ECE5463: Introduction to Robotics

Lecture Note 4: General Rigid Body Motion

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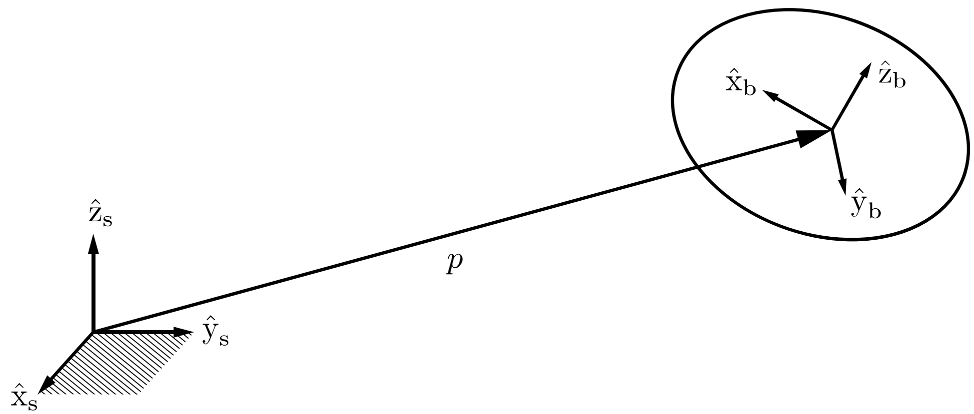
Outline

- Representation of General Rigid Body Motion
- Homogeneous Transformation Matrix
- Twist and $se(3)$
- Twist Representation of Rigid Motion
- Screw Motion and Exponential Coordinate

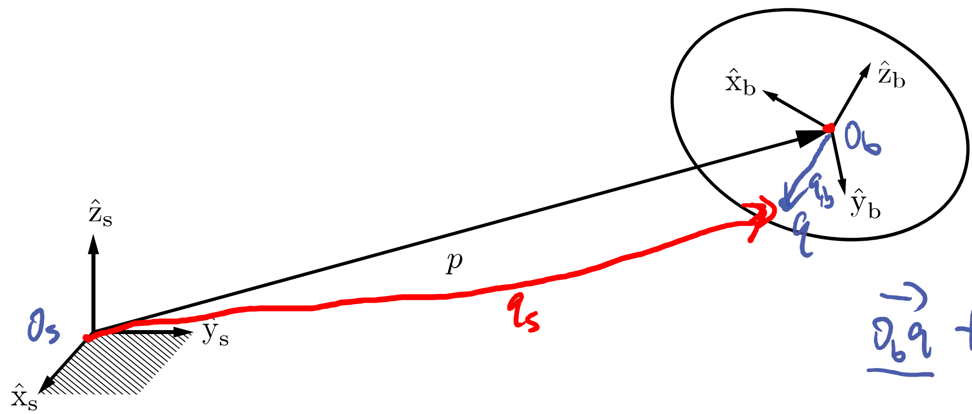
General Rigid Body Configuration



- General rigid body configuration includes both the orientation $R \in SO(3)$ and the position $p \in \mathbb{R}^3$ of the rigid body.
- Rigid body configuration can be represented by the pair (R, p)
- **Definition (Special Euclidean Group):**

$$SE(3) = \{(R, p) : R \in SO(3), p \in \mathbb{R}^3\} = SO(3) \times \mathbb{R}^3$$

Special Euclidean Group



$\underline{\overrightarrow{O_b q}} + \underline{\overrightarrow{O_s O_b}} = \underline{\overrightarrow{O_s q}}$

- Let $(R, p) \in SE(3)$, where p is the coordinate of the origin of {b} in frame {s} and R is the orientation of {b} relative to {s}. Let q_s, q_b be the coordinates of a point q relative to frames {s} and {b}, respectively. Then

$\Rightarrow p + \underbrace{[\hat{x}_b \ \hat{y}_b \ \hat{z}_b]} q_b = q_s$

$\Rightarrow q_s = R q_b + p$

$$q_s = Rq_b + p$$

$$\Rightarrow \begin{bmatrix} q_s \\ 1 \end{bmatrix} = \begin{array}{cc} 3\times3 & 3\times1 \\ \multicolumn{2}{c}{\begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}} \end{array} \begin{bmatrix} q_b \\ 1 \end{bmatrix}$$

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Homogeneous Representation

- For any point $x \in \mathbb{R}^3$, its homogeneous coordinate is $\tilde{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}$ $\in \mathbb{R}^4$

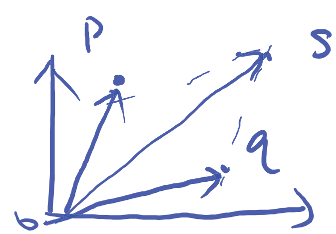
- Similar, homogeneous coordinate for the origin is $\tilde{o} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

- Homogeneous coordinate for a vector v is: $\tilde{v} = \begin{bmatrix} v \\ 0 \end{bmatrix}$

$$v = p - q \Rightarrow \tilde{v} = \begin{bmatrix} p \\ 1 \end{bmatrix} - \begin{bmatrix} q \\ 1 \end{bmatrix} = \begin{bmatrix} p-q \\ 0 \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix}$$

- Some rules of syntax for homogeneous coordinates:

  - diff of pts are vectors
  - sum and diff of vec are vectors
  - sum of a vector and point is a point



$$\tilde{s} = (\tilde{p} - \tilde{o}) + (\tilde{q} - \tilde{o}) + \tilde{o}$$
$$= \begin{bmatrix} p \\ 0 \end{bmatrix} + \begin{bmatrix} q \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} p+q \\ 1 \end{bmatrix}$$

Homogeneous Transformation Matrix

- Associate each $(R, p) \in SE(3)$ with a 4×4 matrix:

$$T \triangleq \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} \text{ with } T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

$$T T^{-1} = \begin{bmatrix} RR^T & -RR^T p + p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} I_{3\times 3} & 0 \\ 0 & 1 \end{bmatrix}$$

- T defined above is called a homogeneous transformation matrix. Any rigid body configuration $(R, p) \in SE(3)$ corresponds to a homogeneous transformation matrix T.

- Equivalently, $SE(3)$ can be defined as the set of all homogeneous transformation matrices.

- Slight abuse of notation: $T = (R, p) \in SE(3)$ and $Tx = Rx + p$ for $x \in \mathbb{R}^3$

Uses of Transformation Matrices

- Representation of rigid body configuration (orientation and position)

orientation ↙ position

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

- Change of reference frame in which a vector or a frame is represented

Same physical point p, Coordinates in {s} and {b} are p_s and p_b

$$p_s = R_{sb} p_b + p_{sb} \Rightarrow \tilde{p}_s = T_{sb} \tilde{p}_b$$

$$\begin{bmatrix} p_s \\ 1 \end{bmatrix} = \begin{bmatrix} R_{sb} & p_{sb} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_b \\ 1 \end{bmatrix}$$

– For frame, let's consider {a}, {b}, {c}. T_{bc}: Configuration of {c} relative to {b}

$$\boxed{T_{ab}} T_{bc} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_{bc} & p_{bc} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{ab} R_{bc} & \underline{R_{ab} p_{bc} + p_{ab}} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_{ac} & p_{ac} \\ 0 & 1 \end{bmatrix}$$

orientation of {c}

change refrence frame for representing orientation of {c} from "relative to {b}" to "relative to {a}"

Uses of Transformation Matrices

- Rigid body motion operator that displaces a vector

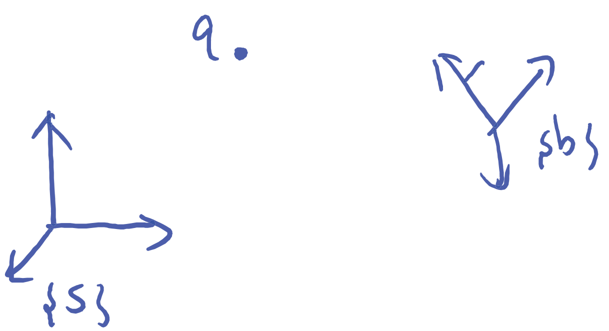
Given $T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$, we can view it as operation $\boxed{T = \mathrm{Tran}(p)\,\mathrm{Rot}(\hat{\omega};\theta)}$

$\Downarrow$

$T(p, \hat{\omega}, \theta)$

$$\mathrm{Tran}(p) \triangleq \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix}, \quad \mathrm{Rot}(\hat{\omega};\theta) = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

we can check: $T = \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & p \\ 0 & 1 \end{bmatrix}$



in {b}: $\tilde{q}_b' = T \cdot \tilde{q}_b = \mathrm{Tran}(p)\,\mathrm{Rot}(\hat{\omega},\theta) \cdot \tilde{q}_b$

in {b}-frame

in {s}: $\tilde{q}_s'' = T \tilde{q}_s = \mathrm{Tran}(p)\,\mathrm{Rot}(\hat{\omega},\theta)\,\tilde{q}_s$

{s}-frame

$\Updownarrow$

Given $\underbrace{T(p, \hat{\omega}, \theta)}_{T}$

Still in {s}: $\tilde{q}_s'' = T\, T_{sb} \tilde{q}_b$

$T_{sb} \tilde{q}_b'' = T\, T_{sb} \tilde{q}_b \Rightarrow \tilde{q}_b'' = (T_{sb}^{-1}\, T\, T_{sb})\, \tilde{q}_b$

Uses of Transformation Matrices

- Rigid body motion operator that displaces a frame

Frame is just 3-axis vectors

Given $T = \mathrm{Trans}(p)\,\mathrm{Rot}(\hat{\omega}, \theta)$

- $T\,T_{bc}$: $\begin{cases}\text{rotate} \\ \text{tran}\end{cases}$ {c} frame where p, $\hat{\omega}$ expressed in {b}

- $(T_{db}^{-1}\,T\,T_{db})\,T_{bc}$: $\boxed{\begin{cases}\text{rotate} \\ \text{trans}\end{cases}}$ {c} where p, $\hat{\omega}$ expressed in {d}

- special case : Let {d} = {c} then $(T_{cb}^{-1}\,T\,T_{cb})\,T_{bc}$

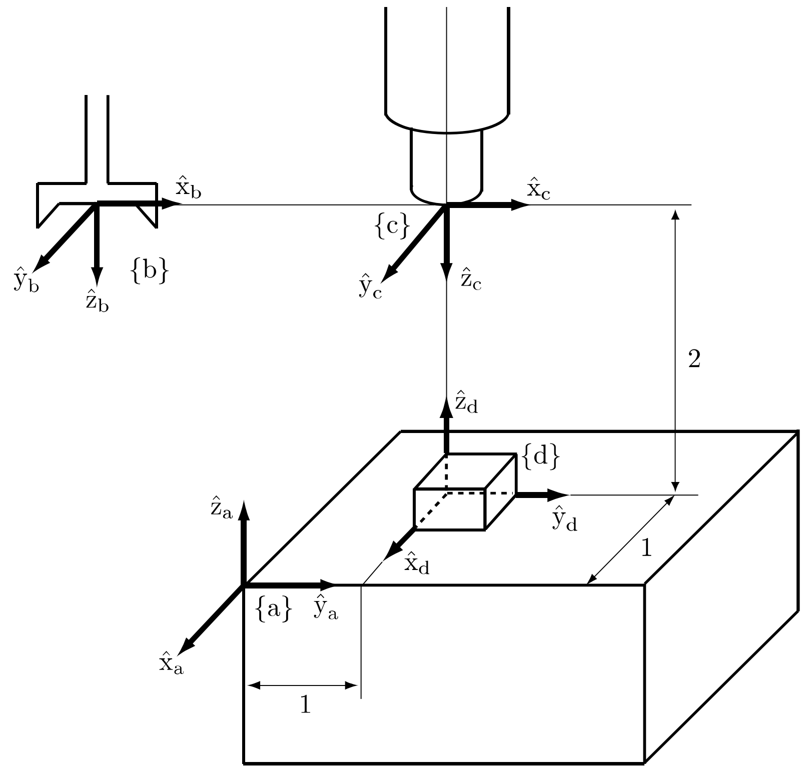
$$= T_{bc}\,T$$

Example of Homogeneous Transformation Matrix

In terms of the coordinates of a fixed space frame $\{s\}$, frame $\{a\}$ has its $\hat{\mathrm{x}}_{\mathrm{a}}$-axis pointing in the direction $(0, 0, 1)$ and its $\hat{\mathrm{y}}_{\mathrm{a}}$-axis pointing $(-1, 0, 0)$, and frame $\{b\}$ has its $\hat{\mathrm{x}}_{\mathrm{b}}$-axis pointing $(1, 0, 0)$ and its $\hat{\mathrm{y}}_{\mathrm{b}}$-axis pointing $(0, 0, -1)$. The origin of $\{a\}$ is at $(3, 0, 0)$ in $\{s\}$ and the origin of $\{b\}$ is at $(0, 2, 0)$ is $\{s\}$.

Example of Homogeneous Transformation Matrix

Fixed frame {a}; end effector frame {b}, the camera frame {c}, and the workpiece frame {d}. Suppose $\|p_c - p_b\| = 4$



What are T_{ad} T_{cd} ?

$$T_{ad} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad \begin{array}{c} \qquad\quad R_{cd} \qquad\quad p_{cd} \\ T_{cd} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

what is T_{ab}?

We can see $T_{bc} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$

e.g: $T_{cd}^{-1} = \left[\begin{array}{c|c} R_{cd}^T & -R_{cd}^T p_{cd} \\ \hline 0\ 0\ 0 & 1 \end{array}\right]$

$$T_{ab} = T_{ad} T_{dc} T_{cb} = T_{ad} T_{cd}^{-1} T_{bc}^{-1} = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & -3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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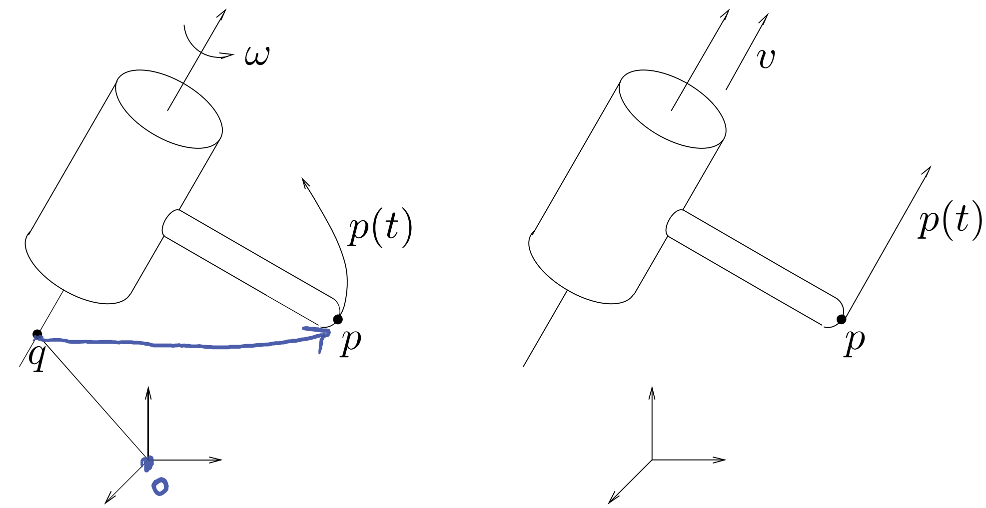
Towards Exponential Coordinate

- Recall: rotation matrix $R \in SO(3)$ can be represented in exponential coordinate $\hat{\omega}\theta$
  - $q(\theta) = \mathrm{Rot}(\hat{\omega}, \theta) q_0$ viewed as a solution to $\dot{q}(t) = [\hat{\omega}] q(t)$ with $q(0) = q_0$ at $t = \theta$.
  - The above relation requires that the rotation axis passes through the origin.
- We can find exponential coordinate for $T \in SE(3)$ using a similar approach (i.e. via differential equation)
- We first need to introduce some important concepts.

Differential Equation for Rigid Body Motion

- Rotation about axis that may not pass through the origin



$$\dot p(t) = \omega \times (p(t) - q) = \omega \times p(t) - \omega \times q$$

$$\tilde p(t) = \begin{bmatrix} p(t) \\ 1 \end{bmatrix} \qquad = [\omega]p(t) - [\omega]q$$

$$\dot{\tilde p}(t) = \begin{bmatrix} \dot p(t) \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} [\omega] & -[\omega]q \\ 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} p(t) \\ 1 \end{bmatrix}}_{\tilde p(t)} \Rightarrow \tilde p(t) = e^{At}\tilde p(0)$$

- Translation:

$$\dot p(t) = v \Longrightarrow \begin{bmatrix} \dot p(t) \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} p(t) \\ 1 \end{bmatrix} \qquad \tilde p(t) = e^{At} p(0)$$

Differential Equation for Rigid Body Motion

- Consider the following differential equation in homogeneous coordinates

$$\dot{p}(t) = \omega \times p(t) + v \quad \Rightarrow \quad \begin{bmatrix} \dot{p}(t) \\ 0 \end{bmatrix} = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p(t) \\ 1 \end{bmatrix} \qquad (1)$$

- The variable v contains all the constant terms (e.g. $-\omega \times q$ in the rotation example); thus, it may NOT be equal to the linear velocity of the origin of the rigid body.

- Solution to (1) is $\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = \exp\left(\begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} t \right) \begin{bmatrix} p(0) \\ 1 \end{bmatrix}$

- Motion of this form is characterized by (ω, v) which is called spatial velocity or Twist.

Twist

- Angular velocity and "linear" velocity can be combined to form the *Spatial Velocity* or *Twist*

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

- Each twist $\mathcal{V}$ corresponds to a motion equation (1).

- For each twist $\mathcal{V} = (\omega, v)$, let $[\mathcal{V}]$ be its matrix representation

$$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$$

- With these notations, solution to (1) is given by

$$\begin{bmatrix} p(t) \\ 1 \end{bmatrix} = e^{[\mathcal{V}]t} \begin{bmatrix} p_0 \\ 1 \end{bmatrix}$$

$se(3)$

- Similar to $so(3)$, we can define $se(3)$:

$$se(3) = \{([\omega], v) : [\omega] \in so(3), v \in \mathbb{R}^3\}$$

$so(3) = \{ [w] : w \in \mathbb{R}^3 \}$

- $se(3)$ contains all matrix representation of twists or equivalently all twists.

- In some references, $[\mathcal{V}]$ is called a twist. We follow the textbook notation to call the spatial velocity $\mathcal{V} = (\omega, v)$ a twist.

- Sometimes, we may abuse notation by writing $\mathcal{V} \in se(3)$.

Example of Twist

- $\mathcal{V} = \left(\begin{array}{c} \omega \\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \end{array}, \begin{array}{c} v \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \end{array} \right)$ can have multiple different physical interpretations

$$\mathcal{V} = (\underline{1,0,0}, \underline{0,1,0}) \xRightarrow{\text{defines ODE}} \dot{p}(t) = \omega \times p(t) + v = \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times p(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

~~Eg~~: This may have multiple physical interpretation:

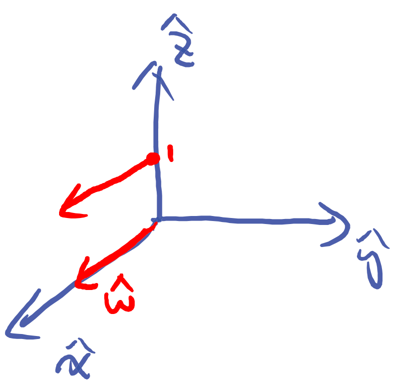
– Rotation axis: charaterized by direction and a point (q) on the axis

1st interpretation: the rotation axis could be the $\hat{x}$-axis

$$\dot{p}(t) = \omega \times (p(t) - q) + v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times p(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\Rightarrow$ rotate about ω + linear motion along $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\hat{y}$-axis



2°: The rotation axis can also be $\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, q = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$

direction

$$\Rightarrow \dot{p}(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \left(p(t) - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times p(t) - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} p(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

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Exponential Map of $se(3)$: From Twist to Rigid Motion

Theorem 1.

For any $\mathcal{V} = (\omega, v)$ *and* $\theta \in \mathbb{R}$, *we have* $e^{[\mathcal{V}]\theta} \in SE(3)$

- *Case 1* ($\omega = 0$): $e^{[\mathcal{V}]\theta} = \begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$
- *Case 2* ($\omega \neq 0$): *without loss of generality assume* $\|\omega\| = 1$. *Then*

$$e^{[\mathcal{V}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & G(\theta)v \\ 0 & 1 \end{bmatrix}, \text{ with } G(\theta) = I\theta + (1-\cos(\theta))[\omega] + (\theta - \sin(\theta))[\omega]^2 \qquad (2)$$

similar to Rodrigue's formula, we can also derive analytical formula for $e^{[\mathcal{V}]\theta}$

Recall: definition: $e^{[\mathcal{V}]\theta} = I_4 + [\mathcal{V}]\theta + [\mathcal{V}]^2 \frac{\theta^2}{2!} + [\mathcal{V}]^3 \frac{\theta^3}{3!} + \cdots$

V.F.Y. the above formula in the theorem. based on the following two facts.

Fact 1:

$[\mathcal{V}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix}$, $[\mathcal{V}]^2 = \begin{bmatrix} [\omega]^2 & [\omega]v \\ 0 & 0 \end{bmatrix}$, $[\mathcal{V}]^3 = \begin{bmatrix} [\omega]^3 & [\omega]^2 v \\ 0 & 0 \end{bmatrix}$, ...

Fact 2: Because $\|\omega\| = 1$, $[\omega]^3 = -[\omega]$

Log of $SE(3)$: from Rigid-Body Motion to Twist

Theorem 2.

Given any $T = (R, p) \in SE(3)$, one can always find twist $\mathcal{V} = (\omega, v)$ and a scalar θ such that

$$e^{[\mathcal{V}]\theta} = T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix}$$

Matrix Logarithm Algorithm:

- If $R = I$, then set $\omega = 0$, $v = p/\|p\|$, and $\theta = \|p\|$.
- Otherwise, use matrix logarithm on $SO(3)$ to determine ω and θ from R. Then v is calculated as $v = G^{-1}(\theta)p$, where

$$G^{-1}(\theta) = \frac{1}{\theta}I - \frac{1}{2}[\omega] + \left(\frac{1}{\theta} - \frac{1}{2}\cos\frac{\theta}{2}\right)[\omega]^2$$

$q' = T \cdot q$

$\Rightarrow$ If we find v such that $e^{[\mathcal{V}]\theta} = T \Rightarrow$ means a trajectory starting from $p(0) = q$ and follows: $\dot{p}(t) = \omega \times p(t) + v$, then at $t = \theta$, we reach q'

Example of Exponential/~~Log~~

Given twist $\mathcal{V} = (0, 0, 1, a, 0, 0) \Rightarrow \omega = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad v = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$

$$[\omega] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\omega]^2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$G(\theta) = I\theta + (1-\cos\theta)\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + (\theta - \sin\theta)\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \sin\theta & \cos\theta - 1 & 0 \\ 1-\cos\theta & \sin\theta & 0 \\ 0 & 0 & \theta \end{bmatrix}$$

$$e^{[\omega]\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow G(\theta)\cdot v = \begin{bmatrix} \Downarrow \end{bmatrix} \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a\sin\theta \\ a(1-\cos\theta) \\ 0 \end{bmatrix}$$

$$\Rightarrow e^{[\mathcal{V}]\theta} = \left[\begin{array}{ccc|c} \cos\theta & -\sin\theta & 0 & a\sin\theta \\ \sin\theta & \cos\theta & 0 & a(1-\cos\theta) \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

Quick Summary

- Angular and linear velocity can be combined to form a spatial velocity or twist $\mathcal{V} = (\omega, v)$
- Each twist $\mathcal{V} = (\omega, v)$ defines a motion such that any point p on the rigid body follows a trajectory generated by the following ODE:

$$\dot{p}(t) = \omega \times p(t) + v$$

- Solution to this ODE (in homogeneous coordinate): $\tilde{p}(t) = e^{[\mathcal{V}]t}\tilde{p}(0)$.
- For any twist $\mathcal{V} = (\omega, v)$ and $\theta \in \mathbb{R}$, its matrix exponential $e^{[\mathcal{V}]\theta} \in SE(3)$, i.e., it corresponds to a rigid body transformation. We have an analytical formula to compute the exponential (Theorem 1)
- For any $T \in SE(3)$, we also have analytical formula (Theorem 2) to find $\mathcal{V} = (\omega, v)$ and $\theta \in \mathbb{R}$ such at $e^{[\mathcal{V}]\theta} = T$.

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Screw Interpretation of Twist

- Given a twist $\mathcal{V} = (\omega, v)$, the associated motion (1) may have different interpretations (different rotation axes, linear velocities).

- We want to impose some nominal interpretable structure on the motion.

- Recall: an angular velocity vector ω can be viewed as $\hat{\omega}\dot{\theta}$, where $\hat{\omega}$ is the unit rotation axis and $\dot{\theta}$ is the rate of rotation about that axis

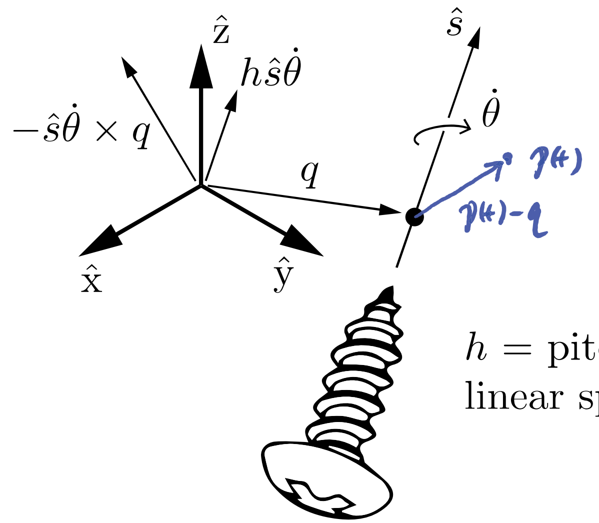
- Similarly, a twist (spatial velocity) $\mathcal{V}$ can be interpreted in terms of a **screw axis** $\mathcal{S}$ and a velocity $\dot{\theta}$ about the screw axis

one motion parameter (just rotation)

not independent rotation + linear motion

Screw Motion: Definition

- Rotating about an axis while also translating along the axis

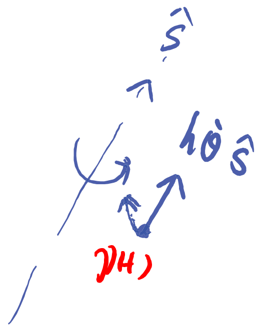


- Represented by screw axis $\{q, \hat{s}, h\}$ and rotation speed $\dot{\theta}$
  - $\hat{s}$: unit vector in the direction of the rotation axis
  - q: any point on the rotation axis
  - h: **screw pitch** which defines the ratio of the linear velocity along the screw axis to the angular velocity about the screw axis

if rotation speed is $\dot{\theta}$ about $\hat{s}$, then linear velocity is just $h \cdot \dot{\theta}\, \hat{s}$

Screw Motion as Solution to ODE

- Consider a point p on a rigid body under a screw motion with (rotation) speed $\dot{\theta}$. Let $p(t)$ be its coordinate at time t. The overall velocity is



$$\dot{p}(t) = \hat{s}\dot{\theta} \times (p(t) - q) + h\hat{s}\dot{\theta} \tag{3}$$

$$= \hat{s}\dot{\theta} \times p(t) - \hat{s}\dot{\theta} \times q + h\dot{\theta} \cdot \hat{s}$$

- Thus, any screw axis $\{q, \hat{s}, h\}$ with rotation speed $\dot{\theta}$ can be represented by a particular twist (ω, v) with $\omega = \hat{s}\dot{\theta}$ and $v = -\hat{s}\dot{\theta} \times q + h\hat{s}\dot{\theta}$.

$$\mathcal{V} = (\omega, v) \quad , \quad \omega = \hat{s}\dot{\theta} \quad , \quad v = -\omega \times q + h\omega$$

From Twist to Screw Axis

- The converse is true as well: given any twist $\mathcal{V} = (\omega, v)$ one can always find $\{q, \hat{s}, h\}$ and $\dot{\theta}$ such that the corresponding screw motion (eq. (3)) coincides with the motion generated by the twist (eq. (1)).
  - If $\omega = 0$, then it is a pure translation ($h = \infty$)

    $h = \infty$, $\hat{s} = v / \|v\|$, $\dot{\theta} = \|v\|$ (convention), q can be arbitrary

    $q = 0$

  - If $\omega \neq 0$: $\hat{s} = \frac{\omega}{\|\omega\|}$, $\dot{\theta} = \|\omega\|$, $h = \frac{\omega^T v}{\|\omega\|^2}$, $q = \frac{\omega \times v}{\|\omega\|^2}$

In this case: the screw motion is governed by

$$\dot{p}(t) = \hat{s}\dot{\theta} \times (p(t) - q) + h\dot{\theta}\hat{s}$$

$$= \omega \times p(t) - \omega \times \left(\frac{\omega \times v}{\|\omega\|^2}\right) + \frac{\omega^T v}{\|\omega\|^2} \|\omega\| \frac{\omega}{\|\omega\|}$$

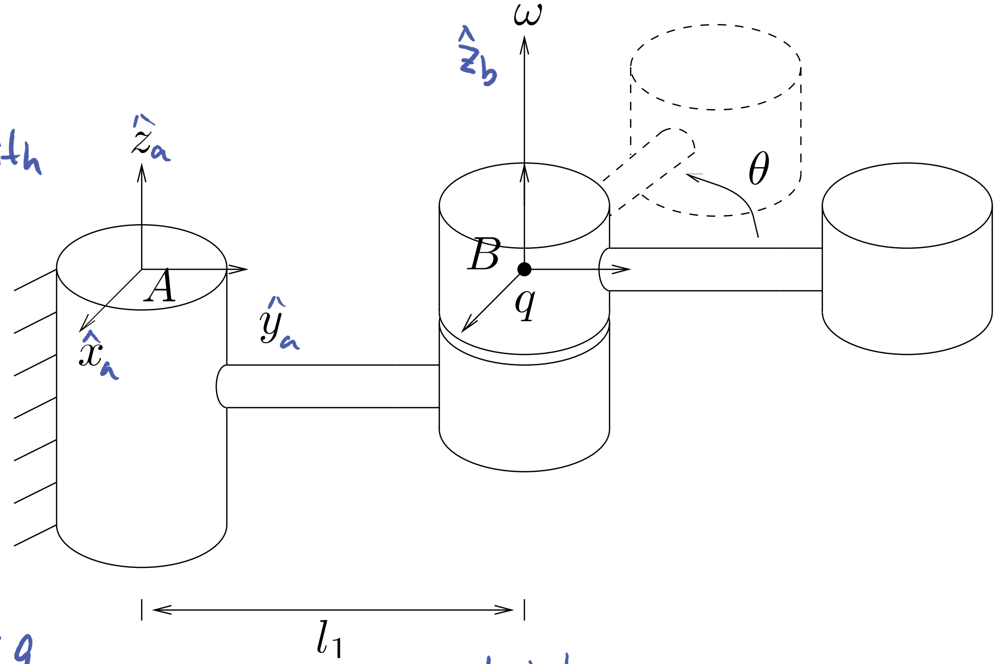
$= v$

Examples Screw Axis and Twist

- What is the twist that corresponds to rotating about $\hat{z}_b$? $\dot{\theta}=1$

In this case, we can view it as screw motion with $h=0$.

screw axis: $\begin{cases} h=0 \\ q=\begin{bmatrix}0\\l_1\\0\end{bmatrix} \\ \hat{s}=\begin{bmatrix}0\\0\\1\end{bmatrix}\end{cases}$



$\Rightarrow \quad w=\begin{bmatrix}0\\0\\1\end{bmatrix}, \quad v=-\hat{s}\dot{\theta}\times q+h\dot{\theta}\hat{s}=-\begin{bmatrix}0\\0\\1\end{bmatrix}\times q$ $\Rightarrow$ twist

$\mathcal{V}=\begin{bmatrix}w\\-w\times q\end{bmatrix}$

- What is the screw axis for twist $\mathcal{V}=(\cancel{0,2,2,4,0,0})$? $(1,1,1,2,0,1)$

$w=\begin{pmatrix}1\\1\\1\end{pmatrix}, \; v=\begin{pmatrix}2\\0\\1\end{pmatrix} \Rightarrow \hat{s}=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \; \dot{\theta}=\sqrt{3}, \; h=\frac{w^Tv}{\|w\|^2}=\frac{3}{3}=1, \quad q=\frac{w\times v}{\|w\|^2}=\frac{1}{3}\begin{bmatrix}1\\1\\-2\end{bmatrix}$

verify: $\dot{p}(t)=w\times\left(p(t)-\frac{1}{3}\begin{bmatrix}1\\1\\-2\end{bmatrix}\right)+1\cdot\sqrt{3}\cdot\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$

$=w\times p(t)-\begin{bmatrix}1\\1\\1\end{bmatrix}\times\left(\frac{1}{3}\right)\begin{bmatrix}1\\1\\-2\end{bmatrix}+\begin{bmatrix}1\\1\\1\end{bmatrix}=w\times p(t)+\begin{bmatrix}2\\0\\1\end{bmatrix}$

Implicit Definition of Screw Axis for a Given a Twist

- For any twist $\mathcal{V} = (\omega, v)$, we can always view it as a "screw velocity" that consists an screw axis $\mathcal{S}$ and the velocity $\dot{\theta}$ about the screw axis.

- Instead of using $\{q, \hat{s}, h\}$ to represent $\mathcal{S}$, we adopt a more convenient representation defined below:

- **Screw axis (corresponding to a twist):** Given any twist $\mathcal{V} = (\omega, v)$, its screw axis is defined as
  - If $\omega \neq 0$, then $\mathcal{S} := \mathcal{V}/\|\omega\| = (\omega/\|\omega\|, v/\|\omega\|)$.

    velocity along S is $\|\omega\|$

  - If $\omega = 0$, then $\mathcal{S} := \mathcal{V}/\|v\| = (0, v/\|v\|)$

    velocity along S is $\|v\|$

$\boxed{S \cdot \dot{\theta}}$ screw velocity

Unit Screw Axis

- **(unit) screw axis** $\mathcal{S}$ can be represented by

$$\mathcal{S} = \begin{bmatrix} \omega \\ v \end{bmatrix} \in \mathbb{R}^6$$

where either (i)$\|\omega\| = 1$ or (ii) $\omega = 0$ and $\|v\| = 1$

for case (i) : $\|w\|=1$, $v = -w \times q + h w$

(ii) : $\|v\|=1$, $h = \infty$,

- We have used (ω, v) to represent both screw axis (where $\|\omega\|$ or $\|v\| = 1$ must be unity) and a twist (where there are no constrains on ω and v)

- $\mathcal{S} = (w, v)$ is called a screw axis, but we typically do not bother to explicitly find the corresponding $\{q, \hat{s}, h\}$. We can find them whenever needed.

Exponential Coordinates of Rigid Transformation

- Screw axis $\mathcal{S} = (\omega, v)$ is just a normalized twist; its matrix representation is

Given $\mathcal{V} = \begin{bmatrix} w \\ v \end{bmatrix}$

$[\mathcal{V}] = \begin{bmatrix} [w] & v \\ 0 & 0 \end{bmatrix}$

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & v \\ 0 & 0 \end{bmatrix} \in se(3)$$

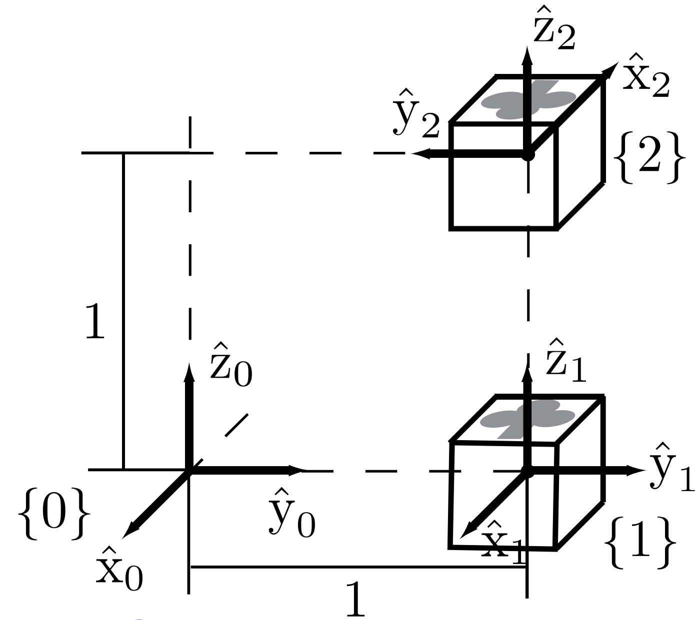
- Therefore, a point started at $p(0)$ at time zero, travel along screw axis $\mathcal{S}$ at unit speed for time t will end up at $p(t) = e^{[\mathcal{S}]t} p(0)$

- Given $\mathcal{S}$ we can use Theorem 1 to compute $e^{[\mathcal{S}]t} \in SE(3)$;

- Given $T \in SE(3)$, we can use Theorem 2 to find $\mathcal{S} = (\omega, v)$ and θ such that $e^{[\mathcal{S}]\theta} = T$. We call $\mathcal{S}\theta$ the **Exponential Coordinate** of the homogeneous transformation $T \in SE(3)$

Example of Exponential Coordinates

What is T_{02}? $\quad T_{02} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.



Given T_{02}, find the exponential coordinate $S\theta$ of T_{02}:

Use theorem 2:

1°: find $\hat{\omega}$ and θ:

1°-(1): $tr(R) = -1 \Rightarrow \theta = \pi$

1°-(2): $\hat{\omega} = \frac{1}{\sqrt{2(1+r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1+r_{33} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

2° $v = G^{-1}(\theta)\, p = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{\pi} \end{bmatrix}$

$$G^{-1}(\theta) = \frac{1}{\pi} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{\pi} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{\pi} \end{bmatrix}$$

More Discussions

$\Rightarrow$ $S = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{1}{2} \\ 0 \\ \frac{1}{\pi} \end{bmatrix}$, $\theta = \pi$ $\Rightarrow$ exponential coordinate for T_{02}

is $S\theta$

$\Rightarrow$ you can find the corresponding $\{h, \hat{s}, q\}$

$$q = \frac{w \times v}{\|w\|^2} \qquad w \times v = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{\pi} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

More Discussions