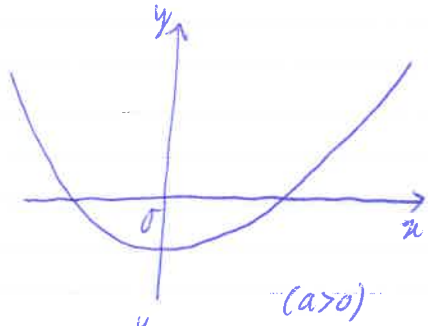
Lecture 3

Quadratic Functions

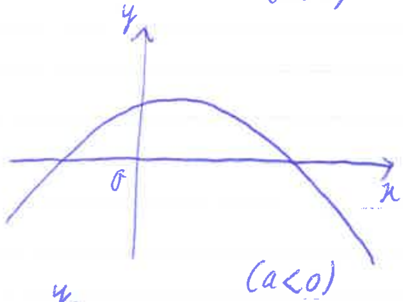


A quadratic function is a function $f: \mathbb{R} \to \mathbb{R}$ whose formula is of the form

$$f(x) = a \cdot x^2 + b \cdot x + c$$

for some $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $c \in \mathbb{R}$ with $a \neq 0$.

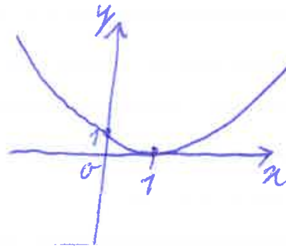
The graph of a quadratic function is a parabola.



If $a > 0$ the parabola opens upward.

If $a < 0$ the parabola opens downward.

(For $a = 0$ we have a linear function, not a quadratic one.)



Example $f(x) = x^2 - 2x + 1$ has a graph.

We can write $f(x) = (x-1)^2$, so $f(x) \geq 0$ for all $x \in \mathbb{R}$ (it is a square of a real number).

We have $f(x) = 0$ only at $x = 1$, so $f(x)$ attains its smallest function value there.

Let $f: D \to \mathbb{R}$ be a function (with D any set) and let $c \in D$. Then:

$f(c)$ is a maximum value for f if $f(c) \geq f(x)$ for all $x \in D$,

$f(c)$ " " minimum " " " " $f(c) \leq f(x)$ " " ",

$f(c)$ is an extreme value for f if it is a maximum or minimum value.

Every quadratic function has an extreme value, which is attained at a unique point in $\mathbb{R}$. It is a minimum value if $a > 0$ and a maximum value if $a < 0$.

To see this we write

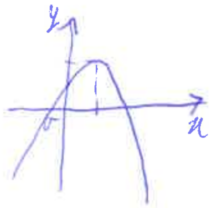
$$f(x) = a \cdot x^2 + b \cdot x + c = a \cdot \left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right).$$

If $a > 0$ we find: $f(x) \geq c - \frac{b^2}{4a}$ because $a \cdot \left(x + \frac{b}{2a}\right)^2 \geq 0$.

If $a < 0$ we find: $f(x) \leq c - \frac{b^2}{4a}$ because $a \cdot \left(x + \frac{b}{2a}\right)^2 \leq 0$.

So $f(x)$ attains the extreme value $c - \frac{b^2}{4a}$ at the point $x = \frac{-b}{2a}$.

Example Find and classify the extreme value of $f(x) = 6x + 7 - 3x^2$.



Solution: $a=-3$, $b=6$ and $c=7$.

Classification: $a=-3$, so $f(x)$ has a maximum value.

The maximum value is attained at $\frac{-b}{2a} = \frac{-6}{2\cdot(-3)} = \frac{-6}{-6} = 1$,

so the maximum value is
$$f(1) = 6\cdot 1 + 7 - 3\cdot 1^2 = 6 + 7 - 3 = 10.$$

$c \in \mathbb{R}$ is called a root (or zero) for $f(x)$ if $f(c) = 0$.

A quadratic function can have 0, 1 or 2 roots. We find them as follows:

the discriminant $D = b^2 - 4ac$ of $f(x)$ only depends on the coefficients a, b and c;

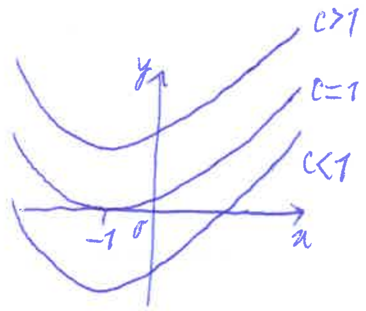
if $D<0$: $f(x)$ has no roots,

if $D=0$: $f(x)$ has a single root at $x = \frac{-b}{2a}$, i.e. at the extreme value,

if $D>0$: $f(x)$ has two roots, which lie symmetrically around $\frac{-b}{2a}$, namely at
$$x = \frac{-b-\sqrt{D}}{2a} \quad \text{and} \quad x = \frac{-b+\sqrt{D}}{2a}.$$

Example Consider $f(x) = x^2 + 2x + c$ for some $c \in \mathbb{R}$.

By changing c we can shift the graph of $f(x)$ up and down.



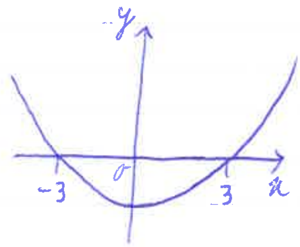
$D = b^2 - 4ac = 2^2 - 4\cdot 1\cdot c = 4 - 4c$.

If $c>1$, then $D<0$ and there are no roots.
If $c=1$, then $D=0$ and there is a single root $x=-1$.
If $c<1$, then $D>0$ and there are two roots.

[Remark: We can prove the formulae above by solving
$$\begin{aligned}
0 &= f(x) \\
0 &= 4\cdot a\cdot f(x) \qquad (\text{as } 4\cdot a \neq 0) \\
&= 4a^2x^2 + 4abx + 4ac \\
&= (2ax+b)^2 - (4ac - b^2) = (2ax+b)^2 - D
\end{aligned}$$
$$(2ax+b)^2 = D$$
if $D<0$ there are no solutions, if $D=0$ we must have $2ax+b=0$ so $x = \frac{-b}{2a}$ and if $D>0$ we find $2ax+b = \sqrt{D}$ or $2ax+b = -\sqrt{D}$.]

To solve an inequality like $f(x) > 0$ or $f(x) < 0$ for a quadratic function $f(x)$,

first solve $f(x) = 0$ (i.e. find the roots)
then sketch $f(x)$ to see if it opens upward or downward
then read off the set of solutions.



Example For what $x \in \mathbb{R}$ is $2x^2 - 18 \leq 0$?

Solution: $f(x) = 2x^2 - 18$ has roots at $x_1 = -3$ and $x_2 = 3$.
The set of solutions is
$$\{x \in \mathbb{R} \mid x \geq -3 \text{ and } x \leq 3\}$$
(between the roots and including them).

Monomials

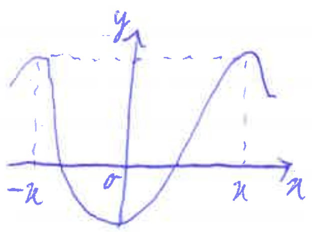
A monomial is a function $f: \mathbb{R} \to \mathbb{R}$ whose formula is of the form

$$f(x) = a \cdot x^n$$ for some exponent $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ (or $n = 0$ with $x^0 = 1$) and some $a \in \mathbb{R}$.

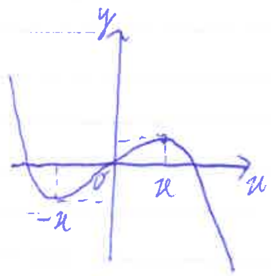
Examples $f(x) = 3 \cdot x^3$, $g(x) = -\frac{1}{2} x^7$ and $h(x) = \pi \cdot x^{101}$ are monomials.

A function $f: \mathbb{R} \to \mathbb{R}$ is called even when $f(-x) = f(x)$ for all $x \in \mathbb{R}$ and
it is called odd when $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

The graph of an even function is symmetric in the y-axis,
" " " " odd " " " under reflection in the origin.



(even)



(odd)

For a monomial we have
$$f(-x) = a \cdot (-x)^n = a \cdot (-1)^n \cdot x^n = (-1)^n \cdot f(x).$$
If n is even, $(-1)^n = 1$, so the monomial is even.
If n is odd, $(-1)^n = -1$, so " " " odd.