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# Lecture Notes
# Quantum Optics, Part II:
# Atom-Light Interaction

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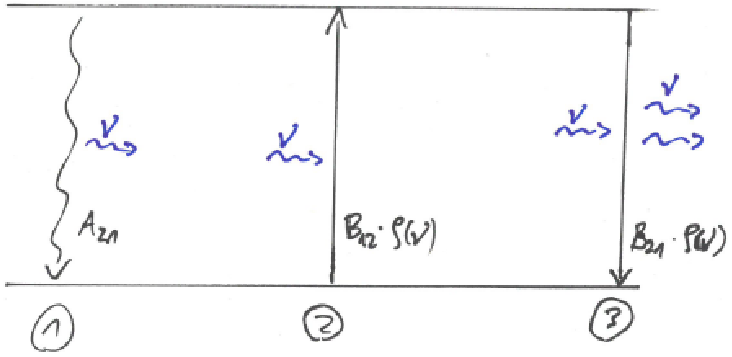
# Contents

# 7 Atom-Light Interaction

## 7.0 Repetition: The Einstein Model

- Three types of processes: (1) spontaneous emission, (2) induced absorption, (3) induced emission




- Rate equation:

$$\frac{dN_2}{dt} = N_1 B_{12}\rho(\nu) - N_2 B_{21}\rho(\nu) - N_2 A_{21}$$

- Plancks spectral energy density:

$$\rho(\nu) = \underbrace{\frac{8\pi\nu^2}{c^3}}_{\text{mode density}} \cdot \underbrace{h\nu}_{\text{energy per photon}} \cdot \underbrace{\frac{1}{e^{\frac{h\nu}{k_B T}} - 1}}_{\text{population of mode}}$$

## 7.1 Hamilton-Operator

- We consider a single electron, which is bound to an atom:

$$H_0 = \frac{1}{2m}\mathbf{p}^2 + V(r) \qquad\qquad r = |\mathbf{r}|$$

- An external electro-magnetic field acts on this electron, described by a scalar ($\phi$) and a vector potential ($\mathbf{A}$):

$$H = \frac{1}{2m}\left(\mathbf{p} + e\mathbf{A}(\mathbf{r},t)\right)^2 - e\phi(\mathbf{r},t) + V(r) \tag{7.1}$$

- The potentials can be transformed without changing the fields. Gauge transformation:

$$\begin{array}{lcl} \phi & \rightarrow & \phi - \partial_t \chi \\ \mathbf{A} & \rightarrow & \mathbf{A} + \nabla \chi \end{array} \qquad \text{invariant:} \qquad \begin{array}{ll} \mathbf{E} & = -\nabla \phi - \frac{\partial}{\partial t}\mathbf{A} \\ \mathbf{B} & = \nabla \times \mathbf{A} \end{array}$$

- We use the Coulomb gauge:
  $\Rightarrow$ choose $\chi$ such that:

$$\nabla \cdot \mathbf{A} = 0 \Rightarrow \phi = 0^{1} \tag{7.2}$$

  $\mathbf{A}$ also satisfies the wave equation:

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0$$

- Now: Consider electro-magnetic waves in the optical domain with typical wave length of $\lambda = 400\,\mathrm{nm}$ to $800\,\mathrm{nm}$. Compared to $\lambda$ the spatial extension of the electron wave function is small.
  $\Rightarrow$ vector potential $\mathbf{A}$ is spatially uniform over the extend of atom:

$$\mathbf{A}(\mathbf{r}, t) \approx \mathbf{A}(t) \qquad \text{and} \qquad \mathbf{E}_0 \cdot e^{i(\mathbf{kr} - \omega_L t)} \rightarrow \mathbf{E}_0 \cdot e^{i\omega_L t}$$

  Where the subscript $L$ stands for *Laser*. This is the so-called **dipole approximation**.

- The Coulomb gauge is not unique. $\rightarrow$ choose the gauge function

$$\chi(\mathbf{r}, t) = -\mathbf{A}(t) \cdot \mathbf{r} \tag{7.3}$$

  With this choice of gauge, the Hamiltonian in Eq.(7.1) reads

$$\begin{aligned} H &= \frac{1}{2m} \left(\mathbf{p} + e(\mathbf{A} + \nabla \chi)\right)^2 - e\left(\phi - \partial_t \chi\right) + V(r) \\ &\overset{(7.2)+(7.3)}{=} \frac{1}{2m} \left(\mathbf{p} + e(\mathbf{A} + \nabla(-\mathbf{A} \cdot \mathbf{r}))\right)^2 - e\left(-\partial_t(-\mathbf{A} \cdot \mathbf{r})\right) + V(r) \\ &= \frac{\mathbf{p}^2}{2m} - e\mathbf{r} \cdot \frac{\partial \mathbf{A}}{\partial t} + V(r) = \frac{\mathbf{p}^2}{2m} + V(r) + e\mathbf{r} \cdot \mathbf{E}(t) \\ \Rightarrow H &\equiv \frac{\mathbf{p}^2}{2m} + V(r) - \mathbf{d} \cdot \mathbf{E} \qquad \text{with} \qquad \mathbf{d} = -e\mathbf{r} \\ H &= H_0 + H_i = H_0 - \mathbf{d} \cdot \mathbf{E} \end{aligned} \tag{7.4}$$

[1] Since the potential is due to an external source, the first of Maxwell laws reads $\nabla \cdot \mathbf{E} = 0$, within the region of interest. Since derivatives can be exchanged, with the choice of gauge $\nabla \cdot \mathbf{A} = 0$, the first law expressed through the potentials reads $\nabla^2 \phi = 0$. With the boundary condition $\phi(\infty) = 0 \Rightarrow \phi \equiv 0$.

## 7.2 Strong Monochromatic Excitation

Note: So far we didn't specify whether the field is classical or quantum-mechanical. The derivation of Equation (7.4) would work in both cases. But for now we choose a classical description of the field. Thus:

$$\begin{aligned} &\mathbf{E}(t) = \mathbf{E}_0 \cdot \cos(\omega_L t) \\ &\Rightarrow \hat{H} = \hat{H}_0 + \hat{H}_i \qquad \text{with} \qquad \hat{H}_i = -\hat{\mathbf{d}} \cdot \mathbf{E} = e\hat{\mathbf{r}} \cdot \mathbf{E} \end{aligned}$$

### 7.2.1 Perturbation Theory Approach

- Consider eigenstates of the unperturbed atomic Hamiltonian:

  $$\hat{H}_0 |\psi_n\rangle = E_n |\psi_n\rangle$$

  with eigenenergy $E_n$ and eigenvectors $|\psi_n\rangle$

- Time evolution in Schrödinger picture:

  $$\begin{aligned} i\hbar\partial_t |\psi_n\rangle &= E_n |\psi_n\rangle \\ \Rightarrow |\psi_n(t)\rangle &= e^{-iE_n t/\hbar} |\psi_n(0)\rangle \end{aligned}$$

- Atomic system under influence of $\hat{H}_i$:
  $\Rightarrow$ Ansatz: expand state vector $|\psi(t)\rangle$ in terms of the complete set of uncoupled atomic states.

  $$\left.\begin{aligned} |\psi(t)\rangle &= \textstyle\sum_n c_n(t) |\psi_n(t)\rangle \\ \Rightarrow i\hbar\partial_t |\psi(t)\rangle &= \hat{H} |\varphi(t)\rangle \end{aligned}\right\} \ldots$$

  $\rightarrow$ obtain set of coupled 1st-order, linear differential equations!

  $$\dot{c}_k(t) = -i \sum_n c_n(t) e^{i\omega_{kn} t} \Omega_{kn} \cos(\omega_L t) \tag{7.5}$$

  with

  $$\omega_{kn} = \frac{E_k - E_n}{\hbar} \qquad \text{and} \qquad \Omega_{kn} = \frac{e\mathbf{E}_0}{\hbar} \underbrace{\langle\psi_k| \hat{\mathbf{r}} |\psi_n\rangle}_{X_{kn}}$$

  $\Omega_{kn}$ is the so-called **Rabi frequency**.

- $X_{kn}$ is the corresponding **dipole matrix element** with the following properties:
  - $X_{kn}$ can be calculated from atomic wave function.
  - $X_{kn} = 0$ for all states with equal parity:

    $$\text{e.g.:} \qquad \langle\psi_n| \hat{\mathbf{r}} |\psi_n\rangle = 0 \qquad \text{as} \qquad \int \psi_n^\dagger \cdot x \cdot \psi_n dx = \int |\psi_n|^2 x dx = 0$$

$\Rightarrow$ In the dipole approximation $\hat{H}_i$ couples states with opposite parity!
$\Rightarrow$ $X_{kn}$ gives rise to certain selection rules[2].
$\Rightarrow$ If $X_{kn} = 0$ the corresponding transition is forbidden!

### 7.2.2 The Rabi Model

- Consider a two-level atom interacting with a classical field :

$$\omega_0 = \frac{E_e - E_g}{\hbar} \qquad \text{and} \qquad \Omega = \langle e| \mathbf{r} |g\rangle \cdot \frac{e\mathbf{E}_0}{\hbar}$$

  Excitation with laser field:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \cos(\omega_L t)$$

  With this field Equation (7.5) reads:

$$\begin{aligned} \dot{c}_g &= -\, i\Omega \cos(\omega_L t) e^{-i\omega_0 t} c_e \\ \dot{c}_e &= -\, i\Omega \cos(\omega_L t) e^{i\omega_0 t} c_g \\ \Rightarrow \dot{c}_e &= -\, i\frac{\Omega}{2} \left(e^{i\omega_L t} + e^{-i\omega_L t}\right) e^{i\omega_0 t} c_g \\ &= -\, \frac{i\Omega}{2} \left(e^{i(\omega_L + \omega_0)t} + e^{-i(\omega_L - \omega_0)t}\right) c_g \end{aligned}$$

- For weak excitation $c_g(t) \approx 1 \Rightarrow \dot{c}_e(t)$ can be integrated!

$$\Rightarrow c_e(t) = -\frac{\Omega}{2} \left[ \cancelto{0}{\frac{e^{i(\omega_L + \omega_0)t}}{\omega_L + \omega_0}} - \frac{e^{-i(\omega_L - \omega_0)t}}{\omega_L - \omega_0} \right]$$

  $\Rightarrow$ fast rotating terms can be neglected.
  This is called the **rotating wave approximation**[3].

$$\Rightarrow \left.\begin{aligned} \dot{c}_g &= -\tfrac{i\Omega}{2} c_e e^{i(\omega_L - \omega_0)t} \\ \dot{c}_e &= -\tfrac{i\Omega}{2} c_g e^{-i(\omega_L - \omega_0)t} \end{aligned}\right\} \ddot{c}_e + i\Delta \dot{c}_e + \frac{\Omega^2}{4} c_e = 0$$

  with $\Delta = \omega_L - \omega_0$.

- Ansatz:

$$c_i(t) = c_i e^{i\lambda t}$$

[2] Notice that these selection rules must not be interpreted as strict rules, but rather as the most probable transitions, since other transitions can still occur in higher multipole expansions.
[3] Since the laser frequency is of the order of THz while the detuning is of the order of MHz, the fast rotating, neglected term is about six orders of magnitude smaller.

- example: initial conditions $c_g(0) = 1$ and $c_e(0) = 0$:

$$c_e(t) = -i\frac{\Omega}{\Omega_{eff}}e^{-i\Delta t/2}\sin\left(\frac{\Omega_{eff}}{2}t\right)$$
$$c_g(t) = e^{i\Delta t/2}\left[\cos\left(\frac{\Omega_{eff}}{2}t\right) - \frac{i\Delta}{\Omega_{eff}}\sin\left(\frac{\Omega_{eff}}{2}t\right)\right]$$

with the **effective Rabi-frequency** $\Omega_{eff} = \sqrt{\Omega^2 + \Delta^2}$.

- Excitation Probability:

$$P_e(t) = |c_e(t)|^2 = \frac{\Omega^2}{\Omega_{eff}^2}\sin^2\left(\frac{\Omega_{eff}}{2}t\right)$$
$$P_e(t) = [1 - \cos(\Omega_{eff}t)]\frac{\Omega^2}{2\Omega_{eff}^2}$$
$$\text{for} \quad \Delta = 0 \quad : \qquad P_e(t) = \sin^2\left(\frac{\Omega}{2}t\right)$$

Coherent atom-light interaction leads to *Rabi flopping* between ground and excited state:

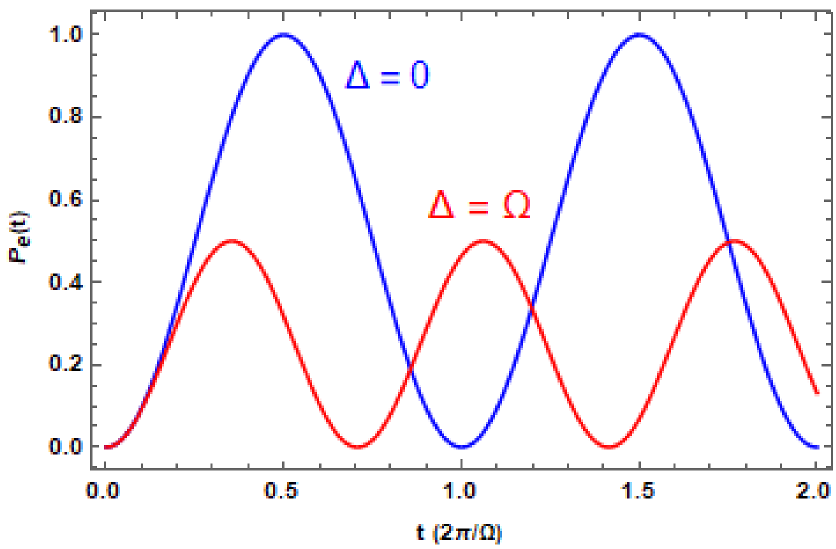



Figure 7.1: Probability to find an atom in the excited state as a function of interaction time $t$. The two curves show Rabi oscillations on resonance (black) and for a detuning $\Delta = \Omega$ (blue).

The excitation probability depends strongly non-linear on the electric field amplitude $E_0$ (related to the laser power).

### 7.2.3 The Optical Bloch Equations (OBEs)

- Goal:
  - general description, also of mixed states
  - simple approach to include damping (spontaneous emission)
- let's consider again a 2-level system

$$|g\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad |e\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{with} \quad E_g = 0$$

- New notation in 2nd quantization:
  also atomic Hamiltonian is expressed by creation and annihilation operators. General description of atom:

$$\hat{H}_0 |i\rangle = \hbar\omega_i |i\rangle$$

  $|i\rangle$ is any eigenstate of the atomic Hamiltonian either bound or unbound.
  From the completeness relation $\sum_i |i\rangle \langle i| = 1$ we obtain

$$\begin{aligned}
\hat{H}_0 &= \underbrace{\sum_i |i\rangle \langle i|}_{\hat{1}} \hat{H}_0 \underbrace{\sum_j |j\rangle \langle j|}_{\hat{1}} \\
&= \sum_{i,j} |i\rangle \underbrace{\langle i| \hat{H}_0 |j\rangle}_{\hbar\omega_j \delta_{ij}} \langle j| \\
&= \sum_i \hbar\omega_i |i\rangle \langle i|
\end{aligned}$$

  which is nothing else than the diagonal form of the Hamiltonian.
  $\Rightarrow$ for a 2-level system ($E_g = \hbar\omega_g = 0$):

$$\hat{H}_0 = \hbar\omega_0 |e\rangle \langle e|$$

  with $\omega_e = \omega_0$.

- At this point it is convenient to introduce the so-called inversion and transition operators.
  Definition:

$$\begin{aligned}
\hat{\sigma}_3 &= |e\rangle \langle e| - |g\rangle \langle g| = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} && \text{inversion operator} \\
\left.\begin{aligned}
\hat{\sigma}_+ &= |e\rangle \langle g| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
\hat{\sigma}_- &= |g\rangle \langle e| = \hat{\sigma}_+^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\end{aligned}\right\} & && \text{transition operators}
\end{aligned}$$

  $\rightarrow$ Pauli-algebra of pseudo-spin:

$$\begin{aligned}
[\hat{\sigma}_+, \hat{\sigma}_-] &= \hat{\sigma}_3 \\
[\hat{\sigma}_3, \hat{\sigma}_\pm] &= \pm\, 2\hat{\sigma}_\pm
\end{aligned}$$

- Interaction with classical field $\mathbf{E}(t) = \mathbf{E}_0 \cos(\omega_L t)$, $\hat{H}_i = -\hat{\mathbf{d}}\mathbf{E}$.
  From completeness relation and from

$$\langle e| \hat{H}_i |g\rangle = e\mathbf{E}_0 \langle e| \mathbf{r} |g\rangle \cos(\omega_L t) = \hbar\Omega \cos(\omega_L t)$$

follows:

$$\begin{aligned}\hat{H}_i &= |e\rangle \langle e| \hat{H}_i |g\rangle \langle g| + |g\rangle \langle g| \hat{H}_i |e\rangle \langle e| \\ &= \hbar\Omega \cos(\omega_L t)(|e\rangle \langle g| + |g\rangle \langle e|)\end{aligned}$$

Now write Hamiltonian in interaction picture:

$$\hat{H}_i = \frac{\hbar\Omega}{2}(e^{-i\omega_L t} + e^{i\omega_L t}) \cdot (e^{-i\omega_0 t}\hat{\sigma}'_+ + e^{i\omega_0 t}\hat{\sigma}'_-)$$

where the $\hat{\sigma}'$ are rotating with $\omega_0$!
Reminder: Interaction Picture:

$$\begin{aligned}|\psi\rangle' &= \hat{U}_0^\dagger |\psi\rangle \qquad \text{with} \qquad \hat{U}_0 = e^{-i\hat{H}_0 t/\hbar} \\ |e\rangle' &= e^{i\omega_0 t} |e\rangle \\ |g\rangle' &= |g\rangle \\ \hat{H}'_i &= \hat{U}_0^\dagger \hat{H}_i \hat{U}_0\end{aligned}$$

- Rotating Wave Approximation

$$\hat{H}'_i \approx \frac{\hbar\Omega}{2}(e^{i\Delta t}\hat{\sigma}_- + e^{-i\Delta t}\hat{\sigma}_+)$$

with $\Delta = \omega_L - \omega_0$.

- Transformation into rotating reference frame
  $\Rightarrow \hat{H}_i$ becomes independent of time

$$\begin{aligned}|\tilde{e}\rangle &= e^{i\Delta t} |e\rangle' \\ \Rightarrow \hat{\tilde{\sigma}}_+ &= |\tilde{e}\rangle \langle g| \text{ , etc}\ldots \\ \Rightarrow \hat{\tilde{H}}_i &= \frac{\hbar\Omega}{2}\left(\hat{\tilde{\sigma}}_+ + \hat{\tilde{\sigma}}_-\right) = \begin{pmatrix} 0 & \frac{\hbar\Omega}{2} \\ \frac{\hbar\Omega}{2} & 0 \end{pmatrix}\end{aligned}$$

$\hat{H}_0$: from $i\hbar\partial_t |\tilde{e}\rangle = -\hbar\Delta |\tilde{e}\rangle \stackrel{!}{=} \hat{\tilde{H}}_0 |\tilde{e}\rangle$

$$\Rightarrow \hat{\tilde{H}}_0 = -\hbar\Delta |\tilde{e}\rangle \langle\tilde{e}| = \begin{pmatrix} 0 & 0 \\ 0 & -\hbar\Delta \end{pmatrix}$$

full Hamiltonian:

$$\hat{\tilde{H}} = \hat{\tilde{H}}_0 + \hat{\tilde{H}}_i = -\hbar\Delta |\tilde{e}\rangle \langle\tilde{e}| + \frac{\hbar\Omega}{2}\left(\hat{\tilde{\sigma}}_+ + \hat{\tilde{\sigma}}_-\right)$$

- Density Matrix Formalism

$$\hat{\rho} = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \qquad \text{for mixed states}$$

$$\hat{\rho} = \left|\psi_i\right\rangle \left\langle\psi_i\right| \qquad \text{for pure state}$$

where $p_i$ stands for a classical probability.
Properties of density matrix:

$$\begin{aligned}
\mathrm{Tr}(\hat{\rho}) &= 1 \\
\mathrm{Tr}\left(\hat{\rho}^2\right) &= 1 \qquad \text{for pure states!} \\
\hat{\rho}^\dagger &= \hat{\rho} \qquad \text{Hermitian} \\
\Rightarrow \rho_{21} &= \rho_{12}^* \qquad \text{self-adjoint matrix} \\
\langle \hat{O} \rangle &= \mathrm{Tr}\left(\hat{\rho}\hat{O}\right)
\end{aligned}$$

- Now consider density matrix of a 2-level system with general wave function

$$\begin{aligned}
\left|\psi\right\rangle &= c_1 \left|g\right\rangle + c_2 \left|e\right\rangle \\
\Rightarrow \hat{\rho} &= \begin{pmatrix} |c_1|^2 & c_1 c_2^* \\ c_2 c_1^* & |c_2|^2 \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}
\end{aligned}$$

$\Rightarrow \rho_{ii}$ describes the population of state $i$ and $\rho_{ij}$ describes the coherences for $i \neq j$.

- density matrix can be deduced from solution of the Rabi model $\rightarrow c_1(t)$, $c_2(t)$
- The dynamics of $\hat{\rho}$ is determined by the **Quantum Liouville equation**

$$-i\hbar\,\partial_t \hat{\tilde{\rho}} = [\hat{\tilde{H}}, \hat{\tilde{\rho}}] \qquad \text{, with } \hat{\tilde{H}} = \hat{\tilde{H}}_0 + \hat{\tilde{H}}_i$$

$$\boxed{\begin{aligned}
\dot{\tilde{\rho}}_{22} &= -\,\dot{\tilde{\rho}}_{11} = \frac{i\Omega}{2}(\tilde{\rho}_{21} - \tilde{\rho}_{12}) \\
\dot{\tilde{\rho}}_{12} &= \dot{\tilde{\rho}}_{21}^* = \frac{i\Omega}{2}(\tilde{\rho}_{11} - \tilde{\rho}_{22}) - i\Delta\tilde{\rho}_{12}
\end{aligned}}$$

which are called **Optical Bloch Equations**

### Solving the Optical Bloch Equations

- Since the OBEs are linear, first order coupled differential equations
$\rightarrow$ Ansatz $\tilde{\rho}_{ij}(t) = \tilde{\rho}_{ij}(0)e^{\lambda t}$
$\rightarrow$ obtain system of linear equations which can be written in vector matrix notation.
Let $A$ be the corresponding matrix, and $\mathbf{x}$ out of the 4 dimensional entry space of $\hat{\tilde{\rho}}$:

$$\Rightarrow A\mathbf{x} \overset{!}{=} \lambda \mathbf{x}$$

$$\Leftrightarrow (A-\lambda I)\mathbf{x} = \begin{pmatrix} -\lambda & 0 & \frac{i\Omega}{2} & -\frac{i\Omega}{2} \\ 0 & -\lambda & -\frac{i\Omega}{2} & \frac{i\Omega}{2} \\ \frac{i\Omega}{2} & -\frac{i\Omega}{2} & -i\Delta-\lambda & 0 \\ -\frac{i\Omega}{2} & \frac{i\Omega}{2} & 0 & i\Delta-\lambda \end{pmatrix} \cdot \begin{pmatrix} \tilde{\rho}_{11}(0) \\ \tilde{\rho}_{22}(0) \\ \tilde{\rho}_{12}(0) \\ \tilde{\rho}_{21}(0) \end{pmatrix} \overset{!}{=} 0$$

$$\Rightarrow \det(A-\lambda I) \overset{!}{=} 0 \Rightarrow \lambda^2(\lambda^2+\Delta^2+\Omega^2) = 0$$

$$\Rightarrow \lambda_{1,2} = 0 \quad , \quad \lambda_3 = i\Omega_{eff} \quad , \quad \lambda_4 = -i\Omega_{eff} \quad \text{with} \quad \Omega_{eff} = \sqrt{\Omega^2+\Delta^2}$$

- general solution

$$\tilde{\rho}_{ij}(t) = \tilde{\rho}_{ij}^{(1)}(0) + \tilde{\rho}_{ij}^{(2)}(0)e^{i\Omega_{eff}t} + \tilde{\rho}_{ij}^{(3)}(0)e^{-i\Omega_{eff}t}$$

- particular solution $\tilde{\rho}_{11}(0) = 1$, rest$= 0$.

$$\begin{aligned} \rightarrow \tilde{\rho}_{22}(t) &= \frac{\Omega^2}{\Omega_{eff}^2}\sin^2\left(\frac{\Omega_{eff}}{2}t\right) = 1-\tilde{\rho}_{11}(t) \\ \tilde{\rho}_{12}(t) &= \frac{\Omega}{\Omega_{eff}^2}\sin\left(\frac{\Omega_{eff}}{2}t\right)\left[\Delta\sin\left(\frac{\Omega_{eff}}{2}t\right) + i\Omega_{eff}\cos\left(\frac{\Omega_{eff}}{2}t\right)\right] \end{aligned}$$

which reproduces the Rabi flopping.
Remember: $\rho_{11}+\rho_{22} \overset{!}{=} 1$ as $\mathrm{Tr}(\tilde{\rho}) = 1$.

### 7.2.4 The Bloch Sphere

- Define 3 variables (coordinates of Bloch vector)

$$\begin{aligned} u &:= \tilde{\rho}_{12}+\tilde{\rho}_{21} = 2\,\mathrm{Re}(\tilde{\rho}_{12}) \\ v &:= -i(\tilde{\rho}_{12}-\tilde{\rho}_{21}) = 2\,\mathrm{Im}(\tilde{\rho}_{12}) \\ w &:= \tilde{\rho}_{11}-\tilde{\rho}_{22} = 1-2\tilde{\rho}_{22} \end{aligned}$$

Write the **Bloch vector** as :

$$\mathbf{R} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$$

- Equation of motion for Bloch vector:

$$\dot{\mathbf{R}} = \mathbf{R}\times\mathbf{W}$$

with "optical torque" $\mathbf{W} = \Omega\mathbf{e}_x + \Delta\mathbf{e}_z$, $|\mathbf{W}| = \sqrt{\Omega^2+\Delta^2} \leftrightarrow$ eff. Rabi frequency!

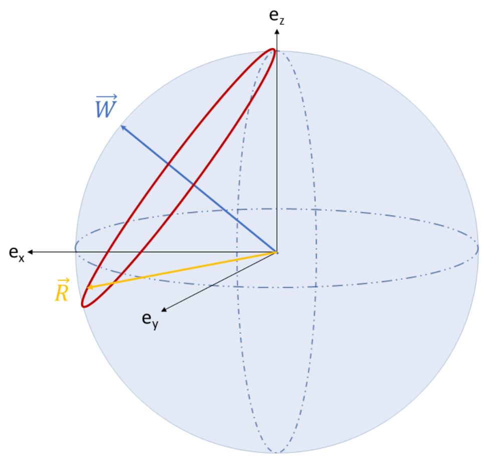



Figure 7.2: Rotation on the Bloch sphere. The Bloch vector **R** (yellow arrow) precesses around **W** (blue arrow), the tip of **R** following a circle lying in the plane perpendicular to **W**.

- <u>The Bloch Sphere</u>
  = geometrical representation of the pure state space of a quantum-mechanical two-level system
  ⇒ examples:
  - atomic and nuclear spin (qubits),
  - states of polarization (Poincaré sphere),
  - ...

- <u>In spherical coordinates:</u>
  angle of the Bloch vector for state $|\psi\rangle = c_1 |g\rangle + c_2 |e\rangle$
  $c_1 = \cos(\theta/2)$ and $c_2 = e^{i\phi}\sin(\theta/2)$

- <u>Remarks:</u>
  - $w = 1 \Rightarrow |\psi\rangle = |1\rangle \, (= |g\rangle)$
  - $w = -1 \Rightarrow |\psi\rangle = |2\rangle \, (= |e\rangle)$
  - $u = 1 \Rightarrow |\psi\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}}$ / $u = -1 \Rightarrow |\psi\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}$
  - $v = 1 \Rightarrow |\psi\rangle = \frac{|1\rangle + i|2\rangle}{\sqrt{2}}$ / $v = -1 \Rightarrow |\psi\rangle = \frac{|1\rangle - i|2\rangle}{\sqrt{2}}$
  - opposing states are orthogonal!

- <u>Rotation of **R**</u>
  - $\Delta = 0 \Rightarrow$ R evolves on great circle

- $\Delta = const$ and $\Omega = const$.
  $\rightarrow \dot{\mathbf{R}} \cdot \mathbf{W} = 0 \Rightarrow \mathbf{RW} = const = RW\cos(\theta) \Rightarrow$ fixed cone!

- How to write density matrix $\hat{\rho}$ in terms of $\mathbf{R}$:
  $\hat{\rho} = \frac{1}{2}(\hat{I} + \mathbf{R} \cdot \hat{\sigma})$, $|\mathbf{R}| \leq 1$.
  for pure states:

$$\begin{aligned} &\mathrm{Tr}\big(\hat{\rho}^2\big) = 1 = \frac{1}{2}(1 + |\mathbf{R}|^2) \stackrel{!}{=} 1 \\ &\Rightarrow |\mathbf{R}| = 1 \end{aligned}$$

- Note: Bloch variables $u, v, w$ are expectation values of $\hat{\sigma}_x$, $\hat{\sigma}_y$, $\hat{\sigma}_z$!

$$\begin{aligned} \text{e.g.} \quad \langle \hat{\sigma}_x \rangle &= \mathrm{Tr}\Big((\hat{\sigma}_- + \hat{\sigma}_+)\hat{\tilde{\rho}}\Big) \\ &= \mathrm{Tr}\Big(|g\rangle \langle e| \,\hat{\tilde{\rho}} + |e\rangle \langle g| \,\hat{\tilde{\rho}}\Big) \\ &= \langle e| \,\hat{\tilde{\rho}}\, |g\rangle + \langle g| \,\hat{\tilde{\rho}}\, |e\rangle \\ &= \tilde{\rho}_{ge} + \tilde{\rho}_{eg} = u \end{aligned}$$

### 7.2.5 Optical Bloch Equation with Damping

#### Open quantum systems:

example spontaneous decay:

- derive from coupling of dipole operator to vacuum fluctuations of the electric field
  $\rightarrow$ Wigner-Weisskopf theory (s. Scully, Meystre)
  $\rightarrow$ rather involved

- in order to describe decay of atom, we need to include a continuum of modes of the electric field

$$|\psi(t)\rangle = c_e(t)\,|e, 0\rangle + \sum_k c_{g,k}\,|g, 1_k\rangle$$

  $\rightarrow$ coupling of a q.m. state to a reservoir
  $\rightarrow$ open quantum system

  (a) Wigner-Weisskopf= example for general class of problems where a small q.m. system couples to a large one
  (b) Born-Approximation = system is large enough that it stays unchanged $\rightarrow$ Reservoir
  (c) Markov-Approximation= environment is uncorrelated (has no memory)
  $\rightarrow$ spontaneous decay is irreversible!
  $\rightarrow$ photon escapes quickly

(d) we are not interested in the field itself, but in dynamics of subsystem $\rightarrow$ reduced density matrix!

Reduced density matrix for a subsystem:

- define reduced density operators for each subsystem by tracing $\hat{\rho}$ over the states of all other systems. Let $j$ be the system of interest and $i$ all other systems

$$\hat{\rho}^{(j)} = \mathrm{Tr}_{(i)}(\hat{\rho}) = \sum_n \left\langle \psi_n^{(i)} \middle| \hat{\rho} \middle| \psi_n^{(i)} \right\rangle$$

  Example:(bipartite system)

$$\begin{aligned}
|\psi\rangle &= c_1 \left|\psi_1^{(1)}\right\rangle \left|\psi_2^{(2)}\right\rangle + c_2 \left|\psi_2^{(1)}\right\rangle \left|\psi_1^{(2)}\right\rangle \\
\hat{\rho} &= |\psi\rangle \langle\psi| \\
&= \left(c_1 \left|\psi_1^{(1)}\right\rangle \left|\psi_2^{(2)}\right\rangle + c_2 \left|\psi_2^{(1)}\right\rangle \left|\psi_1^{(2)}\right\rangle\right) \cdot \left(c_1^* \left\langle\psi_1^{(1)}\right| \left\langle\psi_2^{(2)}\right| + c_2^* \left\langle\psi_2^{(1)}\right| \left\langle\psi_1^{(2)}\right|\right) \\
\Rightarrow \hat{\rho}^{(1)} &= \mathrm{Tr}_2(\hat{\rho}) = \left\langle \psi_1^{(2)} \middle| \hat{\rho} \middle| \psi_1^{(2)} \right\rangle + \left\langle \psi_2^{(2)} \middle| \hat{\rho} \middle| \psi_2^{(2)} \right\rangle \\
&= |c_2|^2 \left|\psi_2^{(1)}\right\rangle \left\langle\psi_2^{(1)}\right| + |c_1|^2 \left|\psi_1^{(1)}\right\rangle \left\langle\psi_1^{(1)}\right|
\end{aligned}$$

  $\rightarrow$ when one particle is considered without regard to the other, it is generally in a mixed state!
  $\rightarrow$ characterize the degree of entanglement according to the degree of purity of either of the subsystems
  $\mathrm{Tr}\big((\hat{\rho}^{(1)})^2\big) < 1$!

- system observable: Let $\hat{\rho} = \hat{\rho}_r \hat{\rho}_s$ , $s$ =system, $r$ =reservoir:

$$\langle \hat{O}_s \rangle = \mathrm{Tr}_{sr}(\hat{O}\hat{\rho}) = \mathrm{Tr}_s(\hat{O}\,\mathrm{Tr}_r(\hat{\rho})) = \mathrm{Tr}_s(\hat{O}\hat{\rho}_s)$$

**Formal introduction of damping:**

- The Master Equation
  = a phenomenologically derived differential equation of 1st order, which describes the time evolution of the probabilities of a system
  e.g. $\frac{d\mathbf{P}}{dt} = M\mathbf{P}$
  Generalization: include off-diagonal elements of density matrix.
  e.g.: OBEs without damping

$$\partial_t \hat{\rho} = +\frac{i}{\hbar}[\hat{H}, \hat{\rho}] = \hat{\mathcal{L}}\hat{\rho}$$

  With $\mathcal{L}$ as the *Liouvillian*

- Master equation in Lindblad form
  ensures that reduced density matrix $\hat{\rho}_s$ has properties of a density matrix

$$\partial_t \hat{\rho}_s = \hat{\mathcal{L}} \hat{\rho}_s$$

  now with superoperator (Liouvillian)

$$\hat{\mathcal{L}}\hat{\rho}_s = +\frac{i}{\hbar}[\hat{H}, \hat{\rho}_s] + \sum_j \underbrace{(\hat{c}_j \hat{\rho}_s \hat{c}_j^\dagger - \frac{1}{2}\hat{\rho}_s \hat{c}_j^\dagger \hat{c}_j - \frac{1}{2}\hat{c}_j^\dagger \hat{c}_j \hat{\rho}_s)}_{\text{interaction with reservoir}}$$

  $\hat{c}_j$: collapse operators
  e.g. $\hat{c} = \sqrt{\Gamma}\hat{\sigma}_-$ for spontaneous emission.
  $\hat{c} = \sqrt{\Gamma_L}\,|g\rangle\langle g|$ for laser linewidth.
  $\Rightarrow$ optical Bloch equation with damping.
  Note: derivation of $\hat{\mathcal{L}}$ see "Meystre, Sargent"

### Solving the OBEs with damping

- extend OBEs by 2 terms describing the decay of the excited state population and of the coherences (phenomenologically!)

$$\boxed{\begin{aligned} \dot{\tilde{\rho}}_{22} = -\dot{\tilde{\rho}}_{11} &= -\frac{i\Omega}{2}\tilde{\rho}_{12} + \frac{i\Omega}{2}\tilde{\rho}_{21} - \Gamma\tilde{\rho}_{22} \\ \dot{\tilde{\rho}}_{12} = \dot{\tilde{\rho}}_{21}^* &= \frac{i\Omega}{2}(\tilde{\rho}_{11} - \tilde{\rho}_{22}) - i\Delta\tilde{\rho}_{12} - \frac{\Gamma}{2}\tilde{\rho}_{12} \end{aligned}}$$

| | |
|---|---|
| $\Gamma = A = 1/\tau_{sp}$: | decay rate of excited state due to spontaneous emission. |
| $\Gamma/2$: | decay rate of coherence |

We identify the two following regimes:

(1) $t \ll \tau_{sp}$: Rabi oscillations can be observed.

(2) $t \gg \tau_{sp}$: The system reaches an equilibrium because of damping.

### Steady State Solution

$$\begin{aligned} \dot{\tilde{\rho}}_{ij} &= 0 \\ \tilde{\rho}_{22}^{(s)} &= \frac{\Omega^2/4}{\Delta^2 + \left(\frac{\Gamma}{2}\right)^2 + \frac{\Omega^2}{2}} = \frac{S_0/2}{1 + S_0 + 4\left(\frac{\Delta}{\Gamma}\right)^2} \\ \tilde{\rho}_{12}^{(s)} &= -\frac{\frac{\Omega}{2}(\Delta - i\frac{\Gamma}{2})}{\Delta^2 + \left(\frac{\Gamma}{2}\right)^2 + \frac{\Omega^2}{2}} \end{aligned} \tag{7.6}$$

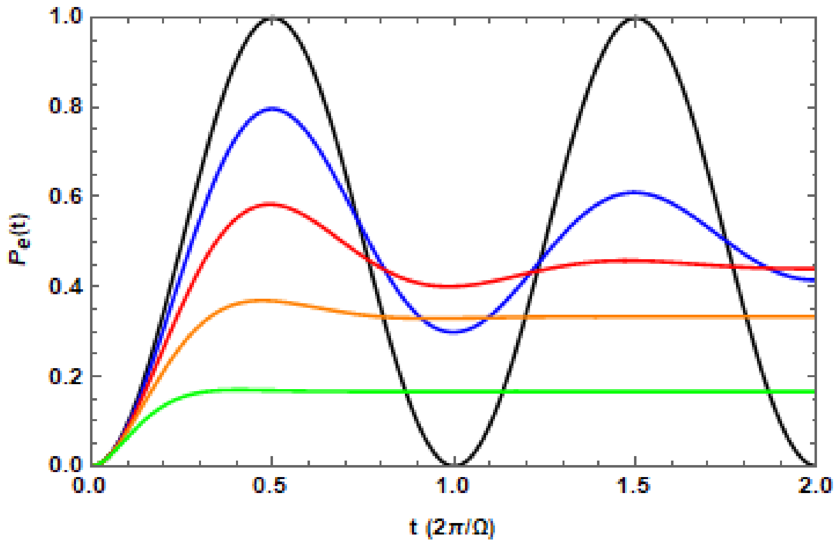



Figure 7.3: Probability to find an atom in the excited state as a function of interaction time $t$. Now we include a decay rate $\Gamma$ in the Rabi model: no damping ($\Gamma = 0$, black curve), $\Gamma = 0.1\Omega$ (blue curve), $\Gamma = 0.25\Omega$ (red curve), $\Gamma = 0.5\Omega$ (orange curve) and $\Gamma = \Omega$ (green curve).

with the saturation parameter:

$$S_0 = \frac{2\Omega^2}{\Gamma^2} = \frac{I}{I_{sat}}$$

$$\Rightarrow \Omega = \Gamma\sqrt{\frac{I}{2I_{sat}}} \qquad \text{and} \qquad I_{sat} = \frac{\pi h c \Gamma}{3\lambda^3}$$

$\rightarrow$ scattering rate for an atom:

$$R = \tilde{\rho}_{22}\Gamma = \frac{\Gamma}{2}\frac{S_0}{1 + S_0 + 4\left(\frac{\Delta}{\Gamma}\right)^2}$$

**Summary of Chapter 7.2**

- dipole transitions dominate atom-light interaction (followed by electric quadrupole and magnetic dipole transition)
- perturbation theory approach: time dependent superposition of atomic wave functions
- in case of near resonant light: rotating wave approximation (RWA)
- Rabi oscillations in 2-level system with $\Omega = \langle e|\,\mathbf{r}\,|g\rangle \cdot \frac{eE_0}{\hbar}$
- Bloch sphere is a geometric representation of states of a 2-level system

- use density matrix formalism to obtain the optical Bloch equations(OBEs)
  $\Rightarrow$ spontaneous emission can be included in a simple way
  $\Rightarrow$ OBEs also apply to multi-level atomic systems

## 7.3 The Jaynes-Cummings Model

### 7.3.1 Interaction with Quantized States of the Electro-Magnetic Field

QO part 1: photons, quantized light fields; $\hat{a}, \hat{a}^\dagger$
QO part 2: so far: atoms quantized; $\hat{\sigma}_+$, $\hat{\sigma}_-$
$\rightarrow$ now: both atoms + field are treated quantum mechanically ; $\hat{a}, \hat{a}^\dagger \leftrightarrow \hat{\sigma}_+, \hat{\sigma}_-$:

- <u>Field operator of a field mode in free space:</u>

$$\mathbf{E} = \left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2} \mathcal{E}[\hat{a} - \hat{a}^\dagger]$$

  with polarization vector $\mathcal{E}$, photon energy $\hbar\omega$, mode volume $V$ and annihilation operator $\hat{a}$. in dipole approximation + Schrödinger picture

- <u>Free Hamiltonian:</u>

$$\hat{H}_0 = \hat{H}_{atom} + \hat{H}_{field}$$

  with $\hat{H}_{field} = \hbar\omega\hat{a}^\dagger\hat{a}$ without zero point energy!

- <u>Interaction:</u>

$$\hat{H}_i = -\mathbf{dE} = -\mathbf{d}\cdot\mathbf{E}_0(\hat{a} - \hat{a}^\dagger) \tag{7.7}$$

  with $\mathbf{E}_0 = \left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2} \mathcal{E}$

- <u>Eigenstates of combined system</u>
  $\Rightarrow$ product states of both system

- Initial state

$$\begin{aligned} |i\rangle = |a\rangle|n\rangle = |a\rangle \otimes |n\rangle \quad &, \qquad E_i = E_a + n\hbar\omega \\ \text{absorption} \rightarrow |f_1\rangle = |b\rangle|n-1\rangle \quad &, \qquad E_{f1} = E_b + (n-1)\hbar\omega \\ \text{emission} \rightarrow |f_2\rangle = |b\rangle|n+1\rangle \quad &, \qquad E_{f2} = E_b + (n+1)\hbar\omega \end{aligned}$$

- <u>How does $\hat{H}_i$ act on these states?</u>

$$\begin{aligned} \hat{H}_i|i\rangle &= -\hat{\mathbf{d}}\mathbf{E}_0(\hat{a} - \hat{a}^\dagger)|i\rangle = -\hat{\mathbf{d}}\mathbf{E}_0(\hat{a} - \hat{a}^\dagger)|a\rangle|n\rangle \\ &= -\hat{\mathbf{d}}\mathbf{E}_0|a\rangle\left(\sqrt{n}|n-1\rangle - \sqrt{n+1}|n+1\rangle\right) \\ \langle f_1|\hat{H}_i|i\rangle &= -\langle n-1|\langle b| - \hat{\mathbf{d}}\mathbf{E}_0|a\rangle\left(\sqrt{n}|n-1\rangle - \sqrt{n+1}|n+1\rangle\right) \\ &= -\langle b|\hat{\mathbf{d}}\mathbf{E}_0|a\rangle \cdot \sqrt{n} \qquad\qquad \text{absorption} \\ \langle f_2|\hat{H}_i|i\rangle &= \langle b|\hat{\mathbf{d}}\mathbf{E}_0|a\rangle\sqrt{n+1} \qquad \text{emission} \end{aligned}$$

  $\Rightarrow$ we have (spontaneous) emission also for $n = 0$!
  $\Rightarrow$ has no classical counterpart

- Ratio of transition rates:

$$\frac{|\langle f_2|\hat{H}_i|i\rangle|^2}{|\langle f_1|\hat{H}_i|i\rangle|^2} = \frac{n+1}{n} = \frac{p_{emission}}{p_{absorption}} > 1$$

$\Rightarrow$ no inversion by strong pumping

### 7.3.2 The Jaynes-Cummings Hamiltonian

JC-model = quantum electrodynamic version of the Rabi model
Assumption: single field mode, e.g. $\mu$-wave or optical resonator

- field operator of a back- and forth propagating field in $z$-direction

$$\mathbf{E} = \mathcal{E}\sqrt{\frac{\hbar\omega}{\epsilon_0 V}}(\hat{a}+\hat{a}^\dagger)\sin(kz)$$

- interaction with atom:

$$\hat{H}_i = -\hat{\mathbf{d}}\hat{\mathbf{E}} = \hat{d}E_0(\hat{a}+\hat{a}^\dagger) \tag{7.8}$$

with $\hat{d} = \hat{\mathbf{d}}\cdot\mathcal{E}$ and $E_0 = -\sqrt{\frac{\hbar\omega}{\epsilon_0 V}}\sin kz$

- dipole operator $\hat{d}$:

$$\hat{d} = \left(\sum_n |n\rangle\langle n|\right)\hat{d}\left(\sum_k |k\rangle\langle k|\right)$$

$$\overset{\text{for 2-level atom}}{\Rightarrow} \hat{d} = d\,|g\rangle\langle e| + d^*\,|e\rangle\langle g|$$

with $d = \langle g|\hat{d}|e\rangle$

$$\Rightarrow \hat{d} = d(\hat{\sigma}_+ + \hat{\sigma}_-) \tag{7.9}$$

without loss of generality $d \in \mathbb{R}$.

- Combining equation (7.8) with equation (7.9) the interaction Hamiltonian reads

$$\hat{H}_i = \hbar g(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a}+\hat{a}^\dagger)$$

with $g = \frac{dE_0}{\hbar}$ $\Rightarrow$ analogon of Rabi frequency $\Omega$.

- Define zero point energy between $E_e$ and $E_g$ and obtain atomic Hamiltonian:

$$\hat{H}_A = \frac{1}{2}(E_e - E_g)\hat{\sigma}_3 = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3$$

- the field Hamiltonian (without zero point energy) is

$$\hat{H}_F = \hbar\omega\hat{a}^\dagger\hat{a}$$

- with this, we write the full Hamiltonian as

$$\begin{aligned}\Rightarrow \hat{H} &= \hat{H}_A + \hat{H}_F + \hat{H}_i \\ &= \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\hat{a}^\dagger\hat{a} + \hbar g(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} + \hat{a}^\dagger)\end{aligned}$$

- now consider the time evolution of the operators

$$\left.\begin{aligned}\hat{a}(t) &= \hat{a}(0)e^{-i\omega t} \\ \hat{a}^\dagger(t) &= \hat{a}^\dagger(0)e^{i\omega t} \\ \hat{\sigma}_+(t) &= \hat{\sigma}_+(0)e^{i\omega_0 t} \\ \hat{\sigma}_-(t) &= \hat{\sigma}_-(0)e^{-i\omega_0 t}\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}\hat{\sigma}_+\hat{a} &\sim e^{i(\omega_0-\omega)t} \\ \hat{\sigma}_-\hat{a}^\dagger &\sim e^{i(-\omega_0+\omega)t} \\ \hat{\sigma}_+\hat{a}^\dagger &\sim e^{i(\omega_0+\omega)t} \\ \hat{\sigma}_-\hat{a} &\sim e^{i(-\omega_0-\omega)t}\end{aligned}\right. \tag{7.10}$$

From Eq.(7.10) one can see that the terms $\hat{\sigma}_-\hat{a}$ and $\hat{\sigma}_+\hat{a}^\dagger$, which violate the conservation of energy (and are therefore only for short times allowed), are the fast rotating terms.

- By omitting the fast rotating terms in Eq.(7.10) we get the
**Jaynes-Cumming Hamiltonian**:

$$\boxed{\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger)} \tag{7.11}$$

- Constants of motion $\leftrightarrow$ conserved quantities, if $[\hat{H}, \hat{A}] = 0$!

  (a) conservation of "electrons":

$$\begin{aligned}\hat{P}_E &= |e\rangle\langle e| + |g\rangle\langle g| = 1 \\ [\hat{H}, \hat{P}_E] &= 0\end{aligned}$$

  (b) conservation of excitations

$$\begin{aligned}\hat{N}_e &= \hat{a}^\dagger\hat{a} + |e\rangle\langle e| \\ [\hat{H}, \hat{N}_e] &= 0\end{aligned}$$

$\Rightarrow$ split JC-Hamiltonian in Eq. (7.11) into 2 commuting parts:

$$\begin{aligned}\hat{H} &= \hat{H}_{\text{I}} + \hat{H}_{\text{II}} \qquad \text{with} \\ \hat{H}_{\text{I}} &= \hbar\omega\hat{N}_e + \hbar(\frac{\omega_0}{2} - \omega)\hat{P}_E \\ \hat{H}_{\text{II}} &= -\hbar\Delta\,|g\rangle\langle g| + \hbar g(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger) \qquad \text{with} \quad \Delta = \omega_0 - \omega\end{aligned}$$

- Notice:
  $\hat{H}_{\mathrm{I}}$ contains only conserved quantities.
  $\Rightarrow \hat{H}_{\mathrm{I}}$ gives only rise to a phase factor
  $\Rightarrow \hat{H}_{\mathrm{II}}$ contains the dynamics!

### 7.3.3 Quantized Rabi Oscillations

- Example:

$$
\begin{aligned}
\Delta &= 0 \\
|i\rangle &= |e\rangle\,|n\rangle \qquad &&\text{with} \quad E_i = \frac{1}{2}\hbar\omega_0 + n\hbar\omega \\
|f\rangle &= |g\rangle\,|n+1\rangle \qquad &&\text{with} \quad E_f = -\frac{1}{2}\hbar\omega_0 + (n+1)\hbar\omega
\end{aligned}
$$

Ansatz:

$$
|\psi(t)\rangle = c_i(t)\,|i\rangle + c_f(t)\,|f\rangle \qquad \text{, with} \quad c_i(0) = 1, c_f(0) = 0
$$

in interaction picture:

$$
\begin{aligned}
i\hbar\partial_t\,|\psi(t)\rangle =& \hat{H}_{\mathrm{II}}\,|\psi(t)\rangle \\
=& \hbar g(\hat{\sigma}_+\hat{a} + \hat{\sigma}_-\hat{a}^\dagger)(c_i(t)\,|i\rangle + c_f(t)\,|f\rangle)
\end{aligned}
$$

with this obtain DE:

$$
\left.\begin{aligned}
\dot{c}_i(t) = &\quad -ig\sqrt{n+1}c_f(t) \\
\dot{c}_f(t) = &\quad -ig\sqrt{n+1}c_i(t)
\end{aligned}\right\} \rightarrow \ddot{c}_i(t) + g^2(n+1)c_i(t) = 0
$$

Solution:

$$
\begin{aligned}
c_i(t) =& \cos\bigl(\sqrt{n+1}gt\bigr) \qquad \text{, for} \quad c_i(0) = 1, c_f(0) = 0 \\
c_f(t) =& -i\sin\bigl(\sqrt{n+1}gt\bigr) \\
|\psi(t)\rangle =& \cos\bigl(\sqrt{n+1}gt\bigr)\,|e\rangle\,|n\rangle - i\sin\bigl(\sqrt{n+1}gt\bigr)\,|g\rangle\,|n+1\rangle
\end{aligned}
$$

Atomic inversion is given by:

$$
\begin{aligned}
w(t) = \langle\psi(t)|\,\hat{\sigma}_3\,|\psi(t)\rangle =& \langle\psi(t)|\,(|e\rangle\langle e| - |g\rangle\langle g|)\,|\psi(t)\rangle \\
=& \cos^2(gt\sqrt{n+1}) - \sin^2(gt\sqrt{n+1}) \\
=& \cos\bigl(2gt\sqrt{n+1}\bigr) \\
=& \cos(\Omega_n t) \qquad \text{with} \quad \Omega_n = 2g\sqrt{n+1}
\end{aligned}
$$

- Remarks:
  - Quantum electrodynamical **Rabi frequency** $\Omega_n = 2g\sqrt{n+1}$

- Rabi-frequency increases in a quantized way with $\sqrt{n} \sim \sqrt{I_{Laser}}$ for $n \gg 1$.
- Rabi frequency $> 0$ for $n = 0$!
  $\rightarrow$ vacuum Rabi oscillations: example for reversible spontaneous emission!
- $|\psi(t)\rangle$ is an entangled state
- Rabi oscillation of a Fock state corresponds to semi-classical case, even though Fock state is non-classical!

### 7.3.4 Rabi Oscillation of a Coherent State

let's extrapolate to the case of an arbitrary superposition of Fock-states (thermal, coherent, squeezed, etc ...)

- Ansatz:

$$\begin{aligned}
|\psi(0)\rangle_{field} =& \sum_{n=0}^{\infty} c_n |n\rangle \\
|\psi(0)\rangle_{atom} =& c_g |g\rangle + c_e |e\rangle \\
|\psi(0)\rangle_{total} =& |\psi(0)\rangle_{atom} \otimes |\psi(0)\rangle_{field} \\
=& \sum_{n=0}^{\infty} (c_n c_g |g\rangle |n\rangle) + \sum_{n=0}^{\infty} (c_n c_e |e\rangle |n\rangle) \\
=& \sum_{n=0}^{\infty} \left( c_{n+1} c_g \underbrace{|g\rangle |n+1\rangle}_{|f\rangle} + c_n c_e \underbrace{|e\rangle |n\rangle}_{|i\rangle} \right) + c_0 c_g |g\rangle |0\rangle \\
\Rightarrow |\psi(t)\rangle_{total} =& \sum_{n=0}^{\infty} c_{n+1} c_g \left( -i \sin\!\big(gt\sqrt{n+1}\big) |e\rangle |n\rangle + \cos\!\big(gt\sqrt{n+1}\big) |g\rangle |n+1\rangle \right) \\
&+ \sum_{n=0}^{\infty} c_n c_e \left( \cos\!\big(gt\sqrt{n+1}\big) |e\rangle |n\rangle - i \sin\!\big(gt\sqrt{n+1}\big) |g\rangle |n+1\rangle \right) \\
&+ c_0 c_g |g\rangle |0\rangle
\end{aligned}$$

now change index $n+1 \rightarrow n$ in second part of summation and include the $|g\rangle |0\rangle$ term:

$$\begin{aligned}
|\psi(t)\rangle_{total} =& \sum_{n=0}^{\infty} \left( -i \sin\!\big(gt\sqrt{n+1}\big) c_{n+1} c_g + \cos\!\big(gt\sqrt{n+1}\big) c_n c_e \right) |e\rangle |n\rangle \\
&+ \sum_{n=0}^{\infty} \left( \cos\!\big(gt\sqrt{n}\big) c_n c_g - i \sin\!\big(gt\sqrt{n}\big) c_{n-1} c_e \right) |g\rangle |n\rangle
\end{aligned}$$

in which $c_{-1} = 0$
$\Rightarrow$ in general: $|\psi(t)\rangle$ is an entangled state!

- Simplify: $c_e = 1, c_g = 0$ (as before)

$$\Rightarrow |\psi(t)\rangle_{total} = |\psi_g(t)\rangle\,|g\rangle + |\psi_e(t)\rangle\,|e\rangle \qquad \text{with}$$

$$\begin{aligned}|\psi_g(t)\rangle &= -\,i\sum_{n=0}^{\infty}\sin\big(gt\sqrt{n}\big)c_{n-1}\,|n\rangle\\ &= -\,i\sum_{n=0}^{\infty}\sin\big(gt\sqrt{n+1}\big)c_n\,|n+1\rangle\\ |\psi_e(t)\rangle &= \sum_{n=0}^{\infty}\cos\big(gt\sqrt{n+1}\big)c_n\,|n\rangle\end{aligned}$$

- Atomic inversion:

$$\begin{aligned}w(t) &= \langle\psi(t)|\,\hat{\sigma}_3\,|\psi(t)\rangle = \langle\psi_e(t)|\psi_e(t)\rangle - \langle\psi_g(t)|\psi_g(t)\rangle\\ w(t) &= \sum_{n=0}^{\infty}|c_n|^2\cos\big(2gt\sqrt{n+1}\big)\end{aligned}$$

  $\rightarrow$ Oscillation consists of initial occupations, oscillating with slightly different frequencies

- Example: coherent state

$$\begin{aligned}c_n &= e^{-\frac{|\alpha|^2}{2}}\frac{\alpha^n}{\sqrt{n!}} \qquad \text{with} \quad |\alpha|^2 = \bar{n}\\ w(t) &= e^{-\bar{n}}\sum_{n=0}^{\infty}\frac{\bar{n}^n}{n!}\cos\big(2gt\sqrt{n+1}\big)\end{aligned}$$

  *How can we deduce $c_n$'s from experiment?*
  $\rightarrow$ perform Fourier transformation of signal

### Collapse and Revival

- Collapse
  - dominant Rabi frequency: $\Omega(\bar{n}) \approx 2g\sqrt{\bar{n}}$, $n \gg 1$.
  - which Fock states contribute?
    $\rightarrow n = \bar{n} \pm \Delta n \Rightarrow$ coherent state: $\Delta n \approx \sqrt{\bar{n}}$.
  - time and frequency uncertainty:

$$\begin{aligned}t_c \cdot \Delta\Omega &\sim 1\\ t_c(\Omega(\bar{n}-\Delta n) - \Omega(\bar{n}+\Delta n)) &\sim 1\end{aligned}$$

with

$$
\begin{aligned}
\Omega(\bar{n} \pm \Delta n) \approx & 2g\sqrt{\bar{n} \pm \sqrt{\bar{n}}} \\
& \vdots \\
\approx & 2g\sqrt{\bar{n}} \pm g \\
\rightarrow \Delta\Omega \approx & 2g
\end{aligned}
$$

$\Rightarrow t_c \approx \frac{1}{2g}$ independent of $\bar{n}$.
As $\Omega_n \sim \sqrt{n} \Rightarrow$ number of unperturbed oscillations increases with $\sqrt{n}$
$\Rightarrow$ transition to classical regime!

- Revivals:
  - neighboring frequencies dephase with $2\pi k$, $k \in \mathrm{N}$

$$
(\Omega(\bar{n}+1) - \Omega(\bar{n}))t_R = 2\pi k \rightarrow t_R \sim 4\pi\sqrt{\bar{n}}t_c \tag{7.12}
$$

  - as photon number is discrete $\Rightarrow$ oscillations revive!
    $\Rightarrow$ signature of quantum electrodynamics!
    $\Rightarrow$ individual field quanta govern atomic evolution
- Applications:
  - CQED: single atom interacting with single photon in resonator
  - trapped ions in trap

#### Summary of 7.3.1 - 7.3.4

- quantized field and atom
- 4 processes, 2 remain after rotating wave approximation

$$
|e\rangle\,|n\rangle \leftrightarrow |g\rangle\,|n+1\rangle
$$

  $\Rightarrow$ Jaynes-Cummings Hamiltonian.
- transition rate: $|e\rangle\,|n\rangle \rightarrow |g\rangle\,|n+1\rangle$, also works for $n = 0 \Rightarrow$ spontaneous and induced emission.
- spontaneous emission occurs via interaction with various modes of the corresponding frequency
- ansatz in the interaction picture with time dependent coefficients $\Rightarrow$ QED Rabi flops on resonance for Fock states
- for the case of general fields one obtains a sum of different Rabi frequencies
  $\rightarrow$ dephasing
- $|c_n|^2$ of Fock states can be determined by Fourier analysis.

### 7.3.5 Dressed States

Alternative way to calculate the dynamics under the $\hat{H}_i$

- so far: solution of the time dependent Schrödinger equation
  - first for Fock state
  - then extension to arbitrary superposition
- now: first find stationary solutions and then look at time evolution.
- Starting point:
$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)$$
  $\hat{H}_i$ couples:
$$|e\rangle|n\rangle \leftrightarrow |g\rangle|n+1\rangle$$
$$|e\rangle|n-1\rangle \leftrightarrow |g\rangle|n\rangle$$
  these product states are "bare states", i.e. eigenstates of field and atom without interaction!
- use the following basis:
$$\begin{aligned} |i\rangle &= |\psi_{1n}\rangle = |e\rangle|n\rangle \\ |f\rangle &= |\psi_{2n}\rangle = |g\rangle|n+1\rangle \\ \Rightarrow \langle\psi_{1n}|\psi_{2n}\rangle &= 0 \end{aligned}$$
  now write the matrix elements of the JC-Hamiltonian:
$$\begin{aligned} H_{ij}^{(n)} &= \langle\psi_{in}|\hat{H}|\psi_{jn}\rangle \\ H_{11}^{(n)} &= \hbar(n\omega + \frac{1}{2}\omega_0) \\ H_{22}^{(n)} &= \hbar((n+1)\omega - \frac{1}{2}\omega_0) \\ H_{12}^{(n)} &= \hbar g\sqrt{n+1} = H_{21}^{(n)} \\ \Rightarrow \hat{H}^{(n)} &= \hbar\begin{pmatrix} n\omega + \frac{1}{2}\omega_0 & g\sqrt{n+1} \\ g\sqrt{n+1} & (n+1)\omega - \frac{1}{2}\omega_0 \end{pmatrix} \end{aligned}$$
  $\Rightarrow$ no other states but $|\psi_{1n}\rangle$, $|\psi_{2n}\rangle$ are occupied!
- Energy eigenvalues of $H^{(n)}$:
$$E_\pm(n) = (n+\frac{1}{2})\hbar\omega \pm \frac{1}{2}\hbar\Omega_n(\Delta)$$
  with $\Omega_n(\Delta) = \sqrt{\Delta^2 + 4g^2(n+1)}$, $\Delta = \omega_0 - \omega$

- <u>Eigenvectors (states) of $H^{(n)}$ $\Rightarrow$ "dressed states":</u>

$$\begin{aligned} |n,+\rangle &= \cos\left(\frac{\Theta_n}{2}\right)|\psi_{1n}\rangle + \sin\left(\frac{\Theta_n}{2}\right)|\psi_{2n}\rangle \\ |n,-\rangle &= -\sin\left(\frac{\Theta_n}{2}\right)|\psi_{1n}\rangle + \cos\left(\frac{\Theta_n}{2}\right)|\psi_{2n}\rangle \end{aligned}$$

with

$$\left.\begin{aligned} \sin\tfrac{\Theta}{2} &= \tfrac{1}{\sqrt{2}}\sqrt{\tfrac{\Omega_n(\Delta)-\Delta}{\Omega_n(\Delta)}} \\ \cos\tfrac{\Theta}{2} &= \tfrac{1}{\sqrt{2}}\sqrt{\tfrac{\Omega_n(\Delta)+\Delta}{\Omega_n(\Delta)}} \end{aligned}\right\} \Rightarrow \Theta_n = \arctan\left(\frac{2g\sqrt{n+1}}{\Delta}\right) = \arctan\left(\frac{\Omega_n(0)}{\Delta}\right)$$

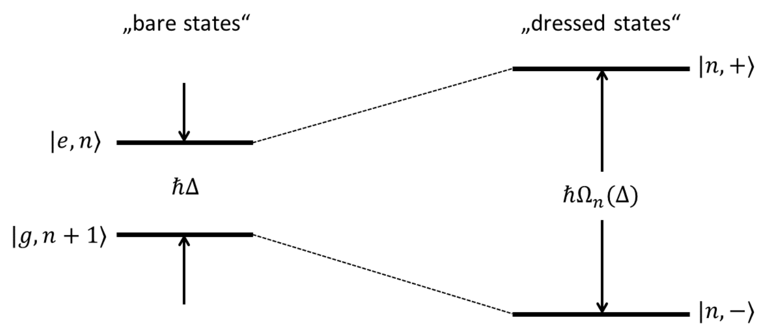



Figure 7.4: Energies of the bare states vs. the dressed states. The latter are a superposition of the bare states and are separated by $\hbar\Omega_n(\Delta)$.

### Remarks

- light field couples the two "bare states"
- the two resulting "dressed states" consist of a linear combination of "bare states" and inherent their properties, e.g. with respect to spontaneous emission
- general concept $\rightarrow$ coupling leads to energy shifts, also called "AC-Stark Shifts"!
- <u>Example:</u> on resonance, $\Delta = 0$:

$$\begin{aligned} |n,+\rangle &= \frac{1}{\sqrt{2}}(|e\rangle|n\rangle + |g\rangle|n+1\rangle) \\ |n,-\rangle &= \frac{1}{\sqrt{2}}(-|e\rangle|n\rangle + |g\rangle|n+1\rangle) \end{aligned}$$

### Dynamics of an arbitrary field state

$$|\psi_f(0)\rangle = \sum_n c_n |n\rangle$$

and atom in excited state $|e\rangle$

$$\Rightarrow |\psi_{af}(0)\rangle = |\psi_f(0)\rangle \otimes |e\rangle = \sum_n c_n |\psi_{1n}\rangle$$

now write $|\psi_{1n}\rangle$ in terms of “dressed states”:

$$|\psi_{1n}\rangle = \cos\left(\frac{\Theta_n}{2}\right) |n,+\rangle - \sin\left(\frac{\Theta_n}{2}\right) |n,-\rangle$$

$\rightarrow$ as $|n,+\rangle$ and $|n,-\rangle$ are eigenstates, there is only a trivial time evolution.

$$\begin{aligned}|\psi_{af}(t)\rangle =& e^{-it\hat{H}/\hbar} |\psi_{af}(0)\rangle \\ =& \sum_n c_n \left(\cos\left(\frac{\Theta_n}{2}\right) |n,+\rangle e^{-iE_+(n)t/\hbar} - \sin\left(\frac{\Theta_n}{2}\right) |n,-\rangle e^{-iE_-(n)t/\hbar}\right)\end{aligned}$$

$\rightarrow$ now transfer back into “bare states”!

### Summary/Discussion of 7.3.5

- “dressed states” are eigenstates of the JC-Hamiltonian
- they possess properties of both states
- AC-Stark Shift

$$\Rightarrow E_\pm(n) = \left(n + \frac{1}{2}\right) \hbar\omega \pm \hbar \left(\frac{\Delta}{2} + \frac{\Omega_n^2(0)}{4\Delta}\right)$$

  for $\frac{\Omega_n(0)}{\Delta} \ll 1$
- “dressed states” explain Mollow triplet

# 8 Experiments with Atoms and Photons

- Applications of the developed theoretical concepts
- Selected examples
  → Laser cooling and trapping
  → Trapped ions

## 8.1 Resonance Fluorescence

→ slides

## 8.2 Laser Cooling and Optical Trapping

### 8.2.1 Introduction

→ slides

### 8.2.2 Spontaneous and Dipole Forces on free Atom

We consider the most simple system, how optical forces can be exerted onto atoms.
→ monochromatic light and two-level atom.
To calculate the light forces, we use the quantum mechanical version of Newton's law:
→ the Ehrenfest theorem says that expectation value of a quantum mechanical operator behaves as its classical counterpart

$$\begin{aligned} \mathbf{F} =& \langle\hat{\mathbf{F}}\rangle = \frac{d}{dt}\langle\hat{\mathbf{p}}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{\mathbf{p}}]\rangle \\ \text{with} \qquad [\hat{H},-i\hbar\nabla]\psi =& -i\hbar\hat{H}(\nabla\psi) + i\hbar(\nabla\hat{H})\psi + i\hbar\hat{H}(\nabla\psi) = i\hbar(\nabla\hat{H})\psi \\ \mathbf{F} \text{ can be written as} \qquad \mathbf{F} =& -\langle\nabla\hat{H}\rangle = -\nabla\langle\hat{H}\rangle \end{aligned} \tag{8.1}$$

For the last step of equation (8.1) the following assumption are made

1. the dipole approximation
2. that for hot atoms the size of the atomic wavepacket $\sigma_\psi \ll \lambda$

Which can be interpreted as conditions for the validity of equation (8.1).
Force of an electric field onto atom (semi-classical):

$$\begin{aligned} \hat{H}_\mathrm{I} =& e\hat{\mathbf{r}}\mathbf{E}(\mathbf{r},t) = -\hat{\mathbf{d}}\mathbf{E}(\mathbf{r},t) \\ \mathbf{F} =& \nabla(\langle\mathbf{E}(\mathbf{r},t)\hat{\mathbf{d}}\rangle) \end{aligned}$$

now calculate expectation value <u>in interaction picture:</u> $|\psi_e\rangle = e^{-i\omega_0 t}|e\rangle$

$$\begin{aligned}\Rightarrow \hat{\mathbf{d}} =& \mathbf{d}_{ge} e^{-i\omega_0 t}|g\rangle\langle e| + \mathbf{d}_{ge}^* e^{i\omega_0 t}|e\rangle\langle g| \\ \langle\hat{\mathbf{d}}\rangle =& \mathrm{Tr}\Big(\hat{\rho}\hat{\mathbf{d}}\Big) = \rho_{ge}\mathbf{d}_{ge}^* e^{i\omega_0 t} + \rho_{ge}^*\mathbf{d}_{ge}e^{-i\omega_0 t}, \qquad as \qquad \rho_{eg} = \rho_{ge}^*\end{aligned}$$

with the light field: $\mathbf{E}(\mathbf{r},t) = \frac{1}{2}\mathbf{E}_0(\mathbf{r})\left(e^{i(\phi(\mathbf{r})-\omega_L t)} + c.c\right)$
the force becomes:

$$\Rightarrow \mathbf{F} \approx \nabla\left(\frac{\mathbf{E}_0(\mathbf{r})}{2}\left(\rho_{ge}\mathbf{d}_{ge}^* e^{i(\phi(\mathbf{r})+\Delta t)} + c.c.\right)\right)$$

with RWA, and $\Delta = \omega_0 - \omega_L$

$$\mathbf{F} = \hbar\nabla\left(\tilde{\rho}_{ge}\Omega(\mathbf{r}) + c.c.\right) \tag{8.2}$$

with $\tilde{\rho}_{ge} = e^{i\Delta t}\rho_{ge}$ and $\Omega(\mathbf{r}) = \frac{\mathbf{d}_{ge}^*\mathbf{E}_0(\mathbf{r})e^{i\phi(\mathbf{r})}}{2\hbar}$

$$\mathbf{F} = \hbar\left(\tilde{\rho}_{ge}\nabla(\Omega(\mathbf{r})) + \tilde{\rho}_{ge}^*\nabla(\Omega^*(\mathbf{r}))\right)$$

let's write $\nabla\Omega$ as:

$$\begin{aligned}\nabla\Omega(\mathbf{r}) =& (\mathbf{q}_r + i\mathbf{q}_i)\cdot\Omega(\mathbf{r}) \quad \text{, remember: } (\ln f)' = \frac{f'}{f} \\ \text{e.g. with} \quad \Omega(\mathbf{r}) =& |\Omega_0(\mathbf{r})|e^{i\phi(\mathbf{r})} \\ \rightarrow \mathbf{q}_r =& \nabla\ln|\Omega(\mathbf{r})| = \frac{\nabla|\Omega(\mathbf{r})|}{|\Omega(\mathbf{r})|} \\ \mathbf{q}_i =& \nabla\phi(\mathbf{r}) \\ \mathbf{F} =& \underbrace{\hbar\mathbf{q}_r(\Omega\tilde{\rho}_{ge} + \Omega^*\tilde{\rho}_{ge}^*)}_{\text{dipole force}} + \underbrace{i\hbar\mathbf{q}_i(\Omega\tilde{\rho}_{ge} - \Omega^*\tilde{\rho}_{ge}^*)}_{\text{spontaneous force}} \\ \mathbf{F} =& \mathbf{F}_{dip} + \mathbf{F}_{sp} \quad \Rightarrow \quad \mathbf{F} \text{ is real!}\end{aligned}$$

**Stationary solution for $\tilde{\rho}_{ge}$ (from OBEs)** for an atom at rest! $(v = 0)$

$$\begin{aligned}\tilde{\rho}_{ge}^{(s)} =& -\frac{\Omega}{2}\frac{(\Delta - i\frac{\Gamma}{2})}{\Delta^2 + \frac{\Gamma^2}{4} + \frac{|\Omega|^2}{2}} \qquad \text{Equation (7.6)} \\ \Rightarrow \mathbf{F} =& \hbar\frac{s}{1+s}(-\Delta\mathbf{q}_r + \frac{\Gamma}{2}\mathbf{q}_i), \qquad \text{with} \quad s = \frac{\frac{2\Omega^2}{\Gamma^2}}{1+\left(\frac{2\Delta}{\Gamma}\right)^2} \\ \text{or} \quad \frac{s}{1+s} =& \frac{s_0}{1+s_0+\left(\frac{2\Delta}{\Gamma}\right)^2} \quad \text{and} \quad s_0 = \frac{2\Omega_0^2}{\Gamma^2}\end{aligned}$$

**Example A: Traveling Wave**

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \mathbf{E}_0 \cos(kz - \omega_L t) \\ \Rightarrow |\Omega(\mathbf{r})| = \Omega_0 = const \Rightarrow \mathbf{q}_r = 0 \Rightarrow \mathbf{F}_{dip} &= 0 \\ \phi(\mathbf{r}) &= kz \\ \Rightarrow \mathbf{q}_i = k\mathbf{e}_z &= \nabla\phi(\mathbf{r}) \\ \Rightarrow \mathbf{F}_{sp} = \mathbf{e}_z \hbar k \underbrace{\frac{\Gamma}{2} \frac{s_0}{1 + s_0 + \left(\frac{2\Delta}{\Gamma}\right)^2}}_{=R=\rho_{ee}\Gamma} &= \mathbf{e}_z \hbar k R \end{aligned}$$

→ spontaneous force = momentum × scattering rate!

$$\mathbf{F}_{sp} = \hbar\mathbf{k}\Gamma\rho_{ee} \tag{8.3}$$

→ for high intensities $s_0 \to \infty$, $\mathbf{F}_{sp}$ saturates at

$$\mathbf{F}_{sp}^{(max)} = \hbar\mathbf{k}\frac{\Gamma}{2}$$

**Discussion:**

1. saturates
2. dissipative, as relies on photons scattering out of the laser beam
   → non reversible process! spontaneous emission!
3. standing wave → $q_i = 0$! average force=0
4. important role in the slowing of atoms and cooling

**Example B: Standing Wave**

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= 2\mathbf{E}_0 \cos(kz) \sin(\omega_L t) \\ |\Omega(\mathbf{r})| &= 2\Omega_0 \cos(kz) \\ \phi(\mathbf{r}) = 0 \Rightarrow \mathbf{F}_{sp} &= 0 \\ \mathbf{q}_r = -\mathbf{e}_z k \frac{\sin(kz)}{\cos(kz)} &= -\mathbf{e}_z k \tan(kz) \\ \mathbf{F}_{dip} = \mathbf{e}_z \hbar k \Delta \frac{\frac{2\Omega^2}{\Gamma^2}}{1 + \frac{2\Omega^2}{\Gamma^2} + \left(\frac{2\Delta}{\Gamma}\right)^2} &\tan(kz) \\ \mathbf{e}_z \frac{2\hbar k s_0 \sin(2kz)\Delta}{1 + 4s_0\cos^2(kz) + \left(\frac{2\Delta}{\Gamma}\right)^2} \quad \text{with} \quad s_0 &= \frac{2\Omega_0^2}{\Gamma^2} \end{aligned}$$

maximal force if $|\Delta| \approx |\Omega|$

$$\Rightarrow \mathbf{F}_{dip}^{(max)} \approx \hbar k \Omega$$

$\mathbf{F}_{dip}$ can be written as the gradient of a potential $U_{dip}$ given by

$$U_{dip} = \frac{1}{2}\hbar\Delta \ln\left(\frac{1 + 4s_0\cos^2(kz) + \left(\frac{2\Delta}{\Gamma}\right)^2}{1 + \left(\frac{2\Delta}{\Gamma}\right)^2}\right)$$

for $|\Delta| \gg \Gamma$:

$$U_{dip} \approx \frac{\hbar\Omega^2(\mathbf{r})}{4\Delta} \quad \text{and} \quad \mathbf{F}_{dip} \approx -\nabla\left(\frac{\hbar|\Omega(\mathbf{r})|^2}{4\Delta^2}\right) \tag{8.4}$$

$U_{dip}$ correspond to energy shift in dressed atom picture!

**Summary of subsection 8.2.2**

- $F_{dip}$ and $F_{sp}$ differ fundamentally:
- $F_{sp}$ saturates at high intensities, $F_{dip}$ does not
- $F_{sp}$ vanishes for standing wave, $F_{dip}$ for running wave
- $F_{sp}$ is dissipative and can be used for cooling
- $F_{dip}$ is conservative and can be used to trap; for this the scattering rate must be suppressed:

  $$\text{for} \quad |\Delta| \gg \Gamma: \qquad R \sim \frac{s_0}{\Delta^2}, \quad \mathbf{F}_{dip} \sim \frac{s_0}{\Delta}$$

  $\Rightarrow$ for large detuning dipole force predominates and stray rate can be suppressed.

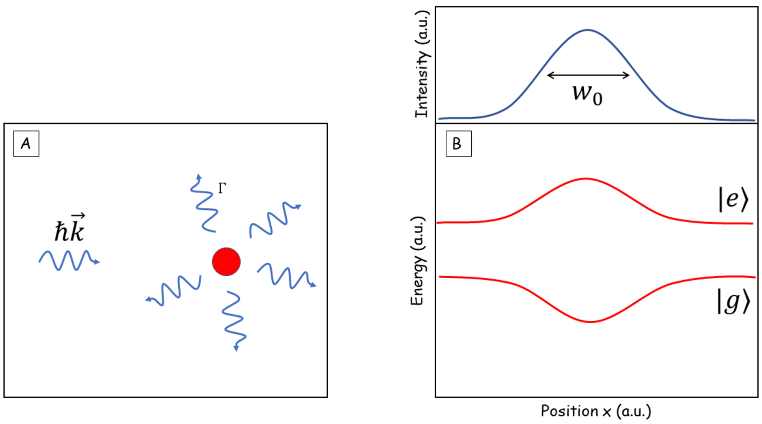



Figure 8.1: Example A: spontaneous light force acting on an atom with decay rate $\Gamma$. Example B: dipole force caused by a laser beam with Gaussian intensity profile.

### 8.2.3 Doppler Cooling

up to here: $v_a = 0$ Now: $v_a \neq 0$

- What is temperature?
  In thermal equilibrium the equipartition theorem applies:
  $$\text{e.g. in 1D:}\quad \frac{1}{2}k_B T = \frac{1}{2}m\langle v_z^2\rangle$$
  Caution: laser cooled atoms are NOT in thermal equilibrium! Different $f(v)$ are possible, having the same $\langle v^2\rangle$.
  Nevertheless, we use the label temperature to describe an atomic sample with average kinetic energy $\langle E_{kin}\rangle$ as
  $$\frac{1}{2}k_B T = \langle E_k\rangle$$
  We use Maxwell-Boltzmann distribution $f(v_z) \sim e^{-\frac{mv_z^2}{2k_BT}}$

- Cooling = reduction of the velocity spread in this distribution $\Rightarrow$ take away energy.

  2 possibilities:
  1. reduce width at constant $\langle v\rangle$
  2. cool to $\langle v\rangle = 0$

  Cooling has to rely on an irreversible process, otherwise cold atoms are reheated.
  $\Rightarrow$ volume of phase space is changed
  $\Rightarrow$ needs dissipative process to redistribute energy

- Energy and Momentum Budget
  1. Absorption:
     conservation of momentum:
     $$\begin{aligned} \hbar\mathbf{k}_1 + m\mathbf{v}_0 &\overset{!}{=} m\mathbf{v} \\ \Rightarrow \Delta\mathbf{v} = \mathbf{v} - \mathbf{v}_0 &= \frac{\hbar k_1}{m} \\ \Rightarrow v_{rec} &= |\Delta\mathbf{v}| \end{aligned}$$
     define recoil energy:
     $$E_{rec} = \frac{1}{2}mv_{rec}^2 = \frac{\hbar^2|\mathbf{k_1}|^2}{2m}$$
     conservation of energy (in 1D):
     $$\hbar\omega_1 + \frac{1}{2}mv_0^2 = \frac{1}{2}mv^2 + \hbar\omega_0$$
     $$\hbar\omega_1 = \underbrace{\frac{\hbar^2 k_1^2}{2m}}_{E_{rec}} + \hbar\omega_0 + \underbrace{v_0\hbar k}_{\hbar\Delta\omega_{Doppler}} \tag{8.5}$$

$\Rightarrow$ for $v_0 = 0$ : energy of absorbed photon is larger by $E_{rec}$ compared to atomic energy $E_0 = \hbar\omega_0$!
In case of moving atoms, the Doppler shift $\Delta\omega_{Doppler} = kv_0$ dominates and must be compensated with detuning of laser.

2. Emission:

$$\hbar\omega_2 = \hbar\omega_0 - \underbrace{\frac{\hbar^2 k_z^2}{2m}}_{\text{recoil shift}} + \underbrace{\hbar k_2 v}_{\text{Doppler shift}}$$

- Frequency of Laser in Frame of Reference of Atom($= \omega'_{atom}$):

$$\omega'_{atom} = \omega_L \left(1 + \frac{\mathbf{kv}}{c|\mathbf{k}|}\right) = \omega + \mathbf{kv}, \qquad \text{as} \quad \omega = c|\mathbf{k}|$$

- Optical Molasses, example (1D):
at weak intensities $(s_0 < 1)$ forces add up:

$$\begin{aligned} F_{mol} =& F_{sp}(\omega_0 - \omega_L + kv) - F_{sp}(\omega_0 - \omega_L - kv) \\ \text{for small } v: \qquad \approx& \frac{\partial F_{sp}}{\partial v} \cdot v = -\alpha v \end{aligned}$$

with friction coefficient $\alpha$:

$$\alpha = -4\hbar k^2 \frac{2\frac{\Delta}{\Gamma} s_0}{\left(1 + \left(\frac{2\Delta}{\Gamma}\right)^2\right)^2}$$

$\rightarrow$ just looking at forces, $v = 0$ would be reached!

- The Doppler cooling limit *Why is T=0 not reached?*
$\Rightarrow$ remaining fluctuations of the velocity of the atom due to discrete absorption and emission processes
$\Rightarrow$ "Random-Walk" of the atom

- mean momentum vanishes $\langle p_z \rangle = 0$.

- mean momentum squared does not: $\langle p_z^2 \rangle = 2N(\hbar k)^2$
after $N \gg 1$ absorption/emission cycles.
With 2 laser beams in 1D:

$$\begin{aligned} N &= 2R \\ \Rightarrow \langle p_z^2 \rangle &= 4\hbar^2 k^2 Rt \end{aligned}$$

Define diffusion coefficient:

$$D = \frac{1}{2} \frac{\partial \langle p_z^2 \rangle}{\partial t} = 2\hbar^2 k^2 R$$

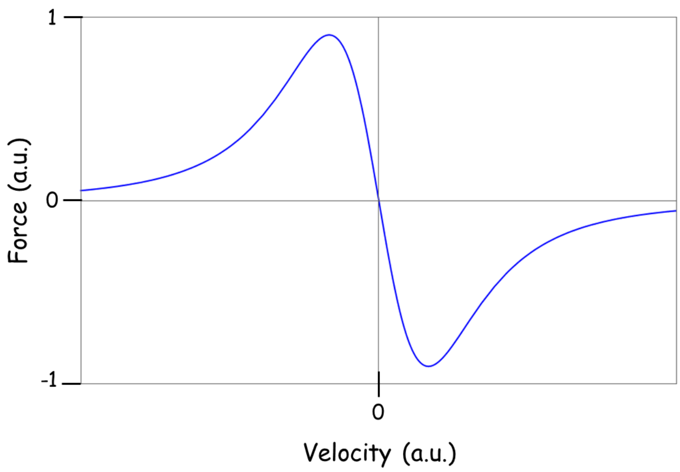



Figure 8.2: Spontaneous force vs. atomic velocity in an optical molasses. The light force is dissipative and is directed opposite to the atomic velocity $v$. Hence, it serves to the reduce the atomic motion.

We express the heating rate in terms of this diffusion coefficient:
Heating Rate:

$$\left(\frac{\partial E}{\partial t}\right)_h = \frac{1}{2m}\frac{\partial \langle p_z^2 \rangle}{\partial t} = \frac{D}{m}$$

While for the cooling rate we have:
Cooling Rate:

$$\left(\frac{\partial E}{\partial t}\right)_c = F_z \cdot v_z = -\alpha v_z^2$$

In equilibrium we have:

$$\left(\frac{\partial E}{\partial t}\right)_h + \left(\frac{\partial E}{\partial t}\right)_c = 0 \Rightarrow v_z^2 = \frac{D}{m\alpha}$$

With the equipartition theorem we write the energy per degree of freedom:

$$\frac{1}{2}k_B T = \frac{1}{2}m\langle v_z^2 \rangle$$

$$\Rightarrow T = \frac{D}{k_B \alpha} = -\frac{\hbar\Gamma}{8k_B}\frac{1+s_0+4\frac{\Delta^2}{\Gamma^2}}{\frac{\Delta}{\Gamma}}$$

$$\text{for } s_0 \ll 1 : T_D = \frac{\hbar\Gamma}{2k_B} \quad \text{at} \qquad \Delta = -\frac{\Gamma}{2} \sim 1\,\text{mK}$$

This is called the **Doppler cooling limit $T_D$!**
or "Doppler temperature"

## 8.3 Cooling and Coherent Manipulation of Trapped Atoms/Ions

### 8.3.1 Paul traps

see slides

### 8.3.2 Laser Cooling of Trapped Ions

- 3 principal axes of harmonic motion in trap (=trap axes)
- 3N eigenmodes of vibration for N ions
- only one laser beam is necessary to cool ions in 3D, if **k**-vector has a projection onto all 3 traps axes
- here we consider 1 dimension
  $\rightarrow$ atom in an harmonic potential with frequency $\omega_{trap}$
- Two Regimes
  1. weak localization ($\Gamma \gg \omega_{trap}$)
     $\Rightarrow$ analogous to Doppler cooling of free atom

     $$\Rightarrow T_D = \frac{\hbar\Gamma}{2k_B} \quad \text{cooling limit}$$

     mean occupation number at $T_D$:

     $$\bar{n} \approx \frac{T_D k_B}{\hbar\omega_{trap}} = \frac{\Gamma}{2\omega_{trap}}$$

     example:

     $$\begin{aligned} {}^{24}Mg^+: \qquad & \Gamma = 2\pi \times 40\,\text{MHz}, \quad \omega_{trap} = 2\pi \times 2\,\text{MHz} \\ & \Rightarrow T_D = 1\,\text{mK}, \quad \bar{n} = 10 \end{aligned}$$

  2. strong localization ($\Gamma \ll \omega_{trap}$)
     $\rightarrow$ sideband cooling

     - requires narrow laser! $\Gamma_{laser} \ll \omega_{trap}$
     - **limit**: non-resonant excitation of carrier(CAR) and blue sideband (BSB)
       $\Rightarrow$ mean occupation number

       $$\bar{n} = \frac{1}{4}\left(\alpha + \frac{1}{4}\right)\frac{\Gamma^2}{\omega_{trap}^2}, \qquad \alpha \approx 1$$

– if a thermal distribution is assumed:

$$T = \frac{\hbar\omega_{trap}}{k_B \ln\left(1 + \frac{1}{\bar{n}}\right)} \approx \text{tens of } \mu K$$

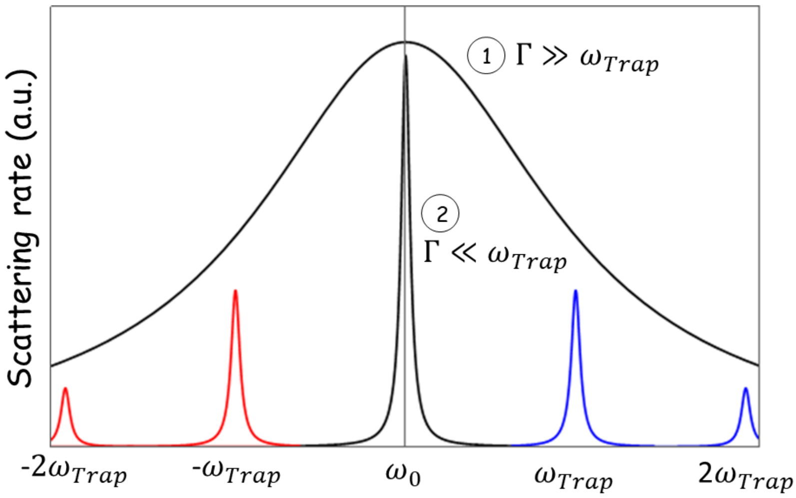



Figure 8.3: Frequency spectrum of an atom with decay rate $\Gamma$, confined in a harmonic potential with frequency $\omega_{\text{Trap}}$. The two curves show the effect of weak and strong localization on the atomic scattering rate.

### 8.3.3 Coherent Manipulation of Ions

2-level atom $\otimes$ harmonic trap + laser(coupling): $E(t,z) = E_0 \cos(kz - \omega_L t + \varphi)$
we work with strong localization, $\omega_z \gg \Gamma$!

- Hamilton operator:

$$\hat{H}_A = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3$$
$$\hat{H}_m = \hbar\omega_z\hat{a}^\dagger\hat{a}$$
$$\hat{H}_\text{I} = \hbar\Omega(\hat{\sigma}_+ + \hat{\sigma}_-)\cos(k\hat{z} - \omega_L t + \varphi)$$

in dipole approximation with $\Omega = \frac{d_{eg}E_0}{\hbar}$ and $\omega_{trap} = \omega_z$

- Re-write $\hat{H}'s$ in Pauli matrices:

$$\hat{H}_A = \frac{1}{2}\hbar\omega_0\hat{\sigma}_z$$
$$\hat{H}_I = \frac{\hbar\Omega}{2}\hat{\sigma}_x(e^{-i(k\hat{z}-\omega_L t+\varphi)} + c.c.)$$

- Transition into Interaction Picture:
(operators have time evolution of unperturbed Hamiltonian)

$$\hat{H} = \underbrace{\hat{H}_A + \hat{H}_m}_{=\hat{H}_0} + \hat{H}_I = \hat{H}_0 + \hat{H}_I$$

with the time evolution operator $U_0 = e^{-i\hat{H}_0 t/\hbar}$
The states are stationary, unless coupled via $\hat{H}_I$.

$$\begin{aligned}\Rightarrow \hat{H}_\mathrm{I}' &= \hat{U}_0^\dagger \hat{H}_\mathrm{I}\hat{U}_0 \\ &= e^{i/\hbar(\hat{H}_A+\hat{H}_m)t}\frac{\hbar\Omega}{2}\hat{\sigma}_x(e^{i(k\hat{z}-\omega_L t+\varphi)} + c.c.)e^{-i/\hbar(\hat{H}_A+\hat{H}_m)t} \qquad (8.6)\end{aligned}$$

with

$$\begin{aligned} e^{i\hat{H}_A t/\hbar}\hat{\sigma}_x e^{-i\hat{H}_A t/\hbar} &= e^{i\omega_0 t\hat{\sigma}_z/2}\hat{\sigma}_x e^{-i\omega_0 t\hat{\sigma}_z/2} \\ &= \cos(\omega_0 t)\hat{\sigma}_x - \sin(\omega_0 t)\hat{\sigma}_y \\ &= e^{i\omega_0 t}\hat{\sigma}_+ + e^{-i\omega_0 t}\sigma_- \qquad (8.7) \\ \text{and} \quad e^{i\hat{H}_m t/\hbar}e^{ik\hat{z}}e^{-i\hat{H}_m t/\hbar} &= e^{i\omega_z t\hat{a}^\dagger\hat{a}}e^{i\eta(\hat{a}+\hat{a}^\dagger)}e^{-i\omega_z t\hat{a}^\dagger\hat{a}} \\ &= e^{i\omega_z t\hat{a}^\dagger\hat{a}}\sum_{n=0}^{\infty}\frac{1}{k!}\left(i\eta(\hat{a}+\hat{a}^\dagger)\right)^k e^{-i\omega_z t\hat{a}^\dagger\hat{a}} \\ (e^{\hat{A}}\hat{B}e^{-\hat{A}})^k = e^{\hat{A}}\hat{B}^k e^{-\hat{A}} \rightarrow &= 1 + \sum_{k=1}^{\infty}\frac{1}{k!}\left(i\eta e^{i\omega_z t\hat{a}^\dagger\hat{a}}(\hat{a}+\hat{a}^\dagger)e^{-i\omega_z t\hat{a}^\dagger\hat{a}}\right)^k \\ &= 1 + \sum_{k=1}^{\infty}\frac{1}{k!}\left(i\eta(e^{i\omega_z t}\hat{a}^\dagger + e^{-i\omega_z t}\hat{a})\right)^k \\ &= e^{i\eta(e^{-i\omega_z t}\hat{a}+ei\omega_z t\hat{a}^\dagger)} \qquad (8.8)\end{aligned}$$

$$\begin{aligned}\overset{(8.6)+(8.7)+(8.8)}{\Rightarrow} \hat{H}_\mathrm{I}' &= \frac{\hbar\Omega}{2}(e^{i\hat{H}_A t/\hbar}\hat{\sigma}_x e^{-i\hat{H}_A t/\hbar})(e^{i\hat{H}_m t/\hbar}(e^{i(k\hat{z}-\omega_L t+\varphi)} + c.c.)e^{-i\hat{H}_m t/\hbar}) \\ &= \frac{\hbar\Omega}{2}(e^{i\omega_0 t}\hat{\sigma}_+ + e^{-i\omega_0 t}\hat{\sigma}_-)(e^{i\eta(e^{-i\omega_z t}\hat{a}+e^{i\omega_z t}\hat{a}^\dagger)-i\omega_L t+i\varphi} + h.c.)\end{aligned}$$

with $\eta = kz_0 = k\sqrt{\frac{\hbar}{2m\omega_z}}$, reminder: $\hat{z} = \sqrt{\frac{\hbar}{2m\omega_z}}(\hat{a}+\hat{a}^\dagger)$

- rotating wave approximation:

$$\hat{H}_{\mathrm{I}}' = \frac{\hbar\Omega}{2}\hat{\sigma}_{+}e^{i\eta(e^{-i\omega_z t}\hat{a}+e^{i\omega_z t}\hat{a}^{\dagger})-i\Delta t+i\varphi} + h.c. \tag{8.9}$$

  with $\Delta = \omega_L - \omega_0$.

- The **Lamb-Dicke parameter**

$$\eta = kz_0 = k\sqrt{\frac{\hbar}{2m\omega_z}} \tag{8.10}$$

  is the ratio of $z_0$, the spread of the atomic wavefunction in the ground state, and the wavelength of the laser $\lambda$, $k = \frac{2\pi}{\lambda}$.

- The Lamb-Dicke Regime
  → size of the wavefunction of atom/ion is much smaller than optical wavelength!
  → for ground state cooled atoms the condition is: $\eta \ll \lambda$

  General expression of equation (8.10):

$$k\sqrt{\langle \hat{z}^2 \rangle} = \eta\sqrt{\langle\phi|\,(\hat{a}^{\dagger}+\hat{a})^2\,|\phi\rangle} \tag{8.11}$$

  ⇒ For the specific case of a Fock state $|\psi\rangle = |n\rangle$: $k\sqrt{\langle \hat{z}^2 \rangle} = \eta\sqrt{2n+1}$

  If $k\sqrt{\langle \hat{z}^2 \rangle} \ll 1$, equation (8.9) can be Taylor expanded!

$$\hat{H}_I' = \frac{\hbar\Omega}{2}\hat{\sigma}_{+}\left(1 + i\eta(e^{-i\omega_z t}\hat{a} + e^{i\omega_z t}\hat{a}^{\dagger})\right)e^{-i\Delta t+i\varphi} + h.c.$$

  **Note:** in three dimensions of space, we have to take into account the angle $\theta$ between wave vector $\mathbf{k}$ and the corresponding trap axis, here $\mathbf{z}$:

$$\eta = \cos\theta k\sqrt{\frac{\hbar}{2m\omega_z}} \quad \text{if} \quad \mathbf{kz} = \cos(\theta)kz. \tag{8.12}$$

  **Note:** $\eta$ in the ground state can also be expressed by $\eta_{|g\rangle} = \sqrt{\frac{\omega_{\mathrm{rec}}}{\omega_z}}$

  There are 3 cases, depending on $\Delta$:

  1. Carrier Transition (CAR), $\Delta = 0$:

$$\Rightarrow \hat{H}_{\mathrm{I}}' = \frac{\hbar\Omega}{2}(\hat{\sigma}_{+}e^{i\varphi} + \hat{\sigma}_{-}e^{-i\varphi})$$

     all other terms vanish (RWA), $e^{\pm i\omega_z t}$ average to zero!
     → laser couples only electronic states
     → motion is unchanged

$$|g,n\rangle \leftrightarrow |e,n\rangle$$

2. Red Sideband Transition (RSB), $\Delta = -\omega_z$:

$$\Rightarrow \hat{H}_{\mathrm{I}}' = \frac{\hbar\Omega}{2}\left(\hat{\sigma}_+ \hat{a} e^{i\varphi} - \hat{\sigma}_- \hat{a}^\dagger e^{-i\varphi}\right) i\eta$$
$$|g,n\rangle \leftrightarrow |e,n-1\rangle$$

$\rightarrow$ when atom is excited, vibrational number n is reduced by 1
$\rightarrow$ coupling strength is reduced, depending on occupation of vibrational modes $|n\rangle$

$$\Omega_{RSB,n} = \Omega\eta\sqrt{n}, \qquad \text{for} \quad |n\rangle \rightarrow |n-1\rangle$$

$\Rightarrow$ analogous to Jaynes-Cummings Hamiltonian!

3. Blue Sideband Transition(BSB), $\Delta = +\omega_z$

$$\Rightarrow \hat{H}_{\mathrm{I}}' = \frac{\hbar\Omega}{2}(\hat{\sigma}_+ \hat{a}^\dagger e^{i\varphi} - \hat{\sigma}_- \hat{a} e^{-i\varphi}) i\eta$$
$$|g,n\rangle \rightarrow |e,n+1\rangle$$

coupling strength

$$\Omega_{BSB,n} = \Omega\eta\sqrt{n+1}, \qquad \text{for} \quad |n\rangle \rightarrow |n+1\rangle$$

$\Rightarrow$ so called Anti-Jaynes-Cummings Hamiltonian!
$\rightarrow$ if the atom is excited, the motional quanta n are increased by 1.

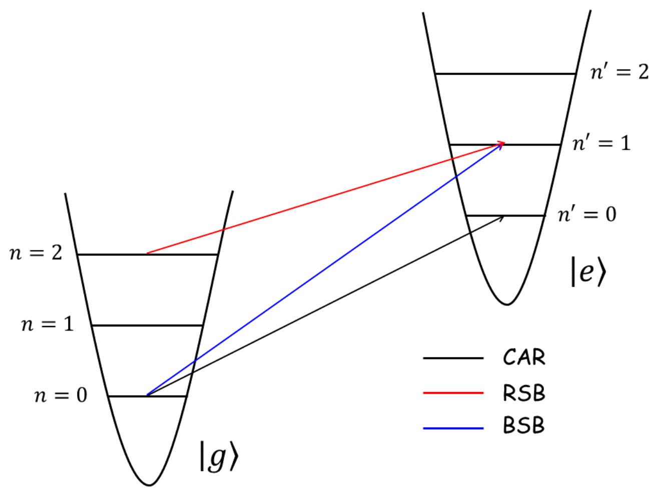



Figure 8.4: Possible transitions from the electronic ground state $|g\rangle$ to an excited state $|e\rangle$, if the atom is localized in the Lamb-Dicke regime: Carrier transitions (black arrow) keep the atomic motion unchanged, red sideband transitions (red arrow) decrease the vibrational quantum number by 1, blue sideband transitions (blue arrow) increase it by 1.