自由电子激光物理学习之五

Discussion of the High Gain FEL Equations

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Outline (I)

- Third-order equation
  - Review of the coupled first-order equations
  - Non-periodic case
  - With/without energy detuning/space charge force
- Gain function of FEL from third-order equation
  - Low gain limit of the high-gain FEL theory
  - Gain function
- FEL bandwidth
- FEL startup by a periodically modulated e-beam

Outline (2)

- Laser saturation
  - Analytical estimation
  - FLASH example
- Microbunching simulation
  - FLASH example
  - Electric and density phase evolution

The coupled first-order Equations

We already have the coupled first-order equations in the periodic model

$$
\begin{cases}
\dfrac{d\psi_n}{dz} = 2k_u\eta_n, n = 1,2, \ldots N \\[2ex]
\dfrac{d\eta_n}{dz} = -\dfrac{e}{m_e c^2 \gamma_r} \mathcal{R}e\left(\left(\dfrac{\hat{K}\tilde{E}_x}{2\gamma_r} - \dfrac{i\mu_0 c^2}{\omega_l} \cdot \tilde{j}_1\right) e^{i\psi_n}\right) \\[2ex]
\dfrac{d\tilde{E}_x}{dz} = -\dfrac{\mu_0 c\hat{K}}{4\gamma_r} \cdot \tilde{j}_1 \\[2ex]
\tilde{j}_1 = j_0 \dfrac{2}{N} \displaystyle\sum_{n=1}^{N} e^{-i\psi_n}
\end{cases}
$$

$$
\begin{cases}
\eta_n = \dfrac{\gamma_n - \gamma_r}{\gamma_r} \\
uniform\ or\ peridic\ model
\end{cases}
$$

Non-periodic First-Order Equations (I)

- Beam current $\tilde{j}_1$ not only depends on z, but the internal bunch coordinate ξ
- The FEL field also depends on the internal longitudinal coordinate u

$$\begin{cases} \tilde{E}_x(z,t) = \hat{E}_x(z,u)e^{i(k_l z - \omega_l t)} \\ \tilde{j}_z(z,t) = j_0(\xi) + \hat{j}_1(z,\xi)e^{i((k_l+k_u)z-\omega_l t)} \end{cases}$$

$$\begin{cases} \xi = \dfrac{\psi}{k_l + k_u} = z - \dfrac{k_l c}{k_l + k_u}t = z - \dfrac{\lambda_u}{\lambda_u + \lambda_l}ct = z - \bar{\beta}ct \\ u = z - ct = \left(1 - \dfrac{1}{\bar{\beta}}\right)z + \dfrac{1}{\bar{\beta}}\xi \end{cases}$$

Non-periodic First-Order Equations (2)

$$\left(\frac{\partial^2}{\partial^2 z} - \frac{1}{c^2}\cdot\frac{\partial^2}{\partial t^2}\right)\tilde{E}_x(z,t) = \mu_0 \frac{\partial \tilde{j}_x}{\partial t}$$

$$\tilde{E}_x(z,t) = \hat{E}_x(z,u)e^{i(k_l z-\omega_l t)}$$

$$\frac{\partial \tilde{E}_x}{\partial z} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial u}\cdot\frac{\partial u}{\partial z}\right)\hat{E}e^{i(k_l z-\omega_l t)} + ik_l\hat{E}e^{i(k_l z-\omega_l t)}$$

$$\begin{aligned}\frac{\partial^2 \tilde{E}_x}{\partial z^2} &= \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial u}\right)\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial u}\right)\hat{E}e^{i(k_l z-\omega_l t)} + 2ik_l\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial u}\right)\hat{E}e^{i(k_l z-\omega_l t)} - {k_l}^2\hat{E}e^{i(k_l z-\omega_l t)} \\ &= \left(\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial u}\right)^2 + 2ik_l\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial u}\right) - {k_l}^2\right)\hat{E}e^{i(k_l z-\omega_l t)}\end{aligned}$$

$$\frac{\partial \tilde{E}_x}{\partial t} = -c\frac{\partial}{\partial u}\hat{E}e^{i(k_l z-\omega_l t)} - i\omega_l\hat{E}e^{i(k_l z-\omega_l t)}$$

$$\begin{aligned}\frac{\partial^2 \tilde{E}_x}{\partial t^2} &= c^2\frac{\partial^2}{\partial u^2}\hat{E}e^{i(k_l z-\omega_l t)} + i\omega_l c\frac{\partial}{\partial u}\hat{E}e^{i(k_l z-\omega_l t)} - i\omega_l\left(-c\frac{\partial}{\partial u}\right)\hat{E}e^{i(k_l z-\omega_l t)} - {\omega_l}^2\hat{E}e^{i(k_l z-\omega_l t)} \\ &= c^2\left(\frac{\partial^2}{\partial u^2} + 2ik_l\frac{\partial}{\partial u} - {k_l}^2\right)\hat{E}e^{i(k_l z-\omega_l t)}\end{aligned}$$

Non-periodic First-Order Equations (3)

$$\left(\frac{\partial^2}{\partial^2 z} - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2}\right) \tilde{E}_x(z,t) = \mu_0 \frac{\partial \tilde{j}_x}{\partial t}$$

$$\tilde{E}_x(z,t) = \hat{E}_x(z,u) e^{i(k_l z - \omega_l t)}$$

$$\frac{\partial^2 \tilde{E}_x}{\partial z^2} = \left(\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial u}\right)^2 + 2ik_l\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial u}\right) - {k_l}^2\right) \hat{E} e^{i(k_l z - \omega_l t)}$$

$$\left|\frac{\partial \tilde{E}_x}{\partial z}\right| \ll k_l \left|\tilde{E}_x\right| \Rightarrow \left|\frac{\partial^2 \tilde{E}_x}{\partial z^2}\right| \ll k_l \left|\frac{\partial \tilde{E}_x}{\partial z}\right|, \left|\frac{\partial^2 \tilde{E}_x}{\partial z \partial u}\right| \ll k_l \left|\frac{\partial \tilde{E}_x}{\partial z}\right|$$

$$\frac{\partial^2 \tilde{E}_x}{\partial z^2} = \left(\frac{\partial^2}{\partial u^2} + 2i\, k_l\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial u}\right) - {k_l}^2\right) \hat{E} e^{i(k_l z - \omega_l t)}$$

$$\frac{\partial^2 \tilde{E}_x}{c^2 \partial t^2} = \left(\frac{\partial^2}{\partial u^2} + 2ik_l \frac{\partial}{\partial u} - {k_l}^2\right) \hat{E} e^{i(k_l z - \omega_l t)}$$

$$\Rightarrow \left(\frac{\partial^2}{\partial^2 z} - \frac{1}{c^2} \cdot \frac{\partial^2}{\partial t^2}\right) \tilde{E}_x(z,t) = 2ik_l \frac{\partial \hat{E}_x(z,u)}{\partial z} e^{i(k_l z - \omega_l t)}$$

Non-periodic First-Order Equations (4)

$$\tilde{j}_z(z,t) = j_0(\xi) + \hat{j}_1(z,\xi)e^{i((k_l+k_u)z-\omega_l t)}$$

$$j_{x,z} = \rho v_{x,z}$$

$$\beta_x = \frac{K}{\gamma}\cos(k_u z)$$

$$\tilde{j}_x = \frac{v_x}{v_z}\tilde{j}_z \simeq \frac{K}{\gamma}\cos(k_u z)\left(\cancel{j_0(\xi)} + \hat{j}_1(z,\xi)e^{i((k_l+k_u)z-\omega_l t)}\right)$$

$$\frac{\partial \tilde{j}_x}{\partial t} = \left(\frac{\partial \hat{j}_1}{\partial \xi}\frac{\partial \xi}{\partial t} - i\omega_l \hat{j}_1\right)\frac{K}{\gamma}e^{i((k_l+k_u)z-\omega_l t)}\cos(k_u z) = -\left(\bar{v_z}\frac{\partial \hat{j}_1}{\partial \xi} + i\omega_l \hat{j}_1\right)\frac{K}{\gamma}e^{i((k_l+k_u)z-\omega_l t)}\cos(k_u z)$$

$$\left|\frac{\partial \hat{j}_1}{\partial \xi}\right| \ll k_l|\hat{j}_1| \Rightarrow \bar{v_z}\left|\frac{\partial \hat{j}_1}{\partial \xi}\right| \ll \omega_l|\hat{j}_1|$$

$$\Rightarrow \frac{\partial \tilde{j}_x}{\partial t} = -i\omega_l\frac{K}{\gamma}\hat{j}_1 e^{i((k_l+k_u)z-\omega_l t)}\cos(k_u z)$$

Non-periodic First-Order Equations (5)

$$\left(\frac{\partial^2}{\partial^2 z} - \frac{1}{c^2}\cdot\frac{\partial^2}{\partial t^2}\right)\tilde{E}_x(z,t) = \mu_0 \frac{\partial \tilde{j}_x}{\partial t}$$

$$\begin{cases} \dfrac{\partial \tilde{j}_x}{\partial t} = -i\omega_l \dfrac{K}{\gamma}\hat{j}_1 e^{i((k_l+k_u)z-\omega_l t)}\cos(k_u z) \\ \left(\dfrac{\partial^2}{\partial^2 z} - \dfrac{1}{c^2}\cdot\dfrac{\partial^2}{\partial t^2}\right)\tilde{E}_x(z,t) = 2ik_l \dfrac{\partial \hat{E}_x(z,u)}{\partial z} e^{i(k_l z-\omega_l t)} \end{cases}$$

$$\Rightarrow 2ik_l \frac{\partial \hat{E}_x(z,u)}{\partial z} e^{i(k_l z-\omega_l t)} = -i\omega_l \mu_0 \frac{K}{\gamma}\hat{j}_1 e^{i((k_l+k_u)z-\omega_l t)}\cos(k_u z)$$

$$\Rightarrow \frac{\partial \hat{E}_x(z,u)}{\partial z} = \frac{-\mu_0 c K}{2\gamma}\hat{j}_1(z,\xi) e^{ik_u z}\cos(k_u z)$$

$$\left|\left\langle e^{ik_u z}\cos(k_u z)\right\rangle_{\lambda_u}\right| = \frac{1}{2}$$

$$\Rightarrow \frac{\partial \hat{E}_x(z,u)}{\partial z} = \frac{-\mu_0 c\widehat{K}}{4\gamma}\hat{j}_1(z,\xi)$$

$$\frac{d\tilde{E}_x}{dz} = -\frac{\mu_0 c\widehat{K}}{4\gamma_r}\cdot\tilde{j}_1(z)$$

Non-periodic First-Order Equations (6)

$$j_z = \frac{-e n_l(\bar{v_z} t)}{t A_b} = -\frac{e\bar{v_z}}{A_b}\sum_{n=1}^{N_t}\delta(\xi - \xi_n(z))$$

Where n_l is the line density of electron, A_b is the cross area of electron bunch, the total charge of bunch is $Q = -N_t e$

In most FEL codes locally periodic conditions are assumed. The bunch is subdivided into slices of length λ_ℓ which are similar the FEL buckets. Within each slice periodic conditions are assumed. The local amplitude of the first harmonic is written as

$$\widehat{j}_1(z, c_m) \approx j_0(c_m)\,\frac{2}{N_m}\sum_{n\in I_m}\exp(-i\,k_\ell \zeta_n) \tag{C.6}$$

where $c_m = m\,\lambda_\ell$ is the center of slice m, N_m the number of particles in that slice, and I_m the index range.

Non-periodic First-Order Equations (7)

$$
\begin{cases}
\dfrac{d\psi_n}{dz} = 2k_u\eta_n, n = 1,2, \dots N \\
\dfrac{d\eta_n}{dz} = -\dfrac{e}{m_e c^2 \gamma_r} \mathcal{R}e\left(\left(\dfrac{\hat{K}\,\tilde{E}_x(z,u_n)}{2\gamma_r} - \dfrac{i\mu_0 c^2}{\omega_l}\cdot \hat{j}_1(z,\xi_n)\right)e^{i\psi_n}\right) \\
\dfrac{d\,\tilde{E}_x(z,u)}{dz} = -\dfrac{\mu_0 c\hat{K}}{4\gamma_r}\cdot \hat{j}_1(z,\xi) \\
\hat{j}_1(z,c_m) = j_0(c_m)\dfrac{2}{N_m}\sum_{n\in I_m} e^{-ik_l\xi_n}
\end{cases}
$$

$$
\begin{cases}
\eta_n = \dfrac{\gamma_n - \gamma_r}{\gamma_r} \\
\xi_n = \dfrac{(\psi_n + \pi/2)}{2\pi}\lambda_l \\
u_n = \left(1 - \dfrac{1}{\bar{\beta}}\right)z + \dfrac{1}{\bar{\beta}}\xi_n \simeq \xi_n - (1-\bar{\beta})z
\end{cases}
$$

$$
\begin{cases}
\tilde{E}_x(z) \rightarrow \hat{E}_x(z,u) \\
\tilde{j}_1(z) \rightarrow \hat{j}_1(z,\xi)
\end{cases}
$$

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Third-Order Equation (An Overview)

Assumptions:

- Small periodic density modulation $\Longrightarrow$ Integro-Differential Equation (4.44)

$$\boxed{\frac{d\tilde{E}_x}{dz} = i\,k_u \frac{\mu_0 \widehat{K} n_e e^2}{2 m_e \gamma_r^2} \int_0^z \left[\frac{\widehat{K}}{2\gamma_r} \tilde{E}_x + i \frac{4\gamma_r c}{\omega_\ell \widehat{K}} \frac{d\tilde{E}_x}{dz} \right] h(z-s) ds}$$

$$\text{with} \quad h(z-s) = \int_{-\delta}^{\delta} (z-s) \exp\left[-i\,2k_u \eta \cdot (z-s)\right] F_0(\eta) d\eta \,.$$

- Mono-energetic beam energy $\Longrightarrow$ Third-Order Equation (4.50)

$$\frac{\tilde{E}_x'''}{\Gamma^3} + 2i \frac{\eta}{\rho_{\mathrm{FEL}}} \frac{\tilde{E}_x''}{\Gamma^2} + \left(\frac{k_p^2}{\Gamma^2} - \left(\frac{\eta}{\rho_{\mathrm{FEL}}} \right)^2 \right) \frac{\tilde{E}_x'}{\Gamma} - i\,\tilde{E}_x = 0$$

  - Beam energy equals to resonant energy, i.e. $\eta = 0$, and without space charge force, i.e. $k_p = 0 \Longrightarrow$ simplest third-order equation
  - $\eta \neq 0$, and/or $k_p \neq 0 \Longrightarrow$ more complicated cases

Third-Order Equation ($\eta \neq 0, k_p \neq 0$)

$$\frac{\tilde{E}_x{}'''}{\Gamma^3} + 2i\frac{\eta}{\rho_{\mathrm{FEL}}}\frac{\tilde{E}_x{}''}{\Gamma^2} + \left(\frac{k_p{}^2}{\Gamma^2} - \left(\frac{\eta}{\rho_{\mathrm{FEL}}}\right)^2\right)\frac{\tilde{E}_x{}'}{\Gamma} - i\tilde{E}_x = 0$$

$$\begin{cases} a \equiv \dfrac{\alpha}{\Gamma} \\ b \equiv \dfrac{\eta}{\rho_{\mathrm{FEL}}} \\ c \equiv \dfrac{k_p}{\Gamma} \end{cases}$$

$$\begin{cases} \Gamma = \left(\dfrac{\mu_0 \hat{K}^2 e^2 k_u n_e}{4\gamma_r{}^3 m_e}\right)^{\frac{1}{3}} \\ \rho_{\mathrm{FEL}} = \dfrac{\Gamma}{2k_u} \end{cases}$$

$$\begin{cases} k_p = \sqrt{\dfrac{2k_u \mu_0 n_e e^2 c}{\gamma_r m_e \omega_l}} = \sqrt{\dfrac{2\lambda_l}{\lambda_u}} \cdot \dfrac{\omega_p^*}{c} \\ \omega_p^* = \sqrt{\dfrac{n_e^* e^2}{\epsilon_0 m_e}} = \sqrt{\dfrac{n_e e^2}{\gamma_r \epsilon_0 m_e}} \end{cases}$$

$$\xrightarrow{\tilde{E}_x \propto e^{\alpha z}} \boxed{a^3 + 2iba^2 + (c^2 - b^2)a - i = 0} \Longrightarrow \begin{cases} a_1 \\ a_2 \\ a_3 \end{cases}$$

$$\boxed{\Longrightarrow \tilde{E}_x(\eta, z) = \sum_{j=1}^{3} c_j(\eta) e^{\alpha_j(\eta) z}}$$

Third-Order Equation (Gain function)

$$\tilde{E}_x(\eta, z) = \sum_{j=1}^{3} c_j(\eta) e^{\alpha_j(\eta) z}$$

$$\begin{pmatrix} \tilde{E}_x(0) \\ \tilde{E}'_x(0) \\ \tilde{E}_x''(0) \end{pmatrix} = A \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \Rightarrow \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} \tilde{E}_x(0) \\ \tilde{E}'_x(0) \\ \tilde{E}_x''(0) \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ {\alpha_1}^2 & {\alpha_2}^2 & {\alpha_3}^2 \end{pmatrix}$$

Define gain function:

$$G(\eta, z) = \left| \frac{\tilde{E}_x(\eta, z)}{E_{\text{in}}} \right|^2 - 1$$

Third-Order Equation (Gain function)

Madey Theorem:

$$G(\xi) = -\frac{\pi e^2 \widehat{K}^2 {N_u}^3 {\lambda_u}^2 n_e}{4\epsilon_0 m_e c^2 {\gamma_r}^3} \cdot \frac{d}{d\xi}\left(\frac{\sin^2\xi}{\xi^2}\right)$$

$$\xi = \pi N_u \frac{(\omega_1 - \omega)}{\omega_1} = \pi N_u \frac{\gamma^2 - {\gamma_r}^2}{{\gamma_t}^2} = 2\pi N_u \eta, \left(\eta = \frac{\gamma - \gamma_r}{\gamma_r}\right)$$

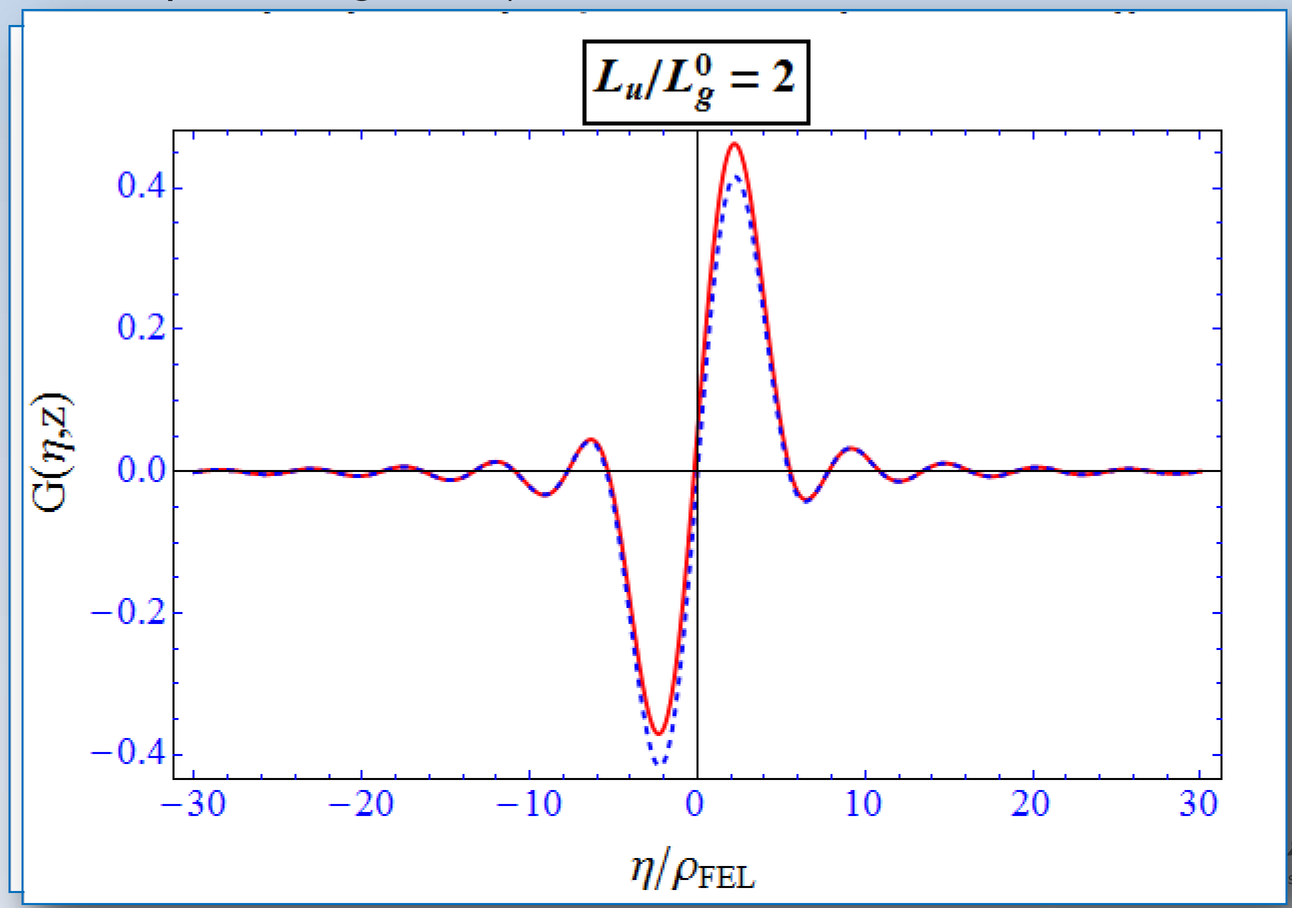
$$\because -\frac{\pi e^2 \widehat{K}^2 {N_u}^3 {\lambda_u}^2 n_e}{4\epsilon_0 m_e c^2 {\gamma_r}^3} = -\frac{\pi e^2 \widehat{K}^2 (N_u \lambda_u)^3 k_u n_e}{2\pi 4\epsilon_0 m_e c^2 {\gamma_r}^3} = -\frac{\mu_0 \widehat{K}^2 e^2 k_u n_e}{4{\gamma_r}^3 m_e} \frac{{L_u}^3}{2} = -\Gamma^3 \frac{{L_u}^3}{2}$$

$$2\pi N_u \eta = 2\pi N_u b \rho_{\mathrm{FEL}} = 2\pi N_u \frac{\Gamma \lambda_u}{4\pi} \cdot b = \frac{\Gamma L_u}{2} \cdot b = \frac{\Gamma L_{g0} \mathrm{Ng}}{2} \cdot b = \frac{1}{2\sqrt{3}} \cdot \mathrm{Ng} \cdot b$$

$$\Rightarrow G(b) = -\frac{\mathrm{Ng}^3}{6\sqrt{3}} \left(\frac{\sin\left(\frac{\mathrm{Ng}}{\sqrt{3}} b\right)}{\left(\frac{\mathrm{Ng}}{2\sqrt{3}} b\right)^2} - \frac{2\left(\sin\left(\frac{\mathrm{Ng}}{2\sqrt{3}} b\right)\right)^2}{\left(\frac{\mathrm{Ng}}{2\sqrt{3}} b\right)^3} \right)$$

Third-Order Equation (Gain function)

Gain function V.S. energy detuning @ undulator length of one 1D power gain length. **Blue** dashed line: calculated form Madey Theorem, **Red** line calculated from third-order equation (without space charge force).



Low gain limit of the high-gain FEL theory (I)

The Madey Theorem at the low gain FEL can be obtained from the low gain limit of the high gain FEL theory as follows:

$$\boxed{\frac{d\tilde{E}_x}{dz} = i\,k_u \frac{\mu_0 \widehat{K} n_e e^2}{2 m_e \gamma_r^2} \int_0^z \left[\frac{\widehat{K}}{2\gamma_r} \tilde{E}_x + i \frac{4\gamma_r c}{\omega_\ell \widehat{K}} \frac{d\tilde{E}_x}{dz} \right] h(z-s) ds}$$

$$\text{with} \quad h(z-s) = \int_{-\delta}^{\delta} (z-s) \exp\left[-i\,2k_u \eta \cdot (z-s)\right] F_0(\eta) d\eta \,.$$

With $\eta = \eta_0, \tilde{E}_x \simeq E_0$

$$\frac{d\tilde{E}_x}{dz} = ik_u \frac{\mu_0 \widehat{K}^2 n_e e^2 E_0}{4 m_e {\gamma_r}^3} \int_0^z (z-s) e^{-i2k_u \eta (z-s)}\, ds$$

Remember $\Gamma = \left(\frac{\mu_0 \widehat{K}^2 e^2 k_u n_e}{4{\gamma_r}^3 m_e}\right)^{1/3} \Longrightarrow \frac{d\tilde{E}_x}{dz} = i\Gamma^3 E_0 \int_0^z (z-s) e^{-i2k_u \eta (z-s)}\, ds$

Low gain limit of the high-gain FEL theory (2)

$$I(z)=\int_0^z (z-s)e^{-i2k_u\eta(z-s)}\,ds=\frac{e^{-2i\eta z k_u}(1+2i\eta z k_u)-1}{(2\eta k_u)^2}$$

$$\frac{d\tilde{E}_x}{dz}=i\Gamma^3 E_0 I(z)$$

$$\Longrightarrow \tilde{E}_x(z)=E_0\left(1+i\Gamma^3\int_0^z I(z)\,dz\right)=E_0(1+A(z))$$

$$A(z)=\frac{-i\Gamma^3}{(2\eta k_u)^2}\left(\left(z+\frac{i}{k_u\eta}\right)+\left(z-\frac{i}{k_u\eta}\right)e^{-i2k_u\eta z}\right)$$

$$\Longrightarrow \left|\frac{\tilde{E}_x(L_u)}{E_0}\right|^2=(1+\mathcal{R}e(A(z)))^2+\mathcal{I}m(A(z))^2$$

$$\xrightarrow{\text{small } z}\quad G(z)\simeq 2\mathcal{R}e(A(z))=\frac{2\Gamma^3}{(2\eta k_u)^2}\left(\frac{1}{k_u\eta}-z\sin(2k_u\eta z)-\frac{\cos(2k_u\eta z)}{k_u\eta}\right)$$

Low gain limit of the high-gain FEL theory (3)

$$L_u = \mathrm{Ng} \cdot L_g^0$$

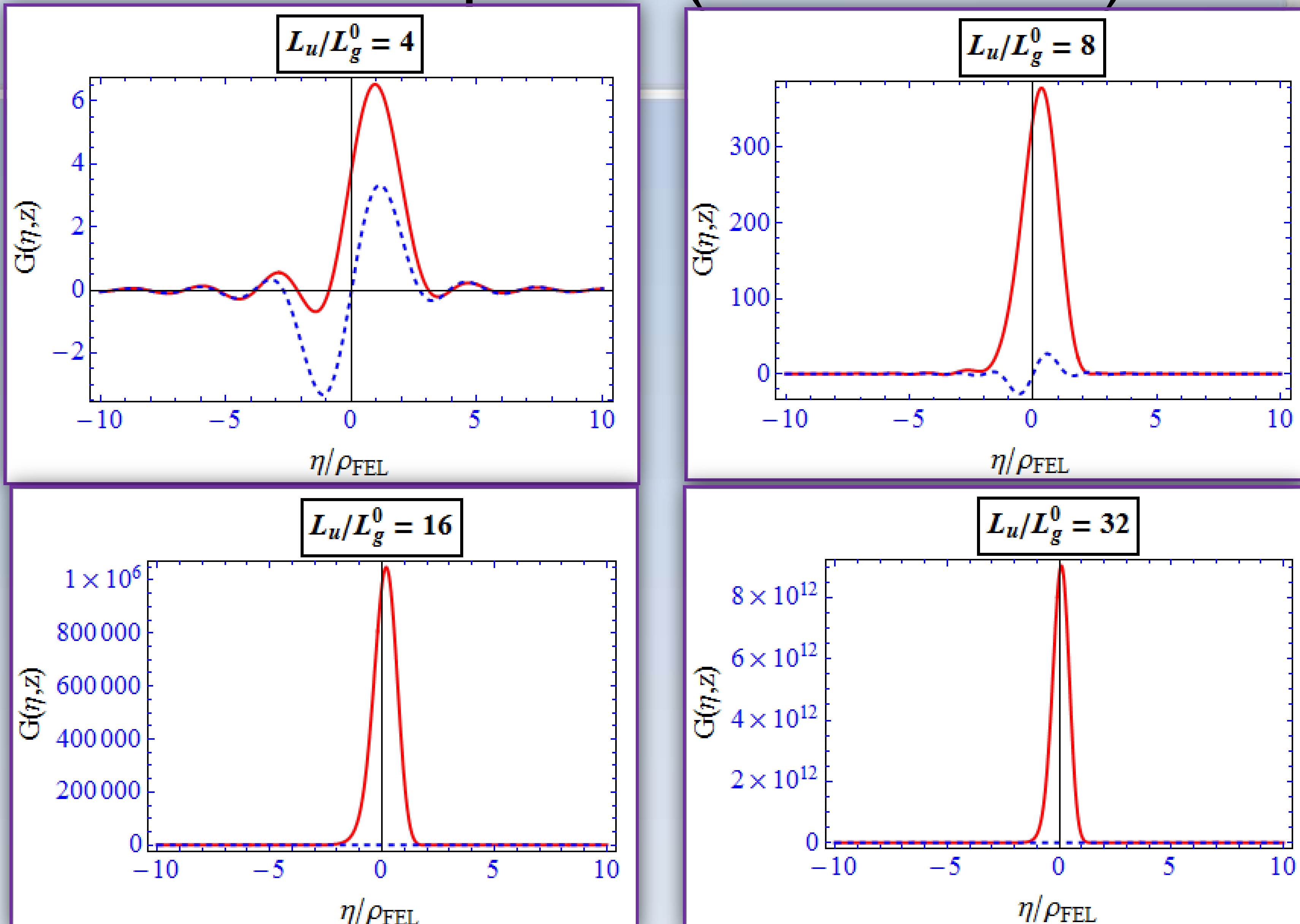
$$2\theta \equiv 2k_u\eta L_u = 2k_u\eta \mathrm{Ng}\, L_g^0 = \mathrm{Ng}\, b\, \rho 2k_u L_g^0 = \mathrm{Ng}\, b\, \Gamma L_g^0 = \frac{\mathrm{Ng}\, b}{\sqrt{3}}$$

$$G(L_u) = \frac{2\Gamma^3}{(2\eta k_u)^2}\left(\frac{1}{k_u\eta} - L_u \sin(2k_u\eta L_u) - \frac{\cos(2k_u\eta L_u)}{k_u\eta}\right)$$
$$= -\frac{\Gamma^3 L_u{}^3}{2}\left(\frac{\sin 2\theta}{\theta^2} - \frac{2\sin^2\theta}{\theta^3}\right) = -\frac{\Gamma^3 L_u{}^3}{2}\frac{d}{d\theta}\left(\frac{\sin^2\theta}{\theta^2}\right)$$

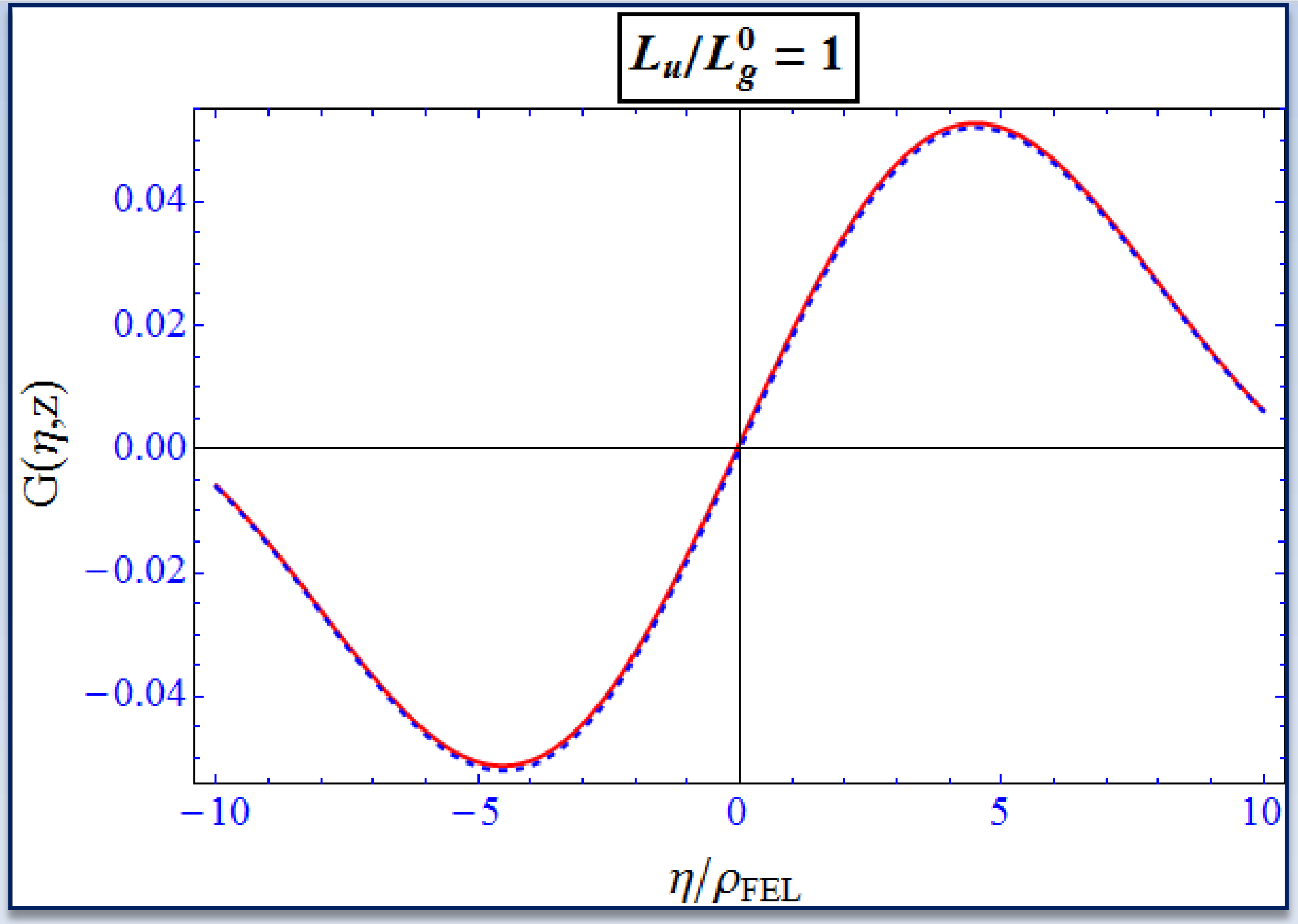
$$or:\ G(\mathrm{Ng}) = -\frac{\mathrm{Ng}^3}{6\sqrt{3}}\left(\frac{\sin\left(\frac{\mathrm{Ng}}{\sqrt{3}}b\right)}{\left(\frac{\mathrm{Ng}}{2\sqrt{3}}b\right)^2} - \frac{2\left(\sin\left(\frac{\mathrm{Ng}}{2\sqrt{3}}b\right)\right)^2}{\left(\frac{\mathrm{Ng}}{2\sqrt{3}}b\right)^3}\right)$$

Which is the same as the simplified gain function from the original Madey Theorem

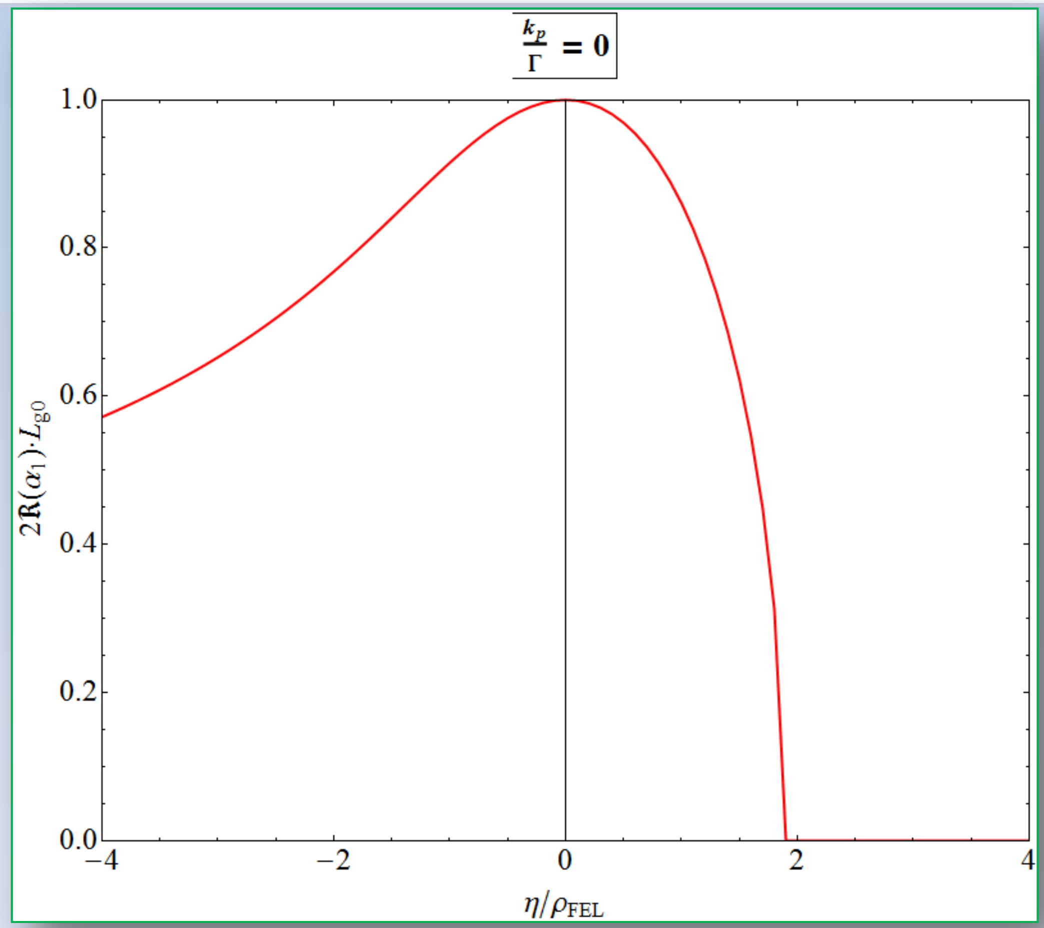
Third-Order Equation (Gain function)



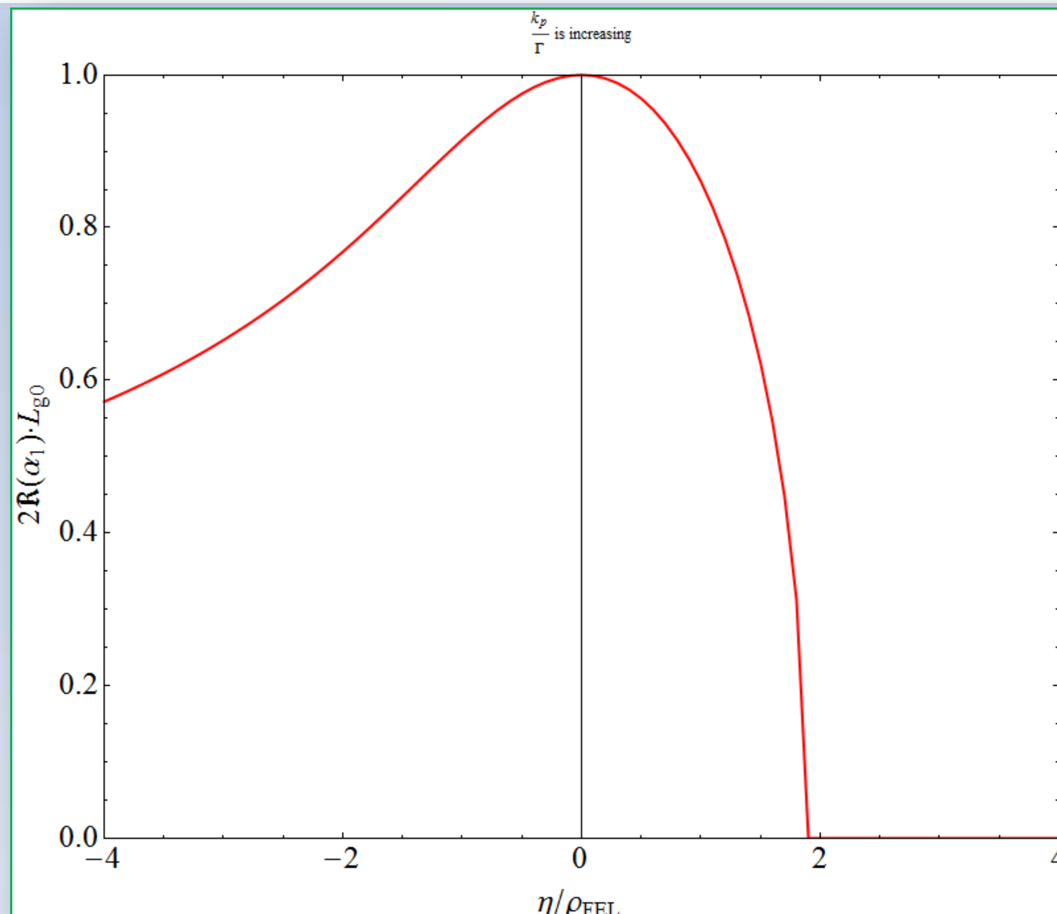
Third-Order Equation (Gain function)



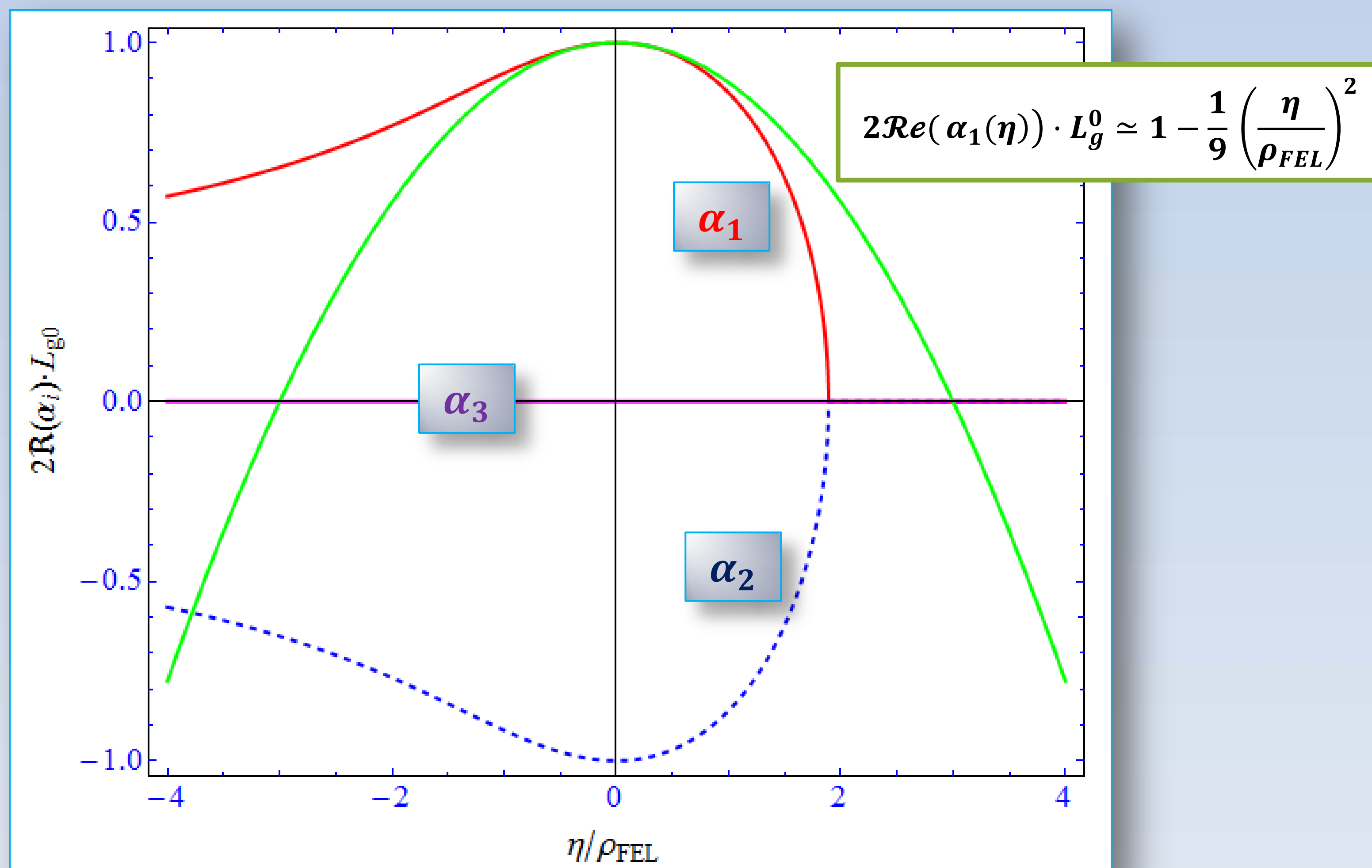
Third-Order Equation (Growth Rate Func)



Third-Order Equation (Growth Rate Func)



FEL Bandwidth (Schematically)

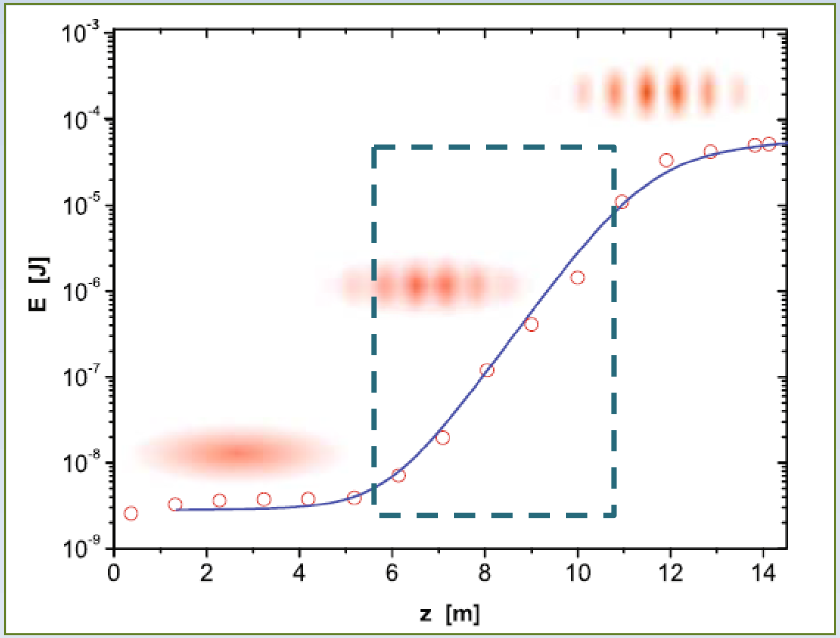


FEL Bandwidth (Theoretically)

$$G(\eta, z) \propto e^{\frac{z}{L_g^0}\left(1-\frac{1}{9}\left(\frac{\eta}{\rho_{\mathrm{FEL}}}\right)^2\right)} = e^{\frac{z}{L_g^0}} \cdot e^{-\frac{\eta^2}{2\tau^2}}$$

Where $\tau = \frac{9\rho_{\mathrm{FEL}}{}^2 L_g^0}{2z}$, then the rms frequency bandwidth of SASE FEL is (not valid in the exponential regime)

$$\sigma_\omega(z) = \tau(z) \cdot 2\omega_l = 3\sqrt{2}\rho_{\mathrm{FEL}}\omega_l\sqrt{\frac{L_g^0}{z}}$$



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FEL Startup by a Periodically Modulated e-beam

We've already known:

$$\tilde{E}_x(\eta, z) = \sum_{j=1}^{3} c_j(\eta) e^{\alpha_j(\eta) z}$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = A^{-1} \cdot \begin{pmatrix} \tilde{E}_x(0) \\ \tilde{E}_x'(0) \\ \tilde{E}_x''(0) \end{pmatrix}, A = \begin{pmatrix} 1 & 1 & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ {\alpha_1}^2 & {\alpha_2}^2 & {\alpha_3}^2 \end{pmatrix}$$

$$\tilde{j}_1(z) = j_0 \frac{2}{N} \sum_{n=1}^{N} e^{-i\psi_n(z)}$$

$$\frac{d\psi_n}{dz} = 2k_u \eta_n$$

$$\tilde{j}_1{}'(0) = \sum_{n=1}^{N} \frac{d\tilde{j}_1}{d\psi_n} \cdot \frac{d\psi_n}{dz} = -i2k_u \eta \tilde{j}_1(0)$$

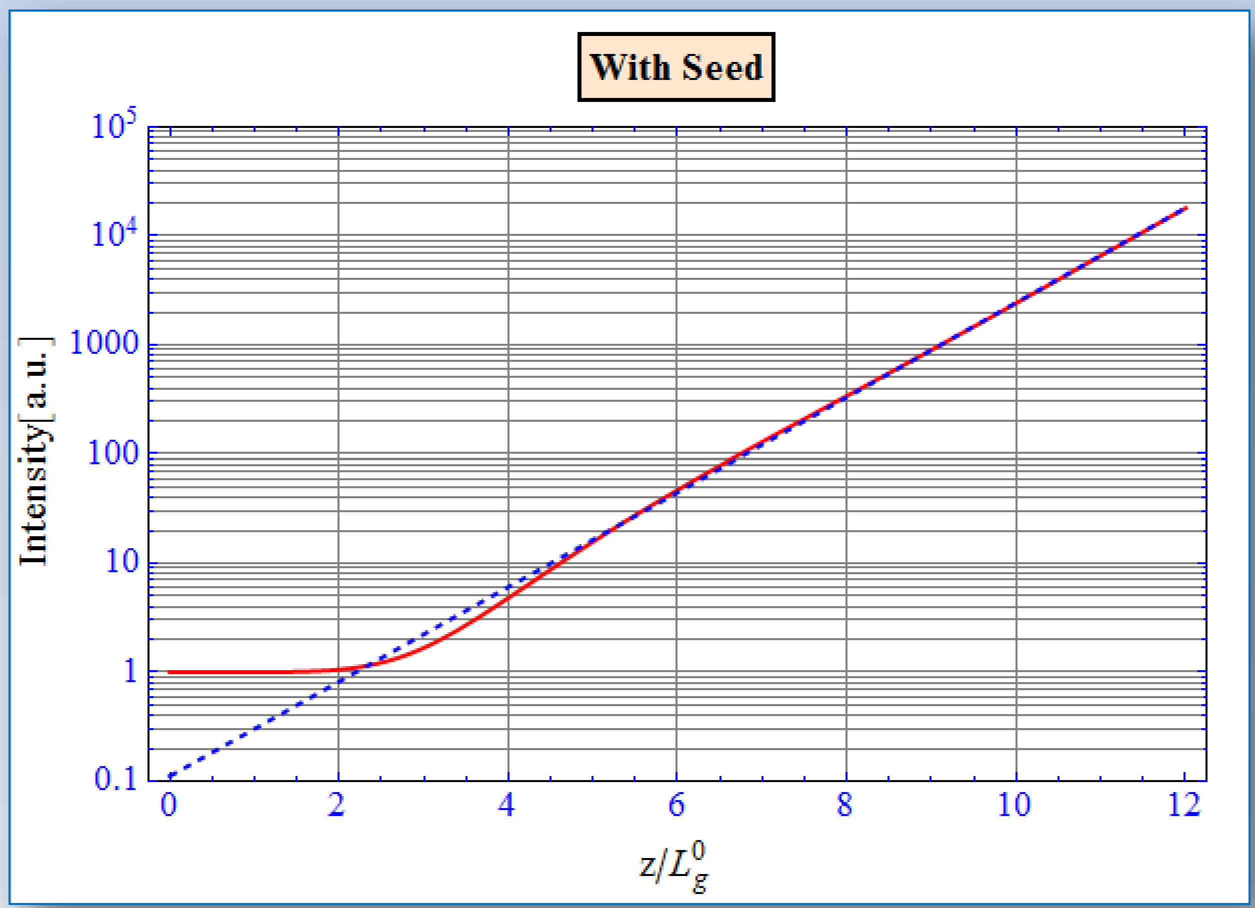
$$E_0 = 0$$

$$E_0{}' \equiv \frac{d\tilde{E}_x}{dz}(0) = -\frac{\mu_0 c\hat{K}}{4\gamma_r} \cdot \tilde{j}_1(0)$$

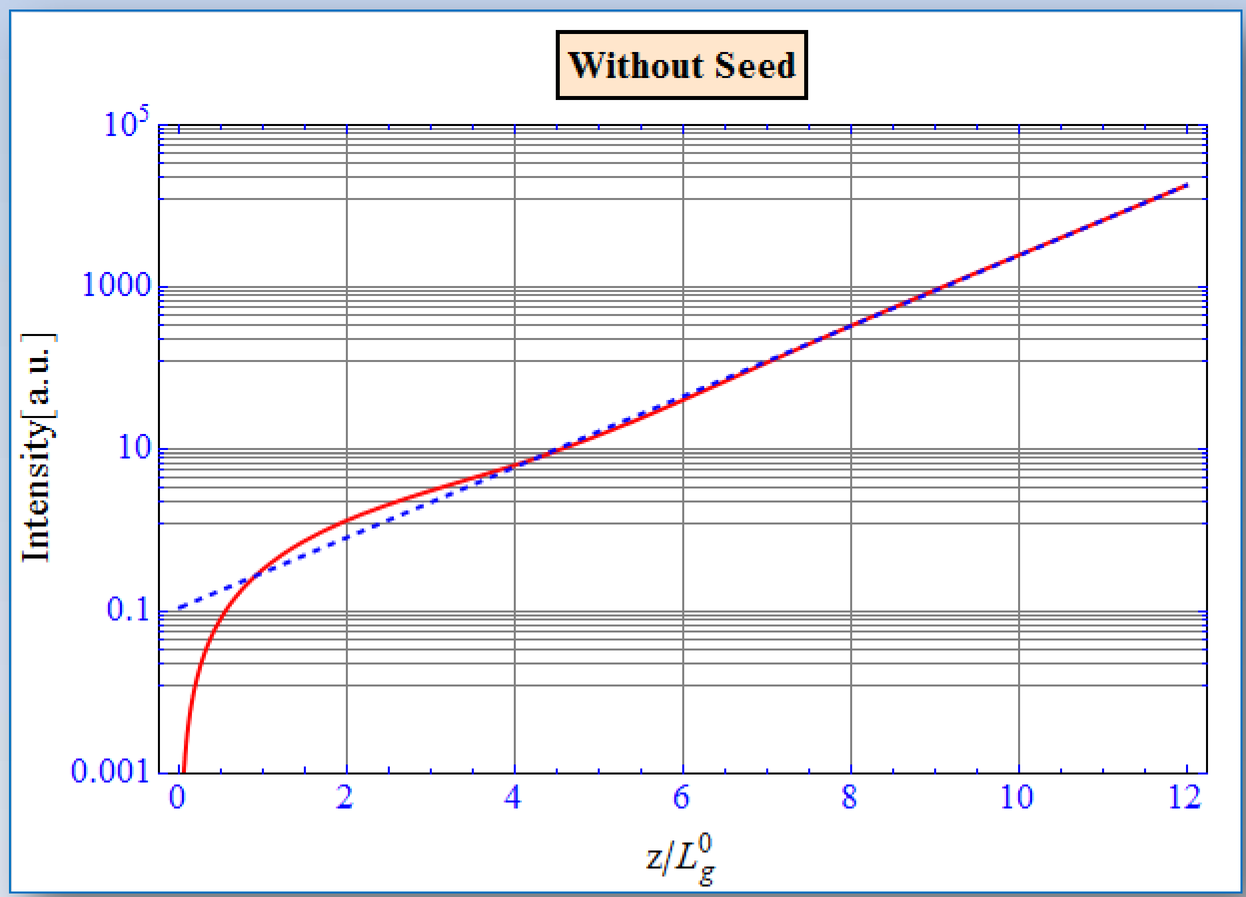
$$E_0{}'' = i2k_u\eta \frac{\mu_0 c\hat{K}}{4\gamma_r} \cdot \tilde{j}_1(0)$$

$$\begin{pmatrix} \tilde{E}_x(0) \\ \tilde{E}_x'(0) \\ \tilde{E}_x''(0) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ i2k_u\eta \end{pmatrix} \frac{\mu_0 c\hat{K}}{4\gamma_r} \cdot \tilde{j}_1(0)$$

Gaincurve (with seed)



Gaincurve (without seed)



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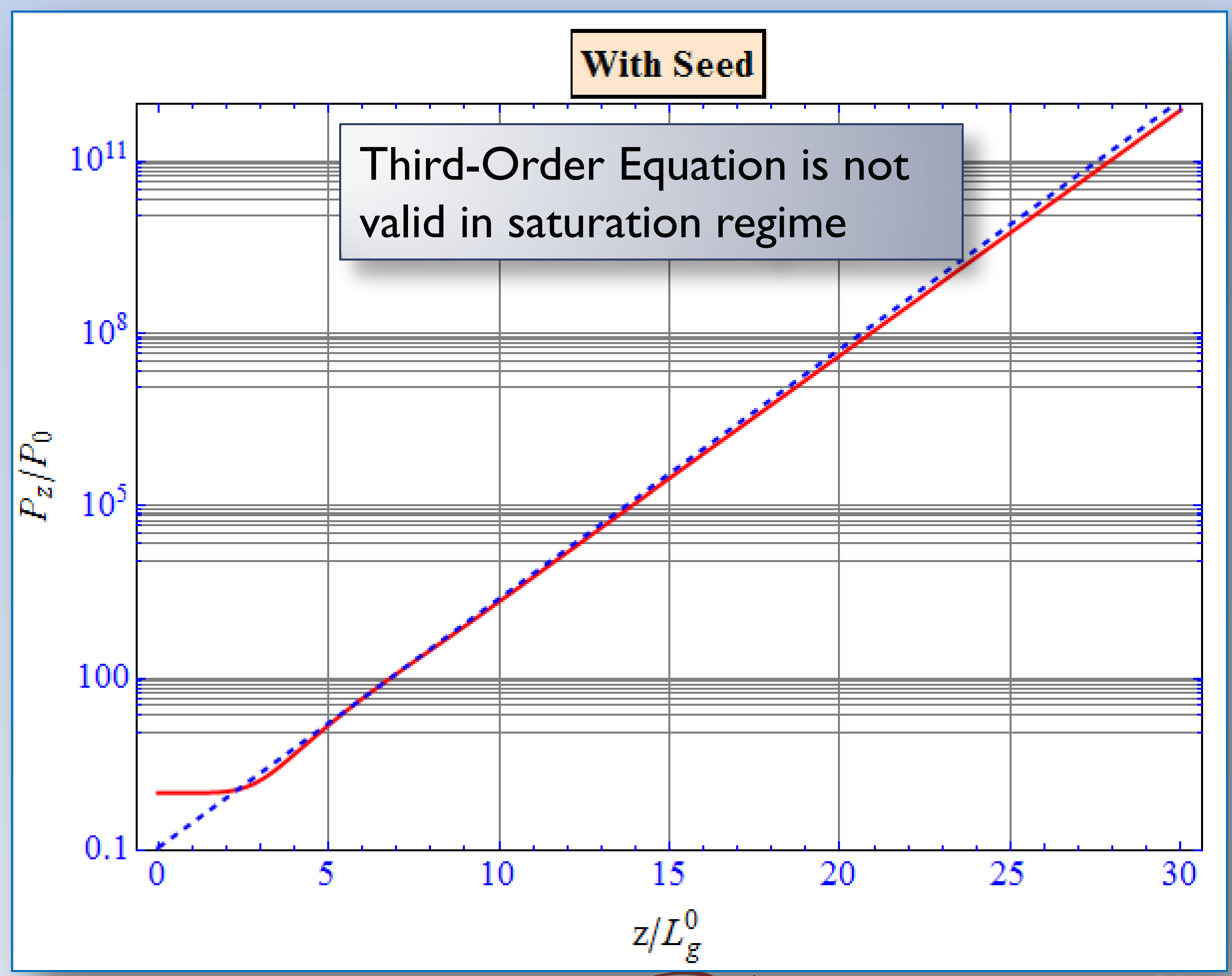
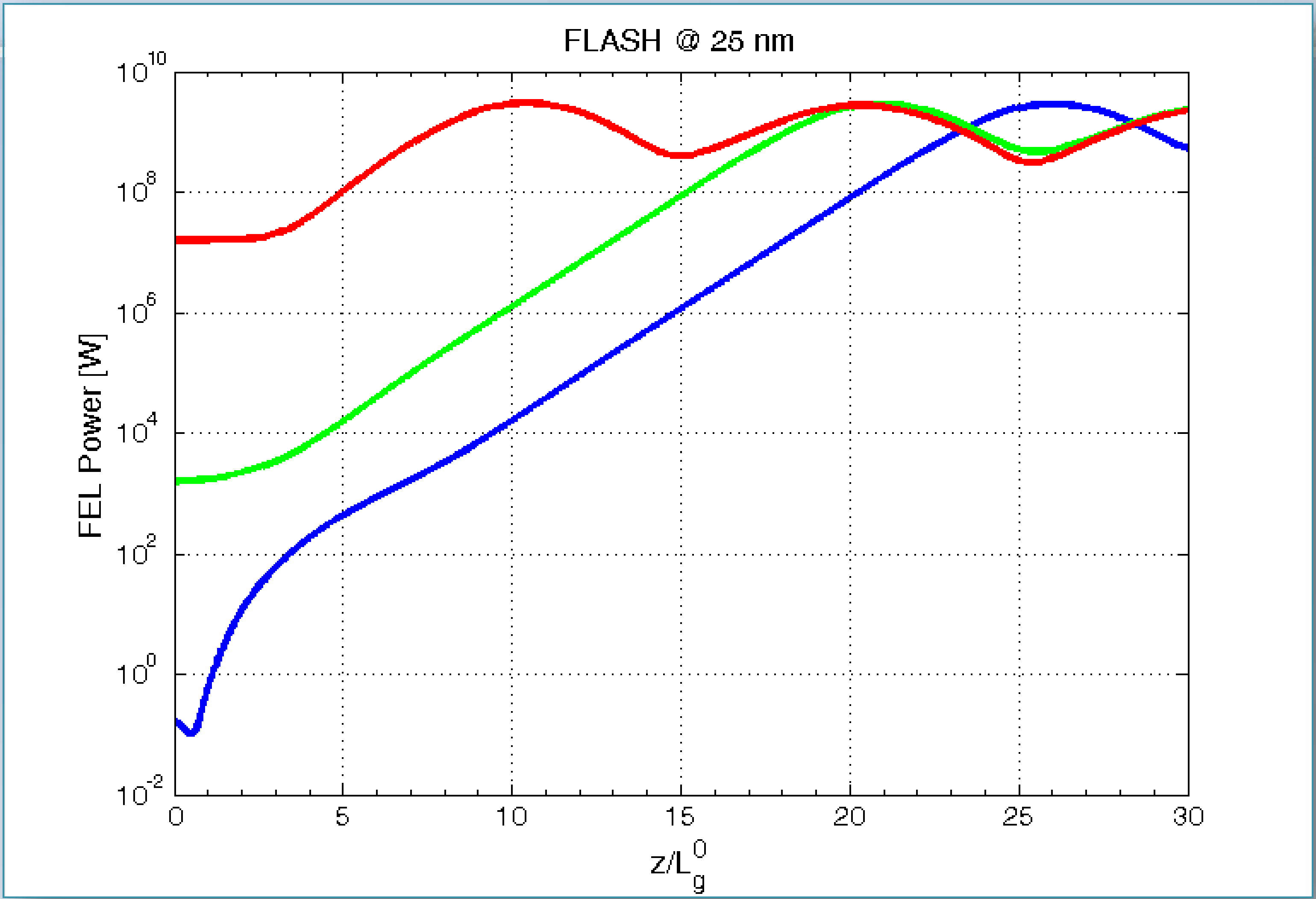
Gaincurve (FLASH example)

Parameter	Value
γ	1000
L_g^0	0.5 m
k_p	0.24 m^{-1}
ρ_{FEL}	0.003
E_{in}	5 MV/m



Laser Saturation (FLASH)



Laser Saturation (Analytical)

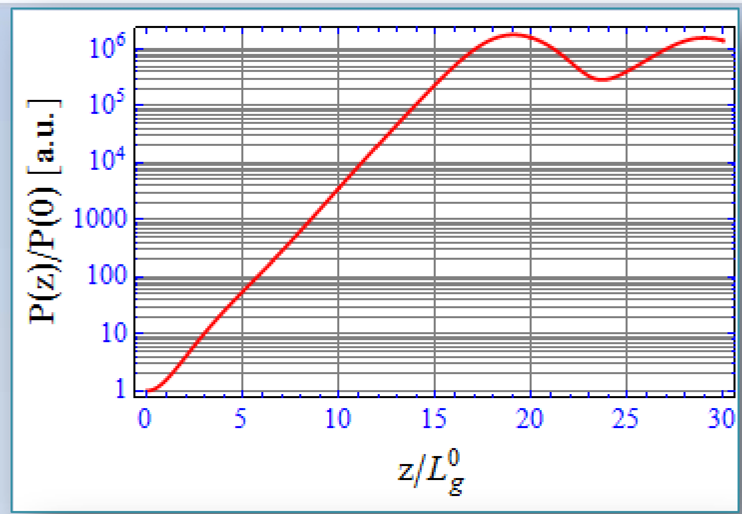
$$|\tilde{j}_1| \sim |j_0|$$

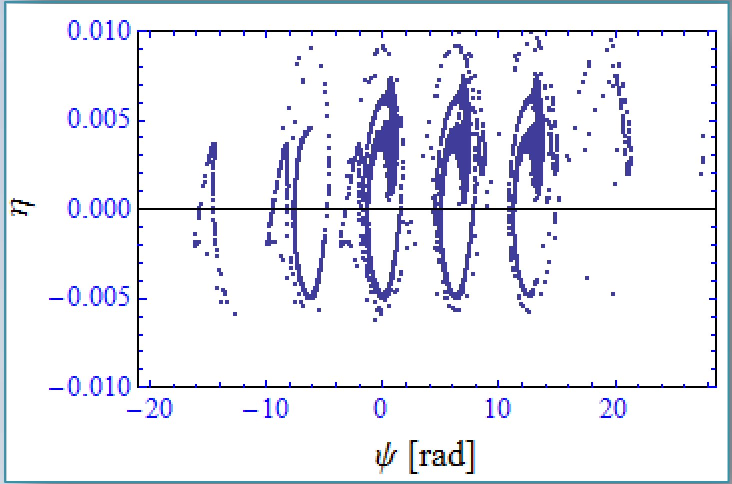
$$|\tilde{E}_x|_{\text{sat}} \approx \left|\frac{d\tilde{E}_x}{dz}\right| \cdot 2L_g^0 = \frac{\mu_0 c\hat{K}}{4\gamma_r}|j_0| \cdot 2L_g^0$$

$$P_{sat} = \frac{1}{2}\epsilon_0 c|\tilde{E}_x|_{sat}^2 = \frac{1}{2}\epsilon_0 c\left(\frac{\mu_0 c\hat{K}}{4\gamma_r}|j_0| \cdot 2L_g^0\right)^2$$
$$= \frac{\rho_{FEL}}{3} \cdot \frac{\gamma_r m_e c^2 I_0}{e} = \frac{\rho_{FEL}}{3} P_{beam}$$

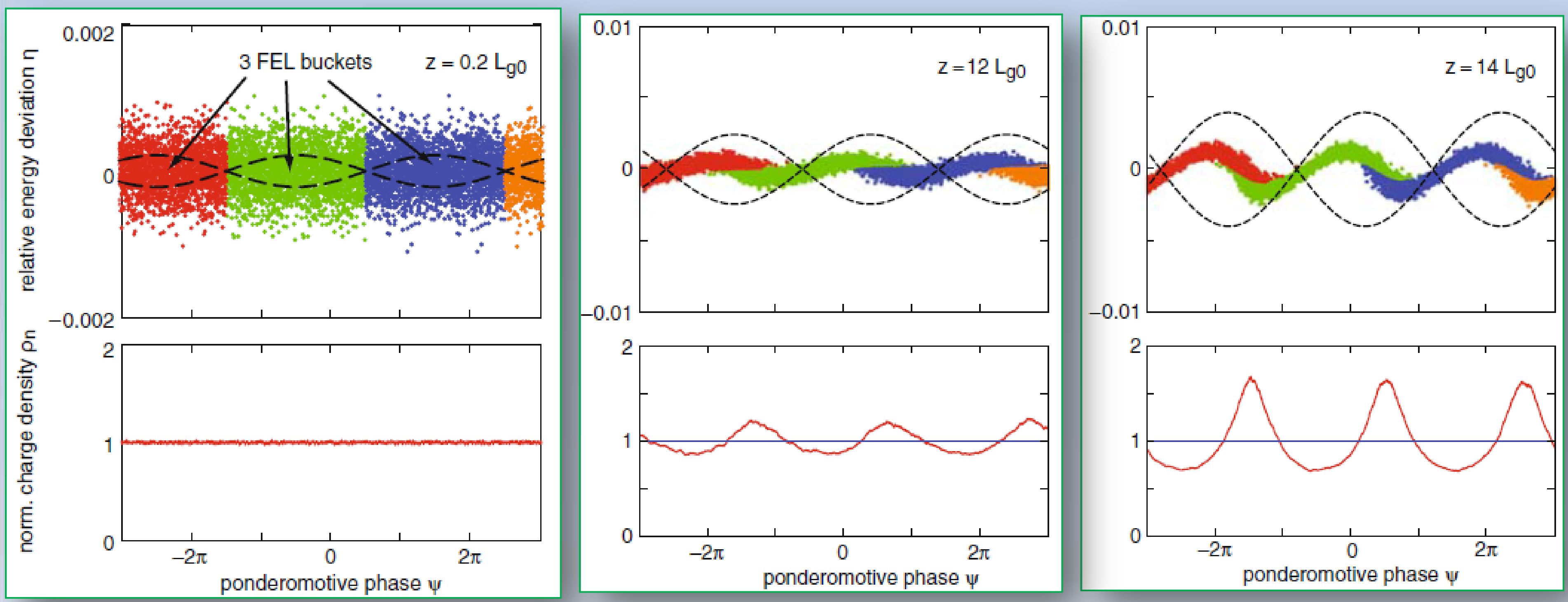
$$P_{\text{sat}} \approx \rho_{\text{FEL}} \cdot P_{\text{beam}}$$

FLASH:
$\rho P_{beam} = 0.003 * 511 * 1600 \approx 2$ GW
Numerical result: ~ 2 GW

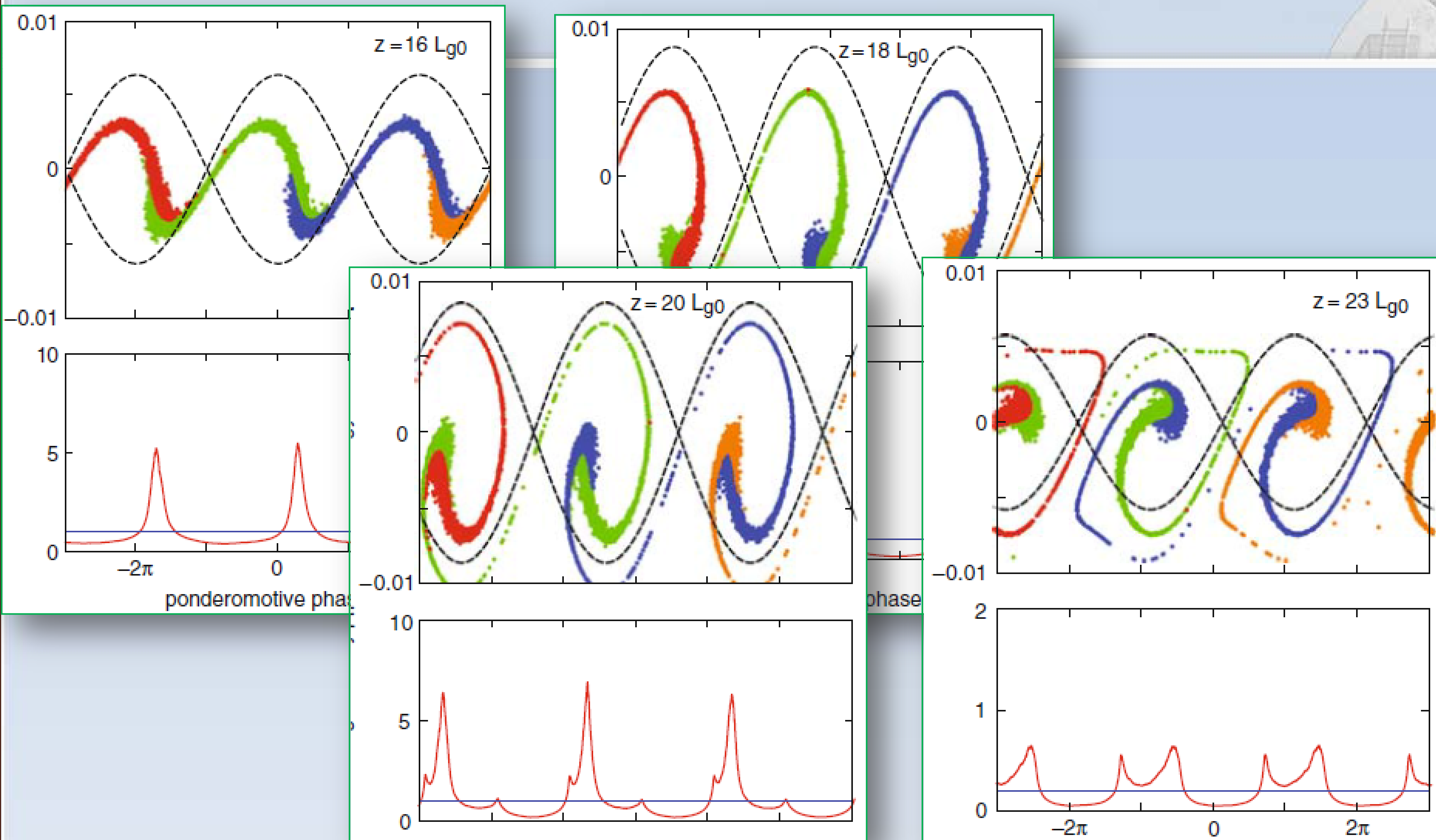




Microbunching evolution (I)



Microbunching evolution (2)



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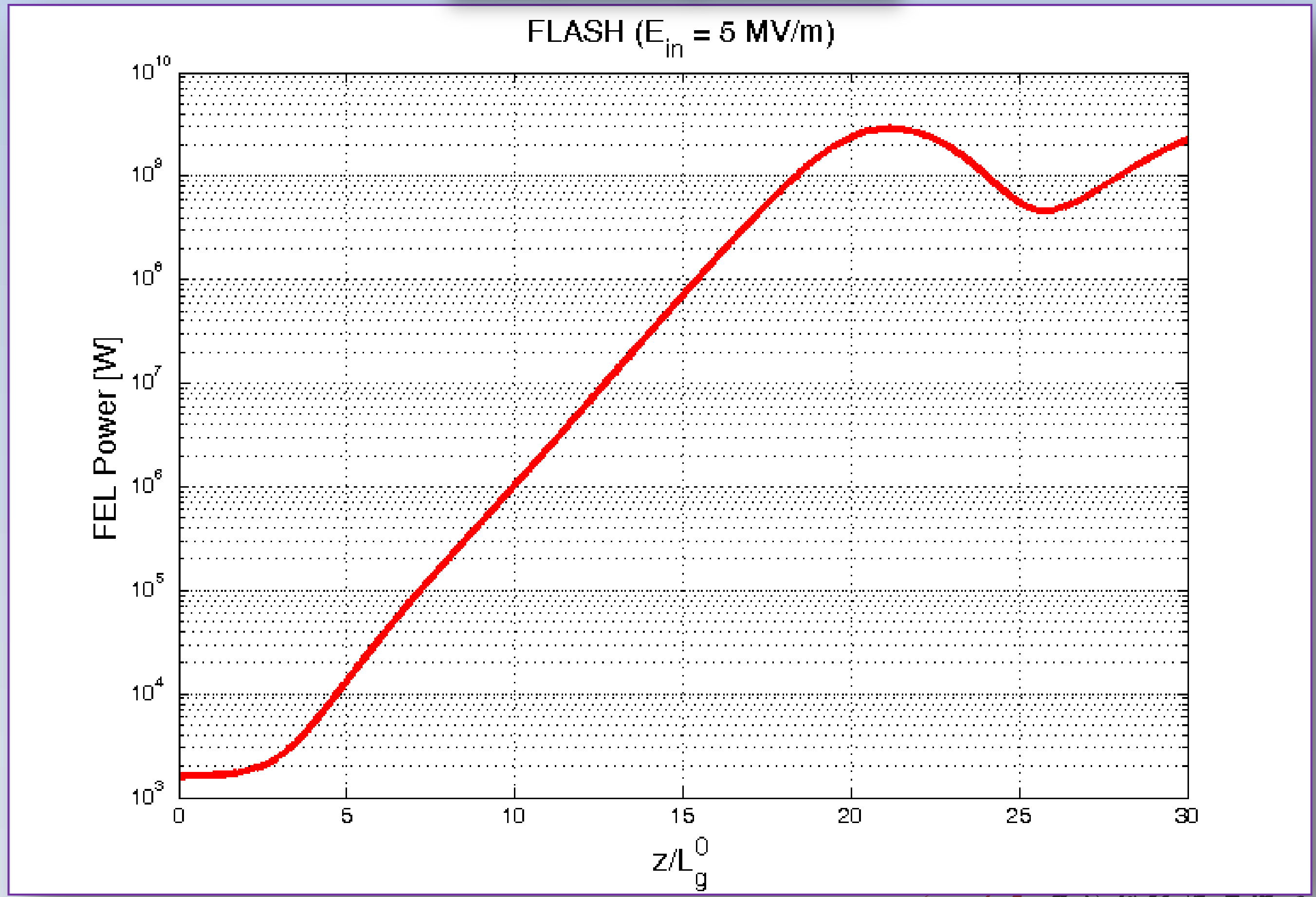
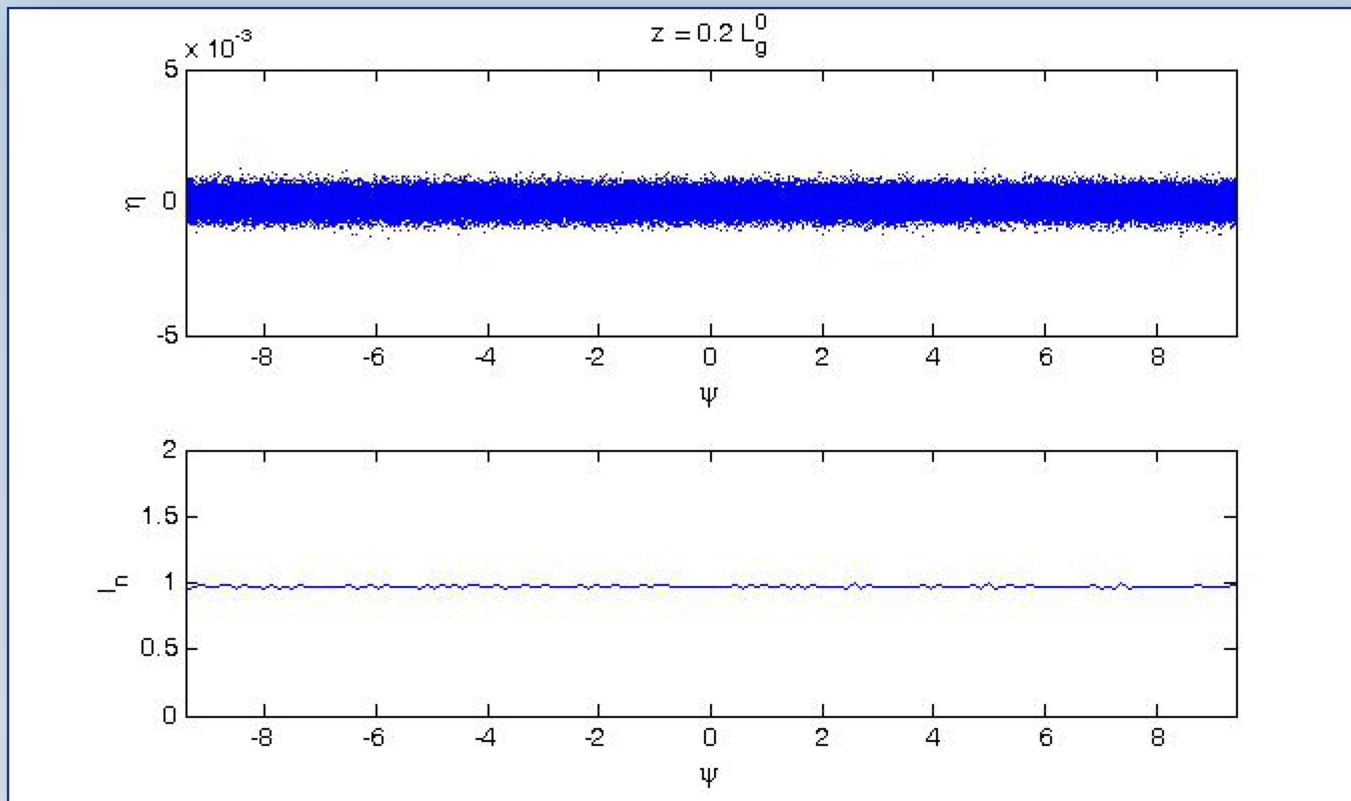
Microbunching simulation (FLASH)

FEL power gain curve



Microbunching simulation (FLASH)

Microbunching evolution along the undulator ($z/z_g^0 = 0.2\sim30$)

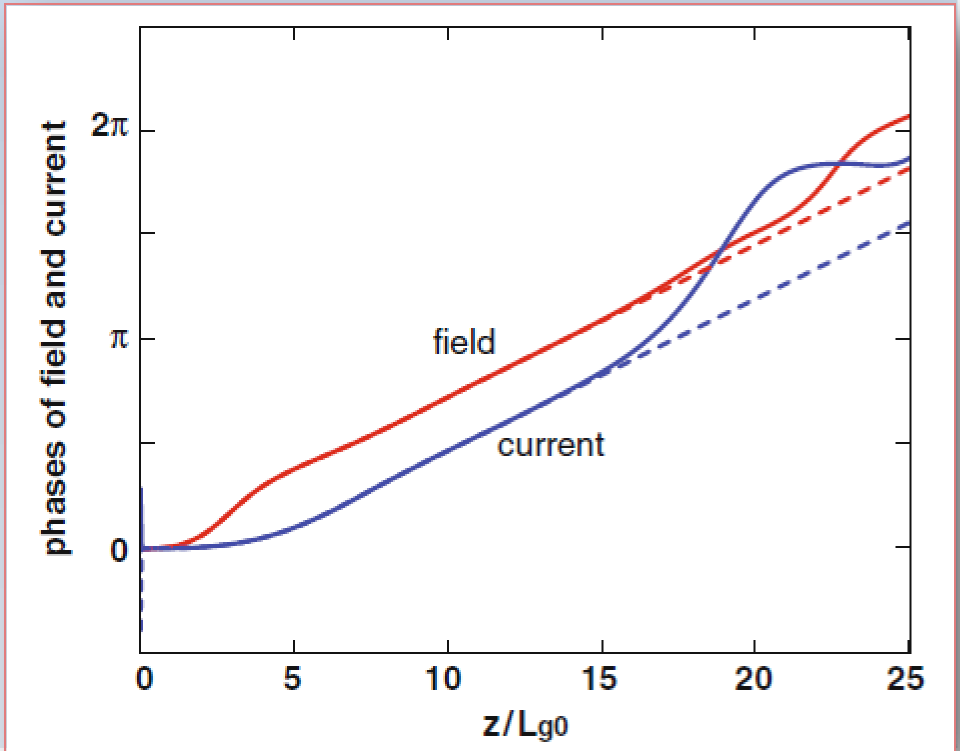


Phase evolution ($\tilde{E}_x \& \tilde{j}_1$)

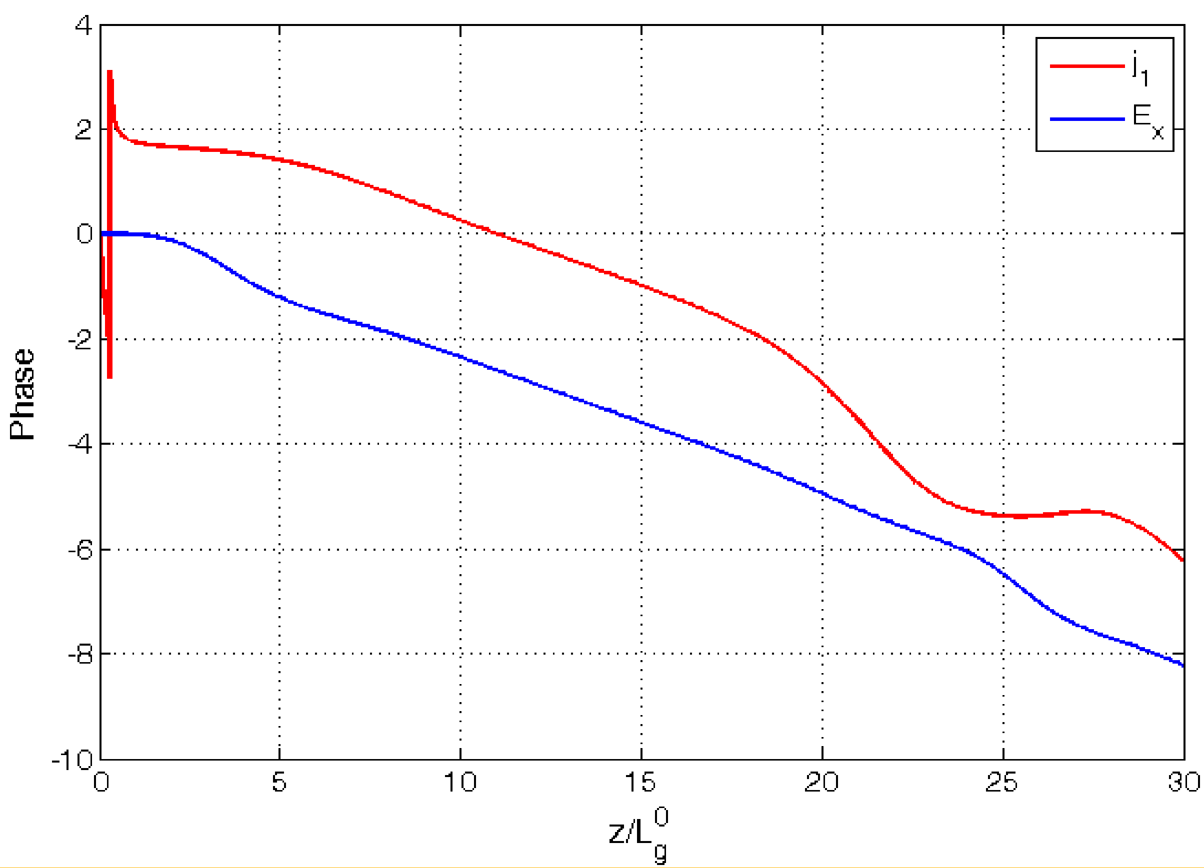
$$\tilde{E}_x(z) = \frac{E_{\rm in}}{3}\sum_{j=1}^{3}\exp(\alpha_j z) \equiv \left|\tilde{E}_x(z)\right| \exp(i\,\varphi_E(z))$$

$$\tilde{j}_1(z) = -\frac{4\gamma_r}{\mu_0 c\,\widehat{K}}\,\tilde{E}'_x(z) = -\frac{4\gamma_r}{\mu_0 c\,\widehat{K}}\sum_{j=1}^{3}\alpha_j\exp(\alpha_j z) \equiv \left|\tilde{j}_1(z)\right| \exp(i\,\varphi_{j1}(z))$$

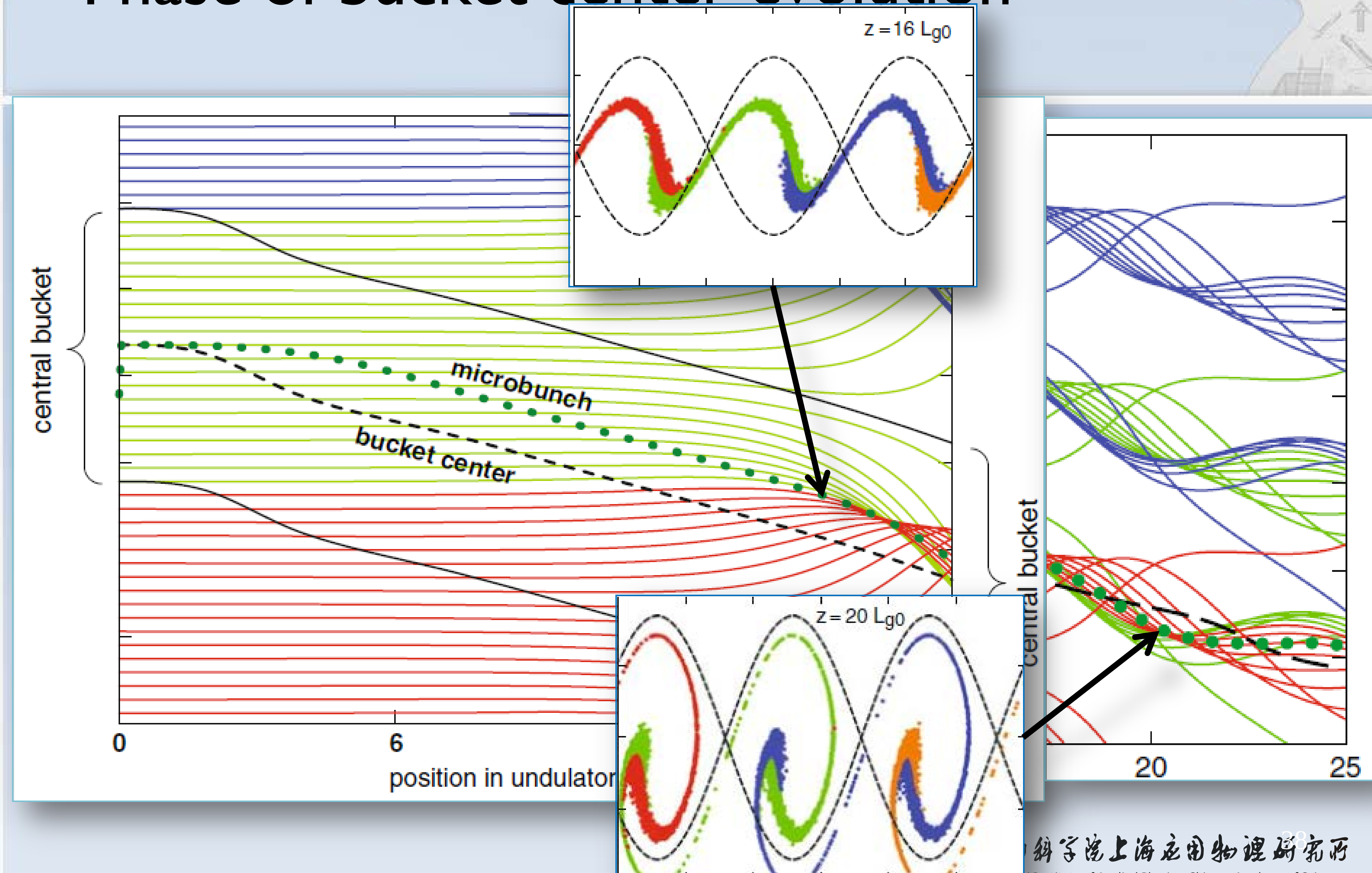
$$\Delta\varphi = [\varphi_E(z) - \varphi_E(0)] - [\varphi_{j1}(z) - \varphi_{j1}(0)]$$



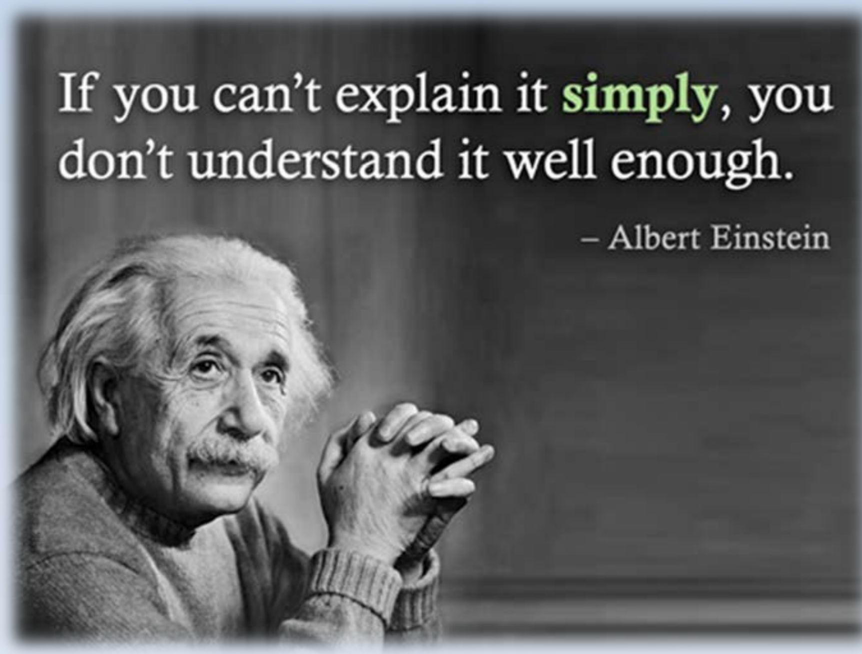
Phase evolution (coupled-first-equations)



Phase of bucket center evolution



1. Peter Schmüser, Martin Dohlus and Jörg Rossbach, ***Ultraviolet and Soft X-Ray Free-Electron Lasers***, Springer, Berlin Heidelberg, 2008, Charpter 4, 5, Appendix B,C,E



Thank you!