Lecture 7

Operations on Functions II

* If $f: A \to B$ is any function and $C \subset A$ a subset, then we can define the restriction $f|_C$ as the function $f|_C : C \to B$ with $f|_C(x) = f(x)$ for $x \in C$.

* For two functions $f: A \to B$ and $g: B \to C$ we can define the composition $g \circ f$ as the function $g \circ f : A \to C$ with $g \circ f(x) = g(f(x))$.
  "g after f"

* For any set D and functions $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R} \setminus \{0\}$ we can define the quotient $\frac{f}{g}$ as the function $\frac{f}{g} : D \to \mathbb{R}$ with $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$.

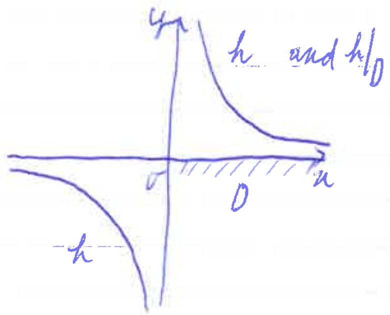
Remarks:

* A restriction of the domain does not change the function values.
* In general $g \circ f \neq f \circ g$. Sometimes only one of the two is defined.
* In a quotient $\frac{f}{g}$ it is important that the range of g does not contain 0, because $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ is not defined at any point x with $g(x) = 0$.
  This is why we require $g: D \to \mathbb{R} \setminus \{0\}$ with codomain $\mathbb{R} \setminus \{0\}$.

Examples

* Consider the function $h: \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$
  $x \mapsto \frac{1}{x}$

  and the set $D = \{x \in \mathbb{R} \mid x > 0\}$.
  The graph of $h|_D$ is a part of the graph of h.



* For the functions $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin(x)$ and $g: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2+1$ we can define

  $\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \frac{\sin(x)}{x^2+1}$ for all $x \in \mathbb{R}$, because $g(x) \neq 0$ for all $x \in \mathbb{R}$,
  i.e. 0 is not in the range of g.

* For f, g and h as before we can write the quotient $\frac{f}{g}$ as a composition:

$$\frac{f}{g} = f \cdot (h \circ g),$$

because

$$\begin{aligned}(f \cdot (h \circ g))(x) &= f(x) \cdot (h \circ g)(x) \\ &= f(x) \cdot h(g(x)) \\ &= f(x) \cdot \frac{1}{g(x)} = \frac{f(x)}{g(x)} \\ &= \frac{\sin(x)}{x^2+1}.\end{aligned}$$

In fact, we have quite generally $\frac{f}{g} = f \cdot (h \circ g)$ for functions f and g with $h(x) = \frac{1}{x}$.

Intervals

Many subsets of $\mathbb{R}$ can nicely be written in terms of intervals.

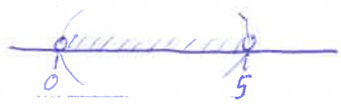
An interval I is a subset of $\mathbb{R}$ with the following property:

if $a \in I$ and $b \in I$ and if $c \in \mathbb{R}$ lies between a and b, then $c \in I$,

i.e. $(a < c$ and $c < b)$ or $(b < c$ and $c < a)$

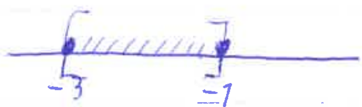
Examples

* $\{x \in \mathbb{R} \mid x > 0$ and $x < 5\}$



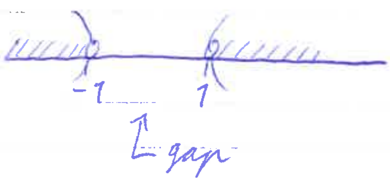
endpoints not included

* $\{x \in \mathbb{R} \mid x \geq -3$ and $x \leq -1\}$



endpoints included

* $\{x \in \mathbb{R} \mid x^2 > 1\}$ is not an interval because it has a gap



An interval is characterised by: (1) its endpoints
(2) whether the endpoints are included or not.

Notation for intervals:

For $a \in \mathbb{R}$ and $b \in \mathbb{R}$ with $a < b$ we write

(a, b) for $\{x \in \mathbb{R} \mid x > a \text{ and } x < b\}$ endpoints not included, <u>open</u> interval

$[a, b]$ for $\{x \in \mathbb{R} \mid x \geq a \text{ and } x \leq b\}$ endpoints included, <u>closed</u> interval

$[a, b)$ for $\{x \in \mathbb{R} \mid x \geq a \text{ and } x < b\}$
$(a, b]$ for $\{x \in \mathbb{R} \mid x > a \text{ and } x \leq b\}$
one endpoint included, <u>half open</u> interval

When $a = b$ we also write $[a, a] = \{a\}$ a set containing only one point.

We also use the following notations: for $a \in \mathbb{R}$

$(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$ an open interval

$(-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$ a closed interval

$(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$ an open interval

$[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}$ a closed interval

$(-\infty, \infty) = \mathbb{R}$ an open and closed interval

<u>Remarks</u>

∞ and $-\infty$ denote infinity and negative infinity. These are <u>not</u> numbers in $\mathbb{R}$. Note that ∞ and $-\infty$ are <u>never</u> included in an interval.