Lecture 4

Operations on Functions I

There are many ways to combine functions into new ones. Here are three of them:

Let D be any set, $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$ functions with the same domain D and $c \in \mathbb{R}$. We define the functions $f+g$, $f \cdot g$ and $c \cdot f$ as follows:

The sum $f+g: D \to \mathbb{R}$ is given by $(f+g)(x) = f(x) + g(x)$,

the product $f \cdot g: D \to \mathbb{R}$ is given by $(f \cdot g)(x) = f(x) \cdot g(x)$,

the multiple $c \cdot f: D \to \mathbb{R}$ is given by $(c \cdot f)(x) = c \cdot f(x)$.

Remark: For any $x \in D$, the right-hand side is a sum or product of real numbers, but f and g are not numbers, so $f+g$, $f \cdot g$ and $c \cdot f$ are operations on functions that we defined in terms of their function values at every $x \in D$.

Examples For $f(x) = x - 3$ and $g(x) = 2x^2 + 1$ (with domain $\mathbb{R}$) and $c = -\frac{1}{2}$:

$$(f+g)(x) = f(x) + g(x) = x - 3 + 2x^2 + 1 = 2x^2 + x - 2,$$
$$(f \cdot g)(x) = f(x) \cdot g(x) = (x-3) \cdot (2x^2+1) = 2x^3 - 6x^2 + x - 3,$$
$$\left(\tfrac{-1}{2} f\right)(x) = \tfrac{-1}{2} \cdot f(x) = \tfrac{-1}{2} \cdot (x-3) = \tfrac{-1}{2} x + 1\tfrac{1}{2}.$$

We have seen that all linear functions are determined by their slope m and intercept b. We can now write this as follows.

Example Consider the identity function $id : \mathbb{R} \to \mathbb{R} : x \mapsto x$
and the constant function $\underline{1} : \mathbb{R} \to \mathbb{R} : x \mapsto 1$.

For any $m \in \mathbb{R}$ and $b \in \mathbb{R}$:

$$\begin{aligned}(m \cdot id + b \cdot \underline{1})(x) &= (m \cdot id)(x) + (b \cdot \underline{1})(x) && \text{(sum of functions)}\\ &= m \cdot id(x) + b \cdot \underline{1}(x) && \text{(multiples of functions)}\\ &= m \cdot x + b \cdot 1 && \text{(evaluate the functions)}\\ &= mx + b,\end{aligned}$$

so the linear function with slope m and intercept b can be written as
$m \cdot id + b \cdot \underline{1}$.

(In practice this is rarely the most convenient name for this function.)

For any function $f: D \to \mathbb{R}$ we often write f^2 instead of $f \cdot f$, f^3 instead of $f \cdot f \cdot f$, etc.

Polynomials

A polynomial is a function $f: \mathbb{R} \to \mathbb{R}$ whose formula is of the form

$$f(x) = a_n \cdot x^n + a_{n-1} \cdot x^{n-1} + \dots + a_2 \cdot x^2 + a_1 \cdot x + a_0$$

for some $n \in \mathbb{N}$ (or $n=0$) and real coefficients $a_n, a_{n-1}, \dots, a_2, a_1, a_0$.

Examples

* $f(x) = 7x^3 - 2x^2 - 4x + 11$,
  $g(x) = 3 - \pi x^{38}$,
  $h(x) = \sqrt{2}\, x^2 + \frac{1}{2} x^4 - 1$ are polynomials.
* Every linear function
  $$f(x) = m \cdot x + b$$
  is a polynomial with $n = 1$, $a_1 = m$ and $a_0 = b$.
* Every quadratic function
  $$f(x) = ax^2 + bx + c$$
  is a polynomial with $n = 2$, $a_2 = a$, $a_1 = b$ and $a_0 = c$.
* Every monomial
  $$f(x) = a \cdot x^n$$
  is a polynomial with $a_n = a$ and $a_{n-1} = \dots = a_2 = a_1 = a_0 = 0$, so only one term is left.

The degree of a polynomial is the largest number $n \in \mathbb{N} \cup \{0\}$ with $a_n \neq 0$.

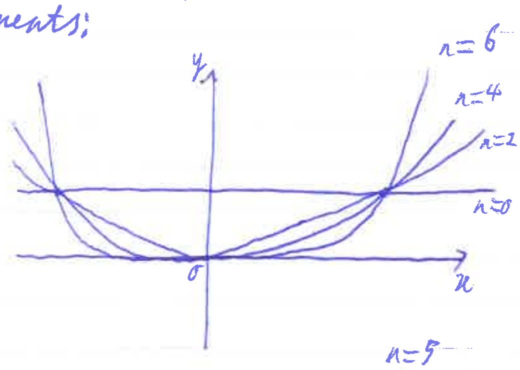
A polynomial of degree 0 is a constant function.
" " " " 1 " " linear function with slope $m \neq 0$.
" " " " 2 " " quadratic function.
" " " " 3 " " cubic function.
" " " " 4 " " quartic function.

Example Using $id(x) = x$ and $1(x) = x$ we can build any polynomial
$$f(x) = a_n \cdot x^n + \dots + a_1 \cdot x + a_0$$
by setting
$$f = a_n \cdot id^n + \dots + a_1 \cdot id + a_0 \cdot 1$$
in terms of operations on functions. E.g., we have
$$id^2(x) = (id \cdot id)(x) = id(x) \cdot id(x) = x \cdot x = x^2,$$
$$id^3(x) = (id \cdot id^2)(x) = id(x) \cdot id^2(x) = x \cdot x^2 = x^3,$$
etc., with the general formula
$$id^n(x) = x^n,$$
for all $x \in \mathbb{R}$. (You can prove this by mathematical induction.)

The graphs of polynomials can get quite complicated if the degree gets large. For monomials we can still make some useful comments:

$f(x) = x^n$ with n even:

this is an even function, $f(-x) = f(x)$,
for $n=0$ we get the constant function 1,
as n increases:
the graphs get flatter near $x=0$, and
" " " steeper for large x.



$f(x) = x^n$ with n odd:

this is an odd function, $f(-x) = -f(x)$,
for $n=1$ we get the identity function id,
as n increases:
the graphs get flatter near $x=0$, and
" " " steeper for large x.