MS 135

Lecture 9

Inverse Functions

A function $f: X \to Y$ is invertible iff there is a function $g: Y \to X$ such that

$$f(g(y)) = y \quad \text{for all } y \in Y,$$
$$g(f(x)) = x \quad \text{for all } x \in X.$$

We call g the inverse function of f and write f^{-1} instead of g.

Warning For numbers $a \in \mathbb{R}$ we write a^{-1} for $\frac{1}{a}$ (if $a \neq 0$).
For functions, f^{-1} does not mean $\frac{1}{f}$, so $f^{-1}(x) \neq \frac{1}{f(x)}$ in general.

Example

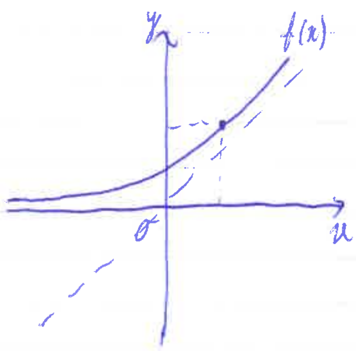
* The identity function $f(x) = x$ is its own inverse function: $f(f(x)) = x$.

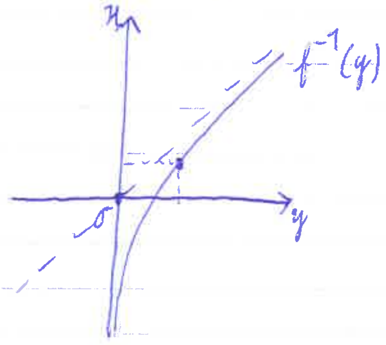
  Note: $f^{-1}(x) = f(x) = x$ is not the same thing as $\frac{1}{f(x)} = \frac{1}{x}$ (unless $x = -1$ or $x = 1$).

* The linear function $f(x) = \frac{1}{2}x - 3$ is invertible with $f^{-1}(y) = 2(y+3)$.
  We can find f^{-1} as follows:

$$y = \tfrac{1}{2}x - 3 \qquad \text{if } y = f(x),$$
$$y + 3 = \tfrac{1}{2}x$$
$$\text{then}$$
$$2(y+3) = x$$
$$x = 2(y+3) = 2y + 6 \qquad x = f^{-1}(y) = 2y + 6.$$

There is a relation between the graphs $y = f(x)$ and $x = f^{-1}(y)$:





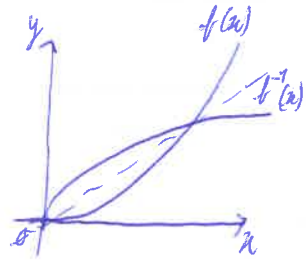
The graphs are each other's reflection in the line $y = x$ (which acts as a mirror).

Examples

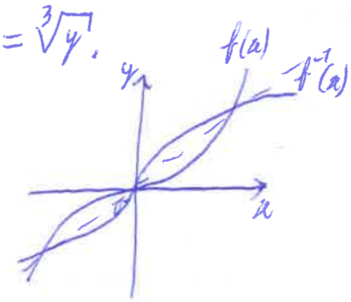
* $f(x) = x^2$ is not invertible, because e.g. $f(-1) = 1 = f(+1)$.

The restriction $f|_{[0,\infty)} : [0,\infty) \to [0,\infty)$ is invertible

and the inverse function is $f^{-1}(y) = \sqrt{y}$.



* $f(x) = x^3$ is invertible and the inverse function is $f^{-1}(y) = \sqrt[3]{y}$.



* For general $n \in \mathbb{N}$:

if n is odd, $f(x) = x^n$ is invertible,

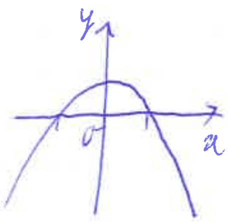
if n is even, $f(x) = x^n$ is not invertible, but $f|_{[0,\infty)} : [0,\infty) \to [0,\infty)$ is.

In both cases we write $f^{-1}(y) = \sqrt[n]{y}$ or also $f^{-1}(y) = y^{\frac{1}{n}}$.

<u>Remark</u>: The fact that these functions are invertible is perhaps not obvious.
It follows from the fact that they are strictly increasing on the given domains.
We will see in a few weeks why this is true.

<u>Example</u> * Find the natural domain of $f(x) = \sqrt[4]{3 - x^2}$. Write the answer in terms of intervals.

Solution: $3 - x^2$ makes sense for all $x \in \mathbb{R}$, but $\sqrt[4]{y}$ only makes sense if $y \geq 0$.
For $f(x)$ to make sense we therefore need:
$$3 - x^2 \geq 0.$$
The quadratic function $3 - x^2$ has roots $x = -\sqrt{3}$ and $x = \sqrt{3}$.

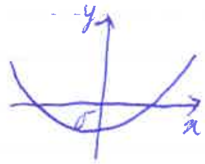


Graphically: $3 - x^2 \geq 0$ iff $x \in [-\sqrt{3}, \sqrt{3}]$.

The natural domain of $f(x)$ is $[-\sqrt{3}, \sqrt{3}]$.

* Find the natural domain of $g(x) = \dfrac{\sqrt{x^2-1}}{x-2}$ and write the answer in terms of intervals.

Solution: x^2-1 and $x-2$ make sense for all $x \in \mathbb{R}$. The quotient $\frac{\cdots}{x-2}$ does not make sense if $x=2$, so 2 is not in the natural domain. $\sqrt{x^2-1}$ only makes sense if $x^2-1 \geq 0$. We have $x^2-1=0$ iff $x=-1$ or $x=1$ and graphically $x^2-1 \geq 0$ iff $x \in (-\infty,-1] \cup [1,\infty)$.



Combining all conditions we find the natural domain

$$(-\infty,-1] \cup [1,2) \cup (2,\infty)$$

number 2 is removed from the domain!