Lecture 4

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1. Operations on Functions 1

There are many ways to combine functions into new ones. Here are three of them:

Let D be any set, $f: D \to R$ and $g: D \to R$ functions with the same domain D and $c \in R$, We define the functions $f + g$, $f . g$ and $c. f$ as follows:

The sum $f + g: D \to R$ is given by $(f + g)(x) = f(x) + g(x)$,
The sum $f . g: D \to R$ is given by $(f . g)(x) = f(x). g(x)$,
The sum $c. g: D \to R$ is given by $(c. g)(x) = c. g(x)$,

Remark: For any $x \in D$, the right-hand side is a sum or product of real numbers, but f and g are not numbers, so $f + g$, $f . g$ and $c. f$ are operations on functions that we defined in terms of their function values at every $x \in D$.

Examples

For $f(x) = x - 3$ and $g(x) = 2x^2 + 1$ (with domain R) and $c = \frac{-1}{2}$:

$$(f + g)(x) = f(x) + g(x) = x - 3 + 2x^2 + 1 = 2x^2 + x - 2$$
$$(f . g)(x) = f(x). g(x) = (x - 3). (2x^2 + 1) = 2x^3 - 6x^2 + x - 3$$
$$(\frac{-1}{2} . f)(x) = \frac{-1}{2} . f(x) = \frac{-1}{2} . (x - 3) = \frac{-1}{2} . x + 1. \frac{1}{2}$$

We have seen that all linear functions are determined by their slope m and intecept b. We can now write this as follows.

Example

Consider the identity function $id: R \to R: x \to x$
and the constant function $1_{c:} R \to R: x \to 1$

For any $m \in R$ and $b \in R$:

$$\begin{aligned}(m. id + b. 1_c)(x) &= (m. id)(x) + (b. 1_c)(x) \\ &= m. id(x) + b. 1_c(x) \\ &= m. x + b. 1_c\end{aligned}$$

so the linear function with slope m and intercept b can be written as:

$$m. id + b. 1_c$$

(In practice this is rarely the most convenient name for this function.)

For any function $f: D \to R$ we often write f^2 intead of $f.\ f$, f^3 instead of $f.\ f.\ f$, etc.

2. Polynomials

A polynomial is a function $f: R \to R$ whose formula is of the form

$$f(x) = a_n.\, x^n + a_{n-1}.\, x^{n-1} + \cdots + a_2.\, x^2 + a_1.\, x + a_0$$

for some $n \in N$ (or $n = 0$) and real coefficients a_n, a_{n-1}, ..., a_2, a_1, a_0.

Examples

* $f(x) = 7x^3 - 2x^2 - 4x + 11$
* $g(x) = 3 - \pi x^{38}$
* $h(x) = \sqrt{2}x^2 + \frac{1}{2}x^4 - 1$ are polynomials

Every linear function $f(x) = m.\, x + b$ is a polynomial with $n = 1$, $a_1 = m$ and $a_0 = b$

Every quadratic function $f(x) = ax^2 + bx + c$ is a polynomial with $n = 2$, $a_2 = a$, $a_1 = b$ and $a_0 = c$

Every monomial $f(x) = a.\, x^n$ is a polynomial with$a_n = a$ and $a_{n-1} = \cdots = a_2 = a_1 = a_0 = 0$, so only one term is left

The **degree** of a polynomial is the largest numbers $n \in N \cup \{°\}$ with $a_n \neq 0$

A polynomial of degree 0 is a constant function

A polynomial of degree 1 is a linear function with slope $m \neq 0$

A polynomial of degree 2 is a quadratic function

A polynomial of degree 3 is a cubic function

A polynomial of degree 4 is a quartic function

Example

Using $id(x) = x$ and $1_c(x) = x$ we can build any polynomial

$$f(x) = a_n.\, x^n + \cdots + a_1.\, x + a_0$$

by setting

$$f = a_n.\, id^n + \cdots + a_1.\, id + a_0.\, 1_c$$

in terms of operations on functions. E.g. we have

$$id^2(x) = (id.\, id)(x) = id(x).\, id(x) = x.\, x = x^2$$

$$id^3(x) = (id.\, id^2)(x) = id(x).\, id^2(x) = x.\, x^2 = x^3$$

etc, with the general formula

$$id^2(x) = x^n$$

for all $x \in R$, (you can prove this by mathematical induction)