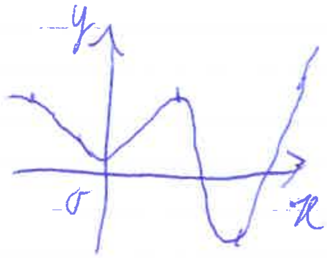
Interpolating polynomials



Theorem

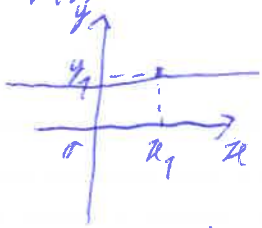
For $n \in \mathbb{N}$ and points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ in $\mathbb{R}^2$
with all the x_i's distinct.
Then there is a unique polynomial of degree $\leq n-1$, $p(x)$, with

$$p(x_1) = y_1, \dots, p(x_n) = y_n.$$

Example

$n=1$: $p(x)$ is a constant function, given point (x_1, y_1),

we need: $p(x) = y_1$.



$n=2$: $p(x)$ is a linear function, given points (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$.

$p(x) = m \cdot x + b$ for some slope m and intercept b

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{and} \quad b = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}.$$

$n=2$, $y_1 = 1$, $y_2 = 0$: the formulae simplify to

$$p(x) = \frac{x - x_2}{x_1 - x_2} = \frac{1}{x_1 - x_2} \cdot x - \frac{x_2}{x_1 - x_2}.$$

$p(x_2) = 0$ because $x_2 - x_2 = 0$,

$p(x_1) = 1$ because $\frac{x_1 - x_2}{x_1 - x_2} = 1$.

$n \geq 2$, $y_1 = 1$, $y_2 = 0$, $y_3 = 0$ ... $y_n = 0$; distinct x_i's.

$$p(x) = \frac{x - x_2}{x_1 - x_2} \cdot \frac{x - x_3}{x_1 - x_3} \cdot \ldots \cdot \frac{x - x_n}{x_1 - x_n}$$

$p(x)$ has degree $\leq n-1$ (product of $n-1$ linear functions)

$p(x_1) = 1 \cdot 1 \cdot \ldots \cdot 1 = 1$

$p(x_i) = 0$ because $\frac{x_i - x_i}{x_1 - x_i} = 0$ if $i = 2, 3, \ldots, n$.

$p(x)$ is called a Lagrange basis polynomial.

General case: $(x_1, y_1), \ldots, (x_n, y_n)$ all x_i's distinct.

To find $p(x)$, first find Lagrange basis polynomials

$p_1(x_1) = 1$, $p_1(x_i) = 0$ if $i \neq 1$

$p_2(x_2) = 1$, $p_2(x_i) = 0$ if $i \neq 2$

etc.

Set $p(x) = y_1 \cdot p_1(x) + y_2 \cdot p_2(x) + \ldots + y_n \cdot p_n(x)$.

Uniqueness: if $p(x)$ and $q(x)$ have degree $\leq n-1$ and $p(x_i) = y_i$ and $q(x_i) = y_i$,

then $p - q$ has degree $n-1$ and $(p-q)(x_i) = p(x_i) - q(x_i) = 0$

so $p - q$ has n roots, ~~so~~ $(p-q)(x) = 0$ for all $x \in \mathbb{R}$

i.e. $p(x) = q(x)$

<u>Example</u> Find the unique quadratic polynomial $p(x)$ through: $(-1,3)$, $(1,-1)$ and $(2,4)$.

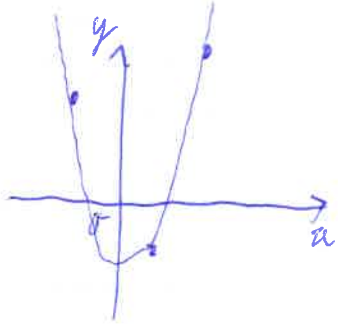
Solution: First find the Lagrange basis polynomials:

$$p_1(x) = \frac{x-1}{-1-1}\cdot\frac{x-2}{-1-2} = \frac{1}{6}(x-1)(x-2) = \frac{1}{6}x^2 - \frac{1}{2}x + \frac{1}{3}$$

$$p_2(x) = \frac{x+1}{1+1}\cdot\frac{x-2}{1-2} = \frac{-1}{2}(x+1)(x-2) = \frac{-1}{2}x^2 + \frac{1}{2}x + 1$$

$$p_3(x) = \frac{x+1}{2+1}\cdot\frac{x-1}{2-1} = \frac{1}{3}(x+1)(x-1) = \frac{1}{3}x^2 - \frac{1}{3}$$

$$\begin{aligned} p(x) &= 3\cdot p_1(x) - p_2(x) + 4\cdot p_3(x) \\ &= 3\cdot\left(\frac{1}{6}x^2 - \frac{1}{2}x + \frac{1}{3}\right) - \left(\frac{-1}{2}x^2 + \frac{1}{2}x + 1\right) \\ &\qquad + 4\left(\frac{1}{3}x^2 - \frac{1}{3}\right) \\ &= 2\tfrac{1}{3}x^2 - 2x - \frac{4}{3}. \end{aligned}$$



check: $p(-1) = 2\tfrac{1}{3} + 2 - \frac{4}{3} = 3$

$p(1) = 2\tfrac{1}{3} - 2 - \frac{4}{3} = -1$

$p(2) = 2\tfrac{1}{3}\cdot 4 - 4 - \frac{4}{3} = 9\tfrac{1}{3} - 4 - 1\tfrac{1}{3} = 4.$