ECE5463: Introduction to Robotics

Lecture Note 10: Generalized Force and Statics of Open Chains

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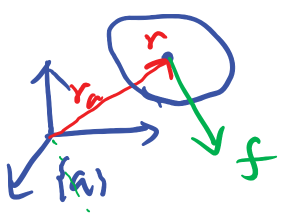
Outline

- Wrench
- Statics of Open Chains

Wrench

- Consider a rigid body with body frame and consider a force f acting on a point r on the rigid body

- Define an arbitrary *stationary frame* $\{a\}$ and let r_a and f_a be the $\{a\}$-frame representations of r and f vectors. This force create a **torque or moment** $m_a \in \mathbb{R}^3$ in frame $\{a\}$



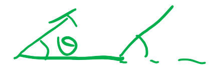
$$m_a = r_a \times f_a$$

- Similar to twist, we can merge the moment and force into a single 6D vector. This vector is called the **spatial force or wrench**.

$$\mathcal{F}_a = \begin{bmatrix} m_a \\ f_a \end{bmatrix}$$

Wrench-Twist Pair and Power

- Recall that for a point mass with linear velocity v and linear force f. Then we know that the power (instantaneous work done by f) is given by $f \cdot v = f^T v$

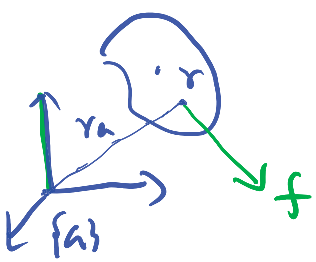


$= |f||v| \cdot \cos\theta \quad = v^T f$

- This relation can be generalized to spatial force (i.e. wrench) and spatial velocity (i.e. twist)

- Suppose a rigid body has a twist $\mathcal{V}_a = (\omega_a, v_a)$ expressed in {a}, and a force f is applied at a point r on the rigid body with wrench $\mathcal{F}_a$. Then the power is simply

$$\mathcal{V}_a \cdot \mathcal{F}_a = \mathcal{V}_a^T \mathcal{F}_a = \omega_a^T m_a + v_a^T f_a$$



Power $= \dot{r}_a^T \cdot f_a$

$\dot{r}_a = w_a \times r_a + v_a \Rightarrow \dot{r}_a^T \cdot f_a = (w_a \times r_a)^T f_a + v_a^T f_a$

$= \underbrace{f_a^T (w_a \times r_a)} + v_a^T f_a$

$= w_a^T (\underbrace{r_a \times f_a}_{m_a}) + v_a^T f_a$

Note:

$a^T (b \times c) = b^T (c \times a)$

Rotational Power

- Consider a point mass with a pure rotational velocity $\omega_a = \dot{\theta}\hat{\omega}_a$, and a moment m_a, relative to frame {a}

- Our previous discussion indicates that its power is

$$\omega_a^T m_a = \dot{\theta} \cdot \underbrace{(\hat{\omega}_a^T m_a)}_{\tau} \triangleq \dot{\theta} \cdot \tau$$

- $\tau = \hat{\omega}_a^T m_a = m_a^T \hat{\omega}_a$ is the projection of the moment onto the rotation axis, i.e. the effective part of the moment.

- Often times, τ is also referred to as "torque" with the understanding that it is a scalar quantifying the effectiveness of a moment (i.e. vector torque) relative to some rotation axis.

angular velicy : $\dot{\theta} \leftrightarrow w = \dot{\theta}\hat{\omega}$

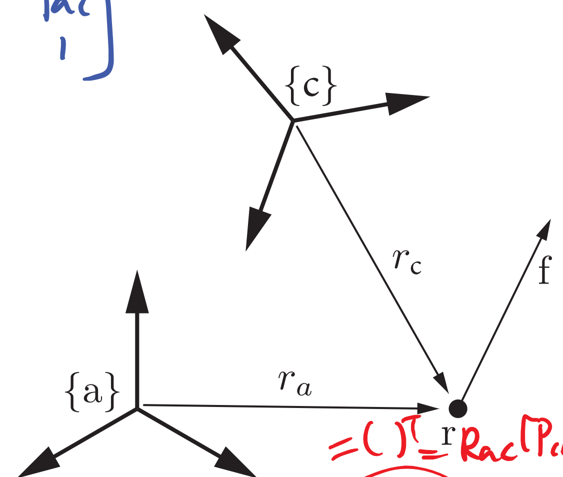
force : $f \leftrightarrow \vec{f}$

torque : $\tau \leftrightarrow m_a$

Wrench Representations in Different Frames

- The wrench $\mathcal{F}_a$ can be expressed in another frame {c}, provided T_{ac} is known
- This is not simply rewriting the coordinates of the vectors m and f in {c}.
- We have to change the vector representation of the point r from r_a (vector from the origin of {a} to r, expressed in {a}) to r_c (vector from the origin of {c} to r, expressed in {c})

$$T_{ac} = \begin{bmatrix} R_{ac} & p_{ac} \\ 0 & 1 \end{bmatrix}$$



$\mathcal{F}_a = \begin{bmatrix} m_a \\ f_a \end{bmatrix}$, we know $m_a = r_a \times f_a$

$\mathcal{F}_c = \begin{bmatrix} m_c \\ f_c \end{bmatrix}$.

$$f_c = R_{ca} f_a, \quad r_c = R_{ca} r_a + p_{ca}$$

$$\Rightarrow m_c = r_c \times f_c = (R_{ca} r_a + p_{ca}) \times (R_{ca} f_a)$$

$$= (R_{ca} r_a) \times (R_{ca} f_a) + p_{ca} \times (R_{ca} f_a)$$

$$= R_{ca} \underbrace{(r_a \times f_a)}_{m_a} + [p_{ca}] R_{ca} f_a$$

$$\Rightarrow \begin{bmatrix} m_c \\ f_c \end{bmatrix} = \begin{bmatrix} R_{ca} & [p_{ca}] R_{ca} \\ 0 & R_{ca} \end{bmatrix} \begin{bmatrix} m_a \\ f_a \end{bmatrix}$$

$= (\ )^T = R_{ac} [p_{ca}]^T$

$$\Rightarrow \mathcal{F}_c = \begin{bmatrix} R_{ac} & 0 \\ R_{ac}[p_{ac}] & R_{ac} \end{bmatrix}^T \mathcal{F}_a \Longleftrightarrow [Ad_{T_{ac}}]^T$$

Wrench Representations in Different Frames

- The power generated by an $(\mathcal{F}, \mathcal{V})$ pair must be the same regardless of the frame in which it is represented.

- Consider two frames {a} and {c}. We must have

$$\boxed{\mathcal{V}_c^T \mathcal{F}_c} = \mathcal{V}_a^T \mathcal{F}_a = (\underbrace{[\mathrm{Ad}_{T_{ac}}] \mathcal{V}_c}_{\mathcal{V}_a})^T \mathcal{F}_a = \boxed{\mathcal{V}_c^T ([\mathrm{Ad}_{T_{ac}}])^T \mathcal{F}_a}$$

- Since the above relation should hold for all possible twist $\mathcal{V}_c$, we must have

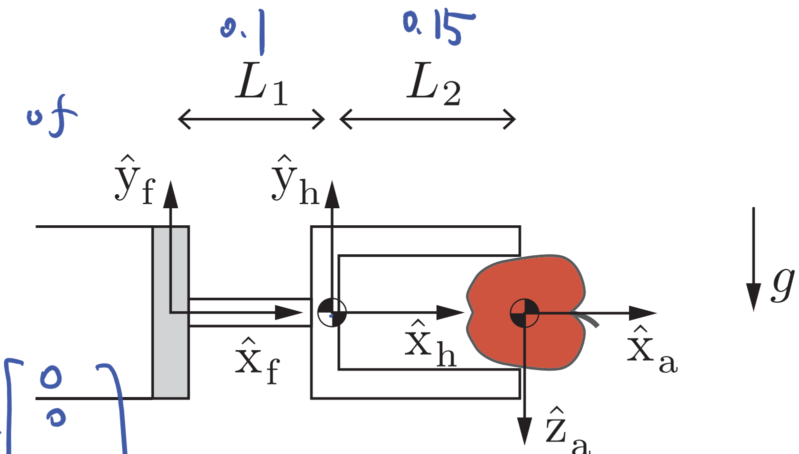
$$\mathcal{F}_c = [\mathrm{Ad}_{T_{ac}}]^T \mathcal{F}_a$$

- We are often interested in fixed space frame {s} and body frame {b}, we can define a **spatial wrench** $\mathcal{F}_s$ and **body wrench** $\mathcal{F}_b$. They are related by

$$\mathcal{F}_b = [\mathrm{Ad}_{T_{sb}}]^T \mathcal{F}_s$$

Example of Wrench

The robot hand is holding an apple with a mass of 0.1kg in a gravitational field $g = 10m/s^2$ (rounded to keep the numbers simple) acting downward on the page. The mass of the hand is 0.5kg. What is the force and torque measured by the six-axis forcetorque sensor between the hand and the robot arm?



{h}: center of mass frame of hand ; {a}: center of Mass of the apple

{f}: torque sensor.

Approach 1: work with {f} frame:

$$f_a = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \quad f_h = \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix}, \quad m_a = \begin{bmatrix} 0.25 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.25 \end{bmatrix}$$

$$m_h = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.5 \end{bmatrix} \Rightarrow \text{overall wrench} = (0,0,-0.25,0,-1,0) + (0,0,-0.5,0,-5,0)$$

$$= (0,0,-0.75,0,-6,0)$$

Approach 2: $\mathcal{F}_h = (0,0,0,0,-5,0) \leftarrow \{h\}$-frame

$\mathcal{F}_a = (0,0,0,0,0,1)$

change reference frame to {f}-frame: $\mathcal{F}_f = [Ad_{T_{hf}}]^T \mathcal{F}_h + [Ad_{T_{af}}]^T \mathcal{F}_a \overset{\text{v.F.T.}}{=}$

$$T_{hf} = \begin{bmatrix} 1 & 0 & 0 & -0.1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{af} = \begin{bmatrix} 1 & 0 & 0 & -0.25 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Statics of Open Chains

- Now consider an open-chain robot with n joints. Let $\tau \in \mathbb{R}^n$ be the joint torques vector.

- Applying torques to joints will result in motion of the robot and forces of the end effector. By conservation of power:

  Power at the joints=(Power to move the robot)+(Power at the end-effector)

- At static equilibrium (i.e. no power is used to move the robot), we have

$$\underbrace{\tau^T\dot{\theta}}_{\tau_1\dot{\theta}_1+\tau_2\dot{\theta}_2+\dots+\tau_n\dot{\theta}_n} = \mathcal{F}_b^T\mathcal{V}_b = \mathcal{F}_b^T J_b(\theta)\dot{\theta}$$

- We can pick $\dot{\theta}$ infinitesimally small, but in arbitrary direction in $\mathbb{R}^n$.

$$\Rightarrow \quad \underbrace{\tau}_{\in\mathbb{R}^n} = \underbrace{J_b^T(\theta)}_{[6\times n]^T}\mathcal{F}_b \in \mathbb{R}^6$$

- If we use the fixed space frame, we will have $\tau = J_s^T(\theta)\mathcal{F}_s$

End-Effector Force Analysis

- If an external wrench $\mathcal{F}$ is applied to the end-effector, the joint torques that can generate opposing wrench $-\mathcal{F}$ is given by

$$\tau = \underbrace{J^T(\theta)}_{(6\times n)^T = (n\times 6)}(-\mathcal{F})$$

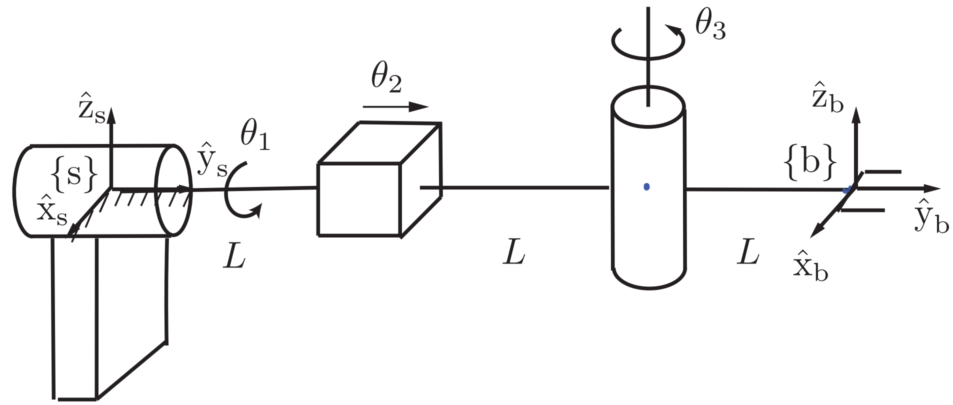
- What is the end-effector wrench generated by a given joint torque vector τ?
  - the answer is $\left(J^T(\theta)\right)^{-1}\tau$ provided $J^T(\theta)$ is invertible
  - If $J^T(\theta)$ is not invertible, the problem is not well defined.
  - An interesting case is when $J^T(\theta)$ has a nontrivial null space:

$$Null(J^T(\theta)) = \{\mathcal{F} \in \mathbb{R}^6 : J^T(\theta)\mathcal{F} = 0\}$$

  - The wrench that lies in the null space causes no torques, i.e., the balance equation is satisfied with $\tau = 0$; the resisting forces are supplied completely by the robot's mechanical structure.

Example of Statics of Open Chains

What are the wrenches that can be resisted by the manipulator with $\tau = 0$?



RPR:

By our discussion from last slide.

the wrench that can be resisted with $\tau=0$

lie in the $\mathrm{Null}(J_b^T(\theta))$

Consider the $\theta=0$ case: $\Rightarrow J_b(0) = [B_1 \ \ B_2 \ \ B_3]$

$B_3 = (w_{b3}, v_{b3})$, $w_{b3} = (0,0,1)$, $q_{b3} = (0,-L,0)$ $\Rightarrow v_{b3} = -w_{b3} \times q_{b3} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -L \\ 0 \end{bmatrix}$

$= \begin{bmatrix} -L \\ 0 \\ 0 \end{bmatrix}$

$B_2 = (w_{b2}, v_{b2})$, $w_{b2} = (0,0,0)$, $v_{b3} = (0,1,0)$

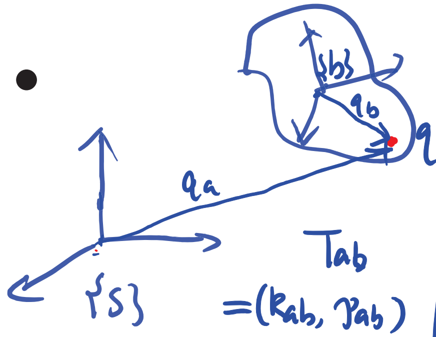
$J_b^T(0) \cdot F = 0$

$B_1 = (w_{b1}, v_{b1})$, $w_{b1} = (0,1,0)$, $v_{b1} = (0,0,0)$

$$\Rightarrow J_b(0) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -L \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \mathrm{Null}(J_b^T(0)) = \mathrm{Null}\left(\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -L & 0 & 0 \end{bmatrix} \right)$$

$$= \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 0 \\ L \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

More Discussions

- 

$\tilde{q}_a = T_{ab}\tilde{q}_b = R_{ab}q_b + p_{ab}$

point coordinate relation

If we a "free" vector a, it {b}-frame coordinate a_b and {s}-coordinate a_s are related by

$a_s = R_{sb}a_b \iff \tilde{a}_s = T_{sb}\tilde{a}_b$, note $\tilde{a}_s = \begin{bmatrix} a_s \\ 0 \end{bmatrix}$

1°: what's velocity of $q_s(t)$? It's just $\dot{q}_s(t)$

2°: what's the body velocity? $(w_s, v_s) \iff \underline{\dot{q}_s(t) = w_s \times q_s(t) + v_s}$

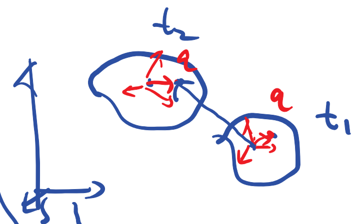
If we choose the q as the origin of {b} $\Rightarrow q(t) = p_{sb}(t)$

$\Rightarrow \dot{p}_{sb} = w_s \times p_{sb} + v_s \Rightarrow v_s = \dot{p}_{sb} + w_s \times (-p_{sb})$

3°: Note: $w_s = R_{sb}w_b$

4°: $q_s(t) \xrightarrow{\{s\}\ \text{velocity}} \dot{q}_s(t) \xrightarrow{\text{accelleration}} \ddot{q}_s(t) = a_q^s$

$\xrightarrow{\{b\}\text{-velocity}} V_q^b = R_{bs}\dot{q}_s(t) \xrightarrow{\text{accelleration}} a_q^b = R_{bs}\ddot{q}_s$



$V_q^b =$

More Discussions

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