MS135

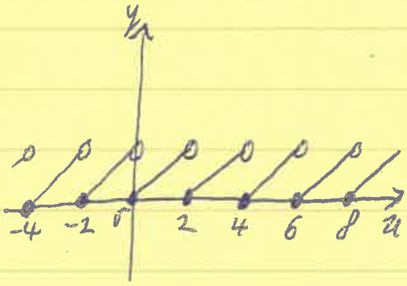
Lecture 13

Discontinuous Functions

Most functions which are given by a formula are continuous on their domain, due to the rules for continuity. It is not so easy to find formulae for discontinuous functions.

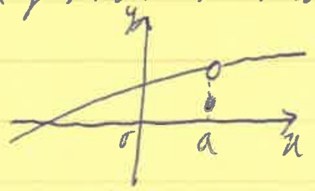
Examples

* $f(x) = x \bmod 2$ for $x \in \mathbb{R}$,
so $f(x) \in [0,2)$ is the unique number such that $x - f(x) = 2 \cdot k$ for some $k \in \mathbb{Z}$.
The graph of $f(x)$ has a sawtooth shape.



$f(x)$ is discontinuous at $x=2$, $x=4$, $x=6$, etc. and at $x=0$, $x=-2$, $x=-4$, etc.
The restriction of $f(x)$ to $\mathbb{R}\setminus\{\ldots,-4,-2,0,2,4,\ldots\}$ is continuous.

* Suppose $f: I \to \mathbb{R}$ is continuous, $I \subset \mathbb{R}$ is an interval and $a \in I$.
If $g: I \to \mathbb{R}$ has $g(a) \neq f(a)$, but $g(x) = f(x)$ if $x \in I\setminus\{a\}$, then g is not continuous at $x=a$.



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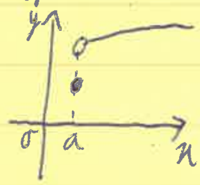

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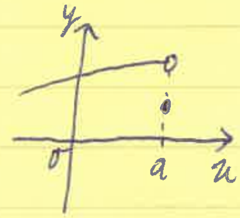
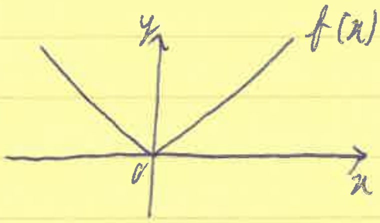


A common way to find discontinuous functions uses <u>piecewise defined</u> functions. These are given by different formulae on different parts of their domain.



Example $f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$

different formulae domain split into two parts

This is the absolute value function, also written as $f(x) = |x|$.
Note: This is not a linear function, nor a polynomial, nor a rational function, because we do not have a single rational formula on the entire domain.

Remarks For general piecewise defined functions:

* The domain can be split into two or more parts.
* The parts partition the domain: different parts must not intersect and their union is the entire domain.
  In the example above the domain is $(-\infty,0) \cup [0,\infty) = \mathbb{R}$.
* The function may be continuous, but it need not be.

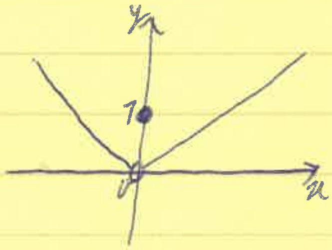
Examples

* $f(x) = \begin{cases} 3 & \text{if } x < -\pi \\ \sin(2x) & \text{if } x \geq -\pi \text{ and } x < 0 \\ 2^x & \text{if } x \geq 1 \end{cases}$

This function appears to be discontinuous at $x = -\pi$.

* $g(x) = \begin{cases} |x| & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

This function is discontinuous at $x=0$.



The discontinuity of $g(x)$ at $x=0$ seems worse than that of $f(x)$ at $x=-\pi$, because the function value $g(0)=1$ does not fit with the left nor with the right branch of the graph.

Left and Right Continuity

A function $f: D \to \mathbb{R}$ is called left continuous at $a \in D$ iff the restriction of f to $(-\infty, a] \cap D$ is continuous at $a \in D$.
Similarly, $f: D \to \mathbb{R}$ is right continuous at $a \in D$ iff the restriction of f to $[a, \infty) \cap D$ is continuous at $a \in D$.

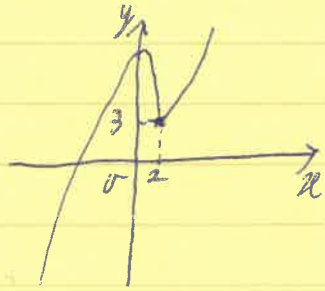
To avoid using continuity at a point we can often use the following test:
If the restriction of f to $(b, a] \cap D$ is continuous for some $b < a$, then f is left continuous at a.
If the restriction of f to $[a, b) \cap D$ is continuous for some $b > a$, then f is right continuous at a.

Example For $f(x)$ and $g(x)$ as in the previous example:

$f(x)$ is right continuous at $x=-\pi$, but not left continuous.
$g(x)$ is neither left nor right continuous at $x=0$.

Remark $f: D \to \mathbb{R}$ is continuous at $a \in D$ iff it is both left and right continuous at a.

Example Determine whether $g(x)=\begin{cases} 7-x^2 & \text{if } x \leq 2 \\ x^2-1 & \text{if } x>2 \end{cases}$ is continuous.



Solution: $7-x^2$ and x^2-1 are continuous,
so $g(x)$ is continuous except possibly at $x=2$,
where the domain is split.

at $x=2$ we check left and right continuity:

$g(x)\big|_{(-\infty,2]} = 7-x^2\big|_{(-\infty,2]}$ is continuous, so g is left continuous at $x=2$;

$$g(x)\big|_{[2,\infty)} = \begin{cases} 7-x^2 & \text{if } x=2 \\ x^2-1 & \text{if } x>2 \end{cases} = x^2-1\big|_{[2,\infty)},$$

because $7-x^2\big|_{x=2} = 7-2^2 = 3 = x^2-1\big|_{x=2}$,

so $g(x)\big|_{[2,\infty)}$ is also continuous and g is right continuous at $x=2$;

We conclude that g is also continuous at $x=2$, so it is continuous.

Remark: We will soon learn how to write the analysis in the example above a bit nicer in terms of limits.