Unique formula for quadratic polynomial

Example

Find the formulas for the unique quadratic polynomial through (1,–3), (2, 2) and (3,–1).

Solution: We find first find the Langrange basis polynomials for the points 1, 2, 3:

$$p_1(x) = \frac{x-2}{1-2} \cdot \frac{x-3}{1-3} = \frac{1}{2}(x^2) - 2\,\frac{1}{2}\,x + 3,$$

$$p_2(x) = \frac{x-1}{2-1} \cdot \frac{x-3}{2-3} = -(x-1)(x-3) = -x^2 + 4x - 3,$$

$$p_3(x) = \frac{x-1}{3-1} \cdot \frac{x-2}{3-2} = \frac{1}{2}(x-1) = \frac{1}{2}x^2 - 1\,\frac{1}{2}\,x + 1,$$

We have

$$p(x) = -3.\ p_1(x) + 2.\ p_2(x) - 1.\ p_3(x)$$

$$= -3.\left(\frac{1}{2}x^2 - 2\,\frac{1}{2}\,x + 3\right) + 2(-x^2 + 4x - 3) - \left(\frac{1}{2}x^2 - 1\,\frac{1}{2}(x) + 1\right)$$

$$= -1\,\frac{1}{2}x^2 + 7\,\frac{1}{2}\,x - 9 - 2x^2 + 8x - 6 - \frac{1}{2}x^2 + 1\,\frac{1}{2}\,x - 1$$

$$= -4x^2 + 17x - 16$$

Check

$$p(1) = -4 + 17 - 16 = -3$$

$$p(2) = -4.\ 4 + 17.\ 2 - 16 = -16 + 34 - 16 = 2$$

$$p(3) = -4.\ 9 + 17.\ 3 - 16 = -36 + 51 - 16 = -1$$

($p(x)$ has a maximum at $2\,\frac{1}{8}$, where $p(2\,\frac{1}{8}) = 2\,\frac{1}{16}$ the roots of p are at $2\,\frac{1}{8} - \frac{\sqrt{33}}{8}$ and $2\,\frac{1}{8} + \frac{\sqrt{33}}{8}$)