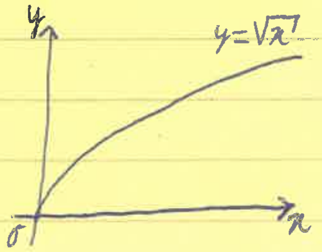
MS135

Lecture 12

Inverse Functions



The graph of the function

$$f : [0,\infty) \to [0,\infty)$$
$$x \mapsto \sqrt{x}$$

shows that it is continuous. However, this does not follow from any of the rules we have seen so far.

Warning: $(f \cdot f)(x) = f(x) \cdot f(x) = \sqrt{x} \cdot \sqrt{x} = x$, so $f \cdot f$ is continuous. However, it does not follow from the product rule that f is continuous.

Note that f is invertible and $f^{-1} : [0,\infty) \to [0,\infty)$, $y \mapsto y^2$ is continuous.

We will show that this gives another way to explain the continuity of f.

Theorem (Continuity of Inverse Functions)

Let $f : I \to J$ be a continuous function, whose domain $I \subset \mathbb{R}$ is an interval and whose range is the set $J \subset \mathbb{R}$. If f is invertible, then f^{-1} is also continuous.

Example For any $n \in \mathbb{N}$ consider $f : [0,\infty) \to [0,\infty)$, $x \mapsto x^n$, which is continuous and invertible. The inverse function $f^{-1} : [0,\infty) \to [0,\infty)$, $y \mapsto \sqrt[n]{y} = y^{\frac{1}{n}}$ is therefore also continuous.

When n is odd we similarly get that $f^{-1}(y) = \sqrt[n]{y}$ is continuous on the domain $\mathbb{R}$.

Remarks: Under the assumptions of the theorem one can also show that

* J is an interval (even if f were not invertible),
* f is either (strictly) increasing or decreasing and
* f^{-1} is also (strictly) increasing or decreasing.

Example Let r be a positive rational number, $r=\frac{m}{n}$ with $m\in\mathbb{N}$ and $n\in\mathbb{N}$, e.g. $r=\frac{7}{8}$.

The function $f(x)=\sqrt[n]{x^m}$ with domain $[0,\infty)$ is continuous, because it is a composition of two continuous functions.
We will write

$$\sqrt[n]{x^m}=(x^m)^{\frac{1}{n}}=x^{\frac{m}{n}}=x^r\ ,\ \text{e.g. } x^{\frac{7}{8}}=\sqrt[8]{x^7}.$$

Exponential Functions

We want a function that we write as $f(x)=2^x$ with the following properties:
1) When $x\in\mathbb{N}$, $2^x=2\cdot2\cdot\ldots\cdot2$ (x times)
2) for all $x\in\mathbb{R}$ and $y\in\mathbb{R}$, $2^x\cdot2^y=2^{x+y}$
3) for all $x\in\mathbb{R}$ and $y\in\mathbb{R}$, $(2^x)^y=2^{x\cdot y}$

This requires in particular: $2^0=1$, $2^{\frac{1}{n}}=\sqrt[n]{2}$, $2^{\frac{m}{n}}=\sqrt[n]{2^m}$ and $2^{-\frac{m}{n}}=\frac{1}{2^{\frac{m}{n}}}$ for all $m\in\mathbb{N}$ and $n\in\mathbb{N}$. This fixes the value 2^x for all rational numbers $x\in\mathbb{Q}$.

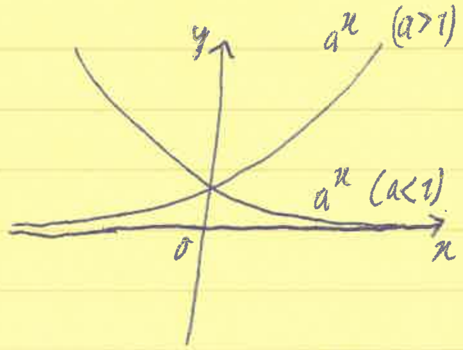
FACT: There is a unique continuous function $f(x)=2^x$ with domain $\mathbb{R}$ and all the properties given above.

More generally, the <u>exponential function</u> with <u>base</u> $a>0$ $(a\in\mathbb{R})$ is the unique continuous function

$$f:\mathbb{R}\to\mathbb{R}:\ x\mapsto a^x$$

with the properties
1) $a^n=a\cdot\ldots\cdot a$ (n times) if $n\in\mathbb{N}$
2) $a^x\cdot a^y=a^{x+y}$ if $x\in\mathbb{R}, y\in\mathbb{R}$
3) $(a^x)^y=a^{x\cdot y}$ if $x\in\mathbb{R}, y\in\mathbb{R}$.

The graph of an exponential function grows very rapidly when $a>1$, decreases very rapidly when $a<1$.



When $a=1$, $a^x=1$ for all $x\in\mathbb{R}$.

Remarks

* $2^x \neq x^2$! An exponential function has the variable (x) in the exponent!
  ↳ except at very special values of x, like $x=2$ or $x=4$.

* $a^x > 0$ for any $a>0$ and any $x \in \mathbb{R}$.

* Note that $a^{\frac{1}{2}} = \sqrt{a}$, $a^0 = 1$, $a^{-1} = \frac{1}{a}$ for any $a>0$.
  We cannot take $a=0$, because $a^{-1} = \frac{1}{a}$ would not make sense.
  We cannot take $a<0$, because $a^{\frac{1}{2}} = \sqrt{a}$ would not make sense.

Exponential functions are often used in science, e.g. to describe the spread of viruses, the growth of populations of plants or animals, or the expansion of the universe shortly after the big bang.