ECE5463: Introduction to Robotics

Lecture Note 6: Forward Kinematics

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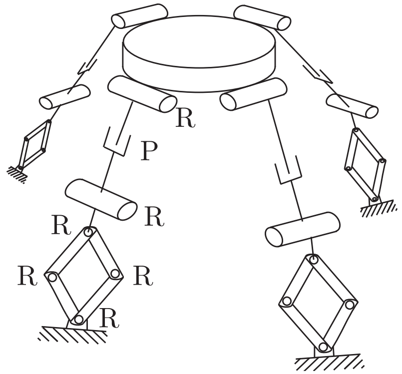
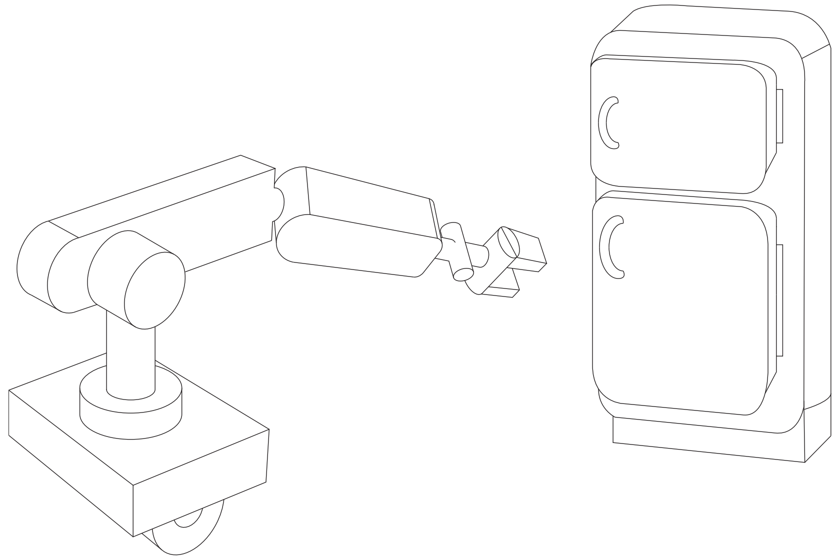
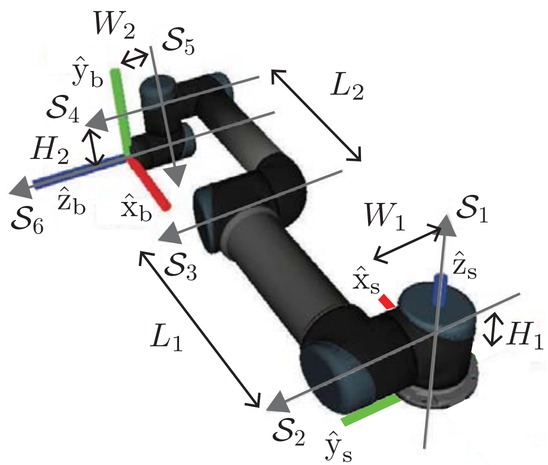
Outline

- Background
- Illustrating Example
- Product of Exponential Formula
- Body Form of the PoE Formula

Kinematics







Kinematics is a branch of classical mechanics that describes the motion of points, bodies (objects), and systems of bodies (groups of objects) without considering the mass of each or the forces that caused the motion

Kinematics of Robotic Manipulator

- **Forward Kinematics**: calculation of the position p and orientation R of the end-effector frame from its joint variables $\theta = (\theta_1, \ldots, \theta_n)$

- Two commonly adopted approaches

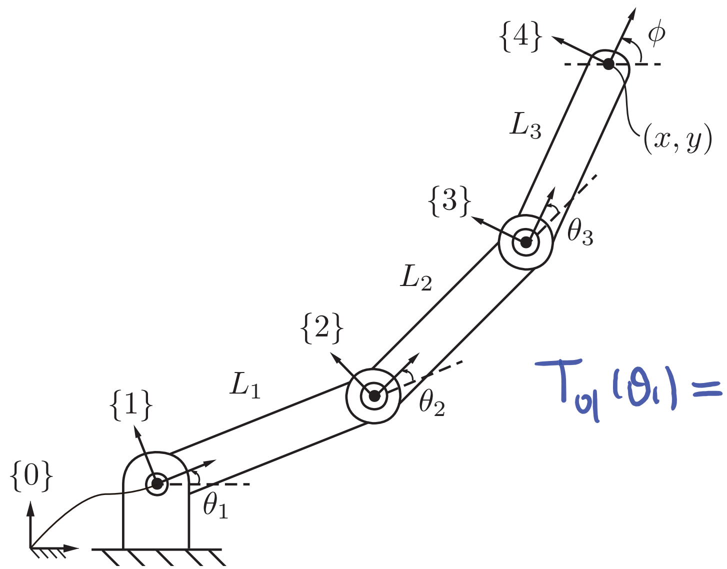
- Approach 1: Assign a frame at each link, typically at the joint axis, then calculate the end-effector configuration through intermediate frames.
  - **Denavit-Hartenberg** parameters: most widely adopted convention for frame assignment

- Approach 2: **Product of Exponential (PoE)** formula: directly compute the end-effector frame configuration relative to the fixed frame through screw motion along the screw axis associated with each joint.

Forward Kinematics: Basic Setup

- Suppose that the robot has n joints and n links. Each joint has one degree of freedom represented by joint variable θ_i , $i=1,\cdots,n$
  - Revolute joint: θ_i represents the joint angle
  - Primatic joint: θ_i represents the joint displacement
- Specify a fixed frame {s}: also referred to as frame {0}
- Attach frame $\{i\}$ to link i at joint i, for $i = 1, \ldots, n$
- Use one more frame {b} at the end-effector: sometimes referred to as frame {n+1}
- **Goal:** Find an analytical expression of $T_{sb}(\theta_1, \ldots, \theta_n)$

A Simple Example: Approach 1 (D-H Parameters)



$n = 3$, {s} frame = {0} frame, {b} = {4}

Goal: find $\boxed{T_{sb}(\theta_1, \theta_2, \theta_3)}$

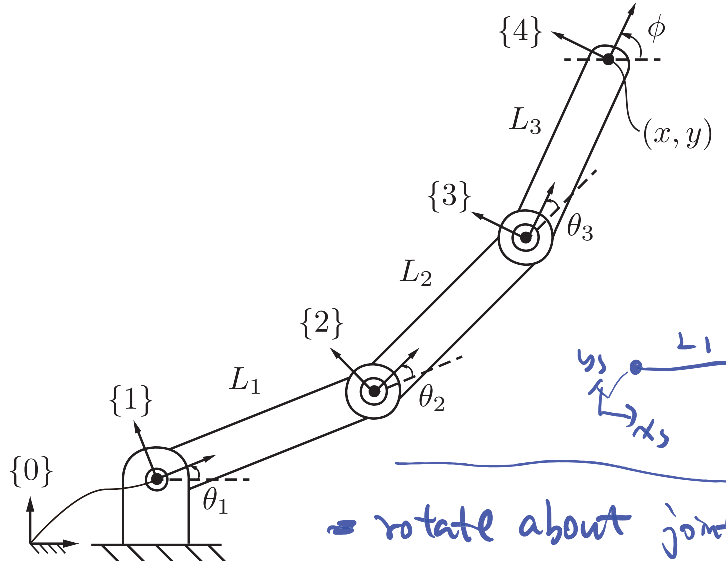
$$T_{sb}(\theta) = T_{01}(\theta_1)\, T_{12}(\theta_2)\, T_{23}(\theta_3)\, T_{34}$$

$$T_{01}(\theta_1) = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathrm{Rot}(\hat{z}, \theta_1)$$

$$T_{12}(\theta_2) = \begin{bmatrix} \mathrm{Rot}(\hat{z}, \theta_2) & \begin{matrix} L_1 \\ 0 \\ 0 \end{matrix} \\ 0 \;\; 0 \;\; 0 & 1 \end{bmatrix}, \quad T_{23}(\theta_3) = \begin{bmatrix} \mathrm{Rot}(z, \theta_3) & \begin{matrix} L_2 \\ 0 \\ 0 \end{matrix} \\ 0 \;\; 0 \;\; 0 & 1 \end{bmatrix}, \quad T_{34} = \begin{bmatrix} I & \begin{matrix} L_3 \\ 0 \\ 0 \end{matrix} \\ 0 \;\; 0 \;\; 0 & 1 \end{bmatrix}$$

Frame assignment & numbering can be quite tricky for 3D robots

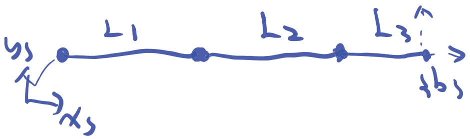
A Simple Example: Approach 2 (PoE Formula)



– start with a simple special case.

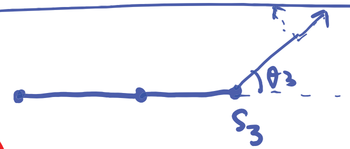
$T_{sb}(\theta_1,\theta_2,\theta_3) = ?$

special case: $\theta_1=\theta_2=\theta_3 \Rightarrow T_{sb}(0,0,0) = \underbrace{\begin{bmatrix}1&0&0&L_1+L_2+L_3\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}}_{\triangleq\, M}$



– rotate about joint 3 by θ_3:

$$T_{sb}(0,0,\theta_3) = e^{[S_3]\theta_3}\underbrace{T(0,0,0)}_{M}$$

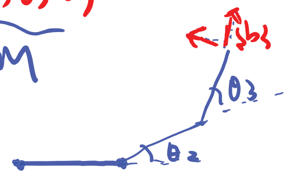


this is a screw motion with axis S_3

$$S_3 = \begin{bmatrix}\omega_3\\ v_3\end{bmatrix}$$

$$\omega_3 = \begin{bmatrix}0\\0\\1\end{bmatrix},\quad v_3 = -\omega_3\times q_3 + 0 = -\begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}L_1+L_2\\0\\0\end{bmatrix} = \begin{bmatrix}0\\-(L_1+L_2)\\0\end{bmatrix}$$

– rotate about joint 2 (screw axis) by θ_2:



$$T_{sb}(0,\theta_2,\theta_3) = e^{[S_2]\theta_2}\,T(0,0,\theta_3)$$

$$= e^{[S_2]\theta_2}e^{[S_3]\theta_3}\,T(0,0,0) = e^{[S_2]\theta_2}e^{[S_3]\theta_3}M$$

what's $S_2 = \begin{pmatrix}\omega_2\\ v_2\end{pmatrix}$, $\omega_2 = \begin{bmatrix}0\\0\\1\end{bmatrix}$, $v_2 = -\omega_2\times q_2 = -\begin{bmatrix}0\\0\\1\end{bmatrix}\times\begin{bmatrix}L_1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\-L_1\\0\end{bmatrix}$

– Similarly, we can obtain:

$$T_{sb}(\theta_1,\theta_2,\theta_3) = e^{[S_1]\theta_1}e^{[S_2]\theta_2}e^{[S_3]\theta_3}M, \text{ where } S_1 = \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}$$

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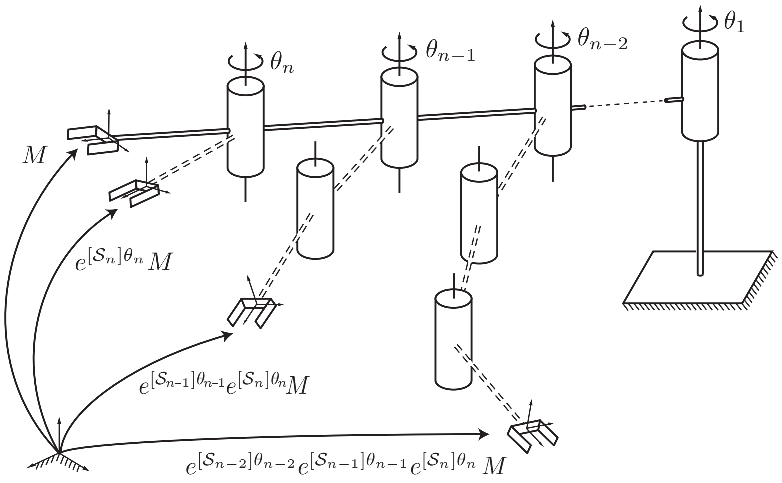
Product of Exponential: Main Idea

- **Goal:** Derive $T_{sb}(\theta_1, \ldots, \theta_n)$

- First choose a zero position ("home" position): $\theta_1 = \theta_1^0, \ldots, \theta_n = \theta_n^0$ such that $M \triangleq T_{sb}(\theta_1^0, \ldots, \theta_n^0)$ can be easily found.

  Without loss of generality, suppose home position is given by $\theta_i^0 = 0$, $i = 1, \ldots, n$.



- Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be the screw axes expressed in {s}, corresponding to the joint motions when the robot is at its home position

- Apply screw motion to joint n: $T_{sb}(0, \ldots, 0, \theta_n) = e^{[\mathcal{S}_n]\theta_n} M$

- Apply screw motion to joint $n-1$ to obtain:

$$T_{sb}(0, \ldots, 0, \theta_{n-1}, \theta_n) = e^{[\mathcal{S}_{n-1}]\theta_{n-1}} e^{[\mathcal{S}_n]\theta_n} M$$

- After n screw motions, the overall forward kinematics:

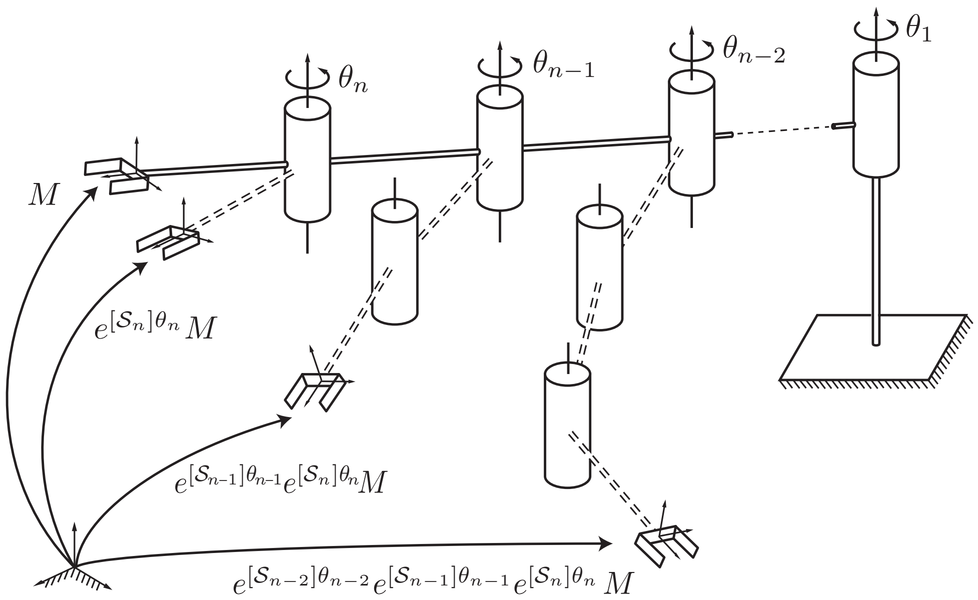
$$T_{sb}(\theta_1, \ldots, \theta_n) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M$$

Forward Kinematics: Steps for PoE

- **Goal:** Derive $T_{sb}(\theta_1, \ldots, \theta_n)$

- **Step 1:** Find the configuration of {b} at the home position

$$M = T_{sb}(0, \ldots, 0)$$



- **Step 2:** Find screw axes $\mathcal{S}_1, \ldots, \mathcal{S}_n$ expressed in {s}, corresponding to the joint motions when the robot is at its home position

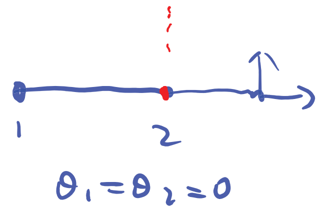
- Forward kinematics:

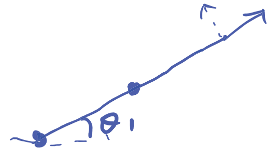
$$T_{sb}(\theta_1, \ldots, \theta_n) = e^{[\mathcal{S}_1]\theta_1} e^{[\mathcal{S}_2]\theta_2} \cdots e^{[\mathcal{S}_n]\theta_n} M$$

PoE: Screw Motions in Different Order (1/2)

- PoE was obtained by applying screw motions along screw axes $\mathcal{S}_n$, $\mathcal{S}_{n-1}$, ... What happens if the order is changed?

- For simplicity, assume that $n = 2$, and let us apply screw motion along $\mathcal{S}_1$ first:
  - $T_{sb}(\theta_1, 0) = e^{[\mathcal{S}_1]\theta_1} M$



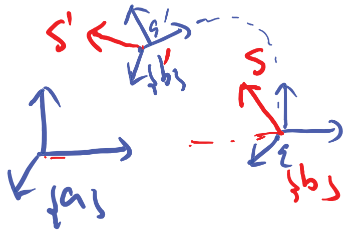


  - Now screw axis for joint 2 has been changed. The new axis $\mathcal{S}_2'$ is given by:

In general: suppose S is rigidly attached to the $\{b\}$

$T_{ab'} = T\, T_{ab}$, where $T = (R, p)$

Question what is S'?



$S = \begin{pmatrix} w \\ v \end{pmatrix}$, $S' = \begin{pmatrix} w' \\ v' \end{pmatrix}$

$\Updownarrow$ $(\hat{s}, h, q)$ $\qquad$ $\Updownarrow$ $(\hat{s}', h', q')$

$\Rightarrow h' = h$, $w' = Rw$, $q' = Rq + p$

$\hat{s}' = R\hat{s}$ $\qquad$ $(\tilde{q}' = T\tilde{q})$

we know $v = -w \times q + hw$

PoE: Screw Motions in Different Order (2/2)

- $T_{sb}(\theta_1, \theta_2) = e^{[\mathcal{S}_2']\theta_2} \underline{T_{sb}(\theta_1, 0)}$

After first screw motion

S_2 becomes $\underline{S_2' = [Ad_T] S_2} \iff [S_2'] = T[S_2]T^{-1}$

where $T = e^{[S_1]\theta_1}$

$\Rightarrow e^{[S_2']\theta_2} = T e^{[S_2]\theta_2} T^{-1}$

$$T_{sb}(\theta_1, \theta_2) = e^{[S_2']\theta_2} \underline{e^{[S_1]\theta_1} M}$$

$$= T e^{[S_2]\theta_2} T^{-1} \underline{e^{[S_1]\theta_1}} M$$

$$= e^{[S_1]\theta_1} e^{[S_2]\theta_2} M$$

$\Rightarrow$ $\underline{w' = Rw, \quad q' = Rq + p, \quad h' = h}$ ①

$v' = -w' \times q' + h' w'$

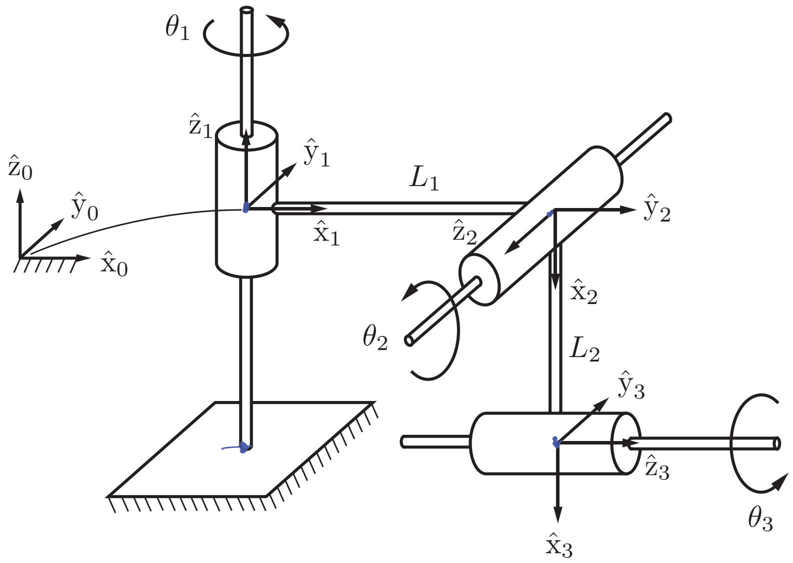
$\underline{\text{V.F.Y.}}$ $Rv + [p]Rw$ ②

① & ② $\Rightarrow$

$$S' = \begin{bmatrix} w' \\ v' \end{bmatrix} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \underbrace{\begin{bmatrix} w \\ v \end{bmatrix}}_{S}$$

$\Rightarrow S' = [Ad_T] S$

PoE Example: 3R Spatial Open Chain



let's pick $\{s\}$ coincide with the initial T_1 frame

pick $\{b\} = \{3\}$ - frame

- step 1: $M = T_{sb}(0,0,0) = \left[\begin{array}{ccc|c} 0 & 0 & 1 & L_1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -L_2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$

- step 2: $S_1 = (w_1, v_1)$: $w_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $v_1 = -w_1 \times q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$S_2 = (w_2, v_2)$: $w_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $v_2 = -w_2 \times q_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} L_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -L_1 \end{bmatrix}$

$S_3 = (w_3, v_3)$: $w_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_3 = -\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ -L_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -L_2 \\ 0 \end{bmatrix}$

$$T_{sb}(\theta_1, \theta_2, \theta_3) = e^{[S_1]\theta_1} e^{[S_2]\theta_2} e^{[S_3]\theta_3} M$$

Body Form of PoE Formula

- Fact: $e^{M^{-1}PM} = M^{-1}e^{P}M \quad \Rightarrow \quad Me^{M^{-1}PM} = e^{P}M$ can be verified using Taylor expansion of matrix exponential
- Let $\mathcal{B}_i$ be the screw axis of joint i expressed in end-effector frame when the robot is at zero position at zero position, we know
- Then body-form of the PoE formula is:

$T_{sb}(0,0,\ldots,0) = M \Rightarrow$

$S_i = [Ad_{T_{sb}}]\mathcal{B}_i$

$\Leftrightarrow \quad = [Ad_M]\mathcal{B}_i$

$[S_i] = M[\mathcal{B}_i]M^{-1}$

$$T_{sb}(\theta_1,\ldots,\theta_n) = Me^{[\mathcal{B}_1]\theta_1}e^{[\mathcal{B}_2]\theta_2}\cdots e^{[\mathcal{B}_n]\theta_n}$$

$$T_{sb}(\theta_1,\theta_2\cdots,\theta_n) = e^{[S_1]\theta_1}\cdots e^{[S_{n-1}]\theta_{n-1}}e^{[S_n]\theta_n}M$$

$$= e^{[S_1]\theta_1}\cdots e^{[S_{n-1}]\theta_{n-1}}Me^{[\mathcal{B}_n]\theta_n}$$

$$\vdots$$

$$= Me^{[\mathcal{B}_1]\theta_1}e^{[\mathcal{B}_2]\theta_2}\cdots e^{[\mathcal{B}_n]\theta_n}$$

$$e^{[S_n]\theta_n}M$$

$$= e^{M[\mathcal{B}_n]\theta_n M^{-1}}M$$

$$= Me^{[\mathcal{B}_n]\theta_n}\cancel{M^{-1}M}$$

More Discussions

- twist $\mathcal{V} = (\omega, v)$: just think of it as "velocity" of rigid body.

  any point q attached to the body $\dot{q} = \omega \times q + v$

- screw motion: particular motion

  $S = \{(\hat{s}, h, q)$

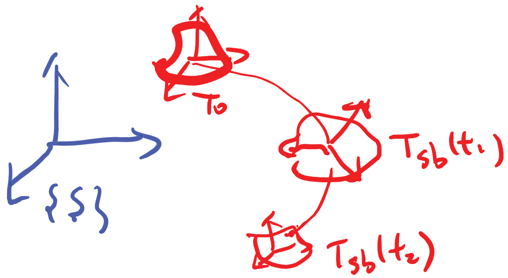
  $S \downarrow$ unit screw axis ; $(\hat{s}, h, q) \Updownarrow (\omega, v)$: twist representation of the screw $\quad v = -\omega \times q + h\omega$

- Any rigid body transformation $T \in SE(3)$, there exists $S\theta$ such that $e^{[S]\theta} = T$

  interpretation: $e^{[S]\theta}$ moves (point or frames) along S axis for amount θ

  or . . . . at speed $\dot{\theta}$ for time $t = \frac{\theta}{\dot{\theta}}$

- Now suppose: rigid body initial conf is $T_0 = (R_0, p_0)$, and follows screw motion with $S\dot{\theta}$ for time t, what's $T_{sb}(t) = e^{[S]\dot{\theta}t} T_0 = \begin{bmatrix} R_{sb}(t) & \underline{p_{sb}(t)} \\ 0 & 1 \end{bmatrix}$



- Now $T_{sb}(t) = \underline{e^{[S]\dot{\theta} t} T_0} \Rightarrow [\mathcal{V}_s] = \dot{T}_{sb}(t) T_{sb}^{-1}(t) = \frac{d}{dt}\left(e^{[S]\dot{\theta} t} T_0\right)\cdot\left(\quad\right)^{-1}$

$$= [S]\dot{\theta}\left(\quad\right)\left(\quad\right)^{-1} = [S]\dot{\theta}$$

$\Leftrightarrow \mathcal{V}_s = S\dot{\theta}$

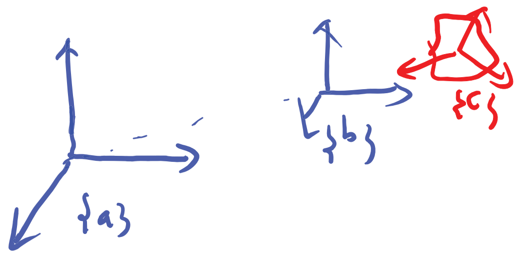
$\Downarrow$

$\mathcal{V}_s = (\omega_s, v_s)$

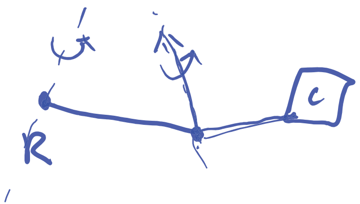
$\mathcal{V}_b$? ① $\mathcal{V}_b = [Ad_{T_{bs}}]\mathcal{V}_s = [Ad_{T_{sb}^{-1}}]\mathcal{V}_s$

② - method: directly use formula $\underline{\omega_b = R_{sb}^T \omega_s, \quad v_b = R_{sb}^T \dot{p}_{sb}}$

- consider rigid body velocity in different frame:



e.g.



$\mathcal{V}_{bc}$: velocity of {c} relative to {b} ; $\mathcal{V}_{ac}$: velocity of {c} relative to {a}

$\mathcal{V}_{ab}$: velocity of {b} relative to {a}

$$[\mathcal{V}_{ac}] = \dot{T}_{ac} T_{ac}^{-1} = \left[(\dot{T}_{ab} T_{bc}) + (T_{ab}\dot{T}_{bc})\right](T_{ab} T_{bc})^{-1} = \dot{T}_{ab} \cancel{T_{bc} T_{bc}^{-1}} T_{ab}^{-1} + T_{ab}(\underline{\dot{T}_{bc} T_{bc}^{-1}}) T_{ab}^{-1}$$

$$= \dot{T}_{ab} T_{ab}^{-1} + T_{ab}(\dot{T}_{bc} T_{bc}^{-1}) T_{ab}^{-1}$$

$$= [\mathcal{V}_{ab}] + T_{ab}[\mathcal{V}_{bc}] T_{ab}^{-1}$$

$\mathcal{V}_{ac} = \mathcal{V}_{ab} + [Ad_{T_{ab}}]\mathcal{V}_{bc} \Leftarrow$

If $\{b\}$ is stationary $\Rightarrow$ $V_{ac} = [Ad_{T_{ab}}] V_{bc}$