# Lecture 6

### 1. Interpolating polynomials

Sometimes we want to find a function that takes some prescribed values at certain given points. Polynomials help us to find such a function, which has no sharp corners.

**Theorem**

For $n \in N$, let $x_1, x_2, \ldots, x_n$ be distinct real numbers and $y_1, \ldots, y_n$ any real numbers.

Then there is a unique polynomial $p(n)$ of degree $\leq n-1$ such that $p(x_i) = y_i$ for all $i \in \{1, \ldots, n\}$

***Remark:*** If the $x_i$'s are not distinct, e.g. $x_1 = x_2$, then there may not be any function with $f(x_1) = y_1$ and $f(x_2) = y_2$, because we may have $y_1 \neq y_2$. (if. "vertical line test")

<u>**Examples**</u>

$n = 1$: Given $x_1 \in R$ and $y_1 \in R$ we can find a contrast function $p(x)$ with $p(x_1) = y_1$,

namely $p(x) = y_1$ for all $x \in R$

$n = 2$: Given $x_1, x_2, y_1, y_2 \in R$ with $x_1 \neq x_2$ we can find a linear function $p(x)$ with $p(x_1) = y_1$ and $p(x_2) = y_2$, Writing

$$p(x) = m.x + b$$

we find

$$m = \frac{y_2 - y_1}{x_2 - x_1} \text{ and } b = \frac{y_1.x_2 - y_2.x_1}{x_2 - x_1}$$

$n = 2$,

$\begin{cases} y_2 = 0 \\ y_1 = 1 \end{cases}$: The formula simplifies to

$$p(x) = \frac{x - x_2}{x_1 - x_2}$$

with $p(x_1) = 1$ and $p(x_2)$

$n \geq 2$: $y_1 = 1$, $y_2 = y_3 = \cdots = y_n = 0$: Taking products of the previous case we can take

$$p(x) = \frac{x - x_2}{x_1 - x_2} \cdot \frac{x - x_3}{x_1 - x_3} \cdots \frac{x - x_n}{x_1 - x_n}$$

Here each factor is a linear function. There are $n-1$ factors, so $p(x)$ has degree $= n-1$.
each factor has value 1 at $x_1$, so $p(x_1) = 1$,
For $x_i$ with $i \geq 2$ there is a f actor with value 0, so $p(x_i) = 0$ if $i \geq 2$.

The polynomial in the last example is called a Lagrange basis polynomial.
Given $n$ distinct numbers $x_1, \ldots, x_n$
there are $n$ such Lagrange basis polynomials, which vanish at all $x_i$'s except for one, where their value is 1.

**Proof of theorem on interpolating polynomials:**

**Uniqueness:** Suppose that $p(x)$ and $q(x)$ are polynomials of degree $\leq n-1$ with $p(x_i) = y_i$ and $q(x_i) = y_i$ for all $i \in \{1, \cdots, n\}$. Then $p - q$ is a polynomial of degree $\leq n-1$ with $(p-q)(x_i) = p(x_i) - q(x_i) = 0$ for all $i \in \{1, \cdots, n\}$ so $p - q$ has $n$ roots. This can only happen when $(p-q)(x) = 0$ for all $x \in R$, i.e. $p(x) = q(x)$, so $p$ and $q$ are the same function.

**Existence:** First consider the Lagrange basis polynomials $p_1(x), \ldots, p_n(x)$ with

$$p_1(x) = \frac{x - x_2}{x_1 - x_2} \cdot \frac{x - x_3}{x_1 - x_3} \cdots \frac{x - x_n}{x_1 - x_n} \qquad p_1(x_1) = 1,\ p(x_2) = \cdots = p(x_n) = 0$$

$$p_2(x) = \frac{x - x_1}{x_2 - x_1} \cdot \frac{x - x_3}{x_2 - x_3} \cdots \frac{x - x_n}{x_2 - x_n} \qquad p_2(x_2) = 1,\ p(x_1) = p(x_3) = \cdots = p(x_n) = 0$$

$$p_n(x) = \frac{x - x_1}{x_n - x_1} \cdot \frac{x - x_2}{x_n - x_2} \cdots \frac{x - x_{n-1}}{x_n - x_{n-1}} \qquad p_n(x_n) = 1,\ p(x_1) = \cdots = p(x_{n-1}) = 0$$

We can take $p(x) = y_1.\, p_1(x) + y_2.\, p_2(x) + \cdots + y_n.\, p_n(x)$, which has degree $\leq 1$
and $p(x_i) = y_i$ for *all* $i \in \{1, \ldots, n\}$.

**<u>Example</u>**

Find the formulas for the unique quadratic polynomial through (1,–3), (2, 2) and (3,–1).

***Solution:*** We find first find the Langrange basis polynomials for the points 1, 2, 3:

$$p_1(x) = \frac{x-2}{1-2} \cdot \frac{x-3}{1-3} = \frac{1}{2}(x^2) - 2\frac{1}{2}x + 3,$$

$$p_2(x) = \frac{x-1}{2-1} \cdot \frac{x-3}{2-3} = -(x-1)(x-3) = -x^2 + 4x - 3,$$

$$p_3(x) = \frac{x-1}{3-1} \cdot \frac{x-2}{3-2} = \frac{1}{2}(x-1) = \frac{1}{2}x^2 - 1\frac{1}{2}x + 1,$$

We have

$$p(x) = -3.\, p_1(x) + 2.\, p_2(x) - 1.\, p_3(x)$$

$$= -3.\,(\frac{1}{2}x^2 - 2\frac{1}{2}x + 3) + 2(-x^2 + 4x - 3) - (\frac{1}{2}x^2 - 1\frac{1}{2}(x) + 1)$$

$$= -1\frac{1}{2}x^2 + 7\frac{1}{2}x - 9 - 2x^2 + 8x - 6 - \frac{1}{2}x^2 + 1\frac{1}{2}x - 1$$

$$= -4x^2 + 17x - 16$$

Check

$p(1) = -4 + 17 - 16 = -3$

$p(2) = -4.\,4 + 17.\,2 - 16 = -16 + 34 - 16 = 2$

$p(3) = -4.\,9 + 17.\,3 - 16 = -36 + 51 - 16 = -1$

($p(x)$ has a maximum at $2\,\frac{1}{8}$, where $p(2\,\frac{1}{8}) = 2\,\frac{1}{16}$ the roots of $p$ are at $2\,\frac{1}{8} - \frac{\sqrt{33}}{8}$ and $2\,\frac{1}{8} + \frac{\sqrt{33}}{8}$)