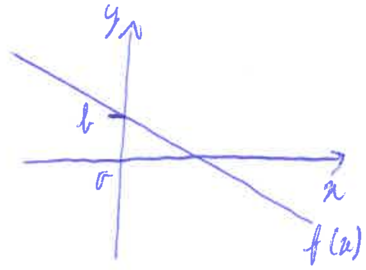
MS135

Lecture 2

Linear Functions

A linear function is a function $f: \mathbb{R} \to \mathbb{R}$ whose graph is a straight line.



The formula for a linear function is of the form

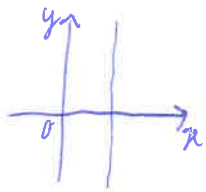
$$f(x) = m \cdot x + b$$

for some fixed real numbers m and b.

m is called the <u>slope</u> of the linear function and b the <u>intercept</u>.

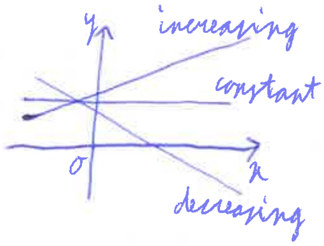
Remarks:

* The graph of a linear function is a straight line, but this line cannot be vertical. A vertical line is not the graph of a function.



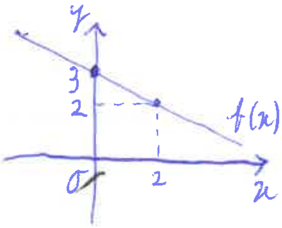
not a function

* Reading the graph from left to right (increasing x) the function can increase (increasing y) or decrease (decreasing y), or remain constant (constant y).



* The intercept is $b = f(0)$, because $f(0) = m \cdot 0 + b = b$.

<u>Example</u> Consider the linear function $f(x) = 3 - \frac{1}{2}x$.



To draw its graph we compute e.g. $f(0) = 3$,
$f(2) = 2$,
draw the points $(0,3)$ and $(2,2)$ and then a straight line through them.

The slope m is the rate of change of a linear function f:

* $m > 0$ if and only if f is increasing,
* $m < 0$ " " " " f is decreasing,
* $m = 0$ " " " " f is constant.

If $x_1 \in \mathbb{R}$ and $x_2 \in \mathbb{R}$ with $x_1 \neq x_2$, then

$$m = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$ the rate of change of f between x_1 and x_2.

This is because

$$\begin{aligned} f(x_2) - f(x_1) &= (m \cdot x_2 + b) - (m \cdot x_1 + b) \\ &= m \cdot x_2 + b - m \cdot x_1 - b \\ &= m \cdot x_2 - m \cdot x_1 = m \cdot (x_2 - x_1) \end{aligned}$$

so $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = \dfrac{m \cdot (x_2 - x_1)}{x_2 - x_1} = m.$

We can draw the graph of a linear function if we know two points on it.
Similarly we can find the formula of a linear function given two points.

__Example__ Find the formula of the linear function $f: \mathbb{R} \to \mathbb{R}$ through the points $(1,2)$ and $(4,3)$.

Solution: $f(x) = m \cdot x + b$ and we must find m and b. Strategy: find m first.

$$m = \frac{f(4) - f(1)}{4 - 1} = \frac{3 - 2}{4 - 1} = \frac{1}{3}$$ so $f(x) = \frac{1}{3} \cdot x + b$ for some b.

To find b we substitute one of the two points:

$$2 = f(1) = \tfrac{1}{3} \cdot 1 + b \quad \Rightarrow \quad b = 2 - \tfrac{1}{3} \cdot 1 = 2 - \tfrac{1}{3} = 1\tfrac{2}{3}.$$

(The other point also works:

$$3 = f(4) = \tfrac{1}{3} \cdot 4 + b \quad \Rightarrow \quad b = 3 - \tfrac{1}{3} \cdot 4 = 3 - 1\tfrac{1}{3} = 1\tfrac{2}{3}$$ same value for b.)

We find: $f(x) = \frac{1}{3} \cdot x + 1\frac{2}{3}.$

Remarks:

* The graphs of two linear functions are parallel when the functions have the same slope.

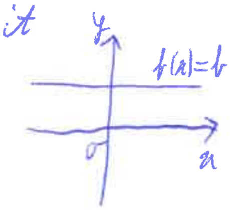
* The graphs of two linear functions intersect perpendicularly (90° angle) if the product of their slopes is -1.

If a linear function $f(x) = m\cdot x + b$ has a slope $m \neq 0$, then its graph intersects the x-axis in a unique point. Algebraically we solve the linear equation

$$\begin{aligned} f(x) &= 0 \\ m\cdot x + b &= 0 \\ m\cdot x &= -b \\ x &= \frac{-b}{m}. \end{aligned}$$

<u>Example</u> $f(x) = 3 - \frac{1}{2}x$ has $m = -\frac{1}{2}$ and $b = 3$, so the graph intersects the x-axis at $x = \frac{-3}{-\frac{1}{2}} = 6$ where $f(6) = 0$.

Remark: When $m = 0$ the graph of $f(x)$ is parallel to the x-axis, so it does not intersect the x-axis anywhere (if $b \neq 0$), or it intersects the x-axis everywhere (if $b = 0$).



To solve a linear inequality like $f(x) > 0$ or $f(x) < 0$ it often helps to first solve $f(x) = 0$ and then consider the graph of $f(x)$.

<u>Example</u> Find all values $x \in \mathbb{R}$ for which $3 - \frac{1}{2}x > 0$.

Solution 1: We first solve

$$\begin{aligned} 3 - \tfrac{1}{2}x &= 0 \\ 3 &= \tfrac{1}{2}x \\ \tfrac{1}{2}x &= 3 \\ x &= 6 \quad \text{(as above)}. \end{aligned}$$

The linear function $f(x) = 3 - \frac{1}{2}x$ has a negative slope, $m = -\frac{1}{2}$, so it is decreasing. Thus: for $x > 6$, $3 - \frac{1}{2}x < 0$
and for $x < 6$, $3 - \frac{1}{2}x > 0$.

The solution set is $\{x \in \mathbb{R} \mid x < 6\}$.

Solution 2: A direct algebraic solution:

$$\begin{aligned} 3 - \tfrac{1}{2}x &> 0 \\ -\tfrac{1}{2}x &> -3 \\ x &< -3\cdot(-2) = 6 \end{aligned}$$

multiply both sides by -2, a negative number, so "$>$" becomes "$<$"

The solution set is $\{x \in \mathbb{R} \mid x < 6\}$ (as before).