ECE5463: Introduction to Robotics

Lecture Note 3: Rotational Motion

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Outline

- Mathematics of Rigid Body Transformation
- Rotation Matrix and $SO(3)$
- Euler Angles and Euler-Like Parameterizations
- Exponential Coordinate of $SO(3)$
- Quaternion Representation of Rotation

Rigid Body Transformation

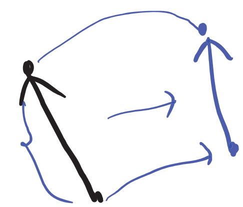
- **Object (Body) in $\mathbb{R}^3$:** a collection of points, represented by a subset $O \subset \mathbb{R}^3$

all pts of body mapped

- **Transformation of a body:** A single mapping $g : O \to \mathbb{R}^3$ which maps the coordinates of points in the body from their initial to final configurations.

- Transformation on points induce an action on vectors in a natural way. Given a transformation $g : O \to \mathbb{R}^3$, define

$$\hat{g}(v) \triangleq g(q) - g(p), \qquad \text{where } v = q - p$$



  - Note: $\hat{g}$ has a different domain than g

Rigid Body Transformation

- **Rigid Body Transformation:** A mapping $g : \mathbb{R}^3 \to \mathbb{R}^3$ is called a *rigid body transformation* if it satisfies the following two properties
  1. Length preserved: $\|g(q) - g(p)\| = \|q - p\|$, for all $p, q \in \mathbb{R}^3$
  2. Cross product is preserved: $\hat{g}(v \times w) = \hat{g}(v) \times \hat{g}(w)$, for all $v, w \in \mathbb{R}^3$.

e.g.: $g: (x,y,z) \mapsto (x,y,-z)$.

For this g, distance is preserved, but orientation is not

- Implications:
  - Inner product is preserved:

$$\hat{g}(v)^T \hat{g}(w) = v^T w, \text{ for all } v, w \in \mathbb{R}^3$$

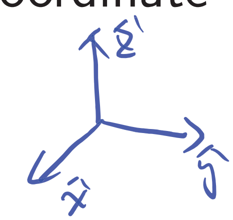
  - Angles between vectors are preserved : v, w are two vector

$\angle(v,w) = \angle(\hat{g}(v), \hat{g}(w))$

  - Orthogonal vectors are transformed to orthogonal vectors
  - Right-handed coordinate frames are transformed to right-handed coordinate frames

$\{\hat{x}, \hat{y}, \hat{z}\}$ unit axis

$\hat{z} = \hat{x} \times \hat{y}$, $\hat{x} = \hat{y} \times \hat{z}$



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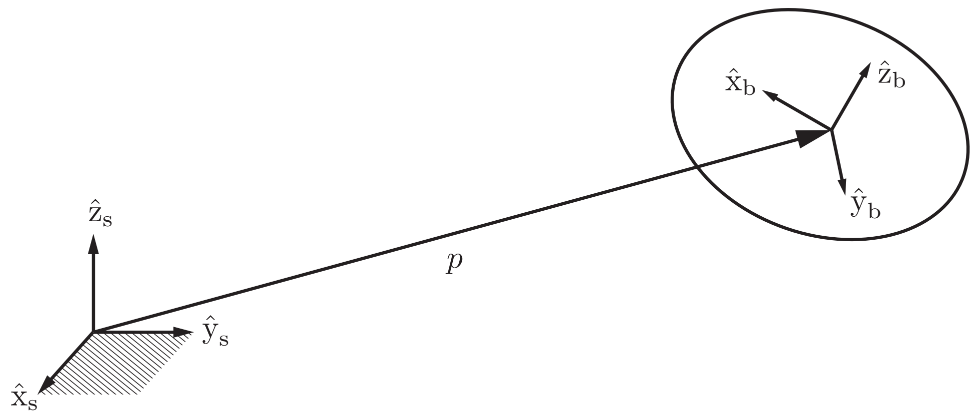
Representation of Orientation



- Basic Reference Frames:
  - **Fixed (or Space) Frame**: $\{s\} = \{\underbrace{\hat{x}_s, \hat{y}_s, \hat{z}_s}\}$ these are free vectors
  - **Body Frame**: $\{b\} = \{\underbrace{\hat{x}_b, \hat{y}_b, \hat{z}_b}\}$ free vectors that have nothing with the origin p
- Let x_{sb}, y_{sb} and z_{sb} be the coordinate of $\hat{x}_b, \hat{y}_b, \hat{z}_b$ in frame $\{s\}$

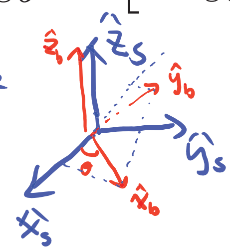
$$x_{sb} = \begin{bmatrix} x_{sb}^1 \\ x_{sb}^2 \\ x_{sb}^3 \end{bmatrix} \Rightarrow \hat{x}_b = x_{sb}^1 \cdot \hat{x}_s + x_{sb}^2 \cdot \hat{y}_s + x_{sb}^3 \hat{z}_s = [\hat{x}_s \ \ \hat{y}_s \ \ \hat{z}_s]\, x_{sb}$$

similarly, we will have $[\hat{x}_b \ \ \hat{y}_b \ \ \hat{z}_b] = [\hat{x}_s \ \ \hat{y}_s \ \ \hat{z}_s]\, \underbrace{[x_{sb} \ \ y_{sb} \ \ z_{sb}]}_{\text{rotation matrix}}$

Rotation Matrix

- Let $R_{sb} = \begin{bmatrix} x_{sb} & y_{sb} & z_{sb} \end{bmatrix}$: completely specifies the orientation of $\{b\}$ relative to $\{s\}$

Example: $\{s\}$ =



$\{b\}$ is obtained by rotating $\{s\}$ about $\hat{z}_s$ through θ

$$R_{sb} = \underbrace{\begin{bmatrix} C_\theta & -S_\theta & 0 \\ S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_z(\theta)}$$

- R_{sb} constructed above is called a **rotation matrix**. We know:
  - $R_{sb}^T R_{sb} = I$

$$R_{sb}^T R_{sb} = \begin{bmatrix} x_{sb}^T \\ y_{sb}^T \\ z_{sb}^T \end{bmatrix} \begin{bmatrix} x_{sb} & y_{sb} & z_{sb} \end{bmatrix} = \begin{bmatrix} x_{sb}^T x_{sb} & x_{sb}^T y_{sb} & x_{sb}^T z_{sb} \\ y_{sb}^T x_{sb} & y_{sb}^T y_{sb} & y_{sb}^T z_{sb} \\ \cdots & & \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow R_{sb}^{-1} = R_{sb}^T$

  - $\det(R_{sb}) = 1$ (because we have right handed frame)

$$\det(R_{sb}) = x_{sb}^T \underbrace{(y_{sb} \times z_{sb})}_{x_{sb}} = 1$$

- $R_{sb} = R_{bs}^{-1} = R_{bs}^T$

Special Orthogonal Group

- **Special Orthogonal Group**: Space of Rotation Matrices in $\mathbb{R}^n$ is defined as

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : RR^T = I, \det(R) = 1\}$$

- $SO(n)$ is a *group*. We are primarily interested in $SO(3)$ and $SO(2)$, rotation groups of $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively.

- **Group** is a set G, together with an operation $\bullet$, satisfying the following group axioms:
  - **Closure:** $a \in G, b \in G \Rightarrow a \bullet b \in G$

    e.g.: Interger $(\mathbb{Z}, +)$

    but $(\mathbb{Z}, \cdot)$ not group
  - **Associativity:** $(a \bullet b) \bullet c = a \bullet (b \bullet c)$, $\forall a, b, c \in G$

    $(\underline{A \in \mathbb{R}^{n\times n}, \det(A) \neq 0}, \cdot)$
  - **Identity element:** $\exists e \in G$ such that $e \bullet a = a$, for all $a \in G$.

    (nonzero rational number, $\cdot$)
  - **Inverse element:** For each $a \in G$, there is a $b \in G$ such that $a \bullet b = b \bullet a = e$, where e is the identity element.

Use of Rotation Matrix

- Representing an orientation: $R_{ab} = \left[\begin{array}{ccc} x_{ab} & y_{ab} & z_{ab} \end{array} \right]$

completely characterize frame {b} orientation relative to frame {a}

- Changing the reference frame:
  - $\boxed{p_a = R_{ab} p_b}$ physical • p, its coordinate in {a} and {b} are p_a, p_b

    $p \Leftrightarrow [\hat{x}_a \ \hat{y}_a \ \hat{z}_a] p_a = [\hat{x}_b \ \hat{y}_b \ \hat{z}_b] p_b$

    $= [\hat{x}_a \ \hat{y}_a \ \hat{z}_a] R_{ab} p_b \Rightarrow p_a = R_{ab} p_b$

  - $R_{ac} = \boxed{R_{ab}} R_{bc}$

    R_{bc}: orientation of {c} relative to {b}

    R_{ab} $\Downarrow$ viewed as changing reference frame from {b} to {a}

- Rotating a vector or a frame:
  - **Theorem (Euler)**:Any orientation $R \in SO(3)$ is equivalent to a rotation about a fixed axis $\hat{\omega} \in \mathbb{R}^3$ through an angle $\theta \in [0, 2\pi)$

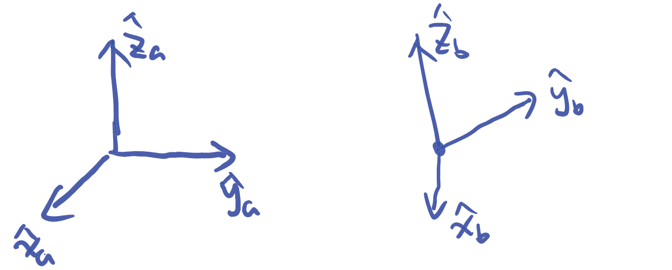
$$R = \mathrm{Rot}(\hat{\omega}, \theta)$$

suppose

$\underline{R_{ab}} = \mathrm{Rot}(\hat{\omega}; \theta)$

$\Downarrow$ viewed as rotation operator that takes {a} to {b}

- suppose $\{a\}$, $\{b\}$; $\{b\}$'s orientation relative $\{a\}$ is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$



rotation operator $R_z(\theta) = \begin{bmatrix} C_\theta & -S_\theta & 0 \\ S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Question is which "z"-axis we are rotating about.

1°: rotating about $\hat{z}_b$, a point $p_b \xrightarrow{R_{z_b}(\theta)} p_b'$ $\Rightarrow$ $\boxed{p_b' = R_z(\theta)\, p_b}$ only one frame is involved

2° . . . . . . $\hat{z}_a$, then what's the representation of rotation matrix in $\{b\}$

a point p with cooridinate p_a, $p_a \xrightarrow{R_{\hat{z}_a}(\theta)} p_a'$ $\Rightarrow$ $p_a' = R_z(\theta)\, p_a$

Now a point p with coordinate p_b in $\{b\}$, but rotate about $\hat{z}_a$

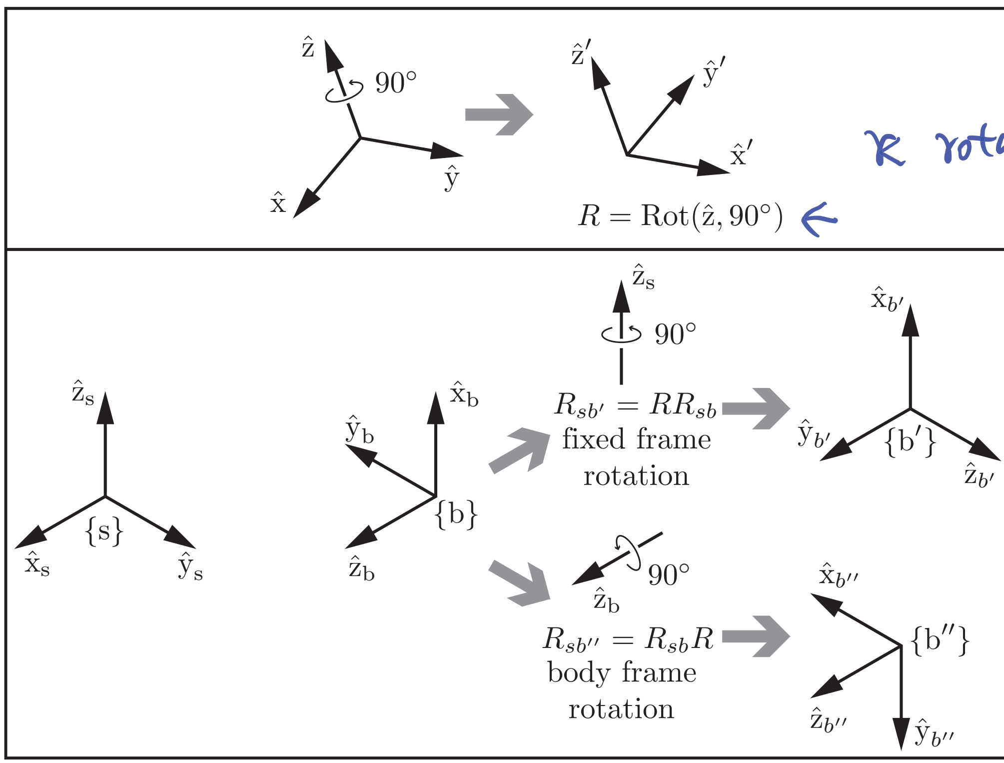
$$p_b \xrightarrow{R_{\hat{z}_a}(\theta)} p_b''$$

$$p_b'' = \overset{R_{ab}^{-1}}{R_{ba}}\, R_z(\theta)\, R_{ab}\, p_b$$

$$p_b'' = \underbrace{\left(R_{ab}^{-1} R_z(\theta) R_{ab}\right)}_{\text{the representation of rotation operator about } \{z_a\} \text{ in frame } \{b\}} \cdot p_b$$

Pre-multiplication vs. Post-multiplication

- Given $R \in SO(3)$, we can always find $\hat{\omega}$ and θ such that $R = \mathrm{Rot}(\hat{\omega}, \theta)$.
- Premultiplying by R yields a rotation about an axis $\hat{\omega}$ considered in the fixed frame;
- Postmultiplying by R yields a rotation about $\hat{\omega}$ considered in the body frame



R rotation about $\hat{z}$ axis for 90°

$$R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_{sb} R$: rotate {b} about $\hat{z}_b$ through 90°

$R_{sb} \cdot (\underline{R_{sb}^{-1} R R_{sb}})$

$= \underline{R_{sb} R_{sb}^{-1}} R R_{sb} = \boxed{R R_{sb}}$

rotate {b} about $\hat{z}_a$ through 90°

Another Example:

rotation operation about y-axis through β : $R_y(\beta) = \begin{bmatrix} C_\beta & 0 & S_\beta \\ 0 & 1 & 0 \\ -S_\beta & 0 & C_\beta \end{bmatrix}$

$R_y(\frac{\pi}{2}) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Suppose we have three frames $\{a\} \xrightarrow{R_{ab}} \{b\} \xrightarrow{R_{bc}} \{c\}$

1°: Frame $\{d\}$: obtained by rotating $\{b\}$ about $\hat{y}_b$ through $\frac{\pi}{2}$

2°: Frame $\{d'\}$: . . . . $\hat{y}_c$ through $\frac{\pi}{2}$

3°: Frame $\{d''\}$: . . . . $\{c\}$ about $\hat{y}_b$ through $\frac{\pi}{2}$

Questions: 1: what's R_{ad}? $\quad R_{ad} = R_{ab} R_y(\frac{\pi}{2}) = R_{ab} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

2 what's $R_{ad'}$? $\quad R_{ad'} = R_{ab} \cdot \big(\underbrace{R_{cb}^{-1} R_y(\frac{\pi}{2}) R_{cb}}_{R_{bd'}} \big)$

3 What's $R_{ad''}$? $\quad R_{ad''} = R_{ac} \underline{R_{cd''}} = \underline{R_{ac}} \left(R_{bc}^{-1} R_y(\frac{\pi}{2}) R_{bc} \right)$

$= R_{ab} \cdot R_{bc} (\; . \; . \; . \; . \;)$

Coordinate System for $SO(3)$

- How to parameterize the elements in $SO(3)$?

- The definition of $SO(3)$ corresponds to implicit representation:

$$R \in \mathbb{R}^{3\times 3}, RR^T = I, det(R) = 1$$

.

  - 6 independent equations with 9 unknowns

  - Dimension of $SO(3)$ is 3.

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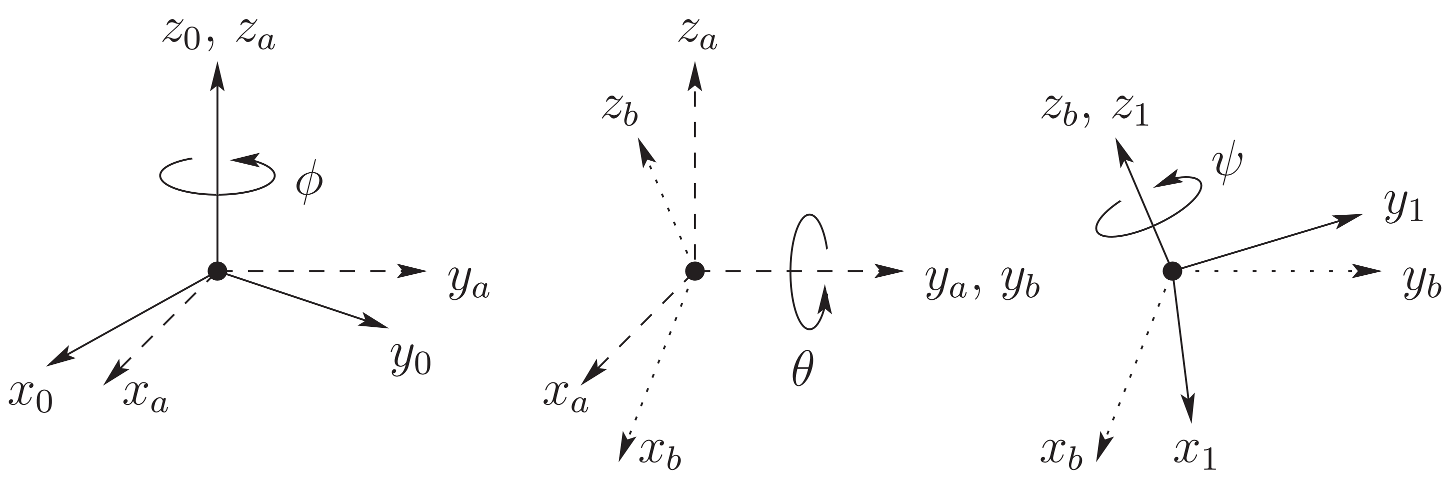
Euler Angle Representation of Rotation



- Euler angle representation:
  - Start with {b} coincident with {s}
  - Rotate {b} about $\hat{\mathrm{z}}_{\mathrm{b}}$ by an angle α, then rotate about the (new) $\hat{\mathrm{y}}_{\mathrm{b}}$ axis by β, and then rotate about the (new) $\hat{\mathrm{z}}_{\mathrm{b}}$ axis by γ. This yields a net orientation $R_{sb}(\alpha, \beta, \gamma)$ parameterized by the ZYZ angles (α, β, γ)
  - $R_{sb}(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma)$

Other Euler-Like Parameterizations

- Other types of Euler angle parameterization can be devised using different ordered sets of rotation axes

- Common choices include:
  - ZYX Euler angles: also called *Fick angles* or yaw, pitch and roll angles
  - YZX Euler angles (Helmholtz angles)

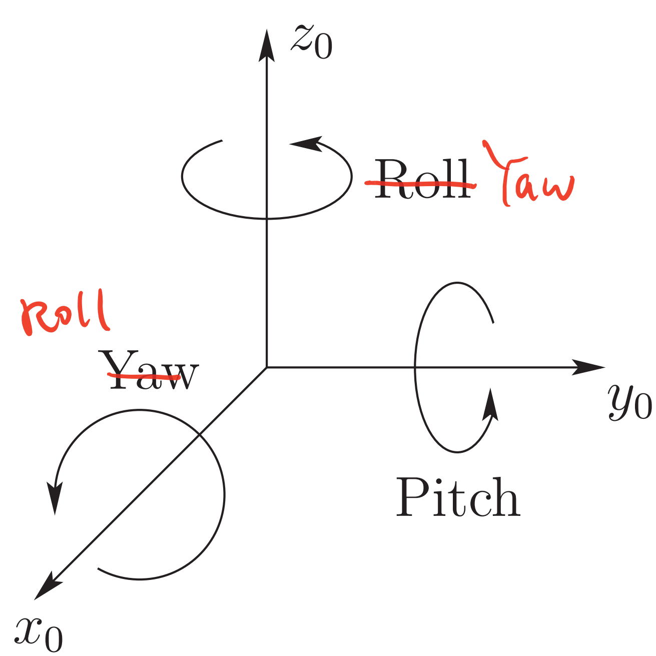
ZYX- Euler angles:

$$R(\alpha,\beta,\gamma) = \underline{R_Z(\alpha)R_Y(\beta)R_X(\gamma)}$$

Rotation hierarchy:

changing α will change the rotation axis for $\hat{y}$ and $\hat{x}$

Changing β will change the rotation axis for just $\hat{x}$



Examples of Euler-Like Representations

$$R_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\gamma & -s_\gamma \\ 0 & s_\gamma & c_\gamma \end{bmatrix}, \quad R_y(\beta) = \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix}, \quad R_z(\alpha) = \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_z(\alpha) R_y(\beta) \neq R_y(\beta) R_z(\alpha)$$

ZYX:

$$R(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_x(\gamma) = \begin{bmatrix} * & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

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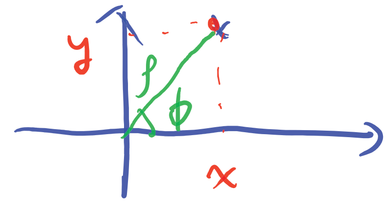
Towards Exponential Coordinate of $SO(3)$

- Recall the polar coordinate system of the complex plane:
  - Every complex number $z = x + jy = \rho e^{j\phi}$



  - Cartesian coordinate $(x, y) \leftrightarrow$ polar coorindate (ρ, ϕ)
  - For some applications, the polar coordinate is preferred due to its geometric meaning.

- For any rotation matrix $R \in SO(3)$, it turned out $R = e^{[\hat{\omega}]\theta}$
  - $\hat{\omega}$: unit vector representing the axis of rotation
  - θ: the degree of rotation
  - $\hat{\omega}\theta$ is called the **exponential coordinate** for $SO(3)$.

e.g. $\hat{\omega}\theta = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ then $\hat{\omega} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}}$, $\theta = \sqrt{2}$

Skew Symmetric Matrices

- Recall that cross product is a special linear transformation.
- For any $\omega \in \mathbb{R}^n$, there is a matrix $[\omega] \in \mathbb{R}^{n \times n}$ such that $\omega \times p = [\omega]p$

$$\omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \leftrightarrow [\omega] = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

- Note that $[\omega] = -[\omega]^T \leftarrow$ skew symmetric
- $[\omega]$ is called a skew-symmetric matrix representation of the vector ω
- The set of skew-symmetric matrices in: $so(n) \triangleq \{S \in \mathbb{R}^{n \times n} : S^T = -S\}$
- We are interested in case $n = 2, 3$

Find Rotation $\mathrm{Rot}(\omega, \theta)$ via Differential Equation

- Consider a point p with coordinate p_0 at time $t = 0$

- Rotate the point with constant unit velocity around fixed axis $\hat{\omega}$. The motion is described by

the sln to this ODE is $p(t)$, we know $\boxed{p(\theta) = p_f}$ ...(**)

$$\underbrace{\dot{p}(t)} = \underbrace{\hat{\omega} \times p(t)}_{v(t)} = [\hat{\omega}]p(t), \text{ with } p(0) = p_0,$$

Note: we assume here that w vector passes through the origin.

- This is a linear ODE with solution: $p(t) = e^{[\hat{\omega}]t}p_0$ (***)

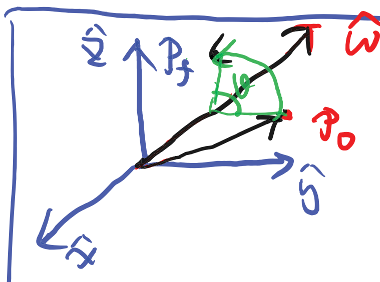
similar to $\dot{x} = Ax$

- Note $p(\theta) = \mathrm{Rot}(\hat{\omega}, \theta)p_0$, therefore

By (*) and (**), $p(\theta) = Rot(\hat{\omega}, \theta)p_0$

then by (***)

$$\boxed{Rot(\omega, \theta) = e^{[\hat{\omega}]\theta}}$$



By previous discussion,

$p_f = Rot(\hat{\omega}; \theta)p_0$ ...(*)

Find Rotation $\mathrm{Rot}(\omega, \theta)$ via Differential Equation

- Exponential Map: By definition

$$e^{[\omega]\theta} = I + \theta[\omega] + \frac{\theta^2}{2!}[\omega]^2 + \frac{\theta^3}{3!}[\omega]^3 + \cdots$$

← where $\hat{\omega}$ is unit vector

- **Rodrigues' Formula**: Given any $[\hat{\omega}] \in so(3)$, we have

$$e^{[\hat{\omega}]\theta} = I + [\hat{\omega}]\sin(\theta) + [\hat{\omega}]^2(1 - \cos(\theta))$$

Fact: for any $\hat{\omega}$ with $\|\hat{\omega}\| = 1$, we have $[\hat{\omega}]^3 = -[\hat{\omega}]$

e.g.: $\hat{\omega} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} \Rightarrow [\hat{\omega}] = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & -1 & 0\\1 & 0 & -1\\0 & 1 & 0\end{bmatrix} \Rightarrow [\hat{\omega}]^2 = \frac{1}{2}\begin{bmatrix}-1 & 0 & 1\\0 & -2 & 0\\1 & 0 & -1\end{bmatrix}$

$[\hat{\omega}]^3 = \frac{1}{2\sqrt{2}}\begin{bmatrix}0 & 2 & 0\\-2 & 0 & 2\\0 & -2 & 0\end{bmatrix} = -[\hat{\omega}]$

$+\frac{\theta^3}{3!}[\hat{\omega}]^3$

$$e^{[\hat{\omega}]\theta} = I + \theta[\hat{\omega}] + \frac{\theta^2}{2!}[\hat{\omega}]^2 - \frac{\theta^3}{3!}[\hat{\omega}] - \frac{\theta^4}{4!}[\hat{\omega}]^2 + \cdots$$

$$= I + \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)}_{\sin\theta}[\hat{\omega}] + \underbrace{\left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)}_{1-\cos\theta}[\hat{\omega}]^2$$

Note: $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$

$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$

Find Rotation $\mathrm{Rot}(\omega, \theta)$ via Differential Equation

- **Proposition:** For any unit vector $[\hat{\omega}] \in so(3)$ and any $\theta \in \mathbb{R}$,

$$\underline{e^{[\hat{\omega}]\theta}} \in SO(3)$$

From slide 19, we know $e^{[\hat{\omega}]\theta} = \underline{\mathrm{Rot}(\hat{\omega};\theta)} \in SO(3)$

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rotation motion

representation of the rotation motion

$$\left.\begin{array}{ll} \text{axis}: & \hat{\omega} \\ \text{angle}: & \theta \end{array}\right\} \xrightarrow{\text{exponential map}} e^{\overbrace{[\hat{\omega}]\theta}} \quad \text{(rotation matrix)}$$

Examples of Forward Exponential Map

- Rotation matrix $R_x(\theta)$ (corresponding to $\hat{x}\theta$) ↙ exponential coordinate

$$\hat{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad R_{\hat{x}}(\theta) = I + \sin\theta \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} + (1-\cos\theta) \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\Downarrow$$

$$[\hat{x}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

- Rotation matrix corresponding to $(1, 0, 1)^T$ exp. coordinate

$$\hat{\omega}\theta = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \hat{\omega} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \theta = \sqrt{2}$$

Based on slide 20: We know $\quad Rot(\hat{\omega};\theta) = e^{[\hat{\omega}]\theta} = I + \frac{\sin\theta}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$

$$+ (1-\cos\theta)\frac{1}{2} \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Let $\theta = \sqrt{2}$

$$= \begin{bmatrix} \; \end{bmatrix}$$

Logarithm of Rotations

- **Proposition:** For any $R \in SO(3)$, there exists $\hat{\omega} \in \mathbb{R}^3$ with $\|\hat{\omega}\| = 1$ and $\theta \in \mathbb{R}$ such that $R = e^{[\hat{\omega}]\theta}$

Inverse problem using Rodrigue's formula.

- If $R = I$, then $\theta = 0$ and $\hat{\omega}$ is undefined.

trace of R sum of diagonals

- If $\mathrm{tr}(R) = -1$, then $\theta = \pi$ and set $\hat{\omega}$ equal to one of the following

$$\frac{1}{\sqrt{2(1+r_{33})}} \begin{bmatrix} r_{13} \\ r_{23} \\ 1 + r_{33} \end{bmatrix}, \frac{1}{\sqrt{2(1+r_{22})}} \begin{bmatrix} r_{12} \\ 1 + r_{22} \\ r_{32} \end{bmatrix}, \frac{1}{\sqrt{2(1+r_{11})}} \begin{bmatrix} 1 + r_{11} \\ r_{21} \\ r_{31} \end{bmatrix}$$

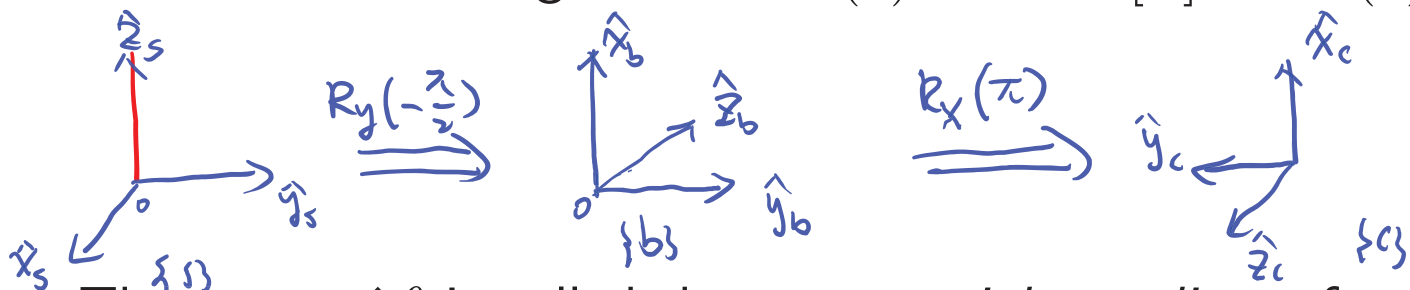
any of these (as long as denominator is not 0) works

- Otherwise, $\boxed{\theta = \cos^{-1}\left(\frac{1}{2}(\mathrm{tr}(R) - 1)\right) \in [0, \pi) \text{ and } [\hat{\omega}] = \frac{1}{2\sin(\theta)}(R - R^T)}$

Exponential Coordinate of $SO(3)$

$$\text{exp:} \quad [\hat{\omega}]\theta \in so(3) \quad \to \quad R \in SO(3)$$

$$\text{log:} \quad R \in SO(3) \quad \to \quad [\hat{\omega}]\theta \in so(3)$$



- The vector $\hat{\omega}\theta$ is called the *exponential coordinate* for R

$R_{sc} = R_y(-\frac{\pi}{2}) R_x(\pi)$

$$= \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

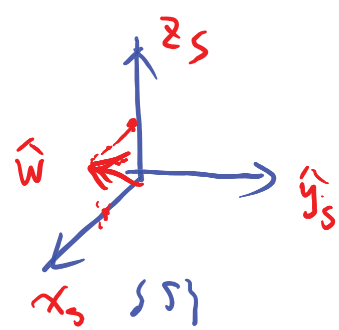
$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

To find exponential coordinate for R_{sc}

$tr(R) = -1$

- The exponential coordinates are also called the canonical coordinates of the rotation group $SO(3)$

then $\theta = \pi$ $\quad \underbrace{\hat{w} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{\text{instantaneous rotation axis}}$



$\Rightarrow$ So exponential coor. is $\frac{\pi}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Outline

- Mathematics of Rigid Body Transformation
- Rotation Matrix and $SO(3)$
- Euler Angles and Euler-Like Parameterizations
- Exponential Coordinate of $SO(3)$
- Quaternion Representation of Rotation

- For rotation matrix, we need 9 number > dof = 3 for SO(3)
- link to axis & angle is hidden
- Composition of rotation requires matrix multiplication

Quaternions

- Quaternions generalize complex numbers and can be used to effectively represent rotations in $\mathbb{R}^3$.
- A quaternion is a vector quantity of the following form:

$$q = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k}$$

  where q_0 is the scalar ("real") component and $\vec{q} = (q_1, q_2, q_3)$ is the vector ("imaginary") component.

- Addition and multiplication operations:
  - $p + q = (p_0 + q_0) + (p_1 + q_1)\boldsymbol{i} + (p_2 + q_2)\boldsymbol{j} + (p_3 + q_3)\boldsymbol{k}$
  - $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$
  - $\boldsymbol{ij} = \boldsymbol{k} = -\boldsymbol{ji}$, $\boldsymbol{jk} = \boldsymbol{i} = -\boldsymbol{kj}$, $\boldsymbol{ki} = \boldsymbol{j} = -\boldsymbol{ik}$

- In scalar-vector form, product of $p = (p_0, \vec{p})$ and $q = (q_0, \vec{q})$ is given by

$$pq = \left(p_0 q_0 - \vec{p}^T\vec{q}, \quad p_0\vec{q} + q_0\vec{p} + \vec{p}\times\vec{q}\right)$$

Conjugate, Norm, and Inverse

- Given a quaternion $q = (q_0, \vec{q})$:
  - **Conjugate**: $q^* = (q_0, -\vec{q})$
  - **Norm**: $\|q\|^2 = qq^* = q^*q = q_0^2 + q_1^2 + q_2^2 + q_3^2$
  - **Inverse:** $q^{-1} \triangleq \frac{q^*}{\|q\|^2}$

Example:

$p = 1 + 2i + 3j + 4k$, $q = j + k$

$pq = j + k + 2ij + 2ik + 3j^2 + 3jk + 4kj + 4k^2$

$= j + k + 2k - 2j \underline{-3} + 3i - 4i \underline{-4}$

$= -7 + (-i - j + 3k) = (-7, (-1, -1, 3))$

Scalar-vector:

$$pq = \Big(\underbrace{0 - \begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{-7} , \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0 + \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}} \Big)$$

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 3 \\ 4 & 0 & -2 \\ -3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

Quaternion Representation of Rotation

- For $\vec{v} \in \mathbb{R}^3$, we can associate it with a 0 scalar component to construct a *purely imaginary* quaternion:
$$\mathring{v} = (0, \vec{v})$$
- Each quaternion $q = (q_0, \vec{q})$ defines an operation on a vector $\vec{v} \in \mathbb{R}^3$:
$$L_q(\vec{v}) = \text{Im}(q\mathring{v}q^*) = (q_0^2 - \|\vec{q}\|^2)\vec{v} + 2(\vec{q}^T\vec{v})\vec{q} + 2q_0(\vec{q} \times \vec{v})$$

take the imaginary part of a quaternion

In fact, you can verify

$$q\mathring{v}q^* = (q_0, \vec{q})(0, \vec{v})(q_0, -\vec{q})$$
$$= (0, \quad)$$

Quaternion Representation of Rotation

- **Unit quaternion:** If $\|q\| = 1$, we can always find $\theta \in [0, 2\pi)$ and unit vector $\hat{\omega} \in \mathbb{R}^3$ such that:
$$q = \left(\cos\left(\frac{\theta}{2}\right), \hat{\omega}\sin\left(\frac{\theta}{2}\right)\right)$$
In this case, $L_q(\vec{v})$ is the vector obtained by rotating $\vec{v} \in \mathbb{R}^3$ about the axis $\hat{\omega}$ for θ degree.

- Given a unit quaternion $q = (q_0, \vec{q})$, we can extract the rotation axis/angle by:
$$\theta = 2\cos^{-1}(q_0), \quad \hat{\omega} = \begin{cases} \frac{\vec{q}}{\sin(\theta/2)} & \text{if } \theta \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

- $L_{q^*}(\vec{v})$ rotates $\vec{v}$ about $\hat{\omega}$ for $-\theta$ degree.

- Quaternion provides global parameterization of $SO(3)$, which does not suffer from singularities

Examples of Quaternions

- Let $\hat{\omega} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$ $\theta = \pi$

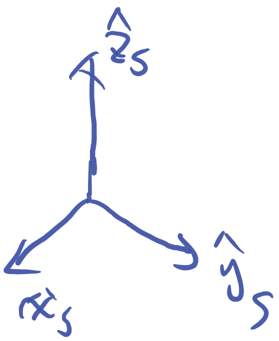
Quaternion for $Rot(\hat{\omega};\pi)$ is $q = \left(\overset{0}{\cos\left(\frac{\pi}{2}\right)}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}.\right)$

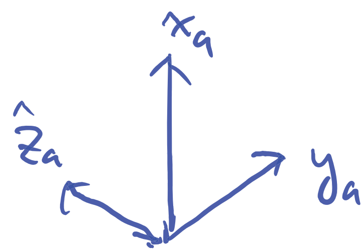
Let $v = \begin{bmatrix}1\\0\\0\end{bmatrix}$ x-axis

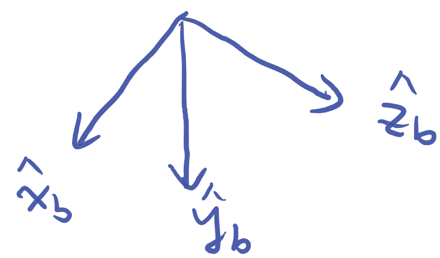
$$L_q(v) = (-1)\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix} + 0$$

$$= \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Exercise 3.1







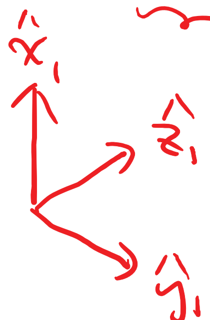
$$R_{sa} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \qquad R_{sb} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

(e)

$$R = R_{sb} = \mathrm{Rot}(\hat{x}, -90)$$

$$R_1 = R_{sa} R = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

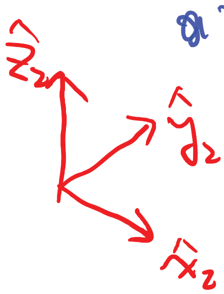
rotate about $\hat{x}_a$



{1}

$$R_2 = R R_{sa} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

rotate about $\hat{x}_s$



(f): $R_{sb} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

If $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = p_b$, then

① $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ can mean coordinate of p in {s}

② $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ can also mean the coordinate of the rotated point p' in {b}

coordinate of p in s

↓

If $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = p_s$, then $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ is the coordinate of the rotated point p'' in {s}

rotation axis is expressed in {s}