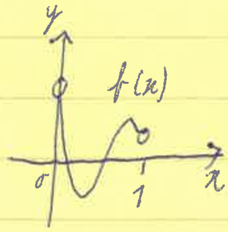
MS135

Lecture 14

Limits

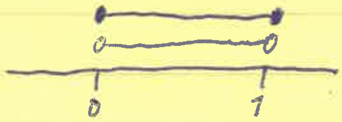
Limits are closely related to continuity. They describe what happens to function values $f(x)$ when the argument x approaches some number a.
We first consider what values we allow for a.



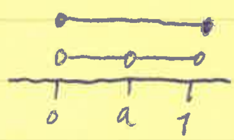
Example Suppose a function has domain $(0,1)$.
We can ask what happens to $f(x)$ when x gets close to e.g. $\frac{1}{2}$ or $\frac{1}{3}$ or any other number in $(0,1)$.
We can also ask what happens when x gets close to 0 or 1.
However, the question makes no sense for numbers <0 or >1, because $x \in (0,1)$ does not get arbitrarily close to such numbers.

For a set $D \subset \mathbb{R}$, the closure $\overline{D}$ is the set of numbers $a \in \mathbb{R}$ such that for all $\varepsilon > 0$ there is an $x \in D$ with $|x-a| < \varepsilon$.

E.g. the closure of $(0,1)$ is $[0,1]$.



FACT: If $D \subset \mathbb{R}$ is a union of open intervals and $a \in D$, then a is in the closure of $D \setminus \{a\}$, i.e. $a \in \overline{D \setminus \{a\}}$



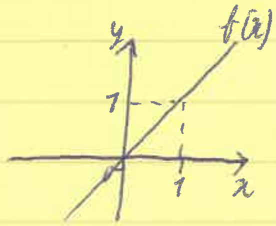
For a function $f: D \to \mathbb{R}$ and real numbers $a \in \mathbb{R}$ and $L \in \mathbb{R}$ we say that the limit of $f(x)$ as x tends to a is L
and we write
$$\lim_{x \to a} f(x) = L$$

when the function $g(x) = \begin{cases} f(x) & \text{if } x \in D\setminus\{a\} \\ L & \text{if } x = a \end{cases}$

is continuous at a and $a \in \overline{D\setminus\{a\}}$.

Remark: $\lim_{x\to a} f(x)$ does not denote a function, but a number (if the limit exists).

Intuitively: $f(x)$ gets as close to L as we like when $x \in D\setminus\{a\}$ gets sufficiently close to a (and $x \in D\setminus\{a\}$ can get arbitrarily close to a).

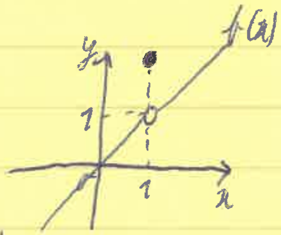


Examples

* $f(x) = x$ has $\lim\limits_{x\to 1} f(x) = 1$, because

$$g(x) = \begin{cases} x & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} = x \text{ is continuous (and hence } g \text{ is continuous at } 1\text{)}$$

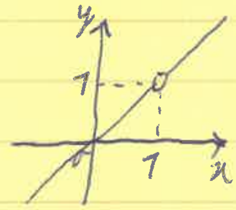
(and $1 \in \overline{\mathbb{R}\setminus\{1\}}$).



* $f(x) = \begin{cases} x & \text{if } x \neq 1 \\ 2 & \text{if } x = 1 \end{cases}$ has $\lim\limits_{x\to 1} f(x) = 1$, because

$$g(x) = \begin{cases} x & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} = x \text{ is continuous (and } 1 \in \overline{\mathbb{R}\setminus\{1\}}\text{)}.$$

Note: The value of $f(x)$ at $x=1$ is irrelevant for the limit at 1.



* $f(x) = x$ on the domain $\mathbb{R}\setminus\{1\}$ has $\lim\limits_{x\to 1} f(x) = 1$, because

$$g(x) = \begin{cases} x & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases} = x \text{ is continuous (and } 1 \in \overline{\mathbb{R}\setminus\{1\}}\text{)}.$$

Note: Whether f is defined at $x=1$ or not is irrelevant for the limit at 1.

For continuous functions it is (usually) easy to compute limits:

Given a function $f: D \to \mathbb{R}$ and $a \in D$ such that $a \in \overline{D\setminus\{a\}}$,
f is continuous at $a \in D$ iff $\lim\limits_{x\to a} f(x) = f(a)$.

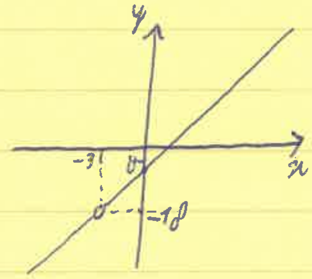
Remark Many textbooks define limits first (intuitively or rigorously) and then define continuity in terms of limits. This leads to essentially the same results. (We won't see exceptions where it doesn't.)

Remark Limits are unique, but they do not always exist:

If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, then $L = M$.

Example $f(x) = \dfrac{4x^2+6x-18}{x+3}$ is a rational function.

-3 is not in the natural domain of $f(x)$. Does $\lim_{x\to -3} f(x)$ exist? If so, what is it?



From the graph we guess: $\lim_{x\to -3} f(x) = -18$.

We can also see this from the formula, because when $x \neq -3$:

$$f(x) = \frac{4x^2+6x-18}{x+3} = \frac{(x+3)(4x-6)}{x+3} = 4x-6$$

which is a linear function (with -3 removed from its domain).

$4\cdot(-3) - 6 = -12 - 6 = -18$, so $g(x) = \begin{cases} f(x) & \text{if } x \neq -3 \\ -18 & \text{if } x = -3 \end{cases} = 4x-6$

is continuous and hence $\lim_{x\to -3} f(x) = -18$.