \dfrac{73}{80}$

Putting x $= \dfrac{73}{80}$ in eqn. (i), we get $y = \dfrac{-15}{4}$

Equation of tangent

$y+\dfrac{15}{4} = 2\left(x-\dfrac{73}{80}\right)$

or 80x – 40y – 223 = 0

12. Evaluate: $\int_1^5 \{|x-1|+|x-2|+|x-3|\}dx$

OR

Evaluate: $\int_0^\pi \dfrac{x\sin x}{1+3\cos^2 x}dx$

Solution:

$\int_1^5 \{|x-1|+|x-2|+|x-3|\}dx$

$= \int_1^5 (x-1)dx + \int_1^2 (2-x)dx + \int_2^5 (x-2)dx + \int_1^3 (3-x)dx + \int_3^5 (x-3)dx$

$$= \left[\frac{x^2}{2} - x\right]_1^5 + \left[2x - \frac{x^2}{2}\right]_1^2 + \left[\frac{x^2}{2} - 2x\right]_2^5 + \left[3x - \frac{x^2}{2}\right]_1^3 + \left[\frac{x^2}{3} - 3x\right]_3^5$$

= 17

OR

Let $I = \int_0^{\pi} \frac{x \sin x}{1 + 3\cos^2 x} dx \qquad ....(i)$

$$I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + 3\cos^2 x(\pi - x)} dx$$

$$= \int_0^{\pi} \frac{\pi \sin x}{1 + 3\cos^2 x} dx - \int_0^{\pi} \frac{x \sin x}{1 + 3\cos^2 x} dx$$

Adding (i) & (ii), we have

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1 + 3\cos^2 x} dx$$

Put cos x = t

–sin x dx = dt, when x = 0 $\Rightarrow$ t = 1, for $x = \pi \Rightarrow t = -1$

$$2I = -\pi \int_1^{-1} \frac{dt}{1 + 3t^2}$$

$$= \frac{\pi}{3} \int_{-1}^{1} \frac{dt}{\left(\frac{1}{\sqrt{3}}\right)^2 + (t)^2}$$

$$= \frac{\pi}{3} \times \sqrt{3} \left[\tan^{-1}\left(\sqrt{3}t\right)\right]_{-1}^{1}$$

$$= \frac{\sqrt{3}\pi}{3} \left[\tan^{-1}\sqrt{3} - (-\tan^{-1}\sqrt{3})\right]$$

$$I = \frac{\sqrt{3}\pi}{3} \cdot \frac{\pi}{3} = \frac{\sqrt{3}\pi^2}{9}$$

13. Find: $\int \frac{2x + 1}{(x^2 + 1)(x^2 + 4)} dx$

Solution:

Let $I = \int \frac{2x + 1}{(x^2 + 1)(x^2 + 4)} dx$

Let $\frac{2x + 1}{(x^2 + 1)(x^2 + 4)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 + 4}$

Getting $A = \frac{2}{3}, B = \frac{1}{3}, C = \frac{-2}{3}, D = \frac{-1}{3}$

$$\therefore\ I = \frac{2}{3}\int \frac{x}{x^2 + 1} dx + \frac{1}{3}\int \frac{x}{x^2 + 1} dx + \frac{-2}{3}\int \frac{xdx}{x^2 + 4} + \frac{-1}{3}\int \frac{dx}{x^2 + 4}$$

$$=\frac{1}{3}\log|x^2+1|+\frac{1}{3}\tan^{-1}x-\frac{1}{3}\log|x^2+4|-\frac{1}{6}\tan^{-1}\frac{x}{2}+C$$

14. Find: $\int(3x+5)\sqrt{5+4x-2x^2}\,dx$

Solution:

$(3x+5)\sqrt{5+4x-2x^2}\,dx$

Let 3x + 5 = A(4 – 4x) + B

$\Rightarrow\ A=-\frac{3}{4}, B=8$

$$I=-\frac{3}{4}\ (4-4x)\sqrt{5+4x+2x^2}\,dx+8\ \sqrt{5+4x-2x^2}\,dx$$

$$=-\frac{3}{4}I_1+8I_2\ \text{(let)}$$

For I_1, put 5 + 4x – $2x_2$ = t

$\Rightarrow$ (4 – 4x) dx = dt

$$-\frac{3}{4}I_1=-\frac{3}{4}\ \sqrt{t}\,\mathrm{dt}=-\frac{3}{4}\times\frac{2}{3}t^{3/2}$$

$$=-\frac{1}{2}(5+4x-2x^2)^{3/2}$$

$$8I_2=8\sqrt{2}\ \sqrt{\frac{7}{2}-(x-1)^2}\,dx$$

$$I=-\frac{1}{2}(5+4x-2x^2)^{3/2}+4\sqrt{2}(x-1)\sqrt{\frac{5}{2}+2x-x^2}+14\sqrt{2}\sin^{-1}\frac{\sqrt{2}(x-1)}{\sqrt{7}}+C$$

15. Solve the differential equation:

$$x\frac{dy}{dx}+y-x+xy\cot x=0;\ x\neq 0.$$

Solution:

$$x\frac{dy}{dx}+y-x+xy\cos x=0$$

$$\Rightarrow\frac{dy}{dx}+\left(\frac{1}{x}+\cot x\right)y=1$$

I.F = $e^{\int\left(\frac{1}{x}+\cot x\right)dx}=e^{\log(x\sin x)}$

= x sin x

$\therefore$ y $\times$ x sin x = x sin x dx

$\Rightarrow$ xy sin x = –x cos x + sin x + C

16. Solve the differential equation :

$(x^2+3xy+y^2)dx-x^2\ dy=0,$ given that y = 0, when x = 1.

Solution:

$(x^2 + 3xy + y^2)\,dx - x^2\,dy = 0$

$$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2}$$

let y = vx

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\therefore \quad v + x\frac{dv}{dx} = 1 + 3v + v^2$$

$$\Rightarrow \quad x\frac{dv}{dx} = v^2 + 2v + 1$$

$$\Rightarrow \quad \frac{dv}{(v+1)^2} = \frac{dx}{x}$$

Integrating both sides

$$\Rightarrow \quad -\frac{1}{v+1} = \log|x| + C$$

$$\Rightarrow \quad \frac{-x}{x+y} = \log|x| + C$$

When x = 1, y = 0 $\Rightarrow$ C = –1

$$\Rightarrow \quad \frac{-x}{x+y} = \log|x| - 1$$

$\Rightarrow$ y = (x + y)log |x|

or $y = \dfrac{x\log|x|}{1 - \log|x|}$

17. Find the angle between the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ if $\vec{a} = 2\hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + \hat{j} - 2\hat{k}$, and hence find a vector perpendicular to both $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.

Solution:

$\vec{a} + \vec{b} = 5\hat{i} + \hat{k}$

$\vec{a} - \vec{b} = -\hat{i} - 2\hat{j} + 5\hat{k}$

Getting $\cos\theta = 0$

$$\Rightarrow \theta = \frac{\pi}{2}$$

a vector perpendicular to both $\vec{a} + \vec{b}$ & $\vec{a} - \vec{b}$ is $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = 2\hat{i} - 26\hat{j} - 10\hat{k}$

18. Show that the lines $\dfrac{x-1}{3} = \dfrac{y-1}{-1} = \dfrac{z+1}{0}$ and $\dfrac{x-4}{2} = \dfrac{y}{0} = \dfrac{z+1}{3}$ intersect. Find their point of intersection.

Solution:

$$\frac{x-1}{3} = \frac{y-1}{-1} = \frac{z+1}{0} = \lambda \ \text{(let)} \qquad \ldots(i)$$

$$\Rightarrow \quad x = 3\lambda + 1,\ y = -\lambda + 1,\ z = -1$$

$$\frac{x-4}{2}=\frac{y}{0}=\frac{z+1}{3}=\mu \qquad ...(ii)$$

$\Rightarrow \quad x=2\mu+4,\ y=0,\ z=3\mu-1$

If the lines intersect, then they have a common point for some value of λ and μ.

So $3\lambda+1=2\mu+4$ ....(iii)

$-\lambda+1=0 \Rightarrow \lambda=1$

$3\mu-1=-1 \Rightarrow \mu=0$

Since $\lambda=1 \,\&\, \mu=0$ satisfy equation (iii) so the given lines intersect and the point of intersection is (4, 0, –1).

19. A committee of 4 students is selected at random from a group consisting of 7 boys and 4 girls. Find the probability that there are exactly 2 boys in the committee, given that at least one girl must be there in the committee.

OR

A random variable X has the following probability distribution :

X	0	1	2	3	4	5	6
P(X)	C	2C	2C	3C	C^2	$2C^2$	$7C^2$ + C

Find the value of C and also calculate mean of the distribution.

Solution:

Let A = exactly 2 boys in the committee

B = at least one girl must be there in the committee.

$$P(B)=\frac{{}^4C_1\times{}^7C_3+{}^4C_2\times{}^7C_2+{}^4C_3\times{}^7C_1+{}^4C_4}{{}^{11}C_4}$$

$$=\frac{59}{66}$$

$$P(A\cap B)=\frac{{}^4C_2\times{}^7C_2}{{}^{11}C_4}=\frac{21}{55}$$

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{21/55}{59/66}=\frac{126}{295}$$

OR

C + 2C + 2C + 3C + C^2 + $2C^2$ + $7C^2$ + C = 1

$\Rightarrow 10C^2+9C=1$

$\Rightarrow 10C^2+9C-1=0$

$\Rightarrow C=\frac{1}{10}$ or $C=-1$ (not possible)

$\therefore \quad C=\frac{1}{10}$

Mean = 0 × C + 1 × 2C + 2 × 2C + 3× 3C + 4 × C^2 + 5 × $2C^2$ + 6($7C^2$ + C)

= $56C^2$ + 21C

$= 56 \times \frac{1}{100} + 21 \times \frac{1}{10}$

= 0.56 + 2.1 = 2.66

SECTION – C

Question numbers 20 to 26 carry 6 marks each.

20. Show that the relation R defined by (a, b) R (c, d) a + d = b + c on the A x A, where A = {1, 2, 3, 10} is an equivalence relation. Hence write the equivalence class [(3, 4)]; a, b, c, d $\in$ A.

Solution:

(a, b) R(c, d) $\Rightarrow$ a + d = b + c

$\because$ a + b = b + a

$\Rightarrow$ (a, b) R(a, b) $\forall$ (a, b) $\in$ A $\times$ A

$\Rightarrow$ R is reflexive

(a, b) R(c, d) $\Rightarrow$ a + d = b + c

$\Rightarrow$ b + c = a + d

$\Rightarrow$ c + b = d + a

$\Rightarrow$ (c, d) R(a, b)

$\Rightarrow$ R is symmetric

For (a, b), (c, d) & (e, f) $\in$ A $\times$ A

(a, b) R(c, d) $\Rightarrow$ a + d = b + c ...(1)

(c, d) R(e, f) $\Rightarrow$ c + f = d + e ...(2)

adding (1) & (2), we get

a + d + c + f = b + c + d + e

$\Rightarrow$ a + f = b + e

$\Rightarrow$ (a, b) R(e, f)

$\therefore$ R is transitive.

Hence R is an equivalence relation.

Now [3, 4] = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6), (6, 7), (7, 8), (8, 9), (9, 10)}

21. Solve for $x: \begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0,$ using properties of determinants.

OR

Using elementary row operations find the inverse of matrix $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$ and hence solve the following system of equations 3x - 3y + 4z = 21, 2x - 3y + 4z = 20, -y + z = 5.

Solution:

$$\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a+x & a-x & a+x \end{vmatrix} = 0$$

Applying $C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} = 0$$

$$\Rightarrow (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix} = 0$$

Applying $R_2 \to R_2 - R_1 \;\&\; R_3 \to -R_1$

$$\Rightarrow (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} = 0$$

$\Rightarrow$ $(3a - x)(4x^2) = 0$

$\Rightarrow$ $x = 0$ or $3a$

OR

$A = I \cdot A$

$$\Rightarrow \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_1 \to R_1 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_2 \to R_2 - 4R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ 0 & 0 & 1 \end{bmatrix} A$$

Applying $R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} A \qquad \left[2\frac{1}{2} \text{ for correct operations to get } A^{-1}\right]$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

The matrix form of given equations

$$\begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 21 \\ 20 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad AX = B, \text{ where } X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 21 \\ 20 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad X = A^{-1}B$$

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} \begin{bmatrix} 21 \\ 20 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

$\therefore \quad x = 1, y = -2, z = 3$

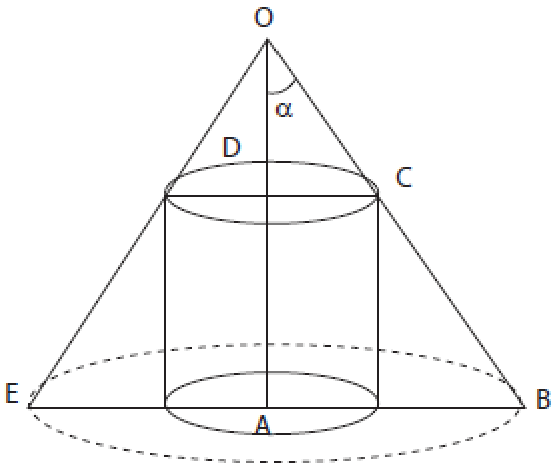
22. Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi-vertical angle a is one-third that of the cone and the greatest volume of the cylinder is $\frac{4}{27}\pi h^3 \tan^2 \alpha$.

OR

Find the intervals in which the function $f(x) = \frac{4\sin x}{2+\cos x} - x; 0 \le x \le 2\pi$ is strictly increasing or strictly decreasing.

Solution:

Let CD = R, AD = x



$\Rightarrow$ OD = h – x

$\because\ ODC \sim \Delta OAB$

$$\Rightarrow \frac{h-x}{h} = \frac{R}{AB} \Rightarrow \frac{h-x}{h} = \frac{R}{h\tan\alpha}$$

$$\Rightarrow R = (h-x)\tan\alpha$$

$$V = \pi R^2 x$$

$$= \pi(h-x)^2 \tan^2\alpha \cdot x$$

$$= \pi\tan^2\alpha(h-x)^2 x$$

$$\frac{dV}{dx} = \pi\tan^2\alpha(h^2 - 4hx + 3x^2)$$

$$\frac{dV}{dx} = 0 \Rightarrow h^2 - 4hx + 3x^2 = 0$$

$$\Rightarrow (h-x)(h-3x) = 0$$

$\Rightarrow$ x = h (not possible) or $x = \dfrac{h}{3}$

$$\frac{d^2V}{dx^2} = \pi\tan^2\alpha(-4h + 6x)$$

$$\left(\frac{d^2V}{dx^2}\right)_{x=h/3} = \pi\tan^2\alpha(-2h) < 0$$

$\Rightarrow$ V is maximum for x = $\dfrac{h}{3}$

So, $V_{\max} = \pi\tan^2\alpha(h-x)^2 x$

$$= \pi\tan^2\alpha\left(h - \frac{h}{3}\right)^2 \frac{h}{3}$$

$$= \frac{4\pi h^3}{27}\tan^2\alpha$$

OR

$$y = \frac{4\sin x}{2+\cos x} - x,\ \ x \in [0, 2\pi]$$

$$\frac{dy}{dx}=\frac{(2+\cos x)4\cos x-4\sin x(-\sin x)}{(2+\cos x)^2}-1$$

$$\frac{dy}{dx}=\frac{\cos x(4-\cos x)}{(2+\cos x)^2}$$

f(x) is strictly increasing for $f'(x) > 0$

i.e., $\cos x > 0 \Rightarrow x\in\left[0,\frac{\pi}{2}\right)\cup\left(\frac{3\pi}{2},2\pi\right]$

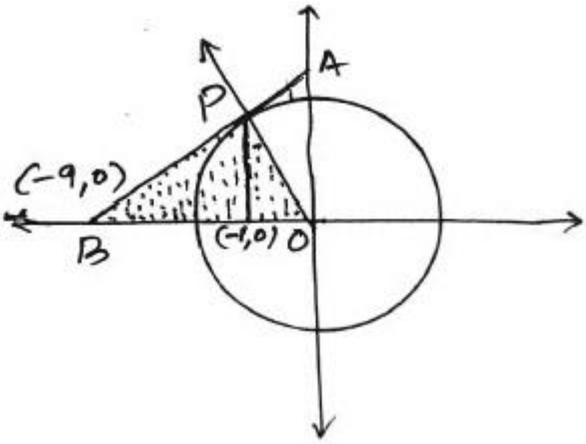
and f(x) is strictly decreasing for $f'(x) < 0$

i.e., $\cos x < 0 \Rightarrow x\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$

23. Using integration, find the area of the triangle formed by negative x-axis and tangent and normal to the circle $x^2 + y^2 = 9$ at $(-1, 2\sqrt{2})$.

Solution:

Correct Figure



Equation of circle $x^2 + y^2 = 9$

Diff. w.r.t x, we have

$$\frac{dy}{dx}=-\frac{x}{y}$$

Slope of tangent at $(-1, 2\sqrt{2})$

$$m_T=\left(-\frac{x}{y}\right)_{(-1,\,2\sqrt{2})}=\frac{1}{2\sqrt{2}}$$

eqn. of tangent

$$y-2\sqrt{2}=\frac{1}{2\sqrt{2}}(x+1)$$

$$\Rightarrow x-2\sqrt{2}y+9=0$$

It cuts x-axis at (–9, 0)

eqn. of normal

$$y-2\sqrt{2}=-2\sqrt{2}(x+1)$$

$\Rightarrow 2\sqrt{2}x + y = 0$

Area of ΔOPB

$$A = \int_{-9}^{-1} \frac{x+9}{2\sqrt{2}} dx + \int_{-1}^{0} -2\sqrt{2}\, dx$$

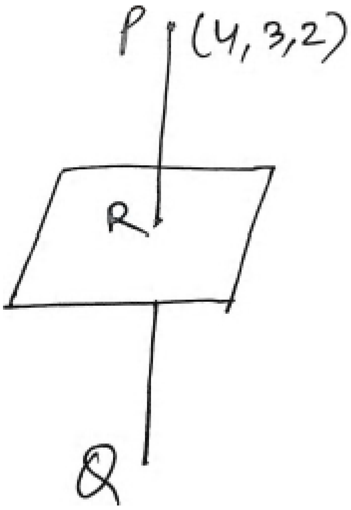
$$= \frac{1}{2\sqrt{2}}\left[\frac{x^2}{2} + 9x\right]_{-9}^{-1} - 2\sqrt{2}\left[\frac{x^2}{2}\right]_{-1}^{0}$$

$= 9\sqrt{2}$ sq. unit

24. Find the coordinates of the foot of perpendicular and perpendicular distance from the point P(4, 3, 2) to the plane x + 2y + 3z = 2. Also find the image of P in the plane.

Solution:

Eqn. of plane x + 2y + 3z = 2



eqn. of PR is $\frac{x-4}{1} = \frac{y-2}{2} = \frac{z-3}{3} = \lambda$ (let)

$\Rightarrow x = \lambda + 4,\ y = 2\lambda + 3, z = 3\lambda + 2$

Let the co-ordinate of R be $(\lambda + 4,\ 2\lambda + 3,\ 3\lambda + 2)$

R also lies on the plane

So, $\lambda + 4 + 2(2\lambda + 3) + 3(3\lambda + 2) = 2$

$\Rightarrow \lambda = -1$

So point R is (3, 1, –1) i.e., foot of perpendicular

let $Q(\alpha, \beta, \gamma)$ be the image of P

$\therefore \quad \frac{4+\alpha}{2} = 3, \frac{3+\beta}{2} = 1, \frac{2+\gamma}{2} = -1$

$\Rightarrow \alpha = 2, \beta = -1, \gamma = -4$

So Image point Q is (2, –1, –4)

Perpendicular distance PR $= \sqrt{14}$

25. A, B and C throw a pair of dice in that order alternately till one of them gets a total of 9 and wins the game. Find their respective probabilities of winning, if A starts first.

Solution:

$$\left.\begin{aligned} &\text{P (winning)} = \frac{1}{9} \\ &\text{P (not winning)} = \frac{8}{9} \end{aligned}\right\}$$

P (A winning) = $P(A) + P(\bar{A}\bar{B}\bar{C}A) + P(\bar{A}\bar{B}\bar{C}\bar{A}\bar{B}\bar{C}A) + ...$

$$= \frac{1}{9} + \left(\frac{8}{9}\right)^3 \frac{1}{9} + \left(\frac{8}{9}\right)^6 \frac{1}{9} + ...$$

$$= \frac{\frac{1}{9}}{1 - \frac{512}{729}} = \frac{81}{217}$$

P(B winning) $= P(\bar{A}B) + P(\bar{A}\bar{B}\bar{C}\bar{A}B) + P(\bar{A}\bar{B}\bar{C}\bar{A}\bar{B}\bar{C}\bar{A}B) + ...$

$$= \frac{8}{9} \times \frac{1}{9} + \left(\frac{8}{9}\right)^4 \times \frac{1}{9} + \left(\frac{8}{9}\right)^7 \times \frac{1}{9} + ...$$

$$\frac{\frac{8}{9} \times \frac{1}{9}}{1 - \frac{512}{729}} = \frac{72}{217}$$

P(C winning) = 1 – [P(A winning) + P(B winning)]

$$= 1 - \left[\frac{81}{217} + \frac{72}{217}\right]$$

$$= 1 - \frac{153}{217} = \frac{64}{217}$$

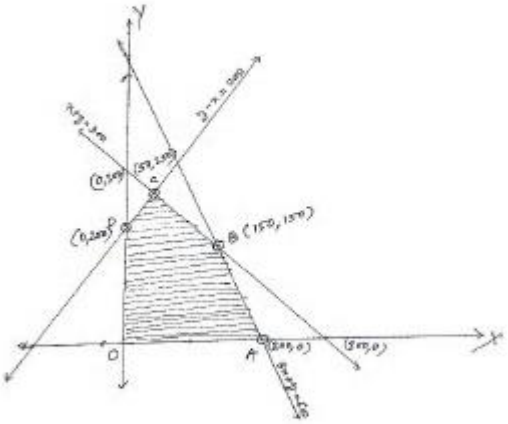
26. A company manufactures two types of cardigans : type A and type B. It costs ₹ 360 to make a type A cardigan and ₹ 120 to make a type B cardigan. The company can make at most 300 cardigans and spend at most ₹ 72,000 a day. The number of cardigans of type B cannot exceed the number of cardigans of type A by more than 200. The company makes a profit of ₹ 100 for each cardigan of type A and ₹ 50 for every cardigan of type B.

Formulate this problem as a linear programming problem to maximise the profit to the company. Solve it graphically and find maximum profit.

Solution:

Let no. of cardigans of type A be x and that of type B by y.

Then, max. Z = 100x + 50y



Subject to constraint,

$x + y \leq 300 \qquad ...(1)$

$360x + 120y \leq 72,000$

$\Rightarrow 3x + y \leq 600 \qquad ...(2)$

$y - x \leq 200 \qquad ....(3)$

$x, y \geq 0$

Correct Figure

Corner points A (200, 0), B(150, 150), C(50, 250), D(0, 200), O(0, 0)

Corner points	**Z = 100x + 50y**
O(0, 0)	0
A(200, 0)	20,000
B(150, 150)	22,500 $\leftarrow$ maximum
C(50, 250)	17,500
D(0, 200)	10,000

Hence no. of cardigans of type A = 150 and of type B = 150 and max. profit is ₹ 22,500.