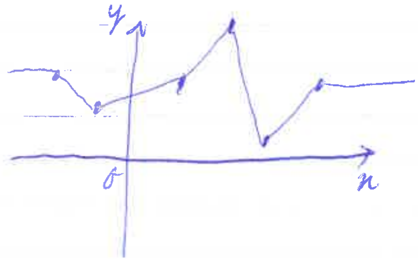
MS 135

Lecture 6

Interpolating polynomials

Sometimes we want to find a function that takes some prescribed values at certain given points. Polynomials help us to find such a function, which has no sharp corners.



Theorem (Interpolating polynomials)

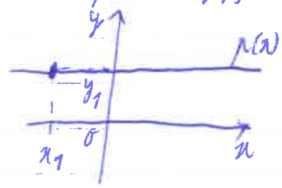
For $n \in \mathbb{N}$, let $x_1, x_2, \dots, x_n$ be distinct real numbers and $y_1, \dots, y_n$ any real numbers.

Then there is a unique polynomial $p(x)$ of degree $\leq n-1$ such that $p(x_i) = y_i$ for all $i \in \{1, \dots, n\}$.

Remark: If the x_i's are not distinct, e.g. $x_1 = x_2$, then there may not be any function with $f(x_1) = y_1$ and $f(x_2) = y_2$, because we may have $y_1 \neq y_2$. (Cf. "vertical line test")

Examples

* $n=1$: Given $x_1 \in \mathbb{R}$ and $y_1 \in \mathbb{R}$ we can find a constant function $p(x)$ with $p(x_1) = y_1$, namely $p(x) = y_1$ for all $x \in \mathbb{R}$.

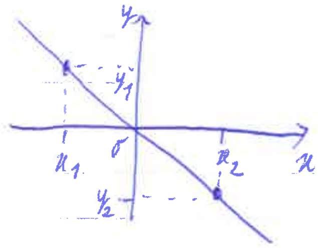


* $n=2$: Given $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \neq x_2$ we can find a linear function $p(x)$ with $p(x_1) = y_1$ and $p(x_2) = y_2$. Writing

$$p(x) = m \cdot x + b$$

we find

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{and} \quad b = \frac{y_1 x_2 - y_2 x_1}{x_2 - x_1}$$



* $n=2$, $\begin{cases} y_2 = 0 \\ y_1 = 1 \end{cases}$: The formula simplifies to

$$p(x) = \frac{x - x_2}{x_1 - x_2}$$

with $p(x_1) = 1$ and $p(x_2)$.

* $n \geq 2$, $y_1 = 1$, $y_2 = y_3 = \dots = y_n = 0$: Taking products of the previous case we can take

$$p(x) = \frac{x - x_2}{x_1 - x_2} \cdot \frac{x - x_3}{x_1 - x_3} \cdot \ldots \cdot \frac{x - x_n}{x_1 - x_n}.$$

Here each factor is a linear function. There are $n-1$ factors, so $p(x)$ has degree $\leq n-1$.
Each factor has value 1 at x_1, so $p(x_1)=1$.
For x_i with $i \geq 2$ there is a factor with value 0, so $p(x_i)=0$ if $i\geq 2$.

The polynomial in the last example is called a Lagrange basis polynomial. Given n distinct numbers $x_1, \ldots, x_n$ there are n such Lagrange basis polynomials, which vanish at all x_i's except for one, where their value is 1.

Proof of theorem on interpolating polynomials:

Uniqueness: Suppose that $p(x)$ and $q(x)$ are polynomials of degree $\leq n-1$ with $p(x_i)=y_i$ and $q(x_i)=y_i$ for all $i\in\{1,\ldots,n\}$. Then $p-q$ is a polynomial of degree $\leq n-1$ with $(p-q)(x_i)=p(x_i)-q(x_i)=0$ for all $i\in\{1,\ldots,n\}$, so $p-q$ has n roots. This can only happen when $(p-q)(x)=0$ for all $x\in\mathbb{R}$, i.e. $p(x)=q(x)$, so p and q are the same function.

Existence: First consider the Lagrange basis polynomials $p_1(x), \ldots, p_n(x)$ with

$$p_1(x)=\frac{x-x_2}{x_1-x_2}\cdot\frac{x-x_3}{x_1-x_3}\cdot\ldots\cdot\frac{x-x_n}{x_1-x_n}, \qquad p_1(x_1)=1,\ p(x_2)=\ldots=p(x_n)=0$$

$$p_2(x)=\frac{x-x_1}{x_2-x_1}\cdot\frac{x-x_3}{x_2-x_3}\cdot\ldots\cdot\frac{x-x_n}{x_2-x_n}, \qquad p_2(x_2)=1,\ p(x_1)=p(x_3)=\ldots=p(x_n)=0$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$p_n(x)=\frac{x-x_1}{x_n-x_1}\cdot\frac{x-x_2}{x_n-x_2}\cdot\ldots\cdot\frac{x-x_{n-1}}{x_n-x_{n-1}}. \qquad p_n(x_n)=1,\ p(x_1)=\ldots=p(x_{n-1})=0.$$

We can take $p(x)=y_1\cdot p_1(x)+y_2\cdot p_2(x)+\ldots+y_n\cdot p_n(x)$, which has degree ≤ 1 and $p(x_i)=y_i$ for all $i\in\{1,\ldots,n\}$.

Q.E.D.

Example Find the formula for the unique quadratic polynomial through $(1,-3)$, $(2,2)$ and $(3,-1)$.

Solution: We first find the Lagrange basis polynomials for the points 1, 2, 3:

$$p_1(x)=\frac{x-2}{1-2}\cdot\frac{x-3}{1-3}=\frac{1}{2}(x-2)(x-3)=\frac{1}{2}x^2-2\tfrac{1}{2}x+3,$$

$$p_2(x)=\frac{x-1}{2-1}\cdot\frac{x-3}{2-3}=-(x-1)(x-3)=-x^2+4x-3,$$

$$p_3(x)=\frac{x-1}{3-1}\cdot\frac{x-2}{3-2}=\frac{1}{2}(x-1)(x-2)=\frac{1}{2}x^2-1\tfrac{1}{2}x+1.$$

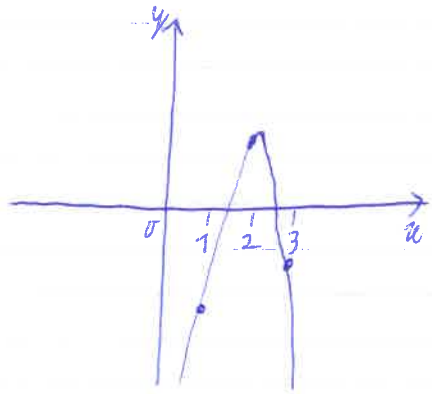
We then have

$$p(x) = -3 \cdot p_1(x) + 2 \cdot p_2(x) - 1 \cdot p_3(x)$$

$$= -3 \cdot \left(\tfrac{1}{2}x^2 - 2\tfrac{1}{2}x + 3\right) + 2 \cdot \left(-x^2 + 4x - 3\right) - \left(\tfrac{1}{2}x^2 - 1\tfrac{1}{2}x + 1\right)$$

$$= -1\tfrac{1}{2}x^2 + 7\tfrac{1}{2}x - 9 \; -2x^2 + 8x - 6 - \tfrac{1}{2}x^2 + 1\tfrac{1}{2}x - 1$$

$$= -4x^2 + 17x - 16.$$



Check: $p(1) = -4 + 17 - 16 = -3,$

$p(2) = -4 \cdot 4 + 17 \cdot 2 - 16 = -16 + 34 - 16 = 2,$

$p(3) = -4 \cdot 9 + 17 \cdot 3 - 16 = -36 + 51 - 16 = -1.$

($p(x)$ has a maximum at $2\frac{1}{8}$, where $p(2\frac{1}{8}) = 2\frac{1}{16}$, the roots of p are at $2\frac{1}{8} - \frac{\sqrt{33}}{8}$ and $2\frac{1}{8} + \frac{\sqrt{33}}{8}$.)