Dual-Learning-Augmented Algorithm for Edge-Weighted Online Bipartite Matching

JIANYI YANG, University of Houston, USA

Edge-weighted Online Bipartite Matching (EOBM) is an important problem applied in many fields such as advertising, computing task assignment, inventory management, etc. Machine Learning (ML) has been developed to improve the empirical performance of EOBM, but suffers from the lack of performance guarantee. In this paper, we design a dual-learning-augmented algorithm for EOBM where we create a dual solution space to guarantee a preset competitive ratio given any ML prediction.

ACM Reference Format:

Jianyi Yang. 2018. Dual-Learning-Augmented Algorithm for Edge-Weighted Online Bipartite Matching. *J. ACM* 37, 4, Article 111 (August 2018), 2 pages. https://doi.org/XXXXXXX.XXXXXXX

1 OVERVIEW

We consider Edge-weighted Online Bipartite Matching (EOBM, a.k.a. Display Ad) [1] which is modeled as the left part of Fig. 1 and has the dual form in the right part of Fig. 1. Given a bipartite graph $\mathcal{G} = \{\mathcal{U}, \mathcal{V}, \mathcal{E}\}$, the problem assigns one offline vertex from the set $\mathcal{U}$ to an online vertex from $\mathcal{V}$ arriving at each round $t \in [T]$. For conciseness, we also denote the online vertex arriving at round t as vertex t. If offline vertex $u \in \mathcal{U}$ is selected for vertex t, a reward of $w_{u,t} \geq 0$ is obtained. For each offline vertex $u \in \mathcal{U}$, we need to ensure that the total number of assigned online vertexes to u does not exceed c_u. The goal of the problem is to maximize the total reward within T rounds, i.e. $\sum_{t=1}^{T} \sum_{u\in\mathcal{U}} w_{u,t} x_{u,t}$ where $x_{u,t} \in \{0, 1\}$ is the binary decision on whether t is assigned to u. EOBM has many application scenarios such as assigning online impressions to advertisers, assigning online computing tasks to servers, etc.

$$\max P := \sum_{t=1}^{T} \sum_{u\in\mathcal{U}} w_{u,t} x_{u,t} \qquad \min D := \sum_{u\in\mathcal{U}} c_u \alpha_u + \sum_{t=1}^{T} \beta_t$$

$$s.t. \forall u \in \mathcal{U}, \sum_{t=1}^{T} x_{u,t} \leq c_u, \qquad s.t. \forall u \in \mathcal{U}, t \in [T], \alpha_u + \beta_t \geq w_{u,t},$$

$$\forall t \in [T], \sum_{u\in\mathcal{U}} x_{u,t} \leq 1. \qquad \forall u \in \mathcal{U}, t \in [T], \beta_t \geq 0, \alpha_u \geq 0.$$

Fig. 1. Primal and Dual Problems.

We exploit ML predictions to improve the expected performance expressed as $\mathbb{E}_{\mathcal{G}}[P(\mathcal{G})]$. Furthermore, we aim to guarantee the worst-case performance of ML-based solutions. The worst-case performance is measured by competitive ratio expressed as $CR = \max_{\mathcal{G}} \frac{P(\mathcal{G})}{P^*(\mathcal{G})}$ where $P^*(\mathcal{G})$ is the reward of the offline-optimal solution. Unfortunately, without further assumption, any algorithm to solve EOBM problem only has zero competitive ratio [1]. Despite that, EOBM is often studied under the **free disposal** setting: Each offline vertex is free to dispose of previously matched vertexes to accept new arrivals with higher edge weights. The free disposal is commonly considered as an important economic concept in fields like advertising.

Contribution. A learning-augmented algorithm for EOBM with a competitive ratio guarantee has been designed in [2], but the algorithm in [2] requires a parallel execution of an expert algorithm

Author's address: Jianyi Yang, jyang66@uh.edu, University of Houston, Houston, Texas, USA.

ACM 0004-5411/2018/8-ART111
https://doi.org/XXXXXXX.XXXXXXX

Algorithm 1 Dual-Learning-Augmented EOBM (DULAM)

1: **Initialization:** $\forall u \in \mathcal{U}, \alpha_u = 0, \beta_0, \cdots, \beta_T = 0$.
2: **for** t=1 to T, a new request $t \in \mathcal{V}$ arrives **do**
3: **Inference**. Get the ML prediction $\tilde{\Delta}_{u,t}, \forall u \in \mathcal{U}$.
4: **Projection**. Project $\tilde{\Delta}_{u,t}$ into $\mathcal{D}_{u,t}$ in (1) and get $\Delta_{u,t}, \forall u \in \mathcal{U}$.
5: **Matching**. $\forall u \in \mathcal{U}$, assign score $s_{u,t} = \max\{0, \eta w_{u,t} - c_u \Delta_{u,t}\}$. If $\forall u \in \mathcal{U}, s_{u,t} = 0$, leave t unmatched. Otherwise, assign t to u_t that maximizes $s_{u,t}$ and set $x_{u_t,t} = 1$.
6: **Free disposal**. If u_t has more than c_{u_t} queries assigned, let v be the query with the least value $l_{u_t,t}$ and set $x_{u_t,v} = 0$.
7: **Dual update**. Set the dual variable as $\beta_t = s_{u_t,t}$, and $\alpha_{u_t} = \alpha_{u_t} + \Delta_{u_t,t} - \eta l_{u_t,t}/c_{u_t}$.
8: **end for**

and its competitive ratio relies on the expert performance. Learning-augmented algorithm without relying on an expert has been designed for the Adwords problem in [3], but it remains a challenge to design expert-free learning-augmented algorithm for the more challenging EOBM. In this paper, we propose the DUal-Learning-Augmented EOBM (DULAM) to guarantee the competitive ratio for ML solutions of EOBM.

2 ALGORITHM

To utilize the ML predictions for improved average performance under the competitive ratio guarantee, we develop DULAM in Alg. 1. The key step of the algorithm is to project the ML prediction into a set D_u and use the projected prediction $\Delta_{u,t}$ to set the score $s_{u,t}$ for each offline vertex $u \in \mathcal{U}$ and update the dual variables. The key to bound the competitive ratio is the design of the set D_u. To give the expression of D_u, we denote $\mathcal{I}_{u,t}$ as the set of online vertexes matched to u at the beginning of round t and denote $\bar{\mathcal{I}}_{u,t} = \mathcal{I}_{u,t} \cup \{w_{u,t}\}$. Rank the weights of the vertexes in $\bar{\mathcal{I}}_{u,t}$ in an non-increasing order and denote $\bar{m}_t(j), j = 1, \cdots, c_u$ as the $j-$th vertex in a non-increasing order. If $j \geq |\bar{\mathcal{I}}_{u,t}|$, then $\bar{m}_t(j)$ represents null vertex with weight $w_{u,\bar{m}_t(j)} = 0$. Given a set of weights $\{\theta_j, j = 1, \cdots, c_u\}$, we denote a weighted average as $E(\{w_{u,\bar{m}_t(j)}, \theta_j\}_{j\in[c_u]}) = \frac{1}{\sum_{j=1}^{c_u} \theta_j} \sum_{j=1}^{c_u} \theta_j w_{u,\bar{m}_t(j)}$. Then, any $\Delta_{u,t}$ in the safe dual set D_u satisfies two inequalities as below.

$$\mathcal{D}_u = \left\{\Delta_{u,t} \mid \eta w_{u,t} - c_u \Delta_{u,t} \geq w_{u,t} - \alpha_u;\ \alpha_u + \Delta_{u,t} - \eta l_{u,t}/c_u \geq E(\{w_{u,\bar{m}_t(j)}, \theta_j\}_{j\in[c_u]})\right\}, \quad (1)$$

where $\eta \geq (1+\frac{1}{c_u})^{c_u}/((1+\frac{1}{c_u})^{c_u} - 1)$ is a preset parameter, $\theta_j = (1+\frac{1}{c_u})^{j-1}$ and $l_{u,t}$ is the smallest weight of the online vertexes in $\bar{\mathcal{I}}_{u,t}$.

3 ANALYSIS

We prove that DULAM always guarantees a competitive ratio given any ML prediction.

Theorem 3.1. *Choose $\eta \geq (1+\frac{1}{c_u})^{c_u}/((1+\frac{1}{c_u})^{c_u} - 1)$ in the set $\mathcal{D}_u$. Given any ML model,* DULAM *in Algorithm 1 achieves a competitive ratio of* $1/\eta$.

Proof sketch. First, we prove that the constraints of the dual problem in Fig. 1 are always satisfied because of the first inequality in $\mathcal{D}_u$. Next, the primal-dual ratio is bounded as $1/\eta$ due to the dual update rule in Line 7 of Algorithm 1. In the end, we prove that $\mathcal{D}_u$ is non-empty thanks to the second inequality in $\mathcal{D}_u$.

REFERENCES

[1] Jon Feldman, Nitish Korula, Vahab Mirrokni, Shanmugavelayutham Muthukrishnan, and Martin Pál. Online ad assignment with free disposal. In *International workshop on internet and network economics*. Springer, 2009.
[2] Pengfei Li, Jianyi Yang, and Shaolei Ren. Learning for edge-weighted online bipartite matching with robustness guarantees. In *ICML*. PMLR, 2023.
[3] Jianyi Yang, Pengfei Li, Adam Wierman, and Shaolei Ren. Online budgeted matching with general bids. *NeurIPS*, 2024.