Lecture 15

Rules for Computing Limits

The following rules are closely related to the rules for continuous functions.

Consider functions $f: D \to \mathbb{R}$, $g: D \to \mathbb{R}$ with the same domain, $c \in \mathbb{R}$ and $a \in \overline{D \setminus \{a\}}$.

* Sum rule: If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = K$ (and in particular both limits exist),

  then $\lim_{x \to a} (f+g)(x) = L + K$ (and in particular the limit exists).

* Product rule: Under the same assumptions as for the sum rule,

  $\lim_{x \to a} (f \cdot g)(x) = L \cdot K$ (and in particular the limit exists).

* Multiple rule: If $\lim_{x \to a} f(x) = L$ (and in particular the limit exists),

  then $\lim_{x \to a} (c \cdot f)(x) = c \cdot L$ (and in particular the limit exists).

* Restriction rule: If $\lim_{x \to a} f(x) = L$ (and in particular the limit exists)

  and if $D' \subset D$ such that $a \in \overline{D' \setminus \{a\}}$, then $\lim_{x \to a} f|_{D'}(x) = L$ (and the limit exists).

* Composition rule: If $\lim_{x \to a} g(x) = K$ (and in particular the limit exists)

  and if $f \circ g$ is well-defined and $f(x)$ is continuous at $x = K$, then

  $\lim_{x \to a} (f \circ g)(x) = f(K)$.

* Quotient rule: If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = K$ with $K \neq 0$ (and in particular the limits exist), then $\lim_{x \to a} \frac{f}{g}(x) = \frac{L}{K}$ (and in particular the limit exists).

Remark Most rules for limits can be written without using L and K, e.g.

$$\lim_{x\to a} (f+g)(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x),$$

$$\lim_{x\to a} (f\cdot g)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x),$$

$$\lim_{x\to a} (c\cdot f)(x) = c\cdot \lim_{x\to a} f(x),$$

$$\lim_{x\to a} f|_{D'}(x) = \lim_{x\to a} f(x),$$

$$\lim_{x\to a} (f\circ g)(x) = f\left(\lim_{x\to a} g(x)\right),$$

$$\lim_{x\to a} \frac{f}{g}(x) = \lim_{x\to a} f(x) \Big/ \lim_{x\to a} g(x).$$

Warning: these formulae hold under the assumptions stated before, but they may fail otherwise. Beware of the direction of the implication.

Example Consider $f(x) = \frac{x-1}{x}$ and $g(x) = \frac{1}{x}$ on the domain $\mathbb{R}\setminus\{0\}$.
We have

$$f(x)+g(x) = \frac{x-1}{x} + \frac{1}{x} = \frac{x-1+1}{x} = \frac{x}{x} = 1 \text{ and } \lim_{x\to 0} (f+g)(x) = \lim_{x\to 0} 1 = 1.$$

However, the limits $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ do not exist!

Left and Right hand limits

For a function $f: D\to\mathbb{R}$, and $a\in\overline{D\setminus\{a\}}$ and $L\in\mathbb{R}$ we write

$$\lim_{x\to a^-} f(x) = L \quad \text{iff} \quad g(x) = \begin{cases} f(x) & \text{if } x\in D \text{ and } x<a \\ L & \text{if } x=a \end{cases} \quad \text{is continuous at } a,$$

$$\lim_{x\to a^+} f(x) = L \quad \text{iff} \quad g(x) = \begin{cases} f(x) & \text{if } x\in D \text{ and } x>a \\ L & \text{if } x=a \end{cases} \quad \text{is continuous at } a.$$

Remarks

* $\lim_{x\to a} f(x) = L$ iff $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

It can happen that the left and right hand limit both exist, but they are not equal. In that case the (two-sided) limit does not exist.

* f is continuous at $a \in D$ iff $\lim_{x\to a} f(x) = f(a)$, i.e.

$f(a)$, $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ must all exist and be equal.

Example Determine whether $f(x) = \begin{cases} x - \frac{1}{2} & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ 2^{-x} & \text{if } x > 1 \end{cases}$ is continuous.

Solution: $x - \frac{1}{2}$ and 2^{-x} are continuous on $\mathbb{R}$, so $f(x)$ is continuous except possibly at $x=1$, where the domain is split.

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x - \tfrac{1}{2} \quad \text{(using formula for } x<1\text{)}$$

$$= \lim_{x\to 1} x - \tfrac{1}{2} = 1 - \tfrac{1}{2} = \tfrac{1}{2} \quad \text{(using continuity of } x - \tfrac{1}{2}\text{)},$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2^{-x} \quad \text{(using formula for } x>1\text{)}$$

$$= \lim_{x\to 1} 2^{-x} = 2^{-1} = \tfrac{1}{2} \quad \text{(using continuity of } 2^{-x}\text{)}.$$

We find that $\lim_{x\to 1} f(x) = \frac{1}{2}$ as a two-sided limit.

However, $f(1) = 1 \neq \frac{1}{2}$, so $f(x)$ is not continuous at $x=1$.