Lecture 8

Rational Functions

A rational function is a function f whose formula has the form

$$f(x) = \frac{p(x)}{q(x)} \quad \text{with } p \text{ and } q \text{ polynomials,}$$

and with domain $D = \{x \in \mathbb{R} \mid q(x) \neq 0\}$.

Examples

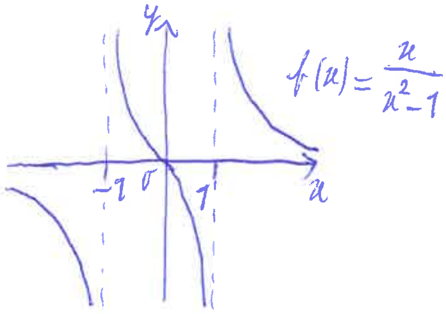
* $f(x) = \dfrac{7 \cdot x^3 - 6}{x^4 + 2}$ is a rational function with domain $\mathbb{R}$, because $x^4 + 2 > 0$ for all $x \in \mathbb{R}$.
* $f(x) = \dfrac{\sin(x)}{x^2+1}$ is not a rational function, as $\sin(x)$ is not a polynomial.
* $f(x) = \dfrac{x}{x^2-1}$ is a rational function.

  Its domain is $\mathbb{R} \setminus \{-1, 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

  -1 and 1 not included

  $f(x)$ is an odd function, $f(-x) = -f(x)$,

  because $p(x) = x$ is odd and $q(x) = x^2 - 1$ is even.



Natural Domains

If a function $f(x)$ is given by a formula, then its natural domain is the largest subset of $\mathbb{R}$ on which the formula makes sense.

E.g.:

polynomials have natural domain $\mathbb{R}$,

$f(x) = \frac{1}{x}$ has natural domain $\mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$,

$\cos(x)$ and $\sin(x)$ have natural domain $\mathbb{R}$,

$\sqrt{x}$ has natural domain $\{x \in \mathbb{R} \mid x \geq 0\} = [0, \infty)$,

$\log(x)$ has natural domain $\{x \in \mathbb{R} \mid x > 0\} = (0, \infty)$.

Note: * Rational functions are taken with their natural domain.
* If no domain is given for a function, we mean its natural domain.

Consider a rational function $f(x) = \frac{p(x)}{q(x)}$ with polynomials p and q.

To find the roots: we need to solve $f(x) = 0$, i.e. $\frac{p(x)}{q(x)} = 0$

so we need $p(x) = 0$ and $q(x) \neq 0$ at the same time.

To solve the inequality $f(x) \geq 0$:

we first find the natural domain, $q(x) \neq 0$, and

there we multiply by the <u>positive</u> function $q(x)^2$ and solve $\overset{f(x)\cdot q(x)^2}{\overset{\parallel}{p(x)\cdot q(x)}} \geq 0$.

<u>Examples</u>

* Find the roots of $f(x) = \frac{x^2-4}{x-2}$.

Solution: $p(x) = x^2-4$ and $q(x) = x-2$, so we need

$$\begin{array}{lcl} p(x) = 0 & \text{and} & q(x) \neq 0 \\ x^2-4 = 0 & & x-2 \neq 0 \\ x=-2 \text{ or } x=2 & & x \neq 2 \end{array}$$

This means that $x=-2$ is the only root.

* Solve $\frac{x^2-4}{2x-6} \geq 0$.

Solution: The natural domain is $\mathbb{R}\setminus\{3\}$, because

$$2x-6=0$$
$$2x=6$$
$$x=3.$$

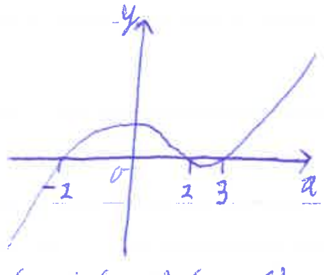
On this domain $(2x-6)^2$ is positive, so we need

$$\frac{x^2-4}{2x-6}\cdot(2x-6)^2 \geq 0\cdot(2x-6)^2 = 0$$

$$(x^2-4)\cdot(2x-6) \geq 0$$

$$(x+2)\cdot(x-2)\cdot(2x-6) \geq 0.$$

The left hand side has roots at -2, 2 and 3.
Sketching the graph we find



$(x+2)\cdot(x-2)\cdot(2x-6)$

$$(x+2)\cdot(x-2)\cdot(2x-6) \geq 0 \quad \text{iff} \quad x \in [-2,2] \cup [3,\infty).$$

Since x must be in the natural domain, the solutions are: $[-2,2] \cup (3,\infty)$

3 not included!

A multiple of a rational function is again a rational function.

The same is true for sums and products of rational functions if we restrict their domains appropriately.

<u>Example</u> Consider the rational functions $f(x) = \frac{x^2}{x-1}$ and $g(x) = \frac{2x}{x+1}$.

Then
$$f(x) + g(x) = \frac{x^2}{x-1} + \frac{2x}{x+1}$$
$$= \frac{x^2}{x-1} \cdot \frac{x+1}{x+1} + \frac{x-1}{x-1} \cdot \frac{2x}{x+1}$$
$$= \frac{x^2 \cdot (x+1)}{(x-1)\cdot(x+1)} + \frac{(x-1)\cdot 2x}{(x-1)\cdot(x+1)}$$
$$= \frac{x^2 \cdot (x+1) + (x-1)\cdot 2x}{(x-1)\cdot(x+1)}$$
$$= \frac{x^3 + 3x^2 - 2x}{x^2 - 1} \longleftarrow \text{rational function}$$

as long as x is in the natural domains of both f and g, i.e. $x \neq 1$ and $x \neq -1$.