Lecture 07: Introduction to Normal-Form Games and Nash Equilibria

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Abstract

In this lecture, we provide a succinct introduction to game theory in preparation for our main topic of learning in games in this second part of the course.

1 Preamble

Game theory is concerned with analyzing situations involving strategic interactions between several optimizing entities (agents, populations, companies, ...) with different objectives, having an influence on each other via their actions.

Unlike optimization, an agent does not control all the variables that affect her.

An agent's choice of their own controlled decision variable also affects other agents.

In strategic games, we identify three important elements:

1. Decision makers: set $\mathcal{I}$ of entities participating.
    - Terminology: agents, players, users, decision makers.
    - Examples: populations, robots, companies, LLMs, ...
2. Choices: Each user $i \in \mathcal{I}$ is associated to a set of choices.
    - Examples: Machine learning model, routes in a transportation network, ...
3. Preferences: Each user $i \in \mathcal{I}$ is associated with a utility function $u_i : \mathcal{X} \to \mathbb{R}$ where $\mathcal{X} := \prod_{i=1}^{N} \mathcal{X}_i$.
    - Terminology: utility, preference, payoff, reward, cost, ...

- (System objective: The system of agents might be associated with a performance metric of the form $W : \mathcal{X} \to \mathbb{R}$.
    - Examples: social welfare, total cost, aggregated metrics,...)

The interactions between distinct agents depend on multiple considerations:

- The number of agents,
- The incentives of agents: aligned or competing interests,
- Whether agents have the same information about the environment,
- Whether agents must act concurrently or sequentially
- Whether agents can directly communicate with each other.

2 Normal-Form Games

In this section, we introduce normal-form games which is a class of simultaneous single-move games.[1]

2.1 Definition

Definition 1 (Normal-form games). *A strategic game in normal-form is defined by a tuple* $\Gamma := (\mathcal{I}, \{\mathcal{X}_i\}_{i\in\mathcal{I}}, \{u_i\}_{i\in\mathcal{I}})$ *where:*

- $\mathcal{I}$ *is a finite set of players* ($|\mathcal{I}| = n \geq 1$),
- $\mathcal{X}_i$ *is a non-empty set of strategies for each player* $i \in \mathcal{I}$,
- $u_i : \mathcal{X} \to \mathbb{R}$ *is a utility function where* $\mathcal{X} := \prod_{i=1}^N \mathcal{X}_i$.

Notation. We use the standard notation $x = (x_i, x_{-i}) \in \mathcal{X}$ where $x_{-i} := (x_j)_{j\in\mathcal{I}\{i\}}$.

Definition 2. *A game* Γ *is said to be finite if each strategy set* $\mathcal{X}_i$ *is finite.*

Definition 3. *A continuous game is a game* $\Gamma = (\mathcal{I}, \{\mathcal{X}_i\}_{i\in\mathcal{I}}, \{u_i\}_{i\in\mathcal{I}})$ *where:*

- $\mathcal{I}$ *is a finite set of players* ($|\mathcal{I}| = n \geq 1$),
- $\mathcal{X}_i$ *is a a compact convex subset of a finite dimensional space* $\mathbb{R}^{d_i}$ *for each* $i \in \mathcal{I}$,
- $u_i : \mathcal{X} \to \mathbb{R}$ *(where* $\mathcal{X} := \prod_{i=1}^N)\mathcal{X}_i$ *is a continuous function mapping an action profile* $x \in \mathcal{X}$ *to its associated payoff* $u_i(x) \in \mathbb{R}$.

We will mostly focus on finite normal-form games.

Definition 4 (Finite normal-form games). *A finite normal-form game* $\Gamma = (\mathcal{I}, \{A_i\}_{i\in\mathcal{I}}, \{u_i\}_{i\in\mathcal{I}})$ *is defined by:*

- *a finite set of players* $\mathcal{I} = \{1, \cdots, N\}$,
- *A finite set of actions* $\mathcal{A}_i$ *for each* $i \in \mathcal{I}$,
- *A payoff function* $u_i : \mathcal{A} \to \mathbb{R}$ *for each* $i \in \mathcal{I}$ *where* $\mathcal{A} := \prod_{i=1}^N \mathcal{A}_i$.

2.2 Best response

Each player is seeking to optimize their utility function. Therefore, it is natural to introduce the following best response map which encodes the optimal choice for a player conditioned on the choices of the other players.

Definition 5 (Best response). *The best response of player* $i \in \mathcal{I}$ *to the actions* $a_{-i} \in \mathcal{A}_{-i}$ *of the other players is a function* $BR^i : \mathcal{A}_{-i} \to 2^{\mathcal{A}_i}$ *(also called a correspondence and denoted* $BR^i : \mathcal{A}_{-i} \rightrightarrows \mathcal{A}_i$*) where*

$$BR^i(a_{-i}) = \{a_i \in \mathcal{A}_i : u_i(a_i', a_{-i}) \leq u_i(a_i, a_{-i}), \forall a_i' \in \mathcal{A}_i\}.$$

For any $\varepsilon \geq 0$, *we define the approximate best response correspondance* $B_\varepsilon : \mathcal{A}_{-i} \rightrightarrows \mathcal{A}_i$ *where*

$$BR^i_\varepsilon(a_{-i}) = \{a_i \in \mathcal{A}_i : u_i(a_i', a_{-i}) \leq u_i(a_i, a_{-i}) + \varepsilon, \forall a_i' \in \mathcal{A}_i\}.$$

The best response map is set-valued. We provide an example below for concreteness.

[1]Sequential-move games will be discussed later on in the course.

Example 6. Consider the following two-player strategic form game with utility functions:

	L	C	R
T	$(4,3)$	$(1,8)$	$(2,5)$
M	$(2,1)$	$(0,0)$	$(-1,1)$
B	$(4,7)$	$(1,1)$	$(+1,-1)$

Examples of best response values: $B_1(L) = \{T, B\}$ and $B_2(T) = \{C\}$.

3 Nash Equilibria

We are now interested in defining a concept to provide a reasonable description of the collective behavior, as optimality is a priori unclear in a game setting where there are multiple selfish agents with different utility functions. One of the central equilibrium concepts in game theory is the concept of Nash equilibrium. A minimal natural requirement is that no player should be able to improve their payoff (given the behavior of other agents) by switching to a different strategy.

3.1 Definition

A Nash equilibrium of a strategic game is a strategy profile such that no player has a unilateral profitable strategy deviation.

Definition 7 (Nash equilibrium). *Let $\Gamma = (\mathcal{I}, \{\mathcal{A}_i\}_{i\in\mathcal{I}}, \{u_i\}_{i\in\mathcal{I}})$ be a finite normal-form game. An action profile $a^\star \in \mathcal{A} = \prod_{i=1}^N \mathcal{A}_i$ is a Nash Equilibrium if:*

$$\forall i \in \mathcal{I}, \quad \forall a_i \in \mathcal{A}_i, \quad u_i(a_i, a^\star_{-i}) \leq u_i(a^\star_i, a^\star_{-i}), \tag{1}$$

or equivalently:

$$\forall i \in \mathcal{I}, \quad \forall a_i \in \mathcal{A}_i, \quad a^\star_i \in BR^i(a^\star_{-i}). \tag{2}$$

For any $\varepsilon \geq 0$, an ε-Nash equilibrium is an action profile $a^\star \in \mathcal{A}$ s.t.

$$\forall i \in \mathcal{I}, \quad \forall a_i \in \mathcal{A}_i, \quad u_i(a_i, a^\star_{-i}) \leq u_i(a^\star_i, a^\star_{-i}) + \varepsilon, \tag{3}$$

or equivalently:

$$\forall i \in \mathcal{I}, \quad \forall a_i \in \mathcal{A}_i, \quad a^\star_i \in BR^i_\varepsilon(a^\star_{-i}). \tag{4}$$

We provide a few comments regarding this definition and the interpretation of NE:

- Each player $i \in \mathcal{I}$ is playing a best response to the actions of the other players.
- In a Nash equilibrium, no player regrets their action choice (i.e. could have played a better action given the actions of others).
- Only *relative* preferences matter in the definition of NE, i.e. deviations $u_i(a_i, a^\star_{-i}) - u_i(a^\star_i, a^\star_{-i})$ for $a_i \in \mathcal{A}_i$.

Remark 8. The definition extends naturally to continuous games by replacing $\mathcal{A}_i$ by $\mathcal{X}_i$.

A few natural questions arise:

- Do NE (always) exist?
- When they do exist, are they unique in general? If not, are they equal in terms of payoffs?

We provide a few answers by looking at a few classical examples.

3.2 Examples and table representation

Example 9 (Prisoner's dilemma)**.** Two suspects are independently interrogated. They have the choice between confessing to committing the crime (including reporting their partner) or not confessing. If both cooperate, both are sentenced to 1 year jail. If they both defect, they are both sentenced to 3 years. Otherwise, if one cooperates and the other defects, the payoffs are respectively -4 and 0.

	C	D
C	$(-1,-1)$	$(-4,0)$
D	$(0,-4)$	$(-3,-3)$

It can be verified that (D,D) is the unique Nash equilibrium for this game. Note for this that action D strictly dominates action C for both players. The suspects' most desirable collective behavior is (C,C) but it is not stable in the sense of NE.

Example 10 (Bach or Stravinsky)**.** A couple would like to go to a music concert to listen either to Bach or Stravinsky. The couple prefers to go together but the partners have different preferences: one prefers Bach whereas the other prefers Stravinsky. A payoff matrix modeling these preferences is given as follows:

	B	S
B	$(2,1)$	$(0,0)$
S	$(0,0)$	$(1,2)$

Notice that there are two Nash equilibria (B,B) and (S,S).

Example 11 (Matching Pennies)**.** Each player simultaneously chooses either Heads (H) or Tails (T). If the choices match (both choose H or both choose T), Player 1 wins and Player 2 loses. If the choices differ, Player 2 wins and Player 1 loses.

	H	T
H	$(1,-1)$	$(-1,1)$
T	$(-1,1)$	$(1,-1)$

Note that there does not exist any NE! There is always a profitable unilateral strategy deviation.

This example motivates the use of a randomized strategy.

3.3 Mixed extension of normal-form games

The mixed extension of a finite game $\Gamma = (\mathcal{I}, \{\mathcal{A}_i\}_{i\in\mathcal{I}}, \{\sqcap_i\}_{i\in\mathcal{I}})$ is a compact continuous (multilinear) game $\tilde{\Gamma}$ (which we will denote $\Delta(\Gamma)$) defined by the tuple $(\mathcal{I}, \{\mathcal{X}_i\}_{i\in\mathcal{I}}, \{\tilde{u}_i\}_{i\in\mathcal{I}})$ where:

- for all $i \in \mathcal{I}, \mathcal{X}_i = \Delta(\mathcal{A}_i)$ where $\Delta(\mathcal{A}_i) = \{x_i \in \mathbb{R}^{|\mathcal{A}_i|} : x_{i,a_i} \geq 0, \sum_{a_i\in\mathcal{A}_i} x_{i,a_i} = 1\}$ is the set of probability distributions over the finite set $\mathcal{A}_i$ with cardinality $|\mathcal{A}_i|$,
- for all $i \in \mathcal{I}$, the payoff of player i in $\tilde{\Gamma}$ is the multilinear extension of the utility function u_i (in the original game Γ) which we denote by $\tilde{u}_i : \mathcal{X} \to \mathbb{R}$defined for any strategy profile $x \in \mathcal{X} = \prod_{i=1}^{N} \Delta(\mathcal{A}_i)$ by:

$$\tilde{u}_i(x) = \sum_{a=(a_1,\cdots,a_N)\in\mathcal{A}} \left(\prod_{i=1}^{N} x_j(a_j) \right) \cdot u_i(a) = \mathbb{E}_{a_1\sim x_1,\cdots,a_n\sim x_n}[u_i(a_1,...,a_N)] \,. \tag{5}$$

With a slight abuse of notation, we will reuse the notation u_i for $\tilde{u}_i$.

Notation. We will use the notation $u_i(a_1, x_{-i})$ for the payoff of player i when player 1 chooses action a_1 while other players select strategies x_{-i}.

Remark 12 (Mixed vs correlated strategies)**.** Note that a mixed strategy as we define it here is a strategy $x \in \prod_{i=1}^N \Delta(\mathcal{A}_i)$ (product of independent distributions) whereas we can define a correlated strategy as $x \in \Delta(\prod_{i=1}^N \mathcal{A}_i)$ (distribution over the product space). We will come back to this distinction later on in the next lectures.

Example 13 (Rock-paper-scissors)**.** For any $a, b > 0$, define the following game with payoffs:

	R	P	S
R	$(0,0)$	$(a,-b)$	$(-b,a)$
P	$(-b,a)$	$(0,0)$	$(a,-b)$
S	$(a,-b)$	$(-b,a)$	$(0,0)$

For the fully-mixed strategy $x_1 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$ and the strategy $x_2 = (\frac{1}{2}, \frac{1}{2}, 0)$, compute the expected payoff $u_1(x_1, x_2)$.

Definition 14 (Mixed Nash equilibrium)**.** *A mixed Nash equilibrium of* Γ *is a Nash equilibrium of the mixed extension* $\Delta(\Gamma)$.

In the RPS game (example 13) with $a = b = 1$, the only Nash equilibrium is the uniform strategy for all players. In particular, there is no NE in pure strategies (pure NE).

Naturally, the definition of the best response correspondence can also be generalized.

Definition 15 (Best response correspondence)**.** *The best response correspondence* $BR : \mathcal{X} \rightrightarrows \mathcal{X}$ *which maps to any strategy* $x \in \mathcal{X}$ *the subset* $\prod_{i=1}^N BR^i(x_{-i})$ *of* $\mathcal{X}$, *i.e.*

$$BR(x) = BR(x_1, \cdots, x_N) = (BR^1(x_{-1}), \cdots, BR^N(x_{-N})). \tag{6}$$

Using this definition we obtain a fixed-point characterization of NE (mixed or pure):

Proposition 16. *A strategy* $x^\star$ *is a NE if and only if* $x^\star \in BR(x^\star)$.

Remark 17 (Computational complexity)**.** It can be shown that computation of fixed points of continuous functions and computation of NE are computationally equivalent, i.e. each problem can be reduced to the other in polynomial time.

4 Existence of Mixed Nash Equilibria

4.1 Nash's theorem

Theorem 18 (Nash (1951))**.** *Every finite game* Γ *has (at least) a mixed Nash equilibrium.*

4.2 Proof

The original proof (Nash, 1950) is based on Kakutani's fixed point theorem (Kakutani, 1941).

Theorem 19 ((Kakutani, 1941))**.** *Let* C *be a non-empty convex and compact subset of a normed vector space and let* F *be a correspondence from* C *to* C *such that:*

1. *For all* $c \in C$, $F(c)$ *is convex, compact, and non-empty;*
2. *The graph* $\Gamma = \{(c, d) \in C \times C : d \in F(c)\}$ *of* F *is closed.*

Then, $\{c \in C : c \in F(c)\}$ *is non-empty and compact.*

Remark 20. In 1952, Glicksberg (1952) and Fan (1952) independently generalized Kakutani's theorem to any Hausdorff locally convex topological vector space.

A year later, Nash (Nash, 1951) proposed an alternative proof based on Brouwer's fixed point theorem. We will present the latter in the following[2].

Theorem 21 (Brouwer's fixed point theorem)**.** *Let C be a non-empty convex and compact subset of a finite-dimensional Euclidean space. Then any continuous function $f : C \to C$ has a fixed point.*

First, define for any $x \in \prod_{i=1}^{N} \Delta\left(A_i\right)$ and any $a_i \in \mathcal{A}_i$,

$$r_{i,a_i}(x) = u_i(a_i, x_{-i}) - u_i(x)\,. \tag{7}$$

We introduce Nash's map $\varphi : \prod_{i=1}^{N} \Delta\left(A_i\right) \to \prod_{i=1}^{N} \Delta\left(A_i\right)$ defined for every $x \in \prod_{i=1}^{N} \Delta\left(A_i\right)$ by:

$$\varphi_{i,a_i}\left(x\right) := \frac{x_{i,a_i} + [r_{i,a_i}(x)]^+}{1 + \sum_{a_i' \in A_i} \left[r_{i,a_i'}(x)\right]^+}\,, \tag{8}$$

for all player $i \in [n]$ and action $a_i \in \mathcal{A}_i$. We use the notation $[r]^+ := \max\{0, r\}$.

Proposition 22 (Utility improvement under Nash map)**.** *For any strategy profile $x \in \prod_{i=1}^{N} \Delta\left(\mathcal{A}_i\right)$, and any player $i \in [n]$,*

$$u_i\left(\varphi_i(x), x_{-i}\right) - u_i(x_i, x_{-i}) = \frac{\sum_{a_i \in \mathcal{A}_i} \left([r_{i,a_i}(x)]^+\right)^2}{1 + \sum_{a_i \in \mathcal{A}_i} [r_{i,a_i}(x)]^+}. \tag{9}$$

- Useful proof technique.
- Interpretation: if a player has an incentive to unilaterally deviate, then the Nash map unilaterally increases that player's utility.
- Even if a single action is profitable, then the Nash map unilaterally strictly increases for the utility of that player.

Proof. Fix $i \in \mathcal{I}$ and introduce the following shorthand notations for $x \in \mathcal{X}, a_i \in \mathcal{A}_i$,

$$r_{i,a_i} := r_{i,a_i}(x)\,, \quad u_{i,a_i} := u_i\left(a_i, x_{-i}\right)\,, \quad x'_{i,a_i} := \varphi_{i,a_i}\left(x_1, \ldots, x_n\right) = \frac{x_{i,a_i} + r^+_{i,a_i}}{1 + \sum_{a_i' \in A_i} r^+_{i,a_i'}}\,. \tag{10}$$

[2] We follow here the exposition in the lecture notes of G. Farina (Farina, 2024)

The utility improvement under Nash's map is computed as follows:

$$
\begin{aligned}
u_i\left(\varphi_i(x), x_{-i}\right) - u_i(x_i, x_{-i}) &= u_i\left(x_i', x_{-i}\right) - u_i(x_i, x_{-i}) \\
&= \sum_{a_i \in A_i} u_{i,a_i} \cdot \left(x'_{i,a_i} - x_{i,a_i}\right) \\
&= \sum_{a_i \in A_i} u_{i,a_i} \cdot \left(\frac{x_{i,a_i} + r^+_{i,a_i}}{1 + \sum_{a_i' \in A_i} r^+_{i,a_i'}} - x_{i,a_i} \right) \\
&= \sum_{a_i \in A_i} u_{i,a_i} \cdot \frac{r^+_{i,a_i} - \sum_{a_i' \in A_i} r^+_{i,a_i'} \cdot x_{i,a_i}}{1 + \sum_{a_i' \in A_i} r^+_{i,a_i'}} \\
&= \frac{1}{1 + \sum_{a_i' \in A_i} r^+_{i,a_i'}} \left(\sum_{a_i \in A_i} r^+_{i,a_i} \cdot u_{i,a_i} - \sum_{a_i' \in A_i} \left(r^+_{i,a_i'} \sum_{a_i \in A_i} x_{i,a_i} \cdot u_{i,a_i} \right) \right) \\
&= \frac{1}{1 + \sum_{a_i' \in A_i} r^+_{i,a_i'}} \left(\sum_{a_i \in A_i} r^+_{i,a_i} \cdot \left(u_{i,a_i} - \sum_{a_i' \in A_i} u_{i,a_i'} \cdot x_{i,a_i'} \right) \right) \\
&= \frac{1}{1 + \sum_{a_i' \in A_i} r^+_{i,a_i'}} \left(\sum_{a_i \in A_i} r^+_{i,a_i} \cdot r_{i,a_i} \right) \\
&= \frac{\sum_{a_i \in \mathcal{A}_i} \left([r_{i,a_i}(x)]^+ \right)^2}{1 + \sum_{a_i \in \mathcal{A}_i} [r_{i,a_i}(x)]^+} ,
\end{aligned}
\tag{11}
$$

where the last inequality follows from observing that $z^+ \cdot z = (z^+)^2$ for all $z \in \mathbb{R}$. $\square$

Proposition 23. *A strategy profile $x \in \mathcal{X}$ is a Nash equilibrium if and only if it is a fixed point of the Nash improvement function φ.*

Proof. ($\Longrightarrow$) If x is a NE, then by definition for all $i \in [n]$ and $a_i \in \mathcal{A}_i$, we have $r_{i,a_i}(x) \leq 0$, hence $[r_{i,a_i}(x)]^+ = 0$. Therefore for all $i \in \mathcal{I}$ and all $a_i \in \mathcal{A}_i$,

$$
\varphi_{i,a_i}(x) = \frac{x_{i,a_i} + [r_{i,a_i}(x)]^+}{1 + \sum_{a_i' \in A_i} \left[r_{i,a_i'}(x) \right]^+} = x_{i,a_i} . \tag{12}
$$

($\Longleftarrow$) Conversely, suppose that x is a fixed point of φ, i.e. $x = \varphi(x)$. Then, for all $i \in \mathcal{I}$, it follows from Proposition 22 that

$$
\frac{\sum_{a_i \in A_i} \left([r_{i,a_i}(x)]^+ \right)^2}{1 + \sum_{a_i \in A_i} [r_{i,a_i}(x)]^+} = u_i\left(\varphi_i(x), x_{-i}\right) - u_i(x) = u_i\left(x_i, x_{-i}\right) - u_i(x) = 0. \tag{13}
$$

Hence for all $i \in \mathcal{I}, a_i \in \mathcal{A}_i$, $[r_{i,a_i}(x)]^+ = 0$ and hence $u_i(a_i, x_{-i}) - u_i(x) \leq 0$. As a consequence, the strategy x is a NE. $\square$

Using Brouwer's fixed-point theorem (with the continuous map ϕ defined over the nonempty convex compact set $\prod_{i=1}^N \Delta(\mathcal{A}_i)$) together with Proposition 23 concludes the proof of Nash's theorem, i.e. guarantees the existence of NE.

5 Game classes

Each agent has their own utility. The preferences of agents might be aligned or misaligned. There is a large spectrum of possible interactions, from full cooperation to full competition. We discuss too fundamental classes of games in the following.

5.1 Two Player Zero-Sum Games

We start by defining one the most common and studied game classes which correspond to the fully competitive setting.

Consider a 2-player normal-form game where n, m are the number of available strategies for both players, respectively. When $u_1 = -u_2$, the game is said to be zero-sum. In this case, it is enough to describe the payoffs of the game using the payoff function of the first player (or minus the second player). The utility function can then be encoded using a single matrix M where $M_{i,j}$ (for $i \in [n], j \in [m]$) is the payoff of player 1 (the row player) when player 1 chooses action $i \in [n]$ and player 2 selects action $j \in [m]$.

The expected payoff of player 1 (row player) when player 1 chooses a mixed strategy $x \in \Delta_n$ and player 2 (column player) chooses a mixed strategy $y \in \Delta_m$ is given by:

$$u_1(x, y) = \sum_{a_i \in \mathcal{A}_1, a'_j \in \mathcal{A}_2} x_{a_i} y_{a'_j} u_1(a_i, a'_j) = x^T M y \,. \tag{14}$$

5.2 Potential Games

We now introduce the celebrated class of potential games which

5.2.1 Identical-interest games

All players share a common utility function, i.e.

$$\forall i, j \in \mathcal{I}, \quad \forall a \in \mathcal{A}, \quad u_i(a) = u_j(a) \,. \tag{15}$$

Example 24 (Identical interest game)**.** Consider the following 2-player identical interest game:

	b_1	b_2	b_3
a_1	$(0,0)$	$(1,1)$	$(3,3)$
a_2	$(4,4)$	$(8,8)$	$(-2,-2)$
a_3	$(6,6)$	$(7,7)$	$(2,2)$

This game has two pure NE: (a_1, b_3), (a_2, b_2) with payoffs 3 and 8 respectively.

- NE are not unique.
- NE have different payoffs, they are not equally efficient.
- Observe that (a_2, b_2) is the strategy maximizing the common payoff over $\mathcal{A}$.

Proposition 25. *In any identical interest game, if $a^\star \in \operatorname{argmax}_{a \in \mathcal{A}} u(a)$ (where u is the common payoff) then $a^\star$ is a pure NE.*

Proof. Since $a^\star \in \operatorname{argmax}_{a \in \mathcal{A}} u(a)$, we have for all $i \in \mathcal{I}, a_i \in \mathcal{A}_i$,

$$u_i(a_i, a^\star_{-i}) = u(a_i, a^\star_{-i}) \leq u(a^\star) = u_i(a^\star_i, a^\star_{-i}) \,. \tag{16}$$

Therefore $a^\star$ is a NE by definition. □

5.2.2 From identical-interest to potential games

Consider the following modified version of the identical interest game introduced in the previous section.[3]

	y_1	y_2	y_3
x_1	$(1+a, 1+A)$	$(2+b, 2+A)$	$(4+c, 4+A)$
x_2	$(5+a, 5+B)$	$(9+b, 9+B)$	$(-1+c, -1+B)$
x_3	$(7+a, 7+C)$	$(8+b, 8+C)$	$(3+c, 3+C)$

Figure 4: An identical interest game with shifted payoffs.

Observe that this game is strategically equivalent to the previous identical interest game, i.e. both games have the exact same NE (e.g. relative preferences of players between their actions remain unchanged even if payoffs might be different).

5.2.3 Definition of potential games

Definition 26 (Potential game)**.** *A game $\Gamma = (\mathcal{I}, \{\mathcal{A}_i\}_{i\in\mathcal{I}}, \{\sqcap_i\}_{i\in\mathcal{I}})$ is said to be potential if there exists a function $\phi : \mathcal{A} \to \mathbb{R}$ such that for every action profile $a \in \mathcal{A}$, agent $i \in N$, and alternative action choice $a_i' \in \mathcal{A}_i$,*

$$u_i(a_i', a_{-i}) - u_i(a_i, a_{-i}) = \phi(a_i', a_{-i}) - \phi(a_i, a_{-i}). \tag{17}$$

We provide a few comments regarding this definition:

- The potential game is not unique in general (just add a constant, only deviations matter).
- In a potential game, it is as if players are playing in a (hidden) identical interest game with a common payoff ϕ (not necessarily known to the players). The relative preferences of any player are the same as they would be if their utility was simply ϕ.
- Any profitable deviation for an agent increases the potential function. This draws an implicit link with optimization.

Remark 27. There are variants of potential games in the literature (e.g. weighted, ordinal).

Remark 28. Important examples of potential games are congestion games (Rosenthal, 1973). In fact, Monderer and Shapley (1996) have shown that potential games and congestion games are isomorphic.

5.2.4 Existence of pure Nash equilibria

Proposition 29. *Every potential game admits a pure NE.*

Proof. Let $a^\star \in \operatorname{argmax}_{a\in\mathcal{A}} \phi(a)$. Then for all $i \in \mathcal{I}, a_i \in \mathcal{A}_i$,

$$u_i(a_i, a_{-i}^\star) - u_i(a_i^\star, a_{-i}^\star) = \phi(a_i, a_{-i}^\star) - \phi(a_i^\star, a_{-i}^\star) \le 0\,. \tag{18}$$

Hence $a^\star$ is a pure NE. □

5.3 Decomposition of games

The interested reader can refer to the work of Candogan et al. (2011) which establishes a canonical direct sum decomposition of an arbitrary game into three components: the potential, harmonic, and nonstrategic.

[3] The exposition and the example in this section are inspired from the lecture notes of Marden (2020).

6 Next lecture

In the next lecture, we will focus on how to learn NE under repeated interactions in the zero-sum and potential games within the online learning setting.

References

Ozan Candogan, Ishai Menache, Asuman Ozdaglar, and Pablo A Parrilo. Flows and decompositions of games: Harmonic and potential games. *Mathematics of Operations Research*, 36(3): 474–503, 2011.

Ky Fan. Fixed-point and minimax theorems in locally convex topological linear spaces. *Proceedings of the National Academy of Sciences*, 38(2):121–126, 1952.

Gabriele Farina. Topics in multi-agent learning (mit 6.s890), lecture 2. https://www.mit.edu/~gfarina/2024/6S890f24_L02_nfg_nash/, 2024.

Irving L Glicksberg. A further generalization of the kakutani fixed point theorem, with application to nash equilibrium points. *Proceedings of the American Mathematical Society*, 3(1): 170–174, 1952.

Shizuo Kakutani. A generalization of Brouwer's fixed point theorem. *Duke Mathematical Journal*, 8(3):457 – 459, 1941. doi: 10.1215/S0012-7094-41-00838-4. URL https://doi.org/10.1215/S0012-7094-41-00838-4.

Rida Laraki, Jérôme Renault, and Sylvain Sorin. *Mathematical foundations of game theory*, volume 202. Springer, 2019.

Jason R. Marden. Game theory and multiagent systems (ece 149), lectures 11, 12, 14. https://web.ece.ucsb.edu/~jrmarden/ece149.html, 2020.

Panayotis Mertikopoulos. *Online optimization and learning in games: Theory and applications*. PhD thesis, Grenoble 1 UGA-Université Grenoble Alpes, 2019.

Dov Monderer and Lloyd S Shapley. Potential games. *Games and economic behavior*, 14(1): 124–143, 1996.

John F. Nash. Equilibrium points in n-person games. *Proceedings of the National Academy of Sciences*, 36(1):48–49, 1950. doi: 10.1073/pnas.36.1.48. URL https://www.pnas.org/doi/abs/10.1073/pnas.36.1.48.

John F. Nash. Non-cooperative games. *Annals of Mathematics*, 54(2):286–295, 1951. URL http://www.jstor.org/stable/1969529.

Robert W Rosenthal. A class of games possessing pure-strategy nash equilibria. *International Journal of Game Theory*, 2(1):65–67, 1973.