MS 135

Lecture 11

Rules for Continuous Functions

Consider functions $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ with domain $D \subset \mathbb{R}$ and $c \in \mathbb{R}$.

* Sum rule: If f and g are continuous, then $f+g$ is a continuous function.
* Product rule: If f and g are continuous, then $f \cdot g$ is a continuous function.
* Multiple rule: If f is continuous, then $c \cdot f$ is a continuous function.

Example $f(x) = 3x^3 + 2x^2 - x + 5$ is continuous.

This is because the constant function $\underline{1}$ and the identity function id are continuous. Taking products, multiples and sums:

$$f(x) = (3 \cdot id \cdot id \cdot id + 2 \cdot id \cdot id - id + 5 \cdot \underline{1})(x),$$
$$= 3 \cdot x \cdot x \cdot x + 2 \cdot x \cdot x - x + 5$$

so f is continuous by the sum rule, multiple rule and product rule.

Remarks

* All polynomials are continuous, using a similar argument as in the example.
* The rules are an implication in one direction. They do not say, e.g.

  if $f+g$ is continuous, then f is continuous and g is continuous. (FALSE)

* The rules still work if we replace "continuous" by "continuous at $a \in \mathbb{R}$" for a single, fixed element $a \in \mathbb{R}$. E.g., if f and g are both continuous at 5, then $f+g$ is continuous at 5.

In order to apply e.g. the Intermediate Value Theorem we can often use:

* Restriction rule: If $f: D \to \mathbb{R}$ is continuous and $D' \subset D$, then the restriction

$$f|_{D'} : D' \to \mathbb{R} \text{ is a continuous function.}$$

Further rules for continuous functions:

* Composition rule: If $f: D \to \mathbb{R}$ and $g: D' \to \mathbb{R}$ are continuous and if the composition $f \circ g$ is defined (D contains the range of g), then $f \circ g$ is a continuous function.

* Quotient rule: If $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ are continuous, then the quotient $\frac{f}{g}$ is a continuous function on its (natural) domain $\{x \in D \mid g(x) \neq 0\}$.

Examples

* The functions $\sin: \mathbb{R} \to \mathbb{R}$ and $\cos: \mathbb{R} \to \mathbb{R}$ are continuous.
  It follows that $\sin(x^3 - 2x + 1)$ is also continuous by the composition rule.

* The rational function $f(x) = \dfrac{2x^2+3}{x^3-1}$ is continuous on its natural domain, which is $\mathbb{R} \setminus \{1\}$.
  This follows from the quotient rule and the fact that the polynomials $2x^2+3$ and x^3-1 are continuous.

Remarks

* The restriction rule still holds if we replace "continuity" by "continuity at a" for a fixed $a \in D'$.
  Similarly for the quotient rule, with a in the natural domain.

* The composition rule still holds if g is continuous at $a \in D'$, f is continuous at $g(a) \in D$ and we conclude that $f \circ g$ is continuous at a.

* The quotient rule can be derived from the restriction, composition and product rules:

  Write D' for the (natural) domain of $\frac{f}{g}$ and let $h(x) = \frac{1}{x}$, then

  $\frac{f}{g} = f|_{D'} \cdot (h \circ g|_{D'})$ and the restrictions, composition and product

  preserve continuity.

* All rational functions are continuous on their natural domain, using a similar argument as in the example.

<u>Example</u> Show that $f(x) = \dfrac{x^3+2x^2-x-1}{x^2-4}$ has a root in the interval $[0,1]$.

Solution: We want to use the Intermediate Value Theorem again.

The denominator of $f(x)$ is 0 when $x^2-4=0$, i.e. $x=-2$ or $x=2$.

The natural domain of $f(x)$ is $\mathbb{R}\setminus\{-2,2\}$, so it contains $[0,1]$.

$f(x)$ is a rational function, so it is continuous on its domain and hence $f|_{[0,1]}$ is continuous.

$f(0) = \frac{-1}{-4} = \frac{1}{4}$ and $f(1) = \dfrac{1+2-1-1}{1-4} = \frac{1}{-3} = \frac{-1}{3}$.

$y=0$ lies between $\frac{1}{4}$ and $\frac{-1}{3}$, so by the Intermediate Value Theorem

there is an $x \in [0,1]$ with $f(x)=0$, i.e. f has a root in the interval $[0,1]$.