Roots of polynomials

Ko Sanders

DCU

1. Generel Notes

** see lecture notes 4 for definition of degree*

For a generel polynomial $f(x)$ it can be hard to solve $f(x) = 0$, i.e. to find its roots. This gets harder if the degree of f increases:

If $f(x)$ has degree ≤ 2 we can find all real roots. There are formulae for these roots in terms of the coefficients of the polynomial.

If $f(x)$ has degree 3 or 4 there are also formulae for the roots, but they are rather complicated and we won't use them.

If $f(x)$ has degree ≥ 5 there is no general formula for its roots. In fact, there is a proof that such a formula cannot exist.

Example Find the roots of $f(x) = x^7 - 9x^3$

Solution: We notice that $f(x) = x^3.(x^4 - 9) = x^3.(x^2 - 3).(x^2 + 3)$

For any $x \in R$ we have

$$f(x) = 0 \text{ if and only if } (x^3 = 0 \text{ or } x^2 - 3 = 0 \text{ or } x^2 + 3 = 0)$$

$x^3 = 0$ has only one solution: $x = 0$, (if $x > 0$, then $x^3 > 0$ and if $x < 0$, then $x^3 < 0$) $x^2 - 3 = 0$ has two solutions: $x = -\sqrt{3}$ or $x = +\sqrt{3}$, $x^2 + 3 = 0$ has no solutions in R.

Therefore, the roots of $f(x)$ are 0, $-\sqrt{3}$ and $\sqrt{3}$.

To find the roots of general polynomials we can try some of these strategies:

1) Have a lucky guess to find a root.

2) Use information about one are several roots to help find more roots.

3) Approximate a root by using a clever algorithm (e.g. the Newton-Raphson method), which can be programmed into a computer.