MS 135

Lecture 5

Roots of polynomials

For a general polynomial $f(x)$ it can be hard to solve $f(x)=0$, i.e. to find its roots. This gets harder if the degree of f increases:

If $f(x)$ has degree ≤ 2 we can find all real roots. There are formulae for these roots in terms of the coefficients of the polynomial.

If $f(x)$ has degree 3 or 4 there are also formulae for the roots, but they are rather complicated and we won't use them.

If $f(x)$ has degree ≥ 5 there is no general formula for its roots. In fact, there is a proof that such a formula cannot exist.

Sometimes we get lucky and the coefficients of $f(x)$ are simple enough to let us find its roots.

Example Find the roots of $f(x) = x^7 - 9x^3$.

Solution: We notice that $f(x) = x^3 \cdot (x^4 - 9) = x^3 \cdot (x^2-3) \cdot (x^2+3)$.
For any $x \in \mathbb{R}$ we have
$$f(x) = 0 \text{ if and only if } (x^3 = 0 \text{ or } x^2-3=0 \text{ or } x^2+3=0).$$
$x^3=0$ has only one solution: $x=0$, (if $x>0$, then $x^3>0$ and if $x<0$, then $x^3<0$)
$x^2-3=0$ has two solutions: $x=-\sqrt{3}$ or $x=+\sqrt{3}$,
$x^2+3=0$ has no solutions in $\mathbb{R}$.

Therefore, the roots of $f(x)$ are 0, $-\sqrt{3}$ and $\sqrt{3}$.

To find the roots of general polynomials we can try some of these strategies:

1) Have a lucky guess to find a root.

2) Use information about one or several roots to help find more roots.

3) Approximate a root by using a clever algorithm (e.g. the Newton-Raphson method), which can be programmed into a computer.

We will consider strategy 2).

Theorem (Dividing out a root of a polynomial)

Let $p(x)$ be a polynomial of degree $n \in \mathbb{N}$ and $c \in \mathbb{R}$ a root of p, so $p(c) = 0$. Then there is a unique polynomial $q(x)$ of degree $n-1$ such that

$$p(x) = (x-c) \cdot q(x) \qquad \text{for all } x \in \mathbb{R}.$$

Example

Consider $p(x) = x^3 - x^2 - 2x + 2$, which has $p(1) = 1 - 1 - 2 + 2 = 0$.
We want to find $q(x)$ such that $p(x) = (x-1) \cdot q(x)$.
p has degree 3, so q has degree 2, i.e.

$$q(x) = ax^2 + bx + c$$

for some real numbers a, b and c, which we want to find. We need:

$$\begin{aligned} x^3 - x^2 - 2x + 2 &= p(x) = (x-1) \cdot q(x) \\ &= (x-1) \cdot (ax^2 + bx + c) \\ &= ax^3 + (b-a)x^2 + (c-b)x - c. \end{aligned}$$

To find a, b and c we start with the highest power of x and compare coefficients on both sides.

x^3: we want $1 = a$ $\rightsquigarrow$ $a = 1$

x^2: " " $-1 = b - a$ $\rightsquigarrow$ $-1 = b - 1$, so $b = 0$

x^1: " " $-2 = c - b$ $\rightsquigarrow$ $-2 = c - 0$, so $c = -2$

x^0: " " $2 = -c$ $\rightsquigarrow$ $2 = -c$, so $c = -2$.

We find $q(x) = 1 \cdot x^2 + 0 \cdot x + (-2) = x^2 - 2$, so $p(x) = (x-1) \cdot (x^2 - 2)$.

Remarks

* We can verify the equality $p(x) = (x-1) \cdot (x^2-2)$ to check for errors.
* We can find the roots for q: $x = -\sqrt{2}$ or $x = +\sqrt{2}$.
  From the product decomposition we then find the roots of p: $-\sqrt{2}$, 1 and $\sqrt{2}$.
* Comparing the coefficients on both sides gives 4 equations, but q only has 3 coefficients that we can adjust. We can still satisfy all 4 equations, because $p(1) = 0$.

Theorem (Real roots for real polynomials)

A polynomial $p(x)$ of degree $n \in \mathbb{N}$ has at most n roots in $\mathbb{R}$.

[Proof: When $n=1$, p is a linear function which is not constant, so p has exactly one root.
We now proceed by mathematical induction and assume the claim for some $n \in \mathbb{N}$.
If p has degree $n+1$ and p has no roots, then the claim holds. On the other hand, if p has a root $c \in \mathbb{R}$, then $p(x) = (x-c) \cdot q(x)$ where q has degree n.

a root for p has to be either c or a root for q. Now q has at most n roots, so together with c, p can have at most $n+1$ roots.
The result now follows from the principle of mathematical induction. Q.E.D.

Proof of dividing out a root of a polynomial:
We can write $p(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \ldots + a_2 \cdot x^2 + a_1 \cdot x + a_0$
for some real coefficients $a_0, \ldots, a_n$. We now want to find real coefficients $b_0, \ldots, b_{n-1}$ such that

$$q(x) = b_{n-1} \cdot x^{n-1} + b_{n-2} \cdot x^{n-2} + \ldots + b_2 \cdot x^2 + b_1 \cdot x + b_0$$

satisfies $p(x) = (x-c) \cdot q(x)$ for all $x \in \mathbb{R}$. We first compute

$$(x-c) \cdot q(x) = b_{n-1} \cdot x^n + (b_{n-2} - c \cdot b_{n-1}) \cdot x^{n-1} + \ldots + (b_1 - c \cdot b_2) \cdot x^2 + (b_0 - c \cdot b_1) x - c\, b_0.$$

We set these coefficients equal to those of $p(x)$, starting at the highest power:

x^n: $b_{n-1} = a_n$
x^{n-1}: $b_{n-2} = a_{n-1} + c\, b_{n-1}$
$\vdots$
x^2: $b_1 = a_2 + c\, b_2$
x^1: $b_0 = a_1 + c\, b_1$.

From these equations we find $b_{n-1}, \ldots, b_0$ one after the other, which fixes $q(x)$.
We then find that

$$p(x) - (x-c) \cdot q(x)$$

is a constant function, because all terms with $x^1, x^2, \ldots, x^n$ cancel out. We still want to show that this constant function is 0 for all x. This follows from the fact that it is 0 at $x = c$:

$$p(c) - (c-c) \cdot q(c) = 0 - 0 \cdot q(c) = 0$$

and hence $p(x) - (x-c)\, q(x) = 0$ for all $x \in \mathbb{R}$, i.e. $p(x) = (x-c) \cdot q(x)$. Q.E.D.]

[proofs are optional reading and will not be examined]