Lecture 10

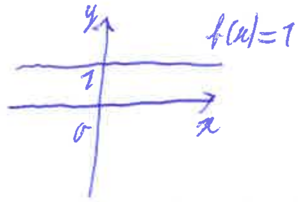
Continuity

Intuitive definition:

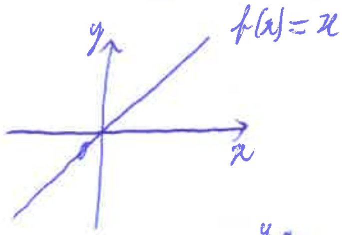
A function $f: I \to \mathbb{R}$ with $I \subset \mathbb{R}$ an interval is called continuous, when we can draw its graph without taking our pen off the paper.

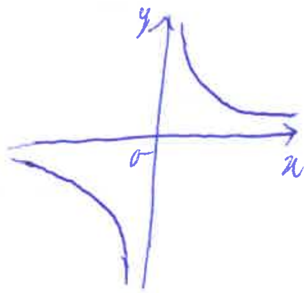
More generally, a function $f: D \to \mathbb{R}$ with domain $D \subset \mathbb{R}$ is called continuous when $f|_I$ is continuous for each interval $I \subset D$.

Basic Examples



* The constant function $f(x)=1$ is continuous.
* The identity function $f(x)=x$ is continuous.



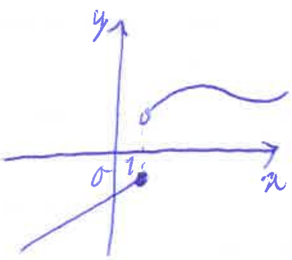


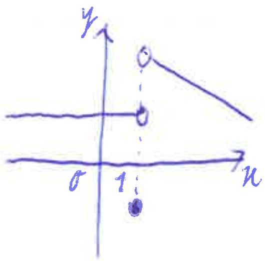
* The function $f(x)=\frac{1}{x}$ is continuous.

  The domain is $\mathbb{R}\setminus\{0\} = (-\infty, 0) \cup (0, \infty)$.

  The graphs of $f|_{(-\infty,0)}$ and $f|_{(0,\infty)}$ can be drawn in a single stroke.

The following are graphs of functions which are not continuous:





Here open circles are not part of the graph, but closed circles are.

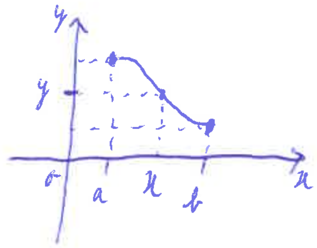
The graphs are not continuous at 1, e.g. on the interval $(0,2)$ we cannot draw the graph in a single stroke without taking our pen of the paper.

Continuous functions have many nice properties, e.g.:

Intermediate Value Theorem

If $f: [a,b] \to \mathbb{R}$ is a continuous function on the closed interval $[a,b]$ and if $y \in \mathbb{R}$ lies between $f(a)$ and $f(b)$, then $y = f(x)$ for some $x \in [a,b]$.

i.e. $f(a) \le y \le f(b)$ or $f(b) \le y \le f(a)$



Remarks: The theorem does not tell us how to find the x with $f(x) = y$.
There may be several numbers $x \in [a,b]$ with $f(x) = y$.

Example: Show that $p(x) = x^5 + 2x^4 - 3x^3 - x^2 + 4x - 1$ has a root in $[0,1]$.
You may assume that $p|_{[0,1]}$ is continuous.

Solution: We use the Intermediate Value Theorem with $a=0$, $b=1$ and $f = p|_{[0,1]}$.

Then $f(a) = p(0) = -1$, $f(b) = p(1) = 1+2-3-1+4-1 = 2$ and $y=0$ lies in between.

The theorem says there is an $x \in [0,1]$ with $f(x) = 0$, so $p(x) = 0$ at this x.

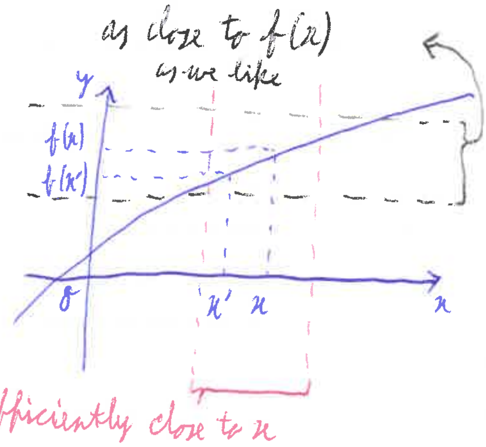
(Finding a value for x that works seems difficult in this case.)

Remark: There is a more precise definition of continuity, which is used to give reliable proofs of the Intermediate Value Theorem and other theorems.

[The following material is not examinable:

A function $f: D \to \mathbb{R}$ with $D \subset \mathbb{R}$ is continuous iff

for all $x \in D$, $f(x')$ is as close to $f(x)$ as we like for all $x' \in D$ which are sufficiently close to x.



"close" means that $|f(x') - f(x)|$ is small (or $|x'-x|$ is small)

"as close as we like" means that for every $\varepsilon > 0$ we can get $|f(x') - f(x)| < \varepsilon$

A precise logical definition of continuity is now as follows:

A function $f: D \to \mathbb{R}$ with $D \subset \mathbb{R}$ is continuous at a point $x \in D$ iff

$$\forall \varepsilon > 0 \;\; \exists \delta > 0 \;\; \forall x' \in D \;\; |x'-x| < \delta \Rightarrow |f(x') - f(x)| < \varepsilon.$$

f is continuous iff it is continuous at all $x \in D$.

Note that we can choose ε as small as we like, but we need to choose ε before δ, so δ may depend on ε.

<u>Example</u> $f(x) = x$ is continuous.

Proof: We fix an arbitrary $x \in \mathbb{R}$ and show that f is continuous at x.

We then fix an arbitrary $\varepsilon > 0$. If we choose $\delta = \varepsilon$ we have for all $x' \in D$ with

$|x'-x| < \delta$: $\quad |f(x') - f(x)| = |x'-x| < \delta = \varepsilon.$

This shows continuity at x and since x was arbitrary, $f(x)$ is continuous. Q.E.D.