Functions

A function is a rule that assigns to every element x of a set A an element y of a set B. Two functions are equal if they have the same domain and the same function values at every element of the domain.

notations: $f: A \to B$ or $f: A \to B$, $x \mapsto f(x)$

important terms and symbols:

f is the name of the function

A is called the domain of the function

B is called the codomain of the function

x is a typical element of A, it is called the independent variable, in $f(x)$ it is called the argument of f.

The range of the function f is the set $\{y \in B \mid y = f(x) \text{ for some } x \in A\}$ which is a subset of the codomain.

Example: (range $\neq$ codomain)

The function $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ has codomain $\mathbb{R}$,

but the range is $\{y \in \mathbb{R} \mid y \geq 0\}$.

Codomain is $\mathbb{R}$: we allow all real numbers as function values.

Range is $\{y \in \mathbb{R} \mid y \geq 0\}$: only real numbers ≥ 0 actually occur as function values.

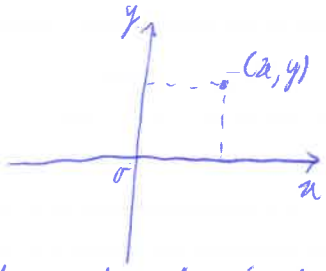
In this module we almost always work with

"real-valued functions of a single real variable",

i.e. domain and codomain are the set $\mathbb{R}$ of real numbers, or subsets thereof.

Graphs

The set $\mathbb{R}$ can be depicted as the real number line.
We can depict domain $\mathbb{R}$ and codomain $\mathbb{R}$ as a plane.



Here: domain is on the horizontal axis,
codomain is on the vertical axis.

For numbers $x \in \mathbb{R}$, $y \in \mathbb{R}$ there is a point in the plane with coordinates (x, y).

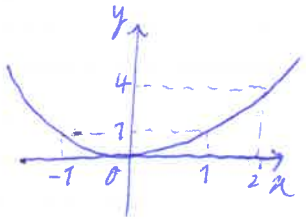
The Cartesian product $\mathbb{R} \times \mathbb{R} = \{(x, y) \mid x \in \mathbb{R},\ y \in \mathbb{R}\}$
corresponds exactly to the plane.

We can draw the graph of a function $f: \mathbb{R} \to \mathbb{R}$ in the plane as follows:

for every $x \in \mathbb{R}$ (↑ domain) we draw a point with coordinates $(x, f(x))$.

Note: this also works if the domain of f is a subset of $\mathbb{R}$.

Example The graph of $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is



To draw or sketch a graph it is often useful to first compute $f(x)$ for a few nice values of x.

Remark: We can think of a function $f: \mathbb{R} \to \mathbb{R}$ as a relation from $\mathbb{R}$ to $\mathbb{R}$, given by the subset

$$\{(x, f(x)) \mid x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}.$$

The graph of f consists of the points in the plane $\mathbb{R} \times \mathbb{R}$ which are elements of this subset.

If we draw a vertical line in the plane, it can never intersect the graph of a function more than once. This "vertical line test" allows us to recognise whether a picture is the graph of a function or not.