ECE5463: Introduction to Robotics

Lecture Note 1: Background

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Outline

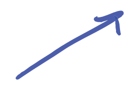
- Linear Algebra

- Linear Differential Equation

- Linear and Angular Motion of Point Mass

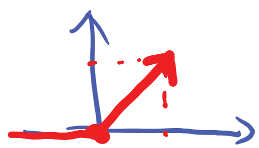
Free Vector

- **Free Vector**: geometric quantify with length and direction

e.g. velocity 

- "free" means not necessarily rooted anywhere; only length and direction matter

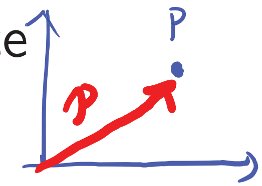
- Given a reference frame, v can be moved to a position such that the base of the arrow is at the origin without changing the orientation. Then the vector v can be represented its coordinates v in the reference frame.



- v denotes the physical quantify while v denote its coordinate wrt some frame.

Point and Its Coordinate

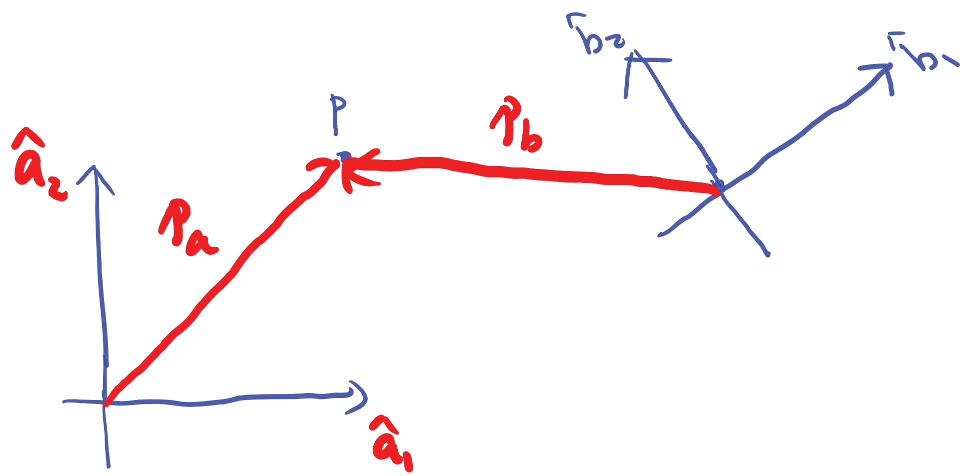
- **Point**: p denotes a point in the physical space



- A point p can be represented by as a vector from frame origin to p

- p denotes the coordinate of a point p

- The coordinate p depends on the choice of reference frame



Vector (Math)

- **Vector:** $p \in \mathbb{R}^n$:

$$p = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{bmatrix}$$

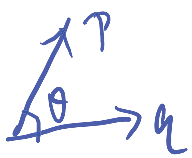
- Inner product of two vectors $p \in \mathbb{R}^n, q \in \mathbb{R}^n$:

$$\langle p, q \rangle = p^T q = p_1 q_1 + p_2 q_2 + \cdots + p_n q_n$$

- Norm of a vector p:

$$\|p\| = \sqrt{\langle p, p \rangle} = \sqrt{p_1^2 + p_2^2 + \cdots + p_n^2}$$

- Angle between two vectors $p, q \in \mathbb{R}^n$:



$$\cos\theta = \frac{\langle p, q \rangle}{\|p\| \, \|q\|}$$

Matrix

- $A \in \mathbb{R}^{n\times m}$:

$$A = \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & \end{array} \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ a_{21} & a_{22} & & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nm} \end{bmatrix} \end{array}$$

- Symmetric matrix :

A is square : $n = m$ A is symmetric : $A = A^T$

- Matrix vector multiplication as linear combination of columns

$y = A\theta$, $\theta \in \mathbb{R}^m$, $y \in \mathbb{R}^n$

$A = [a_1 \ a_2 \ \cdots \ a_m]$

a_j's are columns of A

$\boxed{y = \theta_1 \cdot a_1 + \theta_2 a_2 + \cdots + \theta_m a_m}$

column vectors

e.g.: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\theta = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$y = A\theta = \begin{bmatrix} 8 \\ 18 \end{bmatrix}$

$y = 2 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3 \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \end{bmatrix}$

Change of Basis

basis vectors

- Two Bases for R^n: $\{\text{a}\} = \{\hat{a}_1, \ldots, \hat{a}_n\}$ and $\{\text{b}\} = \{\hat{b}_1, \ldots, \hat{b}_n\}$
- v_a and v_b are the corresponding coordinates of v w.r.t. $\{\text{a}\}$ and $\{\text{b}\}$, how to they relate?

$A = [\hat{a}_1 \ \hat{a}_2 \ \cdots \ \hat{a}_n]$ $\qquad$ $B = [\hat{b}_1 \ \cdots \ \hat{b}_n]$

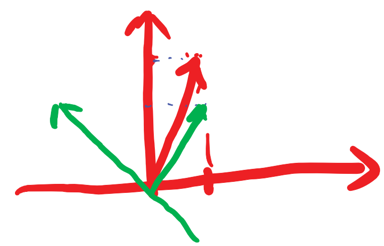
Vector v has coordinate v_a wrt $\{a\}$ means: $v = v_{a_1}\hat{a}_1 + v_{a_2}\hat{a}_2 + \cdots + v_{a_n}\hat{a}_n$

$v_a = \begin{bmatrix} v_{a_1} \\ v_{a_2} \\ \vdots \\ v_{a_n} \end{bmatrix}$ $\qquad = A v_a$

Similarly, we will have $v = B v_b$

$A v_a = B v_b$

Example: $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$



$\{a\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$A v_a = v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow v_a = A^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix} =$

$= \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 0.5 \end{bmatrix}$

Cross Product

- **Cross product** or **vector product** of $a \in \mathbb{R}^3, b \in \mathbb{R}^3$ is defined as

$$a \times b = \begin{bmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{bmatrix} \quad (1)$$

Properties:

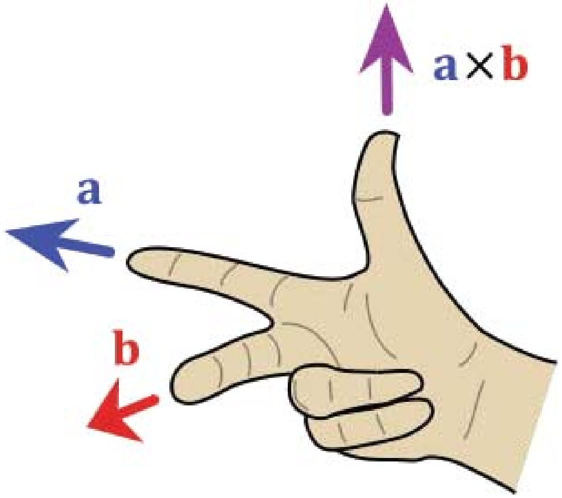
- $\|a \times b\| = \|a\|\|b\| \sin(\theta)$
- $a \times b = -b \times a$
- $a \times a = 0$

$a = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$a \times b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $b = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$a \times b = \begin{bmatrix} 0 - 3 \\ -3 - 0 \\ 1 + 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}$



Skew symmetric representation

- It can be directly verified from definition that $\underline{a \times b} = [a]b$, where

$$[a] \triangleq \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \tag{2}$$

- $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \leftrightarrow [a]$
- $[a] = -[a]^T$ (called skew symmetric)
- Example: Same example: $a = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $b = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$[a] = \begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix} \qquad [a] \cdot b = \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}$$

Positive Semidefinite Matrix

$A = A^T$

- A symmetric square matrix $A \in \mathbb{R}^{n \times n}$ is called *positive semidefinite (p.s.d.)*, denoted by $A \succeq 0$, if $x^T A x \geq 0$, $\forall x \in \mathbb{R}^n$

$\Downarrow$ for all

"$\exists$" there exists

$\forall$ for all

- A symmetric square matrix $A \in \mathbb{R}^{n \times n}$ is called *positive definite (p.d.)*, denoted by $A \succ 0$, if $x^T A x > 0$ for all nonzero $x \in \mathbb{R}^n$

- p.d. matrices characterize positive definite quadratic forms:

e.g.: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$,

$$x^T A x = [x_1 \ x_2] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [x_1 \ x_2] \begin{bmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{bmatrix}$$

$$= 2x_1^2 + 2x_1x_2 + 2x_2^2$$

$$A \succ 0 \iff 2x_1^2 + 2x_1x_2 + 2x_2^2 > 0 \quad \forall \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \setminus \{0\}$$

Positive Semidefinite Matrix II

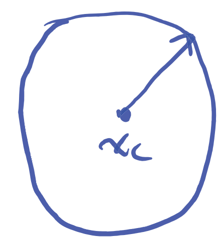
Recall: if A is symmetric
then all eigs of A are real

- Equivalent definitions for p.s.d. matrices:
  - All eigs of A are nonnegative
  - There exists a factorization $A = B^T B$ for some matrix B
- Equivalent definitions for p.d. matrices:
  - All eigs of A are strictly positive
  - There exists a factorization $A = B^T B$ with B square and nonsingular.
- If $A \in \mathbb{R}^{n\times n}$ is p.d., then A^{-1} is also p.d.

$\hookrightarrow A = B^T B \quad \Rightarrow \quad (A^{-1}) = (B^T B)^{-1} = (B^{-1})(B^{-1})^T$

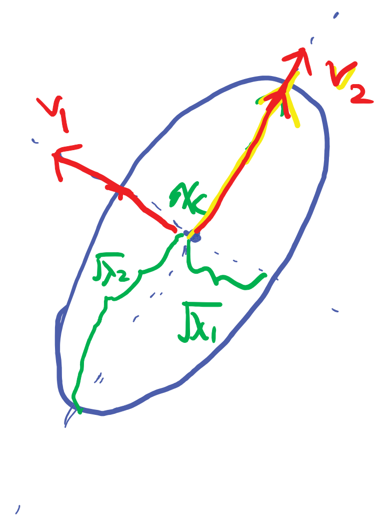
Ellipsoid in $\mathbb{R}^n$

- Unit sphere in $\mathbb{R}^n$: $S = \{x \in \mathbb{R}^n : \overbrace{\|x - x_c\|}^{(x-x_c)^T(x-x_c)} = 1\}$



- Ellipsoid in $\mathbb{R}^n$: $S = \{x \in \mathbb{R}^n : (x - x_c)^T A^{-1} (x - x_c) = 1\}$, for some p.d. $A \in \mathbb{R}^{n \times n}$.

- Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of A with corresponding eigenvectors $v_1, \ldots, v_n$.
  - Principal semi-axis lengths are $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}$
  - Direction of Principal semi-axes are aligned with $v_1, \ldots, v_n$
  - volume of the ellipsoid is proportional to $\sqrt{det(A)}$



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Scalar Linear Differential Equation

$$\dot{x}(t) = ax(t), \quad \text{with initial condition } x(0) = x_0 \tag{3}$$

- $x(t) \in \mathbb{R}$, $a \in \mathbb{R}$ is constant
- The above ODE has a unique solution $x(t) = e^{at}x_0$: $\begin{cases} x(0) = e^0 \cdot x_0 = x_0 \\ \dot{x}(t) = (e^{at})' x_0 = a e^{at} x_0 \\ \quad = a x(t) \end{cases}$
- What is the number "e"?
  - Called Euler number
  - Defined as the particular number such that $(e^x)' = e^x$

$$\Rightarrow \lim_{\delta \to 0} \frac{(e^{x+\delta} - e^x)}{\delta} = e^x \Rightarrow \frac{e^\delta - 1}{\delta} \to 1 \Rightarrow e^\delta \to \delta + 1$$

$$\boxed{e = \lim_{\delta \to 0} (\delta + 1)^{1/\delta}}$$

Complex Exponential

- For real variable $x \in \mathbb{R}$, Taylor series expansion for e^x around $x = 0$:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$$

- This can be extended to complex variables:

$$e^z \triangleq \sum_{k=0}^{\infty} \frac{z^k}{k!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots ,$$

This power series is well defined for all $z \in \mathbb{C}$ for $z \in \mathbb{C} \Rightarrow e^z \in \mathbb{C}$

- In particular, we have let $z = j\theta$ $e^{j\theta} = \underline{1} + j\theta - \underline{\frac{\theta^2}{2}} - j\frac{\theta^3}{3!} + \cdots$

- Comparing with Taylor expansions for $\underline{\cos}(\theta)$ and $\sin(\theta)$ leads to the Euler Identity

$$e^{j\theta} = \cos\theta + j(\sin\theta), \qquad \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Matrix Exponential

- Similar to the real and complex cases, we can define the so-called *matrix exponential*

$$e^A \triangleq \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

For $A \in \mathbb{R}^{n \times n}$, $e^A \in \mathbb{R}^{n \times n}$

- This power series is well defined whenever A is finite and constant.

- One can verify directly from definition:
  - $Ae^A = e^A A$ Recall, in general $AB \neq BA$
  - If $A = PDP^{-1}$, then $e^A = Pe^D P^{-1}$

If $D = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix} \Rightarrow e^D = \begin{bmatrix} e^{d_1} & & & \\ & e^{d_2} & & \\ & & \ddots & \\ & & & e^{d_n} \end{bmatrix}$

Vector Linear Differential Equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x}(t) = Ax(t), \quad \text{with initial condition } x(0) = x_0 \tag{4}$$

- $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is constant matrix, $x_0 \in \mathbb{R}^n$ is given.
- With the definition of matrix exponential, we can show that the solution to (4) is given by

$$\underline{x(t)} = \underline{e^{At}} x_0$$

· proof: $x(0) = e^{A\cdot 0} \cdot x_0 = x_0$

$$\frac{d}{dt} x(t) = \frac{d}{dt}\left(I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots \right) \cdot x_0$$

$$= \left(A + A^2 t + A^3 \frac{t^2}{2!} + \cdots \right) x_0$$

$$= A \underbrace{\left(I + At + A^2t^2/2! + \cdots \right)}_{} x_0 = A\, x(t) \quad \checkmark$$

Example

Find the solution to $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t)$, $\dot{x} = Ax$

By our previous result, the solution is $x(t) = e^{At} x_0$

Suppose you find out $A = \underbrace{\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}^{-1}}_{P^{-1}}$, where $P^{-1} = \begin{bmatrix} 0.5 & -0.5j \\ 0.5 & 0.5j \end{bmatrix}$

$$e^{At} = P \cdot e^{Dt} P^{-1} = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} e^{jt} & 0 \\ 0 & e^{-jt} \end{bmatrix} \begin{bmatrix} 0.5 & -0.5j \\ 0.5 & 0.5j \end{bmatrix} = \begin{bmatrix} e^{jt} & e^{-jt} \\ je^{jt} & -je^{-jt} \end{bmatrix} \begin{bmatrix} 0.5 & -0.5j \\ 0.5 & 0.5j \end{bmatrix}$$

$$= \begin{bmatrix} 0.5(e^{jt} + e^{-jt}) & -0.5j(e^{jt} - e^{-jt}) \\ 0.5j(e^{jt} - e^{-jt}) & 0.5(e^{jt} + e^{-jt}) \end{bmatrix}$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

$$\Rightarrow x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x_0$$

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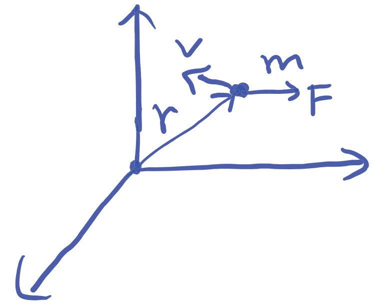
Linear Motion

Consider a particle with mass m



- position : $r \in \mathbb{R}^3$

- velocity/acceleration : $v = \dot{r}$ $\quad a = \dot{v} = \ddot{r}$

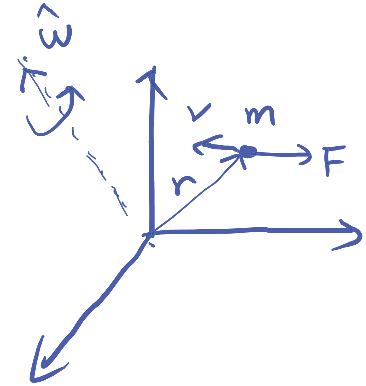
- Force : F

- Momentum : $\phi = m \cdot v$

- Newton's Second Law : $F = ma$

$$F = \frac{d\phi}{dt} = m\dot{v} = ma$$

Angular Motion



- Angle : θ

- Angular velocity : $\omega = \dot{\theta}\hat{\omega}$, $v = \omega \times r$

- Torque (Moment) : $\tau = r \times F$

- Angular Momentum : $L = r \times \phi = r \times (mv) = \overset{\text{Inertia}}{I} \cdot \omega$

  $\phi = m \cdot v$

  we can see that :

  $r \times (mv)$

  $m r \times (\omega \times r)$

  $= m \cdot [r](-r \times \omega)$

  $= \underbrace{m[r][-r]}_{I}\,\omega$

- Newton's Second Law : $\frac{dL}{dt} = \dot{r} \times (mv) + r \times (m\dot{v})$

  $= \underbrace{v \times (mv)}_{=0} + r \times F = \tau$