Dual-Learning-Augmented Algorithm for Edge-Weighted Online Bipartite Matching

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Edge-weighted Online Bipartite Matching (EOBM) is a fundamental problem with applications in areas such as online advertising, task scheduling, and inventory management. While machine learning (ML) techniques have been developed to enhance the empirical performance of EOBM, they often suffers from the lack of worst-case performance guarantee. In this paper, we propose a dual-learning-augmented algorithm for EOBM (DULAM) with the goal of guaranteeing the competitive ratio while exploiting the benefits of ML. The core of the design is a safe dual solution space. DULAM projects the dual learning output into the dual solution space to ensure a preset competitive ratio given any ML prediction.

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1 OVERVIEW

We consider Edge-weighted Online Bipartite Matching (EOBM, a.k.a. Display Ad) [1] which is modeled as the left part of Fig. 1 and has the dual form in the right part of Fig. 1. Given a bipartite graph $\mathcal{G} = \{\mathcal{U}, \mathcal{V}, \mathcal{E}\}$, the problem assigns one offline vertex from the set $\mathcal{U}$ to an online vertex from $\mathcal{V}$ arriving at each round $t \in [T]$. For conciseness, we also denote the online vertex arriving at round t as vertex t. If an offline vertex $u \in \mathcal{U}$ is selected for vertex t, a reward of $w_{u,t} \geq 0$ is obtained. For each offline vertex $u \in \mathcal{U}$, we need to ensure that the total number of assigned online vertexes to u does not exceed c_u. The goal of the problem is to maximize the total reward within T rounds, i.e. $\sum_{t=1}^{T} \sum_{u\in\mathcal{U}} w_{u,t} x_{u,t}$ where $x_{u,t} \in \{0, 1\}$ is the binary decision on whether t is assigned to u. EOBM has many application scenarios such as assigning online impressions to advertisers, assigning online computing tasks to servers, etc.

$$\max P := \sum_{t=1}^{T} \sum_{u\in\mathcal{U}} w_{u,t} x_{u,t} \qquad \min D := \sum_{u\in\mathcal{U}} c_u \alpha_u + \sum_{t=1}^{T} \beta_t$$

$$s.t. \forall u \in \mathcal{U}, \sum_{t=1}^{T} x_{u,t} \leq c_u, \qquad s.t. \forall u \in \mathcal{U}, t \in [T], \alpha_u + \beta_t \geq w_{u,t},$$

$$\forall t \in [T], \sum_{u\in\mathcal{U}} x_{u,t} \leq 1. \qquad \forall u \in \mathcal{U}, t \in [T], \beta_t \geq 0, \alpha_u \geq 0.$$

Fig. 1. Primal and Dual Problems.

We exploit ML predictions to improve the expected performance expressed as $\mathbb{E}_{\mathcal{G}}[P(\mathcal{G})]$. Furthermore, we aim to guarantee the worst-case performance of ML-based solutions. The worst-case performance is measured by competitive ratio expressed as $CR = \max_{\mathcal{G}} \frac{P(\mathcal{G})}{P^*(\mathcal{G})}$ where $P^*(\mathcal{G})$ is the reward of the offline-optimal solution. Unfortunately, without further assumption, any algorithm to solve EOBM problem only has zero competitive ratio [1]. Despite that, EOBM is often studied under the **free disposal** setting: Each offline vertex is free to dispose of previously matched vertexes to accept new arrivals with higher

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Algorithm 1 Dual-Learning-Augmented EOBM (DULAM)

Initialization: $\forall u \in \mathcal{U}, \alpha_u = 0, \beta_0, \cdots, \beta_T = 0$.
for t=1 to T, a new request $t \in \mathcal{V}$ arrives **do**
  Inference. Get the ML prediction $\tilde{\Delta}_{u,t}, \forall u \in \mathcal{U}$.
  Projection. Project $\tilde{\Delta}_{u,t}$ into $\mathcal{D}_{u,t}$ in (1) and get $\Delta_{u,t}, \forall u \in \mathcal{U}$.
  Matching. $\forall u \in \mathcal{U}$, assign score $s_{u,t} = \max\{0, \eta w_{u,t} - c_u \Delta_{u,t}\}$. If $\forall u \in \mathcal{U}, s_{u,t} = 0$, leave t unmatched. Otherwise, assign t to u_t that maximizes $s_{u,t}$ and set $x_{u_t,t} = 1$.
  Free disposal. If u_t has more than c_{u_t} queries assigned, let v be the query with the least value $l_{u_t,t}$ and set $x_{u_t,v} = 0$.
  Dual update. Set the dual variable as $\beta_t = s_{u_t,t}$, and $\alpha_{u_t} = \alpha_{u_t} + \Delta_{u_t,t} - \eta l_{u_t,t}/c_{u_t}$.
end for

edge weights. The free disposal is commonly considered as an important economic concept in fields like advertising.

Contribution. A learning-augmented algorithm for EOBM with a competitive ratio guarantee has been designed in [2], but the algorithm in [2] requires a parallel execution of an expert algorithm and its competitive ratio relies on the expert performance. Learning-augmented algorithm without relying on an expert has been designed for the Adwords problem in [3], but it remains a challenge to design expert-free learning-augmented algorithm for the more challenging EOBM. In this paper, we propose the DUal-Learning-Augmented EOBM (DULAM) to guarantee the competitive ratio while exploiting the benefits of ML for EOBM.

2 ALGORITHM

To utilize the ML predictions for improved average performance under the competitive ratio guarantee, we develop DULAM in Alg. 1. The key step of the algorithm is to project the ML prediction into a dual set $\mathcal{D}_u$ and use the projected prediction $\Delta_{u,t}$ to set the score $s_{u,t}$ for each offline vertex $u \in \mathcal{U}$ and update the dual variables. The key to bound the competitive ratio is the design of the set $\mathcal{D}_u$. To give the expression of $\mathcal{D}_u$, we denote $\mathcal{I}_{u,t}$ as the set of online vertexes matched to u at the beginning of round t and denote $\bar{\mathcal{I}}_{u,t} = \text{Disp}(\mathcal{I}_{u,t} \cup \{t\})$ where Disp means applying a free disposal if $|\mathcal{I}_{u,t}| = c_u$. Rank the weights of the vertexes in $\bar{\mathcal{I}}_{u,t}$ in an non-increasing order and denote $\bar{m}_t(j), j = 1, \cdots, c_u$ as the j−th vertex in a non-increasing order. If $j \geq |\bar{\mathcal{I}}_{u,t}|$, then $\bar{m}_t(j)$ represents null vertex with weight $w_{u,\bar{m}_t(j)} = 0$. Given a set of weights $\{\theta_j, j = 1, \cdots, c_u\}$, we denote a weighted average as $E(\{w_{u,\bar{m}_t(j)}, \theta_j\}_{j \in [c_u]}) = \frac{1}{\sum_{j=1}^{c_u} \theta_j} \sum_{j=1}^{c_u} \theta_j w_{u,\bar{m}_t(j)}$. Then, any $\Delta_{u,t}$ in the safe dual set $\mathcal{D}_u$ satisfies two inequalities as below.

$$\mathcal{D}_u = \left\{\Delta_{u,t} \mid \eta w_{u,t} - c_u \Delta_{u,t} \geq w_{u,t} - \alpha_u;\ \Delta_{u,t} - \eta l_{u,t}/c_u \geq \max\{E(\{w_{u,\bar{m}_t(j)}, \theta_j\}_{j \in [c_u]}) - \alpha_u, 0\}\right\}, \tag{1}$$

where $\eta \geq (1 + \frac{1}{c_u})^{c_u} / ((1 + \frac{1}{c_u})^{c_u} - 1)$ is a preset parameter, $\theta_j = (1 + \frac{1}{c_u})^{j-1}$ and $l_{u,t}$ is the smallest weight of the online vertices in $\bar{\mathcal{I}}_{u,t}$.

3 ANALYSIS

We prove that DULAM always guarantees a competitive ratio given any ML prediction.

Theorem 3.1. *Choose $\eta \geq (1 + \frac{1}{c_u})^{c_u} / ((1 + \frac{1}{c_u})^{c_u} - 1)$ in the set $\mathcal{D}_u$. Given any ML model,* DULAM *in Algorithm 1 achieves a competitive ratio of $1/\eta$.*

Theorem 1 shows that there exists a safe dual set $\mathcal{D}_{u,t}$ in (1) and by projecting the dual prediction into the safe dual set $\mathcal{D}_{u,t}$, DULAM can always guarantee a preset competitive ratio η for EOBM. The preset competitive ratio in DULAM does not exceed the competitive ratio by [1] which matches the optimal competitive ratio as $c_u \to \infty$. However, if a smaller preset competitive ratio η is selected, the dual set $\mathcal{D}_u$ becomes larger and DULAM gets more freedom to exploit the benefit of ML for expected performance.

Proof sketch. The proof Theorem 1 relies on the primal-dual conditions of the competitive ratio and the feasibility of the dual set. First, we prove that the constraints of the dual problem in Fig. 1 are always satisfied if the inequalities in $\mathcal{D}_u$ are satisfied. Next, the primal-dual ratio is bounded as $1/\eta$ due to the dual update rule in Line 7 of Algorithm 1. Finally, we show the dual set $\mathcal{D}_u$ is not empty by providing a nominal $\Delta_{u,t}^{\dagger}$ which always satisfies the constraints in $\mathcal{D}_u$ and has the expression as

$$\Delta_{u,t}^{\dagger} = E(\{w_{u,\bar{m}_t(j)}, \theta_j\}_{j\in[c_u]}) - E(\{w_{u,m_t(j)}, \theta_j\}_{j\in[c_u]}) + \eta l_{u,t}/c_u, \tag{2}$$

where $\theta_j = (1+\frac{1}{c_u})^{j-1}$, $l_{u,t}$ is the smallest weight of the online vertices in $\bar{\mathcal{I}}_{u,t}$, $m_t(j)$ is the jth online node in $\mathcal{I}_{u,t}$ and $\bar{m}_t(j)$ is the jth online node in $\bar{\mathcal{I}}_{u,t} = \mathrm{Disp}(\mathcal{I}_{u,t} \cup \{t\})$ (the vertices in $\mathcal{I}_{u,t}$ and $\bar{\mathcal{I}}_{u,t}$ are ranked based on a non-increasing order of the corresponding weights.).

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